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Chapter 2

Tropical limits of decimated polynomials¹

2.1 Introduction

Our reader at some point might have encountered the following mathematical puzzle:

Is it possible to tile the 8×8 chessboard without two opposite corners with 2×1 dominoes?

A simple parity argument shows that it is, indeed, not possible. A more difficult question is in how many ways we can tile the usual 8×8 , and, more generally, $2n \times 2n$ chessboard with the 2×1 dominos?

In 1961, a Dutch physicist Piet Kasteleyn found complete solutions of several ‘arrangement problems’ of such nature [53]. In particular, he showed that the number of domino tilings of a chessboard of a size $2n \times 2n$ is given by

$$Z_n = \prod_{m=0}^{n-1} \prod_{k=0}^{n-1} \left(4 - 2 \cos \left(\frac{2m+1}{2n+1} \pi \right) - 2 \cos \left(\frac{2k+1}{2n+1} \pi \right) \right). \quad (2.1)$$

There is a one-to-one correspondence between the domino tilings and dimer configurations, or, in other words, the perfect matchings of a corre-

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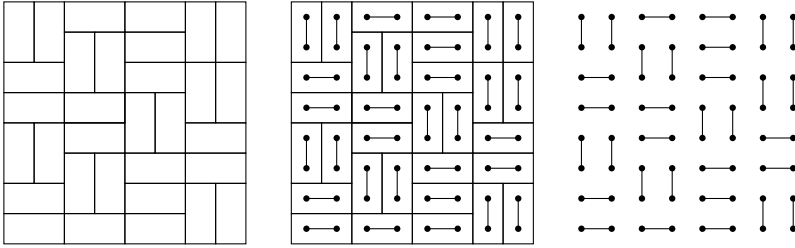


Figure 2.1: Correspondence between the domino tilings and the dimer matchings on boxes $2n \times 2n$.

sponding graph, as Figure 2.1 demonstrates. A *perfect matching* of a graph is a subset of its edges such that every vertex of the graph is incident to exactly one edge of the subset.

The method developed by Kasteleyn is not only applicable to counting the number of domino tilings, or equivalently, dimer configurations, but can also be used to compute the weighted sum of the form

$$Z_n = \sum_{M \in \mathcal{M}(G_{2n \times 2n})} w(M), \quad (2.2)$$

where $\mathcal{M}(G_{2n \times 2n})$ is a collection of all dimer configurations of the square box of size $2n \times 2n$. For any dimer configuration (matching) M , its weight is given by

$$w(M) = \prod_{e \in M} w(e), \quad w(e) = \begin{cases} u, & \text{for horizontal edges,} \\ v & \text{for vertical edges,} \end{cases}, \quad u, v > 0.$$

Obtaining explicit expressions for the *partition function* Z_n (2.2) is important in Statistical Physics, as it allows the computation of the *free energy* $F(u, v)$ given by

$$F(u, v) = - \lim_{n \rightarrow \infty} \frac{1}{n^2} \log Z_n, \quad (2.3)$$

and, by analysing the free energy, one is able to determine some important macroscopic properties of the systems in the thermodynamic limit. In particular, the singularities of the free energy function indicate the presence of the phase transitions.

It turns out [22, 55], that it is easier to compute a weighted partition function of the form (2.2) if we embed the $2n \times 2n$ square grid on a torus (see Fig. 2.2).

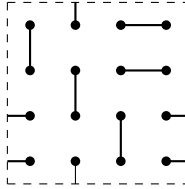


Figure 2.2: A dimer configuration (matching) on the 4×4 box on a torus.

The partition function $Z_n^{\mathbb{T}}$ for weighted dimer matchings on a torus possesses an elegant explicit expression:

$$Z_n^{\mathbb{T}} = \frac{1}{2} (-f_n(1, 1) + f_n(-1, 1) + f_n(1, -1) + f_n(-1, -1)), \quad (2.4)$$

where, for every integer $n \geq 1$,

$$f_n(x, y) = \prod_{m=0}^{n-1} \prod_{k=0}^{n-1} f(e^{2\pi i \frac{m}{n}} x, e^{2\pi i \frac{k}{n}} y). \quad (2.5)$$

with f being a Laurent polynomial in two variables x, y , namely,

$$f(x, y) = 4(u^2 + v^2) - u^2(x + x^{-1}) - v^2(y + y^{-1}).$$

The limiting free energy F does not depend on whether we consider the partition function Z_n or $Z_n^{\mathbb{T}}$. In both cases,

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \log Z_n = \lim_{n \rightarrow \infty} \frac{1}{n^2} \log Z_n^{\mathbb{T}} = \int_0^1 \int_0^1 \log |f(e^{2\pi i \theta_1}, e^{2\pi i \theta_2})| d\theta_1 d\theta_2. \quad (2.6)$$

More generally, expressions of a form (2.4) and (2.6) hold for all planar simple bipartite \mathbb{Z}^2 -periodic graphs and some associated polynomials f . For example, for the honeycomb lattice, $f(x, y) = a + bx + cy$, where a, b, c are the weights of the horizontal, north-east, and south-east edges, respectively.

Kenyon, Okounkov, and Sheffield (see [55]) obtained a beautiful limiting shape result for the coefficients of $f_n(x, y)$ in (2.5) for the special ‘dimer’ polynomials f . In the present note we discuss the generalization of the result of Kenyon, Okounkov, and Sheffield to arbitrary Laurent polynomials in d variables, where $d \geq 2$.

2.2 Decimation of polynomials

A Laurent polynomial in d commuting variables z_1, \dots, z_d , can be presented as a sum

$$f(\mathbf{z}) = \sum_{\mathbf{m} \in \mathbb{Z}^d} f^{\mathbf{m}} \mathbf{z}^{\mathbf{m}},$$

where we use the multi-index notation $\mathbf{m} = (m_1, \dots, m_d)$. The sum has a finite number of terms: there are only finitely many \mathbf{m} 's with $f^{\mathbf{m}} \neq 0$. The multi-indices $\mathbf{m} \in \mathbb{Z}^d$ with $f^{\mathbf{m}} \neq 0$ are called the *exponent vectors*. The set of all exponent vectors is called *the support* of f and is denoted by $\text{supp}(f)$.

Now, fix an arbitrary integer $n \geq 1$. Then, the n -th *decimation* of a Laurent polynomial $f(z_1, \dots, z_d)$ is defined as

$$f_n(z_1, \dots, z_d) = \prod_{k_1=1}^n \dots \prod_{k_d=1}^n f\left(e^{2\pi i k_1/n} z_1, \dots, e^{2\pi i k_d/n} z_d\right). \quad (2.7)$$

Our interest in these polynomials arose when studying decimations (renormalization transformation) of the so-called principle algebraic actions – a natural class of algebraic dynamics, see [1] for more details. Such polynomials have been considered earlier by Purbhoo [92] who studied approximations of amoebas. In the present paper, we will discuss properties of the decimated polynomials.

For every n , a decimated polynomial f_n is again a Laurent polynomial

$$f_n(\mathbf{z}) = \sum_{\mathbf{m} \in \mathbb{Z}^d} f_n^{\mathbf{m}} \mathbf{z}^{\mathbf{m}}.$$

Moreover, the resulting exponent vectors of f_n are entry-wise divisible by n . In other words, f_n is a Laurent polynomial in z_1^n, \dots, z_d^n .

Example 2.2.1. Consider a polynomial $f(x, y) = 1 - x - y$. Then, the first three decimations are (see Fig. 2.3):

1. $f_1(x, y) = 1 - x - y$;
2. $f_2(x, y) = 1 - 2x^2 - 2y^2 - 2x^2y^2 + y^4 + x^4$;
3. $f_3(x, y) = 1 - 3x^3 - 3y^3 + 3x^6 + 3y^6 - 3x^6y^3 - 3x^3y^6 - 21x^3y^3 - y^9 - x^9$.

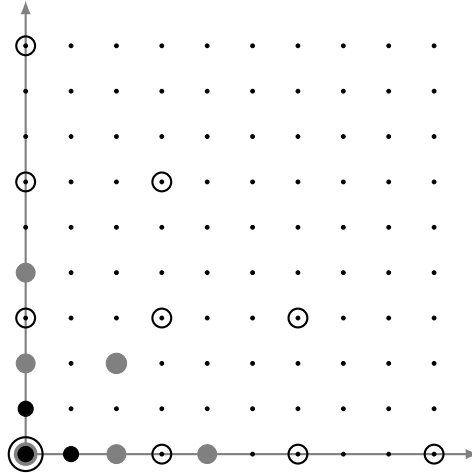


Figure 2.3: Black circles correspond to the exponent vectors of $f = f_1$, namely, $(0, 0)$, $(1, 0)$, and $(0, 1)$; the exponent vectors of f_2 and f_3 are denoted by the grey and the white circles, respectively.

We remind the reader that the *Newton polytope* $\mathcal{N}(f)$ of a Laurent polynomial $f(z_1, \dots, z_d)$ is a subset of \mathbb{R}^d which is a convex hull of the exponent vectors of $f(z_1, \dots, z_d)$. Note that the Newton polytopes of f_n and of f satisfy the following relation:

$$\mathcal{N}(f_n) = n^d \mathcal{N}(f).$$

Therefore, when n increases, the Newton polytope of f_n grows, and so do the absolute values of the non-zero coefficients $f_n^{\mathbf{m}}$. In fact, their growth rate is exponential in n . The natural question is whether there is a scaling limit of the coefficients of f_n . Namely, whether the limits

$$\lim_{n \rightarrow \infty} \frac{1}{n^d} \log |f_n^{\mathbf{m}_n}|, \quad (2.8)$$

exist for sequences $\mathbf{m}_n \in \text{supp}(f_n)$ such that $\frac{\mathbf{m}_n}{n^d} \rightarrow \mathbf{u} \in \mathcal{N}(f)$.

2.3 The scaling limit

In tropical algebra, the standard addition and multiplication of real numbers are redefined as follows:

- tropical addition: $a \oplus b = \max\{a, b\}$;

- tropical multiplication: $a \odot b = a + b$.

Hence, $2 \oplus 5 = 5$ and $2 \odot 5 = 7$. The tropical operations allow to define tropical polynomials. For example, consider $f(x, y) = 2x^2 - 4x^2y + y$; its tropical analogue is then

$$F(x, y) = (2 \odot x \odot x) \oplus (-4 \odot x \odot x \odot y) \oplus (1 \odot y).$$

Using the tropical operations, defined above, one can easily evaluate F at any $(x, y) \in \mathbb{R}^2$

$$F(x, y) = \max\{2 + 2x, -4 + 2x + y, y\}.$$

Tropical geometry incorporates many facets of algebraic geometry and convex analysis [74].

Let us consider a Laurent polynomial $f(z) = \sum_m f^m z^m$. The *tropicalization* of $f(z)$, denoted by $\text{trop}(f)(\mathbf{t})$, is a function on \mathbb{R}^d defined as follows: for any $\mathbf{t} = (t_1, \dots, t_d) \in \mathbb{R}^d$, take

$$\begin{aligned} \text{trop}(f)(\mathbf{t}) &= \bigoplus_{m \in \text{supp}(f)} \log |f^m| \odot \mathbf{t}^m \\ &= \max_{m \in \text{supp}(f)} \left(\log |f^m| + m_1 t_1 + \dots + m_d t_d \right) \\ &= \max_{m \in \text{supp}(f)} \left(\log |f^m| + \langle m, \mathbf{t} \rangle \right), \end{aligned} \tag{2.9}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product on \mathbb{R}^d .

Tropicalization of f is thus a tropical analogue of the Laurent polynomial $\sum_m \log |f^m| z^m$.

The *tropical variety* of $f(z)$ is the set of all points $\mathbf{t} \in \mathbb{R}^d$ such that the maximum in (2.9) is achieved at at least two monomials. Therefore, the tropicalization of f is a piecewise affine convex function on \mathbb{R}^d ; each component of the complement of the tropical variety defines a domain where a certain monomial of f is maximal, c.f. (2.9). Figure 2.4 shows the tropical varieties of the first 4 decimations of a polynomial $1 + x + y$.

In a joint work with Doug Lind and Klaus Schmidt [1] (see Chapter 3), we established the following result on the existence of scaling limits of tropicalizations of the decimated polynomials f_n .

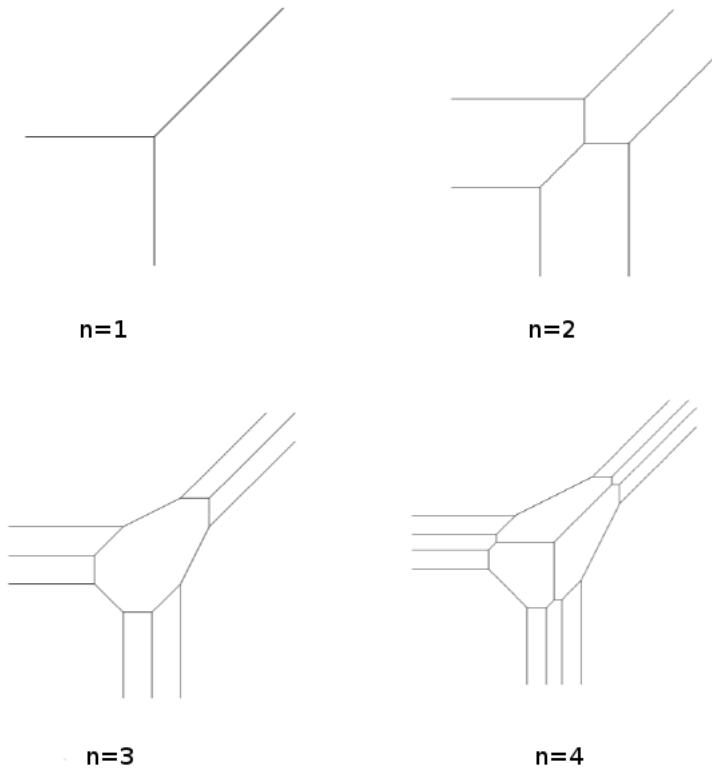


Figure 2.4: Tropical varieties of $\text{trop}(f_n)$ for $f = 1 + x + y$ and $n = 1, 2, 3, 4$.

Theorem 2.3.1. For every non-zero Laurent polynomial $f(z)$ and all $\mathbf{t} \in \mathbb{R}^d$, there exists a limit

$$\lim_{n \rightarrow \infty} \frac{1}{n^d} \text{trop}(f_n)(\mathbf{t}) = R_f(\mathbf{t}), \quad (2.10)$$

where $R_f : \mathbb{R}^d \rightarrow \mathbb{R}$ is the Ronkin function of f , given by

$$\begin{aligned} R_f(\mathbf{t}) &= \int_0^1 \cdots \int_0^1 \log |f(e^{t_1+2\pi i\theta_1}, \dots, e^{t_d+2\pi i\theta_d})| d\theta_1 \dots d\theta_d \\ &= \int_{\mathbf{t} \in \mathbb{T}^d} \log |f(e^{\mathbf{t}+2\pi i\theta})| d\theta. \end{aligned} \quad (2.11)$$

Sketch of the proof. (The full proof can be found in [1], Chapter 3.) The first observation relates the Ronkin functions of f and f_n ; namely, $R_{f_n}(\mathbf{t}) = n^d R_f(\mathbf{t})$. Hence, it suffices to compare $\text{trop}(f_n)$ and R_f .

The Ronkin function can be easily bounded from above: indeed, let us denote by $\#\text{supp}(f_n)$ the number of integer points inside the support of f_n . Then,

$$\begin{aligned}
R_{f_n}(\mathbf{t}) &= \int_{\mathbb{T}^d} \log \left| \sum_{\mathbf{m}} f_n^{\mathbf{m}} e^{\langle \mathbf{t}, \mathbf{m} \rangle} e^{2\pi i \langle \mathbf{m}, \mathbf{t} \mathbf{h} \rangle} \right| d\mathbf{t} \mathbf{h} \\
&\leq \int_{\mathbb{T}^d} \log \sum_{\mathbf{m} \in \text{supp}(f_n)} \left| f_n^{\mathbf{m}} e^{\langle \mathbf{t}, \mathbf{m} \rangle} e^{2\pi i \langle \mathbf{m}, \mathbf{t} \mathbf{h} \rangle} \right| d\mathbf{t} \mathbf{h} \\
&= \int_{\mathbb{T}^d} \log \sum_{\mathbf{m} \in \text{supp}(f_n)} \left| f_n^{\mathbf{m}} e^{\langle \mathbf{t}, \mathbf{m} \rangle} \right| \\
&\leq \log \left(\#\text{supp}(f_n) \max_{\mathbf{m} \in \text{supp}(f_n)} \exp(\log |f_n^{\mathbf{m}}| + \langle \mathbf{t}, \mathbf{m} \rangle) \right) \\
&= \log \#\text{supp}(f_n) + \max_{\mathbf{m} \in \text{supp}(f_n)} (\log |f_n^{\mathbf{m}}| + \langle \mathbf{t}, \mathbf{m} \rangle) \\
&= \log \#\text{supp}(f_n) + \text{trop}(f_n)(\mathbf{t}).
\end{aligned}$$

Hence, for all $n \in \mathbb{Z}$ and $\mathbf{t} \in \mathbb{R}^d$ one has

$$R_f(\mathbf{t}) = \frac{1}{n^d} R_{f_n}(\mathbf{t}) \leq \frac{1}{n^d} \log \#\text{supp}(f_n) + \frac{1}{n^d} \text{trop}(f_n)(\mathbf{t}).$$

Since the cardinality of the support of f_n grows at most as $\text{const} \cdot n^d$, we immediately conclude that

$$R_f(\mathbf{t}) \leq \liminf_{n \rightarrow \infty} \frac{1}{n^d} \text{trop}(f_n)(\mathbf{t}). \quad (2.12)$$

Let us start the discussion of the lower bound of $R_f(\mathbf{t})$ with the following observation. Suppose that $\mathbf{z} = (z_1, \dots, z_d) \in (\mathbb{C}^*)^d$ is a d -tuple of non-zero complex numbers; denote by $t_j \in \mathbb{R}$ and $\varphi_j \in \mathbb{T} \sim [0, 1)$ the modulus and the argument of z_j for every $j = 1, \dots, d$, i.e., $z_j = e^{t_j + 2\pi i \varphi_j}$. Note that

$$\frac{1}{n^d} |\log f_n(\mathbf{z})| = \frac{1}{n^d} \sum_{k_1=0}^{n-1} \dots \sum_{k_d=0}^{n-1} \log \left| f \left(e^{t_1 + 2\pi i \varphi_1 + 2\pi i k_1/n}, \dots, e^{t_d + 2\pi i \varphi_d + 2\pi i k_d/n} \right) \right|. \quad (2.13)$$

The expression on the right hand side is a Riemann sum for the integral (2.11) defining the Ronkin function $R_f(\mathbf{t})$. Note also, that despite the fact f may have zeros on a torus $\{z : |z| = e^{\mathbf{t}}\}$, the function $\log |f|$ is still integrable since the singularities are only logarithmic. One naturally expects that for almost all \mathbf{z} , the Riemann sums in (2.13) converge to $R_f(\mathbf{t})$. However, establishing such convergence turns out to be a rather intricate Diophantine problem [25, 65].

Fortunately, in order to establish the lower bound, one does not have to deal with a convergence problem in a complete generality. It suffices to prove that the Riemann sums are bounded from above by $R_f(\mathbf{t})$.

We say that the initial value $\mathbf{z} = (e^{t_1+2\pi i\varphi_1}, \dots, e^{t_d+2\pi i\varphi_d})$ is good if the points of the set

$$Q_n(\mathbf{z}) = \left\{ \left(e^{t_1+2\pi i\varphi_1+2\pi i k_1/n}, \dots, e^{t_d+2\pi i\varphi_d+2\pi i k_d/n} \right), \quad k_1, \dots, k_d = 0, \dots, n-1 \right\}$$

do not fall or come too close (depending on n) to the variety of f : $V_f = \{\mathbf{z} \in (\mathbb{C}^*)^d : f(\mathbf{z}) = 0\}$. For good points, it is relatively easy to show that the Riemann sums are close to the value of the integral $R_f(\mathbf{t})$. On the contrary, for the 'bad' initial values \mathbf{z} , the points in $Q_n(\mathbf{z})$, which are close to V_f , give a negative contribution to the sum (2.13). Hence, one is able to conclude that for all \mathbf{z} with $|z_1| = e^{t_1}, \dots, |z_d| = e^{t_d}$,

$$\limsup_{n \rightarrow \infty} \frac{1}{n^d} \log |f_n(\mathbf{z})| \leq R_f(\mathbf{t}), \quad (2.14)$$

or, equivalently,

$$|f_n(\mathbf{z})| \leq \exp \left(n^d (R_f(\mathbf{t}) + o(1)) \right). \quad (2.15)$$

The final part of the argument is based on a relatively simple statement from Fourier analysis: if the absolute value of a complex (trigonometric) polynomial is bounded from above by some constant M on a torus $\mathbb{T}_t = \{\mathbf{z} : |\mathbf{z}| = e^{\mathbf{t}}\}$ then the absolute values of all of its monomials are also bounded from above by the same constant. Therefore, applying this result to f_n and the inequality (2.15), we conclude that that for all \mathbf{m}

$$|f_n^{\mathbf{m}} e^{\langle \mathbf{t}, \mathbf{m} \rangle}| \leq \exp \left(n^d (R_f(\mathbf{t}) + o(1)) \right),$$

and, hence,

$$\text{trop}(f_n)(\mathbf{t}) = \max_{\mathbf{m}} (\log |f_n^{\mathbf{m}}| + \langle \mathbf{t}, \mathbf{m} \rangle) \leq n^d (R_f(\mathbf{t}) + o(1)). \quad (2.16)$$

Combining the inequalities (2.12) and (2.16), we obtain the desired result. \square

Remark 2.3.2. Theorem 2.3.1 provides some insight on the geometry of tropical varieties of f_n . In Figure 2.4, the similarity between the shapes of tropical varieties of decimations of $f = 1 + x + y$ for various n is not accidental. Let us recall the notion of an amoeba of a Laurent polynomial f of d variables that was first suggested by Gelfand, Kapranov, and Zelevinsky in [36]. An *amoeba* of f denoted by A_f is an image of the variety V_f under the map $\text{Log}: V_f \mapsto \mathbb{R}^d$ given by the formula $\text{Log}(z_1, \dots, z_d) = (\log |z_1|, \dots, \log |z_d|)$.

The amoeba A_f is a closed subset of \mathbb{R}^d with a non-empty convex complement (see Figure 2.5). The Ronkin function R_f is strictly convex over A_f and affine on each component of $\mathbb{R}^d \setminus A_f$ [83].

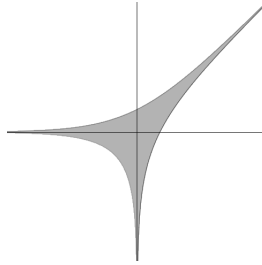


Figure 2.5: Amoeba A_f for $f = 1 + x + y$.

It is easy to see that A_f coincides with A_{f_n} for every positive n . According to Theorem 2.3.1, $\frac{1}{n^d} \text{trop}(f_n)(\mathbf{t})$ is a sequence of piecewise-affine convex functions converging to the Ronkin function R_f which is affine on the complement of A_f and strictly convex inside A_f . Therefore, the outer boundary of the tropical varieties $\frac{1}{n^d} \text{trop}(f_n)$ converge to the boundary of A_f as n approaches infinity.

2.3.1 Surface tension

The *Legendre transform* (or a dual) of a function $F : \mathbb{R}^d \rightarrow \mathbb{R}$ is defined as

$$F^*(\mathbf{t}) = \sup_{\mathbf{u} \in \mathbb{R}^d} (\langle \mathbf{t}, \mathbf{u} \rangle - F(\mathbf{u})).$$

The Legendre transform F^* is always a convex function; moreover, for a convex closed function F , the Legendre transform is an involution

$$F^{**}(\mathbf{u}) = F(\mathbf{u}).$$

Suppose f is a Laurent polynomial. Then, the tropicalization of f is, in fact, a Legendre transform of the function F defined as follows:

$$F(\mathbf{u}) = \begin{cases} -\log |f^{\mathbf{m}}|, & \text{if } \mathbf{u} = \mathbf{m} \in \text{supp}(f), \\ +\infty, & \text{otherwise,} \end{cases}.$$

Indeed, one has

$$F^*(\mathbf{t}) = \sup_{\mathbf{u} \in \mathbb{R}^d} (\langle \mathbf{t}, \mathbf{u} \rangle - F(\mathbf{u})) = \sup_{\mathbf{m} \in \text{supp}(f)} (\langle \mathbf{t}, \mathbf{m} \rangle - (-\log |f^{\mathbf{m}}|)) = \text{trop}(f)(\mathbf{t}).$$

Applying the Legendre transform once again, we obtain $F^{**} = \text{trop}(f)^*$. Since F , in general, is not convex, $F^{**} \neq F$. However, F^{**} is easy to describe; namely, $F^{**} = \text{conv}(F)$, where $\text{conv}(F)$ is the so-called *greatest convex minorant* or a *convex hull* of F : the largest convex function satisfying $\text{conv}(F)(\mathbf{u}) \leq F(\mathbf{u})$ for all \mathbf{u} . Clearly, $\text{conv}(F)(\mathbf{u}) = +\infty$ for all $\mathbf{u} \notin \mathcal{N}(f)$ and is finite on $\mathcal{N}(f)$.

Using the above arguments for the polynomials f_n and the result of the Theorem 2.3.1, one immediately obtains the following result:

Corollary 2.3.3. Let $\text{conv}(F_n)$ be the greatest convex minorant of

$$F^{(n)} = \begin{cases} -\log |f_n^m|, & \text{if } \mathbf{u} = \mathbf{m} \in \text{supp}(f_n), \\ +\infty, & \text{otherwise.} \end{cases}$$

Then, for all $\mathbf{u} \in \mathbb{R}^d$,

$$\sigma_f(\mathbf{u}) := \lim_{n \rightarrow \infty} \frac{1}{n^d} \text{conv}(F_n)(n^d \mathbf{u}) = -R_f^*(\mathbf{u}).$$

By analogy with [55], we refer to the function σ_f as to the *surface tension of f* .

Corollary 2.3.3 should be seen as a weaker, but at the same time, a more general version of the result established in [55] for polynomials appearing in dimer problems. It turns out that these polynomials are rather special in the following sense: for such polynomials, one is able to define the surface tension using the coefficients of f_n without the need to resort to convex hulls of the coefficients. In other words, some form of convexity is already present in the coefficients of f_n . It is, of course, very interesting to identify such polynomials. Okounkov and Kenyon [87] showed that for every Harnack curve, one can construct a polynomial of 2 variables, whose algebraic variety is the given Harnack curve, and for which the surface tension is well defined. At the present moment, it is not clear which conditions could play a similar role in higher dimensions.

Finally, we would like to remark that the use of tropical geometric methods to study limits of partition functions or similar quantities is very natural, and has been proposed in [34].

