# Chapter 1

# Introduction

*"The theory of dynamical systems is a major mathematical discipline closely intertwined with most of the main areas of mathematics. Its mathematical core is the study of the global orbit structure of maps and flows with emphasis on properties invariant under coordinate changes."*
A. Katok, B. Hasselblat, Introduction to the modern theory of dynamical systems

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A dynamical system is a system that undergoes time evolution. Formally speaking, it is a three-tuple consisting of:

1. A phase space $X$ where each point describes some configuration of the system. This space is equipped with a measure $\mu$ together with a $\sigma$-algebra $\mathcal{B}$.

2. A parameter $t$ that is often referred to as time. It typically belongs to the real numbers $\mathbb{R}$ (continuous time) or the integers $\mathbb{Z}$ (discrete time). However, this parameter can also be multi-dimensional (for example, algebraic $\mathbb{Z}^d$-actions).

3. The time evolution law, i.e., a family of transformations $\{T^t\}_t$ of the phase space $X$ that are parametrised by time $t$. The law allows to obtain the state of the system $X_t$ at any time $t$ from the initial state $X_0$ by computing $T^t X_0$. The measure $\mu$ is called $T$-invariant if $\mu(T^{-t}A) = \mu(A)$ for any measurable $A \subset X$.

As the epigraph suggests, dynamical systems can be observed in many areas of mathematics and physics. The theory of dynamical systems unites these areas by studying similar problems associated to time evolution.

An important direction of the theory of dynamical systems is their classification, or, in other words, the identification of whether two dynamical systems are isomorphic. One can tackle the classification problem by studying invariants, i.e., properties or quantities associated to a system which are invariant under system isomorphisms. One of such invariant quantities is the Kolmogorov-Sinai entropy [58, 104]. It measures the limiting complexity of a system and is the complete invariant for Bernoulli systems. Let us discuss entropies of some solvable models investigated in the present thesis.

Consider the lattice $\mathbb{Z}^2$ as a graph, where the vertices are the nodes of the lattice, and the edges connect vertices which are adjacent in vertical and horizontal directions. A dimer configuration on $\mathbb{Z}^2$ is a subset of edges which covers each node exactly once. The set of all dimer configurations on $\mathbb{Z}^2$ is invariant under the $\mathbb{Z}^2$-action by shifts, and together with this action it defines a dynamical system. The dimer model has been extensively studied in the context of symbolic dynamics and statistical mechanics (see, for example, [55]). In particular, its topological entropy is given by

$$h_d = \frac{1}{4}\int_0^1 \int_0^1 \log\left(4 - 2(\cos 2\pi x_1 + \cos 2\pi x_2)\right)\, dx_1\, dx_2. \tag{1.1}$$

The uniform spanning forest model on $\mathbb{Z}^2$ is similar to the dimer model. It turns out that there exists a bijection between the set of spanning trees of an $n \times n$ box and the set of dimer matchings of a $(2n-1)\times(2n-1)$ box with a corner removed (the Temperley-Fisher bijection, see [107]). The entropy of the uniform spanning forest model $h_s$ on $\mathbb{Z}^2$ is directly related to the dimer entropy [16]:

$$h_s = 4h_d = \int_0^1 \int_0^1 \log\left(4 - 2(\cos 2\pi x_1 + \cos 2\pi x_2)\right) dx_1\, dx_2. \tag{1.2}$$

Remarkably, a number of other systems share the same expression for entropy, for example, the sandpile model [24]. Principal algebraic actions considered in the present thesis also provide an example of a system with the same entropy. Denote the additive torus $\mathbb{R}/\mathbb{Z}$ by $\mathbb{T}$ and consider a group

$$X = \left\{ x \in \mathbb{T}^{\mathbb{Z}^2} : 4x^{n,m} - x^{n+1,m} - x^{n-1,m} - x^{n,m+1} - x^{n,m-1} = 0, (n,m) \in \mathbb{Z}^2 \right\}.$$

The group $\mathbb{T}^{\mathbb{Z}^2}$ admits an action of $\mathbb{Z}^2$ by shifts. Denote by $\alpha$ the restriction of this action to the compact group $X$. The principal algebraic action

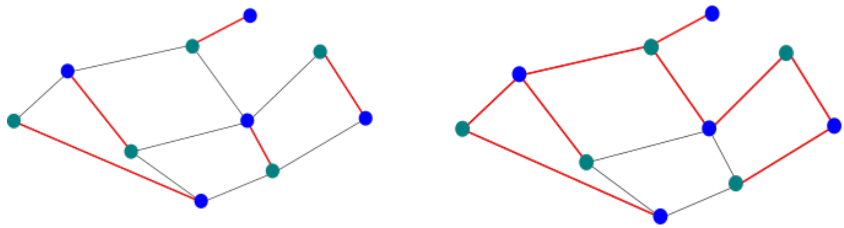

Figure 1.1: Left: a dimer configuration is a subset of edges covering each vertex exactly once. Right: a spanning tree is a subgraph that is a tree and connects all vertices.

$(X, \alpha)$ has the same entropy expression as (1.2) [64]. Moreover, this algebraic system is measure-theoretically isomorphic both to the double shift dimer model and to the uniform spanning forest model [99]. It is conjectured that the link between these models is, in fact, even stronger [100].

In the present thesis we investigate the limiting behaviour of some models. In particular, we consider the following problems:

- Existence of a scaling limit of principal actions (Chapters 2, 3).
- Existence and properties of a limiting measure or a limiting distribution (Chapters 4, 5).
- Properties of a foliation of a moduli space (Chapter 6).

## 1.1 Dimer configurations and decimations

Consider a planar weighted simple bipartite graph $G = (V, E)$, where $V$ is the set of vertices and $E$ is the set of unoriented edges of $G$. By definition, a *dimer configuration* of $G$ is a subset of $E$ that covers each vertex exactly once (see Figure 1.1).

One defines a *partition function* $Z_G$ of a graph $G$ as a formal sum of weights of dimer configurations, namely,

$$Z_G = \sum_{M \in \mathcal{M}(G)} \nu(M) = \sum_{M \in \mathcal{M}(G)} \prod_{e \in M} \nu(e),$$

where $\mathcal{M}(G)$ is the collection of all possible dimer configurations of $G$ and $\nu(e)$ is the weight of $e \in E$. Suppose that $G$ is an infinite weighted $\mathbb{Z}^2$-

periodic bipartite planar graph, and let $G_N = G/N\mathbb{Z}^2, N \geqslant 1$, be its finite periodic factor graph, which we view as a graph on the torus $\mathbb{T}^2$. It turns out that for any $N$, the partition function $Z_{G_N}$ can be computed explicitly (see Section 6 in [93]).

In order to illustrate how $Z_{G_N}$ is computed, let us consider the example of the honeycomb lattice (see Figure 1.2 for $G_3$). The graph $G_N$ has $N^2$ black vertices, $N^2$ white vertices, and $3N^2$ edges. The positive weights $a, b, c$ are assigned periodically to the edges of $G_N$ as indicated in Figure 1.2. Denote by $N_b(M)$ and $N_c(M)$ the number of edges of type $b$ and $c$, respectively, of a dimer configuration $M$ of $G_N$. The partition function $Z_{G_N}$ is then a sum of the weights of all possible dimer configurations of $G_N$:

$$Z_{G_N} = \sum_{M \in \mathcal{M}(G_N)} a^{3N^2 - N_b(M) - N_c(M)} b^{N_b(M)} c^{N_c(M)}.$$

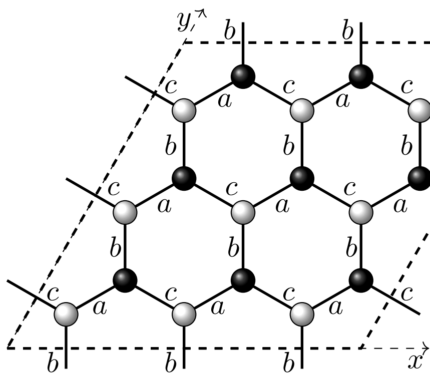


Figure 1.2: Periodic weighted hexagonal lattice.

It turns out that the partition function $Z_{G_N}$ can be calculated by using the *characteristic polynomial* of two variables $P(z, w)$ that only depends on $G_1$. Namely,

$$Z_{G_N} = \frac{1}{2}\Big(-\mathcal{P}_N(1,1) + \mathcal{P}_N(-1,1) + \mathcal{P}_N(1,-1) + \mathcal{P}_N(-1,-1)\Big), \tag{1.3}$$

where $\mathcal{P}_N(\pm 1, \pm 1) = \prod_{z^N = \pm 1} \prod_{w^N = \pm 1} P(z, w)$. In case of the hexagonal lattice $P(z, w) = a - bz - cw$ [55]. What is surprising is that not only the partition function $Z_{G_N}$ (the weighted number of configurations), but also the coefficients of $Z_{G_N}(z, w)$ have a physical meaning. Namely, if $Z_{G_N}(z, w) = \sum_{\mathbf{n}} Z_{G_N}(\mathbf{n}) z^{n_1} w^{n_2}$, then $Z_{G_N}(\mathbf{n}) = \sum_{M \in \mathcal{M}_{\mathbf{n}}(G_N)} \nu(M)$, where $\mathcal{M}_{\mathbf{n}}(G_N)$, $\mathbf{n} = (n_1, n_2)$, is the collection of dimer configurations of $G_N$ that have precisely

$Nn_1$ edges with weight $b$ and $Nn_2$ edges with weight $c$. This means that $Z_{G_N}(\mathbf{n})$ is also a partition function. It turns out that the restricted partition functions $Z_{G_N}(\mathbf{n})$ have scaling limits: for $(s,t) \in \Delta_P$, i.e., in the Newton polytope of the polynomial $P$,

$$\lim_{N\to\infty} \frac{1}{N^2} \log Z_{G_N}([Ns],[Nt]) = -R_P^*(s,t), \tag{1.4}$$

where $[\cdot]$ is the integer part and $R_P^*(s,t)$ is the Legendre dual of the Ronkin function of $P$ [55]. The coefficients of $Z_{G_N}$ are directly related to the coefficients of the $N$th decimation $P_N$ of $P(z,w)$, which is defined as $P_N(z,w) = \prod_{q_1^N=1}\prod_{q_2^N=1} P(q_1 z, q_2 w)$. The latter can be generalised to polynomials with any number of variables. The $N$-th decimation $f_N$ of a polynomial $f$ is given by the formula

$$f_N(x_1,\ldots,x_d) = \prod_{q_1^N=1} \cdots \prod_{q_d^N=1} f(q_1x_1,\ldots,q_dx_d).$$

Note that $f_N$ is a polynomial with integer coefficients with all powers of monomials pointwise divisible by $N$. A natural question is to understand whether the coefficients of $f_N$ for arbitrary $f$ also have scaling limits similar to (1.4). The methods used in [55] can only be applied for a class of polynomials in two variables that arise as characteristic polynomials of dimer models and cannot be generalised outside of this context. It is not difficult to demonstrate that (1.4) does not hold for arbitrary decimated polynomials (see, for instance, Example 3.2.3 of Chapter 3). However, in Chapters 2 and 3 we show that the *convex hulls* of decimated scaled coefficients always exist (Theorems 2.3.1 and 3.1.1). Chapter 2 is a geometric introduction to the problem of decimations of polynomials. It contains a proof of the existence of convex hulls of decimated scaled coefficients that relies on an analytic argument to bound Riemann sums. Chapter 3 views the problem of decimations from the algebraic point of view of $\mathbb{Z}^d$-actions. Even though the statement of its Theorem 3.1.1 of Chapter 3 coincides with the statement of Theorem 2.3.1 of Chapter 2, it features a different proof that only uses Mahler's estimates. Moreover, Chapter 3 provides a link between the decimation of a polynomial $f$ and the decimation of a principal action associated to $f$ stated in Theorem 3.8.12.

## 1.2 Spanning trees and determinantal point processes

Given a set $E$, a point process $X = \{0,1\}^E$ is called *determinantal* if the probability $\mathbb{P}(X(e_1) = 1,\ldots,X(e_n) = 1)$ is given by a determinant of an

$n \times n$ matrix with entries given by a correlation function $K$. Namely, the $(i, j)$-th entry is equal to $K(e_i, e_j)$. Determinantal point processes (DPP) first emerged in 1960s in the framework of mathematical physics (see [30,38] for the first well known examples) and in 1975 the general notion of DPP was first introduced for modelling fermion distributions at thermal equilibrium [79]. Even though DPPs have been studied for many years, most applications are unexplored yet promising (for instance, to statistics and machine learning).

In Chapter 4 we are concerned with a probabilistic problem of a DPP associated to uniform spanning forest measures. A spanning tree on a graph $G = (V, E)$ is a subset of edges $E' \subset E$ such that the graph $(V, E')$ is a connected tree (see Figure 1.1). The subject of random spanning trees of a graph goes back to Kirchhoff in 1847, who showed its surprising relation to electrical networks. One of Kirchhoff's results expresses the probability that a uniformly chosen spanning tree contains a given edge in terms of the electrical current in the graph.The number of spanning trees $\tau(G)$ of a finite graph $G$ can be calculated by applying the Kirchhoff's Matrix-Tree theorem: $\tau(G) = |V(G)|^{-1} \det' \Delta_G$, where $\det' \Delta_G$ is the product of all non-zero eigenvalues of the Laplacian of $G$.

Given a $\mathbb{Z}^d$-periodic infinite graph $G$, one can consider its approximation by finite graphs: $G_1 \subset G_2 \subset \ldots \subset G$, where $V(G_n) = \{v \in \mathbb{Z}^d : \max |v_i| \leqslant n\}$, with either free or wired boundary conditions. A number of properties of the spanning forest (a collection of spanning trees) structure on $G$ has been established: for instance, the asymptotic limit of the number of spanning trees of $G_n$ on $\mathbb{Z}^2$ is the entropy $\int_0^1 \int_0^1 \log\left(4 - 2(\cos 2\pi x_1 + \cos 2\pi x_2)\right) dx_1 dx_2$ as discussed above. Moreover, the probability measures associated to spanning trees on the finite graphs $G_n$ converge to a uniform spanning forest measure on $G$ that depends on the chosen boundary conditions. The resulting measures are determinantal (see Theorems 4.1.1 and 4.1.2 in Chapter 4, and [16,89]), and the corresponding correlation kernels are expressed in terms of operator projections (Theorem 4.1.6 in Chapter 4). Despite the simple projection formulas of Theorem 4.1.6, the corresponding correlations kernels are difficult to compute explicitly even for simple graphs (see Example 4.1.3 of Chapter 4 for $\mathbb{Z}^d$ and [57] for computations on ladder-like graphs). Our result – Theorem 4.4.3, uses a functional-analytic generalisation of a simple linear algebra statement, which allows to effectively calculate correlation kernels of DPPs and correlations associated to uniform spanning forest measures on infinite graphs with $\mathbb{Z}^d$-symmetry. The goal of Chapter 4 is not only to provide a new method of computing the correlations, but also to demonstrate the application of the method. Therefore, Chapter 4 is equipped with a number of examples of computations for different $\mathbb{Z}^d$-periodic graphs.

## 1.3 Central Limit Theorem for dynamical systems

A probabilistic question closely related to the existence of a limiting measure is the question of existence of a limiting distribution. Consider a sequence of independent and identically distributed random variables $Y_1, Y_2, \ldots$ with zero mean and variance $0 < \sigma^2 < \infty$. These satisfy the *Central Limit Theorem* (CLT)

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i \xrightarrow{d} \mathcal{N}(0, \sigma^2),$$

i.e., $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i$ converges in distribution to a normal law. The same question can be asked in the framework of a dynamical system $(X, \mu, T)$ (where $\mu$ is $T$-invariant), for a sequence of centered functions $f \circ T^i$, $f \in L^2(X, \mu)$, that can be treated as random variables. We say that $f$ satisfies the CLT if $\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} f \circ T^i \to \mathcal{N}(0, \sigma^2)$ for $0 < \sigma^2 < \infty$. There are relatively straightforward methods to prove the CLT for independent identically distributed random variables, but the variables $Y_i = f \circ T^i$ might not be independent. However, the CLT still holds in certain cases, for example, when the sequence of random variables possesses sufficiently strong mixing properties (see, for instance, [51, 97]) or forms a martingale (see [40]).

The models considered in Chapters 2, 3, 4 are the so-called solvable models, i.e, their free energy can be expressed in terms of some known function, usually a polynomial. The polynomials appearing in solvable models are typically non-expansive, i.e., some roots have unit absolute value. It is well known that expansive and non-expansive actions can have very different dynamical properties. In particular, in Chapter 5 we consider the CLT for ergodic (both expansive and non-expansive) toral automorphisms: the CLT has been established for ergodic toral automorphisms by using the martingale method (see [10, 62] for the CLT and its refinements for sufficiently smooth functions and [44] for general nilmanifords and Hölder functions).

When considering the CLT, we find that the difference between the hyperbolic (expansive) and non-hyperbolic (non-expansive) automorphisms be-

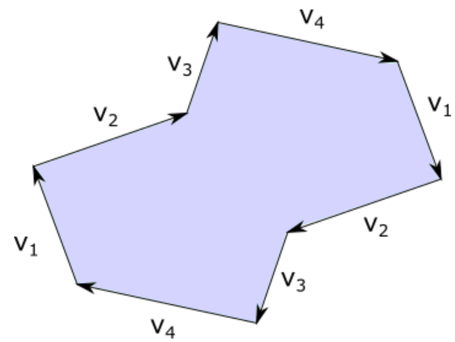


Figure 1.3: Flat surface - each pair of parallel sides is identified by translation. The resulting surface is a 2-torus (a sphere with 2 handles) with a conical singularity of angle $6\pi$.

comes vivid. The hyperbolic toral automorphisms possess certain dynamical properties, such as a Markov partition and a spectral gap (in some Banach spaces), which allow one to prove the CLT in a variety of ways that do not apply in the context of the whole family of ergodic toral automorphisms (see Section 5.2 of Chapter 5 for detailed comparison of methods). In Chapter 5 we prove the CLT and obtain a new result – the rates of convergence – for Hölder functions in a special family of ergodic toral automorphisms, without applying the classic martingale method but rather by using Stein's method as in [49] and the mixing properties of Hölder functions [26, 90]. Moreover, it seems that the proof of the CLT and the rates of convergence in Chapter 5 can be generalised for the whole class of ergodic toral automorphisms, and also for the case of non-linear toral automorphisms and multivariate Hölder observables.

## 1.4 Dynamics on moduli space

In Chapter 5 we discussed dynamics on a torus, which is the simplest example of a flat surface, i.e., a surface obtained by pairwise identification of parallel sides of a collection of polygons in Euclidean plane (for an example less trivial than a torus, see Figure 1.3). Flat surfaces naturally appear in many areas of mathematics and physics, including billiards in rational polygons and electron transport in metal [85]. Despite the apparent simplicity of the definition, flat surfaces pose a number of mathematical questions that are still not answered, for example, questions on typical behaviour of a generic geodesic, ergodicity of the geodesic flow, and existence, number and length of closed geodesics [108]. It turns out that the dynamics on a flat surface is closely connected to the dynamics in the stratum of the moduli space of flat surfaces. Let us illustrate the latter statement with two examples. A famous theorem called Masur's criterion implies that if the vertical flow on a flat surface $S$ is minimal but not ergodic, then the Teichmüller geodesic

$$\{g_t S\}_{t\in\mathbb{R}} = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} S \right\}$$

eventually leaves any fixed compact subset $K \subset \mathcal{M}_g$ in the moduli space and never visits it again [80]. Another example of the connection between the dynamics on a flat surface and on the moduli space is (a version of) Veech's dichotomy: if the $\mathrm{GL}^+(2,\mathbb{R})$-orbit of $S$ is closed in the stratum $H(\kappa) \subset M_g$, then any directional flow on S is either completely periodic or uniquely ergodic. Therefore, it is reasonable to associate problems of dynamics on a flat surface with corresponding problems of dynamics on the moduli space $\mathcal{M}_g$ or in the stratum. In particular, a full understanding of the dynamics on $\mathcal{M}_g$ is crucial for the study of the dynamical properties of the geodesic flow on flat surfaces.

The moduli space $\mathcal{M}_g$ normally has a complicated topological structure being a non-compact orbifold, so it is often convenient to consider its compactification. For $g \geqslant 1$, consider the bundle $\Omega\mathcal{M}_{g,n} \to \mathcal{M}_{g,n}$ whose fiber over $(S, x_1, \ldots, x_n)$ is the space of meromorphic forms $\omega$ on $S$ having simple poles at $x_1, \ldots, x_n$. The moduli space $\Omega\mathcal{M}_{g,n}$ appears naturally as a boundary component in the Deligne-Mumford compactification of moduli spaces $\Omega\mathcal{M}_g$ of holomorphic 1-forms. The moduli spaces of 1-forms have a natural period coordinate system, namely, to each pair $(S,\omega) \in \Omega\mathcal{M}_{g,n}$ one can associate a period map $p \in \mathrm{Hom}(H_1(\Sigma_{g,n^*}), \mathbb{C})$, where $p(\gamma) = \int_{f^*\gamma} \omega$ and $f$ is a map from the reference surface $\Sigma_{g,n^*}$ to $S$. Therefore period coordinates appear naturally as a coordinate system of $\Omega\mathcal{M}_{g,n}$. However, knowing the period coordinates of $(S,\omega)$ does not allow us to recover the pair $(S,\omega)$ even infinitesimally. In other words, it is always possible to find non-trivial isoperiodic deformations on $\Omega\mathcal{M}_{g,n}$ that give rise to the isoperiodic foliation on $\Omega\mathcal{M}_{g,n}$ [17]. When considering Deligne-Mumford compactification of $\Omega\mathcal{M}_g$, we see that the different isoperiodic foliations glue together and define a global algebraic foliation of a resulting compact space.

It is known that for degree at least 3 the isoperiodic sets of $\Omega\mathcal{M}_g, g \geqslant 2$, with no marked points are connected (see [17] for the proof and other dynamic properties, such as ergodicity of the foliation). However, it turns out that the same methods do not apply to the study of the isoperiodic foliations of moduli spaces of meromorphic 1-forms. Notably, not much is known about the isoperiodic foliations on $\Omega\mathcal{M}_{g,n}, n \geqslant 1$ (an overview can be found in the introduction of Chapter 6). The methods that were proposed to study the isoperiodic foliations of $\Omega\mathcal{M}_{g,n}, n \geqslant 1$, heavily depend on the values of $g, n$ (known results are restricted to $n \leqslant 2$; see [18]). In Chapter 6 we propose a new geometric method of studying the isoperiodic sets of $\Omega\mathcal{M}_{g,n}$ that can be applied to $\Omega\mathcal{M}_{g,n}$ for any (small) value of $g, n$. We demonstrate the method by proving a new result, namely, that the real isoperiodic sets of $\Omega\mathcal{M}_{1,3}$ are connected (Theorem 6.1.4). The significance of the result of Chapter 6 is not only the novelty of the method: it seems that the general statements for arbitrary $g, n$ can be proved by induction upon providing a sufficient induction base (similarly to [17]). Thus, Theorem 6.1.4 serves as the base of induction for our further research.

In conclusion, the present thesis addresses problems that arise in different mathematical areas - algebra, probability, geometry, statistical mechanics, graph theory. Nevertheless, the problems that we treat are similar in nature and are aimed at understanding limiting behaviour associated to dynamical systems. As this thesis demonstrates, the methods that apply in different mathematical contexts are diverse, and range from Diophantine approximations to combinatorics, linear algebra, functional analysis, and more.