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Francesca Arici and Jens Kaad

Abstract

Motivated by the study of symmetries of C^*-algebras, as well as by multivariate operator theory, we introduce the notion of an $SU(2)$-equivariant subproduct system of Hilbert spaces. We analyse the resulting Toeplitz and Cuntz–Pimsner algebras and provide results about their topological invariants through Kasparov's bivariant K-theory. In particular, starting from an irreducible representation of $SU(2)$, we show that the corresponding Toeplitz algebra is equivariantly KK-equivalent to the algebra of complex numbers. In this way, we obtain a six-term exact sequence of K-groups containing a noncommutative analogue of the Euler class.

Contents

Motivated by the study of symmetries of C^*-algebras, as well as by multivariate operator theory, in this paper we introduce the notion of an $SU(2)$-equivariant subproduct system of Hilbert spaces. Starting from a unitary representation of the Lie group $SU(2)$ on a finite-dimensional Hilbert space, we give an algorithm for constructing such an equivariant subproduct system and describe the associated Toeplitz–Pimsner and Cuntz–Pimsner algebras.

In the spirit of noncommutative topology, we compute topological invariants through Kasparov's bivariant K-theory [**24**]. In particular, we provide, for our class of algebras, a partial answer to Open Question 3 in [**40**, Section 6] concerning the computation of the K-theory groups of the Cuntz–Pimsner and Toeplitz–Pimsner algebras of a subproduct system. Note in this respect that the paper [**16**] also contains valuable computations of K-theory groups relating to Viselter's question. The present text offers a completely new approach, which exploits topological features like the existence of higher dimensional Gysin sequences. More precisely, our main result, Theorem 6.1, concerns the-equivalence between the Toeplitz algebra of the subproduct system of an irreducible $SU(2)$-representation and the C^*-algebra of

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complex numbers. We further use this equivalence result to prove that the defining extension for the Cuntz–Pimsner algebra of a subproduct system induces an exact sequence in operator K-theory which contains a noncommutative Euler class and hence resembles a Gysin sequence. Using the exact sequence, we are able to compute the K-theory groups of the Cuntz–Pimsner algebra of our $SU(2)$-subproduct system.

Our work fits into the framework of noncommutative topology, building on representation theoretic techniques, as well as Kasparov's bivariant K-theory. One of our driving motivations lies in the noncommutative description of principal fibre bundles through Hopf–Galois extensions, a theory which works both algebraically and topologically [**6**]. This approach allows one to extend the scope to consider symmetries implemented by compact quantum groups.

It is natural to try to extend this analogy to bundles with fibres other than quantum groups, as described in [**10**], where the authors initiated the development of an algebraic framework for noncommutative bundles with quantum homogeneous fibres. Here, however, we still focus on the group case and set the basis for an *operator theoretic* approach to the study of sphere bundles with fibre the three-dimensional sphere. We are following the bottom-up approach offered by both the classical construction of the associated principal G-bundle to a fibre bundle with structure group G, and the construction of the sphere bundle of a Hermitian vector bundle.

We build on the earlier work [**4**], where we observed how the Cuntz–Pimsner algebra [**32**] of a noncommutative line bundle can be interpreted as the algebra of functions on a noncommutative circle bundle. This analogy also works at the level of topological invariants: Pimsner's construction naturally yields an exact sequence in K-theory, which mimics the classical Gysin sequence for circle bundles [**19, 23**].

The generalisation of this construction to structure groups different from $U(1)$ is not so straightforward and has, to our knowledge, escaped a satisfactory treatment. For instance, when applying Pimsner's construction to the module of sections of a complex n-dimensional vector bundle, possibly carrying the action of a compact group G, the resulting C^*-algebra has the structure of a bundle of algebras with fibres the Cuntz algebra $\mathbb{O}_n$ [**38**], a very different object from the algebra of functions of the associated principal G-bundle. Nevertheless, understanding the properties and symmetries of such C^*-algebras is an interesting question, which was recently addressed in [**13**], where the author studied the Cuntz–Pimsner algebras constructed starting from the action of a compact group G on a complex Hermitian vector bundle and their crossed products by G.

Inspired by the representation theory of the group $SU(2)$, in particular by the Clebsch–Gordan theory, we adopt a novel approach, which relies on the theory of subproduct systems of C^*-correspondences. Subproduct systems were first described by Shalit and Solel in [**36**], inspired by the dilation theory of semigroups of completely positive maps, and independently by Bhat and Mukherjee [**7**] in the Hilbert space setting, under the name of *inclusion systems*. Motivated by examples in quantum electrodynamics, the related notion of *interacting Fock spaces* was investigated in [**1, 2**]. The theory of subproduct systems was further developed by Viselter, who extended the notions of covariant representation and of Cuntz–Pimsner algebras of a C^*-correspondence to this more general framework [**39, 40**]. More recently, Dor-On and Markiewicz [**15, 16**] applied the theory of subproduct systems to the study of stochastic matrices.

Another motivation for our work can be found in the question of understanding operator and C^*-algebras arising from zeros of polynomials in noncommutative variables. This relates to the programme of studying noncommutative domains initiated by Popescu [**33, 34**]. In [**36**, Section 7], Shalit and Solel established a noncommutative *Nullstellensastz*: every homogeneous ideal I in the algebra of noncommutative polynomials corresponds to a unique subproduct system, and vice versa. In our case, for every $n \in \mathbb{N}$, we consider noncommutative varieties whose defining ideal in the free algebra $\mathbb{C}\langle X_0, \dots, X_n\rangle$ is generated by a single degree-two homogeneous polynomial arising from the *determinant* of an $SU(2)$-representation. From a purely algebraic perspective, our setting is closely related to the one-relator quadratic regular Koszul algebras of global dimension two studied in [**41, 42**].

The outline of the paper is as follows. Section 1 is devoted to preliminaries on the theory of subproduct systems: we introduce the notion of G-equivariant subproduct system of C^*-correspondences, which we then specialise to the Hilbert space case. At the end of the section, we recall the one-to-one correspondence between subproduct systems of Hilbert spaces and ideals in the algebra of noncommutative polynomials.

In Section 2, we show how, starting from a unitary representation of the Lie group $SU(2)$ on a finite-dimensional Hilbert space, one can construct an $SU(2)$-*equivariant subproduct system* of Hilbert spaces over the semi-group $\mathbb{N}_0$. An essential ingredient in our construction is what we call the *determinant* of the representation. This determinant will resurface later in our computations in KK-theory as one of the summands in the Euler class of the representation.

We proceed to studying the fusion rules of our equivariant $SU(2)$-subproduct system in Section 3. This section contains several lemmas containing explicit computations and showcasing interesting combinatorial properties, on which our later analysis relies. In particular, the structural properties of our subproduct systems naturally lead us to the commutation relations in the Toeplitz algebras, described in Section 4.

Finally, we focus on K-theoretic invariants: Section 6 is dedicated to the proof of KK-equivalence between the Toeplitz algebra of an irreducible $SU(2)$-representation and the algebra of complex numbers $\mathbb{C}$. In Section 7, we present our main application: we establish a Gysin sequence in operator K-theory and employ it to compute the K-theory groups of the Cuntz–Pimsner algebra of the subproduct system. In the final section, we conclude the paper by mentioning a few open questions that we would like to address in the future.

1. *Preliminaries on subproduct systems*

In this section, we review the theory of subproduct systems of correspondences, specialising to the Hilbert space case. From the point of view of multivariate operator theory, subproduct systems of Hilbert space provide the natural framework for the study of row-contractive tuples of operators subject to polynomial constraints. We shall elaborate on this analogy in the last part of the section.

For a pair of C^*-correspondences X and Y over the same C^*-algebra B, we let $X\widehat{\otimes}_B Y$ denote their interior tensor product, which is again a C^*-correspondence over the C^*-algebra B (see, for instance, [**28**, Section 4]). In the case where G is a locally compact group and both X and Y are G-C^*-correspondences over the same G-C^*-algebra B, we turn $X\widehat{\otimes}_B Y$ into a G-C^*-correspondence as well by equipping it with the diagonal action $g(\xi \otimes \eta) := g(\xi) \otimes g(\eta)$.

We recall that a G-C^*-correspondence for a locally compact group G consists of a C^*-correspondence X from a G-C^*-algebra A to a G-C^*-algebra B such that X is furthermore equipped with a strongly continuous group homomorphism $U : G \to \text{Isom}(X)$ (where $\text{Isom}(X)$ denotes the group of invertible isometries on X). This data has to be compatible in the sense that

$$U(g)(\xi \cdot b) = U(g)(\xi) \cdot g(b) \qquad U(g)(a \cdot \xi) = g(a) \cdot U(g)(\xi) \qquad \text{and}$$
$$\langle U(g)(\xi), U(g)(\eta)\rangle = g(\langle \xi, \eta\rangle)$$

for all $\xi, \eta \in X$, $a \in A$, $b \in B$ and $g \in G$. Remark that $U(g)$ is in general not adjointable (since it is in general not even linear over B). For more details on these matters, we refer to [**25**].

We say that a C^*-correspondence X over B is *faithful* when the left action $B \to \mathbb{L}(X)$ is an injective $*$-homomorphism and *essential* when $B \cdot X$ is a norm-dense B-submodule of X.

DEFINITION 1.1 [**36, 40**]. Suppose that $\{E_m\}_{m\in\mathbb{N}_0}$ is a sequence of essential and faithful C^*-correspondences over a C^*-algebra B and that $\iota_{k,m} : E_{k+m} \to E_k\widehat{\otimes}_B E_m$ is a bounded adjointable isometry for every $k, m \in \mathbb{N}_0$. We say that (E, ι) is a *subproduct system* over B when the following holds for all $k, l, m \in \mathbb{N}_0$.

(i) $E_0 = B$.

(ii) $\iota_{0,m} : E_m \to E_0\widehat{\otimes}_B E_m$ and $\iota_{m,0} : E_m \to E_m\widehat{\otimes}_B E_0$ are the canonical identifications (so that the adjoints are induced by the bimodule structure on E_m).

(iii) The two bounded adjointable isometries $(1_k \otimes \iota_{l,m}) \circ \iota_{k,l+m}$ and $(\iota_{k,l} \otimes 1_m) \circ \iota_{k+l,m} : E_{k+l+m} \to E_k\widehat{\otimes}_B E_l\widehat{\otimes}_B E_m$ agree, where 1_k and 1_m denote the identity operators on E_k and E_m, respectively.

We refer to the bounded adjointable isometries $\iota_{k,m} : E_{k+m} \to E_k\widehat{\otimes}_B E_m$, $k, m \in \mathbb{N}_0$, as the structure maps of our subproduct system.

Note that for every $k, m \in \mathbb{N}_0$, we have the orthogonal projection

$$p_{k,m} = \iota_{k,m}\iota_{k,m}^* : E_k\widehat{\otimes}_B E_m \to E_k\widehat{\otimes}_B E_m. \tag{1.1}$$

Clearly the image of $p_{k,m}$ is then unitarily isomorphic to E_{k+m} via the bounded adjointable isometry $\iota_{k,m} : E_{k+m} \to E_k\widehat{\otimes}_B E_m$, see also [**36**, Lemma 6.1].

DEFINITION 1.2. Let G be a locally compact group and let (E, ι) be a subproduct system over a C^*-algebra B. We say that (E, ι) is a *G-subproduct system* when B is a G-C^*-algebra and E_m is a G-C^*-correspondence for all $m \in \mathbb{N}$, such that the structure maps $\iota_{k,m} : E_{k+m} \to E_k\widehat{\otimes}_B E_m$ are G-equivariant for all $k, m \in \mathbb{N}_0$.

EXAMPLE 1. If (X, ϕ) is an essential and faithful C^*-correspondence over a C^*-algebra B, then the sequence $\{X^{\widehat{\otimes}_B m}\}_{m=0}^\infty$ defines a subproduct system over B, where the structure maps are given by the canonical identifications $X^{\widehat{\otimes}_B (m+k)} \cong X^{\widehat{\otimes}_B m}\widehat{\otimes}_B X^{\widehat{\otimes}_B k}$.

DEFINITION 1.3. Given a subproduct system (E, ι), one defines its *Fock correspondence* as the infinite Hilbert C^*-module direct sum $F := \oplus_{m=0}^\infty E_m$.

In the case where G is a locally compact group and (E, ι) is a G-subproduct system, it holds that the Fock correspondence F is a G-Hilbert C^*-module where the action of G on F is given by

$$g(\{\xi_m\}_{m=0}^\infty) := \{g(\xi_m)\}_{m=0}^\infty$$

for all $g \in G$ and $\{\xi_m\}_{m=0}^\infty \in F$.

For each $\xi \in E_k$, we define the *creation operator* $T_\xi \in \mathbb{L}(F)$ as

$$T_\xi : F \to F \qquad T_\xi(\zeta) := \iota_{k,m}^*(\xi \otimes \zeta), \qquad \zeta \in E_m \subseteq F.$$

DEFINITION 1.4. Let (E, ι) be a subproduct system. We define the *Toeplitz algebra* of the subproduct system E, denoted $\mathbb{T}_E$, as the smallest C^*-subalgebra of $\mathbb{L}(F)$ that contains all the creation operators, that is,

$$T_\xi \in \mathbb{T}_E \qquad \text{for all } \xi \in E_k\ ,\ \ k \in \mathbb{N}_0.$$

LEMMA 1.5. *Let G be a locally compact group and suppose that (E, ι) is a G-subproduct system. Then there is a strongly continuous action of G on the Fock correspondence, which induces a strongly continuous action of G on the Toeplitz algebra $\mathbb{T}_E$ satisfying $g(T_\xi) := T_{g(\xi)}$.*

Proof. Since E_k is a G-C^*-correspondence for each $k \in \mathbb{N}_0$, we obtain that the Fock correspondence F is also a G-C^*-correspondence. For every $\xi \in E_k$, we record that the map $G \to E_k$ given by $g \mapsto g(\xi)$ is continuous.

Let us now consider the Toeplitz algebra $\mathbb{T}_E$. Since the structure maps of our subproduct system are G-equivariant, we have that

$$gT_\xi g^{-1}(\eta) = g\iota^*_{k,m}(\xi \otimes g^{-1}\eta) = \iota^*_{k,m}(g\xi \otimes \eta) = T_{g(\xi)}(\eta),$$

so we have a well-defined action of G on $\mathbb{T}_E$.

Note that since $\|T_\xi\| \leqslant \|\xi\|$, we obtain that the map $G \to \mathbb{T}_E$ given by $g \mapsto T_{g(\xi)}$ is continuous for every $\xi \in E_k$. Strong continuity of our G-action follows since the Toeplitz algebra is generated by the creation operators T_ξ, $\xi \in E_k$. □

Covariant representations of subproduct systems of C^*-correspondences inducing a C^*-representation of the Toeplitz algebra were studied in [**39**]. In the subsequent work [**40**], the author described how one can associate a Cuntz–Pimsner algebra to every subproduct system. Due to [**40**, Theorem 2.5], one can define the Cuntz–Pimsner algebra of a subproduct system as the quotient of the Toeplitz algebra by a suitable gauge-invariant ideal. We recall the construction here.

For each $m \in \mathbb{N}_0$, we let $Q_m : F \to F$ denote the orthogonal projection with image $E_m \subseteq F$.

Definition 1.6. Let (E, ι) be a subproduct system. The *Cuntz–Pimsner algebra* of the subproduct system (E, ι), denoted $\mathbb{O}_E$, is the unital C^*-algebra obtained as the quotient of the Toeplitz algebra $\mathbb{T}_E$ by the ideal

$$\mathbb{I}_E := \big\{x \in \mathbb{T}_E \mid \lim_{m\to\infty} \|Q_m x\| = 0\big\}.$$

Thus, $\mathbb{O}_E := \mathbb{T}_E/\mathbb{I}_E$.

In the case where G is a locally compact group acting on a subproduct system (E, ι), we obtain that our strongly continuous action of G on the Toeplitz algebra $\mathbb{T}_E$ descends to a strongly continuous action of G on the Cuntz–Pimsner algebra. Indeed, for $g \in G$, let $U(g) : F \to F$ denote the invertible isometry implementing the $*$-automorphism $g : \mathbb{T}_E \to \mathbb{T}_E$. Remark that $U(g)$ is in general not adjointable since it can fail to be linear over the base algebra B. For each $x \in \mathbb{I}_E$ and each $m \in \mathbb{N}_0$, we then have that $\|Q_m g(x)\| = \|Q_m U(g) x U(g^{-1})\| = \|U(g) Q_m x U(g^{-1})\| = \|Q_m x\|$ and hence that $g(x) \in \mathbb{I}_E$ as well.

Viselter furthermore proved that, if (E, ι) is a subproduct system of finite-dimensional Hilbert spaces, then the ideal $\mathbb{I}_E$ is isomorphic to $\mathbb{K}(F)$ (cf. [**40**, Corollary 3.2]). Thus, in this case, we have that $\mathbb{O}_E = \mathbb{T}_E/\mathbb{K}(F)$.

1.1. *Subproduct systems and zeros of polynomials in noncommutative variables*

We conclude this section by recalling how subproduct systems offer a framework for studying row-contractive tuples of operators satisfying relations given by homogeneous polynomials. Our main reference is [**36**, Section 7]. In what follows, we will restrict our attention to the finite-dimensional case.

Let $X := \{x_0, \dots, x_n\}$ be a finite set of $n+1$ variables. We shall denote the free monoid generated by X by $\langle X\rangle$, with unit the empty word, denoted by 1. We denote by X^m the set of all words of length m in $\langle X\rangle$, so that the free monoid $\langle X\rangle$ is naturally graded by length.

Let $\mathbb{C}\langle X\rangle := \mathbb{C}\langle x_0, \dots, x_n\rangle$ denote the complex free associative unital algebra generated by X. Similarly to the free monoid, the free associative unital algebra $\mathbb{C}\langle X\rangle$ is also graded by length. An element of $\mathbb{C}\langle X\rangle$ is called a noncommutative polynomial. A noncommutative polynomial $f \in \mathbb{C}\langle X\rangle$ is *homogeneous of degree* m if $f \in \mathbb{C}X^m$. By a homogeneous ideal in $\mathbb{C}\langle X\rangle$, we mean a two-sided ideal which is generated by a set of homogeneous polynomials.

Let $T = (T_0, T_1, \dots, T_n)$ be an $(n+1)$-tuple of operators acting on a Hilbert space H. If $\alpha = (\alpha_1, \dots, \alpha_m) \in X^m$ is a word of length m, then we shall use the multi-index notation to indicate the product

$$T^\alpha := T_{\alpha_1} \dots T_{\alpha_m},$$

with the convention that $T^1 = 1_H$.

If $p(x) = \sum c_\alpha x^\alpha \in \mathbb{C}\langle X\rangle$ is a noncommutative polynomial, then $p(T)$ refers to the linear combination of operators $p(T) := \sum c_\alpha T^\alpha$.

We recall that a *standard subproduct system* (in the case where the base C^*-algebra agrees with $\mathbb{C}$) is a subproduct system satisfying that $E_{k+m} \subseteq E_k \widehat{\otimes} E_m$ for all $k, m \in \mathbb{N}$ and where the corresponding linear isometry $\iota_{k,m} : E_{k+m} \to E_k \widehat{\otimes} E_m$ agrees with the inclusion. We refer the reader to [**36**, Lemma 6.1] for more details.

Proposition 1.7 [**36**, Proposition 7.2]. *Let H be an $(n+1)$-dimensional Hilbert space with orthonormal basis $\{e_i\}_{i=0}^n$. Then there is a bijective inclusion-reversing correspondence between proper homogeneous ideals $J \subseteq \mathbb{C}\langle x_0, \dots, x_n\rangle$ and standard subproduct systems $\{E_m\}_{m\in\mathbb{N}_0}$ with $E_1 \subseteq H$ (all structure maps are given by canonical inclusions).*

The correspondence works as follows: for a noncommutative polynomial $p = \sum c_\alpha x^\alpha \in \mathbb{C}\langle X\rangle$, we write $p(e) = \sum c_\alpha e_{\alpha_1} \otimes \dots \otimes e_{\alpha_m}$. To any proper homogeneous ideal $J \subseteq \mathbb{C}\langle X\rangle$, we associate the standard subproduct system with fibres $E_m^J := H^{\otimes m} \ominus \{p(e) | p \in J^{(m)}\}$, for every $m \geqslant 0$, where $J^{(m)}$ denotes the degree m component of the ideal J.

Conversely, given a standard subproduct system of Hilbert spaces $\{E_m\}_{m\in\mathbb{N}_0}$ with $E_1 \subseteq H$, we associate to it the proper homogeneous ideal $J_E = \operatorname{span}_{\mathbb{C}}\{p \in \mathbb{C}\langle X\rangle \mid \exists m > 0 : p(e) \in H^{\otimes m} \ominus E_m\}$.

The fact that the two maps are inverses to each other follows from the properties of the structure maps of a subproduct system outlined in Definition 1.1.

Following [**36**, Definition 7.3], we refer to E^J and J_E as the *subproduct system associated to the ideal* J, and the *ideal associated to the subproduct system* E, respectively.

Note that, while the subproduct system E^J associated to a proper homogeneous ideal $J \subseteq \mathbb{C}\langle X\rangle$ depends on the choice of orthonormal basis for the Hilbert space H, different choices give rise to isomorphic subproduct systems (cf. [**36**, Proposition 7.4]).

In this work, we will be considering subproduct systems arising from a homogeneous ideal generated by a single degree two homogeneous polynomial. From an algebraic viewpoint, these ideals are examples of the defining ideals for the one-relator quadratic regular Koszul algebras of global dimension two studied in [**41, 42**].

2. *Subproduct systems from $SU(2)$-actions*

Let $\tau : SU(2) \to U(H)$ be a unitary representation of the Lie group $SU(2)$ on a finite-dimensional Hilbert space H.

We shall in this section see how every such representation $\tau : SU(2) \to U(H)$ gives rise to an $SU(2)$-subproduct system of finite-dimensional Hilbert spaces. These subproduct systems and their associated Cuntz–Pimsner algebras are the main focus of the present paper. To our knowledge, these Cuntz–Pimsner algebras have so far only been studied in the particular case where the representation agrees with the fundamental representation of $SU(2)$ on $\mathbb{C}^2$.

In that case, our procedure recovers the symmetric subproduct system on $\mathbb{C}^2$ (cf. [**36**, Example 1.3; **40**, Example 2.3]).

Definition 2.1. We define the *determinant* of H with respect to the representation τ as the subspace of invariant elements with respect to the diagonal action $\tau \otimes \tau$ on the tensor product $H \otimes H$:

$$\det(\tau, H) = \{\xi \in H \otimes H \mid (\tau(g) \otimes \tau(g))\xi = \xi \quad \forall g \in SU(2)\}.$$

For each $m \in \{2, 3, \ldots\}$ and each $i \in \{1, 2, \ldots, m-1\}$, we define the unitary representation

$$\Delta_m(i) : SU(2) \to U(H^{\otimes m}) \qquad \Delta_m(i) := 1^{\otimes(i-1)} \otimes (\tau \otimes \tau) \otimes 1^{\otimes(m-i-1)}.$$

We then have the subspace $K_m(i) \subseteq H^{\otimes m}$ of invariant elements given by

$$K_m(i) := \{\xi \in H^{\otimes m} \mid \Delta_m(i)(g)(\xi) = \xi, \quad \forall g \in SU(2)\}, \tag{2.1}$$

and we consider the vector space span:

$$K_m := \operatorname{span}_{\mathbb{C}}\big\{\xi \mid \xi \in K_m(i) \text{ for some } i \in \{1, 2, \ldots, m-1\}\big\} = \sum_{i=1}^{m-1} K_m(i) \subseteq H^{\otimes m}. \tag{2.2}$$

In particular, we remark that $K_2 = K_2(1) = \det(\tau, H)$.

Note that we have the following isomorphisms of vector spaces:

$$K_m = K_2 \otimes H^{\otimes(m-2)} + H \otimes K_2 \otimes H^{\otimes(m-3)} + \cdots + H^{\otimes(m-2)} \otimes K_2 \subseteq H^{\otimes m}. \tag{2.3}$$

For each $m \in \mathbb{N}_0$, we put

$$E_m(\tau, H) := \begin{cases} K_m^\perp \subseteq H^{\otimes m} & \text{for} \quad m \geqslant 2 \\ H & \text{for} \quad m = 1 \\ \mathbb{C} & \text{for} \quad m = 0 \end{cases}.$$

When the representation $\tau : SU(2) \to U(H)$ is clear from the context we will suppress it from the notation and put $E_m := E_m(\tau, H)$.

We record the following:

Lemma 2.2. *Let $m \in \{2, 3, \ldots\}$. The diagonal representation*

$$\tau^{\otimes m} : SU(2) \to U(H^{\otimes m})$$

restricts to a unitary representation of $SU(2)$ on the subspace $E_m \subseteq H^{\otimes m}$.

Proof. Since $\tau^{\otimes m}$ is a unitary representation, it suffices to show that each $K_m(i) \subseteq H^{\otimes m}$ is an invariant subspace for $\tau^{\otimes m}$. Thus, let $\xi \in K_m(i)$ for some $i \in \{1, 2, \ldots, m-1\}$ and let $g, h \in SU(2)$. We then have that

$$\begin{aligned} \Delta_m(i)(h)\tau(g)^{\otimes m}(\xi) &= \Big(\tau(g)^{\otimes(i-1)} \otimes 1^{\otimes 2} \otimes \tau(g)^{\otimes(m-i-1)}\Big)\Delta_m(i)(h)\Delta_m(i)(g)(\xi) \\ &= \Big(\tau(g)^{\otimes(i-1)} \otimes 1^{\otimes 2} \otimes \tau(g)^{\otimes(m-i-1)}\Big)(\xi) = \tau(g)^{\otimes m}(\xi). \end{aligned}$$

This proves the lemma. □

For each $m \geqslant 2$, we denote the representation of $SU(2)$ on E_m by

$$\tau_m : SU(2) \to U(E_m).$$

Clearly, $SU(2)$ also acts on $E_1 = H$ (via the representation τ) and on $\mathbb{C}$ (via the trivial representation).

We consider the sequence $E = \{E_m\}_{m=0}^\infty$ of finite-dimensional $SU(2)$-Hilbert spaces together with the structure maps $\iota_{k,m} : E_{k+m} \to E_k \otimes E_m$, $k, m \in \mathbb{N}_0$, induced by the canonical identification $H^{\otimes(k+m)} \cong H^{\otimes k} \otimes H^{\otimes m}$.

PROPOSITION 2.3. *The pair (E,ι) is an $SU(2)$-subproduct system.*

Proof. Consider $k,m\in\mathbb{N}_0$, we need to verify that $E_{k+m}\subseteq E_k\otimes E_m$. We assume that $k,m\geqslant 2$ and leave the remaining (easier) cases to the reader. We recall that $E_l=K_l^\perp$ for all $l\in\{2,3,\ldots\}$, so we need to show that

$$K_{k+m}^\perp\subseteq K_k^\perp\otimes K_m^\perp,$$

but this is equivalent to showing that

$$K_k\otimes H^{\otimes m}+H^{\otimes k}\otimes K_m=(K_k^\perp\otimes K_m^\perp)^\perp\subseteq K_{k+m}.$$

The inclusion $K_k\otimes H^{\otimes m}+H^{\otimes k}\otimes K_m\subseteq K_{k+m}$ is an immediate consequence of the definition of K_l for $l\in\{2,3,\ldots\}$, see (2.1) and (2.2).

By definition of the involved $SU(2)$-actions, we obtain that the inclusions $\iota_{k,m}:E_{k+m}\to E_k\otimes E_m$ are $SU(2)$-equivariant. □

REMARK 1. Note that our subproduct system is by construction isomorphic to the *maximal* subproduct system with prescribed fibres $E_1=H$ and $E_2:=\det(\tau,H)^\perp$, as defined in [**36**, Section 6.1]. However, the context in [**36**, Section 6.1] does not in general yield the extra structure of an $SU(2)$-subproduct system.

We denote the Fock space of our $SU(2)$-equivariant subproduct system by

$$F:=F(\tau,H):=\oplus_{m=0}^\infty E_m(\tau,H)=\oplus_{m=0}^\infty E_m$$

and the associated strongly continuous action of $SU(2)$ on F by

$$\tau_\infty:=\oplus_{m=0}^\infty\tau_m:SU(2)\to U(F). \tag{2.4}$$

For each $m\in\mathbb{N}_0$, we recall that the orthogonal projection onto $E_m\subseteq F$ is denoted by $Q_m: F\to F$.

We apply the notation

$$\mathbb{T}:=\mathbb{T}(\tau,H)\subseteq\mathbb{L}(F)\qquad\text{and}\qquad\mathbb{O}:=\mathbb{O}(\tau,H):=\mathbb{T}/\mathbb{K}(F).$$

for the associated Toeplitz algebra and Cuntz–Pimsner algebra. By the observations carried out in Section 1, we see that both the Toeplitz algebra and the Cuntz–Pimsner algebra carry a gauge action of $SU(2)$.

We let $F_{\text{alg}}\subseteq F$ denote the algebraic direct sum of the subspaces $E_m\subseteq F$:

$$F_{\text{alg}}:=F_{\text{alg}}(\tau,H):=\operatorname{span}\{\xi\in F\mid\xi\in E_m\text{ for some }m\in\mathbb{N}_0\}.$$

We also define $F_+\subseteq F$ as the Hilbert space direct sum

$$F_+:=\oplus_{m=1}^\infty E_m$$

and denote the vacuum vector by $\omega:=1\in E_0=\mathbb{C}\subseteq F$, so that F_+ identifies with the orthogonal complement $(\mathbb{C}\omega)^\perp\subseteq F$. In particular, we have that

$$F_+=(1-Q_0)F\quad\text{and}\quad\mathbb{C}\omega=Q_0F.$$

REMARK 2. Since the Hilbert space H is finite dimensional, it follows from the definition of the determinant as a subspace of $H\otimes H$ that the correspondence from Proposition 1.7 maps the generators of $\det(\tau,H)$ to a finite number of quadratic polynomials. Therefore, our subproduct system corresponds to an ideal generated by a finite collection of quadratic polynomials, and this ideal in turn corresponds to a quadratic algebra (through the correspondence described, for instance, in [**30**, Chapter 4]).

It is therefore not surprising that we make use of the identity (2.3) when inductively constructing our subproduct system: the same formula is used in algebra for realising any given quadratic algebra as a quotient of the tensor algebra.

2.1. *Example: the case of the fundamental representation*

We are now going to describe the subproduct system coming from the fundamental representation $\rho : SU(2) \to U(\mathbb{C}^2)$. We let $\{f_0, f_1\}$ denote the standard basis for $\mathbb{C}^2$.

We have that

$$\det(\rho, \mathbb{C}^2) = \mathbb{C} \cdot (f_0 \otimes f_1 - f_1 \otimes f_0) \subseteq \mathbb{C}^2 \otimes \mathbb{C}^2$$

and thus that

$$K_m(i) = (\mathbb{C}^2)^{\otimes(i-1)} \otimes \mathbb{C} \cdot (f_0 \otimes f_1 - f_1 \otimes f_0) \otimes (\mathbb{C}^2)^{\otimes m-i-1}$$

for all $m \in \{2, 3, \ldots\}$ and all $i \in \{1, 2, \ldots, m-1\}$. Remark in particular that $\det(\rho, \mathbb{C}^2)$ agrees with the usual determinant of $\mathbb{C}^2$ namely the wedge-product $\mathbb{C}^2 \wedge \mathbb{C}^2 \subseteq \mathbb{C}^2 \otimes \mathbb{C}^2$.

Let now $m \in \mathbb{N}$. We recall that the m-fold symmetric tensor product of a finite-dimensional Hilbert space H may be defined as the invariant subspace

$$H^{\otimes_S m} := \big\{\xi \in H^{\otimes m} \mid \sigma(\xi) = \xi \quad \forall \sigma \in S_m\big\},$$

where the symmetric group S_m acts unitarily on $H^{\otimes m}$ via the rule

$$\Phi_\sigma(\xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_m) := \xi_{\sigma^{-1}(1)} \otimes \xi_{\sigma^{-1}(2)} \otimes \ldots \otimes \xi_{\sigma^{-1}(m)}.$$

In particular, we have the identity of vector spaces

$$(\mathbb{C}^2)^{\otimes_S m} = E_m(\rho, \mathbb{C}^2).$$

This follows from the Clebsch–Gordan theory for the representations of $SU(2)$ (cf. [**20**, Appendix C]) and from the properties of the symmetric subproduct system [**36**, Examples 1.3, 6.4].

For each $m \in \mathbb{N}$, we define the vectors

$$f_0^k f_1^{m-k} := p_m(f_0^{\otimes k} \otimes f_1^{\otimes(m-k)}), \quad k = 0, \ldots, m,$$

where $p_m : (\mathbb{C}^2)^{\otimes m} \to (\mathbb{C}^2)^{\otimes m}$ denotes the orthogonal projection onto the symmetric tensor product $(\mathbb{C}^2)^{\otimes_S m} \subseteq (\mathbb{C}^2)^{\otimes m}$. The vectors $\{f_0^k f_1^{m-k},\ k = 0, \ldots, m\}$ form an orthogonal vector space basis for $E_m(\rho, \mathbb{C}^2)$ and their norm is given by the combinatorial expression

$$\|f_0^k f_1^{m-k}\|^2 = \frac{k!(m-k)!}{m!} \tag{2.5}$$

as described in [**5**, Lemma 3.8].

Due to the identification between symmetric tensors and homogeneous polynomials, we obtain a unitary isomorphism between the resulting Fock space $F(\rho, \mathbb{C}^2)$ and the Drury–Arveson space H_2^2, see [**5, 17, 35**].

On our Fock space, we introduce the unbounded selfadjoint operator $N : \mathrm{Dom}(N) \to F(\rho, \mathbb{C}^2)$ defined by $N(\xi) = m \cdot \xi$ for every homogeneous $\xi \in E_m$. The domain of N is given explicitly by

$$\mathrm{Dom}(N) := \big\{\{\xi_m\}_{m=0}^\infty \in F \mid \{m \cdot \xi_m\}_{m=0}^\infty \in F\big\}.$$

The unbounded selfadjoint operator N is referred to as the *number operator*.

THEOREM 2.4 (cf. [5, Proposition 5.3; 36, Example 6.4]). *The Toeplitz algebra $\mathbb{T}(\rho, \mathbb{C}^2)$ associated to the fundamental representation is the C^*-subalgebra of $\mathbb{L}(F)$ generated by the two operators $T_0 := T_{f_0}$ and $T_1 := T_{f_1}$. These satisfy the commutation relations*

$$T_0 T_1 = T_1 T_0, \tag{2.6}$$

$$T_0^* T_0 + T_1^* T_1 = (2 + N)(1 + N)^{-1}, \tag{2.7}$$

$$T_i^* T_j - T_j T_i^* = (1 + N)^{-1}(\delta_{ij} 1 - T_j T_i^*). \tag{2.8}$$

In other words, the pair of operators (T_0, T_1) is a commuting, essentially normal row contraction. We remark that the two operators also satisfy $T_0 T_0^* + T_1 T_1^* = 1 - Q_0$, that is, the contraction is pure.

THEOREM 2.5 [5, Theorem 5.7]. *The Toeplitz algebra $\mathbb{T}(\rho, \mathbb{C}^2)$ contains the algebra of compact operators on the Drury–Arveson space H_2^2, and we have an exact sequence of C^*-algebras*

$$0 \longrightarrow \mathbb{K}(H_2^2) \longrightarrow \mathbb{T}(\rho, \mathbb{C}^2) \longrightarrow C(S^3) \longrightarrow 0\ , \tag{2.9}$$

where $C(S^3)$ is the commutative C^*-algebra of continuous functions on the 3-sphere $S^3 \subseteq \mathbb{C}^2$. In particular, we have that the Cuntz–Pimsner algebra $\mathbb{O}(\rho, \mathbb{C}^2)$ is isomorphic to $C(S^3)$.

The above Toeplitz extension is well studied and understood (see, for instance, [29] for an index-theoretic perspective on Toeplitz extensions). Moreover, the Toeplitz algebra is known to be KK-equivalent to the complex numbers. We are going to prove that this is a general feature of the Toeplitz algebras of the $SU(2)$-subproduct systems constructed from irreducible $SU(2)$-representations.

2.2. *Computation of determinants*

We now provide a computation of the subspace $\det(\tau, H) \subseteq H \otimes H$, starting with the case where the representation $\tau : SU(2) \to U(H)$ is irreducible. Recall from [20, Example 4.10, Proposition 4.11] that for every fixed positive integer, there exists a unique irreducible representation of the group $SU(2)$ on an complex vector space of that dimension. Uniqueness follows from the orthogonality relations for characters of representations [9, Proposition 5.3].

In what follows, we will disregard the case where τ is (unitarily equivalent to) the trivial representation on $\mathbb{C}$. We put $n := \dim(H) - 1 \in \mathbb{N}$ and we let $L_n = (\mathbb{C}^2)^{\otimes_S n}$ denote the n-fold symmetric tensor product $\mathbb{C}^2$. We let $\rho_n : SU(2) \to U(L_n)$ denote the irreducible representation obtained by restriction of the n-fold tensor product of the fundamental representation. For each $k \in \{0, 1, \ldots, n\}$, we define the unit vector

$$e_k := \sqrt{\frac{n!}{k!(n-k)!}} \cdot f_0^k f_1^{n-k} \in L_n, \tag{2.10}$$

so that $\{e_k\}_{k=0}^n$ is an orthonormal basis for L_n, see Subsection 2.1.

PROPOSITION 2.6. *Suppose that $\tau : SU(2) \to U(H)$ is irreducible and let $V : L_n \to H$ be a unitary operator intertwining τ with ρ_n. Then the determinant $\det(\tau, H) \subseteq H \otimes H$ is a one-dimensional vector space spanned by the vector*

$$(V \otimes V)\left((n+1)^{-1/2} \sum_{k=0}^{n} (-1)^{n-k} e_k \otimes e_{n-k}\right).$$

Proof. Using the representation theory for $SU(2)$, we know that we may find a unitary operator W from $\oplus_{m=0}^n L_{2m}$ to $L_n \otimes L_n$ intertwining the representations $\oplus_{m=0}^n \rho_{2m}$ and $\rho_n \otimes \rho_n$. The structure of this unitary operator is determined by the Clebsch–Gordan coefficients and on $L_0 = \mathbb{C}$, it is given by

$$W(1) = (n+1)^{-1/2} \sum_{k=0}^n (-1)^{n-k} e_k \otimes e_{n-k} = \sum_{k,l=0}^n C^{0,0}_{n/2,k-n/2,n/2,l-n/2} \cdot e_k \otimes e_l,$$

with $C^{0,0}_{n/2,k-n/2,n/2,l-n/2}$ denoting the Clebsch–Gordan coefficients, as described, for instance, in [**37**]. □

REMARK 3. Going back to the correspondence described in Subsection 1.1, the homogeneous ideal associated to the subproduct system of the irreducible representation $\rho_n : SU(2) \to U(L_n)$ is the proper homogeneous ideal in the free algebra on $(n+1)$ generators $\mathbb{C}\langle x_0, \ldots, x_n \rangle$ generated by the single degree two homogeneous polynomial $p(x_0, \ldots, x_n) = \sum_{i=0}^n (-1)^i x_i x_{n-i}$.

In the more general case where $\tau : SU(2) \to U(H)$ need not be irreducible, we choose a unitary operator $V : \bigoplus_{m=0}^\infty L_m^{\oplus k_m} \to H$ intertwining the representations $\bigoplus_{m=0}^\infty \rho_m^{\oplus k_m}$ and τ, where $k_m \in \mathbb{N}_0$ for all $m \in \mathbb{N}_0$ and we identify $L_n^{\oplus 0}$ with $\{0\}$. Of course, since H is finite dimensional, there exists an $M \in \mathbb{N}_0$ such that $k_m = 0$ for all $m \geqslant M$.

PROPOSITION 2.7. *The determinant* $\det(\tau, H) \subseteq H \otimes H$ *has dimension* $\sum_{m=0}^\infty k_m^2$ *and is unitarily isomorphic to the Hilbert space*

$$\bigoplus_{m=0}^\infty \det(\rho_m, L_m)^{\oplus k_m^2} \subseteq \bigoplus_{m=0}^\infty (L_m \otimes L_m)^{\oplus k_m^2}$$

via the isometry

$$\textstyle\bigoplus_{m=0}^\infty (L_m \otimes L_m)^{\oplus k_m^2} \cong \bigoplus_{m=0}^\infty (L_m^{\oplus k_m} \otimes L_m^{\oplus k_m}) \xrightarrow{\iota} H \otimes H,$$

where ι *is defined in degree* m *by* $\iota(\xi_m \otimes \eta_m) := V(\xi_m \delta_m) \otimes V(\eta_m \delta_m)$.

Proof. Using the unitary operator $V : \oplus_{m=0}^\infty L_m^{\oplus k_m} \to H$. we identify $H \otimes H$ with

$$\left(\oplus_{m=0}^\infty L_m^{\oplus k_m}\right) \otimes \left(\oplus_{l=0}^\infty L_l^{\oplus k_l}\right) \cong \oplus_{m,l=0}^\infty (L_m \otimes L_l)^{\oplus k_m \cdot k_l}.$$

Under this unitary isomorphism the representation $\tau \otimes \tau$ identifies with the representation $\oplus_{m,l=0}^\infty (\rho_m \otimes \rho_l)^{\oplus k_m \cdot k_l}$. Since the tensor product of representations $\rho_m \otimes \rho_l$ contains no copy of the trivial representation for $m \neq l$, the determinant in question identifies with $\oplus_{m=0}^\infty \det(\rho_m, L_m)^{\oplus k_m^2}$. The claim concerning the dimension of the determinant now follows immediately from Proposition 2.6. □

3. *Fusion rules for an* $SU(2)$*-equivariant subproduct system*

From now on, we fix a strictly positive integer $n \in \mathbb{N}$ and consider the irreducible representation $\rho_n : SU(2) \to U(L_n)$. We write $\{e_k\}_{k=0}^n$ for the orthonormal basis for the Hilbert space $L_n = (\mathbb{C}^2)^{\otimes_S n}$ introduced in (2.10). We put

$$D := \det(\rho_n, L_n) \subseteq L_n \otimes L_n$$

so that D is a one-dimensional vector space spanned by the unit vector

$$\delta := \frac{1}{\sqrt{n+1}} \cdot \sum_{k=0}^{n} (-1)^k e_k \otimes e_{n-k} \in D, \tag{3.1}$$

as shown in Proposition 2.6.

We have an associated sequence of finite-dimensional Hilbert spaces $\{E_m\}_{m=0}^{\infty} := \{E_m(\rho_n, L_n)\}_{m=0}^{\infty}$ defined as in Section 2. Each of these Hilbert spaces carries a unitary representation of $SU(2)$ which in degree $m \in \mathbb{N}_0$ is induced by the tensor product $\rho_n^{\otimes m} : L_n^{\otimes m} \to L_n^{\otimes m}$. We emphasise that these representations are in general not irreducible (unless $n = 1$, in which case each E_m agrees with the unique irreducible $(m+1)$-dimensional representation space L_m).

The main result of this section is the following orthogonal decomposition of the tensor products:

THEOREM 3.1. *For each $k, l \in \mathbb{N}_0$, there exists an explicit $SU(2)$-equivariant unitary isomorphism*

$$E_k \otimes E_l \cong E_{k+l} \oplus E_{k+l-2} \oplus \ldots \oplus E_{|k-l|}.$$

We view Theorem 3.1 as an expression of the *fusion rules* for our $SU(2)$-equivariant subproduct system. Moreover, for $n > 1$, one may interpret Theorem 3.1 as a non-irreducible solution to the fusion rules of $SU(2)$. For $n = 1$, we exactly recover the (irreducible) fusion rules of $SU(2)$ (see, for instance [**14**]). The fusion rules presented in Theorem 3.1 play a key role in our later computation of the K-theory of the Toeplitz algebra $\mathbb{T}(\rho_n, L_n)$.

For every $k, m \in \mathbb{N}_0$, we remind the reader of the notation

$$\iota_{k,m} : E_{k+m} \to E_k \otimes E_m \qquad \text{and} \qquad p_{k,m} := \iota_{k,m}\iota_{k,m}^* : E_k \otimes E_m \to E_k \otimes E_m$$

for the inclusion and the associated orthogonal projection.

3.1. *Preliminaries on integer sequences*

We consider the sequence of strictly positive integers $\{d_m\}_{m=0}^{\infty}$ defined recursively by the formula:

$$d_0 := 1, \;\; d_1 := n+1, \;\; d_m := d_1 \cdot d_{m-1} - d_{m-2}, \;\; m \geqslant 2. \tag{3.2}$$

We furthermore put $d_{-1} := 0$. These sequences are well studied and understood and we refer the reader to the Online Encyclopaedia of Integer Sequences [**31**], where examples are given.

Later on, in Lemma 3.6, we shall see that $d_m = \dim(E_m)$ for all $m \in \mathbb{N}_0$. Towards this goal, we start out by summarising various identities involving the numbers $d_m \in \mathbb{N}$, $m \in \mathbb{N}_0$.

LEMMA 3.2. *Let $m, k, l \in \mathbb{N}_0$. We have the identities*

$$d_m^2 - d_{m-1}d_{m+1} = 1 \qquad \text{and} \qquad \sum_{i=0}^{l} d_{k+m+2i} = d_{k+l}d_{m+l} - d_{k-1}d_{m-1}.$$

Proof. For the convenience of the reader, we provide a proof of the second of the two identities. The proof runs by induction on $l \in \mathbb{N}_0$ but the only tricky part is the induction start. So suppose that $l = 0$. We shall prove by induction on $m \in \mathbb{N}_0$ that

$$d_{k+m} = d_k d_m - d_{k-1}d_{m-1}, \tag{3.3}$$

whenever $k \in \mathbb{N}_0$ is fixed. For $m = 0, 1$, there is nothing to prove, so supposing that the identity in (3.3) is verified for all $m \in \{0, 1, \ldots, m_0\}$ for some $m_0 \in \mathbb{N}$, we compute that

$$\begin{aligned} d_{k+m_0+1} &= d_{k+m_0} d_1 - d_{k+m_0-1} = (d_k d_{m_0} - d_{k-1} d_{m_0-1}) d_1 - d_k d_{m_0-1} + d_{k-1} d_{m_0-2} \\ &= d_k (d_{m_0} d_1 - d_{m_0-1}) - d_{k-1}(d_{m_0-1} d_1 - d_{m_0-2}) = d_k d_{m_0+1} - d_{k-1} d_{m_0}. \end{aligned}$$

This proves the lemma. □

We remind the reader that $n \in \mathbb{N}$, that is, we are excluding the case of the trivial representation. This is essential for our results, which do not hold for $n = 0$.

LEMMA 3.3. *The sequence of quotients* $\{d_{m-1}/d_m\}_{m=0}^{\infty}$ *is strictly increasing and converges to the limit* $\gamma_n = (n+1-\sqrt{(n+1)^2-4})/2 \in (0, 1]$.

Proof. We first remark that $d_{m+1} > d_m$ for all $m \in \mathbb{N}_0$, and hence that $d_{m+1} \geqslant m+1$ (because $d_0 = 1$). Indeed, assuming that $d_m > d_{m-1}$ for some $m \in \mathbb{N}$, we obtain that

$$d_{m+1} - d_m = d_m \cdot n - d_{m-1} > d_{m-1} \cdot (n-1) \geqslant 0,$$

since $n \in \mathbb{N}$ by our standing assumptions. The claimed result now follows by induction (remark that the assumption $n \in \mathbb{N}$ translates into the strict inequality $d_1 > d_0$.

We also observe that Lemma 3.2 implies

$$\frac{d_{m-1}}{d_m} = \sum_{j=1}^{m} \left(\frac{d_{j-1}}{d_j} - \frac{d_{j-2}}{d_{j-1}} \right) = \sum_{j=1}^{m} \frac{1}{d_{j-1} d_j}.$$

This shows that our sequence is strictly increasing and moreover, our lower bound on the dimensions imply that the infinite sum $\sum_{j=1}^{\infty} 1/(d_{j-1} d_j)$ converges.

In order to compute the limit γ_n, we apply (3.2) to see that

$$\frac{d_{m-1}}{d_m} = \frac{d_m + d_{m-2}}{d_m d_1} = \frac{1}{d_1} + \frac{d_{m-2}}{d_1 d_m} = \frac{1}{n+1} + \frac{1}{n+1} \cdot \frac{d_{m-2}}{d_m},$$

for all $m \in \mathbb{N}$, implying by taking limits that

$$\gamma_n = \frac{1}{n+1} + \frac{1}{n+1} \cdot \gamma_n^2.$$

The above quadratic equation has only one solution in the interval (0,1], which yields

$$\gamma_n = \frac{n+1-\sqrt{(n+1)^2-4}}{2}. \tag{3.4}$$

This proves the claim. □

REMARK 4. Note that d_m agrees with the number of length m words in the alphabet $\{0, 1, 2, \ldots, n\}$ that do not contain the string $(0, n)$ (cf. [**21**, Corollary 37]). In particular, our sequences are an example of cardinality sequences of word systems: due to [**18**, Proposition 3.2], for every finite-dimensional subproduct systems of Hilbert spaces $\{H_m\}_{m \in \mathbb{N}_0}$, there exists a word system $\{X_m\}_{m \in \mathbb{N}_0}$ such that $\dim(H_m) = |X_m|$ for all $m \in \mathbb{N}_0$ (see also [**3**, Lemma 1.1] for a noncommutative algebraic version of this claim). However, the subproduct system associated to the word system described above is, in general, not isomorphic to the original one.

For $n \geqslant 2$, the constant γ_n in (3.4) equals the Perron–Frobenius eigenvalue of the $(n+1) \times (n+1)$-matrix with all entries equal to 1 and except for a single 0 in position $(1, n+1)$. See, for instance, [**27**, Observation 1.4.2]. For $n = 1$, we cannot use the Perron–Frobenius theory because the matrix associated to the set of words in the alphabet is not an *irreducible* one. Still the above ratio converges to the highest eigenvalue of said 2×2-matrix.

To end this subsection, we define the strictly positive integers

$$\mu_m := \frac{d_m d_{m-1}}{d_1} \ , \ \ m \in \mathbb{N}. \tag{3.5}$$

Using the recursive definition (3.2), it can be verified that the sequences $\{\mu_m\}_{m=1}^{\infty}$ and $\{d_m\}_{m=0}^{\infty}$ are connected via the identity

$$d_m^2 = \mu_m + \mu_{m+1} \ , \ \ m \in \mathbb{N}. \tag{3.6}$$

This can be used to prove that the sequence $\{\mu_m\}_{m=1}^{\infty}$ can also be obtained using the recurrence relation

$$\mu_{m+1} = ((n+1)^2 - 2)\mu_m - \mu_{m-1} + 1, \quad \mu_1 = 1, \quad \mu_2 = (n+1)^2 - 1. \tag{3.7}$$

For $n = 1, 2, 3$, we recover known combinatorial sequences, see [**31**], but at the moment of writing this paper, the sequences $\{\mu_m\}_{m=1}^{\infty}$ for $n \geqslant 4$ were not listed in the OEIS.

3.2. *Decomposing tensor products by E_1 from the right*

We start out by proving the decomposition result in Theorem 3.1 in the case where the second representation space is just E_1. Thus, for every $m \in \mathbb{N}$, we are going to show that $E_m \otimes E_1 \cong E_{m+1} \oplus E_{m-1}$ via an $SU(2)$-equivariant unitary.

We recall that $K_2 = \mathbb{C} \cdot \delta$ and for every $m \geqslant 2$, we have that

$$K_m = \sum_{i=0}^{m-2} L_n^{\otimes i} \otimes K_2 \otimes L_n^{\otimes (m-2-i)}.$$

We also put $K_1 = K_0 := \{0\}$ and define $E_m = K_m^{\perp} \subseteq L_n^{\otimes m}$ for all $m \in \mathbb{N}_0$. As in Definition 1.1, we denote the identity operator on the Hilbert space E_m, with the symbol 1_m.

We recursively define a linear map $G_m : E_{m-1} \to K_{m+1}$ for each $m \in \mathbb{N}$:

$$G_1(1) := \delta \ , \ \ G_m := G_{m-1} \otimes 1 + (-1)^{(n+1)(m-1)} d_{m-1} \cdot 1_{m-1} \otimes G_1 \qquad \text{for } m \geqslant 2, \tag{3.8}$$

where we are suppressing the inclusion $\iota_{m-2,1} : E_{m-1} \to E_{m-2} \otimes E_1$ and the obvious identification $\iota_{m-1,0} : E_{m-1} \xrightarrow{\cong} E_{m-1} \otimes E_0$.

Lemma 3.4. *Let $m \in \mathbb{N}$. The linear map $G_m : E_{m-1} \to K_{m+1}$ is equivariant meaning that*

$$\rho_n^{\otimes (m+1)}(g) G_m = G_m \rho_n^{\otimes (m-1)}(g) \qquad \text{for all } g \in SU(2).$$

Proof. The proof runs by induction on $m \in \mathbb{N}$. The case where $m = 1$ holds since $\rho_n(g)^{\otimes 2}(\delta) = \delta = G_1(1)$. Suppose now that the equivariance condition holds for some $m \in \mathbb{N}$. For $\xi \in E_m$, the recursive definition of the maps G_m in (3.8) implies

$$\begin{aligned} \rho_n^{\otimes (m+2)}(g) G_{m+1}(\xi) &= \rho_n^{\otimes (m+2)}(g)(G_m \otimes 1)(\xi) + (-1)^{(n+1)m} d_m \cdot \rho_n^{\otimes (m+2)}(g)(\xi \otimes \delta) \\ &= (G_m \otimes 1)\rho_n^{\otimes m}(g)(\xi) + (-1)^{(n+1)m} d_m \cdot \rho_n^{\otimes m}(g)(\xi) \otimes \delta \\ &= G_{m+1} \rho_n^{\otimes m}(g)(\xi). \end{aligned}$$

This proves the lemma. □

Lemma 3.5. *Let $m \in \mathbb{N}$. We have:*

(i) $\langle (G_m \otimes 1)(\xi), \eta \otimes \delta \rangle = (-1)^{(n+1)m+1} \frac{d_{m-1}}{d_1} \cdot \langle \xi, \eta \rangle$ *for all* $\xi \in E_{m-1} \otimes E_1$, $\eta \in E_m$;
(ii) $\langle G_m(\xi), G_m(\eta) \rangle = \mu_m \cdot \langle \xi, \eta \rangle$ *for all* $\xi, \eta \in E_{m-1}$;
(iii) $\langle (G_m \otimes 1)(\xi), G_{m+1}(\eta) \rangle = 0$ *for all* $\xi \in E_{m-1} \otimes E_1, \eta \in E_m$.

Proof. (1) We focus on the case where $m \geqslant 2$. Let $\xi = \sum_{j=0}^n \xi_j \otimes e_j \in E_{m-1} \otimes E_1$ and $\eta \in E_m$ be given. We compute that

$$\begin{aligned}
\big\langle (G_m \otimes 1)(\xi), \eta \otimes \delta \big\rangle &= \sum_{j=0}^n \big\langle G_m(\xi_j) \otimes e_j, \eta \otimes \delta \big\rangle \\
&= \sum_{j=0}^n \big\langle (G_{m-1} \otimes 1)(\xi_j) \otimes e_j, \eta \otimes \delta \big\rangle \\
&\quad + (-1)^{(n+1)(m-1)} d_{m-1} \cdot \sum_{j=0}^n \langle \xi_j \otimes \delta \otimes e_j, \eta \otimes \delta \rangle \\
&= (-1)^{(n+1)(m-1)} d_{m-1} \cdot \sum_{j=0}^n \frac{(-1)^n}{n+1} \cdot \langle \xi_j \otimes e_j \otimes e_{n-j} \otimes e_j, \eta \otimes e_{n-j} \otimes e_j \rangle \\
&= (-1)^{(n+1)m+1} \frac{d_{m-1}}{d_1} \cdot \langle \xi, \eta \rangle,
\end{aligned}$$

where the third identity follows from the structure of the vector $\delta = \frac{1}{\sqrt{n+1}} \cdot \sum_{j=0}^n (-1)^j e_j \otimes e_{n-j}$ and from the inclusion $\mathrm{Im}(G_{m-1}) \subseteq K_m = E_m^\perp$.

(2) The proof runs by induction on $m \in \mathbb{N}$. For $m = 1$, the result follows since $\langle \delta, \delta \rangle = 1$. Next, given $m \geqslant 1$ we assume that (2) holds and for $\xi, \eta \in E_m$, we then compute that

$$\begin{aligned}
\big\langle G_{m+1}(\xi), G_{m+1}(\eta) \big\rangle &= \big\langle (G_m \otimes 1)(\xi), (G_m \otimes 1)(\eta) \big\rangle + d_m^2 \cdot \langle \xi \otimes \delta, \eta \otimes \delta \rangle \\
&\quad + (-1)^{(n+1)m} d_m \cdot \Big(\big\langle (G_m \otimes 1)(\xi), \eta \otimes \delta \big\rangle \\
&\quad + \big\langle \xi \otimes \delta, (G_m \otimes 1)(\eta) \big\rangle \Big) \\
&= \mu_m \cdot \langle \xi, \eta \rangle + d_m^2 \cdot \langle \xi, \eta \rangle \\
&\quad + (-1)^{(n+1)m} d_m \cdot (-1)^{(n+1)m+1} \frac{d_{m-1}}{d_1} \cdot 2 \langle \xi, \eta \rangle \\
&= \mu_m \cdot \langle \xi, \eta \rangle + d_m^2 \cdot \langle \xi, \eta \rangle - 2 \frac{d_m d_{m-1}}{d_1} \cdot \langle \xi, \eta \rangle \\
&= (d_m^2 - \mu_m) \cdot \langle \xi, \eta \rangle = \mu_{m+1} \cdot \langle \xi, \eta \rangle,
\end{aligned}$$

where the second identity follows from the induction hypothesis and (1) and the fifth identity follows from (3.6).

(3) Let $\xi \in E_{m-1} \otimes E_1$ and $\eta \in E_m$ be given. Using (1) and (2), we compute that

$$\begin{aligned}
\big\langle (G_m \otimes 1)(\xi), G_{m+1}(\eta) \big\rangle &= \big\langle (G_m \otimes 1)(\xi), (G_m \otimes 1)(\eta) \big\rangle \\
&\quad + (-1)^{(n+1)m} d_m \cdot \big\langle (G_m \otimes 1)(\xi), \eta \otimes \delta \big\rangle \\
&= \mu_m \cdot \langle \xi, \eta \rangle - \frac{d_m d_{m-1}}{d_1} \cdot \langle \xi, \eta \rangle = 0.
\end{aligned}$$

This proves the lemma. □

LEMMA 3.6. *The vector space sum yields a unitary isomorphism of Hilbert spaces*

$$(K_m \otimes E_1) \oplus G_m(E_{m-1}) \cong K_{m+1},$$

for all $m \geqslant 1$.

Proof. For $m = 1$, the vector space decomposition follows immediately from the identities $G_1(E_0) = \mathbb{C} \cdot \delta = K_2$ and $K_1 = \{0\}$.

Suppose thus that $m \geqslant 2$. We start out by proving that the vector space sum yields a surjective map from $(K_m \otimes E_1) \oplus G_m(E_{m-1})$ to K_{m+1} or, in other words, that $K_{m+1} = (K_m \otimes E_1) + G_m(E_{m-1})$. Let thus $\xi \in K_{m+1}$ be given. Remark that

$$\begin{aligned} K_{m+1} &= K_m \otimes E_1 + E_1^{\otimes(m-1)} \otimes K_2 = K_m \otimes E_1 + K_{m-1} \otimes K_2 + E_{m-1} \otimes K_2 \\ &= K_m \otimes E_1 + E_{m-1} \otimes K_2. \end{aligned}$$

We may therefore choose $\eta \in K_m \otimes E_1$ and $\zeta \in E_{m-1}$ such that $\xi = \eta + \zeta \otimes \delta$. Using (3.8), we then obtain that

$$\xi = \eta + \frac{(-1)^{(n+1)(m-1)}}{d_{m-1}} \cdot (G_m(\zeta) - (G_{m-1} \otimes 1)(\zeta))$$

Since $\mathrm{Im}(G_{m-1}) \subseteq K_m$, this proves the surjectivity claim.

To prove that the Hilbert space direct sum in question is isometrically isomorphic to K_{m+1}, we apply induction on $m \geqslant 1$. The case $m = 1$ has already been discussed, so suppose that the vector space sum yields an isometry for some $m \geqslant 1$ and let $\eta \in K_{m+1} \otimes E_1$ and $\zeta \in E_m$ be given. We need to show that $\langle \eta, G_{m+1}(\zeta) \rangle = 0$. By the surjectivity part, we may find $\xi \in K_m \otimes E_1 \otimes E_1$ and $\rho \in E_{m-1} \otimes E_1$ such that $\eta = \xi + (G_m \otimes 1)(\rho)$. By Lemma 3.5 part (3), the induction hypothesis, and the fact that $K_m = E_m^\perp$, we then have the identities

$$\begin{aligned} \big\langle \eta, G_{m+1}(\zeta) \big\rangle &= \big\langle \xi, G_{m+1}(\zeta) \big\rangle + \big\langle (G_m \otimes 1)(\rho), G_{m+1}(\zeta) \big\rangle = \big\langle \xi, G_{m+1}(\zeta) \big\rangle \\ &= \big\langle \xi, (G_m \otimes 1)(\zeta) \big\rangle + (-1)^{(n+1)m} d_m \cdot \big\langle \xi, \zeta \otimes \delta \big\rangle = 0. \end{aligned}$$

This proves the lemma. □

LEMMA 3.7. *We have* $\dim(E_m) = d_m$ *for all* $m \in \mathbb{N}_0$.

Proof. This is a consequence of Lemma 3.6, yielding the following identities of dimensions:

$$\begin{aligned} \dim(E_{m+1}) &= (n+1)^{m+1} - \dim(K_{m+1}) \\ &= (n+1)^{m+1} - (n+1) \cdot \dim(K_m) - \dim(E_{m-1}) \\ &= (n+1) \cdot \dim(E_m) - \dim(E_{m-1}). \end{aligned}$$

Since $d_0 = \dim(E_0)$ and $d_1 = \dim(E_1)$ and since the sequences $\{d_m\}_{m=0}^\infty$ and $\{\dim(E_m)\}_{m=0}^\infty$ satisfy the same recursion formula, they must necessarily agree. □

REMARK 5. Note that a subproduct system of Hilbert spaces $\{E_m\}_{m \in \mathbb{N}_0}$ is called *commutative* if the corresponding Fock space is a subspace of the symmetric Fock space on E_1 or, equivalently, if $E_m \subseteq E_1^{\otimes_S m}$ for all $m \in \mathbb{N}_0$. It follows from Lemma 3.7 that our subproduct systems are noncommutative for every $n > 1$, as we have $\dim(E_2) = (n+1)^2 - 1 > \binom{n+2}{2} = \dim((\mathbb{C}^{n+1})^{\otimes_S 2})$.

Lemma 3.6 has the important consequence that the image of $G_m : E_{m-1} \to K_{m+1}$ is in fact equal to the intersection $K_{m+1} \cap (E_m \otimes E_1)$. Moreover, Lemma 3.5 implies that the induced $SU(2)$-equivariant linear map

$$V_m := \frac{(-1)^{(n+1)(m-1)}}{\sqrt{\mu_m}} \cdot G_m : E_{m-1} \to E_m \otimes E_1 \tag{3.9}$$

is an isometry for all $m \geqslant 1$. We have therefore established the announced main result of this subsection:

Proposition 3.8. *Let $m \in \mathbb{N}$. The linear map*

$$\begin{pmatrix} \iota_{m,1} & V_m \end{pmatrix} : E_{m+1} \oplus E_{m-1} \to E_m \otimes E_1$$

is an $SU(2)$-equivariant unitary isomorphism.

3.3. *Decomposing tensor products by E_1 from the left*

The result of Proposition 3.8 provides us with an $SU(2)$-equivariant unitary isomorphism $E_{m+1} \oplus E_{m-1} \to E_1 \otimes E_m$, for every $m \in \mathbb{N}_0$, obtained by composing $(\iota_{m,1} \quad V_m)$ with the flip map $E_m \otimes E_1 \to E_1 \otimes E_m$. In this subsection, we shall provide an alternative $SU(2)$-equivariant unitary isomorphism $E_{m+1} \oplus E_{m-1} \to E_1 \otimes E_m$, where the relevant isometry $E_{m-1} \to E_1 \otimes E_m$ is given by a recursive formula which is similar to (3.8). This alternative $SU(2)$-equivariant unitary isomorphism will play an essential role in the rest of our work, as one of the building blocks for our proof of the KK-equivalence between the Toeplitz algebra and the complex numbers.

We define the linear maps $G'_m : E_{m-1} \to K_{m+1}$, $m \in \mathbb{N}_0$, recursively by the formulae

$$G'_1(1) := \delta \ , \ \ G'_m := 1 \otimes G'_{m-1} + (-1)^{(n+1)(m-1)} d_{m-1} \cdot G'_1 \otimes 1_{m-1} \ , \ \ m \geqslant 2, \tag{3.10}$$

where the vector $\delta \in K_2$ and the constant d_{m-1} are defined in (3.1) and (3.2).

Again, note that we are suppressing the inclusion $\iota_{1,m-2} : E_{m-1} \to E_1 \otimes E_{m-2}$ (for $m \geqslant 2$) and the obvious identification $\iota_{0,m-1} : E_{m-1} \xrightarrow{\cong} E_0 \otimes E_{m-1}$.

Lemma 3.9. *Let $m \in \mathbb{N}$. The linear map $G'_m : E_{m-1} \to K_{m+1}$ is equivariant meaning that*

$$\rho_n^{\otimes(m+1)}(g) G'_m = G'_m \rho_n^{\otimes(m-1)}(g) \qquad \text{for all } g \in SU(2).$$

Proof. The proof runs by induction on $m \in \mathbb{N}$, using the same argument as in the proof of Lemma 3.4. □

Lemma 3.10. *Let $m \in \mathbb{N}$. We have the identities:*

(i) $\langle (1 \otimes G'_m)(\xi), \delta \otimes \eta \rangle = (-1)^{(n+1)m+1} \frac{d_{m-1}}{d_1} \cdot \langle \xi, \eta \rangle$ *for all $\xi \in E_1 \otimes E_{m-1}$, $\eta \in E_m$;*
(ii) $\langle G'_m(\xi), G'_m(\eta) \rangle = \mu_m \cdot \langle \xi, \eta \rangle$ *for all $\xi, \eta \in E_{m-1}$;*
(iii) $\langle (1 \otimes G'_m)(\xi), G'_{m+1}(\eta) \rangle = 0$ *for all $\xi \in E_1 \otimes E_{m-1}$, $\eta \in E_m$.*

Proof. The proof follows the proof of Lemma 3.5 *verbatim.* □

Lemma 3.11. *For each $m \in \mathbb{N}$, the vector space sum yields a unitary isomorphism of Hilbert spaces*

$$(E_1 \otimes K_m) \oplus G'_m(E_{m-1}) \cong K_{m+1}.$$

Proof. The proof is *mutatis mutandis* the same as the proof of Lemma 3.6. □

In analogy with the previous subsection, we obtain from Lemma 3.11 that the image of G'_m agrees with the intersection $K_{m+1} \cap (E_1 \otimes E_m)$ and, moreover, we see from Lemma 3.10 that the induced $SU(2)$-equivariant linear map

$$V'_m := \frac{(-1)^{(n+1)(m-1)}}{\sqrt{\mu_m}} \cdot G'_m : E_{m-1} \to E_1 \otimes E_m \tag{3.11}$$

is an isometry for all $m \geqslant 1$. We announce the following:

PROPOSITION 3.12. *Let $m \in \mathbb{N}$. The linear map*

$$\begin{pmatrix} \iota_{1,m} & V'_m \end{pmatrix} : E_{m+1} \oplus E_{m-1} \to E_1 \otimes E_m$$

is an $SU(2)$-equivariant unitary isomorphism.

3.4. *Orthogonal decomposition of tensor products of representations*

As we saw in Lemmas 3.6 and 3.11, we may change the codomains of the linear maps defined in (3.8) and (3.10) and instead consider the $SU(2)$-equivariant linear maps

$$G_m : E_{m-1} \to E_m \otimes E_1 \qquad \text{and} \qquad G'_m : E_{m-1} \to E_1 \otimes E_m$$

for all $m \in \mathbb{N}$. These linear maps then satisfy the recursive relations

$$\begin{aligned} (\iota_{m-1,1} \otimes 1) \circ G_m &= (G_{m-1} \otimes 1) \circ \iota_{m-2,1} + (-1)^{(n+1)(m-1)} d_{m-1} \cdot 1_{m-1} \otimes G_1 \qquad \text{and} \\ (1 \otimes \iota_{1,m-1}) \circ G'_m &= (1 \otimes G'_{m-1}) \circ \iota_{1,m-2} + (-1)^{(n+1)(m-1)} d_{m-1} \cdot G'_1 \otimes 1_{m-1} \end{aligned} \tag{3.12}$$

for all $m \geqslant 2$. We recall that $G'_1(1) = G_1(1) = \delta$, where the unit vector $\delta \in K_2$ was introduced in (3.1).

For every $k, m \in \mathbb{N}_0$, we introduce the $SU(2)$-equivariant linear map

$$\sigma_{k,m} : E_k \otimes E_m \to E_{k+1} \otimes E_{m+1} \qquad \sigma_{k,m} := (1_{k+1} \otimes \iota^*_{1,m})(G_{k+1} \otimes 1_m). \tag{3.13}$$

For $k = -1$ or $m = -1$, we put $\sigma_{k,m} := 0 : \{0\} \to E_{k+1} \otimes E_{m+1}$. These linear maps are going to play a key role in establishing the main result of this section, namely the fusion rules for our $SU(2)$-equivariant subproduct system as announced in Theorem 3.1. Before we can study these maps in more detail, we need a few preliminary lemmas.

LEMMA 3.13. *Let $m \in \mathbb{N}$. We have*

$$\begin{aligned} G_m &= (-1)^{(n+1)(m-1)} d_{m-1} \cdot (\iota^*_{m-1,1} \otimes 1)(1_{m-1} \otimes G_1) \qquad \textit{and} \\ G'_m &= (-1)^{(n+1)(m-1)} d_{m-1} \cdot (1 \otimes \iota^*_{1,m-1})(G_1 \otimes 1_{m-1}). \end{aligned}$$

Proof. We focus on proving the claim for $G_m : E_{m-1} \to E_m \otimes E_1$. To this end, we compute that

$$\begin{aligned} G_m &= (\iota^*_{m-1,1} \iota_{m-1,1} \otimes 1) G_m \\ &= (\iota^*_{m-1,1} \otimes 1)(G_{m-1} \otimes 1)\iota_{m-2,1} \\ &\quad + (-1)^{(n+1)(m-1)} d_{m-1} \cdot (\iota^*_{m-1,1} \otimes 1)(1_{m-1} \otimes G_1) \\ &= (-1)^{(n+1)(m-1)} d_{m-1} \cdot (\iota^*_{m-1,1} \otimes 1)(1_{m-1} \otimes G_1), \end{aligned}$$

where the last identity follows from $\mathrm{Im}(G_{m-1}) = K_m \cap (E_{m-1} \otimes E_1)$ and from the fact that $\iota_{m-1,1}\iota^*_{m-1,1} : E_{m-1} \otimes E_1 \to E_{m-1} \otimes E_1$ is the orthogonal projection onto the subspace $E_m = K_m^\perp$. □

LEMMA 3.14. *Let $m \in \mathbb{N}$. We have*

$$\iota_{m-1,1}^* = (-1)^{(n+1)m+1} \frac{d_1}{d_{m-1}} \cdot (1_m \otimes G_1^*)(G_m \otimes 1) : E_{m-1} \otimes E_1 \to E_m,$$

$$\iota_{1,m-1}^* = (-1)^{(n+1)m+1} \frac{d_1}{d_{m-1}} \cdot ((G_1')^* \otimes 1_m)(1 \otimes G_m') : E_1 \otimes E_{m-1} \to E_m.$$

Proof. We focus on proving the claim for $\iota_{m-1,1}^* : E_{m-1} \otimes E_1 \to E_m$. Using Lemmas 3.10 (1) and 3.13, we obtain that

$$\begin{aligned}
&(-1)^{(n+1)m+1} \frac{d_1}{d_{m-1}} \cdot (1_m \otimes G_1^*)(G_m \otimes 1) \\
&\quad = (-1)^n d_1 \cdot (1_m \otimes G_1^*)(\iota_{m-1,1}^* \otimes 1 \otimes 1)(1_{m-1} \otimes G_1 \otimes 1) \\
&\quad = (-1)^n d_1 \cdot \iota_{m-1,1}^* (1_{m-1} \otimes 1 \otimes G_1^*)(1_{m-1} \otimes G_1 \otimes 1) = \iota_{m-1,1}^*.
\end{aligned}$$

□

LEMMA 3.15. *Let $m \in \mathbb{N}$. We have*

$$\begin{aligned}
p_{m-1,1} &= 1_{m-1} \otimes 1 + (-1)^{(n+1)m+1} \frac{d_1}{d_{m-1}} \cdot (G_{m-1} \otimes G_1^*)(\iota_{m-2,1} \otimes 1) \\
&\qquad : E_{m-1} \otimes E_1 \to E_{m-1} \otimes E_1 \qquad \text{and} \\
p_{1,m-1} &= 1 \otimes 1_{m-1} + (-1)^{(n+1)m+1} \frac{d_1}{d_{m-1}} (G_1^* \otimes G_{m-1}')(1 \otimes \iota_{1,m-2}) \\
&\qquad : E_1 \otimes E_{m-1} \to E_1 \otimes E_{m-1}.
\end{aligned}$$

Proof. We focus on the orthogonal projection $p_{m-1,1} : E_{m-1} \otimes E_1 \to E_{m-1} \otimes E_1$. Using Lemma 3.5 (1), Lemma 3.14, and the recursive relation from (3.12), we compute that

$$\begin{aligned}
p_{m-1,1} &= \iota_{m-1,1} \iota_{m-1,1}^* \\
&= (-1)^{(n+1)m+1} \frac{d_1}{d_{m-1}} \cdot (1_{m-1} \otimes 1 \otimes G_1^*)(\iota_{m-1,1} \otimes 1 \otimes 1)(G_m \otimes 1) \\
&= (-1)^{(n+1)m+1} \frac{d_1}{d_{m-1}} \cdot (G_{m-1} \otimes G_1^*)(\iota_{m-2,1} \otimes 1) \\
&\qquad + (-1)^{(n+1)m+1} \frac{d_1}{d_{m-1}} \cdot (-1)^{(n+1)(m-1)} d_{m-1} \cdot (1_{m-1} \otimes 1 \otimes G_1^*)(1_{m-1} \otimes G_1 \otimes 1) \\
&= (-1)^{(n+1)m+1} \frac{d_1}{d_{m-1}} \cdot (G_{m-1} \otimes G_1^*)(\iota_{m-2,1} \otimes 1) + 1_{m-1} \otimes 1.
\end{aligned}$$

□

PROPOSITION 3.16. *Let $k, m \in \mathbb{N}_0$. We have the identity*

$$\begin{aligned}
\sigma_{k,m}^* \sigma_{k,m} &= \frac{d_k d_{k+m+1}}{d_1 d_m} \cdot 1_k \otimes 1_m + \frac{d_k d_{m-1}}{d_{k-1} d_m} \cdot \sigma_{k-1,m-1} \sigma_{k-1,m-1}^* \\
&\qquad : E_k \otimes E_m \to E_k \otimes E_m.
\end{aligned} \tag{3.14}$$

Proof. We focus on the case where $k, m \in \mathbb{N}$. Using Lemmas 3.5 and 3.15, we see that

$$\begin{aligned}\sigma_{k,m}^*\sigma_{k,m} &= (G_{k+1} \otimes 1_m)^*(1_{k+1} \otimes p_{1,m})(G_{k+1} \otimes 1_m) \\ &= \mu_{k+1} \cdot 1_k \otimes 1_m \\ &\quad + (-1)^{(n+1)m+n}\frac{d_1}{d_m} \cdot (G_{k+1} \otimes 1_m)^*(1_{k+1} \otimes G_1^* \otimes G_m')(G_{k+1} \otimes \iota_{1,m-1}).\end{aligned}$$

We continue by analysing the second term in this sum by applying Lemma 3.13 and the recursive relation from (3.12):

$$\begin{aligned}&(-1)^{(n+1)m+n}\frac{d_1}{d_m} \cdot (G_{k+1} \otimes 1_m)^*(1_{k+1} \otimes G_1^* \otimes G_m')(G_{k+1} \otimes \iota_{1,m-1}) \\ &\quad = (-1)^{(n+1)(m+k)+n}\frac{d_1 d_k}{d_m} \cdot (1_k \otimes G_1^* \otimes 1_m)(1_k \otimes 1 \otimes G_1^* \otimes G_m') \\ &\quad \circ (\iota_{k,1} \otimes 1 \otimes \iota_{1,m-1})(G_{k+1} \otimes 1_m) \qquad (3.15)\\ &\quad = (-1)^{(n+1)(m+k)+n}\frac{d_1 d_k}{d_m} \cdot (1_k \otimes G_1^* \otimes 1_m)(G_k \otimes G_1^* \otimes G_m')(\iota_{k-1,1} \otimes \iota_{1,m-1}) \\ &\quad + (-1)^{(n+1)m+n}\frac{d_1 d_k^2}{d_m} \cdot (1_k \otimes G_1^* \otimes 1_m)(1_k \otimes 1 \otimes G_1^* \otimes G_m')(1_k \otimes G_1 \otimes \iota_{1,m-1}).\end{aligned}$$

Using Lemmas 3.13 and 3.14, we then obtain that the first term in the above sum is given by

$$\begin{aligned}&(-1)^{(n+1)(m+k)+n}\frac{d_1 d_k}{d_m} \cdot (1_k \otimes G_1^* \otimes 1_m)(G_k \otimes G_1^* \otimes G_m')(\iota_{k-1,1} \otimes \iota_{1,m-1}) \\ &\quad = (-1)^{(n+1)(k+1)}\frac{d_k d_{m-1}}{d_m} \cdot (1_k \otimes \iota_{1,m-1}^*)(G_k \otimes G_1^* \otimes 1_{m-1})(\iota_{k-1,1} \otimes \iota_{1,m-1}) \\ &\quad = \frac{d_k d_{m-1}}{d_m d_{k-1}} \cdot \sigma_{k-1,m-1}\sigma_{k-1,m-1}^*,\end{aligned}$$

corresponding to the second term in (3.14) (in the case where $k, m \in \mathbb{N}$). We continue with the remaining term in (3.15) and apply Lemma 3.10, Lemma 3.14:

$$\begin{aligned}&(-1)^{(n+1)m+n}\frac{d_1 d_k^2}{d_m} \cdot (1_k \otimes G_1^* \otimes 1_m)(1_k \otimes 1 \otimes G_1^* \otimes G_m')(1_k \otimes G_1 \otimes \iota_{1,m-1}) \\ &\quad = (-1)^{(n+1)m}\frac{d_k^2}{d_m} \cdot (1_k \otimes G_1^* \otimes 1_m)(1_k \otimes 1 \otimes G_m')(1_k \otimes \iota_{1,m-1}) \\ &\quad = -\frac{d_k^2 d_{m-1}}{d_1 d_m} \cdot 1_k \otimes 1_m.\end{aligned}$$

The result of the proposition now follows by an application of Lemma 3.2 in the case where $l = 0$, yielding that

$$\mu_{k+1} - \frac{d_k^2 d_{m-1}}{d_1 d_m} = \frac{d_k d_{k+m+1}}{d_1 d_m}.$$

□

The following lemmas contain further properties of the operators $\sigma_{k,m} : E_k \otimes E_m \to E_{k+1} \otimes E_{m+1}$, $k, m \in \mathbb{N}_0$. For ease of notation, we omit the subscripts.

LEMMA 3.17. *Let $k, m \in \mathbb{N}_0$ and $j \in \mathbb{N}$. We have the identity*

$$\sigma^* \sigma^j = \mu_{k+j} \cdot \left(1 - \frac{d_k d_{m-1}}{d_{k+j} d_{m+j-1}}\right) \cdot \sigma^{j-1} + \frac{d_{m-1} d_{k+j-1}}{d_{k-1} d_{m+j-1}} \cdot \sigma^j \sigma^*$$

$$: E_k \otimes E_m \to E_{k+j-1} \otimes E_{m+j-1}.$$

Proof. Applying Proposition 3.16, we obtain by induction on $j \in \mathbb{N}$ that

$$\sigma^* \sigma^j = \frac{d_{k+j-1}(d_{k+m+2j-1} + d_{k+m+2j-3} + \cdots + d_{k+m+1})}{d_1 d_{m+j-1}} \sigma^{j-1} + \frac{d_{k+j-1} d_{m-1}}{d_{k-1} d_{m+j-1}} \sigma^j \sigma^*.$$

The result of the present lemma then follows by an application of Lemma 3.2:

$$\frac{d_{k+j-1}(d_{k+m+2j-1} + d_{k+m+2j-3} + \cdots + d_{k+m+1})}{d_1 d_{m+j-1}} = \frac{d_{k+j-1}(d_{k+j} d_{m+j-1} - d_k d_{m-1})}{d_1 d_{m+j-1}} = \mu_{k+j} \cdot \left(1 - \frac{d_k d_{m-1}}{d_{k+j} d_{m+j-1}}\right).$$

□

LEMMA 3.18. *Let $k, m \in \mathbb{N}_0$ and $j \in \mathbb{N}$. We have the identities*

$$\sigma^* \iota_{k,m} = 0 : E_{k+m} \to E_{k-1} \otimes E_{m-1} \qquad \text{and}$$

$$(\sigma^*)^j \sigma^j \iota_{k,m} = \prod_{i=1}^{j} \mu_{k+i} \left(1 - \frac{d_k d_{m-1}}{d_{k+i} d_{m+i-1}}\right) \cdot \iota_{k,m} : E_{k+m} \to E_k \otimes E_m.$$

Proof. By Lemma 3.17, it suffices to show that $\sigma^*_{k-1,m-1} \iota_{k,m} = 0$. This is a triviality for $k = 0$ or $m = 0$ and for $k, m \in \mathbb{N}$, we have that $\sigma^*_{k-1,m-1} \iota_{k,m} = (G_k^* \otimes 1_{m-1})(1_k \otimes \iota_{1,m-1})\iota_{k,m} : E_{k+m} \to E_{k-1} \otimes E_{m-1}$. However, by Lemma 3.13, this linear map is a scalar multiple of the inclusion $E_{k+m} \to E_{k-1} \otimes E_1 \otimes E_1 \otimes E_{m-1}$ composed with $1_{k-1} \otimes \langle \delta, \cdot \rangle \otimes 1_{m-1}$. Since $E_{k-1} \otimes D \otimes E_{m-1}$ lies in the orthogonal complement of $E_{k+m} \subseteq E_{k-1} \otimes E_1 \otimes E_1 \otimes E_{m-1}$, we have proved the lemma. □

Our computations culminate in the following important result concerning the decomposition of the tensor product of two elements of our subproduct system of Hilbert spaces.

THEOREM 3.19. *Let $k, m \in \mathbb{N}_0$ and put $l := \min\{k, m\}$. We have an $SU(2)$-equivariant unitary isomorphism*

$$W_{k,m} = \begin{pmatrix} W^0_{k,m} & W^1_{k,m} & \dots & W^l_{k,m} \end{pmatrix} : \bigoplus_{j=0}^{l} E_{k+m-2j} \to E_k \otimes E_m$$

defined component-wise by

$$W^j_{k,m} = \prod_{i=1}^{j} \frac{1}{\sqrt{\mu_{k-j+i}}} \left(1 - \frac{d_{k-j} d_{m-j-1}}{d_{k-j+i} d_{m-j+i-1}}\right)^{-1/2} \cdot \sigma^j \iota_{k-j,m-j}$$

$$: E_{k+m-2j} \to E_k \otimes E_m$$

for all $j \in \{1, \dots, l\}$ and $W^0_{k,m} := \iota_{k,m} : E_{k+m} \to E_k \otimes E_m$.

Proof. By Lemma 3.18, we have that $W^j_{k,m} : E_{k+m-2j} \to E_k \otimes E_m$ is an isometry for all $j \in \{0, 1, \ldots, l\}$. Moreover, it follows from Lemma 3.18 that $(W^i_{k,m})^* W^j_{k,m} = 0 : E_{k+m-2j} \to E_{k+m-2i}$ whenever $0 \leqslant j < i \leqslant l$. These two observations establish that $W_{k,m} : \bigoplus_{j=0}^l E_{k+m-2j} \to E_k \otimes E_m$ is an isometry. The fact that $W_{k,m}$ is surjective now follows by dimension considerations since Lemma 3.2 implies $d_k d_m = \sum_{j=0}^l d_{k+m-2l+2j}$. The $SU(2)$-equivariance of $W_{k,m}$ is a consequence of the $SU(2)$-equivariance of the structure maps of our subproduct system and the definition in (3.13) together with Lemma 3.4. □

4. *Commutation relations for the Toeplitz algebra*

Throughout this section, we fix an $n \in \mathbb{N}$ and consider the Toeplitz algebra coming from the irreducible representation $\rho_n : SU(2) \to U(L_n)$. We let $\{e_j\}_{j=0}^n$ denote the orthonormal basis for L_n introduced in (2.10). In particular, we have the associated Toeplitz operators

$$T_j := T_{e_j} : F \to F \qquad j \in \{0, 1, \ldots, n\}.$$

For each $j \in \{0, 1, 2, \ldots, n\}$, we also introduce the bounded operator $T'_j : F \to F$ defined by

$$T'_j(\xi) := \iota^*_{m,1}(\xi \otimes e_j) \qquad \text{for all } \xi \in E_m.$$

In other words, T'_j is the *right* creation operator associated to the vector $e_j \in E_1 = L_n$.

We define the $SU(2)$-equivariant bounded operators $\iota_L : F \to E_1 \otimes F$ and $\iota_R : F \to F \otimes E_1$ by $\iota_L(\xi) := \iota_{1,m-1}(\xi)$ and $\iota_R(\xi) := \iota_{m-1,1}(\xi)$ for homogeneous elements $\xi \in E_m$ with $m \geqslant 1$ and for $\xi \in E_0$ we put $\iota_L(\xi) = 0$ and $\iota_R(\xi) = 0$.

Lemma 4.1. *We have the identities*

$$\iota^*_L = \sum_{j=0}^n \langle e_j, \cdot \rangle \otimes T_j : E_1 \otimes F \to F \qquad \text{and}$$

$$\iota^*_R = \sum_{j=0}^n T'_j \otimes \langle e_j, \cdot \rangle : F \otimes E_1 \to F.$$

Proof. Let $\xi \in E_m$ and $i \in \{0, 1, \ldots, n\}$ be given. We compute that

$$\iota^*_L(e_i \otimes \xi) = \iota^*_{1,m}(e_i \otimes \xi) = T_i(\xi) = \sum_{j=0}^n (\langle e_j, \cdot \rangle \otimes T_j)(e_i \otimes \xi).$$

The identity involving $\iota^*_R : F \otimes E_1 \to F$ is proved in the same way. □

We are now going to further analyse the structural properties of the $SU(2)$-equivariant isometries $V_m : E_{m-1} \to E_m \otimes E_1$ and $V'_m : E_{m-1} \to E_1 \otimes E_m$ defined in (3.9) and (3.11).

Lemma 4.2. *Let $m \in \mathbb{N}$. For every $\xi \in E_{m-1}$, we have the identities*

$$V'_m(\xi) = \sqrt{d_{m-1}/d_m} \cdot \sum_{j=0}^n (-1)^j \cdot e_j \otimes T_{n-j}(\xi) \qquad \text{and}$$

$$V_m(\xi) = \sqrt{d_{m-1}/d_m} \cdot \sum_{j=0}^n (-1)^{n-j} \cdot T'_{n-j}(\xi) \otimes e_j.$$

Proof. By definition of $V_m' : E_{m-1} \to E_1 \otimes E_m$ and by Lemma 3.13, it holds that

$$V_m'(\xi) = \frac{(-1)^{(n+1)(m-1)}}{\sqrt{\mu_m}} \cdot G_m'(\xi) = \frac{d_{m-1}}{\sqrt{\mu_m}}(1 \otimes \iota_{1,m-1}^*)(\delta \otimes \xi)$$
$$= \frac{d_{m-1}}{\sqrt{\mu_m \cdot (n+1)}} \cdot \sum_{j=0}^{n} (-1)^j \cdot e_j \otimes T_{n-j}(\xi) = \sqrt{\frac{d_{m-1}}{d_m}} \cdot \sum_{j=0}^{n} (-1)^j \cdot e_j \otimes T_{n-j}(\xi),$$

where the last equality follows from the definition of the constant μ_m in (3.5).

The proof of the second identity follows *mutatis mutandis* the proof of the first one. □

4.1. *The dimension operator*

Recall that $F_{\text{alg}} \subseteq F$ denotes the algebraic Fock space defined as the vector space direct sum of the vector spaces E_m, $m \in \mathbb{N}_0$.

Definition 4.3. We define the *dimension operator* $D : \text{Dom}(D) \to F$ as the closure of the unbounded operator $\mathcal{D} : F_{\text{alg}} \to F$, given by $\mathcal{D}(\xi) = d_m \cdot \xi$ for $\xi \in E_m$.

Observe that the dimension operator is positive and invertible and that the inverse $D^{-1} : F \to F$ is an $SU(2)$-equivariant compact operator. In particular, $D^{-1} \in \mathbb{T}$.

In the special case of the fundamental representation, the operator D equals $N + 1$, where N is the number operator.

We further define the $SU(2)$-equivariant bounded positive invertible operator

$$\Phi : F \to F \qquad \Phi\xi = \frac{d_m}{d_{m+1}}\xi \quad \text{for all } \xi \in E_m. \tag{4.1}$$

Lemma 4.4. *The bounded invertible operator $\Phi : F \to F$ belongs to the Toeplitz algebra $\mathbb{T}$.*

Proof. Let $\gamma_n \in (0, 1]$ be the constant defined in Lemma 3.3. Since $\Phi - \gamma_n \cdot 1_F$ is a compact operator on F and $\mathbb{K}(F) \subseteq \mathbb{T}$, we obtain the result of the lemma. □

We define the $SU(2)$-equivariant isometries $V_R : F \to F \otimes E_1$ and $V_L : F \to E_1 \otimes F$ by

$$V_R(\xi) = V_m(\xi) \qquad \text{and} \qquad V_L(\xi) = V_m'(\xi)$$

for all $\xi \in E_{m-1} \subseteq F$. We may then restate the result of Lemma 4.2 as follows:

Proposition 4.5. *For every $\xi \in F$, we have the identities*

$$V_L(\xi) = \sum_{j=0}^{n} (-1)^j \cdot e_j \otimes T_{n-j}\Phi^{1/2}(\xi) \qquad \textit{and}$$

$$V_R(\xi) = \sum_{j=0}^{n} (-1)^{n-j} \cdot T_{n-j}'\Phi^{1/2}(\xi) \otimes e_j.$$

4.2. *Commutation relations*

We now present the commutation relations for our Toeplitz algebras in the general case of an irreducible representation $\rho_n : SU(2) \to U(L_n)$ for $n \geqslant 1$. These commutation relations can be used to recover the commutation relations in Theorem 2.4 in the case of the fundamental representation. For the time being, we do not know whether there are any further relations in the Toeplitz algebra $\mathbb{T}(\rho_n, L_n)$.

We start out by remarking that

$$\sum_{i=0}^{n} T_i T_i^* = \iota_L^* \iota_L = 1_F - Q_0. \tag{4.2}$$

THEOREM 4.6. *Let $n \in \mathbb{N}$, and consider the irreducible representation $\rho_n : SU(2) \to U(L_n)$. Then the Toeplitz operators T_i, with $i = 0, \ldots, n$ satisfy the following commutation relations:*

$$\sum_{i=0}^{n} (-1)^i T_i T_{n-i} = 0, \tag{4.3}$$

$$T_i^* T_j = \delta_{ij} \cdot 1_F + (-1)^{i+j+1} \big((n+1) \cdot 1_F - \Phi^{-1} \big) T_{n-i} T_{n-j}^* \tag{4.4}$$

$$\sum_{i=0}^{n} T_i^* T_i = \Phi^{-1}. \tag{4.5}$$

Proof. The relation in (4.3) follows from our computation of the determinant in Proposition 2.6 (cf. [**36**, § 10]).

We now move on to establishing the relation in (4.4). Consider $i, j \in \{0, 1, \ldots, n\}$. By Proposition 3.12, we have that $\iota_L \iota_L^* + V_L V_L^* = 1 \otimes 1_F : E_1 \otimes F \to E_1 \otimes F$ and hence that

$$T_i^* T_j = (\langle e_i, \cdot \rangle \otimes 1_F) \iota_L \iota_L^* (e_j \otimes 1_F) = \delta_{ij} \cdot 1_F - (\langle e_i, \cdot \rangle \otimes 1_F) V_L V_L^* (e_j \otimes 1_F).$$

Then, on using Proposition 4.5, we obtain that $(\langle e_i, \cdot \rangle \otimes 1_F) V_L = (-1)^i T_{n-i} \Phi^{1/2}$ and hence that

$$T_i^* T_j = \delta_{ij} \cdot 1_F + (-1)^{i+j+1} T_{n-i} \Phi T_{n-j}^*.$$

The relation in (4.4) now follows by the definition of $\Phi : F \to F$ from (4.1) on noting that $T_{n-i}(E_m) \subseteq E_{m+1}$ and $d_1 - d_{m+2}/d_{m+1} = d_m/d_{m+1}$ for all $m \in \mathbb{N}_0$.

We are now left with proving the relation in (4.5). From the identities in (4.2) and (4.4), we obtain that

$$\begin{aligned} \sum_{i=0}^{n} T_i^* T_i &= (n+1) \cdot 1_F - \big((n+1) \cdot 1_F - \Phi^{-1} \big) \sum_{i=0}^{n} T_{n-i} T_{n-i}^* \\ &= (n+1) \cdot 1_F - \big((n+1) \cdot 1_F - \Phi^{-1} \big) (1_F - Q_0) = \Phi^{-1}. \end{aligned}$$

This ends the proof of the theorem. □

5. *A quasi-homomorphism from the Toeplitz algebra to the complex numbers*

Let $n \in \mathbb{N}$ be given and consider the irreducible representation $\rho_n : SU(2) \to U(L_n)$. We denote the corresponding Toeplitz algebra by $\mathbb{T} \subseteq \mathbb{L}(F)$, where $F = \bigoplus_{m=0}^{\infty} E_m$ denotes the Fock space. In this section, we start relating the K-theory of the Toeplitz algebra to the K-theory of the complex numbers using the quasi-isomorphism picture introduced by Cuntz [**12**]: We shall construct an $SU(2)$-equivariant quasi-homomorphism (ψ_+, ψ_-) from $\mathbb{T}$ to $\mathbb{C}$. Such an $SU(2)$-equivariant quasi-homomorphism from $\mathbb{T}$ to $\mathbb{C}$ consists of a Hilbert space H which is equipped with a strongly continuous action $U : SU(2) \to U(H)$ together with two $*$-homomorphisms $\psi_+, \psi_- : \mathbb{T} \to \mathbb{L}(H)$. These data have to satisfy that $\psi_+(x) - \psi_-(x)$ is a compact operator for all $x \in \mathbb{T}$ and that both ψ_+ and ψ_- are $SU(2)$-equivariant in the sense that $U(g)\psi_\pm(x)U(g^{-1}) =$

$\psi_\pm(g(x))$ for all $x \in \mathbb{T}$ and all $g \in SU(2)$. For more information on KK-theory we refer to the standard text books on the subject, [**8, 22**].

In our specific case, both of the $*$-homomorphisms ψ_+ and ψ_- act on the Hilbert space direct sum $F \oplus F$ and we define $\psi_+ : \mathbb{T} \to \mathbb{L}(F \oplus F)$ by $\psi_+(x) := x \oplus x$ for all $x \in \mathbb{T}$. The construction of $\psi_- : \mathbb{T} \to \mathbb{L}(F \oplus F)$ uses the representation theoretic considerations from Section 3.

Recall that $V_R : F \to F \otimes E_1$ denotes the $SU(2)$-equivariant isometry defined by

$$V_R(\xi) := V_{m+1}(\xi) = \frac{(-1)^{(n+1)m}}{\sqrt{\mu_{m+1}}} \cdot G_{m+1}(\xi) \in E_{m+1} \otimes E_1 \subseteq F \otimes E_1$$

for every homogeneous $\xi \in E_m \subseteq F$, $m \in \mathbb{N}_0$. Moreover, we have the $SU(2)$-equivariant linear map $\iota_R : F \to F \otimes E_1$ defined by

$$\iota_R(\xi) := \iota_{m-1,1}(\xi) \in E_{m-1} \otimes E_1 \subseteq F \otimes E_1$$

for every homogeneous $\xi \in E_m \subseteq F$, $m \in \mathbb{N}$ and $\iota_R(\xi) = 0$ for $\xi \in E_0$. It follows from Proposition 3.8 that the $SU(2)$-equivariant linear map

$$W_R : F \otimes E_1 \to F \oplus F \qquad W_R = \begin{pmatrix} \iota_R^* \\ V_R^* \end{pmatrix} \tag{5.1}$$

is an isometry and that the image agrees with the subspace $F_+ \oplus F \subseteq F \oplus F$. We may thus define the $*$-homomorphism

$$\psi_- : \mathbb{T} \to \mathbb{L}(F \oplus F) \qquad \psi_-(x) := W_R(x \otimes 1)W_R^*.$$

We also recall that we have the $SU(2)$-equivariant linear map $\iota_L : F \to E_1 \otimes F$ defined by the formula

$$\iota_L(\xi) := \iota_{1,m-1}(\xi) \in E_1 \otimes E_{m-1} \subseteq E_1 \otimes F$$

for homogeneous elements $\xi \in E_m \subseteq F$ with $m \in \mathbb{N}$ ad $\iota_L(\xi) = 0$ for $\xi \in E_0$.

We announce the following result:

Proposition 5.1. *The pair of $*$-homomorphisms (ψ_+, ψ_-) defines an $SU(2)$-equivariant quasi-homomorphism from $\mathbb{T}$ to $\mathbb{C}$ and hence a class $[\psi_+, \psi_-] \in KK_0^{SU(2)}(\mathbb{T}, \mathbb{C})$.*

Proof. The $SU(2)$-equivariance of the two $*$-homomorphisms follows from the $SU(2)$-equivariance of $W_R : F \otimes E_1 \to F \oplus F$ together with the observation that the action of $SU(2)$ on the Toeplitz algebra is obtained via conjugation with the corresponding action on the Fock space F, see Lemma 1.5.

For each $x \in \mathbb{T}$, we have to show that the difference $\psi_+(x) - \psi_-(x) = x \oplus x - W_R(x \otimes 1)W_R^*$ is a compact operator on $F \oplus F$. Since $\mathbb{T}$ is generated as a C^*-algebra by the operators $T_j^* : F \to F$, $j \in \{0, 1, \ldots, n\}$ together with the unit $1_F : F \to F$, it suffices to prove compactness when $x \in \mathbb{T}$ agrees with one of these operators. For the case of the unit $1_F : F \to F$ we have that $1_F \oplus 1_F - W_R W_R^*$ agrees with the orthogonal projection onto the one-dimensional subspace $(F_+ \oplus F)^\perp \cong \mathbb{C}$ so we focus on the operator $T_j^* : F \to F$ for a fixed $j \in \{0, 1, \ldots, n\}$. We compute that

$$W_R(T_j^* \otimes 1)W_R^* = \begin{pmatrix} \iota_R^*(T_j^* \otimes 1)\iota_R & \iota_R^*(T_j^* \otimes 1)V_R \\ V_R^*(T_j^* \otimes 1)\iota_R & V_R^*(T_j^* \otimes 1)V_R \end{pmatrix}.$$

Applying the identities $(T_j^* \otimes 1)\iota_R = \iota_R T_j^*$, $V_R^* \iota_R = 0$ (see Proposition 3.8), and using the fact that $\iota_R^* \iota_R$ is the orthogonal projection onto $F_+ \subseteq F$, we obtain that

$$W_R(T_j^* \otimes 1)W_R^* \sim \begin{pmatrix} T_j^* & \iota_R^*(T_j^* \otimes 1)V_R \\ 0 & V_R^*(T_j^* \otimes 1)V_R \end{pmatrix}$$

modulo compact operators. Now, by Proposition 5.4 here below in Subsection 5.1 we have that the operator $(T_j^* \otimes 1)V_R$ agrees with $V_R T_j^*$ modulo compact operators. But this implies the result of this proposition, using that $V_R^* V_R = 1_F$ and $\iota_R^* V_R = 0$. □

We are eventually going to show that the Toeplitz algebra $\mathbb{T}$ is KK-equivalent to $\mathbb{C}$ and the class $[\psi_+, \psi_-] \in KK_0^{SU(2)}(\mathbb{T}, \mathbb{C})$ provides us with one of the two relevant morphisms. The other morphism is given by the unital inclusion $i : \mathbb{C} \to \mathbb{T}$, which defines a class $[i] \in KK_0^{SU(2)}(\mathbb{C}, \mathbb{T})$.

PROPOSITION 5.2. *The interior Kasparov product* $[i]\widehat{\otimes}_{\mathbb{T}}[\psi_+, \psi_-]$ *agrees with the unit* $\mathbf{1}_{\mathbb{C}} \in KK_0^{SU(2)}(\mathbb{C}, \mathbb{C})$.

Proof. The interior Kasparov product $[i]\widehat{\otimes}_{\mathbb{T}}[\psi_+, \psi_-]$ is represented by the $SU(2)$-equivariant quasi-homomorphism $(\psi_+ \circ i, \psi_- \circ i)$. The $*$-homomorphism $\psi_+ \circ i : \mathbb{C} \to \mathbb{L}(F \oplus F)$ is unital, whereas $(\psi_- \circ i)(1) = W_R W_R^* : F \oplus F \to F \oplus F$. Since $1_{F \oplus F} - W_R W_R^* : F \oplus F \to F \oplus F$ is the orthogonal projection onto the one-dimensional subspace $\mathbb{C}\omega \oplus \{0\} \subseteq F \oplus F$, this proves the proposition. □

5.1. *Compactness of commutators*

In this subsection, we provide the remaining ingredient for the proof of Proposition 5.1. More precisely, we shall see in Proposition 5.4 that the difference $V_R T_j^* - (T_j^* \otimes 1)V_R : F \to F \otimes E_1$ is indeed a compact operator.

LEMMA 5.3. *For each* $m \geqslant 2$, *we have the identity*

$$\iota_{1,m-2}^*(1 \otimes V_{m-1})^*(\iota_{1,m-1} \otimes 1)V_m = \left(1 - \frac{1}{d_{m-1}^2}\right)^{1/2} \cdot 1_{m-1}.$$

Proof. Using Lemma 3.13 and (3.9), we see that

$$\begin{aligned}(1 \otimes V_{m-1})^* &= \frac{(-1)^{(n+1)m}}{\sqrt{\mu_{m-1}}} \cdot (1 \otimes G_{m-1}^*) \\ &= \frac{d_{m-2}}{\sqrt{\mu_{m-1}}} \cdot (1 \otimes 1_{m-2} \otimes G_1^*)(1 \otimes \iota_{m-2,1} \otimes 1).\end{aligned} \tag{5.2}$$

Next, by associativity of the subproduct system, we have

$$(1 \otimes \iota_{m-2,1})\iota_{1,m-1} = (\iota_{1,m-2} \otimes 1)\iota_{m-1,1} : E_m \to E_1 \otimes E_{m-2} \otimes E_1,$$

which combined with (5.2) yields that

$$\begin{aligned}&\iota_{1,m-2}^*(1 \otimes V_{m-1})^*(\iota_{1,m-1} \otimes 1)V_m \\ &= \frac{d_{m-2}}{\sqrt{\mu_{m-1}}} \cdot \iota_{1,m-2}^*(1 \otimes 1_{m-2} \otimes G_1^*)(\iota_{1,m-2} \otimes 1 \otimes 1)(\iota_{m-1,1} \otimes 1)V_m \\ &= \frac{d_{m-2}}{\sqrt{\mu_{m-1}}} \cdot (1_{m-1} \otimes G_1^*)(\iota_{m-1,1} \otimes 1)V_m.\end{aligned}$$

Using Lemmas 3.2 and 3.13, the identity (5.2), and the fact that $V_m : E_{m-1} \to E_m \otimes E_1$ is an isometry, we then get that

$$\begin{aligned}\frac{d_{m-2}}{\sqrt{\mu_{m-1}}} &\cdot (1_{m-1} \otimes G_1^*)(\iota_{m-1,1} \otimes 1)V_m \\ &= \frac{d_{m-2} \cdot \sqrt{\mu_m}}{\sqrt{\mu_{m-1}} \cdot d_{m-1}} \cdot V_m^* V_m = \frac{\sqrt{d_{m-2} d_m}}{d_{m-1}} \cdot 1_{m-1} = \Big(1 - \frac{1}{d_{m-1}^2}\Big)^{1/2} \cdot 1_{m-1}. \qquad \square\end{aligned}$$

PROPOSITION 5.4. *The difference*

$$(T_j^* \otimes 1)V_R - V_R T_j^* : F \to F \otimes E_1$$

is a compact operator.

Proof. Since $T_j^* = (\langle e_j, \cdot \rangle \otimes 1_F)\iota_L : F \to F$, it is enough to show that the difference

$$(\iota_L \otimes 1)V_R - (1 \otimes V_R)\iota_L : F \to E_1 \otimes F \otimes E_1$$

is a compact operator.

Note first that the Hilbert space $E_{m-1} \subseteq F$ is finite-dimensional for each $m \in \mathbb{N}$ and that both $(\iota_L \otimes 1)V_R$ and $(1 \otimes V_R)\iota_L$ map E_{m-1} into $E_1 \otimes E_{m-1} \otimes E_1$. The corresponding restrictions are given by $(\iota_{1,m-1} \otimes 1)V_m$ and $(1 \otimes V_{m-1})\iota_{1,m-2} : E_{m-1} \to E_1 \otimes E_{m-1} \otimes E_1$. It therefore suffices to show that the sequence of operator norms

$$\big\{\|(\iota_{1,m-1} \otimes 1)V_m - (1 \otimes V_{m-1})\iota_{1,m-2}\|\big\}_{m=1}^\infty$$

converges to zero.

Let $m \geqslant 2$. Using Lemma 5.3 together with the fact that $(\iota_{1,m-1} \otimes 1)V_m$ and $(1 \otimes V_{m-1})\iota_{1,m-2}$ are isometries, we obtain that

$$\begin{aligned}&((\iota_{1,m-1} \otimes 1)V_m - (1 \otimes V_{m-1})\iota_{1,m-2})^*((\iota_{1,m-1} \otimes 1)V_m - (1 \otimes V_{m-1})\iota_{1,m-2}) \\ &\quad = 2\left(1 - \Big(1 - \frac{1}{d_{m-1}^2}\Big)^{1/2}\right) \cdot 1_{m-1},\end{aligned}$$

which implies

$$\|(\iota_{1,m-1} \otimes 1)V_m - (1 \otimes V_{m-1})\iota_{1,m-2}\| = \sqrt{2} \cdot \left(1 - \Big(1 - \frac{1}{d_{m-1}^2}\Big)^{1/2}\right)^{1/2}. \tag{5.3}$$

The result of the lemma now follows since the sequence $\{1/d_{m-1}^2\}_{m=1}^\infty$ converges to zero (using again the global assumption that $n \geqslant 1$). $\square$

In fact, we can do slightly better than the above proposition:

LEMMA 5.5. *Let $p \in [0,1]$. The operator*

$$(D^p \otimes 1)\big((T_j^* \otimes 1)V_R - V_R T_j^*\big)D^{1-p} : F_{\text{alg}} \to F \otimes E_1$$

extends to a bounded operator.

Proof. We first remark that the unbounded operator $(D^p \otimes 1)((T_j^* \otimes 1)V_R - V_R T_j^*)D^{1-p} : F_{\text{alg}} \to F \otimes E_1$ maps the subspace E_m into $E_m \otimes E_1$ for each $m \in \mathbb{N}_0$. It therefore suffices to show that the supremum over $m \in \mathbb{N}_0$ of the corresponding operator norms is finite.

Let $m \in \mathbb{N}$ be given. We compute that

$$(D^p \otimes 1)\big((T_j^* \otimes 1)V_R - V_R T_j^*\big) D^{1-p}|_{E_m}$$
$$= d_m \cdot \big(((\langle e_j, \cdot\rangle \otimes 1_m \otimes 1)(\iota_{1,m} \otimes 1)V_{m+1} - (\langle e_j, \cdot\rangle \otimes V_m)\iota_{1,m-1}\big)$$
$$= d_m \cdot (\langle e_j, \cdot\rangle \otimes 1_m \otimes 1)((\iota_{1,m} \otimes 1)V_{m+1} - (1 \otimes V_m)\iota_{1,m-1}).$$

The result of the present lemma then follows from (5.3) by noting that

$$d_m^2 \cdot \|(\iota_{1,m} \otimes 1)V_{m+1} - (1 \otimes V_m)\iota_{1,m-1}\|^2$$
$$= 2d_m^2 \cdot (1 - \sqrt{1 - 1/d_m^2}) \leqslant 2. \qquad \square$$

6. *The K-theory of the Toeplitz algebra*

Recall from Section 5 that we have an $SU(2)$-equivariant isometry $W_R : F \otimes E_1 \to F \oplus F$ (cf. (5.1)), which we use to define the $*$-homomorphism

$$\psi_- : \mathbb{T} \to \mathbb{L}(F \oplus F) \qquad \psi_-(x) := W_R(x \otimes 1)W_R^*.$$

We clearly also have the $*$-homomorphism $\psi_+ : \mathbb{T} \to \mathbb{L}(F \oplus F)$, $\psi_+(x) := x \oplus x$.

We saw in Proposition 5.1 that the pair (ψ_+, ψ_-) yields an $SU(2)$-equivariant quasi-homomorphism form $\mathbb{T}$ to $\mathbb{C}$ and we therefore have a class $[\psi_+, \psi_-] \in KK_0^{SU(2)}(\mathbb{T}, \mathbb{C})$. We moreover saw in Proposition 5.2 that the interior Kasparov product $[i]\widehat{\otimes}_{\mathbb{T}}[\psi_+, \psi_-] \in KK_0^{SU(2)}(\mathbb{C}, \mathbb{C})$ agrees with the unit $\mathbf{1}_{\mathbb{C}}$, where we recall that $[i] \in KK_0^{SU(2)}(\mathbb{C}, \mathbb{T})$ is the class associated with the unital inclusion $i : \mathbb{C} \to \mathbb{T}$.

In this section, we are going to prove the following main result:

THEOREM 6.1. *The interior Kasparov product $[\psi_+, \psi_-]\widehat{\otimes}_{\mathbb{C}}[i]$ agrees with the unit $\mathbf{1}_{\mathbb{T}} \in KK_0^{SU(2)}(\mathbb{T}, \mathbb{T})$. In particular, we have that $\mathbb{T}$ and $\mathbb{C}$ are KK-equivalent in an $SU(2)$-equivariant way.*

We let $F\widehat{\otimes}\mathbb{T}$ denote the standard module over $\mathbb{T}$, defined as the exterior tensor product of the Fock space F and the Toeplitz algebra $\mathbb{T}$ viewed as a right Hilbert C^*-module over itself. The standard module becomes an $SU(2)$-Hilbert-C^*-module via the diagonal representation of $SU(2)$ on $F\widehat{\otimes}\mathbb{T}$ given explicitly by

$$g(\xi \otimes T_\eta) := g(\xi) \otimes T_{g(\eta)}$$

for every $g \in SU(2)$, $\xi \in F$ and $\eta \in E_k$.

We remark that the interior Kasparov product $[\psi_+, \psi_-]\widehat{\otimes}_{\mathbb{C}}[i]$ is represented by the $SU(2)$-equivariant quasi-homomorphism $(\psi_+ \otimes 1_{\mathbb{T}}, \psi_- \otimes 1_{\mathbb{T}})$, where $\psi_+ \otimes 1_{\mathbb{T}} : \mathbb{T} \to \mathbb{L}((F \oplus F)\widehat{\otimes}\mathbb{T})$ and $\psi_- \otimes 1_{\mathbb{T}} : \mathbb{T} \to \mathbb{L}((F \oplus F)\widehat{\otimes}\mathbb{T})$ are $SU(2)$-equivariant $*$-homomorphisms.

We let $M_{\mathbb{T}} : \mathbb{T} \to \mathbb{L}(\mathbb{T})$ denote the $SU(2)$-equivariant $*$-homomorphism obtained by letting the Toeplitz algebra act as bounded adjointable operators on itself via left-multiplication. Recall moreover that $Q_0 : F \to F$ is the orthogonal projection onto the vacuum subspace $E_0 \subseteq F$.

Our proof of Theorem 6.1 amounts to showing that the $SU(2)$-equivariant quasi-homomorphism $(\psi_+ \otimes 1_{\mathbb{T}}, \psi_- \otimes 1_{\mathbb{T}})$ is homotopic to the $SU(2)$-equivariant quasi-homomorphism $(\psi_- \otimes 1_{\mathbb{T}} + (Q_0 \oplus 0) \otimes M_{\mathbb{T}}, \psi_- \otimes 1_{\mathbb{T}})$. Indeed, we would then obtain the following identities inside $KK_0^{SU(2)}(\mathbb{T}, \mathbb{T})$:

$$[\psi_+, \psi_-]\widehat{\otimes}_{\mathbb{C}}[i] = [\psi_+ \otimes 1_{\mathbb{T}}, \psi_- \otimes 1_{\mathbb{T}}] = [\psi_- \otimes 1_{\mathbb{T}} + (Q_0 \oplus 0) \otimes M_{\mathbb{T}}, \psi_- \otimes 1_{\mathbb{T}}] = \mathbf{1}_{\mathbb{T}}.$$

The proof of the $SU(2)$-equivariant homotopy

$$(\psi_+ \otimes 1_{\mathbb{T}}, \psi_- \otimes 1_{\mathbb{T}}) \sim_h (\psi_- \otimes 1_{\mathbb{T}} + (Q_0 \oplus 0) \otimes M_{\mathbb{T}}, \psi_- \otimes 1_{\mathbb{T}})$$

is divided into three steps and occupies the remainder of this section.

It will sometimes be convenient to view the standard module $F \widehat{\otimes} \mathbb{T}$ as a closed subspace of bounded operators from F to the Hilbert space tensor product $F \widehat{\otimes} F$. Indeed, for every $\xi \in F$ and $x \in \mathbb{T}$, we have the bounded operator

$$\xi \otimes x : F \to F \widehat{\otimes} F \qquad (\xi \otimes x)(\eta) := \xi \otimes x(\eta)$$

and $F \widehat{\otimes} \mathbb{T}$ does in fact agree with the smallest closed subspace of $\mathbb{L}(F, F \widehat{\otimes} F)$ containing the bounded operators of the form $\xi \otimes x$ for all $\xi \in F$ and $x \in \mathbb{T}$. The inner product on $F \widehat{\otimes} \mathbb{T}$ then agrees with the operation

$$\langle \xi, \eta \rangle := \xi^* \cdot \eta \qquad \xi, \eta \in F \widehat{\otimes} \mathbb{T}$$

using only products and adjoints of bounded operators. Moreover, the right action of $\mathbb{T}$ on $F \widehat{\otimes} \mathbb{T}$ is simply induced by the composition of bounded operators $\mathbb{L}(F, F \widehat{\otimes} F)$ and $\mathbb{L}(F)$. Any bounded operator $T : F \widehat{\otimes} F \to F \widehat{\otimes} F$ acts on the operator space $\mathbb{L}(F, F \widehat{\otimes} F)$ via the composition of bounded operators in $\mathbb{L}(F \widehat{\otimes} F)$ and $\mathbb{L}(F, F \widehat{\otimes} F)$. In this fashion, the unital C^*-algebra of bounded adjointable operators on $F \widehat{\otimes} \mathbb{T}$ identifies with the unital C^*-subalgebra of $\mathbb{L}(F \widehat{\otimes} F)$ consisting of those bounded operators $T : F \widehat{\otimes} F \to F \widehat{\otimes} F$ with the property that both T and T^* preserves the closed subspace $F \widehat{\otimes} \mathbb{T} \subseteq \mathbb{L}(F, F \widehat{\otimes} F)$. To wit,

$$\mathbb{L}(F \widehat{\otimes} \mathbb{T}) \cong \big\{ T \in \mathbb{L}(F \widehat{\otimes} F) \mid T \cdot (F \widehat{\otimes} \mathbb{T}) \, , \, T^* \cdot (F \widehat{\otimes} \mathbb{T}) \subseteq F \widehat{\otimes} \mathbb{T} \big\}.$$

6.1. *Intertwining representations of the Toeplitz algebra*

Before we can construct our homotopy we need some preliminaries, explaining better the relationship between the $SU(2)$-equivariant $*$-homomorphisms $\psi_+ \otimes 1_{\mathbb{T}}$ and $\psi_- \otimes 1_{\mathbb{T}} + Q_0^T \otimes M_{\mathbb{T}} : \mathbb{T} \to \mathbb{L}((F \oplus F) \widehat{\otimes} \mathbb{T})$.

We are in this respect particularly interested in the $SU(2)$-equivariant bounded operator

$$W : (F \widehat{\otimes} F)^{\oplus 2} \to (F \widehat{\otimes} F)^{\oplus 2}$$

defined as the composition

$$(F \widehat{\otimes} F)^{\oplus 2} \xrightarrow{\begin{pmatrix} \iota_R^* \otimes 1_F \\ V_R^* \otimes 1_F \end{pmatrix}^*} (F \otimes E_1) \widehat{\otimes} F \cong F \widehat{\otimes} (E_1 \otimes F) \xrightarrow{\begin{pmatrix} 1_F \otimes \iota_L^* \\ 1_F \otimes V_L^* \end{pmatrix}} (F \widehat{\otimes} F)^{\oplus 2}.$$

We express this bounded operator in the following matrix form:

$$W = \begin{pmatrix} v^{TT} & v^{TB} \\ v^{BT} & v^{BB} \end{pmatrix} = \begin{pmatrix} (1 \otimes \iota_L^*)(\iota_R \otimes 1) & (1 \otimes \iota_L^*)(V_R \otimes 1) \\ (1 \otimes V_L^*)(\iota_R \otimes 1) & (1 \otimes V_L^*)(V_R \otimes 1) \end{pmatrix}, \tag{6.1}$$

where all the entries belong to $\mathbb{L}(F \widehat{\otimes} F)$.

We moreover let $\Sigma : F \widehat{\otimes} F \to F \widehat{\otimes} F$ denote the flip map $\Sigma(\xi \otimes \eta) = \eta \otimes \xi$ and remark that Σ is an $SU(2)$-equivariant unitary operator.

Using Propositions 3.8 and 3.12, we see that the $SU(2)$-equivariant operators

$$W_R := \begin{pmatrix} \iota_R^* \\ V_R^* \end{pmatrix} : F \otimes E_1 \to F \oplus F \qquad \text{and}$$

$$W_L := \begin{pmatrix} \iota_L^* \\ V_L^* \end{pmatrix} : E_1 \otimes F \to F \oplus F,$$

are isometric with $W_R W_R^*$ and $W_L W_L^*$ both being the orthogonal projection onto $F_+ \oplus F$. It moreover holds that

$$W = (1_F \otimes W_L)(W_R^* \otimes 1_F) \in \mathbb{L}\big((F \widehat{\otimes} F) \oplus (F \widehat{\otimes} F)\big).$$

LEMMA 6.2. *The $SU(2)$-equivariant operator W is a partial isometry with*

$$1 - WW^* = \begin{pmatrix} 1_F \otimes Q_0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad 1 - W^*W = \begin{pmatrix} Q_0 \otimes 1_F & 0 \\ 0 & 0 \end{pmatrix}.$$

Moreover, we have

$$W^*(\psi_+(x) \otimes 1_F)W = \psi_-(x) \otimes 1_F$$

for all $x \in \mathbb{T}$.

Proof. The first claim follows immediately from the above remarks and the computations

$$\begin{aligned} WW^* &= (1_F \otimes W_L)(W_R^* \otimes 1_F)(W_R \otimes 1_F)(1_F \otimes W_L^*) = 1_F \otimes W_L W_L^* \qquad \text{and} \\ W^*W &= (W_R \otimes 1_F)(1_F \otimes W_L^*)(1_F \otimes W_L)(W_R^* \otimes 1_F) = W_R W_R^* \otimes 1_F. \end{aligned}$$

Let now $x \in \mathbb{T}$ be given. The second claim follows from the computations

$$\begin{aligned} W^*(\psi_+(x) \otimes 1_F)W &= (W_R \otimes 1_F)(1_F \otimes W_L^*)(x \otimes 1_{F \oplus F})(1_F \otimes W_L)(W_R^* \otimes 1_F) \\ &= (W_R \otimes 1_F)(x \otimes 1 \otimes 1_F)(W_R^* \otimes 1_F) = \psi_-(x) \otimes 1_F, \end{aligned}$$

using that $W_L : F \otimes E_1 \to F \oplus F$ is an isometry. □

LEMMA 6.3. *The operator*

$$H_0 := -W + \begin{pmatrix} \Sigma(Q_0 \otimes 1_F) & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{L}\big((F \widehat{\otimes} F) \oplus (F \widehat{\otimes} F)\big)$$

is an $SU(2)$-equivariant unitary operator and we have the identity

$$H_0^*(\psi_+(x) \otimes 1_F)H_0 = \psi_-(x) \otimes 1_F + \begin{pmatrix} Q_0 \otimes x & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{L}\big((F \widehat{\otimes} F) \oplus (F \widehat{\otimes} F)\big)$$

for all $x \in \mathbb{T}$.

Proof. The fact that H_0 is a unitary operator follows by noting that both W and $\left(\begin{smallmatrix} \Sigma(Q_0 \otimes 1_F) & 0 \\ 0 & 0 \end{smallmatrix}\right)$ are partial isometries satisfying

$$\begin{aligned} WW^* &+ \begin{pmatrix} \Sigma(Q_0 \otimes 1_F)(Q_0 \otimes 1_F)\Sigma & 0 \\ 0 & 0 \end{pmatrix} \\ &= 1 = W^*W + \begin{pmatrix} (Q_0 \otimes 1_F)\Sigma\Sigma(Q_0 \otimes 1_F) & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Since all the involved operators are $SU(2)$-equivariant, it holds that H_0 is $SU(2)$-equivariant as well.

Let now $x \in \mathbb{T}$ be given. Using that $W_R : F \otimes E_1 \to F \oplus F$ is an isometry together with the definitions of the involved operators, we compute that

$$\begin{aligned}(\psi_+(x) \otimes 1_F)H_0 &= -(x \otimes 1_{F\oplus F})W + \begin{pmatrix}(x \otimes 1_F)\Sigma(Q_0 \otimes 1_F) & 0\\ 0 & 0\end{pmatrix}\\ &= -(x \otimes 1_{F\oplus F})(1_F \otimes W_L)(W_R^* \otimes 1_F) + \begin{pmatrix}\Sigma(Q_0 \otimes x) & 0\\ 0 & 0\end{pmatrix}\\ &= -W(\psi_-(x) \otimes 1_F) + \begin{pmatrix}\Sigma(Q_0 \otimes x) & 0\\ 0 & 0\end{pmatrix}.\end{aligned}$$

This computation and the first part of the present proof imply the intertwining identity stated in the lemma. □

Let us apply the notation $j : \mathbb{T} \to \mathbb{L}(F)$ for the inclusion $\mathbb{T} \subseteq \mathbb{L}(F)$ so that j becomes a unital $*$-homomorphism. The above lemma then shows that the two $SU(2)$-equivariant $*$-homomorphisms

$$\psi_+ \otimes 1_F \text{ and } \psi_- \otimes 1_F + (Q_0 \oplus 0) \otimes j : \mathbb{T} \to \mathbb{L}\big((F\widehat{\otimes}F) \oplus (F\widehat{\otimes}F)\big)$$

are unitarily equivalent via the $SU(2)$-equivariant unitary operator $H_0 \in \mathbb{L}((F\widehat{\otimes}F) \oplus (F\widehat{\otimes}F))$. We emphasise that H_0 does *not* define a bounded adjointable operator on $(F\widehat{\otimes}\mathbb{T}) \oplus (F\widehat{\otimes}\mathbb{T})$ (because of the part containing the flip map). The two $*$-homomorphisms

$$\psi_+ \otimes 1_{\mathbb{T}} \text{ and } \psi_- \otimes 1_{\mathbb{T}} + (Q_0 \oplus 0) \otimes M_{\mathbb{T}} : \mathbb{T} \to \mathbb{L}\big((F\widehat{\otimes}\mathbb{T}) \oplus (F\widehat{\otimes}\mathbb{T})\big)$$

are therefore most likely *not* unitarily equivalent.

In any case, we now start analysing the unitary operator $H_0 \in \mathbb{L}((F\widehat{\otimes}F) \oplus (F\widehat{\otimes}F))$ in more details, paying particular attention to the partial isometry $W \in \mathbb{L}((F\widehat{\otimes}F) \oplus (F\widehat{\otimes}F))$.

Recall that the invertible element $\Phi \in \mathbb{T}$ was introduced in (4.1).

LEMMA 6.4. *The partial isometry W defines a bounded adjointable operator on $(F\widehat{\otimes}\mathbb{T}) \oplus (F\widehat{\otimes}\mathbb{T})$. In fact, we explicitly have that*

$$\begin{aligned}W &= \begin{pmatrix} v^{TT} & v^{TB}\\ v^{BT} & v^{BB}\end{pmatrix}\\ &= \sum_{j=0}^n \begin{pmatrix}(T_j')^* \otimes T_j & (-1)^{n-j}T'_{n-j}\Phi^{1/2} \otimes T_j\\ (-1)^j (T_j')^* \otimes \Phi^{1/2}T^*_{n-j} & (-1)^n T'_{n-j}\Phi^{1/2} \otimes \Phi^{1/2}T^*_{n-j}\end{pmatrix}.\end{aligned}$$

Proof. This follows from Lemma 4.4 and the matrix description of W from (6.1) together with the formulae provided in Lemma 4.1 and Proposition 4.5. □

Remark that it follows from Lemma 6.4 that

$$\begin{aligned}v^{BT} &= (\Phi^{-1/2} \otimes \Phi^{1/2}) \cdot (v^{TB})^* \qquad \text{and}\\ v^{BB} &= (-1)^n(1_F \otimes \Phi^{1/2}) \cdot (v^{TT})^* \cdot (\Phi^{1/2} \otimes 1_{\mathbb{T}}).\end{aligned} \tag{6.2}$$

For later use, we now relate the bounded operator $v^{TB} : F\widehat{\otimes}F \to F\widehat{\otimes}F$ to the bounded operators $\sigma_{k,m} : E_k \otimes E_m \to E_{k+1} \otimes E_{m+1}$ introduced in (3.13) for $k, m \in \mathbb{N}_0$.

LEMMA 6.5. *We have the identity*

$$v^{TB}(\xi) = \frac{(-1)^{(n+1)k}}{\sqrt{\mu_{k+1}}} \cdot \sigma_{k,m}(\xi) = \frac{(-1)^{(n+1)k}\sqrt{n+1}}{\sqrt{d_k d_{k+1}}} \cdot \sigma_{k,m}(\xi).$$

for all $\xi \in E_k \otimes E_m$.

Proof. This follows immediately from the definition of the involved operators, see (3.9), (3.13) and (6.1). Recall also from (3.5) that $\mu_{k+1} = (d_k d_{k+1})/d_1$ for all $k \in \mathbb{N}_0$. □

PROPOSITION 6.6. *For every* $x \in \mathbb{T}$, *we have that the commutator* $[\psi_+(x) \otimes 1_{\mathbb{T}}, W]$ *belongs to the algebra* $M_2(\mathbb{K}\widehat{\otimes}\mathbb{T})$.

Proof. Let $x \in \mathbb{T}$ be given. We know from Proposition 5.1 that the difference

$$\psi_-(x) - \psi_+(x) : F \oplus F \to F \oplus F$$

is a compact operator. Note also that it follows from Lemma 6.2 that $WW^*(\psi_+(x) \otimes 1_{\mathbb{T}}) = (\psi_+(x) \otimes 1_{\mathbb{T}})WW^*$. Using these facts together with one more application of Lemmas 6.2 and 6.4, we may compute the above commutator modulo compact operators in the following way:

$$\begin{aligned}[\psi_+(x) \otimes 1_{\mathbb{T}}, W] &\sim (\psi_+(x) \otimes 1_{\mathbb{T}})W - W(\psi_-(x) \otimes 1_{\mathbb{T}}) \\ &= (\psi_+(x) \otimes 1_{\mathbb{T}})W - WW^*(\psi_+(x) \otimes 1_{\mathbb{T}})W = 0.\end{aligned}$$

This proves the present proposition. □

We now present a more refined estimate on the commutator between the generator $T_j^* : F \to F$ and the intertwining partial isometry $W \in M_2(\mathbb{L}(F\widehat{\otimes}\mathbb{T}))$.

PROPOSITION 6.7. *Let* $p \in [0, 1]$ *and* $j \in \{0, 1, \dots, n\}$. *The unbounded operators*

$$(D^p \otimes 1_{\mathbb{C}^2\otimes\mathbb{T}})[\psi_+(T_j^*) \otimes 1_{\mathbb{T}}, W](D^{1-p} \otimes 1_{\mathbb{C}^2\otimes\mathbb{T}}) \qquad \text{and}$$

$$(D^p \otimes 1_{\mathbb{C}^2\otimes\mathbb{T}})[\psi_+(T_j^*) \otimes 1_{\mathbb{T}}, W^*](D^{1-p} \otimes 1_{\mathbb{C}^2\otimes\mathbb{T}}) : (F_{\text{alg}} \otimes \mathbb{C}^2 \otimes \mathbb{T}) \to (F \otimes \mathbb{C}^2)\widehat{\otimes}\mathbb{T}$$

both extend to elements in $M_2(\mathbb{L}(F\widehat{\otimes}\mathbb{T}))$.

Proof. We start with the claim regarding the commutator with $W : (F \oplus F)\widehat{\otimes}\mathbb{T} \to (F \oplus F)\widehat{\otimes}\mathbb{T}$. By the identity in (6.1) and the fact that $(T_j^* \otimes 1)\iota_R = \iota_R T_j^*$, we have that

$$[\psi_+(T_j^*) \otimes 1_{\mathbb{T}}, W] = \begin{pmatrix} 0 & (1_F \otimes \iota_L^*)\big(\big((T_j^* \otimes 1)V_R - V_R T_j^*\big) \otimes 1_{\mathbb{T}}\big) \\ 0 & (1_F \otimes V_L^*)\big(\big((T_j^* \otimes 1)V_R - V_R T_j^*\big) \otimes 1_{\mathbb{T}}\big) \end{pmatrix}. \tag{6.3}$$

Now, from Lemma 4.1 and Proposition 4.5, we obtain that the bounded operators

$$1_F \otimes \iota_L^* \text{ and } 1_F \otimes V_L^* : F\widehat{\otimes}(E_1 \otimes F) \to F\widehat{\otimes}F$$

both define elements in $\mathbb{L}((F \otimes E_1)\widehat{\otimes}\mathbb{T}, F\widehat{\otimes}\mathbb{T})$. It therefore suffices to show that

$$(D^p \otimes 1)((T_j^* \otimes 1)V_R - V_R T_j^*)D^{1-p} : F_{\text{alg}} \to F \otimes E_1$$

extends to a bounded operator. But this was already proved in Lemma 5.5.

We continue with the claim regarding the commutator with $W^* : (F \oplus F)\widehat{\otimes}\mathbb{T} \to (F \oplus F)\widehat{\otimes}\mathbb{T}$. We are going to suppress the extra '$\otimes 1_{\mathbb{C}^2\otimes\mathbb{T}}$' from the notation, for example, writing D^p instead of $D^p \otimes 1_{\mathbb{C}^2\otimes\mathbb{T}}$. Note first that the unbounded operator

$$D^r W^* D^{-r} : (F_{\text{alg}} \otimes \mathbb{C}^2 \otimes \mathbb{T}) \to (F \otimes \mathbb{C}^2)\widehat{\otimes}\mathbb{T}$$

extends to a bounded adjointable operators on $(F \oplus F)\widehat{\otimes}\mathbb{T}$ for all $r \in \mathbb{R}$. To see this, we remark that

$$D^r \iota_R^*(D^{-r} \otimes 1)(\xi) = \iota_R^*(\Phi^{-r} \otimes 1)(\xi) \qquad \text{and}$$
$$D^r V_R^*(D^{-r} \otimes 1)(\xi) = \Phi^r V_R^*(\xi)$$

for all $\xi \in F_{\mathrm{alg}} \otimes E_1$ and hence, on using (6.1), Lemma 4.1 and Proposition 4.5, we obtain that $D^r W^* D^{-r}$ extends to the bounded adjointable operator

$$\begin{pmatrix} (v^{TT})^*(\Phi^{-r} \otimes 1_{\mathbb{T}}) & (v^{BT})^*(\Phi^{-r} \otimes 1_{\mathbb{T}}) \\ (\Phi^r \otimes 1_{\mathbb{T}})(v^{TB})^* & (\Phi^r \otimes 1_{\mathbb{T}})(v^{BB})^* \end{pmatrix} \in \mathbb{L}\big((F \oplus F)\widehat{\otimes}\mathbb{T}\big).$$

Next, remark that $T_j^* W W^* = W W^* T_j^*$ since $1 - WW^* = (1_F \otimes Q_0) \oplus 0$. Then, for every $\xi \in F_{\mathrm{alg}} \otimes \mathbb{C}^2 \otimes \mathbb{T}$, we have that

$$\begin{aligned} D^p[T_j^*, W^*]D^{1-p}(\xi) &= (1 - W^*W)D^p T_j^* W^* D^{1-p}(\xi) \\ &\qquad + D^p W^* W T_j^* W^* D^{1-p} - D^p W^* T_j^* D^{1-p}(\xi) \\ &= (1 - W^*W)D^p T_j^* W^* D^{1-p}(\xi) \\ &\qquad + D^p W^* D^{-p} \cdot (D^p[W, T_j^*]D^{1-p}) \cdot D^{p-1} W^* D^{1-p}(\xi). \end{aligned}$$

Each of the terms in this sum extends to a bounded adjointable operator on $(F \oplus F)\widehat{\otimes}\mathbb{T}$. For the first term, this follows since $1 - W^*W = (Q_0 \otimes 1_{\mathbb{T}}) \oplus 0$, and for the second term this follows from the argument carried out earlier in this proof. □

6.2. *Decomposition of the standard module*

We define the Hilbert space $G \subseteq F\widehat{\otimes}F$ as the closure of the subspace

$$\operatorname{span}\big\{\iota_{k,m}(\xi) \mid k, m \in \mathbb{N}_0\,,\ \xi \in E_{k+m}\big\} \subseteq F\widehat{\otimes}F. \tag{6.4}$$

Our strategy for constructing our homotopy is to work separately on the closed subspace

$$(G \oplus \{0\}) \subseteq (F\widehat{\otimes}F) \oplus (F\widehat{\otimes}F)$$

and the orthogonal complement $G^\perp \oplus (F\widehat{\otimes}F)$. In fact, it turns out that our homotopy behaves very much like the classical $U(1)$-case (cf. [**32**, Section 4]) on the closed subspace $G \oplus \{0\}$, whereas the remaining part (taking place on $G^\perp \oplus (F\widehat{\otimes}F)$) requires a separate argument. We therefore need to understand the orthogonal projection $\Pi : F\widehat{\otimes}F \to F\widehat{\otimes}F$ onto the orthogonal complement $G^\perp \subseteq F\widehat{\otimes}F$. We show here below that Π defines a bounded adjointable operator on $F\widehat{\otimes}\mathbb{T}$ and that the commutator $[x \otimes 1_{\mathbb{T}}, \Pi]$ is a compact operator for every $x \in \mathbb{T}$.

It turns out that the orthogonal projection $\Pi : F\widehat{\otimes}F \to F\widehat{\otimes}F$ is related to the bounded operator $v^{TB} : F\widehat{\otimes}F \to F\widehat{\otimes}F$ and a proper description of this relationship requires a better understanding of the polar decomposition of $v^{TB} : F\widehat{\otimes}F \to F\widehat{\otimes}F$.

We are going to apply Proposition A.1 with $X := F\widehat{\otimes}\mathbb{T}$ and $y := v^{TB} : F\widehat{\otimes}\mathbb{T} \to F\widehat{\otimes}\mathbb{T}$. The relevant dense submodule is the algebraic tensor product $\mathscr{X} := F_{\mathrm{alg}} \otimes \mathbb{T}$. We fix $j \in \{0, 1, \ldots, n\}$ and put $x_j := T_j^* \otimes 1_{\mathbb{T}} : X \to X$. We immediately remark that

$$x_j(\mathscr{X})\,,\ x_j^*(\mathscr{X})\,,\ y^*(\mathscr{X}) \subseteq \mathscr{X}\,,$$

where the last inclusion follows from Lemma 6.4.

We now compute the bounded adjointable operator $y^*y = (v^{TB})^* v^{TB} : F\widehat{\otimes}\mathbb{T} \to F\widehat{\otimes}\mathbb{T}$. To this end, we apply Theorem 3.19 and define positive invertible operators

$$\Gamma_{k,m} : E_k \otimes E_m \to E_k \otimes E_m \qquad k, m \in \mathbb{N}_0$$

using the prescription

$$\Gamma_{k,m}(\sigma^j \iota_{k-j,m-j}\xi) := \Big(1 - \frac{d_{k-j}d_{m-j-1}}{d_{k+1}d_m}\Big)(\sigma^j \iota_{k-j,m-j}\xi), \tag{6.5}$$

for all $\xi \in E_{k+m-2j}$ and $0 \leqslant j \leqslant k, m$. A quick computation shows that

$$\|\Gamma_{k,m}\| = 1 - \frac{d_{k-l}d_{m-l-1}}{d_{k+1}d_m} \leqslant 1, \tag{6.6}$$

where $l = \min\{k, m\}$ and we therefore obtain a positive bounded operator

$$\Gamma : F\widehat{\otimes}F \to F\widehat{\otimes}F \qquad \Gamma|_{E_k \otimes E_m} := \Gamma_{k,m}$$

with dense image. We are here applying our standing convention that $n \in \mathbb{N}$ so that the irreducible representation $\rho_n : SU(2) \to U(L_n)$ is non-trivial.

Lemma 6.8. *We have the identity*

$$(v^{TB})^* v^{TB} = \Gamma : F\widehat{\otimes}F \to F\widehat{\otimes}F.$$

Proof. Let $k, m \in \mathbb{N}_0$, let $j \in \{0, 1, \ldots, \min\{k, m\}\}$ and let $\xi \in E_{k+m-2j}$ be given. Using Theorem 3.19, it suffices to show that

$$(v^{TB})^* v^{TB}(\sigma^j \iota_{k-j,m-j}\xi) = \Gamma(\sigma^j \iota_{k-j,m-j}\xi).$$

However, by Lemma 6.5, we have that

$$(v^{TB})^* v^{TB}(\eta) = \frac{1}{\mu_{k+1}} \sigma^*_{k,m}\sigma_{k,m}(\eta)$$

for every $\eta \in E_k \otimes E_m$. Hence we see from Lemmas 3.17 and 3.18 that

$$\begin{aligned}(v^{TB})^* v^{TB}(\sigma^j \iota_{k-j,m-j}\xi) &= \frac{1}{\mu_{k+1}} \sigma^* \sigma^{j+1} \iota_{k-j,m-j}\xi = \Big(1 - \frac{d_{k-j}d_{m-j-1}}{d_{k+1}d_m}\Big) \cdot \sigma^j \iota_{k-j,m-j}\xi \\ &= \Gamma_{k,m}(\sigma^j \iota_{k-j,m-j}\xi).\end{aligned}$$

This proves the present lemma. □

It follows from Lemmas 6.4 and 6.8 that the positive bounded operator $\Gamma : F\widehat{\otimes}F \to F\widehat{\otimes}F$ defines a positive bounded adjointable operator $\Gamma : F\widehat{\otimes}\mathbb{T} \to F\widehat{\otimes}\mathbb{T}$.

Lemma 6.9. *The image of the positive bounded adjointable operator* $\Gamma : F\widehat{\otimes}\mathbb{T} \to F\widehat{\otimes}\mathbb{T}$ *contains the dense submodule* $\mathscr{X} = F_{\text{alg}} \otimes \mathbb{T} \subseteq F\widehat{\otimes}\mathbb{T}$.

Proof. Let us fix a $k \in \mathbb{N}_0$ and show that $E_k \otimes \mathbb{T} \subseteq \text{Im}(\Gamma)$. We recall that $Q_k : F \to F$ denotes the orthogonal projection with image $E_k \subseteq F$. It then follows from the definition of $\Gamma : F\widehat{\otimes}F \to F\widehat{\otimes}F$ that the bounded operator

$$\begin{aligned}&\Gamma(Q_k \otimes 1_F) + (1_F - Q_k) \otimes 1_F \\ &\quad = (Q_k \otimes 1_F)\Gamma(Q_k \otimes 1_F) + (1_F - Q_k) \otimes 1_F : F\widehat{\otimes}F \to F\widehat{\otimes}F\end{aligned}$$

has a bounded inverse. Indeed, for every $m \in \mathbb{N}_0$, it holds that $\Gamma_{k,m} : E_k \otimes E_m \to E_k \otimes E_m$ is invertible with

$$\|\Gamma_{k,m}^{-1}\| = \sup_{j=0,1,\ldots,\min\{k,m\}} \left(1 - \frac{d_{k-j}d_{m-j-1}}{d_{k+1}d_m}\right)^{-1}$$
$$\leqslant \left(1 - \frac{d_k d_{m-1}}{d_{k+1}d_m}\right)^{-1} \leqslant \left(1 - \frac{d_k}{d_{k+1}}\right)^{-1}.$$

Now, since the invertible bounded operator $\Gamma(Q_k \otimes 1_F) + (1_F - Q_k) \otimes 1_F \in \mathbb{L}(F\widehat{\otimes}F)$ belongs to the unital C^*-subalgebra $\mathbb{L}(F\widehat{\otimes}\mathbb{T}) \subseteq \mathbb{L}(F\widehat{\otimes}F)$, we obtain that the bounded adjointable operator $\Gamma(Q_k \otimes 1_{\mathbb{T}}) + (1_F - Q_k) \otimes 1_{\mathbb{T}} : F\widehat{\otimes}\mathbb{T} \to F\widehat{\otimes}\mathbb{T}$ is invertible as well. But this shows that

$$E_k \otimes \mathbb{T} = \mathrm{Im}(\Gamma(Q_k \otimes 1_{\mathbb{T}})) \subseteq \mathrm{Im}(\Gamma). \qquad \square$$

As a consequence of Lemma 6.9, we obtain that $\Gamma^{-1} : \mathrm{Im}(\Gamma) \to F\widehat{\otimes}\mathbb{T}$ is an unbounded positive and regular operator on the Hilbert C^*-module $F\widehat{\otimes}\mathbb{T}$. Moreover, we see from the proof of Lemma 6.9 that the domain of Γ^{-1} contains the algebraic tensor product $\mathscr{X} = F_{\mathrm{alg}} \otimes \mathbb{T}$.

Lemma 6.10. *The closure of $v^{TB}\Gamma^{-1/2} : Im(\Gamma^{1/2}) \to F\widehat{\otimes}\mathbb{T}$ is a bounded adjointable isometry $\Theta : F\widehat{\otimes}\mathbb{T} \to F\widehat{\otimes}\mathbb{T}$ and the associated orthogonal projection $\Theta\Theta^* \in \mathbb{L}(F\widehat{\otimes}\mathbb{T})$ agrees with $\Pi \in \mathbb{L}(F\widehat{\otimes}F)$ (on suppressing the inclusion $\mathbb{L}(F\widehat{\otimes}\mathbb{T}) \subseteq \mathbb{L}(F\widehat{\otimes}F)$).*

Proof. Since $\Gamma = (v^{TB})^* v^{TB}$ and the domains of both $v^{TB}\Gamma^{-1/2}$ and $(v^{TB}\Gamma^{-1/2})^*$ contain the dense submodule $F_{\mathrm{alg}} \otimes \mathbb{T}$, we obtain that $\Theta : F\widehat{\otimes}\mathbb{T} \to F\widehat{\otimes}\mathbb{T}$ is a well-defined bounded adjointable isometry. We now compute the image of Θ considered as a bounded operator on $F\widehat{\otimes}F$. This image clearly agrees with the closure of the image of v^{TB} restricted to the algebraic tensor product $F_{\mathrm{alg}} \otimes F_{\mathrm{alg}}$. For each $k, m \in \mathbb{N}_0$, we know that the image of $v^{TB}|_{E_k \otimes E_m} : E_k \otimes E_m \to E_{k+1} \otimes E_{m+1}$ agrees with the image of $\sigma_{k,m} : E_k \otimes E_m \to E_{k+1} \otimes E_{m+1}$. However, from Theorem 3.19, we see that the image of $\sigma_{k,m} : E_k \otimes E_m \to E_{k+1} \otimes E_{m+1}$ agrees with the orthogonal complement of $\iota_{k+1,m+1}(E_{k+m+2}) \subseteq E_{k+1} \otimes E_{m+1}$. These observations entail that the image of $\Theta : F\widehat{\otimes}F \to F\widehat{\otimes}F$ agrees with

$$\mathrm{span}\{\iota_{k,m}(\xi) \mid k, m \in \mathbb{N}_0\,,\ \xi \in E_{k+m}\}^\perp \subseteq F\widehat{\otimes}F.$$

In other words, we have that $\mathrm{Im}(\Theta : F\widehat{\otimes}F \to F\widehat{\otimes}F) = G^\perp = \mathrm{Im}(\Pi)$. This proves the present lemma. $\square$

Let us introduce the compact operator

$$K := D^{-1} \otimes 1_{\mathbb{T}} : F\widehat{\otimes}\mathbb{T} \to F\widehat{\otimes}\mathbb{T},$$

recalling that the dimension operator $D : \mathrm{Dom}(D) \to F$ was introduced in Definition 4.3.

Recall that $x_j := T_j^* \otimes 1_{\mathbb{T}}$ and $y := v^{TB} : F\widehat{\otimes}\mathbb{T} \to F\widehat{\otimes}\mathbb{T}$.

Lemma 6.11. *There exist bounded adjointable operators $L, \overline{L}, M, \overline{M} : F\widehat{\otimes}\mathbb{T} \to F\widehat{\otimes}\mathbb{T}$ such that*

$$K^{1/2}LK^{1/2} = [x_j, y] = MK \qquad \text{and} \qquad K^{1/2}\overline{L}K^{1/2} = [x_j, y^*] = K\overline{M}.$$

Proof. This follows immediately from Proposition 6.7. Firstly, L and $\overline{L}$ are the bounded adjointable extensions of $(D^{1/2} \otimes 1_{\mathbb{T}})[T_j^* \otimes 1_{\mathbb{T}}, v^{TB}](D^{1/2} \otimes 1_{\mathbb{T}})$ and $(D^{1/2} \otimes 1_{\mathbb{T}})[T_j^* \otimes 1_{\mathbb{T}}, (v^{TB})^*](D^{1/2} \otimes 1_{\mathbb{T}})$, respectively. Secondly, M and $\overline{M}$ are the bounded adjointable extensions of $[T_j^* \otimes 1_{\mathbb{T}}, v^{TB}](D \otimes 1_{\mathbb{T}})$ and $(D \otimes 1_{\mathbb{T}})[T_j^* \otimes 1_{\mathbb{T}}, (v^{TB})^*]$, respectively. It is here understood that all the involved unbounded operators are defined on the algebraic tensor product $F_{\text{alg}} \otimes \mathbb{T}$ even though this is not properly reflected in the notation. □

In order to apply Proposition A.1, we still have to control the growth of the resolvent $R_\lambda := (\lambda + (v^{TB})^* v^{TB})^{-1}$ as the parameter $\lambda > 0$ approaches zero.

LEMMA 6.12. *The identity* $(D^{-1} \otimes 1_{\mathbb{T}})(v^{TB})^* v^{TB} = (v^{TB})^* v^{TB}(D^{-1} \otimes 1_{\mathbb{T}})$ *holds. Moreover, there exists a constant* $C > 0$ *such that*

$$\|(D^{-1} \otimes 1_{\mathbb{T}})R_\lambda\| \leqslant C \qquad \text{and} \qquad \|(D^{-1/2} \otimes 1_{\mathbb{T}})v^{TB} R_\lambda\| \leqslant C$$

for all $\lambda > 0$.

Proof. It follows from the definitions of $\Gamma = (v^{TB})^* v^{TB}$ and $D^{-1} \otimes 1_{\mathbb{T}} : F \widehat{\otimes} \mathbb{T} \to F \widehat{\otimes} \mathbb{T}$ that these two operators commute. Moreover, similarly to the proof of Proposition 6.7, we obtain that $(D^{-1/2} \otimes 1_{\mathbb{T}})v^{TB}(D^{1/2} \otimes 1_{\mathbb{T}}) : F_{\text{alg}} \otimes \mathbb{T} \to F \widehat{\otimes} \mathbb{T}$ extends to the bounded adjointable operator $v^{TB}(\Phi^{1/2} \otimes 1_{\mathbb{T}})$. This implies

$$(D^{-1/2} \otimes 1_{\mathbb{T}})v^{TB} R_\lambda = v^{TB}(\Phi^{1/2} D^{-1/2} \otimes 1_{\mathbb{T}})R_\lambda = v^{TB} R_\lambda (D^{-1/2}\Phi^{1/2} \otimes 1_{\mathbb{T}}). \tag{6.7}$$

It therefore suffices to estimate the quantity $\|(D^{-1} \otimes 1_{\mathbb{T}})R_\lambda\|$ for all $\lambda > 0$. Indeed, from (6.7) and the fact that $D^{-1} \otimes 1_{\mathbb{T}}$ and R_λ commute, we obtain that

$$\begin{aligned}
\big\|(D^{-1/2} \otimes 1_{\mathbb{T}})v^{TB} R_\lambda\big\| &= \big\|v^{TB} R_\lambda (D^{-1/2}\Phi^{1/2} \otimes 1_{\mathbb{T}})\big\| \\
&\leqslant \|v^{TB} R_\lambda^{1/2}\| \cdot \big\|R_\lambda^{1/2}(D^{-1/2} \otimes 1_{\mathbb{T}})\big\| \\
&\leqslant \big\|R_\lambda^{1/2}(D^{-1/2} \otimes 1_{\mathbb{T}})\big\| = \|R_\lambda(D^{-1} \otimes 1_{\mathbb{T}})\|^{1/2}.
\end{aligned}$$

Let $\lambda > 0$ and $k, m \in \mathbb{N}_0$ be given. We remark that $E_k \otimes E_m$ is an invariant subspace for the selfadjoint operator $(D^{-1} \otimes 1_F)R_\lambda : F \widehat{\otimes} F \to F \widehat{\otimes} F$. The restriction to this subspace is given by

$$d_k^{-1}(\lambda + \Gamma_{k,m})^{-1} : E_k \otimes E_m \to E_k \otimes E_m.$$

Using the description of $\Gamma_{k,m} : E_k \otimes E_m \to E_k \otimes E_m$ from (6.5), we then obtain that

$$\begin{aligned}
\|d_k^{-1}(\lambda + \Gamma_{k,m})^{-1}\| \leqslant \|d_k^{-1}\Gamma_{k,m}^{-1}\| &= d_k^{-1} \cdot \left(1 - \frac{d_k d_{m-1}}{d_{k+1} d_m}\right)^{-1} \\
&\leqslant d_k^{-1} \cdot \left(1 - \frac{d_k}{d_{k+1}}\right)^{-1} = \frac{d_{k+1}}{d_k} \cdot (d_{k+1} - d_k)^{-1} \leqslant n + 1.
\end{aligned}$$

Remark that we are here applying the recursive definition of the sequence $\{d_l\}_{l=0}^\infty$ from (3.2) together with Lemma 3.3 which ensures that $d_{k+1} - d_k \geqslant 1$ for all $k \in \mathbb{N}_0$. □

We are now ready to establish the main result of this subsection:

PROPOSITION 6.13. *The unbounded operator* $v^{TB}|v^{TB}|^{-1} : \mathrm{Im}(|v^{TB}|) \to F \widehat{\otimes} \mathbb{T}$ *extends to a bounded adjointable isometry* $\Theta : F \widehat{\otimes} \mathbb{T} \to F \widehat{\otimes} \mathbb{T}$ *satisfying that:*

(i) *the commutator* $[\Theta, x \otimes 1_{\mathbb{T}}] : F \widehat{\otimes} \mathbb{T} \to F \widehat{\otimes} \mathbb{T}$ *is a compact operator for all* $x \in \mathbb{T}$*;*
(ii) *the composition* $\Theta\Theta^*$ *agrees with the orthogonal projection* $\Pi : F \widehat{\otimes} \mathbb{T} \to F \widehat{\otimes} \mathbb{T}$.

In particular, we obtain that $[x \otimes 1_{\mathbb{T}}, \Pi] \in \mathbb{K}(F \widehat{\otimes} \mathbb{T})$ *for all* $x \in \mathbb{T}$.

Proof. The claim in (2) was already verified in Lemma 6.10. The claim regarding the commutator with Π follows immediately from (i) and (ii) and the fact that Θ is a bounded adjointable operator. So we focus on the claim in (i). It suffices to establish this claim for the generators T_j^* and T_j, $j \in \{0, 1, \ldots, n\}$. But this is a consequence of Proposition A.1 on applying Lemmas 6.8, 6.9, 6.11 and 6.12. □

REMARK 6. For $n > 1$, it can be proved that $\Gamma : F\widehat{\otimes}F \to F\widehat{\otimes}F$ has a bounded inverse. It then follows from Lemmas 6.4 and 6.8 that $\Gamma^{-1} \in \mathbb{L}(F\widehat{\otimes}F)$ defines a positive bounded adjointable operator on the standard module $F\widehat{\otimes}\mathbb{T}$. We therefore immediately obtain that the isometry $\Theta = v^{TB}\Gamma^{-1/2}$ lies in $\mathbb{L}(F\widehat{\otimes}\mathbb{T})$ as well. Remark now that the set of bounded adjointable operators on $F\widehat{\otimes}\mathbb{T}$ which commutes up to compact operators with all operators of the form $x \otimes 1_{\mathbb{T}}$ for $x \in \mathbb{T}$ form a unital C^*-subalgebra of $\mathbb{L}(F\widehat{\otimes}\mathbb{T})$. This observation together with Lemma 6.8 and Proposition 6.6 then allow us to conclude that $\Theta \in \mathbb{L}(F\widehat{\otimes}\mathbb{T})$ has this property as well. The situation is more complicated for $n = 1$ since the inverse of $\Gamma : F\widehat{\otimes}F \to F\widehat{\otimes}F$ is in fact unbounded. Our present approach treats both the (well-understood) case where $n = 1$ and the novel case where $n > 1$ in a unified fashion.

6.3. *First step: the classical part*

Let $\mathrm{inc} : \mathbb{T} \to \mathbb{L}(F)$ denote the inclusion of the Toeplitz C^*-algebra into the bounded operators on the Fock Hilbert space F. In the first step of our homotopy between the two quasi-homomorphisms $(\psi_- \otimes 1_{\mathbb{T}} + (Q_0 \oplus 0) \otimes M_{\mathbb{T}}, \psi_- \otimes 1_{\mathbb{T}})$ and $(\psi_+ \otimes 1_{\mathbb{T}}, \psi_- \otimes 1_{\mathbb{T}})$, we create a homotopy between the two $*$-homomorphisms

$$(Q_0 \oplus 0) \otimes M_{\mathbb{T}} \text{ and } (\mathrm{inc} \oplus 0) \otimes Q_0 : \mathbb{T} \to \mathbb{L}\big((F \oplus F)\widehat{\otimes}\mathbb{T}\big).$$

This part of the homotopy behaves very much like the classical $U(1)$-case corresponding to Cuntz–Pimsner algebras associated with C^*-correspondences, see, for instance, [**32**, Theorem 4.4]. However, since we are working with an $SU(2)$-gauge action instead of a $U(1)$-gauge action, it is unreasonable to expect that the $U(1)$-argument would entirely carry over to our situation. Therefore, after this initial step there is still a quite complicated homotopy argument left and this is mainly carried out in Subsection 6.4.

We recall the definition of the closed subspace $G \subseteq F\widehat{\otimes}F$ from (6.4) and we apply the notation

$$P := \Pi \oplus 1_{F\widehat{\otimes}F} \in \mathbb{L}\big((F\widehat{\otimes}F) \oplus (F\widehat{\otimes}F)\big) \tag{6.8}$$

for the orthogonal projection onto the closed subspace $G^\perp \oplus (F\widehat{\otimes}F)$. We emphasise that it follows from the definition of the closed subspace $G \subseteq F\widehat{\otimes}F$ that the orthogonal projection Π onto $G^\perp \subseteq F\widehat{\otimes}F$ is $SU(2)$-equivariant.

LEMMA 6.14. *We have that $[W, P] = 0$ and the restriction $W|_{\mathrm{Im}(P)} : Im(P) \to Im(P)$ is a unitary operator. In fact, we have the identities*

$$v^{TT}(1 - \Pi) = (1 - \Pi)v^{TT} \qquad \text{and} \qquad v^{BT}(1 - \Pi) = 0 = (1 - \Pi)v^{TB} \tag{6.9}$$

among bounded operators on $F\widehat{\otimes}F$.

Proof. Let $k, m \in \mathbb{N}_0$ and $\xi \in E_{k+m}$ be given and consider the vector $\iota_{k,m}(\xi) \in \mathrm{Im}(1 - P)$. Remark that this kind of vectors span a dense subspace of $\mathrm{Im}(1 - P)$. Using the properties of the structure maps for our subproduct system, we obtain that

$$(\iota_R \otimes 1_F)\iota_{k,m}(\xi) = (1_F \otimes \iota_L)\iota_{k-1,m+1}(\xi) \qquad \text{and}$$
$$(1_F \otimes \iota_L)\iota_{k,m}(\xi) = (\iota_R \otimes 1_F)\iota_{k+1,m-1}(\xi),$$

where we apply the convention $\iota_{l,-1} = 0 = \iota_{-1,l}$ for all $l \in \mathbb{N}_0$. Since $V_L^* \iota_L = 0 = V_R^* \iota_R$ and $\iota_R^* \iota_R = 1_F - Q_0 = \iota_L^* \iota_L$, we then obtain that

$$W \begin{pmatrix} \iota_{k,m}(\xi) \\ 0 \end{pmatrix} = \begin{pmatrix} \iota_{k-1,m+1}(\xi) \\ 0 \end{pmatrix} \in \mathrm{Im}(1-P) \text{ and}$$
$$W^* \begin{pmatrix} \iota_{k,m}(\xi) \\ 0 \end{pmatrix} = \begin{pmatrix} \iota_{k+1,m-1}(\xi) \\ 0 \end{pmatrix} \in \mathrm{Im}(1-P), \tag{6.10}$$

proving the first claim of the lemma together with the identities in (6.9). The fact that the restriction $W|_{\mathrm{Im}(P)} : \mathrm{Im}(P) \to \mathrm{Im}(P)$ is a unitary operator now follows since both $1 - W^*W = (Q_0 \otimes 1_F) \oplus 0$ and $1 - WW^* = (1_F \otimes Q_0) \oplus 0$ restrict to the zero operator on $\mathrm{Im}(P) \subseteq (F \widehat{\otimes} F) \oplus (F \widehat{\otimes} F)$. □

For ease of notation, we put

$$p_R := 1 - W^*W = \begin{pmatrix} Q_0 \otimes 1_{\mathbb{T}} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } p_L := 1 - WW^* = \begin{pmatrix} 1_F \otimes Q_0 & 0 \\ 0 & 0 \end{pmatrix}.$$

For each $t \in (0, \pi/2]$, we then define the $SU(2)$-equivariant bounded adjointable operator

$$U_t := -\cos(t)W + (p_L + \sin(t)WW^*)(1 - \cos(t)W^*)^{-1}(p_R + \sin(t)W^*W)$$
$$\in M_2\big(\mathbb{L}(F \widehat{\otimes} \mathbb{T})\big). \tag{6.11}$$

Note that $U_{\pi/2} = 1$. Moreover, we define the $SU(2)$-equivariant bounded adjointable operator

$$H_t := U_t(1 - P) - WP \in M_2\big(\mathbb{L}(F \widehat{\otimes} \mathbb{T})\big) \subseteq M_2\big(\mathbb{L}(F \widehat{\otimes} F)\big).$$

For $t = 0$, we recall from Lemma 6.3 that

$$H_0 = -W + \begin{pmatrix} \Sigma(Q_0 \otimes 1_F) & 0 \\ 0 & 0 \end{pmatrix} \in M_2\big(\mathbb{L}(F \widehat{\otimes} F)\big).$$

LEMMA 6.15. *The $SU(2)$-equivariant bounded operator $H_t \in M_2(\mathbb{L}(F \widehat{\otimes} F))$ is unitary for all $t \in [0, \pi/2]$.*

Proof. For $t = 0$, this was already proved in Lemma 6.3. Thus, let $t \in (0, \pi/2]$ be given. We start by noting that $U_t \in M_2(\mathbb{L}(F \widehat{\otimes} \mathbb{T}))$ is a unitary operator. In fact, a unitary operator like U_t can be constructed from an arbitrary partial isometry W in a unital C^*-algebra. It is in this respect crucial that $t \neq 0$ since $(1 - \cos(t)W^*)^{-1}$ would otherwise not be a well-defined bounded operator. Using Lemma 6.14, we then see that

$$H_t^* H_t = U_t^* U_t (1 - P) + W^*WP = 1 = U_t U_t^* (1 - P) + WW^*P = H_t H_t^*. \qquad \square$$

PROPOSITION 6.16. *Let $j \in \{0, 1, \dots, n\}$. For each $t \in [0, \pi/2]$, we have that*

$$H_t^* \cdot (\psi_+(T_j^*) \otimes 1_F) \cdot H_t(1 - P)$$
$$= (W^*W + p_R \cdot \sin(t)) \cdot \big(\psi_+(T_j^*) \otimes 1_F\big) \cdot (1 - P) + \cos(t) \cdot (1_{F \oplus F} \otimes T_j^*) \cdot p_R. \tag{6.12}$$

In particular, the map

$$t \mapsto H_t^* \cdot (\psi_+(T_j^*) \otimes 1_F) \cdot H_t(1 - P)$$

is continuous in operator norm on the interval $[0, \pi/2]$.

Proof. We start by remarking that

$$(T_j^* \otimes 1_F)\iota_{k,m}(\xi) = \iota_{k-1,m}(T_j^* \xi) \tag{6.13}$$

for all $k, m \in \mathbb{N}_0$ and all $\xi \in E_{k+m}$.

For the rest of this proof, we sometimes use the shorthand notation T_j^* for $\psi_+(T_j^*) \otimes 1_F$. It follows from (6.13) that $T_j^*(1-P) = (1-P)T_j^*(1-P)$ and hence we obtain from Lemma 6.14 and (6.11) that

$$H_t^* \cdot T_j^* \cdot H_t(1-P) = U_t^* T_j^* U_t(1-P)$$

for all $t \in (0, \pi/2]$.

Using the identities in (6.13) and (6.10), we moreover see that

$$\begin{aligned} T_j^* W \cdot (1-P) &= W T_j^* \cdot (1-P) \qquad \text{and} \\ T_j^* W^* \cdot (1-P) &= W^* T_j^* \cdot (1-P) + (1 \otimes T_j^*) \cdot p_R. \end{aligned} \tag{6.14}$$

Indeed, for the second identity, let $k, m \in \mathbb{N}_0$ and $\xi \in E_{k+m}$ be given. For $k > 0$, we then have that

$$T_j^* W^* \begin{pmatrix} \iota_{k,m}(\xi) \\ 0 \end{pmatrix} = \begin{pmatrix} T_j^* \iota_{k+1,m-1}(\xi) \\ 0 \end{pmatrix} = \begin{pmatrix} \iota_{k,m-1} T_j^*(\xi) \\ 0 \end{pmatrix} = \begin{pmatrix} W^* T_j^* \iota_{k,m}(\xi) \\ 0 \end{pmatrix}$$

and for $k = 0$, we get that

$$T_j^* W^* \begin{pmatrix} \iota_{0,m}(\xi) \\ 0 \end{pmatrix} = \begin{pmatrix} \iota_{0,m-1} T_j^*(\xi) \\ 0 \end{pmatrix} = \begin{pmatrix} (1 \otimes T_j^*) p_R \cdot \iota_{0,m}(\xi) \\ 0 \end{pmatrix}.$$

For $t = 0$, we then know from Lemmas 6.2 and 6.3 that

$$\begin{aligned} H_0^* \cdot T_j^* \cdot H_0(1-P) &= (\psi_-(T_j^*) \otimes 1_F) \cdot (1-P) + (1 \otimes T_j^*) \cdot p_R \\ &= W^* T_j^* W \cdot (1-P) + (1 \otimes T_j^*) \cdot p_R \\ &= W^* W T_j^* \cdot (1-P) + (1 \otimes T_j^*) \cdot p_R. \end{aligned}$$

This proves the identity in (6.12) for $t = 0$.

For $t \in (0, \pi/2]$, we record that

$$\begin{aligned} & T_j^* W W^* = W W^* T_j^* \qquad \text{and} \\ & T_j^* (1 - \cos(t) W^*)^{-1} \cdot (1-P) \\ & \qquad = (1 - \cos(t) W^*)^{-1} \cdot \big(T_j^* \cdot (1-P) + \cos(t)(1 \otimes T_j^*) \cdot p_R \big), \end{aligned}$$

where the first identity relies on Lemma 6.2 and the second identity uses (6.14) together with the fact that $p_R W^* = 0$. We also remark that

$$T_j^* \cdot (p_R + \sin(t) W^* W) = \sin(t) \cdot T_j^* = (p_R + \sin(t) W^* W) \cdot (\sin(t) p_R + W^* W) T_j^*,$$

where we are using that $p_R = 1 - W^* W$ and $T_j^* p_R = 0$. For $t \in (0, \pi/2]$, the identity in (6.12) then follows from the computation

$$\begin{aligned} T_j^* U_t \cdot (1-P) = &- \cos(t) W \cdot T_j^* \cdot (1-P) \\ & + (p_L + \sin(t) W W^*)(1 - \cos(t) W^*)^{-1} \\ & \cdot \big(T_j^* (1-P) + \cos(t)(1 \otimes T_j^*) p_R \big)(p_R + \sin(t) W^* W) \\ = &- \cos(t) W \cdot T_j^* \cdot (1-P) + U_t \cdot \cos(t)(1 \otimes T_j^*) p_R \end{aligned}$$

$$+ (p_L + \sin(t)WW^*)(1 - \cos(t)W^*)^{-1}(p_R + \sin(t)W^*W)$$
$$\cdot (\sin(t)p_R + W^*W)T_j^* \cdot (1 - P)$$
$$= U_t \cdot (\sin(t)p_R + W^*W)T_j^* \cdot (1 - P) + U_t \cdot \cos(t)(1 \otimes T_j^*) \cdot p_R. \qquad \square$$

LEMMA 6.17. *Let $K \in M_2(\mathbb{K}\widehat{\otimes}\mathbb{T}) \subseteq M_2(\mathbb{L}(F\widehat{\otimes}F))$. The map $t \mapsto H_t^* K$ is operator norm-continuous on the interval $[0, \pi/2]$.*

Proof. Since the map $t \to H_t$ is operator norm-continuous on the interval $(0, \pi/2]$, it is enough to check continuity at $t = 0$.

We recall that

$$\begin{aligned} H_t^* &= U_t^*(1 - P) - W^*P \\ &= \big(-\cos(t)W^* + (p_R + \sin(t)W^*W)(1 - \cos(t)W)^{-1}(p_L + \sin(t)WW^*)\big)(1 - P) \\ &\quad - W^*P \end{aligned}$$

for $t \in (0, \pi/2]$, whereas

$$H_0^* = -W^*P - W^*(1 - P) + \begin{pmatrix} (Q_0 \otimes 1_F)\Sigma & 0 \\ 0 & 0 \end{pmatrix}.$$

We remark that $\lim_{N\to\infty}(\sum_{k=0}^N Q_k \otimes 1_{F\oplus F})K = K$, where the convergence takes place in operator norm. Next, we recall from Proposition 6.13 that $P \in M_2(\mathbb{L}(F\widehat{\otimes}\mathbb{T}))$ and moreover that $M_2(\mathbb{K}\widehat{\otimes}\mathbb{T}) \subseteq M_2(\mathbb{L}(F\widehat{\otimes}\mathbb{T}))$ is an ideal. Because of the structure of the involved operators, we may then focus on proving that

$$\begin{aligned} &\lim_{t\to 0}(p_R + \sin(t)W^*W)(1 - \cos(t)W)^{-1}(p_L + \sin(t)WW^*) \cdot (Q_k \otimes 1_{F\oplus F}) \cdot (1 - P) \\ &\quad = (Q_0 \otimes 1_F)\Sigma(Q_k \otimes 1_F) \oplus 0. \end{aligned}$$

for every fixed $k \in \mathbb{N}_0$. However, by (6.10), we have that

$$\begin{aligned} &\lim_{t\to 0}(p_R + \sin(t)W^*W)(1 - \cos(t)W)^{-1}(p_L + \sin(t)WW^*)(Q_k \otimes 1_{F\oplus F}) \cdot (1 - P) \\ &= \lim_{t\to 0}(p_R + \sin(t)W^*W)\sum_{j=0}^k(\cos(t)W)^j(p_L + \sin(t)WW^*)(Q_k \otimes 1_{F\oplus F}) \cdot (1 - P) \\ &= p_R\sum_{j=0}^k W^j(Q_k \otimes 1_{F\oplus F})p_L = p_R W^k(Q_k \otimes 1_{F\oplus F})p_L \\ &= (Q_0 \otimes 1_F)\Sigma(Q_k \otimes 1_F) \oplus 0. \end{aligned}$$

This proves the result of the lemma. $\square$

PROPOSITION 6.18. *Let $x \in \mathbb{T}$. The difference*

$$H_t^*(\psi_+(x) \otimes 1_F)H_t - (\psi_-(x) \otimes 1_F)$$

defines a compact operator on $(F \oplus F)\widehat{\otimes}\mathbb{T}$ for all $t \in [0, \pi/2]$ and the map

$$[0, \pi/2] \to \mathbb{L}((F \oplus F)\widehat{\otimes}\mathbb{T}) \qquad t \mapsto H_t^*(\psi_+(x) \otimes 1_F)H_t$$

is norm-continuous. In particular, we have the identity

$$\mathbf{1}_{\mathbb{T}} = \Big[H^*_{\pi/2}(\psi_+ \otimes 1_{\mathbb{T}})H_{\pi/2}, \psi_- \otimes 1_{\mathbb{T}}\Big]$$

inside $KK_0^{SU(2)}(\mathbb{T},\mathbb{T})$.

Proof. We start by proving the statement on compactness. For $t = 0$, we know from Lemma 6.3 that

$$H_0^*(\psi_+(x) \otimes 1_F)H_0 - (\psi_-(x) \otimes 1_F) = p_R(1_{F\oplus F} \otimes x),$$

which belongs to $M_2(\mathbb{K}(F\widehat{\otimes}\mathbb{T}))$ since $p_R = (Q_0 \otimes 1_{\mathbb{T}}) \oplus 0$. For $t \in (0, \pi/2]$, we see from Lemma 6.4, Propositions 6.6 and 6.13 that $[\psi_+(x) \otimes 1_{\mathbb{T}}, H_t] \in M_2(\mathbb{K}(F\widehat{\otimes}\mathbb{T}))$. An application of Lemma 6.15 and Proposition 5.1 then yields that

$$H_t^*(\psi_+(x) \otimes 1_{\mathbb{T}})H_t \sim \psi_+(x) \otimes 1_{\mathbb{T}} \sim \psi_-(x) \otimes 1_{\mathbb{T}}$$

hence proving the statement regarding compactness.

We now focus on proving norm-continuity. Using standard density arguments, we may restrict our attention to the case where x is one of the generators $x = T_j^*$ for some $j \in \{0, 1, \ldots, n\}$. Once more, we use the shorthand notation $T_j^* := \psi_+(T_j^*) \otimes 1_F$. We already know from Proposition 6.16 that the path $t \mapsto H_t^* T_j^* H_t(1 - P)$ is continuous in operator norm on $[0, \pi/2]$. Now, for $t \in [0, \pi/2]$, we have that

$$H_t^* T_j^* H_t P = -H_t^* T_j^* W P = -H_t^* W P T_j^* - H_t^*[T_j^*, WP] = T_j^* - H_t^*[T_j^*, WP].$$

Since the commutator $[T_j^*, WP]$ belongs to $M_2(\mathbb{K}\widehat{\otimes}\mathbb{T})$ by Propositions 6.6 and 6.13, it follows from Lemma 6.17 that $t \mapsto H_t^* T_j^* H_t P$ is norm-continuous as well. This proves the statement regarding continuity.

The remaining claim on classes in $SU(2)$-equivariant KK-theory now follows from the above considerations on remarking that all the involved quasi-homomorphisms are $SU(2)$-equivariant. Indeed, we then have the string of identities

$$\begin{aligned}\mathbf{1}_{\mathbb{T}} &= [\psi_- \otimes 1_{\mathbb{T}} + p_R(1_{F\oplus F} \otimes M_{\mathbb{T}}), \psi_- \otimes 1_{\mathbb{T}}] = [H_0^*(\psi_+ \otimes 1_F)H_0, \psi_- \otimes 1_{\mathbb{T}}] \\ &= [H^*_{\pi/2}(\psi_+ \otimes 1_{\mathbb{T}})H_{\pi/2}, \psi_- \otimes 1_{\mathbb{T}}]\end{aligned}$$

inside $KK_0^{SU(2)}(\mathbb{T},\mathbb{T})$. □

6.4. *Second step: everything else*

For each $t \in [0, 1]$, we define the $SU(2)$-equivariant bounded adjointable operator

$$y_t := 1 - P + \begin{pmatrix} (1-t)^{1/2} v^{TT} & v^{TB} \\ v^{BT} & (1-t)^{1/2} v^{BB} \end{pmatrix} P : (F \oplus F)\widehat{\otimes}\mathbb{T} \to (F \oplus F)\widehat{\otimes}\mathbb{T}.$$

Since the assignment $t \mapsto y_t$ is continuous in operator norm, we obtain a bounded adjointable operator

$$y : (F \oplus F)\widehat{\otimes}C([0, 1], \mathbb{T}) \to (F \oplus F)\widehat{\otimes}C([0, 1], \mathbb{T}),$$

which acts as y_t on the fibre $(F \oplus F)\widehat{\otimes}\mathbb{T}$ associated with the evaluation at the point $t \in [0, 1]$.

We shall see in this subsection that both y and y^* have dense images and that the corresponding unitary operator (obtained via polar decomposition)

$$I : (F \oplus F)\widehat{\otimes}C([0, 1], \mathbb{T}) \to (F \oplus F)\widehat{\otimes}C([0, 1], \mathbb{T})$$

yields the next step of our homotopy.

More precisely, it is the aim of this subsection to prove the following:

PROPOSITION 6.19. *For each $x \in \mathbb{T}$, the path $t \mapsto I_t^*(\psi_+(x) \otimes 1_{\mathbb{T}})I_t - (\psi_+(x) \otimes 1_{\mathbb{T}})$ is a norm-continuous path of compact operators on $(F \oplus F)\widehat{\otimes}\mathbb{T}$. In particular, we have the identity*

$$\mathbf{1}_{\mathbb{T}} = [I_1^*(\psi_+ \otimes 1_{\mathbb{T}})I_1, \psi_- \otimes 1_{\mathbb{T}}]$$

inside the $SU(2)$-equivariant KK-group, $KK_0^{SU(2)}(\mathbb{T}, \mathbb{T})$.

The proof of this proposition relies on the results in the Appendix. Aligning with the notation applied there, we define

$$\begin{aligned} X &:= (F \oplus F)\widehat{\otimes}C([0,1],\mathbb{T}) \qquad & \mathscr{X} &:= (F_{\mathrm{alg}} \oplus F_{\mathrm{alg}}) \otimes C([0,1],\mathbb{T}) \\ x_j &:= P\big(\psi_+(T_j^*) \otimes 1_{C([0,1],\mathbb{T})}\big)P \qquad & K &:= (D^{-1} \oplus D^{-1}) \otimes 1_{C([0,1],\mathbb{T})}, \end{aligned} \tag{6.15}$$

for all $j \in \{0, 1, 2, \ldots, n\}$. Remark here that $P : X \to X$ is the orthogonal projection which agrees with $P \in \mathbb{L}((F \oplus F)\widehat{\otimes}\mathbb{T})$ in each fibre (corresponding to the evaluations at the points $t \in [0,1]$). We note that $x_j : X \to X$ is a bounded adjointable operator for every $j \in \{0, 1, 2 \ldots, n\}$, whereas $K : X \to X$ is a compact operator.

LEMMA 6.20. *The bounded adjointable operators y and $y^* : X \to X$ both have norm-dense image. Moreover, for each $j \in \{0, 1, 2 \ldots, n\}$ we have $x_j(\mathscr{X}), x_j^*(\mathscr{X}), y(\mathscr{X}), y^*(\mathscr{X}) \subseteq X$.*

Proof. We first remark that $\Pi(Q_k \otimes Q_m) = (Q_k \otimes Q_m)\Pi$ for all $k, m \in \mathbb{N}_0$ and this implies that Π preserves the dense submodule $F_{\mathrm{alg}} \otimes \mathbb{T} \subseteq F\widehat{\otimes}\mathbb{T}$. The fact that x_j, x_j^*, y and y^* all preserve the dense submodule $\mathscr{X} = (F_{\mathrm{alg}} \oplus F_{\mathrm{alg}}) \otimes C([0,1],\mathbb{T})$ is then a consequence of Lemma 6.4 and the definition of the Toeplitz operators T_j and $T_j^* \in \mathbb{T}$.

We continue by focusing on the claim regarding the images of y and y^*. Since the path $t \mapsto y_t$ is norm-continuous, it suffices to verify that y_t and $y_t^* : (F \oplus F)\widehat{\otimes}\mathbb{T} \to (F \oplus F)\widehat{\otimes}\mathbb{T}$ both have norm-dense image for each $t \in [0,1]$. Applying Lemma 6.14, we obtain that

$$\begin{aligned} y_t^* y_t &= \begin{pmatrix} (1-\Pi) + (1 - t + t \cdot (v^{BT})^* v^{BT})\Pi & 0 \\ 0 & 1 - t + t \cdot (v^{TB})^* v^{TB} \end{pmatrix} \quad \text{and} \\ y_t y_t^* &= \begin{pmatrix} (1-\Pi) + (1 - t + t \cdot v^{TB}(v^{TB})^*)\Pi & 0 \\ 0 & 1 - t + t \cdot v^{BT}(v^{BT})^* \end{pmatrix} \end{aligned} \tag{6.16}$$

for all $t \in [0,1]$. For $t \in [0,1)$, we see from these identities that y_t and y_t^* are in fact invertible as bounded adjointable operators (and they are therefore in particular surjective).

For $t = 1$, we obtain from (6.2) that

$$\begin{aligned} y_1 &= \begin{pmatrix} 1-\Pi & v^{TB} \\ v^{BT} & 0 \end{pmatrix} = \begin{pmatrix} 1-\Pi & v^{TB} \\ (\Phi^{-1/2} \otimes \Phi^{1/2}) \cdot (v^{TB})^* & 0 \end{pmatrix} \quad \text{and} \\ y_1^* &= \begin{pmatrix} 1-\Pi & (v^{BT})^* \\ (v^{TB})^* & 0 \end{pmatrix} = \begin{pmatrix} 1-\Pi & v^{TB} \cdot (\Phi^{-1/2} \otimes \Phi^{1/2}) \\ (v^{TB})^* & 0 \end{pmatrix}. \end{aligned}$$

We recall that $\Phi : F \to F$ is an invertible element in $\mathbb{T} \subseteq \mathbb{L}(F)$. The fact that y_1 and y_1^* have dense images then follows from an application of Lemmas 6.9 and 6.10. □

In order to achieve a better understanding of the bounded adjointable operator $y^*y : X \to X$, we apply the decomposition from Theorem 3.19. This decomposition allows us for each $k, m \in \mathbb{N}_0$ to introduce the bounded operator

$$\Delta_{k,m} : E_k \otimes E_m \to E_k \otimes E_m$$

$$\Delta_{k,m}\big(\sigma^j \iota_{k-j,m-j}(\xi)\big) := \begin{cases} 0 & \text{for} \quad j = 0 \\ \frac{d_k d_{m-1}}{d_{k-1} d_m} \cdot (1 - \frac{d_{k-j} d_{m-j-1}}{d_k d_{m-1}}) \cdot \sigma^j \iota_{k-j,m-j}(\xi) & \text{for} \quad 0 < j \leqslant k, m \end{cases}$$

defined whenever $0 \leqslant j \leqslant k, m$ and $\xi \in E_{k+m-2j}$. We note that

$$\|\Delta_{k,m}\| \leqslant \frac{d_k d_{m-1}}{d_{k-1} d_m} \leqslant n + 1$$

for all $k, m \in \mathbb{N}$ and we therefore obtain a bounded operator

$$\Delta : F \widehat{\otimes} F \to F \widehat{\otimes} F \qquad \Delta(\xi) := \Delta_{k,m}(\xi) \ , \quad k, m \in \mathbb{N}_0.$$

Remark also that $\Delta_{k,m} = 0$ for $k = 0$ or $m = 0$.

Lemma 6.21. *We have the identity $(v^{BT})^* v^{BT} = \Delta$. In particular, $\Delta \in \mathbb{L}(F \widehat{\otimes} F)$ belongs to the unital C^*-subalgebra $\mathbb{L}(F \widehat{\otimes} \mathbb{T}) \subseteq \mathbb{L}(F \widehat{\otimes} F)$.*

Proof. The identity holds trivially on $E_k \otimes E_m$ for $k = 0$ or $m = 0$. This may also be seen as a consequence of Lemma 6.4. Thus, let $k, m \in \mathbb{N}$. From the identities in (6.2), Lemma 6.5 and the definition in (4.1), we obtain that

$$((v^{BT})^* v^{BT})(\eta) = v^{TB}(\Phi^{-1} \otimes \Phi)(v^{TB})^*(\eta) = \frac{d_k d_{m-1}}{\mu_k \cdot d_{k-1} d_m} \sigma_{k-1,m-1} \sigma^*_{k-1,m-1}(\eta) \tag{6.17}$$

for all $\eta \in E_k \otimes E_m$. Let $0 \leqslant j \leqslant k, m$ and let $\xi \in E_{k+m-2j}$ be given. Using Theorem 3.19, we only need to verify that

$$((v^{BT})^* v^{BT})(\sigma^j \iota_{k-j,m-j}(\xi)) = \Delta_{k,m}(\sigma^j \iota_{k-j,m-j}).$$

The case where $j = 0$ follows since $v^{BT}(1 - \Pi) = 0$ and the remaining cases follow from (6.17) by applying Lemmas 3.17 and 3.18. □

The next lemma is a straightforward consequence of Lemma 6.8, Lemma 6.21 and (6.16).

Lemma 6.22. *Let $t \in [0, 1]$. We have the identity*

$$y_t^* y_t = \begin{pmatrix} 1 - \Pi + ((1 - t) + t \cdot \Delta) \cdot \Pi & 0 \\ 0 & 1 - t + t \cdot \Gamma \end{pmatrix}.$$

Lemma 6.23. *The norm-dense submodule $\mathscr{X} = (F_{\text{alg}} \oplus F_{\text{alg}}) \otimes C([0, 1], \mathbb{T}) \subseteq X$ is contained in the image of $y^*y : X \to X$.*

Proof. For $k \in \mathbb{N}_0$, we sometimes apply the identification $Q_k := Q_k \otimes 1_{\mathbb{T}} : F \widehat{\otimes} \mathbb{T} \to F \widehat{\otimes} \mathbb{T}$. It follows from Lemma 6.22 and the definition of the involved operators that

$$y_t^* y_t (Q_k \oplus Q_k) = (Q_k \oplus Q_k) y_t^* y_t$$

for all $t \in [0, 1]$. In particular, on identifying $Q_k \in \mathbb{L}(F \widehat{\otimes} \mathbb{T})$ with the constant path with value Q_k for all $t \in [0, 1]$, we obtain that

$$\operatorname{Im}((Q_k \oplus Q_k) \cdot (y^* y (Q_k \oplus Q_k) + (1 - Q_k) \oplus (1 - Q_k))) \subseteq \operatorname{Im}(y^* y)$$

for all $k \in \mathbb{N}_0$. Since $\text{Im}(Q_k \oplus Q_k) = (E_k \oplus E_k) \otimes C([0,1], \mathbb{T})$, it therefore suffices to show that

$$y^* y(Q_k \oplus Q_k) : (Q_k \oplus Q_k)X \to (Q_k \oplus Q_k)X$$

is invertible. In other words, we have to show that the fibre

$$(y_t^* y_t)(Q_k \oplus Q_k) : Q_k(F \widehat{\otimes} \mathbb{T}) \oplus Q_k(F \widehat{\otimes} \mathbb{T}) \to Q_k(F \widehat{\otimes} \mathbb{T}) \oplus Q_k(F \widehat{\otimes} \mathbb{T})$$

is invertible for each $t \in [0,1]$ and that

$$\sup_{t \in [0,1]} \left\| \left((y_t^* y_t)(Q_k \oplus Q_k) \right)^{-1} \right\| < \infty.$$

As we did in Lemma 6.9, we may switch over and solve the corresponding problem on the Hilbert space $(Q_k F \widehat{\otimes} F) \oplus (Q_k F \widehat{\otimes} F)$. We apply Lemma 6.22 and deal with each component separately, namely

$$1 - \Pi + ((1-t) + t \cdot \Delta) \cdot \Pi \qquad \text{and} \qquad 1 - t + t \cdot \Gamma : F \widehat{\otimes} F \to F \widehat{\otimes} F.$$

Let $k \in \mathbb{N}_0$ be fixed. We saw in the proof of Lemma 6.9 that $(1 - t + t \cdot \Gamma)(Q_k \otimes 1_F) : E_k \otimes F \to E_k \otimes F$ for all $t \in [0,1]$ is invertible and that

$$\sup_{t \in [0,1]} \left\| \left((1 - t + t \cdot \Gamma)(Q_k \otimes 1_F) \right)^{-1} \right\| < \infty.$$

Remark that we are here also applying that $\Gamma : F \widehat{\otimes} F \to F \widehat{\otimes} F$ is a positive bounded operator.

We now consider the problematic part of the other component of $y_t^* y_t (Q_k \oplus Q_k)$:

$$(1 - t + t\Delta)\Pi(Q_k \otimes 1_F) : \Pi(E_k \otimes F) \to \Pi(E_k \otimes F).$$

Remark in this respect that $(Q_k \otimes Q_m)\Pi = \Pi(Q_k \otimes Q_m)$ for all $m \in \mathbb{N}_0$.

For each $t \in [0,1]$ and $m \in \mathbb{N}_0$, we are interested in the invertible operator

$$(1 - t + t\Delta)\Pi(Q_k \otimes Q_m) : \Pi(E_k \otimes E_m) \to \Pi(E_k \otimes E_m).$$

For $k = 0$ or $m = 0$, we have that $\Pi(E_k \otimes E_m) = \{0\}$ so suppose that $k, m \in \mathbb{N}$. In this case, we have that the bounded operator $\Delta_{k,m}\Pi : \Pi(E_k \otimes E_m) \to \Pi(E_k \otimes E_m)$ is invertible with

$$\begin{aligned} \|(\Delta_{k,m}\Pi)^{-1}\| &= \frac{d_{k-1} d_m}{d_k d_{m-1}} \left(1 - \frac{d_{k-1} d_{m-2}}{d_k d_{m-1}} \right)^{-1} \leqslant \frac{d_{k-1} d_1}{d_k} \left(1 - \frac{d_{k-1}}{d_k} \right)^{-1} \\ &= d_1 \cdot \left(\frac{d_k}{d_{k-1}} - 1 \right)^{-1}. \end{aligned} \tag{6.18}$$

Since this norm bound is independent of $m \in \mathbb{N}$, we conclude that

$$(t + (1-t)\Delta)\Pi(Q_k \otimes 1_F) : \Pi(E_k \otimes F) \to \Pi(E_k \otimes F)$$

is invertible for all $t \in [0,1]$ and that

$$\sup_{t \in [0,1]} \left\| \left((t + (1-t)\Delta)\Pi(Q_k \otimes 1_F) \right)^{-1} \right\| < \infty.$$

We are here also relying on the positivity of the bounded operator $\Delta : F \widehat{\otimes} F \to F \widehat{\otimes} F$. □

Recall the definition of the bounded adjointable operators x_j, y and $K : X \to X$ from (6.15).

Lemma 6.24. *Let $j \in \{0, 1, 2, \ldots, n\}$. There exist bounded adjointable operators $L, \overline{L}, M, \overline{M} : X \to X$ such that*

$$K^{1/2} L K^{1/2} = [x_j, y] = MK \qquad \text{and} \qquad K^{1/2} \overline{L} K^{1/2} = [x_j, y^*] = K\overline{M}.$$

Proof. To ease the notation, we put $T_j^* := T_j^* \otimes 1_{\mathbb{T}}$. For each $t \in [0,1]$, we apply Lemma 6.14 and compute that

$$[x_j, y_t] = \begin{pmatrix} (1-t)^{1/2}\Pi \cdot [T_j^*, v^{TT}] \cdot \Pi & \Pi[T_j^*, v^{TB}] \\ [T_j^*, v^{BT}] \cdot \Pi & (1-t)^{1/2}[T_j^*, v^{BB}] \end{pmatrix} \quad \text{and}$$

$$[x_j, y_t^*] = \begin{pmatrix} (1-t)^{1/2}\Pi \cdot [T_j^*, (v^{TT})^*] \cdot \Pi & \Pi \cdot [T_j^*, (v^{BT})^*] \\ [T_j^*, (v^{TB})^*] \cdot \Pi & (1-t)^{1/2}[T_j^*, (v^{BB})^*] \end{pmatrix}.$$

We consider the inverses $K^{-1/2}$ and K^{-1}. These positive and regular unbounded operators both have $\mathscr{X}$ as a core and on this core they are given by

$$D^{1/2} \otimes 1_{\mathbb{C}^2 \otimes C([0,1],\mathbb{T})} \text{ and } D \otimes 1_{\mathbb{C}^2 \otimes C([0,1],\mathbb{T})} : \mathscr{X} \to X,$$

respectively. The result of the lemma now follows from Proposition 6.7. Indeed, L and $\overline{L}$ are the bounded adjointable extensions of $D^{1/2}[x_j, y]D^{1/2}$ and $D^{1/2}[x_j, y^*]D^{1/2}$, respectively. Whereas M and $\overline{M}$ are the bounded adjointable extensions of $[x_j, y]D$ and $D[x_j, y^*]$, respectively. We remark that all of these four unbounded operators are understood to be defined on the algebraic tensor product $\mathscr{X} = (F_{\text{alg}} \oplus F_{\text{alg}}) \otimes C([0,1],\mathbb{T})$. Indeed, this algebraic tensor product works well in this respect since it is a core for both D and $D^{1/2}$ and since it is invariant under x_j, y and y^*. □

For each $\lambda > 0$, we put $R_\lambda := (\lambda + y^*y)^{-1}$. We remark that it follows from Lemma 6.22 that

$$R_\lambda \begin{pmatrix} 1-\Pi & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (\lambda+1)^{-1}(1-\Pi) & 0 \\ 0 & 0 \end{pmatrix}.$$

Lemma 6.25. *The identity $Ky^*y = y^*yK$ holds. Moreover, there exists a constant such that*

$$\|KR_\lambda\| \leqslant C \qquad \text{and} \qquad \|K^{1/2}yR_\lambda\| \leqslant C$$

for all $\lambda > 0$.

Proof. The fact that $Ky^*y = y^*yK$ follows since y^*y leaves the submodule $(E_k \oplus E_l) \otimes C([0,1],\mathbb{T})$ invariant for all $k, l \in \mathbb{N}_0$. Moreover, writing $y : X \to X$ as a 2×2-matrix in the following fashion

$$y = \begin{pmatrix} y^{TT} & y^{TB} \\ y^{BT} & y^{BB} \end{pmatrix} \in M_2\big(\mathbb{L}(F \widehat{\otimes} C([0,1],\mathbb{T}))\big),$$

we see from the argument given in the proof of Proposition 6.7 that $K^{1/2}yK^{-1/2} : \mathscr{X} \to X$ extends to the bounded adjointable operator

$$\begin{pmatrix} 1 - \Pi + (\Phi^{-1/2} \otimes 1)y^{TT}\Pi & y^{TB}(\Phi^{1/2} \otimes 1) \\ (\Phi^{-1/2} \otimes 1)y^{BT} & y^{BB}(\Phi^{1/2} \otimes 1) \end{pmatrix}$$

on X. Moreover, since each component in

$$R_\lambda = \begin{pmatrix} R_\lambda^{TT} & 0 \\ 0 & R_\lambda^{BB} \end{pmatrix} = \begin{pmatrix} (1-\Pi)R_\lambda^{TT}(1-\Pi) + \Pi R_\lambda^{TT}\Pi & 0 \\ 0 & R_\lambda^{BB} \end{pmatrix}$$
$$= \begin{pmatrix} (\lambda+1)^{-1}(1-\Pi) + \Pi R_\lambda^{TT}\Pi & 0 \\ 0 & R_\lambda^{BB} \end{pmatrix}$$

commutes with $\Phi \otimes 1$, we obtain that

$$K^{1/2}yR_\lambda = \begin{pmatrix} 1 - \Pi + (\Phi^{-1/2} \otimes 1)y^{TT}\Pi & y^{TB}(\Phi^{1/2} \otimes 1) \\ (\Phi^{-1/2} \otimes 1)y^{BT} & y^{BB}(\Phi^{1/2} \otimes 1) \end{pmatrix} R_\lambda K^{1/2}$$
$$= \begin{pmatrix} (1-\Pi)(\lambda+1)^{-1} & 0 \\ 0 & 0 \end{pmatrix} K^{1/2}$$
$$+ \begin{pmatrix} (\Phi^{-1/2} \otimes 1)y^{TT}(R_\lambda^{TT})^{1/2}\Pi & y^{TB}(R_\lambda^{BB})^{1/2}(\Phi^{1/2} \otimes 1) \\ (\Phi^{-1/2} \otimes 1)y^{BT}(R_\lambda^{TT})^{1/2} & y^{BB}(R_\lambda^{BB})^{1/2}(\Phi^{1/2} \otimes 1) \end{pmatrix} R_\lambda^{1/2} K^{1/2}.$$

It therefore suffices to find a constant $C > 0$ such that $\|KR_\lambda\| \leqslant C$ for all $\lambda > 0$. Using the description of $y^*y : X \to X$ from Lemma 6.22, together with the definitions of Γ and $\Delta : F\widehat{\otimes}F \to F\widehat{\otimes}F$, we may focus on showing that

$$\sup_{k,m\in\mathbb{N}_0} \|d_k^{-1}\Gamma_{k,m}^{-1}\| < \infty \qquad \text{and} \qquad \sup_{k,m\in\mathbb{N}} \|d_k^{-1}(\Delta_{k,m}\Pi)^{-1}\| < \infty,$$

where we consider $\Delta_{k,m}\Pi$ as a bounded invertible operator on the Hilbert space $\Pi(E_k \otimes E_m)$ for $k, m \in \mathbb{N}$. The first estimate was already established in the proof of Lemma 6.12 and the second estimate follows from Lemma 3.3 and the estimate in (6.18). Indeed, we have that

$$\|d_k^{-1}(\Delta_{k,m}\Pi)^{-1}\| \leqslant \frac{d_1}{d_k}\left(\frac{d_k}{d_{k-1}} - 1\right)^{-1} \leqslant d_1 \cdot \frac{d_{k-1}}{d_k} \leqslant (n+1)\cdot\gamma_n$$

for all $k, m \in \mathbb{N}$. □

For each $t \in [0,1]$, define $I_t : (F \oplus F)\widehat{\otimes}\mathbb{T} \to (F \oplus F)\widehat{\otimes}\mathbb{T}$ as the bounded adjointable extension of

$$y_t|y_t|^{-1} : \mathrm{Im}(|y_t|) \to (F \oplus F)\widehat{\otimes}\mathbb{T}.$$

Note that such extension is indeed a well-defined unitary operator on $(F \oplus F)\widehat{\otimes}\mathbb{T}$, since both y_t and $y_t^* : (F \oplus F)\widehat{\otimes}\mathbb{T}$ have dense images (cf. Lemma 6.20 and [**28**, Proposition 3.8]).

We emphasise that

$$I_0 = y_0 = H_{\pi/2} \text{ and } I_1 = \begin{pmatrix} 1-\Pi & \Theta \\ \Theta^* & 0 \end{pmatrix} : (F \oplus F)\widehat{\otimes}\mathbb{T} \to (F \oplus F)\widehat{\otimes}\mathbb{T}, \tag{6.19}$$

where the bounded adjointable isometry $\Theta : F\widehat{\otimes}\mathbb{T} \to F\widehat{\otimes}\mathbb{T}$ was introduced in Lemma 6.10.

We are now ready to prove the main result of this subsection:

PROPOSITION 6.26. *The map $t \mapsto I_t$ is a strictly continuous path of $SU(2)$-equivariant unitary operators on $(F \oplus F)\widehat{\otimes}\mathbb{T}$. Moreover, for every $x \in \mathbb{T}$, the map $t \mapsto I_t^*(\psi_+(x) \otimes 1_\mathbb{T})I_t - \psi_+(x) \otimes 1_\mathbb{T}$ is a norm-continuous path of compact operators on $(F \oplus F)\widehat{\otimes}\mathbb{T}$. In particular, we have the identity*

$$\mathbf{1}_\mathbb{T} = [I_1^*(\psi_+ \otimes 1_\mathbb{T})I_1, \psi_- \otimes 1_\mathbb{T}]$$

inside $KK_0^{SU(2)}(\mathbb{T},\mathbb{T})$.

Proof. By Lemma 6.20, the operator $y|y|^{-1} : \mathrm{Im}(|y|) \to X$ extends to a unitary operator I on $X = (F \oplus F)\widehat{\otimes}C([0,1],\mathbb{T})$. The fibres of this unitary operator are exactly the unitary operators $I_t : (F \oplus F)\widehat{\otimes}\mathbb{T} \to (F \oplus F)\widehat{\otimes}\mathbb{T}$, $t \in [0,1]$. This means that the path $t \mapsto I_t$ is a strictly continuous path of unitary operators on $(F \oplus F)\widehat{\otimes}\mathbb{T}$. Moreover, since $y_t \in \mathbb{L}((F \oplus F)\widehat{\otimes}\mathbb{T})$ is $SU(2)$-equivariant, we obtain that $I_t \in \mathbb{L}((F \oplus F)\widehat{\otimes}\mathbb{T})$ is $SU(2)$-equivariant as well.

Next, a combination of Proposition A.1, Lemmas 6.20, 6.23, 6.24, and 6.25 shows that the commutators $[x_j, I]$ and $[x_j^*, I]$ belong to the compact operators on $(F \oplus F)\widehat{\otimes}C([0,1],\mathbb{T})$ for every $j \in \{0,1,2,\dots,n\}$. Now, put $T_j^* := \psi_+(T_j^*) \otimes 1_{C([0,1],\mathbb{T})}$ and remark that

$$T_j^* = x_j + (1-P)T_j^*(1-P) + (1-P)T_j^*P.$$

We know from Proposition 6.13 and from the definition of P in (6.8) that

$$(1-P)T_j^*P = (1-P)[T_j^*, P] = \begin{pmatrix} 1-\Pi & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} [T_j^* \otimes 1_{C([0,1],\mathbb{T})}, \Pi] & 0 \\ 0 & 0 \end{pmatrix}$$

is a compact operator on $(F \oplus F)\widehat{\otimes}C([0,1],\mathbb{T})$. Indeed, applying the point evaluations at $t \in [0,1]$ yields a constant path of compact operators on $F\widehat{\otimes}\mathbb{T}$:

$$t \mapsto (1-\Pi)[T_j^* \otimes 1_{\mathbb{T}}, \Pi].$$

We moreover have that

$$[I, T_j^*] = [I, x_j] + [I, (1-P)T_j^*P]$$

and similarly with I^* instead of I. This shows that $[I, T_j^*]$ and $[I^*, T_j^*]$ are compact operators on $(F \oplus F)\widehat{\otimes}C([0,1],\mathbb{T})$ for all $j \in \{0,1,2,\dots,n\}$ and hence that

$$I^*(\psi_+(x) \otimes 1_{C([0,1],\mathbb{T})})I - \psi_+(x) \otimes 1_{C([0,1],\mathbb{T})}$$

is a compact operator on $(F \oplus F)\widehat{\otimes}C([0,1],\mathbb{T})$ for all $x \in \mathbb{T}$. But this means that the path

$$t \mapsto I_t^*(\psi_+(x) \otimes 1_{\mathbb{T}})I_t - \psi_+(x) \otimes 1_{\mathbb{T}}$$

is a norm-continuous path of compact operators on $(F \oplus F)\widehat{\otimes}\mathbb{T}$. Since $\psi_+(x) \otimes 1_{\mathbb{T}} - \psi_-(x) \otimes 1_{\mathbb{T}}$ is a compact operator as well (for every $x \in \mathbb{T}$), we obtain the identity

$$[I_0^*(\psi_+ \otimes 1_{\mathbb{T}})I_0, \psi_- \otimes 1_{\mathbb{T}}] = [I_1^*(\psi_+ \otimes 1_{\mathbb{T}})I_1, \psi_- \otimes 1_{\mathbb{T}}]$$

inside the $SU(2)$-equivariant KK-group $KK_0^{SU(2)}(\mathbb{T},\mathbb{T})$. Since $I_0 = H_{\pi/2}$, we obtain the result of the present proposition by an application of Proposition 6.18. □

REMARK 7. For $n > 1$, it can be established that both y^*y and yy^* are invertible as bounded adjointable operators on X and a more straightforward proof of Proposition 6.26 can therefore be given. For $n = 1$, it only holds that y^*y and yy^* have dense images in X and this is one of the reasons for carrying out some of the more detailed analysis presented here. Our present approach treats both cases on an equal footing and might be applicable in a wider range of examples.

6.5. *Third step: proof of KK-equivalence*

We are now ready to finish the proof of Theorem 6.1 establishing that $\mathbb{T}$ and $\mathbb{C}$ are $KK^{SU(2)}$-equivalent.

Proof of Theorem 6.1. From Proposition 6.26, we have the identity

$$\mathbf{1}_{\mathbb{T}} = [I_1^*(\psi_+ \otimes 1_{\mathbb{T}})I_1, \psi_- \otimes 1_{\mathbb{T}}]$$

inside the $SU(2)$-equivariant KK-group $KK_0^{SU(2)}(\mathbb{T},\mathbb{T})$. Thus in order to prove the identity

$$\mathbf{1}_{\mathbb{T}} = [\psi_+, \psi_-]\widehat{\otimes}_{\mathbb{C}}[i],$$

we only need to show that

$$[I_1^*(\psi_+ \otimes 1_{\mathbb{T}})I_1, \psi_- \otimes 1_{\mathbb{T}}] = [\psi_+ \otimes 1_{\mathbb{T}}, \psi_- \otimes 1_{\mathbb{T}}]. \tag{6.20}$$

We recall from (6.19) that

$$I_1 = \begin{pmatrix} 1-\Pi & \Theta \\ \Theta^* & 0 \end{pmatrix}$$

and hence that $I_1 \in \mathbb{L}((F \oplus F)\widehat{\otimes}\mathbb{T})$ is an $SU(2)$-equivariant selfadjoint unitary operator.

For each $t \in [0,1]$, define

$$J_t := \frac{1+I_1}{2} + \exp(\pi i t) \cdot \frac{I_1 - 1}{2}$$

so that $J_t \in \mathbb{L}((F \oplus F)\widehat{\otimes}\mathbb{T})$ is an $SU(2)$-equivariant unitary operator and $t \mapsto J_t$ is a norm continuous path with $J_0 = I_1$ and $J_1 = 1$. Moreover, for every $x \in \mathbb{T}$, the assignment

$$[0,1] \ni t \mapsto J_t^*(\psi_+(x) \otimes 1_{\mathbb{T}})J_t - \psi_+(x) \otimes 1_{\mathbb{T}}$$

yields a norm continuous path of compact operators on the module $(F \oplus F)\widehat{\otimes}\mathbb{T}$. Indeed, the last claim on compactness follows immediately from Proposition 6.26.

The existence of the path $t \mapsto J_t$ with the above properties establishes the identity in (6.20) and we have proved our main theorem. $\square$

7. *The Gysin sequence*

Throughout this section, we fix a strictly positive integer n and consider the irreducible representation $\rho_n : SU(2) \to U(L_n)$. We apply the notation

$$\mathbb{K}(F) := \mathbb{K}(F(\rho_n, L_n))\,, \ \mathbb{T} := \mathbb{T}(\rho_n, L_n) \text{ and } \mathbb{O} := \mathbb{T}(\rho_n, L_n)/\mathbb{K}(F(\rho_n, L_n))$$

for the associated compact operators, Toeplitz algebra and Cuntz–Pimsner algebra. By construction, we have the exact sequence

$$0 \to \mathbb{K}(F(\rho_n, L_n)) \xrightarrow{j} \mathbb{T}(\rho_n, L_n) \xrightarrow{q} \mathbb{O}(\rho_n, L_n) \to 0$$

of C^*-algebras. This exact sequence in turn results in the following six-term exact sequence of K-groups:

$$\begin{array}{ccccc} K_0(\mathbb{K}(F)) & \xrightarrow{j_*} & K_0(\mathbb{T}) & \xrightarrow{q_*} & K_0(\mathbb{O}) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(\mathbb{O}) & \xleftarrow{q_*} & K_1(\mathbb{T}) & \xleftarrow{j_*} & K_1(\mathbb{K}(F)). \end{array}$$

We recall that the compact operators $\mathbb{K}(F)$ are strongly Morita equivalent to the complex numbers via the C^*-correspondence $F = F(\rho_n, L_n)$ from $\mathbb{K}(F(\rho_n, L_n))$ to $\mathbb{C}$. In particular, this C^*-correspondence together with its dual $F(\rho_n, L_n)^*$ implements a KK-equivalence between $\mathbb{K}(F)$ and $\mathbb{C}$. We denote the corresponding classes in KK-theory by

$$[F] \in KK_0(\mathbb{K}(F), \mathbb{C}) \qquad \text{and} \qquad [F^*] \in KK_0(\mathbb{C}, \mathbb{K}(F)).$$

Combining these observations with the KK-equivalence from Theorem 6.1, we obtain the exact sequence

$$\begin{array}{ccccccc} K_1(\mathbb{O}) & \xrightarrow{([F]\widehat{\otimes}_{\mathbb{K}(F)}\cdot)\circ\partial} & K_0(\mathbb{C}) & \xrightarrow{[F^*]\widehat{\otimes}_{\mathbb{K}(F)}[j]\widehat{\otimes}_{\mathbb{T}}[\psi_+,\psi_-]} & K_0(\mathbb{C}) & \xrightarrow{(q\circ i)_*} & K_0(\mathbb{O}). \\ & \nwarrow & & \{0\} & & \swarrow & \end{array}$$

We recall that $i : \mathbb{C} \to \mathbb{T}$ denotes the unital inclusion of $\mathbb{C}$ into the Toeplitz algebra and remark that $q \circ i : \mathbb{C} \to \mathbb{O}$ agrees with the unital inclusion of the complex numbers into $\mathbb{O}$. We will abuse notation and denote the latter inclusion with the same symbol i.

In the next proposition, we compute the composition $[F^*]\widehat{\otimes}_{\mathbb{K}(F)}[j]\widehat{\otimes}_{\mathbb{T}}[\psi_+, \psi_-]$, which we identify with the Euler class of the irreducible representation $\rho_n : SU(2) \to U(L_n)$, that is, the alternating sum of KK-classes $\mathbf{1}_{\mathbb{C}} - [L_n] + [\det(\rho_n, L_n)] \in KK_0(\mathbb{C}, \mathbb{C})$.

Proposition 7.1. *We have the identity*

$$[j]\widehat{\otimes}_{\mathbb{T}}[\psi_+, \psi_-] = [F]\widehat{\otimes}_{\mathbb{C}}(\mathbf{1}_{\mathbb{C}} - [L_n] + [det(\rho_n, L_n)])$$

in $KK_0(\mathbb{K}(F), \mathbb{C})$.

Proof. By Proposition 2.6, we have that $\det(\rho_n, L_n)$ is a one-dimensional complex vector space and hence that $[\det(\rho_n, L_n)] = \mathbf{1}_{\mathbb{C}}$ inside $KK_0(\mathbb{C}, \mathbb{C})$. Hence we have to show that

$$[j]\widehat{\otimes}_{\mathbb{T}}[\psi_+, \psi_-] = 2 \cdot [F] - [F]\widehat{\otimes}_{\mathbb{C}}[E_1]. \tag{7.1}$$

Since $j : \mathbb{K}(F) \to \mathbb{T}$ is the inclusion, we have that both $\psi_+ \circ j$ and $\psi_- \circ j : \mathbb{K}(F) \to \mathbb{L}(F \oplus F)$ factorises through the compact operators on $F \oplus F$ and the left-hand side of (7.1) is therefore given by

$$[j]\widehat{\otimes}_{\mathbb{T}}[\psi_+, \psi_-] = [\psi_+ \circ j, 0] - [\psi_- \circ j, 0].$$

Now, letting $\phi : \mathbb{K}(F) \to \mathbb{L}(F)$ denote the inclusion of the compact operators into the bounded operators, we have that $\psi_+ \circ j = \phi \oplus \phi : \mathbb{K}(F) \to \mathbb{L}(F \oplus F)$ and hence that $[\psi_+ \circ j, 0] = 2 \cdot [F]$ inside $KK_0(\mathbb{K}(F), \mathbb{C})$.

Next, recall that $\psi_-(x) = W_R(x \otimes 1_{E_1})W_R^* : F \oplus F \to F \oplus F$ for all $x \in \mathbb{T}$, where $W_R : F \otimes E_1 \to F \oplus F$ is the isometry defined in (5.1). In particular, we have that W_R implements a unitary isomorphism between $F \otimes E_1$ and $W_R W_R^*(F \oplus F)$.

We define the $*$-homomorphism $\phi_- : \mathbb{K}(F) \to \mathbb{L}(W_R W_R^*(F \oplus F))$ by

$$\phi_-(x)(\xi) = (\psi_- \circ j)(\xi)$$

for all $\xi \in W_R W_R^*(F \oplus F)$. We then have that $(\phi_-, 0)$ is unitarily equivalent to the quasi-homomorphism $(\phi \otimes 1_{E_1}, 0)$. Moreover, we see that the quasi-homomorphisms $(\psi_- \circ j, 0)$ and $(\phi_-, 0)$ agree up to addition of a degenerate quasi-homomorphism. We therefore obtain the identities

$$[\psi_- \circ j, 0] = [\phi_-, 0] = [\phi \otimes 1_{E_1}, 0] = [F]\widehat{\otimes}_{\mathbb{C}}[E_1]$$

inside the KK-group $KK_0(\mathbb{K}(F), \mathbb{C})$. □

Combining the above results, we obtain the KK-theoretic Gysin sequence associated with the irreducible representation $\rho_n : SU(2) \to U(L_n)$:

Theorem 7.2. *The following sequence of K-groups is exact:*

$$K_1(\mathbb{O}) \xrightarrow{([F]\widehat{\otimes}_{\mathbb{K}(F)} \cdot) \circ \partial} K_0(\mathbb{C}) \xrightarrow{\mathbf{1}_{\mathbb{C}} - [L_n] + [\det(\rho_n, L_n)]} K_0(\mathbb{C}) \xrightarrow{i_*} K_0(\mathbb{O}) \longrightarrow \{0\} \longrightarrow K_1(\mathbb{O})$$

Corollary 7.3. *For every $n \in \mathbb{N}$, we have*

$$K_0(\mathbb{O}(\rho_n, L_n)) \cong \mathbb{Z}/(n-1)\mathbb{Z} \qquad K_1(\mathbb{O}(\rho_n, L_n)) \cong \begin{cases} \mathbb{Z} & n = 1, \\ \{0\} & \textit{otherwise.} \end{cases} \tag{7.2}$$

7.1. *Concluding remarks and open problems*

The present paper raises a number of questions and open problems and we would like to conclude by listing a few of them.

(1) It is relevant to consider the case where the representation $\tau : SU(2) \to U(H)$ is no longer irreducible, but where H remains finite dimensional. We expect, however, that a lot of the considerations appearing in the present paper could be carried over to this more general context without too much trouble. In this direction, we have so far only computed the determinant of the representation, see Proposition 2.7.

(2) In the present work, we have only been studying $SU(2)$-subproduct systems in a Hilbert space context, meaning that we have in some sense been looking at $SU(2)$-bundles with a one-point parameter space. In order to find a noncommutative analogue of the classical K-theoretic Gysin sequence for the sphere bundle of a complex Hermitian vector bundle of rank 2, [**23**, Subsection IV.1.13], it is necessary to extend our work to $SU(2)$-subproduct systems with a non-trivial parameter space. This means that an interesting starting point could be a general $SU(2)$-C^*-correspondence where the left action factorises through the compact operators. In this context, it could be relevant to compare the corresponding extension class with the class appearing in [**11**].

(3) We have here been focusing on representations and subproduct systems relating to $SU(2)$ since this object has the nice property of being both a Lie group and an odd-dimensional sphere at the same time. Classical results from algebraic topology (for example, the Leray–Serre spectral sequence) suggest that we cannot expect the existence of a six-term exact sequence in K-theory, like the Gysin exact sequence, when looking at noncommutative fibre bundles where the fibre is not some analogue of a sphere.

(4) In analogy with the case of Cuntz–Pimsner algebras arising from a C^*-correspondence, it is an important problem to settle the universal properties both for the Toeplitz algebras and the Cuntz–Pimsner algebras coming from our $SU(2)$-equivariant data.

(5) Finally, it would be worthwhile to look for an $SU(2)$-gauge invariant uniqueness theorem as obtained in the $U(1)$-setting by Katsura in [**26**, Theorem 6.4]. □

Appendix. *Commutators and polar decompositions*

Throughout this appendix, we let X be a countably generated Hilbert C^*-module over a C^*-algebra B.

PROPOSITION A.1. *Suppose that $x, y : X \to X$ are bounded adjointable operator and that there exists a norm-dense submodule $\mathscr{X} \subseteq X$ such that*

$$\mathscr{X} \subseteq \mathrm{Im}(y^* y) \text{ and } x(\mathscr{X}), x^*(\mathscr{X}), y^*(\mathscr{X}) \subseteq \mathscr{X}.$$

Suppose moreover that $K : X \to X$ is a positive compact operator and that $L, \overline{L}, M, \overline{M} : X \to X$ are bounded adjointable operators such that:

(i) $K^{1/2} L K^{1/2} = [x, y]$ *and* $K^{1/2} \overline{L} K^{1/2} = [x, y^*]$*;*
(ii) $MK = [x, y]$ *and* $K\overline{M} = [x, y^*]$*.*

Suppose finally that there exists a constant $C > 0$ such that

$$\|K^{1/2}(\lambda + y^* y)^{-1/2}\|\,,\ \|K(\lambda + y^* y)^{-1}\|\,,\ \|K^{1/2} y(\lambda + y^* y)^{-1}\| \leqslant C$$

for all $\lambda > 0$. Then the unbounded operator $y|y|^{-1} : \mathrm{Im}(|y|) \to X$ extends to a bounded adjointable isometry $\theta : X \to X$ satisfying that $[x, \theta]$ and $[x^*, \theta]$ both lie in $\mathbb{K}(X)$.

Proof. We start by recording that since $|y| : X \to X$ is positive and has dense image, we know that $|y|^{-1} : \mathrm{Im}(|y|) \to X$ is a well-defined unbounded positive and regular operator. The unbounded operator $y|y|^{-1} : \mathrm{Im}(|y|) \to X$ then extends to an isometry $\theta : X \to X$ and this isometry is adjointable since $|y|^{-1}y^*$ is densely defined as well (the domain of $|y|^{-1}y^*$ contains $\mathscr{X}$ and the adjoint $\theta^* : X \to X$ is the unique bounded extension of $|y|^{-1}y^*$).

It follows from the identities in (1) and compactness of $K : X \to X$ that both $[x, y]$ and $[x, y^*]$ lie in $\mathbb{K}(X)$.

For each $\lambda > 0$, we put $R_\lambda := (\lambda + y^*y)^{-1}$. For every $\xi \in \mathrm{Im}(y^*y)$, we have that $|y|^{-1}\xi = \frac{1}{\pi}\int_0^\infty \lambda^{-1/2} R_\lambda \xi d\lambda$ where the integral converges absolutely (using the norm on X). We compute that

$$\begin{aligned} [x, R_\lambda] &= -R_\lambda[x, y^*y]R_\lambda = -R_\lambda[x, y^*]yR_\lambda - R_\lambda y^*[x, y]R_\lambda \\ &= -R_\lambda K^{1/2}\overline{L}K^{1/2}yR_\lambda - R_\lambda y^*MKR_\lambda. \end{aligned}$$

This in particular implies that $[x, R_\lambda] \in \mathbb{K}(X)$. Note now that $\|y^*yR_\lambda\| \leqslant 1$ for all $\lambda > 0$. Combining this estimate with our assumptions, we obtain that

$$\begin{aligned} \big\|y[x, R_\lambda]\big\| &\leqslant \|yR_\lambda K^{1/2}\| \cdot \|\overline{L}\| \cdot \|K^{1/2}yR_\lambda\| + \|yR_\lambda y^*\| \cdot \|M\| \cdot \|KR_\lambda\| \\ &\leqslant C^2 \cdot \|\overline{L}\| + C \cdot \|M\| \end{aligned} \tag{A.1}$$

for all $\lambda > 0$.

Remark now that the integral $\int_1^\infty \lambda^{-1/2} y[x, R_\lambda] d\lambda$ converges absolutely in operator norm since $\|R_\lambda\| \leqslant \lambda^{-1}$ for all $\lambda > 0$. Moreover, we obtain from the estimate in (A.1) that the integral $\int_0^1 \lambda^{-1/2} y[x, R_\lambda] d\lambda$ converges absolutely in operator norm as well. The whole integral

$$\int_0^\infty \lambda^{-1/2} y[x, R_\lambda] d\lambda$$

therefore converges absolutely in operator norm and since the integrand is a continuous map $(0, \infty) \to \mathbb{K}(X)$, we conclude that

$$\frac{1}{\pi}\int_0^\infty \lambda^{-1/2} y[x, R_\lambda] d\lambda \in \mathbb{K}(X).$$

We may likewise show that the integral

$$\frac{1}{\pi}\int_0^\infty \lambda^{-1/2} KR_\lambda d\lambda$$

converges absolutely to a compact operator.

The claim that $[x, \theta]$ is a compact operator is now verified by noting that

$$\begin{aligned} [x, \theta]\xi &= [x, y]|y|^{-1}\xi + y[x, |y|^{-1}]\xi \\ &= M\frac{1}{\pi}\int_0^\infty \lambda^{-1/2} KR_\lambda(\xi) d\lambda + \frac{1}{\pi}\int_0^\infty \lambda^{-1/2} y[x, R_\lambda](\xi) d\lambda, \end{aligned}$$

for all $\xi \in \mathscr{X}$.

Since our assumptions are symmetric in x and x^*, it follows immediately that $[x^*, \theta]$ is a compact operator as well. □

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