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MATHEMATISCHE NACHRICHTEN

Diffusive stability for periodic metric graphs

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Abstract
We consider a nonlinear diffusion equation on an infinite periodic metric graph. We prove that the terms which are irrelevant w.r.t. linear diffusion on the real line are irrelevant w.r.t. linear diffusion on the periodic metric graph, too. The proof is based on L^1-L^∞ estimates combined with Bloch wave analysis for periodic metric graphs.

KEYWORDS
asymptotic behavior, Bloch waves, irrelevant nonlinearities

MSC (2010)
35B35, 35B40, 35K55, 35R02

1 | INTRODUCTION

It is well known that on the real line the nonlinear terms u^p are irrelevant w.r.t. linear diffusive behavior if $p > 3$. In detail we have: Consider

$$\partial_t u = \partial_x^2 u + u^p, \qquad u|_{t=0} = u_0, \tag{1.1}$$

with $t \geq 0$, $x \in \mathbb{R}$, $p \in \mathbb{N}$, and $u(x,t) \in \mathbb{R}$. Then for $p > 3$ and $C > 0$ there exists a $\delta > 0$ such that

$$\|u_0\|_{L^1} + \|u_0\|_{L^\infty} \leq \delta$$

implies

$$\|u(\cdot,t)\|_{L^1} \leq C \qquad \text{and} \qquad \|u(\cdot,t)\|_{L^\infty} \leq C(1+t)^{-1/2} \tag{1.2}$$

for all $t \geq 0$. It is the goal of this paper to prove that a similar result holds true, if (1.1) is posed on an infinite periodic metric graph.

A metric graph is a network of edges connected at vertices. The mathematical analysis of nonlinear PDEs on such graphs attracted recently a lot of interest, cf. [1,5,11,13]. The use of graphs is motivated for instance by nano-technological objects with a similar geometric structure, such as nano-tubes or graphene. See [2] for further motivations. In order to explain our approach without too many technical details we restrict the subsequent presentation to the necklace graph shown in Figure 2 which has already been used in [7] for other purposes. We will discuss at the end of Section 2 how to handle more general one-dimensional periodic graphs.

Stability and blow-up results for (1.1) on the real line have been discussed by a number of authors, cf. [6,20]. The idea has been transferred to more complicated problems such as the stability of spatially periodic equilibria in the Ginzburg–Landau equation, cf. [3,4], in pattern forming systems, cf. [16,17], or in pattern forming systems with a conservation law, cf. [9]. There are various approaches to establish such results. These are the discrete and continuous renormalization approach, the use of

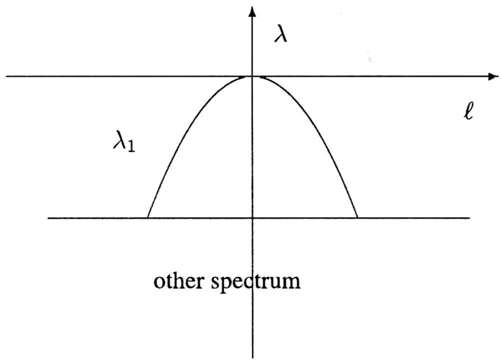

FIGURE 1 The eigenvalues $\lambda(\ell)$ plotted versus the Bloch wave numbers ℓ. The diffusive behavior comes from the parabola-like curve λ_1 through $(\ell, \lambda) = (0, 0)$. The rest of the spectrum leads to exponential decay. Due to the periodicity w.r.t. the Bloch wave number ℓ we can restrict ourselves to $\ell \in [-1/2, 1/2]$

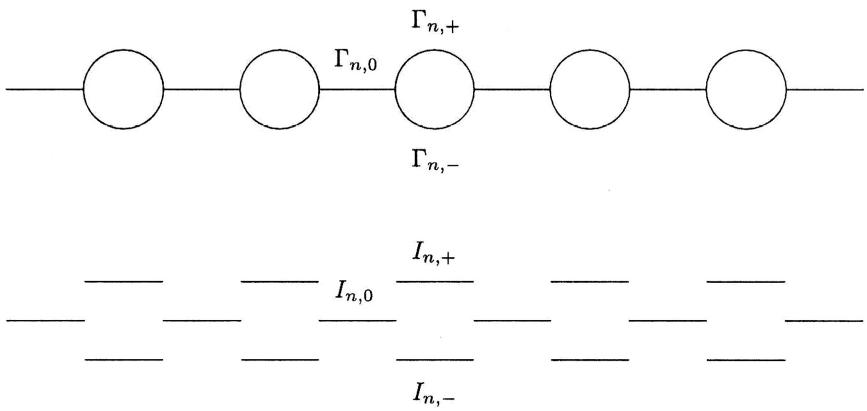

FIGURE 2 The periodic metric graph Γ shown in the upper panel is of the form $\Gamma = \bigoplus_{n\in\mathbb{Z}} \Gamma_n$, with $\Gamma_n = \Gamma_{n,0} \oplus \Gamma_{n,+} \oplus \Gamma_{n,-}$, where the $\Gamma_{n,0}$ are the horizontal links between the circles and the $\Gamma_{n,\pm}$ the upper and lower semicircles, all of the same length π, for $n \in \mathbb{Z}$. The part $\Gamma_{n,0}$ is identified isometrically with the interval $I_{n,0} = [2\pi n, 2\pi n + \pi]$ and the $\Gamma_{n,\pm}$ with the intervals $I_{n,\pm} = [2\pi n, 2\pi(n+1)]$. See the lower panel. For a function $u : \Gamma \to \mathbb{C}$, we denote the part on the interval $I_{n,0}$ with $u_{n,0}$ and the parts on the intervals $I_{n,\pm}$ with $u_{n,\pm}$

Lyapunov functions, and L^1-L^∞-estimates. See [18, Chapter 14] for more details. Although not explictely stated in the literature, Equation (1.1) with smooth spatially periodic coefficients can be handled like these more advanced problems. Problem (1.1) posed on the necklace graph is a new challenge in the sense that we are in a very irregular situation. Restricting to solutions which are symmetric in the lower and upper semi-circle, cf. Figure 2, our problem can be mapped to a problem on real line with a periodic δ'-potential, cf. Remark 2.2.

We follow the L^1-L^∞-approach. In a first step the spectral picture necessary for diffusive behavior has to be computed. Since we have a spatially periodic problem the solutions of the linearized system are of Bloch wave form

$$u(x, t) = e^{\lambda_n(\ell)t} e^{i\ell x} f_n(\ell, x),$$

with $\lambda_n(\ell) \in \mathbb{R}$, $n \in \mathbb{N}$, Bloch wave number $\ell \in \mathbb{R}$, and f_n having the same periodicity w.r.t. x as the metric graph. For (1.1) posed on the necklace graph we obtain a spectral picture as sketched in Figure 1.

The fact that the spectrum can be estimated from above by $-C\ell^2$ near the origin is a necessary condition that a $t^{-1/2}$ decay can be established. However, the approach presented in [12] and its generalization, in case of additional exponentially damped modes, presented in [18, Chapter 14], cannot be used directly since both do not fit together with the local existence and uniqueness theory for (1.1) posed on infinite periodic metric graphs. In Remark 6.8 we explain why we think that a pure L^1-L^∞ approach will fail. As in [18, Chapter 14] we separate the diffusive modes from the exponentially damped modes with some projections which are defined for every fixed ℓ. However, in contrast to [18, Chapter 14] only the diffusive modes, corresponding to λ_1, cf. Figure 1, are handled with L^1 and L^∞. For the linearly exponentially damped modes we use the domain of definition $\mathcal{H}^2 = D(\partial_x^2|_\Gamma)$, from the local existence and uniqueness theory, as suitable function space. This space is closed under pointwise multiplication. Hence the L^1-L^∞-estimates for the exponentially damped modes used in [18, Chapter 14] are replaced by L^2-estimates for these modes. We believe that the presented approach in Section 6 is conceptually more easy and more easy to apply in other situations. The detailed formulation of our stability result requires a number of notations and is therefore postponed to Section 6. Under a number of smallness assumptions on the initial conditions the solution will satisfy $\sup_{x\in\Gamma} |u(x, t)| \leq C(1 + t)^{-1/2}$.

The present paper is a first step in answering similar problems for dispersive equations, cf. [19], such as Klein–Gordon or NLS equations. To our knowldege, so far, global existence results, which are based on dispersive estimates, do not exist for equations posed on non-trivial infinite periodic metric graphs. In [10] dispersive estimates for finitely many spectral bands for a problem on the real line with a periodic δ-potential has been shown. The situation in [10] is therefore one derivative more regular than the situation on the graph which corresponds in the symmetric case to a periodic δ'-potential, cf. Remark 2.2.

The plan of the paper is as follows. We transfer (1.1) into a vector-valued problem on the real line with boundary conditions at the vertices. In order to do so we recall and use the notation from [7] and explain in Section 2 what is meant exactly by posing (1.1) on an infinite periodic metric graph. In Section 3 we discuss the spectral problem associated to the linear diffusion operator ∂_x^2 defined on the metric graph Γ. In Section 4 we introduce the functional analytic set-up, in particular some function spaces and Bloch transform. In Section 5 we separate the diffusive modes u_c from the exponentially damped modes u_s. Then we establish linear L^1-L^∞ estimates for the diffusive part u_c and L^2-estimates for the exponentially damped part u_s. In Section 6 we prove the irrelevance of the nonlinear terms u^p w.r.t. linear diffusive behavior, i.e., we prove that the decay rates from Section 5 for u_c in the linear system hold in the nonlinear system, too.

Notation. Throughout this paper, many different constants are denoted by C if they can be chosen independently of time $t \geq 0$.

2 | THE PDE ON THE METRIC GRAPH

Considering (1.1) on the periodic metric graph Γ shown in Figure 2 means the following: For a function $u : \Gamma \to \mathbb{C}$, we denote the part on the interval $I_{n,0}$ associated to $\Gamma_{n,0}$ with $u_{n,0}$ and the parts on the intervals $I_{n,\pm}$ associated to $\Gamma_{n,\pm}$ with $u_{n,\pm}$. The scalar PDE problem on the periodic metric graph Γ is transferred to a vector-valued PDE problem on the real line by introducing the functions

$$u_0(x) = \begin{cases} u_{n,0}(x), & x \in I_{n,0}, \\ 0, & x \in I_{n,\pm}, \end{cases} \tag{2.1}$$

and

$$u_\pm(x) = \begin{cases} u_{n,\pm}(x), & x \in I_{n,\pm}, \\ 0, & x \in I_{n,0}. \end{cases} \tag{2.2}$$

We collect the functions u_0 and $u_\pm$ in the vector $U = (u_0, u_+, u_-)$ and rewrite the evolutionary problem (1.1) as

$$\partial_t U = \partial_x^2 U + U^p, \quad t \geq 0, \quad x \in \mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\}, \tag{2.3}$$

where the nonlinear term U^p stands for the vector $U^p = \left(u_0^p, u_+^p, u_-^p\right)$, and where at the vertex points $\{x = n\pi : n \in \mathbb{Z}\}$ we have Kirchhoff boundary conditions. These are given by the continuity of the functions at the vertices

$$\begin{aligned} u_{n,0}(2\pi n + \pi, t) &= u_{n,+}(2\pi n + \pi, t) = u_{n,-}(2\pi n + \pi, t), \\ u_{n+1,0}(2\pi(n+1), t) &= u_{n,+}(2\pi(n+1), t) = u_{n,-}(2\pi(n+1), t), \end{aligned} \tag{2.4}$$

and the continuity of the fluxes at the vertices

$$\begin{aligned} \partial_x u_{n,0}(2\pi n + \pi, t) &= \partial_x u_{n,+}(2\pi n + \pi, t) + \partial_x u_{n,-}(2\pi n + \pi, t), \\ \partial_x u_{n+1,0}(2\pi(n+1), t) &= \partial_x u_{n,+}(2\pi(n+1), t) + \partial_x u_{n,-}(2\pi(n+1), t). \end{aligned} \tag{2.5}$$

Remark 2.1. Alternatively, the problem could be considered as a scalar problem on the real line. In order to do so we identify $\Gamma_{n,0}$ with $(3\pi n, 3\pi n + \pi)$, $\Gamma_{n,-}$ with $(3\pi n + \pi, 3\pi n + 2\pi)$ and $\Gamma_{n,+}$ with $(3\pi n + 2\pi, 3\pi(n+1))$. The transfer of the boundary conditions is straightforward, we have for instance

$$\lim_{x \to \pi, x < \pi} u(x) = \lim_{x \to \pi, x > \pi} u(x) = \lim_{x \to 2\pi, x > 2\pi} u(x).$$

The values in $x = n\pi$ with $n \in \mathbb{Z}$ are arbitrary, we can choose for instance $u(n\pi) = \lim_{x\to n\pi, x<n\pi} u(x)$. It is obvious that every reasonable periodic graph can be brought info this form. Nevertheless, we think that our approach is more natural. In [7, Section 7] it is explained how to handle other one-dimensional infinite periodic metric graphs with our approach. The spectral pictures for the examples of other periodic metric graphs, presented in [7, Section 7], have to be rotated by an angle of π. Moreover, the eigenvalues are real and no longer imaginary.

Remark 2.2. The subspace of solutions with $u_+ = u_-$ is invariant. In the subspace our problem is equivalent to the scalar problem

$$\partial_t u = \partial_x^2 u + Vu + u^p,$$

with $V(x) = -\delta'(x)/2$ for $x = n\pi$ with n odd and $V(x) = \delta'(x)$ for $x = n\pi$ with n even, which has to be understood in the distributional sense

3 | SPECTRAL ANALYSIS

We start with the discussion of the linear problem

$$\partial_t U = \partial_x^2 U, \quad t \geq 0, \quad x \in \mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\}, \tag{3.1}$$

with the Kirchhoff boundary conditions (2.4) and (2.5). It is solved by $U(x,t) = W(x)e^{\lambda t}$ leading to the spectral problem

$$\partial_x^2 W = \lambda W, \qquad x \in \mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\}, \tag{3.2}$$

with Kirchhoff boundary conditions at the vertices. The spectral problem has already been discussed in [7]. We recall some details, which are necessary for the subsequent analysis. Since (3.2) with the Kirchhoff boundary conditions at the vertices is a periodic ODE problem, there is a basis of eigenfunctions W which are given by Bloch waves

$$W(x) = e^{i\ell x} f(\ell, x), \quad \ell, x \in \mathbb{R}, \tag{3.3}$$

where $f(\ell,\cdot) = (f_0, f_+, f_-)(\ell,\cdot)$ is a 2π-periodic function for every $\ell \in \mathbb{R}$. Since these functions satisfy the continuation conditions

$$f(\ell, x) = f(\ell, x + 2\pi), \quad f(\ell, x) = f(\ell + 1, x)e^{ix}, \quad \ell, x \in \mathbb{R}, \tag{3.4}$$

we can restrict the definition of $f(\ell, x)$ to $x \in \mathbb{T}_{2\pi} = \mathbb{R}/(2\pi\mathbb{Z})$ and $\ell \in \mathbb{T}_1 = \mathbb{R}/\mathbb{Z}$. The torus $\mathbb{T}_{2\pi}$ is isometrically parameterized with $x \in [0, 2\pi]$ and the torus $\mathbb{T}_1$ with $\ell \in [-1/2, 1/2]$, where the endpoints of the intervals are identified to be the same. Hence, f can be found as a solution of the eigenvalue problem

$$(\partial_x + i\ell)^2 f = \lambda(\ell) f, \quad x \in \mathbb{T}_{2\pi}, \tag{3.5}$$

with associated boundary conditions

$$\begin{cases} f_0(\ell, \pi) = f_+(\ell, \pi) = f_-(\ell, \pi), \\ f_0(\ell, 0) = f_+(\ell, 2\pi) = f_-(\ell, 2\pi) \end{cases} \tag{3.6}$$

and

$$\begin{cases} (\partial_x + i\ell) f_0(\ell, \pi) = (\partial_x + i\ell) f_+(\ell, \pi) + (\partial_x + i\ell) f_-(\ell, \pi), \\ (\partial_x + i\ell) f_0(\ell, 0) = (\partial_x + i\ell) f_+(\ell, 2\pi) + (\partial_x + i\ell) f_-(\ell, 2\pi). \end{cases} \tag{3.7}$$

The functions $f_0(\ell,\cdot)$ and $f_\pm(\ell,\cdot)$ have supports in $I_{0,0} = [0,\pi] \subset \mathbb{T}_{2\pi}$ and $I_{0,\pm} = [\pi, 2\pi] \subset \mathbb{T}_{2\pi}$. The spectrum consists of two parts. There is a sequence of eigenvalues at $\{-m^2\}_{m\in\mathbb{N}}$ of infinite multiplicity. The associated eigenfunctions are supported compactly in each circle. Moreover, there are countably many curves of eigenvalues which correspond to the eigenvalues of the monodromy matrix M. They can be computed via roots $\rho_{1,2} = e^{2\pi i\ell}$ of $\rho^2 - \mathrm{tr}(M)(-\lambda)\rho + 1 = 0$, where

$$\mathrm{tr}(M)(-\lambda) = \frac{1}{4}\Big[9\cos(2\pi\sqrt{-\lambda}) - 1\Big]. \tag{3.8}$$

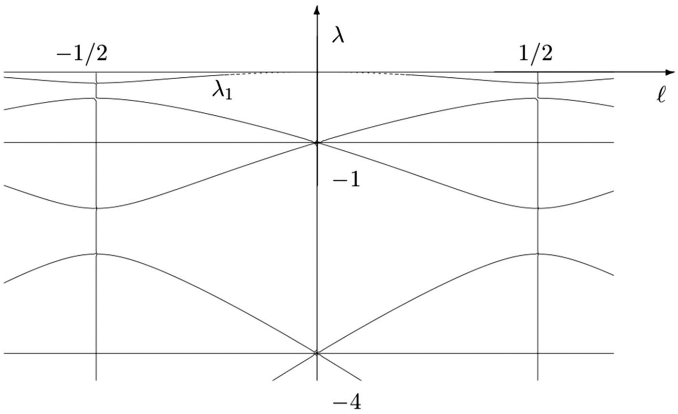

FIGURE 3 The eigenvalues $\lambda(\ell)$ plotted versus the Bloch wave numbers ℓ. The diffusive behavior comes from the parabola-like curve λ_1 through $(\ell,\lambda)=(0,0)$. All other curves lead to exponential decay

The spectral bands of the periodic eigenvalue problem (3.5) are shown on Figure 3.

Remark 3.1. At a first view it seems that the infinitely many localized eigenfunctions $\underline{W}^{m,k}$, indexed with $m\in\mathbb{N}$ and $k\in\mathbb{Z}$, with the explicit representation

$$\underline{W}^{m,k}_{n,0}=0,\qquad \underline{W}^{m,k}_{n,\pm}=\pm\delta_{n,k}\sin(mx),$$

cannot be written in Bloch form (3.3). However, the linear combination

$$w^m_{n,0}=e^{i\ell x}f^m_{n,0}(\ell,x),\qquad w^m_{n,\pm}=e^{i\ell x}f^m_{n,\pm}(\ell,x),$$

with $f^m_{n,0}(\ell,x)=0$ for $x\in[2\pi n,2\pi n+\pi)$ and

$$f^m_{n,\pm}(\ell,x)=\pm e^{-i\ell(x-2\pi n)}\sin(mx)$$

for $x\in[2\pi n+\pi,2\pi(n+1))$ is of the required form. It is an easy exercise to show that the f^m satisfy (3.4).

4 | THE FUNCTIONAL ANALYTIC SET-UP

The diffusive behavior can be seen in the spectral picture plotted in Figure 3. The parabola-like curve touching $\lambda=0$ at $\ell=0$ leads to such diffusive behavior. Hence, a natural approach to establish the diffusive decay estimates (1.2) is an expansion of (2.3) w.r.t. the associated eigenfunctions, the Bloch modes.

4.1 | The system in Bloch space

Bloch transform $\mathcal{T}$ generalizes Fourier transform $\mathcal{F}$ from spatially homogeneous problems to spatially periodic problems, cf. [15]. It is defined by

$$\widetilde{u}(\ell,x)=(\mathcal{T}u)(\ell,x)=\sum_{n\in\mathbb{Z}}u(x+2\pi n)e^{-i\ell x-2\pi in\ell}. \tag{4.1}$$

The inverse of Bloch transform is given by

$$u(x)=\left(\mathcal{T}^{-1}\widetilde{u}\right)(x)=\int_{-1/2}^{1/2}e^{i\ell x}\widetilde{u}(\ell,x)\,d\ell. \tag{4.2}$$

By construction, $\widetilde{u}(\ell,x)$ is extended from $(\ell,x)\in\mathbb{T}_1\times\mathbb{T}_{2\pi}$ to $(\ell,x)\in\mathbb{R}\times\mathbb{R}$ according to

$$\widetilde{u}(\ell,x)=\widetilde{u}(\ell,x+2\pi)\quad\text{and}\quad\widetilde{u}(\ell,x)=\widetilde{u}(\ell+1,x)e^{ix}. \tag{4.3}$$

Multiplication of two functions $u(x)$ and $v(x)$ in x-space corresponds to the convolution integral in Bloch space:

$$(\tilde{u} * \tilde{v})(\ell, x) = \int_{-1/2}^{1/2} \tilde{u}(\ell - m, x)\tilde{v}(m, x)\, dm, \tag{4.4}$$

where (4.3) has to be used for $|\ell - m| > 1/2$.

We apply the Bloch transform $\mathcal{T}$ to all components of $U = (u_0, u_+, u_-)$ and obtain

$$\partial_t \widetilde{U}(t, \ell, x) = \widetilde{L}(\ell)\widetilde{U}(t, \ell, x) + \widetilde{N}\big(\widetilde{U}\big)(t, \ell, x), \tag{4.5}$$

where $\widetilde{U} = \mathcal{T}U$, $\widetilde{L} = \mathcal{T}L\mathcal{T}^{-1}$, with L the linear operator defined on the right-hand side of (3.1), and $\widetilde{N}\big(\widetilde{U}\big) = \mathcal{T}N\big(\mathcal{T}^{-1}\widetilde{U}\big)$ with $N(U) = U^p$. By definition $\widetilde{U}(t, \ell, x) = (\widetilde{u}_0, \widetilde{u}_+, \widetilde{u}_-)(t, \ell, x)$ satisfies

$$\widetilde{U}(t, \ell, x) = \widetilde{U}(t, \ell, x + 2\pi) \quad \text{and} \quad \widetilde{U}(t, \ell, x) = \widetilde{U}(t, \ell + 1, x)e^{ix}. \tag{4.6}$$

4.2 | The function spaces

In order analyse (4.5), we define L^2-based spaces for fixed $\ell \in \mathbb{T}_1$.

Definition 4.1. Let

$$L^2_\Gamma := \Big\{ \widetilde{U} = (\widetilde{u}_0, \widetilde{u}_+, \widetilde{u}_-) \in \big(L^2(\mathbb{T}_{2\pi})\big)^3 : \quad \operatorname{supp}\big(\widetilde{u}_j\big) = I_{0,j}, \quad j \in \{0, +, -\} \Big\} \tag{4.7}$$

and

$$H^2_\Gamma(\ell) := \Big\{ \widetilde{U} \in L^2_\Gamma : \quad \widetilde{u}_j \in H^2\big(I_{0,j}\big), \quad j \in \{0, +, -\}, \quad (3.6)\text{–}(3.7) \quad \text{are satisfied} \Big\},$$

equipped with the norm

$$\Big\| \widetilde{U} \Big\|_{H^2_\Gamma(\ell)} = \Big(\|\widetilde{u}_0\|^2_{H^2(I_{0,0})} + \|\widetilde{u}_+\|^2_{H^2(I_{0,+})} + \|\widetilde{u}_-\|^2_{H^2(I_{0,-})} \Big)^{1/2}.$$

The parameter ℓ is defined in $H^2_\Gamma(\ell)$ by means of the ℓ-dependent boundary conditions (3.6)–(3.7). We recall [7, Lemma 2.1].

Lemma 4.2. *For fixed $\ell \in \mathbb{T}_1$, the operator $-\widetilde{L}(\ell) := -(\partial_x + i\ell)^2$ is a self-adjoint, positive semi-definite operator in L^2_Γ.*

The domain of definition of the operator $\widetilde{L}(\ell)$ is given by:

Definition 4.3. We define

$$\widetilde{\mathcal{H}}^2 = \Big\{ \widetilde{U} \in L^2\left(\mathbb{T}_1, L^2_\Gamma\right) : \quad \widetilde{u}_j \in L^2\left(\mathbb{T}_1, H^2\left(I_{0,j}\right)\right), \quad j \in \{0, +, -\}, \quad (3.6)\text{–}(3.7) \quad \text{are satisfied} \Big\},$$

equipped with the norm

$$\Big\| \widetilde{U} \Big\|_{\widetilde{\mathcal{H}}^2} = \left(\int_{-1/2}^{1/2} \Big(\|\widetilde{u}_0(\ell, \cdot)\|^2_{H^2(I_{0,0})} + \|\widetilde{u}_+(\ell, \cdot)\|^2_{H^2(I_{0,+})} + \|\widetilde{u}_-(\ell, \cdot)\|^2_{H^2(I_{0,-})} \Big)\, d\ell \right)^{1/2}.$$

We have the following local existence and uniqueness result:

Theorem 4.4. *For every $\widetilde{U}_0 \in \widetilde{\mathcal{H}}^2$, there exists a $T_0 = T_0\Big(\Big\|\widetilde{U}_0\Big\|_{\widetilde{\mathcal{H}}^2}\Big) > 0$ and a unique mild solution $\widetilde{U} \in C\big([0, T_0], \widetilde{\mathcal{H}}^2\big)$ of (4.5) with $\widetilde{U}|_{t=0} = \widetilde{U}_0$.*

Proof. In Appendix A we prove the local existence and uniqueness of solutions in $\mathcal{H}^2$, the domain of definition of the operator $L := \partial_x^2$ in physical space. For the definition of $\mathcal{H}^2$ see Appendix A. Theorem 4.4 follows immediately from Theorem A.4 since, according with [7, Lemma 4.2], Bloch transform $\mathcal{T}$ is an isomorphism between the spaces $\mathcal{H}^2$ and $\widetilde{\mathcal{H}}^2$. □

We close this section with the remark that the global existence in $\widetilde{\mathcal{H}}^2$ of the solutions in which we are interested follows with the subsequent diffusive estimates. They can be used as a priori estimates such that Theorem 4.4 can be applied again and again. Hence, it remains to establish the polynomial decay rates (1.2).

5 | DIFFUSIVE AND EXPONENTIALLY DAMPED MODES

As explained in the introduction we separate the diffusive from the exponentially damped modes. In the following we use the abbreviation $\widetilde{\varphi}_1(\ell) = f^1(\ell,\cdot)$ and the normalization $\left\langle \widetilde{\varphi}_1(\ell), \widetilde{\varphi}_1(\ell) \right\rangle = 1$, where $\langle u, v\rangle = \int_0^{2\pi} \overline{u(x)} v(x)\, dx$.

The curve of the eigenvalues $\lambda_1(l)$ is separated from the rest of the spectrum, cf. Figure 3. Since $\widetilde{L}(\ell)$ is a self-adjoint operator for fixed ℓ, cf. Lemma 4.2, we define the orthogonal projection on the diffusive modes by

$$\widetilde{P}_c(\ell)\widetilde{U}(\ell) = \left\langle \widetilde{\varphi}_1(\ell), \widetilde{U}(\ell) \right\rangle \widetilde{\varphi}_1(\ell).$$

Moreover, we let $\widetilde{P}_s(\ell)\widetilde{U}(\ell) = \widetilde{U}(\ell) - \widetilde{P}_c(\ell)\widetilde{U}(\ell)$ be the orthogonal projection on the exponentially damped modes. We use the projections to separate (4.5) in two parts, namely

$$\partial_t \widetilde{v}_c(\ell,t) = \widetilde{L}_c(\ell)\widetilde{v}_c(\ell,t) + \widetilde{P}_c(\ell)\widetilde{N}\left(\widetilde{U}\right)(\ell,t), \tag{5.1}$$

$$\partial_t \widetilde{v}_s(\ell,x,t) = \widetilde{L}_s(\ell)\widetilde{v}_s(\ell,x,t) + \widetilde{P}_s(\ell)\widetilde{N}\left(\widetilde{U}\right)(\ell,x,t). \tag{5.2}$$

where $\widetilde{L}_c(\ell) = \widetilde{L}(\ell)\widetilde{P}_c(\ell)$ and $\widetilde{L}_s(\ell) = \widetilde{L}(\ell)\widetilde{P}_s(\ell)$. By construction the operators $\widetilde{P}_s(\ell)$ and $\widetilde{P}_c(\ell)$ commute with $\widetilde{L}(\ell)$. System (5.1)–(5.2) is solved with initial conditions $\widetilde{v}_c|_{t=0} = \widetilde{P}_c\left(\widetilde{U}|_{t=0}\right)$ and $\widetilde{v}_s|_{t=0} = \widetilde{P}_s\left(\widetilde{U}|_{t=0}\right)$. Then $\widetilde{v}_c$ and $\widetilde{v}_s$ are defined via the solutions of (5.1)–(5.2).

Since the sectorial operator $\widetilde{L}_s$ has spectrum in the left half plane strictly bounded away from the imaginary axis, we obviously have the following result, cf. [8, Theorem 1.5.3.].

Lemma 5.1. *For the analytic semigroup generated by $\widetilde{L}_s$ we have the estimate*

$$\left\| e^{t\widetilde{L}_s} \right\|_{\widetilde{\mathcal{H}}^2 \to \widetilde{\mathcal{H}}^2} \le C e^{-\sigma_s t/2},$$

for a $\sigma_s > 0$ and all $t \ge 0$.

For the handling of the $\widetilde{v}_c$ we introduce the following two spaces.

Definition 5.2. Let

$$\widetilde{\mathcal{X}}^1 = \left\{ \widetilde{U} \in L^1\left(\mathbb{T}_1, L^2_\Gamma\right) : \quad \widetilde{u}_j \in L^1\left(\mathbb{T}_1, H^2(I_{0,j})\right), \quad j \in \{0,+,-\}, \quad \text{(3.6)–(3.7)} \quad \text{are satisfied} \right\},$$

equipped with the norm

$$\left\|\widetilde{U}\right\|_{\widetilde{\mathcal{X}}^1} = \int_{-1/2}^{1/2} \left(\|\widetilde{u}_0(\ell,\cdot)\|_{H^2(I_{0,0})} + \|\widetilde{u}_+(\ell,\cdot)\|_{H^2(I_{0,+})} + \|\widetilde{u}_-(\ell,\cdot)\|_{H^2(I_{0,-})} \right) d\ell$$

and

$$\widetilde{\mathcal{X}}^\infty = \left\{ \widetilde{U} \in L^\infty\left(\mathbb{T}_1, L^2_\Gamma\right) : \quad \widetilde{u}_j \in L^\infty\left(\mathbb{T}_1, H^2(I_{0,j})\right), \quad j \in \{0,+,-\}, \quad \text{(3.6)–(3.7)} \quad \text{are satisfied} \right\},$$

equipped with the norm

$$\left\|\widetilde{U}\right\|_{\widetilde{\mathcal{X}}^\infty} = \sup_{\ell\in\mathbb{T}_1} \left(\|\widetilde{u}_0(\ell,\cdot)\|_{H^2(I_{0,0})} + \|\widetilde{u}_+(\ell,\cdot)\|_{H^2(I_{0,+})} + \|\widetilde{u}_-(\ell,\cdot)\|_{H^2(I_{0,-})} \right).$$

Moreover, we set $\widetilde{\mathcal{X}}^2 = \widetilde{\mathcal{H}}^2$.

For analyzing the $\widetilde{v}_c$-part we use its representation

$$\widetilde{v}_c(\ell, x) = \widehat{v}_{c,j}(\ell)\varphi_{1,j}(\ell, x), \tag{5.3}$$

with $\widehat{v}_{c,j}(\ell) \in \mathbb{C}$, $j \in \{0, +, -\}$. Since $\varphi_{1,j} \in C^{\infty}_{with(4.3)}\big(\mathbb{T}_1, H^2(I_{0,j})\big)$ we have

$$\|\widetilde{v}_c\|_{\widetilde{\mathcal{X}}^1} = \int_{-1/2}^{1/2} \left(\sum_{j\in\{0,+,-\}} \|\widehat{v}_c(\ell)\varphi_{1,j}(\ell,\cdot)\|_{H^2(I_{0,j})} \right) d\ell \le C\|\widehat{v}_c\|_{L^1(\mathbb{T}_1)}.$$

Since the norm $\|\varphi_{1,j}(\ell,\cdot)\|_{H^2(I_{0,j})}$ is not only bounded from above, but also from below, we also have

$$\|\widehat{v}_c\|_{L^1(\mathbb{T}_1)} \le C\|\widetilde{v}_c\|_{\widetilde{\mathcal{X}}^1}.$$

Similarly, we find

$$\|\widetilde{v}_c\|_{\widetilde{\mathcal{X}}^\infty} \le C\|\widehat{v}_c\|_{L^\infty(\mathbb{T}_1)} \qquad \text{and} \qquad \|\widehat{v}_c\|_{L^\infty(\mathbb{T}_1)} \le C\|\widetilde{v}_c\|_{\widetilde{\mathcal{X}}^\infty}.$$

For the $\widetilde{v}_c$-part we therefore obtain

Lemma 5.3. *For the analytic semigroup generated by $\widetilde{L}_c$ we have the estimates*

$$\left\|e^{t\widetilde{L}_c}\right\|_{\widetilde{\mathcal{X}}^1\to\widetilde{\mathcal{X}}^1} \le C, \quad \left\|e^{t\widetilde{L}_c}\right\|_{\widetilde{\mathcal{X}}^\infty\to\widetilde{\mathcal{X}}^\infty} \le C, \quad \left\|e^{t\widetilde{L}_c}\right\|_{\widetilde{\mathcal{X}}^\infty\to\widetilde{\mathcal{X}}^1} \le Ct^{-1/2}.$$

Proof. Since $\lambda_1(\ell) \le -C\ell^2$ for small ℓ and $e^{t\widetilde{L}_c(\ell)}\widetilde{v}_c(\ell) = e^{\lambda_1(\ell)t}\widetilde{v}_c(\ell)$ we obviously have

$$\begin{aligned}
\left\|e^{t\widetilde{L}_c}\widetilde{v}_c\right\|_{\widetilde{\mathcal{X}}^1} &\le \left\|e^{\lambda_1(k)t}\right\|_{L^\infty} \|\widetilde{v}_c\|_{\widetilde{\mathcal{X}}^1} \le C\|\widetilde{v}_c\|_{\widetilde{\mathcal{X}}^1},\\
\left\|e^{t\widetilde{L}_c}\widetilde{v}_c\right\|_{\widetilde{\mathcal{X}}^\infty} &\le \left\|e^{\lambda_1(k)t}\right\|_{L^\infty} \|\widetilde{v}_c\|_{\widetilde{\mathcal{X}}^\infty} \le C\|\widetilde{v}_c\|_{\widetilde{\mathcal{X}}^\infty},\\
\left\|e^{t\widetilde{L}_c}\widetilde{v}_c\right\|_{\widetilde{\mathcal{X}}^1} &\le \left\|e^{\lambda_1(k)t}\right\|_{L^1} \|\widetilde{v}_c\|_{\widetilde{\mathcal{X}}^\infty} \le Ct^{-1/2}\|\widetilde{v}_c\|_{\widetilde{\mathcal{X}}^\infty}.
\end{aligned}$$

□

6 | THE NONLINEAR DECAY ESTIMATES

6.1 | Preliminiaries

For the proof of the diffusive decay estimates (1.2) we first need a number of inequalities. From [7, Lemma 3.1 and Lemma 4.2] we know that $\widetilde{\mathcal{X}}^2 = \widetilde{\mathcal{H}}^2$ is closed under the convolution (4.4).

Lemma 6.1. *There exists a $C > 0$ such that for all $\widetilde{u}, \widetilde{v} \in \widetilde{\mathcal{X}}^2$ we have*

$$\|\widetilde{u} * \widetilde{v}\|_{\widetilde{\mathcal{X}}^2} \le C\|\widetilde{u}\|_{\widetilde{\mathcal{X}}^2}\|\widetilde{v}\|_{\widetilde{\mathcal{X}}^2}. \tag{6.1}$$

Next we need:

Lemma 6.2. *There exists a $C > 0$ such that for all $\widetilde{u} \in \widetilde{\mathcal{X}}^1$ and $\widetilde{v} \in \widetilde{\mathcal{X}}^p$ with $p \in \{2, \infty\}$ we have*

$$\|\widetilde{u} * \widetilde{v}\|_{\widetilde{\mathcal{X}}^p} \le C\|\widetilde{u}\|_{\widetilde{\mathcal{X}}^1}\|\widetilde{v}\|_{\widetilde{\mathcal{X}}^p}. \tag{6.2}$$

Proof. Since the $H^2\big(I_{0,j}\big)$ are closed under multiplication, by classical Young's inequality $\|\widehat{u} * \widehat{v}\|_{L^p} \le C\|\widehat{u}\|_{L^1}\|\widehat{v}\|_{L^p}$ for convolutions it follows

$$\begin{aligned}
\|\widetilde{u} * \widetilde{v}\|_{\widetilde{\mathcal{X}}^p} &= \left\| \sum_{j\in\{0,+,-\}} \left\| \int_{-1/2}^{1/2} \widetilde{u}_j(\ell-\ell_1, x)\widetilde{v}_j(\ell_1, x)\, d\ell_1 \right\|_{H^2(I_{0,j})(dx)} \right\|_{L^p(d\ell)}\\
&\le C \left\| \int_{-1/2}^{1/2} \sum_{j\in\{0,+,-\}} \|\widetilde{u}_j(\ell-\ell_1, x)\widetilde{v}_j(\ell_1, x)\|_{H^2(I_{0,j})(dx)}\, d\ell_1 \right\|_{L^p(d\ell)}
\end{aligned}$$

$$\leq C \sum_{j\in\{0,+,-\}} \left\| \int_{-1/2}^{1/2} \|\widetilde{u}_j(\ell-\ell_1,x)\|_{H^2(I_{0,j})(dx)} \|\widetilde{v}_j(\ell_1,x)\|_{H^2(I_{0,j})(dx)}\, d\ell_1 \right\|_{L^p(d\ell)}$$
$$\leq C \sum_{j\in\{0,+,-\}} \|\widetilde{u}_j\|_{\widetilde{\mathcal{X}}^1} \|\widetilde{v}_j\|_{\widetilde{\mathcal{X}}^p} \leq C\|\widetilde{u}\|_{\widetilde{\mathcal{X}}^1}\|\widetilde{v}\|_{\widetilde{\mathcal{X}}^p}.$$

□

Lemma 6.3. *There exists a $C > 0$ such that for all $\widetilde{u}, \widetilde{v} \in \widetilde{\mathcal{X}}^2$ we have*

$$\|\widetilde{u} * \widetilde{v}\|_{\widetilde{\mathcal{X}}^\infty} \leq C\|\widetilde{u}\|_{\widetilde{\mathcal{X}}^2}\|\widetilde{v}\|_{\widetilde{\mathcal{X}}^2}. \tag{6.3}$$

Proof. We proceed as above, but use $\|\hat{u} * \hat{v}\|_{L^\infty} \leq C\|\hat{u}\|_{L^2}\|\hat{v}\|_{L^2}$. We find

$$\|\widetilde{u} * \widetilde{v}\|_{\widetilde{\mathcal{X}}^\infty} = \left\| \sum_{j\in\{0,+,-\}} \left\| \int_{-1/2}^{1/2} \widetilde{u}_j(\ell-\ell_1,x)\widetilde{v}_j(\ell_1,x)\, d\ell_1 \right\|_{H^2(I_{0,j})(dx)} \right\|_{L^\infty(d\ell)}$$
$$\leq C \left\| \int_{-1/2}^{1/2} \sum_{j\in\{0,+,-\}} \|\widetilde{u}_j(\ell-\ell_1,x)\widetilde{v}_j(\ell_1,x)\|_{H^2(I_{0,j})(dx)}\, d\ell_1 \right\|_{L^\infty(d\ell)}$$
$$\leq C \sum_{j\in\{0,+,-\}} \left\| \int_{-1/2}^{1/2} \|\widetilde{u}_j(\ell-\ell_1,x)\|_{H^2(I_{0,j})(dx)} \|\widetilde{v}_j(\ell_1,x)\|_{H^2(I_{0,j})(dx)}\, d\ell_1 \right\|_{L^\infty(d\ell)}$$
$$\leq C \sum_{j\in\{0,+,-\}} \|\widetilde{u}_j\|_{\widetilde{\mathcal{X}}^2} \|\widetilde{v}_j\|_{\widetilde{\mathcal{X}}^2} \leq C\|\widetilde{u}\|_{\widetilde{\mathcal{X}}^2}\|\widetilde{v}\|_{\widetilde{\mathcal{X}}^2}.$$

□

Moreover, we need the following embedding.

Lemma 6.4. *There exists a $C > 0$ such that for all $\widetilde{u} \in \widetilde{\mathcal{X}}^2$ we have*

$$\|\widetilde{u}\|_{\widetilde{\mathcal{X}}^1} \leq C\|\widetilde{u}\|_{\widetilde{\mathcal{X}}^2}. \tag{6.4}$$

Proof. Since $\ell \in \mathbb{T}^1$ we have

$$\|\widetilde{u}\|_{\widetilde{\mathcal{X}}^1} = \left\| \sum_{j\in\{0,+,-\}} \|\widetilde{u}_j(\ell,x)\|_{H^2(I_{0,j})(dx)} \right\|_{L^1(d\ell)}$$
$$\leq C \sum_{j\in\{0,+,-\}} \left\| \|\widetilde{u}_j(\ell,x)\|_{H^2(I_{0,j})(dx)} \cdot 1 \right\|_{L^1(d\ell)}$$
$$\leq C \sum_{j\in\{0,+,-\}} \left\| \|\widetilde{u}_j(\ell,x)\|_{H^2(I_{0,j})(dx)} \right\|_{L^2(d\ell)} \|1\|_{L^2(d\ell)}$$
$$\leq C \sum_{j\in\{0,+,-\}} \left\| \|\widetilde{u}_j(\ell,x)\|_{H^2(I_{0,j})(dx)} \right\|_{L^2(d\ell)} \leq C\|\widetilde{u}\|_{\widetilde{\mathcal{X}}^2}.$$

□

Finally, we need the following interpolation inequality.

Lemma 6.5. *There exists a $C > 0$ such that for all $\widetilde{u} \in \widetilde{\mathcal{X}}^1 \cap \widetilde{\mathcal{X}}^\infty$ we have*

$$\|\widetilde{u}\|^2_{\widetilde{\mathcal{X}}^2} \leq C\|\widetilde{u}\|_{\widetilde{\mathcal{X}}^\infty}\|\widetilde{u}\|_{\widetilde{\mathcal{X}}^1}. \tag{6.5}$$

Proof. We estimate

$$\begin{aligned}
\|\widetilde{u}\|_{\widetilde{\mathcal{X}}^2}^2 &= \left\| \sum_{j\in\{0,+,-\}} \|\widetilde{u}_j(\ell,x)\|_{H^2(I_{0,j})(dx)} \right\|_{L^2(d\ell)}^2 \\
&= \left\| \sum_{j\in\{0,+,-\}} \|\widetilde{u}_j(\ell,x)\|_{H^2(I_{0,j})(dx)}^2 \right\|_{L^1(d\ell)} \\
&\leq \sum_{j\in\{0,+,-\}} \left\| \left\| \|\widetilde{u}_j(\ell,x)\|_{H^2(I_{0,j})(dx)} \right\|_{L^\infty(d\ell)} \|\widetilde{u}_j(\ell,x)\|_{H^2(I_{0,j})(dx)} \right\|_{L^1(d\ell)} \\
&\leq C\|u\|_{\widetilde{\mathcal{X}}^\infty} \|\widetilde{u}\|_{\widetilde{\mathcal{X}}^1}.
\end{aligned}$$

□

6.2 | Irrelevance of the nonlinear terms

We proceed as in [12] and consider the variation of constant formula

$$\widetilde{v}_c(t) = e^{t\widetilde{L}_c}\widetilde{v}_c(0) + \int_0^t e^{(t-\tau)\widetilde{L}_c}\widetilde{P}_c\widetilde{N}\left(\widetilde{v}_c,\widetilde{v}_s\right)(\tau)\,\mathrm{d}\tau, \tag{6.6}$$

$$\widetilde{v}_s(t) = e^{t\widetilde{L}_s}\widetilde{v}_s(0) + \int_0^t e^{(t-\tau)\widetilde{L}_s}\widetilde{P}_s\widetilde{N}\left(\widetilde{v}_c,\widetilde{v}_s\right)(\tau)\,\mathrm{d}\tau \tag{6.7}$$

for (5.1)–(5.2). In the following we use the abbreviations

$$\begin{aligned}
a(t) &= \sup_{0\leq\tau\leq t} \|\widetilde{v}_c(\tau)\|_{\widetilde{\mathcal{X}}^\infty}, \\
b(t) &= \sup_{0\leq\tau\leq t} \left\|(1+\tau)^{1/2}\widetilde{v}_c(\tau)\right\|_{\widetilde{\mathcal{X}}^1}, \\
c(t) &= \sup_{0\leq\tau\leq t} \left\|(1+\tau)^{3/4}\widetilde{v}_s(\tau)\right\|_{\widetilde{\mathcal{X}}^2}
\end{aligned}$$

and

$$r(t) = a(t) + b(t) + c(t).$$

Moreover, many different constants are now denoted with the same symbol C, if they can be chosen independently of $r(t)$ and t.

a) Lemma 6.2 and Lemma 6.4 imply

$$\begin{aligned}
\left\|\left(\widetilde{v}_c+\widetilde{v}_s\right)^{*p}\right\|_{\widetilde{\mathcal{X}}^1} &\leq C\|\widetilde{v}_c+\widetilde{v}_s\|_{\widetilde{\mathcal{X}}^1}^p \leq C\left(\|\widetilde{v}_c\|_{\widetilde{\mathcal{X}}^1}+\|\widetilde{v}_s\|_{\widetilde{\mathcal{X}}^1}\right)^p \leq C\left(\|\widetilde{v}_c\|_{\widetilde{\mathcal{X}}^1}+\|\widetilde{v}_s\|_{\widetilde{H}^2}\right)^p \\
&\leq C\left((1+t)^{-1/2}b(t)+(1+t)^{-3/4}c(t)\right)^p \leq C(1+t)^{-p/2}r(t)^p,
\end{aligned}$$

where $\widetilde{v}^p$ denotes the $p-1$-times convolution of $\widetilde{v}$.

b) In the same way we obtain

$$\begin{aligned}
\left\|\left(\widetilde{v}_c+\widetilde{v}_s\right)^{*p}\right\|_{\widetilde{\mathcal{X}}^2} &\leq C\|\widetilde{v}_c+\widetilde{v}_s\|_{\widetilde{\mathcal{X}}^1}^{p-1}\|\widetilde{v}_c+\widetilde{v}_s\|_{\widetilde{\mathcal{X}}^2} \\
&\leq C\left(\|\widetilde{v}_c\|_{\widetilde{\mathcal{X}}^1}+\|\widetilde{v}_s\|_{\widetilde{\mathcal{X}}^1}\right)^{p-1}\left(\|\widetilde{v}_c\|_{\widetilde{\mathcal{X}}^2}+\|\widetilde{v}_s\|_{\widetilde{\mathcal{X}}^2}\right) \\
&\leq C\left(\|\widetilde{v}_c\|_{\widetilde{\mathcal{X}}^1}+\|\widetilde{v}_s\|_{\widetilde{\mathcal{X}}^1}\right)^{p-1}\left(\|\widetilde{v}_c\|_{\widetilde{\mathcal{X}}^1}^{1/2}\|\widetilde{v}_c\|_{\widetilde{\mathcal{X}}^\infty}^{1/2}+\|\widetilde{v}_s\|_{\widetilde{\mathcal{X}}^2}\right) \\
&\leq C\left((1+t)^{-1/2}b(t)+(1+t)^{-3/4}c(t)\right)^{p-1}\left((1+t)^{-1/4}a(t)^{1/2}b(t)^{1/2}+(1+t)^{-3/4}c(t)\right) \\
&\leq C(1+t)^{(1-p)/2-1/4}r(t)^p,
\end{aligned}$$

where we additionally used Lemma 6.5.

c) The estimates for $\left\|\left(\widetilde{v}_c+\widetilde{v}_s\right)^{*p}\right\|_{\widetilde{L}^\infty}$ are more involved. We use that

$$\left(\widetilde{v}_c+\widetilde{v}_s\right)^{*p}=\sum_{j=0}^{p}\binom{p}{j}\widetilde{v}_c^{*j}*\widetilde{v}_s^{*(p-j)},$$

use the triangle inequality, and estimate $\left\|\widetilde{v}_c^{*j}*\widetilde{v}_s^{*(p-j)}\right\|_{\widetilde{L}^\infty}$.

i) Using Lemma 6.2, Lemma 6.4, and Lemma 6.3 yields

$$\|\widetilde{v}_s^{*p}\|_{\widetilde{X}^\infty}\le\|\widetilde{v}_s\|_{\widetilde{X}^1}^{p-2}\|\widetilde{v}_s^{*2}\|_{\widetilde{X}^\infty}\le\|\widetilde{v}_s\|_{\widetilde{\mathcal{X}}^2}^{p-2}\|\widetilde{v}_s\|_{\widetilde{\mathcal{X}}^2}^{2}\le C(1+t)^{-3p/4}r(t)^p.$$

ii) Using Lemma 6.2 and Lemma 6.3 yields

$$\left\|\widetilde{v}_c^{*(p-2)}*\widetilde{v}_s^{*2}\right\|_{\widetilde{X}^\infty}\le\|\widetilde{v}_c\|_{\widetilde{X}^1}^{p-2}\left\|\widetilde{v}_s^{*2}\right\|_{\widetilde{X}^\infty}\le\|\widetilde{v}_c\|_{\widetilde{X}^1}^{p-2}\|\widetilde{v}_s\|_{\widetilde{\mathcal{X}}^2}^{2}\le C(1+t)^{(2-p)/2-3/2}r(t)^p.$$

iii) All terms $\widetilde{v}_c^{*j}*\widetilde{v}_s^{*(p-j)}$ with $j\ge 2$ obviously can be estimated as in i) and ii). Hence two terms remain.

iv) Using Lemma 6.2, Lemma 6.3, and Lemma 6.5, the first of these terms is estimated by

$$\begin{aligned}\left\|\widetilde{v}_c^{*(p-1)}*\widetilde{v}_s\right\|_{\widetilde{X}^\infty}&\le C\|\widetilde{v}_c\|_{\widetilde{X}^1}^{p-2}\|\widetilde{v}_c*\widetilde{v}_s\|_{\widetilde{X}^\infty}\\&\le C\|\widetilde{v}_c\|_{\widetilde{X}^1}^{p-2}\|\widetilde{v}_c\|_{\widetilde{\mathcal{X}}^2}\|\widetilde{v}_s\|_{\widetilde{\mathcal{X}}^2}\\&\le C\|\widetilde{v}_c\|_{\widetilde{X}^1}^{p-2}\|\widetilde{v}_c\|_{\widetilde{X}^\infty}^{1/2}\|\widetilde{v}_c\|_{\widetilde{X}^1}^{1/2}\|\widetilde{v}_s\|_{\widetilde{\mathcal{X}}^2}\\&\le C(1+t)^{(2-p)/2-1/4-3/4}r(t)^p=(1+t)^{-p/2}r(t)^p.\end{aligned}$$

v) Using Lemma 6.2 the last term is estimated by

$$\|\widetilde{v}_c^{*p}\|_{\widetilde{X}^\infty}\le C\|\widetilde{v}_c\|_{\widetilde{X}^1}^{p-1}\|\widetilde{v}_c\|_{\widetilde{X}^\infty}\le C(1+t)^{(1-p)/2}r(t)^p.$$

Summarizing all estimates i)-v) yields

$$\left\|\left(\widetilde{v}_c+\widetilde{v}_s\right)^{*p}\right\|_{\widetilde{X}^\infty}\le C(1+t)^{(1-p)/2}r(t)^p.$$

We have:

Aa) We estimate

$$\begin{aligned}\left\|\int_0^t e^{(t-\tau)\widetilde{L}_c}\widetilde{P}_c\widetilde{N}\left(\widetilde{v}_c,\widetilde{v}_s\right)(\tau)\,\mathrm{d}\tau\right\|_{\widetilde{X}^\infty}&\le\int_0^t\left\|e^{(t-\tau)\widetilde{L}_c}\right\|_{\widetilde{X}^\infty\to\widetilde{X}^\infty}\left\|\widetilde{N}\left(\widetilde{v}_c,\widetilde{v}_s\right)(\tau)\right\|_{\widetilde{X}^\infty}\mathrm{d}\tau\\&\le C\int_0^t(1+\tau)^{-(p-1)/2}\,\mathrm{d}\tau\cdot r(t)^p\le Cr(t)^p,\end{aligned}$$

where the integral is bounded w.r.t. t since $p>3$.

Ab) Next we estimate

$$\begin{aligned}(1+t)^{1/2}\left\|\int_0^t e^{(t-\tau)\widetilde{L}_c}\widetilde{P}_c\widetilde{N}\left(\widetilde{v}_c,\widetilde{v}_s\right)(\tau)\,\mathrm{d}\tau\right\|_{\widetilde{X}^1}&\le(1+t)^{1/2}\int_0^t\left\|e^{(t-\tau)\widetilde{L}_c}\right\|_{\widetilde{X}^\infty\to\widetilde{X}^1}\left\|\widetilde{N}\left(\widetilde{v}_c,\widetilde{v}_s\right)(\tau)\right\|_{\widetilde{X}^\infty}\mathrm{d}\tau\\&\le C(1+t)^{1/2}\int_0^t(t-\tau)^{-1/2}(1+\tau)^{(1-p)/2}\,\mathrm{d}\tau\cdot r(t)^p\\&\le C(1+t)^{1/2}\int_0^{t/2}(t/2)^{-1/2}(1+\tau)^{(1-p)/2}\,\mathrm{d}\tau\cdot r(t)^p\\&\quad+C(1+t)^{1/2}\int_{t/2}^t(t-\tau)^{-1/2}(1+t/2)^{(1-p)/2}\,\mathrm{d}\tau\cdot r(t)^p\le Cr(t)^p.\end{aligned}$$

B) For the linearly exponentially damped part we estimate

$$\begin{aligned}(1+t)^{3/4}\left\|\int_0^t e^{(t-\tau)\widetilde{L}_s}\widetilde{P}_s\widetilde{N}(\widetilde{v}_c,\widetilde{v}_s)(\tau)\,\mathrm{d}\tau\right\|_{\widetilde{\mathcal{X}}^2} &\le (1+t)^{3/4}\int_0^t \left\|e^{(t-\tau)\widetilde{L}_s}\right\|_{\widetilde{\mathcal{X}}^2\to\widetilde{\mathcal{X}}^2}\left\|\widetilde{N}(\widetilde{v}_c,\widetilde{v}_s)(\tau)\right\|_{\widetilde{\mathcal{X}}^2}\mathrm{d}\tau \\ &\le C(1+t)^{3/4}\int_0^t e^{-\sigma_s(t-\tau)}(1+\tau)^{(1-p)/2-1/4}\,\mathrm{d}\tau\cdot r(t)^p \\ &\le Cr(t)^p\end{aligned}$$

due to the uniform boundedness of

$$\begin{aligned}&(1+t)^{3/4}\int_0^t e^{-\sigma_s(t-\tau)}(1+\tau)^{(1-p)/2-1/4}\,\mathrm{d}\tau \\ &\le (1+t)^{3/4}\int_0^{t/2} e^{-\sigma_s t/2}(1+\tau)^{(1-p)/2-1/4}\,\mathrm{d}\tau + (1+t)^{3/4}\int_{t/2}^t e^{-\sigma_s(t-\tau)}(1+t/2)^{(1-p)/2-1/4}\,\mathrm{d}\tau.\end{aligned}$$

Remark 6.6. In the last estimate the pre-factor $(1+t)^{3/4}$ can be replaced by the pre-factor $(1+t)^{(p-1)/2+1/4}$ without destroying the uniform boundedness w.r.t. t. Therefore, in principle for our problem the decay for $\widetilde{v}_s(\cdot,t)$ can be improved to

$$\|\widetilde{v}_s(\cdot,t)\|_{\widetilde{\mathcal{X}}^2} \le C(1+t)^{(1-p)/2-1/4}.$$

We used the slower decay $(1+t)^{-3/4}$ to show that the result is true even if quadratic terms, i.e. $p=2$, are present in the linearly exponentially damped part. Hence, the approach used in the proof of Theorem 6.7 is an alternative to the approach given in [18, Chapter 14] and thus allows to redo the stability proofs for spatially periodic equilibria in a number of pattern forming systems in a slightly different manner.

C) Summing up all estimates yields an inequality

$$r(t) \le Cr(0) + Cr(t)^p.$$

Comparing the curves $r \mapsto r$ and $r \mapsto C\delta + Cr^p$ for $\delta > 0$ small, it is easy to see that r cannot go beyond $2C\delta$. Hence, if $r(0) < \delta$, with $\delta > 0$ sufficiently small, we have the existence of a $C > 0$ such that $R(t) \le 2C\delta$ for all $t \ge 0$.
Therefore, we have established:

Theorem 6.7. *Consider* (5.1)–(5.2) *with* $p > 3$. *Then for all* $C > 0$ *there exists a* $\delta > 0$ *such that*

$$\|\widetilde{v}_c|_{t=0}\|_{\widetilde{\mathcal{X}}^1} + \|\widetilde{v}_c|_{t=0}\|_{\widetilde{\mathcal{X}}^\infty} + \|\widetilde{v}_s|_{t=0}\|_{\widetilde{\mathcal{X}}^2} \le \delta$$

implies

$$\|\widetilde{v}_c(\cdot,t)\|_{\widetilde{\mathcal{X}}^\infty} \le C, \qquad \|\widetilde{v}_c(\cdot,t)\|_{\widetilde{\mathcal{X}}^1} \le C(1+t)^{-1/2} \qquad \textit{and} \qquad \|\widetilde{v}_s(\cdot,t)\|_{\widetilde{\mathcal{X}}^2} \le C(1+t)^{-3/4}$$

for all $t \ge 0$.

A direct consequence of Theorem 6.7 is

$$\sup_{x\in\Gamma}|u(x,t)| \le C(1+t)^{-1/2},$$

since $\widetilde{u} \in \widetilde{\mathcal{X}}^1$ is mapped continuously to $u \in L^\infty(\Gamma)$, and $\widetilde{u}_s \in \widetilde{\mathcal{X}}^2$ implies $u_s \in \mathcal{X}^2$, due to [7, Lemma 4.2], which again is mapped continuously to $u \in L^\infty(\Gamma)$ due to Sobolev's embedding theorem.

Remark 6.8. As far as we can see, we cannot solely work in $\widetilde{\mathcal{X}}^1$ and $\widetilde{\mathcal{X}}^\infty$. The space $\widetilde{\mathcal{X}}^2$ is used in Lemma 5.1. Establishing this lemma in $\widetilde{\mathcal{X}}^1$ and $\widetilde{\mathcal{X}}^\infty$ would require additional knowledge about the eigenfunctions $\varphi_k(\ell,\cdot)$.

Remark 6.9. We expect that for (1.1) the decay rates for $\widetilde{v}_s(\cdot,t)$ can be improved further, at least by an additional factor $t^{-1/2}$ due to the fact that $\varphi_1(0,x)=1$. Therefore, $\varphi_1(0,x)^p = 1$ and the projection on the $\widetilde{v}_s$-part vanishes at $\ell = 0$. This vanishing corresponds to a derivative which gives a factor $t^{-1/2}$ in case of diffusive behavior, cf. [18, Chapter 14].

Remark 6.10. In the critical case $p = 3$ stability and instability depend on the sign of the nonlinearity. In the unstable case $+u^3$ we have decay on an exponentially long time scale $\mathcal{O}(e^{-1/\delta})$ with δ the size of the initial condition.

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APPENDIX A: LOCAL EXISTENCE AND UNIQUENESS

For completeness we show the local existence and uniqueness of solutions of (2.3). In order to do so we consider the operator $L = \partial_x^2$ in the space

$$\mathcal{L}^2 = \left\{ U = (u_0, u_+, u_-) \in \left(L^2(\mathbb{R})\right)^3 : \quad \operatorname{supp}\left(u_{n,j}\right) = I_{n,j}, \quad n \in \mathbb{Z}, \quad j \in \{0, +, -\} \right\}$$

with the domain of definition

$$\mathcal{H}^2 := \left\{ U \in \mathcal{L}^2 : \quad u_{n,j} \in H^2\left(I_{n,j}\right), \quad n \in \mathbb{Z}, \quad j \in \{0,+,-\}, \quad (2.4)-(2.5) \text{ are satisfied} \right\},$$

equipped with the norm

$$\|U\|_{\mathcal{H}^2} := \left(\sum_{n\in\mathbb{Z}} \|u_{n,0}\|^2_{H^2(I_{n,0})} + \|u_{n,+}\|^2_{H^2(I_{n,+})} + \|u_{n,-}\|^2_{H^2(I_{n,-})} \right)^{1/2}.$$

We recall [7, Lemma 3.1] and [7, Lemma 3.2].

Lemma A.1. *The space $\mathcal{H}^2$ is closed under pointwise multiplication.*

Lemma A.2. *The operator $-L$ with the domain $\mathcal{H}^2$ is self-adjoint and positive semi-definite in $\mathcal{L}^2$.*

As a direct consequence of the last lemma, abstract semigroup theory [14] implies:

Corollary A.3. *The sectorial operator L is the generator of an analytic semigroup $\left(e^{-Lt}\right)_{t\geq 0}$. In particular there exists a constant C_L such that*

$$\left\| e^{Lt} \right\|_{\mathcal{H}^2 \to \mathcal{H}^2} \leq C_L.$$

We are now ready to prove the local existence and uniqueness of solutions of the initial value problem associated with (2.3) in $\mathcal{H}^2$.

Theorem A.4. *For every $U_0 \in \mathcal{H}^2$, there exists a $T_0 = T_0\left(\|U_0\|_{\mathcal{H}^2}\right) > 0$ and a unique mild solution $U \in C\left([0,T_0],\mathcal{H}^2\right)$ of* (2.3) *with $U|_{t=0} = U_0$.*

Proof. The estimates from Lemma A.1 and Corollary A.3 allow us to proceed with the general theory for semilinear dynamical systems [14]. We rewrite the initial value problem associated with (2.3) as the integral equation

$$U(t,\cdot) = e^{Lt}U(0,\cdot) + \int_0^t e^{L(t-\tau)}U(\tau,\cdot)^p \, d\tau. \tag{A.1}$$

We search for solutions in the space

$$\mathcal{M} := \left\{ U \in C\left([0,T_0],\mathcal{H}^2\right) : \sup_{t\in[0,T_0]} \|U(t,\cdot)\|_{\mathcal{H}^2} \leq 2C_L\|U(0,\cdot)\|_{\mathcal{H}^2} \right\},$$

where the constant C_L is from Corollary A.3. For every $U_0 \in \mathcal{H}^2$, there is a sufficiently small $T_0 = T_0\left(\|U_0\|_{\mathcal{H}^2}\right) > 0$ such that the right-hand side of (A.1) is a contraction in the space $\mathcal{M}$. Therefore, the existence of a unique mild solution $U \in C\left([0,T_0],\mathcal{H}^2\right)$ follows from Banach's fixed-point theorem. □