Spatial populations with seed-bank

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APPENDIX B

Appendix Part II

§B.1 Computation of scaling coefficients

In Appendices B.1.1–B.1.2 we spell out a technical computation for the tail of the wake-up time defined in (4.40)–(4.41) in the two parameter regimes given by (4.52)–(4.53). In Appendix B.1.3 we carry out a computation that is needed in Section 5.1.

§B.1.1 Regularly varying coefficients

In (4.40), note that for large t in the sum over m only small values of e_m/N^m contribute, which means large m. Hence, by the Euler-MacLaurin approximation formula, we have

$$P(\tau > t) = \frac{1}{\chi} \sum_{m\in\mathbb{N}_0} K_m \frac{e_m}{N^m}\, \mathrm{e}^{-(e_m/N^m)t} \sim \frac{1}{\chi} \int_c^\infty \mathrm{d}m\, K_m \frac{e_m}{N^m}\, \mathrm{e}^{-(e_m/N^m)t}, \tag{B.1}$$

where c is a constant that identifies from which value of m onward terms contribute significantly. Make the change of variable $s = \frac{e_m}{N^m}$. Since $e_m \sim Bm^{-\beta}$ and $K_m \sim Am^{-\alpha}$ as $m \to \infty$, we have

$$s \sim Bm^{-\beta} N^{-m} \tag{B.2}$$

and hence

$$\begin{aligned} &\log s \sim \log B - \beta \log m - m \log N,\\ &\log \frac{1}{s} = m \log N \left(-\frac{B}{m \log N} + \frac{\beta \log m}{m \log N} + 1 \right) = [1 + o(1)]\, m \log N, \end{aligned} \tag{B.3}$$

which gives

$$m = [1 + o(1)]\, \frac{\log(\frac{1}{s})}{\log N}. \tag{B.4}$$

Thus,

$$\frac{1}{s} \frac{\mathrm{d}s}{\mathrm{d}m} = -\log N - \frac{\beta}{m} = -[1 + o(1)]\, \log N, \tag{B.5}$$

which implies

$$\frac{\mathrm{d}s}{\mathrm{d}m} = -[1 + o(1)]\, s \log N, \tag{B.6}$$

so that $s(m)$ is asymptotically decreasing in m, and

$$\frac{\mathrm{d}m}{\mathrm{d}s} = -[1+o(1)]\,(s\log N)^{-1}. \tag{B.7}$$

Note that if $c \leq m < \infty$, then asymptotically $0 < m^{-\beta}N^{-m} < c^{-\beta}N^{-c} = C_2$. Doing the substitution, we get

$$\begin{aligned}
\mathbb{P}(\tau > t) &\sim \frac{1}{\chi}\int_0^{C_2} \mathrm{d}s\, K_m s\,(s\log N)^{-1}\,\mathrm{e}^{-st} \\
&\sim \frac{1}{\chi}\int_0^{C_2} \mathrm{d}s\, Am^{-\alpha}\,(\log N)^{-1}\,\mathrm{e}^{-st} \\
&\sim \frac{1}{\chi}\int_0^{C_2} \mathrm{d}s\, A\left(\frac{\log(\frac{1}{s})}{\log N}\right)^{-\alpha}(\log N)^{-1}\,\mathrm{e}^{-st} \\
&\sim \frac{A}{\chi}\left(\frac{1}{\log N}\right)^{-\alpha+1}\int_0^{C_2} \mathrm{d}s\,\log\left(\tfrac{1}{s}\right)^{-\alpha}\,\mathrm{e}^{-st}.
\end{aligned} \tag{B.8}$$

Next, put $st = u$, so $s = \frac{u}{t}$ and $\frac{\mathrm{d}s}{\mathrm{d}u} = \frac{1}{t}$ and $0 < u < tC_2$. Then

$$\mathbb{P}(\tau > t) \sim \frac{A}{\chi}\left(\frac{1}{\log N}\right)^{-\alpha+1}\frac{1}{t}\int_0^{C_2 t} \mathrm{d}u\,\log\left(\tfrac{t}{u}\right)^{-\alpha}\,\mathrm{e}^{-u}. \tag{B.9}$$

We will show that

$$\frac{A}{\chi}\left(\frac{1}{\log N}\right)^{-\alpha+1}\frac{1}{t}\int_0^{C_2 t} \mathrm{d}u\,\log\left(\tfrac{t}{u}\right)^{-\alpha}\,\mathrm{e}^{-u} \asymp \frac{A}{\chi}\left(\frac{1}{\log N}\right)^{-\alpha+1}\frac{1}{t}\int_0^{C_2 t} \mathrm{d}u\,\log t^{-\alpha}\,\mathrm{e}^{-u}. \tag{B.10}$$

For $\alpha = 0$ this claim is immediate. For $\alpha \in (-\infty, 0)$, note that $\log\left(\frac{t}{u}\right)^{-\alpha}$ is a decreasing function on $(0, C_2 t)$. Therefore we can reason as follows:

$$\begin{aligned}
&\int_0^{C_2 t} \mathrm{d}u\,\log\left(\tfrac{t}{u}\right)^{-\alpha}\,\mathrm{e}^{-u} \\
&= \int_0^{1} \mathrm{d}u\,\log\left(\tfrac{t}{u}\right)^{-\alpha}\,\mathrm{e}^{-u} + \int_1^{C_2 t} \mathrm{d}u\,\log\left(\tfrac{t}{u}\right)^{-\alpha}\,\mathrm{e}^{-u} \\
&\leq \int_0^{1} \mathrm{d}u\,\log\left(\tfrac{t}{u}\right)^{-\alpha} + \int_1^{C_2 t} \mathrm{d}u\,\log t^{-\alpha}\mathrm{e}^{-u} \\
&\leq 2^{-\alpha}\int_0^{\frac{1}{t}} \mathrm{d}u\,\log\left(\tfrac{1}{u}\right)^{-\alpha} + 2^{-\alpha}\int_{\frac{1}{t}}^{1} \mathrm{d}u\,\log t^{-\alpha} + \log t^{-\alpha}\left[1-\mathrm{e}^{-1}\right] \\
&\leq 2^{-\alpha}\Gamma(-\alpha+1) + 2^{-\alpha}\log t^{-\alpha}\left[1-\tfrac{1}{t}\right] + \log t^{-\alpha}\left[1-\mathrm{e}^{-1}\right] \\
&= \log t^{-\alpha}\left[2^{-\alpha}\tfrac{\Gamma(-\alpha+1)}{\log t^{-\alpha}} + 2^{-\alpha}\left[1-\tfrac{1}{t}\right] + \left[1-\mathrm{e}^{-1}\right]\right] \\
&\asymp \log t^{-\alpha}.
\end{aligned} \tag{B.11}$$

For the lower bound, note that

$$\begin{aligned}
\int_0^{C_2 t} \mathrm{d}u\,\log\left(\tfrac{t}{u}\right)^{-\alpha}\mathrm{e}^{-u} &\geq \log(t)^{-\alpha}\int_0^1 \mathrm{d}u\,\mathrm{e}^{-u} + \log(\tfrac{1}{C_2})^{-\alpha}\int_1^{C_2 t}\mathrm{d}u\,\mathrm{e}^{-u} \\
&= \log t^{-\alpha}\left[1 - e^{-1} + \frac{\log(\frac{1}{C_2})^{-\alpha}}{\log t^{-\alpha}}\right] \asymp \log t^{-\alpha}.
\end{aligned} \tag{B.12}$$

For $\alpha \in (0,1]$, note that the function $\log\left(\frac{t}{u}\right)^{-\alpha}$ is increasing in u. For the lower bound estimate

$$\begin{aligned}\int_0^{C_2 t} \mathrm{d}u \log\left(\tfrac{t}{u}\right)^{-\alpha} \mathrm{e}^{-u} &\geq \lim_{u\to 0} \log\left(\tfrac{t}{u}\right)^{-\alpha}[1-\mathrm{e}^{-1}] + \log t^{-\alpha}[\mathrm{e}^{-1} - e^{-C_2 t}] \\ &= \log t^{-\alpha}\left[0 + \mathrm{e}^{-1} - \mathrm{e}^{-C_2 t}\right] \asymp \log t^{-\alpha}.\end{aligned} \tag{B.13}$$

For the upper bound estimate

$$\begin{aligned}&\int_0^{C_2 t} \mathrm{d}u \log\left(\tfrac{t}{u}\right)^{-\alpha} \mathrm{e}^{-u} \\ &\leq \log t^{-\alpha}[1-\mathrm{e}^{-1}] + \log\left(\frac{t}{\sqrt{C_2 t}}\right)^{-\alpha} \int_1^{\sqrt{C_2 t}} \mathrm{d}u\, e^{-u} + \log\left(\frac{1}{C_2}\right)^{-\alpha} \int_{\sqrt{C_2 t}}^{C_2 t} \mathrm{d}u\, \mathrm{e}^{-u} \\ &= \log t^{-\alpha}[1-\mathrm{e}^{-1}] + (\tfrac{1}{2})^{-\alpha} \log\left(\frac{t}{C_2}\right)^{-\alpha}\left[\mathrm{e}^{-1} - \mathrm{e}^{-\sqrt{C_2 t}}\right] \\ &\qquad + \log\left(\frac{1}{C_2}\right)^{-\alpha}\left[\mathrm{e}^{-\sqrt{C_2 t}} - \mathrm{e}^{-C_2 t}\right] \\ &= \log t^{-\alpha}\left[1 - \mathrm{e}^{-1} + (\tfrac{1}{2})^{-\alpha}\left(\frac{\log t - \log C_2}{\log t}\right)^{-\alpha}\left[\mathrm{e}^{-1} - \mathrm{e}^{-\sqrt{C_2 t}}\right]\right. \\ &\qquad\qquad \left. + \log\left(\frac{1}{C_2}\right)^{-\alpha} \frac{\left[\mathrm{e}^{-\sqrt{C_2 t}} - \mathrm{e}^{-C_2 t}\right]}{\log t^{-\alpha}}\right] \asymp \log t^{-\alpha}.\end{aligned} \tag{B.14}$$

§B.1.2 Pure exponential coefficients

In order to satisfy condition in (4.12), we must assume that $Ke < N$. Since $K \geq 1$ for $\rho = \infty$, we also have $e < N$. We again use that for large t only large m contribute to the sum. Hence, again by the Euler-MacLaurin approximation formula, we have

$$P(\tau > t) = \frac{1}{\chi} \sum_{m\in\mathbb{N}_0} K_m \frac{e_m}{N^m} \mathrm{e}^{-(e_m/N^m)t} \sim \int_M^\infty \mathrm{d}m\, K_m \frac{e_m}{N^m} \mathrm{e}^{-(e_m/N^m)t}. \tag{B.15}$$

Again we put $s = \frac{e^m}{N^m}$. Hence

$$\log s = m \log\left(\frac{e}{N}\right), \qquad m = \frac{\log s}{\log\frac{e}{N}}, \qquad \frac{\mathrm{d}m}{\mathrm{d}s} = \frac{1}{s\log\frac{e}{N}}, \tag{B.16}$$

and

$$K_m \sim K^m \sim \mathrm{e}^{\log s \frac{\log K}{\log \frac{e}{N}}} \sim s^{\frac{\log K}{\log\frac{e}{N}}}. \tag{B.17}$$

Since $s(m)$ is decreasing in m, putting $C = (\frac{e}{N})^M$ we obtain

$$\mathbb{P}(\tau > t) \sim \int_0^C \mathrm{d}s\, K_m \frac{s}{s\log\frac{e}{N}} \mathrm{e}^{-st} \sim \frac{1}{\log\frac{e}{N}} \int_0^C \mathrm{d}s\, s^{\frac{\log K}{\log\frac{e}{N}}} \mathrm{e}^{-st}. \tag{B.18}$$

Substitute $u = st$, i.e., $\frac{u}{t} = s$, to get

$$\begin{aligned}\mathbb{P}(\tau > t) &\sim \frac{1}{\log \frac{e}{N}} t^{-1-\frac{\log K}{\log \frac{e}{N}}} \int_0^{Ct} \mathrm{d}u\, u^{\frac{\log K}{\log \frac{e}{N}}} \mathrm{e}^{-u} \\ &\sim \frac{1}{\log \frac{e}{N}} t^{\frac{-\log(\frac{e}{N})-\log K}{\log \frac{e}{N}}} \int_0^{Ct} \mathrm{d}u\, u^{\frac{\log K}{\log \frac{e}{N}}} \mathrm{e}^{-u} \sim \frac{1}{\log \frac{e}{N}} t^{-\frac{\log(\frac{N}{Ke})}{\log \frac{N}{e}}} \int_0^{Ct} \mathrm{d}u\, u^{\frac{\log K}{\log \frac{e}{N}}} \mathrm{e}^{-u}.\end{aligned} \tag{B.19}$$

The last integral converges because $\frac{\log K}{\log(\frac{e}{N})} > -1$, and

$$\int_0^{Ct} \mathrm{d}u\, u^{\frac{\log K}{\log \frac{e}{N}}} \mathrm{e}^{-u} \leq \int_0^{\infty} \mathrm{d}u\, u^{\frac{\log K}{\log \frac{e}{N}}} \mathrm{e}^{-u} = \Gamma\left(\frac{\log K}{\log(\frac{e}{N})} + 1\right). \tag{B.20}$$

§B.1.3 Slowly varying functions

Return to Section 5.1. Note that $t(s) = \varphi(s)^{-1} s^{\gamma}$. Since this is the total time two lineages are active up to time s, $t(s)$ must be smaller than s. By (4.49), we have

$$\frac{\varphi(t)}{\varphi(s)} = \exp\left[-\int_{t(s)}^{s} \frac{\mathrm{d}u}{u} \psi(u)\right]. \tag{B.21}$$

Since we are interested in $s \to \infty$, we may assume that $s \gg 1$ and $t(s) > 1$, and estimate

$$\begin{aligned}\frac{\varphi(t)}{\varphi(s)} &\leq \exp\left[\int_{t(s)}^{s} \frac{\mathrm{d}u}{u} \frac{C}{\log u}\right] = \exp\left[C(\log\log s - \log\log t(s))\right] \\ &= \exp\left[C \log\left(\frac{\log s}{\log\left(\varphi(s)^{-1} s^{\gamma}\right)}\right)\right] = \exp\left[-C \log\left(\frac{\gamma \log s - \log \varphi(s)}{\log s}\right)\right].\end{aligned} \tag{B.22}$$

A similar lower bound holds with the sign reversed. Using that $\lim_{s\to\infty} \frac{\log \varphi(s)}{\log s} = 0$, we get

$$\gamma^{C} \leq \liminf_{s\to\infty} \frac{\varphi(t)}{\varphi(s)} \leq \limsup_{s\to\infty} \frac{\varphi(t)}{\varphi(s)} \leq \gamma^{-C}. \tag{B.23}$$

Both bounds above are positive, so indeed $\frac{\varphi(t)}{\varphi(s)} \asymp 1$.

§B.2 Meyer-Zheng topology

§B.2.1 Basic facts about the Meyer-Zheng topology

In the Meyer-Zheng topology we assign to each real-valued Borel measurable function $(w(t))_{t\geq 0}$ a probability law on $[0,\infty] \times \bar{\mathbb{R}}$ that is called the pseudopath ψ_w. Note that the Borel-σ algebra on $[0,\infty] \times \bar{\mathbb{R}}$ is generated by sets of the form $[a,b] \times B$ for $B \in \mathcal{B}$ and $0 < a < b$. For $A = [a,b] \times B$, set

$$\psi_w(A) = \int 1_A(t, w(t)) \mathrm{e}^{-t} \mathrm{d}t = \int_a^b 1_B(w(t)) e^{-t} \mathrm{d}t, \tag{B.24}$$

i.e., ψ_w is the image measure of the mapping $t \to (t, w(t))$ under the measure $\lambda(\mathrm{d}t) = \mathrm{e}^{-t}\mathrm{d}t$. The set of all pseudopaths is denoted by Ψ. Note that a pseudopath corresponding to $(w(t))_{t>0}$ is simply its occupation measure. The following important facts are stated in [59]:

- If two paths w_1 and w_2 are the same Lebesgue a.e., then $\psi_{w_1} = \psi_{w_2}$.
- Denote by $\mathbf{D}$ the space of càdlàg paths on $[0, \infty] \times \mathbb{R}$. The mapping $\psi\colon \mathbf{D} \to \Psi$, $w \mapsto \psi_w$ is one-to-one on $\mathbf{D}$ and hence gives an embedding of $\mathbf{D}$ into the compact space $\bar{\mathcal{P}}$, the space of probability measures on $[0, \infty] \times \bar{\mathbb{R}}$.
- Note if f is a function on $[0, \infty] \times \mathbb{R}$ and $w \in \mathbf{D}$, then

  $$\psi_w(f) = \int_0^\infty f(t, w(t))\, \mathrm{e}^{-t}\mathrm{d}t. \tag{B.25}$$

  Therefore we say that the sequence of pseudopaths induced by $(w_n) \subset \mathbf{D}$ converges to a pseudopath w if, for all continuous bounded function $f(t, w(t))$ on $[0, \infty] \times \bar{\mathbb{R}}$,

  $$\lim_{n\to\infty} \int_0^\infty f(t, w_n(t))\, \mathrm{e}^{-t}\mathrm{d}t = \int_0^\infty f(t, w(t))\, \mathrm{e}^{-t}\mathrm{d}t. \tag{B.26}$$

  Since a pseudopath is a measure, convergence of pseudopaths is convergence of measures.
- $\mathbf{D}$ endowed with the pseudopath topology is *not* a Polish space. Ψ endowed with the pseudopath topology is a Polish space.
- According to [59][Lemma 1], the pseudopath topology on Ψ is convergence in Lebesgue measure on $\mathbf{D}$.

§B.2.2 Pseudopaths of stochastic processes on a general metric separable space

In [53] the results of [59] on state space $\mathbb{R}$ are generalised to a general metric separable space E. Let $(Z(t))_{t>0}$ be a stochastic process with state space E. Then we assign a random pseudopath to $(Z(t))$ as follows: for $\omega \in \Omega$ and $A = [a, b] \times B$, $0 \leq a < b$ and $B \in \mathcal{B}(E)$,

$$\psi_{(Z(t,\omega))_{t\geq 0}}(A) = \int_a^b 1_B(Z(t,\omega))\, \mathrm{e}^{-t}\mathrm{d}t. \tag{B.27}$$

Hence $\psi_{(Z(t))_{t\geq 0}}$ is a random variable with state space Ψ, i.e., $\psi_{(Z(t))_{t\geq 0}} \in \mathcal{M}(\Psi)$, the set of probability measures on pseudopaths. Note that

$$\mathbb{E}\left[\psi_{(Z(t))_{t\geq 0}} f\right] = \mathbb{E}\left[\int_0^\infty f(t, Z(t,\omega))\, \mathrm{e}^{-t}\mathrm{d}t\right] = \mathbb{E}\left[\int_0^\infty f(t, Z(t))\, \mathrm{e}^{-t}\mathrm{d}t\right]. \tag{B.28}$$

Appendix B

Weak convergence in the Meyer-Zheng topology. Let $(Z_n(t))_{t\geq 0}$ and $(Z(t))_{t\geq 0}$ be stochastic processes with state-space E. We say that

$$\mathcal{L}\left[(Z_n(t))_{t\geq 0}\right] = \mathcal{L}\left[(Z(t))_{t\geq 0}\right] \text{ in the Meyer-Zheng topology} \tag{B.29}$$

if, for all $f \in \mathcal{C}_b(\Psi)$,

$$\lim_{n\to\infty} \mathbb{E}[f(\psi_{(Z^n(t))_{t\geq 0}})] = \mathbb{E}[f(\psi_{(Z(t))_{t\geq 0}})]. \tag{B.30}$$

Let $\mathcal{C}_m([0,\infty)\times E) \subset \mathcal{C}_b([0,\infty)\times E)$ be the set of functions of the form

$$\begin{aligned}\mathcal{C}_m([0,\infty)\times E) = \Big\{ & F \in \mathcal{C}_b([0,\infty)\times E) : F(t,x(t)) = \prod_{i=1}^m \int_0^{T_i} f_i(t,x(t))\mathrm{d}t, \\ & m\in\mathbb{N},\ \forall 1\leq i\leq m,\ f_i\in\mathcal{C}_b([0,\infty)\times E),\ T_i>0\Big\}.\end{aligned} \tag{B.31}$$

Note that $\mathcal{C}_m$ is an algebra. Let $M_E[0,\infty)$ be the space of measurable processes from $[0,\infty)$ to $\mathbb{E}$, so $\mathbf{D} \subset M_E[0,\infty)$. Note that $\mathcal{C}_m$ separates points in $M_E[0,\infty)$. By [53][Proposition 4.5], the set $\mathcal{C}_m$ is separating in the set of measures on $M_E[0,\infty)$. This means that if two stochastic processes $(Z_1(t))_{t>0}$ and $(Z_2(t))_{t\geq 0}$ satisfy

$$\mathbb{E}[F(Z_1)] = \mathbb{E}[F(Z_2)] \qquad \forall F \in \mathcal{C}_m, \tag{B.32}$$

then $\mathcal{L}[Z_1] = \mathcal{L}[Z_2]$.

Define

$$F(\psi) = \int \mathrm{d}\psi \prod_{i=1}^m \int_0^{T_i} f_i(t,x(t))\,\mathrm{d}t. \tag{B.33}$$

Recall that a pseudopath ψ is associated with a path $w \in M_E[0,\infty)$. Hence this becomes

$$F(\psi_w) = \prod_{i=1}^m \int_0^{T_i} f_i(t,w(t))\,\mathrm{d}t. \tag{B.34}$$

Since each pseudopath $\psi \in \Psi$ is associated with a path in $M_E[0\infty)$, $\mathcal{C}_m$ also separates points on Ψ and hence $\mathcal{C}_m$ separates measures on Ψ. This implies that if

$$\mathbb{E}[F(\psi_{Z_1})] = \mathbb{E}[F(\psi_{Z_2})] \qquad \forall F \in \mathcal{C}_m, \tag{B.35}$$

then $\mathcal{L}[\psi_{Z_1}] = \mathcal{L}[\psi_{Z_2}]$. Therefore $\mathcal{L}[Z_1] = \mathcal{L}[Z_2]$ if and only if $\mathcal{L}[\psi_{Z_1}] = \mathcal{L}[\psi_{Z_2}]$.

The Meyer-Zheng topology is a weaker than the Skohorod topology.

Lemma B.2.1. *Let $(Z_n(t))_{t\geq 0}$ $n \in \mathbb{N}$ and $(Z(t))_{t\geq 0}$ be stochastic processes with Polish state-space E. If*

$$\lim_{n\to\infty} \mathcal{L}\left[(Z_n(t))_{t\geq 0}\right] = \mathcal{L}\left[(Z(t))_{t\geq 0}\right] \text{ in the Skohorod topology,} \tag{B.36}$$

then

$$\lim_{n\to\infty} \mathcal{L}\left[(Z_n(t))_{t\geq 0}\right] = \mathcal{L}\left[(Z(t))_{t\geq 0}\right] \text{ in the Meyer-Zheng topology.} \tag{B.37}$$

Proof. Since we do not know whether Ψ is compact, the set $\mathcal{C}_m$ does not have to be convergence determining. Therefore, via Skorohod's theorem we construct the process $\tilde{Z}^n$ and $\tilde{Z}$ on one probability space, such that $\mathcal{L}[\tilde{Z}^n] = \mathcal{L}[Z^n]$ and $\mathcal{L}[\tilde{Z}] = \mathcal{L}[Z]$, and

$$\lim_{n\to\infty} \tilde{Z}_n = \tilde{Z} \qquad a.s. \tag{B.38}$$

This implies

$$\lim_{n\to\infty} \psi_{\tilde{Z}^n} = \psi_{\tilde{Z}} \qquad a.s. \tag{B.39}$$

Consequently, for all $f \in \mathcal{C}_b(\Psi)$,

$$\lim_{n\to\infty} \mathbb{E}[f(\psi_{\tilde{Z}^n})] = \mathbb{E}[f(\psi_{\tilde{Z}})]. \tag{B.40}$$

Note that, since $\mathcal{L}[\tilde{Z}^n] = \mathcal{L}[Z^n]$ and $\mathcal{L}[\tilde{Z}] = \mathcal{L}[Z]$, we can use (B.32) and (B.35) to see that the latter implies $\mathcal{L}[\psi_{Z^n}] = \mathcal{L}[\psi_{\tilde{Z}^n}]$ and $\mathcal{L}[\psi_Z] = \mathcal{L}[\psi_{\tilde{Z}}]$. Hence (B.40) indeed implies that

$$\lim_{n\to\infty} \mathcal{L}[\psi_{Z^n}] = \mathcal{L}[\psi_Z]. \tag{B.41}$$

□

Convergence in probability in the Meyer-Zheng topology. Let (S, d) be a metric space, $\mathcal{B}(S)$ denote the Borel-σ algebra on S, and $\mathcal{P}(S)$ the set of probability measures on $\mathcal{B}(S)$. Recall (see e.g. [32, Chapter 3]) that the Prohorov metric d_P on the space $\mathcal{P}(S)$ is given by

$$d_P(\mathbb{P}, \mathbb{Q}) = \inf\left\{\epsilon > 0\colon\ \mathbb{P}(A) \leq \mathbb{Q}(A^\epsilon) + \epsilon\ \forall\, A \in \mathcal{C}\right\}, \tag{B.42}$$

where $\mathcal{C} \subset \mathcal{B}(S)$ is the set of all closed sets in S and $A^\epsilon = \{x \in S\colon\ \inf_{y\in A} d(x, y) < \epsilon\}$.

Recall the following theorem (see e.g.[[32, Theorem 3.1.2]])

Theorem B.2.2. *Let (S, d) be separable and let $\mathbb{P}, \mathbb{Q} \in \mathcal{P}(S)$. Define $\mathcal{M}(\mathbb{P}, \mathbb{Q})$ to be the set of all $\mu \in \mathcal{P}(S \times S)$ with marginals $\mathbb{P}$ and $\mathbb{Q}$, i.e., $\mu(A \times S) = \mathbb{P}(A)$ and $\mu(S \times A) = \mathbb{Q}(A)$ for all $A \in \mathcal{B}(S)$. Then*

$$d_P(\mathbb{P}, \mathbb{Q}) = \inf_{\mu\in\mathcal{M}(\mathbb{P},\mathbb{Q})} \inf\{\epsilon > 0\colon\ \mu(\{(x, y)\colon\ d(x, y) \geq \epsilon\}) \leq \epsilon\}. \tag{B.43}$$

Moreover, [32, Theorem 3.3.1] states that convergence of measures in the Prohorov distance, $\lim_{n\to\infty} d_P(\mathbb{P}_n, \mathbb{P}) = 0$, is the same as weak convergence $\mathbb{P}_n \Rightarrow \mathbb{P}$. Hence, since convergence of pseudopaths is weak convergence, we can endow the space of pseudopaths Ψ with the metric d_P.

Let (Ψ, d_P) be the pseudopath space metrized by the Prohorov distance. Let $(Z^n(t))_{t>0}$, $(Z(t))_{t>0}$ be stochastic processes on the state space E, where E is endowed with the metric $d(\cdot,\cdot)$. Note that convergence in probability in the Meyer-Zheng topology means that

$$\forall\, \delta > 0\colon \qquad \lim_{n\to\infty} \mathbb{P}\left[d_P\left(\psi_{Z^n}, \psi_Z\right) > \delta\right] = 0. \tag{B.44}$$

Tightness. Define the *conditional variation* for an $\mathbb{R}$-valued process $(U(t))_{t\geq 0}$ with natural filtration $(\mathcal{F}(t))_{t\geq 0}$ as follows. For a subdivision
$\tau\colon\ 0 = t_0 < t_1 < \cdots < t_n = \infty$, set

$$V_\tau(U) = \sum_{0\leq i<n} \mathbb{E}\Big[\big|\mathbb{E}[U(t_{i+1}) - U(t_i) \mid F(t_i)]\big|\Big] \tag{B.45}$$

(with $U(\infty) = 0$) and

$$V(U) = \sup_\tau V_\tau(U). \tag{B.46}$$

If $V(U) < \infty$, then U is called a *quasi-martingale.* Note that we can always stop the process at some finite time and work with compact time intervals.

Lemma B.2.3 (Tightness in the Meyer-Zheng topology).
If $(P_n)_{n\in\mathbb{N}}$ is a sequence of probability laws on $D([0,T],\mathbb{R})$ such that under P_n the coordinate process $(U(t))_{t\geq 0}$ is a quasi-martingale with a conditional variation $V_n(U)$ that is bounded uniformly in n, then there exists a subsequence $(P_{n_k})_{k\in\mathbb{N}}$ that converges weakly in the Meyer-Zheng topology on $D([0,T],\mathbb{R})$ to a probability law P, and $(U(t))_{t\geq 0}$ is a quasi-martingale under P.

(See [59, Theorem 7] for the identification of the limiting semi-martingale.)

§B.2.3 Proof of key lemmas

- **Proof of Lemma 6.2.19.**

Proof. Fix $\delta > 0$. Then

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}\left[d_P(\psi_{Z_n},\psi_Z) > \delta\right]\\
&= \lim_{n\to\infty} \mathbb{P}\left[\inf_{\mu\in\mathcal{M}(\psi_{Z_n},\psi_Z)} \inf\{\epsilon > 0\colon\ \mu(\{(x,y)\colon\ d(x,y)\geq\epsilon\}) \leq \epsilon\} > \delta\right]\\
&= \lim_{n\to\infty} \mathbb{P}\left[\forall\mu\in\mathcal{M}(\psi_{Z_n},\psi_Z),\ \inf\{\epsilon > 0\colon\ \mu(\{(x,y)\colon\ d(x,y)\geq\epsilon\}) \leq \epsilon\} > \delta\right]\\
&= \lim_{n\to\infty} \mathbb{P}\left[\forall\mu\in\mathcal{M}(\psi_{Z_n},\psi_Z),\ \mu(\{(x,y)\colon\ d(x,y)\geq\delta\}) > \delta\right].
\end{aligned} \tag{B.47}$$

Let $\mu_n \in \mathcal{P}(([0,\infty]\times E)^2)$ be the measure defined by

$$\mu_n(A) = \int_0^\infty 1_A\left((t,Z_n(t)),(t,Z(t))\right) \mathrm{e}^{-t}\mathrm{d}t, \qquad A\in\mathcal{B}(([0,\infty]\times E)^2), \tag{B.48}$$

such that, for $B\in\mathcal{B}([0,\infty]\times E)$,

$$\mu_n(B\times S) = \int_0^\infty 1_B(t,Z_n(t))1_S((t,Z(t))\,\mathrm{e}^{-t}\mathrm{d}t = \psi_{Z^n}(B), \tag{B.49}$$

and similarly $\mu_n(S\times B) = \psi_Z(B)$. Hence $\mu_n\in\mathcal{M}(\psi_{Z_n},\psi_Z)$ for all $n\in\mathbb{N}$, and we obtain from (B.47) that

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}\left[d_P(\psi_{Z_n},\psi_Z) > \delta\right] \\
&\leq \lim_{n\to\infty} \mathbb{P}\left[\mu_n(\{(x,y) : d(x,y) \geq \delta\}) > \delta\right] \\
&\leq \lim_{n\to\infty} \mathbb{P}\left[\int_0^\infty 1_{\{(x,y):d(x,y)\geq\delta\}}\left((t,Z_n(t)),(t,Z(t))\right)\mathrm{e}^{-t}\mathrm{d}t > \delta\right] \\
&\leq \lim_{n\to\infty} \mathbb{P}\left[\int_0^\infty 1_{\{d(Z_n(t),Z(t))\geq\delta\}}\,\mathrm{e}^{-t}\mathrm{d}t > \delta\right] \\
&\leq \lim_{n\to\infty} \frac{1}{\delta}\mathbb{E}\left[\int_0^\infty d(Z_n(t),Z(t))\,\mathrm{e}^{-t}\mathrm{d}t\right] \\
&= \lim_{n\to\infty} \frac{1}{\delta}\int_0^\infty \mathbb{E}\left[d(Z_n(t),Z(t))\right]\,\mathrm{e}^{-t}\mathrm{d}t = 0.
\end{aligned} \tag{B.50}$$

□

- **Proof of Lemma 6.2.20.**

Proof. We have to show that

$$\lim_{n\to\infty} \mathcal{L}\left[\psi_{(X_n,Y_n)}\right] = \mathcal{L}\left[\psi_{(X,c)}\right]. \tag{B.51}$$

Hence we must show that, for all $f \in \mathcal{C}_b(\Psi)$,

$$\lim_{n\to\infty} \mathbb{E}[f(\psi_{(X_n,Y_n)})] = \mathbb{E}[f(\psi_{(X,c)})]. \tag{B.52}$$

We can write

$$\begin{aligned}
&|\mathbb{E}[f(\psi_{(X_n,Y_n)}) - f(\psi_{(X,c)})]| \\
&\leq |\mathbb{E}[f(\psi_{(X_n,Y_n)}) - f(\psi_{(X_n,c)})]| + |\mathbb{E}[f(\psi_{(X_n,c)}) - f(\psi_{(X,c)})]|.
\end{aligned} \tag{B.53}$$

Since $\lim_{n\to\infty}\mathbb{E}[d(Y_n(t),c)] = 0$ implies $\lim_{n\to\infty}\mathbb{E}[d((X_n(t),Y_n(t)),(X_n(t),c))] = 0$, it follows from Lemma 6.2.19 that, for all $\delta > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left[d_P\left(\psi_{(X_n,Y_n)},\psi_{(X_n,c)}\right)\right] = 0. \tag{B.54}$$

Hence, for all $f \in \mathcal{C}_b(\Psi)$,

$$\lim_{n\to\infty} |\mathbb{E}[f(\psi_{(X_n,Y_n)}) - f(\psi_{(X_n,c)})]| = 0. \tag{B.55}$$

To see that the second term in the right-hand side of (B.53) tends to zero, note that we can define

$$\tilde{f}(\psi_x) = f(\psi_{x,c}). \tag{B.56}$$

We show that $\tilde{f}$ is continuous.

Recall that convergence in the Meyer-Zheng topology is simply convergence in Lebesgue measure. Hence, for two paths $(t,x_n(t))$ and $(t,x(t)) \in M_E[0\infty)$ we have $\psi_{x_n} \to \psi_x$ if and only if, for all $\delta > 0$,

$$\lim_{n\to\infty} \int_0^\infty 1_{\{d(x_n(t),x(t))>\delta\}}\,\mathrm{e}^{-t}\mathrm{d}t = 0. \tag{B.57}$$

Therefore $\psi_{x_n} \to \psi_x$ implies that, for all $\delta > 0$,

$$\lim_{n\to\infty} \int_0^\infty 1_{\{d((x_n(t),c),(x(t),c))>\delta\}}\, \mathrm{e}^{-t} \mathrm{d}t = 0, \tag{B.58}$$

and hence $\psi_{x_n,c} \to \psi_{x,c}$. Therefore

$$\lim_{n\to\infty} \tilde{f}(\psi_{x_n}) = \lim_{n\to\infty} f(\psi_{(x_n,c)}) = f(\psi_{(x,c)}) = \tilde{f}(\psi_x) \tag{B.59}$$

and we conclude that $f \in \mathcal{C}_b(\Psi)$. Since $\mathcal{L}[X_n] = \mathcal{L}[X]$ in the Meyer-Zheng topology, we have, for all $f \in \mathcal{C}_b(\Psi)$,

$$\lim_{n\to\infty} |\mathbb{E}[f(\psi_{(X_n,c)}) - f(\psi_{(X,c)})]| = \lim_{n\to\infty} |\mathbb{E}[\tilde{f}(\psi_{(X_n)}) - \tilde{f}(\psi_{(X)})]| = 0. \tag{B.60}$$

Therefore also the second term on the right-hand side of (B.53) tends to 0. □

- **Proof of Lemma 6.2.21.**

Proof. For part (a), suppose that $\lim_{n\to\infty} \psi_{x_n} = \psi_x$. Then, since convergence in pseudopath space is convergence in measure, we have, for all $\delta > 0$,

$$\lim_{n\to\infty} \int_0^\infty 1_{\{d(x_n(t),x(t))>\delta\}}\, \mathrm{e}^{-t} \mathrm{d}t = 0. \tag{B.61}$$

Since f is a continuous function, this implies that, for all $\epsilon > 0$,

$$\lim_{n\to\infty} \int_0^\infty 1_{\{d(f(x_n(t)),f(x(t)))>\epsilon\}}\, \mathrm{e}^{-t} \mathrm{d}t = 0. \tag{B.62}$$

and we conclude that $\lim_{n\to\infty} \psi_{f(x_n)} = \psi_{f(x)}$. Hence h is indeed continuous.

For part (b), recall that

$$\lim_{n\to\infty} \mathcal{L}[X_n] = \mathcal{L}[X] \text{ in the Meyer-Zheng topology} \tag{B.63}$$

implies that, for all $g \in \mathcal{C}_b(\Psi)$,

$$\lim_{n\to\infty} \mathbb{E}[g(\psi_{X_n})] = \mathbb{E}[g(\psi_X)]. \tag{B.64}$$

Since $h\colon\ \Psi \to \Psi$ is continuous, we have for all $g \in \mathcal{C}_b(\Psi)$ that $g \circ h \in \mathcal{C}_b(\Psi)$. Hence

$$\lim_{n\to\infty} \mathbb{E}[g(\psi_{f(X_n)}] = \lim_{n\to\infty} \mathbb{E}[g \circ h(\psi_{X_n})] = \mathbb{E}[g \circ h(\psi_X)] = \mathbb{E}[g(\psi_{f(X)})]. \tag{B.65}$$

We conclude that

$$\lim_{n\to\infty} \mathcal{L}[f(X_n)] = \mathcal{L}[f(X)] \text{ in the Meyer-Zheng topology.} \tag{B.66}$$

□

- **Proof of Lemma 7.2.14.**

Proof. Suppose that $\lim_{n\to\infty}\psi_{(x_n,y_n)}=\psi_{(x,y)}$. Since convergence of pseudopaths is convergence in Lebesgue measure, we have

$$\lim_{n\to\infty}\int_0^\infty 1_{\{d[(x_n,y_n),(x,y)]>\delta\}}\mathrm{e}^{-t}\mathrm{d}t=0 \tag{B.67}$$

and, consequently,

$$\lim_{n\to\infty}\int_0^\infty 1_{\{d[x_n,x]>\delta\}}\mathrm{e}^{-t}\mathrm{d}t=0. \tag{B.68}$$

Therefore $\lim_{n\to\infty}\psi_{x_n}=\psi_x$. Suppose that $f\in\mathcal{C}_b(\Psi(E))$, so f is bounded continuous function on the space of pseudopaths on $[0,\infty]\times E$. Define the function $\tilde{f}$ on the space of pseudopaths on $[0,\infty]\times E^2$, i.e., $\tilde{f}$ is a function on $\Psi(E^2)$, by

$$\tilde{f}(\psi_{(x,y)})=f(\psi_x). \tag{B.69}$$

Then $\tilde{f}\in\mathcal{C}_b(\Psi(E^2))$ and

$$\lim_{n\to\infty}\tilde{f}(\psi_{(x_n,y_n)})=\lim_{n\to\infty}f(\psi_{x_n})=f(\psi_x)=\tilde{f}(\psi_{(x_n,y_n)}). \tag{B.70}$$

Hence $\tilde{f}$ is indeed a continuous function on $\Psi(E^2)$. Moreover, since f is bounded, it follows that $\tilde{f}$ is bounded and we conclude that $\tilde{f}\in\mathcal{C}_b(\Psi(E^2))$.

Therefore, if X_n,Y_n are continuous-time stochastic processes on E and

$$\lim_{n\to\infty}\mathcal{L}\left[(X_n(s),Y_n(s))_{s>0}\right]=\mathcal{L}\left[(X(s),Y(s))_{s>0}\right]\ \text{ in Meyer Zheng topology,} \tag{B.71}$$

then for all $f\in\mathcal{C}_b(\Psi(E^2))$ we have

$$\lim_{n\to\infty}\mathbb{E}[f(\psi_{(X_n,Y_n)})]=\mathbb{E}[f(\psi_{(X,Y)})]. \tag{B.72}$$

Since for each $f\in\mathcal{C}_b(\Psi(E))$ we can construct a function $\tilde{f}\in\mathcal{C}_b(\Psi(E^2))$ as in (B.69), we obtain for all $f\in\mathcal{C}_b(\Psi(E))$ that

$$\lim_{n\to\infty}\mathbb{E}[f(\psi_{(X_n)})]=\lim_{n\to\infty}\mathbb{E}[\tilde{f}(\psi_{(X_n,Y_n)})]=\mathbb{E}[\tilde{f}(\psi_{(X,Y)})]=\mathbb{E}[\tilde{f}(\psi_X)]. \tag{B.73}$$

We conclude that

$$\lim_{n\to\infty}\mathcal{L}[(X_n(s))]=\mathcal{L}[(X(s))_{s>0}]\ \text{ in Meyer-Zheng topology} \tag{B.74}$$

and, similarly,

$$\lim_{n\to\infty}\mathcal{L}[(Y_n(s))]=\mathcal{L}[(Y(s))_{s>0}]\ \text{ in Meyer-Zheng topology.} \tag{B.75}$$

□