Spatial populations with seed-bank

Oomen, M.

CHAPTER 10

Orbit of the renormalisation transformation

In this chapter we analyse the orbit of the renormalisation transformation and show that it has the Fisher-Wright diffusion as a global attractor. In Section 10.1 we write down moment relations for the equilibrium defined in (4.67) for single colonies (Proposition 10.1.1) and for block averages (Proposition 10.1.2). In Section 10.2 we derive the iterates of these moment relations for single colonies (Proposition 10.2.1) and for blocks (Proposition 10.2.2). In Section 10.3 we prove clustering (Propositions 10.3.1–10.3.2). In Section 10.4 we prove Theorems 4.5.1 and 4.5.3, and work out the dichotomy of a finite seed-bank ($\rho < \infty$) versus infinite seed-bank ($\rho = \infty$).

§10.1 Moment relations

We use Itô-calculus to compute the mixed moments. Recall $\theta_x, (\theta_{y_m})$ as defined in (4.21), ϑ_k as defined in (4.62) and $\bar{\vartheta}^{(k)}$ (4.135). Also recall E_k as defined in (4.64). Abbreviate

$$A_0^n = \frac{1}{2}\sum_{k=0}^{n}\frac{E_k}{c_k}\frac{(E_k c_k + e_k)}{(E_k c_k + e_k) + E_k K_k e_k}, \qquad n \in \mathbb{N}, \tag{10.1}$$

and

$$B_0 = \frac{1}{2}\frac{E_0^2}{(E_0 c_0 + e_0) + E_0 K_0 e_0}. \tag{10.2}$$

In the following proposition, the first five equations are first and second moment relations, while the last equation is the definition of the renormalisation transformation. Later we will see that this set of equations can be iterated.

Proposition 10.1.1 (Moment relations: single colonies).
Let ϑ_0 be as defined in (4.62), and let $\Gamma^{(0)}_{(\vartheta_0,y_l)} = \Gamma^{g,c_0,E_0,K_0,e_0}_{(\vartheta_0,y_l)}$ be the equilibrium of (4.67) measure defined in (4.73) with $k = 0$, with $g \in \mathcal{G}$, $c_0 \in (0,\infty)$, $E_0 \in [0,1]$ and

$K_0, e_0 \in (0, \infty)$. *Then the following moment relations hold:*

$$\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} x_0\, \Gamma^{g,E_0,c_0,K_0,e_0}_{(\vartheta_0,y_l)}(\mathrm{d}z_0) = \vartheta_0, \tag{10.3}$$

$$\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} y_{0,0}\, \Gamma^{g,E_0,c_0,K_0,e_0}_{(\vartheta_0,y_l)}(\mathrm{d}z_0) = \vartheta_0, \tag{10.4}$$

$$\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} x_0 y_{0,0}\, \Gamma^{g,E_0,c_0,K_0,e_0}_{(\vartheta_0,y_l)}(\mathrm{d}z_0) = \int_{[0,1]\times[0,1]^{\mathbb{N}_0}} y_{0,0}^2\, \Gamma^{g,E_0,c_0,K_0,e_0}_{(\vartheta_0,y_l)}(\mathrm{d}z_0), \tag{10.5}$$

$$\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} x_0^2\, \Gamma^{g,E_0,c_0,K_0,e_0}_{(\vartheta_0,y_l)}(\mathrm{d}z_0) = \vartheta_0^2 + A_0^0(\mathcal{F}g)(\vartheta_0), \tag{10.6}$$

$$\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} y_{0,0}^2\, \Gamma^{g,E_0,c_0,K_0,e_0}_{(\vartheta_0,y_l)}(\mathrm{d}z_0) = \vartheta_0^2 + (A_0^0 - B_0)(\mathcal{F}g)(\vartheta_0), \tag{10.7}$$

$$\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} g(x_0)\, \Gamma^{g,E_0,c_0,K_0,e_0}_{(\vartheta_0,y_l)}(\mathrm{d}z_0) = (\mathcal{F}g)(\vartheta_0). \tag{10.8}$$

Proof. For ease of notation we write x, y_0 instead of $x_0, y_{0,0}$ for the single colonies. We use Itô's formula to compute the first and second moments, and invoke the equilibrium condition to get the above formulas, except for the last formula, which is the definition of $\mathcal{F}$ in (4.75).

1. We begin with the first moments of x and y_0. For $k = 0$, (4.67) becomes

$$\begin{aligned} \mathrm{d}x(t) &= E_0\Big[c_0\big[\vartheta_0 - x(t)\big]\,\mathrm{d}t + \sqrt{g(x(t))}\,\mathrm{d}w_0(t) \\ &\qquad\qquad + K_0 e_0\,[y_0(t) - x(t)]\,\mathrm{d}t\Big], \end{aligned} \tag{10.9}$$

$$\mathrm{d}y_0(t) = e_0\,[x(t) - y_0(t)]\,\mathrm{d}t, \tag{10.10}$$

$$\mathrm{d}y_m(t) = 0.$$

In equilibrium the distribution of $x(t)$ is constant in time, and so $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[x(t)] = 0$, where $\mathbb{E}$ denotes expectation w.r.t. $\Gamma^{g,c_0,E_0,K_0,e_0}_{\theta}$. Integrating (10.9) and taking the expectation, we get

$$\begin{aligned} \mathbb{E}[x(t) - x(0)] &= E_0\left[\mathbb{E}\left[\int_0^t \mathrm{d}s\, c_0\big[\vartheta_0 - x(s)\big] + K_0 e_0 \int_0^t \mathrm{d}s\,[y_0(s) - x(s)]\right]\right], \\ &= E_0\left[\int_0^t \mathrm{d}s\, \mathbb{E}\Big[c_0\big[\vartheta_0 - x(s)\big] + K_0 e_0\big[y_0(s) - x(s)\big]\Big]\right], \end{aligned} \tag{10.11}$$

where in the second equation we use Fubini. Turning back to differential notation, we see from (10.11) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}[x(0)] = 0 = E_0\Big\{\mathbb{E}\Big[c_0\big[\vartheta_0 - x(t)\big] + K_0 e_0\big[y_0(t) - x(t)\big]\Big]\Big\}, \tag{10.12}$$

and it follows that

$$\mathbb{E}\Big[c_0\big[\vartheta_0 - x\big] + K_0 e_0\big[y_0 - x\big]\Big] = 0. \tag{10.13}$$

In the same way it follows from (10.10) that

$$\mathbb{E}\Big[e_0\big[x - y_0\big]\Big] = 0. \tag{10.14}$$

Therefore we obtain from (10.13)–(10.14) that

$$\mathbb{E}[x] = \mathbb{E}[y_0] = \vartheta_0. \tag{10.15}$$

2. We next compute the second moments. By Itô's formula,

$$\begin{aligned} \mathrm{d}(x(t))^2 &= 2x(t)\,\mathrm{d}x(t) + (\mathrm{d}x(t))^2 \\ &= 2c_0 x(t) E_0[\vartheta_0 - x(t)]\,\mathrm{d}t + 2x(t)E_0\sqrt{g(x(t))}\,\mathrm{d}w_0(t) \\ &\quad + E_0\big[2K_0e_0\,x(t)y_0(t) - 2K_0e_0\,x_t^2\big]\,\mathrm{d}t + E_0^2 g(x(t))\,\mathrm{d}t. \end{aligned} \tag{10.16}$$

Taking expectations and using that we are in equilibrium, we get

$$0 = 2c_0\vartheta_0^2 - 2c_0\mathbb{E}[x^2] + 2K_0e_0\mathbb{E}[xy_0] - 2K_0e_0\mathbb{E}[x^2] + E_0\mathbb{E}[g(x)]. \tag{10.17}$$

Using $\mathbb{E}[g(x)] = (\mathcal{F}g)(\vartheta_0)$, we find

$$\mathbb{E}[x^2] = \frac{c_0}{(c_0+K_0e_0)}\vartheta_0^2 + \frac{K_0e_0}{(c_0+K_0e_0)}\mathbb{E}[xy_0] + \frac{E_0}{2(c_0+K_0e_0)}(\mathcal{F}g)(\vartheta_0). \tag{10.18}$$

In the same way we find

$$\mathbb{E}[y_0^2] = \mathbb{E}[xy_0], \tag{10.19}$$

and for the mixed second moment

$$\mathbb{E}[xy_0] = \frac{E_0c_0}{(E_0c_0+e_0)}\vartheta_0^2 + \frac{e_0}{(E_0c_0+e_0)}\mathbb{E}[x^2]. \tag{10.20}$$

Substituting (10.20) into (10.18), we find $\mathbb{E}[x^2]$ and hence also $\mathbb{E}[y_0^2]$ and $\mathbb{E}[x_0y_0]$. This finishes the proof of Proposition 10.1.1. □

Similar moment relations can be derived for the equilibrium measures of the block averages. Define

$$A_m^n = \frac{1}{2}\sum_{k=m}^{n} \frac{E_k}{c_k}\,\frac{(E_kc_k+e_k)}{(E_kc_k+e_k)+E_kK_ke_k}, \qquad m\in\mathbb{N}_0,\ n\in\mathbb{N}, \tag{10.21}$$

and

$$B_m = \frac{1}{2}\,\frac{E_m^2}{(E_mc_m+e_m)+E_mK_me_m}, \qquad m\in\mathbb{N}_0. \tag{10.22}$$

Recall the definition of $\mathcal{F}^{(n)}$ in (4.76).

Proposition 10.1.2 (Moment relations: blocks).
Let ϑ_m be as defined in (4.62), and let $\Gamma^{(m)}_{(\vartheta_m,y_m)} = \Gamma^{\mathcal{F}^{(m)}g,c_m,E_m,K_m,e_m}_{(\vartheta_m,y_m)}$ be the equilibrium measure of (4.67) with $k=m$, with $g\in\mathcal{G}$, $c_0\in(0,\infty)$, $E_0\in[0,1]$ and

CHAPTER 10

$K_0, e_0 \in (0,\infty)$. *Then the following moment relations hold:*

$$\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} x_m \, \Gamma^{\mathcal{F}^{(m)}g,E_m,c_m,K_m,e_m}_{(\vartheta_m,y_m)}(\mathrm{d}z_m) = \vartheta_m, \tag{10.23}$$

$$\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} y_{m,m} \, \Gamma^{\mathcal{F}^{(m)}g,E_m,c_m,K_m,e_m}_{(\vartheta_m,y_m)}(\mathrm{d}z_m) = \vartheta_m, \tag{10.24}$$

$$\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} x_m y_{m,m} \, \Gamma^{\mathcal{F}^{(m)}g,E_m,c_m,K_m,e_m}_{(\vartheta_m,y_m)}(\mathrm{d}z_m) \tag{10.25}$$

$$= \int_{[0,1]\times[0,1]^{\mathbb{N}_0}} y^2_{m,m} \Gamma^{\mathcal{F}^{(m)}g,E_m,c_m,K_m,e_m}_{(\vartheta_m,y_m)}(\mathrm{d}z_m),$$

$$\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} x^2_m \, \Gamma^{\mathcal{F}^{(m)}g,E_m,c_m,K_m,e_m}_{(\vartheta_m,y_m)}(\mathrm{d}z_m) \tag{10.26}$$

$$= \vartheta^2_m + A^m_m(\mathcal{F}^{(m+1)}g)(\vartheta_m), \tag{10.27}$$

$$\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} y^2_{m,m} \, \Gamma^{\mathcal{F}^{(m)}g,E_m,c_m,K_m,e_m}_{(\vartheta_m,y_m)}(\mathrm{d}z_m) \tag{10.28}$$

$$= \vartheta^2_m + (A^m_m - B_m)(\mathcal{F}^{(m+1)}g)(\vartheta_m),$$

$$\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} (\mathcal{F}^{(m)}g)(x_m) \, \mathrm{d}\Gamma^{\mathcal{F}^{(m)}g,E_m,c_m,K_m,e_m}_{(\vartheta_m,y_m)}(\mathrm{d}z_m) \tag{10.29}$$

$$= (\mathcal{F}^{(m+1)}g)(\vartheta_m). \tag{10.30}$$

Proof. The proof follows the same line of argument as the proof of Proposition 10.1.1. □

§10.2 Iterate moment relations

Recall the kernels defined in (4.79), the iterates of the kernels defined in (4.134) and $\bar{\vartheta}^{(n)}$. Recall that $Q^{(n)}(\bar{\vartheta}^{(n)}, \mathrm{d}z_0)$ is the probability density to see the population of a single colony in state z_0 given that the $(n+1)$-block averages equal $\bar{\vartheta}^{(n)}$.

Proposition 10.2.1 (Iterated moment relations: single components). *For* $n \in \mathbb{N}_0$,

$$\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} x_0 \, Q^{(n)}(\bar{\vartheta}^{(n)}, \mathrm{d}z_0) = \vartheta_n, \tag{10.31}$$

$$\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} y_{0,0} \, Q^{(n)}(\bar{\vartheta}^{(n)}, \mathrm{d}z_0) = \vartheta_n, \tag{10.32}$$

$$\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} x^2_0 \, Q^{(n)}(\bar{\vartheta}^{(n)}, \mathrm{d}z_0) = \vartheta^2_n + A^n_0 \, (\mathcal{F}^{(n+1)}g)(\vartheta_n), \tag{10.33}$$

$$\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} y^2_{0,0} \, Q^{(n)}(\bar{\vartheta}^{(n)}, \mathrm{d}z_0) = \vartheta^2_n + (A^n_0 - B_0)(\mathcal{F}^{(n+1)}g)(\vartheta_n), \tag{10.34}$$

$$\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} x_0 y_{0,0} \, Q^{(n)}(\bar{\vartheta}^{(n)}, \mathrm{d}z_0) = \int_{[0,1]\times[0,1]^{\mathbb{N}_0}} y^2_{0,0} \, Q^{(n)}(\vartheta_n, \mathrm{d}z_0), \tag{10.35}$$

$$\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} g(x) \, Q^{(n)}(\bar{\vartheta}^{(n)}, \mathrm{d}z_0) = (\mathcal{F}^{(n+1)}g)(\vartheta_n). \tag{10.36}$$

Proof. We prove the claim for x_0^2 only. The other relations follow in a similar way. The proof proceeds by induction. The result for $n = 0$ follows directly from Proposition 10.1.1. Assume the result holds true for $n = n$, for $n = n + 1$ we write

$$\begin{aligned}
&\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} x_0^2\, Q^{(n+1)}(\bar{\vartheta}^{(n+1)}, \mathrm{d}z_0) \\
&= \int_{[0,1]\times[0,1]^{\mathbb{N}_0}} x_0^2\, (Q^{[n+1]} \circ Q^{(n)})(\bar{\vartheta}^{(n+1)}, \mathrm{d}z_0) \\
&= \int_{[0,1]\times[0,1]^{\mathbb{N}_0}} \int_{[0,1]\times[0,1]^{\mathbb{N}_0}} x_0^2\, Q^{[n+1]}(\bar{\vartheta}^{(n+1)}, \mathrm{d}z_{n+1})\, Q^{(n)}(z_{n+1}, \mathrm{d}z_0) \\
&= \int_{[0,1]\times[0,1]^{\mathbb{N}_0}} \left[\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} x_0^2\, Q^{(n)}(z_{n+1}, \mathrm{d}z_0)\right] Q^{[n+1]}(\bar{\vartheta}^{(n+1)}, \mathrm{d}z_{n+1}) \\
&= \int_{[0,1]\times[0,1]^{\mathbb{N}_0}} \left[x_{n+1}^2 + A_0^n\, (\mathcal{F}^{(n+1)}g)(x_{n+1})\right] \Gamma_{\bar{\vartheta}^{(n+1)}}(\mathrm{d}z_{n+1}) \\
&= \vartheta_{n+1}^2 + A_0^n\, (\mathcal{F}^{(n+2)}g)(\vartheta_{n+1}) + A_{n+1}^{n+1}\, (\mathcal{F}^{(n+2)}g)(\vartheta_{n+1}) \\
&= \theta_{n+1}^2 + A_0^{n+1}\, (\mathcal{F}^{(n+2)}g)(\vartheta_{n+1}).
\end{aligned} \tag{10.37}$$

The first and second equality use the definition in (4.134), the third equality uses Fubini, the fourth equality is the induction step, the fifth equality uses Proposition 10.1.2, in particular, (10.26) and (10.29). □

Similar iterate moment relations hold for blocks. Define, for $m, n \in \mathbb{N}_0$ with $n \geq m$,

$$Q_m^{(n)} = Q^{[n]} \circ \cdots \circ Q^{[m]}. \tag{10.38}$$

Proposition 10.2.2 (Iterated moment relations: blocks of components). *For $n, m \in \mathbb{N}_0$ with $n \geq m$,*

$$\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} x_m\, Q_m^{(n)}(\bar{\vartheta}^{(n)}, \mathrm{d}z_m) = \vartheta_n, \tag{10.39}$$

$$\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} y_{m,m}\, Q_m^{(n)}(\bar{\vartheta}^{(n)}, \mathrm{d}z_m) = \vartheta_n, \tag{10.40}$$

$$\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} x_m^2\, Q_m^{(n)}(\bar{\vartheta}^{(n)}, \mathrm{d}z_m) = \vartheta_n^2 + A_m^n\, (\mathcal{F}^{(n+1)}g)(\vartheta_n), \tag{10.41}$$

$$\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} y_{m,m}^2\, Q_m^{(n)}(\bar{\vartheta}^{(n)}, \mathrm{d}z_m) = \vartheta_n^2 + (A_m^n - B_m)(\mathcal{F}^{(n+1)}g)(\vartheta_n), \tag{10.42}$$

$$\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} x_m y_{m,m}\, Q_m^{(n)}(\bar{\vartheta}^{(n)}, \mathrm{d}z_m) = \int_{[0,1]\times[0,1]^{\mathbb{N}_0}} y_{m,m}^2\, Q_m^{(n)}(\bar{\vartheta}^{(n)}, \mathrm{d}z_m), \tag{10.43}$$

$$\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} (\mathcal{F}^{(m)}g)(x_m)\, Q_m^{(n)}(\bar{\vartheta}^{(n)}, \mathrm{d}z_m) = (\mathcal{F}^{(n+1)}g)(\vartheta_n). \tag{10.44}$$

Proof. Follow a similar induction argument as in the proof of Proposition 10.2.1. □

§10.3 Clustering

To prove Theorem 4.5.1, we proceed as in [5]. The following clustering property holds for the kernels associated with single colonies.

Proposition 10.3.1 (Clustering: single colonies). *Assume*

$$\lim_{n\to\infty} \vartheta_n = \theta. \tag{10.45}$$

Then

$$\begin{aligned}\lim_{n\to\infty} Q^{(n)}\big(\bar\vartheta^{(n)}, \{(x_0, y_{(0,0)}) = (0,0)\}\big) &= 1-\theta,\\ \lim_{n\to\infty} Q^{(n)}\big(\bar\vartheta^{(n)}, \{(x_0, y_{(0,0)}) = (1,1)\}\big) &= \theta,\end{aligned} \tag{10.46}$$

if and only if

$$\lim_{n\to\infty} A_0^n = \infty. \tag{10.47}$$

Consequently,

$$\lim_{n\to\infty} Q^{(n)}\big(\bar\vartheta^{(n)}, \{(x_0, y_{0,0})\} \notin \{(0,0),(1,1)\}\big) = 0. \tag{10.48}$$

Proof. The proof exploits the iterated moment relations. First assume (10.47)

1. By Proposition 10.2.1

$$\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} x_0(1-x_0)\, Q^{(n)}(\bar\vartheta^{(n)}, \mathrm{d}z_0) = \vartheta_n(1-\vartheta_n) - A_0^n(\mathcal{F}^{(n+1)}g)(\vartheta_n). \tag{10.49}$$

Because $x_0(1-x_0) \geq 0$ for $x \in [0,1]$, we have

$$\vartheta_n(1-\vartheta_n) \geq A_0^n(\mathcal{F}^{(n+1)}g)(\vartheta_n) \qquad \forall\, n \in \mathbb{N}. \tag{10.50}$$

Since $\lim_{n\to\infty} A_0^n = \infty$, it follows that $\lim_{n\to\infty}(\mathcal{F}^{(n+1)}g)(\vartheta_n) = 0$. On the other hand,

$$(\mathcal{F}^{(n+1)}g)(\vartheta_n) = \int_{[0,1]\times[0,1]^{\mathbb{N}_0}} g(x_0)\, Q^{(n)}(\bar\vartheta^{(n)}, \mathrm{d}z_0) \tag{10.51}$$

and, because $g(x) > 0$ for $x \in (0,1)$, $Q^{(n)}(\vartheta_n, \mathrm{d}z_0)$ puts all its mass on $x_0 = 0$ and $x_0 = 1$ in the limit as $n \to \infty$. Let

$$Q^{(n)}\big(\bar\vartheta^{(n)}, \{x_0 = 0 \text{ or } x_0 = 1\}\big) = \int_{[0,1]\times[0,1]^{\mathbb{N}_0}} \mathbf{1}_{\{x_0=0 \text{ or } x_0=1\}}(z_0)\, Q^{(n)}(\bar\vartheta^{(n)}, \mathrm{d}z_0), \tag{10.52}$$

then

$$\lim_{n\to\infty} Q^{(n)}\big(\bar\vartheta^{(n)}, \{x_0 = 0 \text{ or } x_1 = 1\}\big) = 1. \tag{10.53}$$

The first moment of x_0 converges to (recall (4.63))

$$\lim_{n\to\infty} \int_{[0,1]\times[0,1]^{\mathbb{N}_0}} x_0\, Q^{(n)}(\bar\vartheta^{(n)}, \mathrm{d}z_0) = \lim_{n\to\infty} \vartheta_n = \theta. \tag{10.54}$$

Hence

$$\begin{aligned}\lim_{n\to\infty} Q^{(n)}(\bar\vartheta^{(n)}, \{x_0 = 0\}) &= 1-\theta,\\ \lim_{n\to\infty} Q^{(n)}(\bar\vartheta^{(n)}, \{x_0 = 1\}) &= \theta,\end{aligned} \tag{10.55}$$

and

$$\lim_{n\to\infty}\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} x_0^2\, Q^{(n)}(\bar{\vartheta}^{(n)},\mathrm{d}z_0) = \lim_{n\to\infty}\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} x_0^2\Big(\mathbf{1}_{\{1\}}(x_0)+\mathbf{1}_{\{0\}}(x_0)+\mathbf{1}_{\{(0,1)\}}(x_0)\Big)\, Q^{(n)}(\bar{\vartheta}^{(n)},\mathrm{d}z_0)=\theta. \tag{10.56}$$

On the other hand,

$$\lim_{n\to\infty}\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} x_0^2\, Q^{(n)}(\bar{\vartheta}^{(n)},\mathrm{d}z_0) = \theta^2 + \lim_{n\to\infty} A_0^n(\mathcal{F}^{(n+1)}g)(\theta), \tag{10.57}$$

and so, combining (10.56)–(10.57), we obtain

$$\lim_{n\to\infty} A_0^n(\mathcal{F}^{(n+1)}g)(\theta) = \theta(1-\theta). \tag{10.58}$$

2. We know also that

$$\begin{aligned}\lim_{n\to\infty}&\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} x_0 y_{0,0}\, Q^{(n)}(\bar{\vartheta}^{(n)},\mathrm{d}z_0) \\ &= \lim_{n\to\infty} \vartheta_n^2 + \left(A_0^n - \frac{E_0^2}{E_0c_0+e_0+E_0K_0e_0}\right)(\mathcal{F}^{(n+1)}g)(\vartheta_n) \\ &= \theta^2+\theta(1-\theta)=\theta\end{aligned} \tag{10.59}$$

and

$$\begin{aligned}\lim_{n\to\infty}&\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} x y_0\, Q^{(n)}(\bar{\vartheta}^{(n)},\mathrm{d}z_0) \\ &= \lim_{n\to\infty}\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} x_0 y_{0,0}\Big(\mathbf{1}_{\{1\}}(x_0)+\mathbf{1}_{\{0\}}(x_0)+\mathbf{1}_{\{(0,1)\}}(x_0)\Big)\, Q^{(n)}(\bar{\vartheta}^{(n)},\mathrm{d}z_0) \\ &= \lim_{n\to\infty}\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} y_{0,0}\,\mathbf{1}_{\{1\}}(x_0)\, Q^{(n)}(\bar{\vartheta}^{(n)},\mathrm{d}z_0).\end{aligned} \tag{10.60}$$

Therefore

$$\lim_{n\to\infty}\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} y_{0,0}\,\mathbf{1}_{\{1\}}(x_0)\, Q^{(n)}(\bar{\vartheta}^{(n)},\mathrm{d}z_0) = \lim_{n\to\infty}\vartheta_n = \theta, \tag{10.61}$$

and hence

$$\lim_{n\to\infty}\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} (1-y_{(0,0)})\,\mathbf{1}_{\{1\}}(x_0)\, Q^{(n)}(\bar{\vartheta}^{(n)},\mathrm{d}z_0) = \theta-\theta=0. \tag{10.62}$$

Since $1-y_{(0,0)}\geq 0$, we conclude that if $x_0=1$, then $Q^{(n)}(\bar{\vartheta}^{(n)},\mathrm{d}z_0)$ puts all its mass on $y_{0,0}=1$ in the limit as $n\to\infty$. Hence

$$\lim_{n\to\infty} Q^{(n)}\big(\bar{\vartheta}^{(n)},\{(x_0,y_{0,0})=(1,1)\}\big)=\theta. \tag{10.63}$$

From Proposition 10.2.2 it also follows that

$$\begin{aligned}
&\lim_{n\to\infty}\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} (1-x_0)(1-y_{0,0})\, Q^{(n)}(\bar{\vartheta}^{(n)},\mathrm{d}z_0) \\
&\quad = 1-\theta-\theta+\theta^2+\lim_{n\to\infty}\left(A_0^n-\frac{E_0^2}{E_0c_0+e_0+E_0K_0e_0}\right)(\mathcal{F}^{(n+1)}g)(\vartheta_n) \\
&\quad = 1-\theta.
\end{aligned} \tag{10.64}$$

On the other hand,

$$\begin{aligned}
&\lim_{n\to\infty}\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} (1-x_0)(1-y_{0,0})\, Q^{(n)}(\bar{\vartheta}^{(n)},\mathrm{d}z_0) \\
&= \lim_{n\to\infty}\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} (1-x_0)(1-y_{0,0})\, \Big(\mathbf{1}_{\{1\}}(x_0)+\mathbf{1}_{\{0\}}(x_0)+\mathbf{1}_{\{(0,1)\}}(x_0)\Big) \\
&\qquad\qquad \times Q^{(n)}(\bar{\vartheta}^{(n)},\mathrm{d}z_0) \\
&= \lim_{n\to\infty}\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} (1-y_{0,0})\,\mathbf{1}_{\{0\}}(x_0)\, Q^{(n)}(\bar{\vartheta}^{(n)},\mathrm{d}z_0) \\
&= 1-\theta.
\end{aligned} \tag{10.65}$$

Since $y\in[0,1]$ and

$$\lim_{n\to\infty}\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} \mathbf{1}_{\{0\}}(x_0)\, Q^{(n)}(\bar{\vartheta}^{(n)},\mathrm{d}z_0) = 1-\theta, \tag{10.66}$$

it follows that

$$\lim_{n\to\infty}\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} y_{0,0}\mathbf{1}_{\{0\}}(x_0)\, Q^{(n)}(\bar{\vartheta}^{(n)},\mathrm{d}z_0) = 0. \tag{10.67}$$

This implies that if $x_0=0$, then $Q^{(n)}(\bar{\vartheta}^{(n)},\mathrm{d}z_0)$ puts all its mass on $y_{0,0}=0$ in the limit as $n\to\infty$. Hence

$$\begin{aligned}
&\lim_{n\to\infty} Q^{(n)}\big(\bar{\vartheta}^{(n)},\{(x_0,y_{0,0})=(0,0)\}\big) = 1-\theta, \\
&\lim_{n\to\infty} Q^{(n)}\big(\bar{\vartheta}^{(n)},\{(x_0,y_{0,0})=(1,1)\}\big) = \theta.
\end{aligned} \tag{10.68}$$

Now assume (10.46). Then (10.56) still holds. On the other hand, also (10.57) still holds by Proposition 10.2.1. Therefore we obtain (10.58). On the other hand, by (10.46)

$$\lim_{n\to\infty}\mathcal{F}^{(n+1)}g = \lim_{n\to\infty}\int_{[0,1]\times[0,1]^{\mathbb{N}_0}} g(x_0)Q^{(n)}(\bar{\vartheta}^{(n)},\mathrm{d}z_0) = 0. \tag{10.69}$$

Hence (10.47) holds. □

A similar clustering property holds for the kernels associated with blocks.

Proposition 10.3.2 (Clustering: blocks). *Assume*

$$\lim_{n\to\infty} \vartheta_n = \theta. \tag{10.70}$$

Then

$$\begin{aligned} &\lim_{n\to\infty} Q_m^{(n)}\big(\bar\vartheta^{(n)}, \{(x_m, y_{(m,m)}) = (0,0)\}\big) = 1-\theta,\\ &\lim_{n\to\infty} Q_m^{(n)}\big(\bar\vartheta^{(n)}, \{(x_m, y_{(m,m)}) = (1,1)\}\big) = \theta, \end{aligned} \tag{10.71}$$

if and only if

$$\lim_{n\to\infty} A_m^n = \infty. \tag{10.72}$$

Consequently,

$$\lim_{n\to\infty} Q_m^{(n)}\big(\bar\vartheta^{(n)}, \{(x_m, y_{m,m})\} \notin \{(0,0),(1,1)\}\big) = 0. \tag{10.73}$$

Proof. We can proceed exactly as in the proof of Proposition 10.3.1. □

Finally, we can prove Theorem 4.5.1.

Proof. Note that (10.47) implies (10.72). Recall that single colonies of deep seed-banks that have already interacted and reached their quasi-equilibrium equal the block average of the level on which they interact (see Theorem 4.4.4). It follows that, for $m \in \mathbb{N}_0$,

$$\begin{aligned} &\lim_{n\to\infty} Q^{(n)}\big(\bar\vartheta^{(n)}, (x_0, y_{0,m}) = (1,1)\big) = \theta,\\ &\lim_{n\to\infty} Q^{(n)}\big(\bar\vartheta^{(n)}, (x_0, y_{0,m}) = (0,0)\big) = 1-\theta. \end{aligned} \tag{10.74}$$

Therefore, for $N \in \mathbb{N}_0$,

$$\begin{aligned} &\lim_{n\to\infty} Q^{(n)}\left(\bar\vartheta^{(n)}, \bigcap_{m=0}^{N} \{(x_0, y_{0,m}) = (1,1) \text{ or } (x_0, y_{0,m}) = (0,0)\}\right)\\ &\quad = 1 - \lim_{n\to\infty} Q^{(n)}\left(\bar\vartheta^{(n)}, \bigcup_{m=0}^{N} \{(x_0, y_{0,m}) \in [0,1]^2 \backslash \{(0,0),(1,1)\}\}\right)\\ &\quad \geq 1 - \lim_{n\to\infty} \sum_{m=0}^{N} Q^{(n)}\Big(\bar\vartheta^{(n)}, \big\{(x_0, y_{0,m}) \in [0,1]^2 \backslash \{(0,0),(1,1)\}\big\}\Big) = 1-0 = 1. \end{aligned} \tag{10.75}$$

Note that

$$\begin{aligned} &\lim_{n\to\infty} Q^{(n)}\left(\bar\vartheta^{(n)}, \bigcap_{m=0}^{N} \big\{(x_0, y_{0,m}) = (1,1) \text{ or } (x_0, y_{0,m}) = (0,0)\big\}\right)\\ &\quad = \lim_{n\to\infty} Q^{(n)}\Big(\bar\vartheta^{(n)}, \big\{(x_0, (y_{0,m})_{0\le m\le N}) = (0, 0^{N+1})\\ &\qquad\qquad \text{or } (x_0, (y_{0,m})_{0\le m\le N}) = (1, 1^{N+1})\big\}\Big) = 1. \end{aligned} \tag{10.76}$$

On the other hand,

$$\begin{aligned} &\lim_{n\to\infty} Q^{(n)}\Big(\bar\vartheta^{(n)}, \{(x_0, (y_{0,m})_{0\le m\le N}) = (1, 1^{N+1})\}\Big)\\ &\quad \le \lim_{n\to\infty} Q^{(n)}\big(\bar\vartheta^{(n)}, \{(x_0, y_{0,0}) = (1,1)\}\big) = \theta \end{aligned} \tag{10.77}$$

and

$$\lim_{n\to\infty} Q^{(n)}\Big(\bar{\vartheta}^{(n)}, \{(x_0,(y_{0,m})_{0\le m\le N}) = (0,0^{N+1})\}\Big) \le \lim_{n\to\infty} Q^{(n)}\big(\bar{\vartheta}^{(n)}, \{(x_0,y_{0,0}) = (0,0)\}\big) = 1-\theta. \tag{10.78}$$

Hence we conclude that

$$\lim_{n\to\infty} Q^{(n)}\Big(\bar{\vartheta}^{(n)}, \{(x_0,(y_{0,m})_{0\le m\le N}) = (1,1^{N+1})\}\Big) = \theta \tag{10.79}$$

and

$$\lim_{n\to\infty} Q^{(n)}\Big(\bar{\vartheta}^{(n)}, \{(x_0,(y_{0,m})_{0\le m\le N}) = (0,0^{N+1})\}\Big) = 1-\theta. \tag{10.80}$$

We can do the same for all finite-dimensional distributions. Since $[0,1]\times[0,1]^{\mathbb{N}_0}$ is compact, the process $z_0 = (x_0,(y_{0,m})_{m\in\mathbb{N}_0})$ is tight. Therefore, by (10.79)–(10.80) we find for every converging subsequence

$$\lim_{k\to\infty} Q^{(n_k)}\Big(\bar{\vartheta}^{(n_k)}, \cdot\Big) = (1-\theta)\,\delta_{(0,0^{\mathbb{N}_0})} + \theta\,\delta_{(1,1^{\mathbb{N}_0})}. \tag{10.81}$$

We conclude that

$$\lim_{n\to\infty} Q^{(n)}(\bar{\vartheta}^{(n)}, \mathrm{d}z_0) = (1-\theta)\,\delta_{(0,0^{\mathbb{N}_0})} + \theta\,\delta_{(1,1^{\mathbb{N}_0})}, \tag{10.82}$$

which is the claim in (4.137). □

§10.4 Dichotomy finite versus infinite seed-bank

In this section we prove Theorem 4.5.3.

Proof. We investigate for what choices of the sequences $\underline{c}, \underline{K}, \underline{e}$ defined in (4.5) and (4.10) we meet the *clustering criterion* $\lim_{n\to\infty} A_n \to \infty$ in (4.138). Recall from (4.64) and (4.136) that

$$A_n = \frac{1}{2}\sum_{k=0}^{n-1} \frac{E_k}{c_k}\frac{(E_k c_k + e_k)}{(E_k c_k + e_k) + E_k K_k e_k}, \qquad E_k = \frac{1}{1+\sum_{m=0}^{k-1} K_m}. \tag{10.83}$$

We distinguish between three regimes as $k\to\infty$:

(a) $E_k c_k + e_k \gg E_k K_k e_k$.

(b) $E_k c_k + e_k \asymp E_k K_k e_k$.

(c) $E_k c_k + e_k \ll E_k K_k e_k$.

These regimes correspond to the following scaling for A_n as $n\to\infty$:

(a) $A_n \sim \frac{1}{2}\sum_{k=0}^{n-1}\frac{E_k}{c_k}$.

(b) $A_n \asymp \sum_{k=0}^{n-1}\frac{E_k}{c_k}$.

(c) $A_n \sim \frac{1}{2}\sum_{k=0}^{n-1}\frac{E_k c_k + e_k}{c_k K_k e_k}$.

Recall from that (4.14) that

$$\rho = \sum_{m\in\mathbb{N}_0} K_m. \tag{10.84}$$

Different behaviour shows up for finite seed-bank ($\rho<\infty$) and infinite seed-bank ($\rho=\infty$).

(I) $\rho < \infty$**.** Note that $k \mapsto E_k$ is non-increasing and converges to $1/(1+\rho) > 0$. Since $E_k \le 1$, we have $\lim_{k\to\infty} E_k K_k = 0$ and hence we are in regime 1. Therefore

$$A_n \sim \frac{1}{2(1+\rho)} \sum_{k=0}^{n-1} \frac{1}{c_k}, \qquad n \to \infty, \tag{10.85}$$

and clustering occurs if and only if $\sum_{k\in\mathbb{N}_0} \frac{1}{c_k} = \infty$, which is the same criterion as for the system without seed-bank.

(II) $\rho = \infty$**.** We focus on the settings in (4.52) and (4.53), which fall in regimes 1 and 2.

Asymptotically polynomial. Suppose that

$$K_k \sim A k^{-\alpha}, \quad k \to \infty,\, A \in (0,\infty),\, \alpha \in (-\infty, 1). \tag{10.86}$$

Then

$$E_k \sim \frac{1-\alpha}{A} k^{-(1-\alpha)}, \qquad E_k K_k \sim \frac{1-\alpha}{A} k^{-1}, \qquad k \to \infty. \tag{10.87}$$

Hence we are in regime 1. Suppose that

$$c_k \sim F k^{-\phi}, \quad k \to \infty,\, F \in (0,\infty),\, \phi \in \mathbb{R}. \tag{10.88}$$

Then

$$A_n \sim \frac{1-\alpha}{2AF} \sum_{k=1}^{n-1} k^{-1+\alpha+\phi}, \qquad n \to \infty, \tag{10.89}$$

and clustering occurs if and only if $-\phi \le \alpha < 1$. In this case

$$\begin{aligned} -\phi < \alpha\colon &\quad A_n \sim \frac{1-\alpha}{2AF(\alpha+\phi)}\, n^{\alpha+\phi}, \\ -\phi = \alpha\colon &\quad A_n \sim \frac{1-\alpha}{2AF}\, \log n. \end{aligned} \tag{10.90}$$

The case $\alpha = 1$ can be included. Then (10.87) becomes

$$E_k \sim \frac{1}{A \log k}, \qquad E_k K_k \sim \frac{1}{k \log k}, \qquad k \to \infty, \tag{10.91}$$

so that we are again in regime 1. Now (10.89) becomes

$$A_n \sim \tfrac{1}{2AF} \sum_{k=1}^{n-1} \frac{k^\phi}{\log k}, \qquad n \to \infty, \tag{10.92}$$

and clustering occurs if and only if $-\phi \le 1$. In this case

$$\begin{aligned} -\phi < 1\colon &\quad A_n \sim \frac{1}{2AF(1+\phi)}\, \frac{n^{1+\phi}}{\log n}, \\ -\phi = 1\colon &\quad A_n \sim \frac{1}{2AF}\, \log\log n. \end{aligned} \tag{10.93}$$

Pure exponential. Suppose that

$$K_k = K^k, \quad k \in \mathbb{N}_0,\, K \in (1,\infty). \tag{10.94}$$

Then

$$E_k = \frac{1}{1+\sum_{m=0}^{k-1} K^m} = \frac{1}{1+\frac{K^k-1}{K-1}} = \frac{K-1}{K^k+K-2}. \tag{10.95}$$

Suppose that

$$e_k = e^k, \quad c_k = c^k, \quad k \in \mathbb{N}_0,\, e,c \in (0,\infty). \tag{10.96}$$

Then

$$E_k c_k + e_k = \frac{K-1}{K^k+K-2}c^k + e^k, \qquad E_k K_k e_k = \frac{K-1}{K^k+K-2}K^k e^k, \tag{10.97}$$

and so

$$E_k c_k + e_k \sim (K-1)\left(\frac{c}{K}\right)^k + e^k, \qquad E_k K_k e_k \sim (K-1)e^k, \qquad k \to \infty. \tag{10.98}$$

For $c \leq Ke$ we are in regime 2, and hence

$$A_n \sim \frac{1}{2}\sum_{k=0}^{n-1} \frac{K-1}{(Kc)^k}\,\frac{(K-1)(\frac{c}{K})^k + e^k}{(K-1)(\frac{c}{K})^k + Ke^k}, \qquad n \to \infty, \tag{10.99}$$

which simplifies to

$$\begin{aligned} c < Ke\colon \quad A_n &\sim \frac{1}{2K}\sum_{k=0}^{n-1}\frac{K-1}{(Kc)^k}, \\ c = Ke\colon \quad A_n &\sim \frac{K-1}{2(2K-1)}\sum_{k=0}^{n-1}\frac{K-1}{(Kc)^k}. \end{aligned} \tag{10.100}$$

Clustering occurs if and only if $Kc \leq 1$. In this case

$$\begin{aligned} Kc < 1\colon \quad &\sum_{k=0}^{n-1}\frac{1}{(Kc)^k} \sim \frac{1}{1-Kc}\,(Kc)^{-(n-1)}, \\ Kc = 1\colon \quad &\sum_{k=0}^{n-1}\frac{1}{(Kc)^k} \sim n. \end{aligned} \tag{10.101}$$

For $c > Ke$, on the other hand, we are in regime 1, and hence

$$A_n \sim \frac{1}{2}\sum_{k=0}^{n-1}\frac{K-1}{(Kc)^k}, \qquad n \to \infty, \tag{10.102}$$

for which we can again use (10.101).

The case $K=1$ can be included. Then $E_k = 1/k$ and (10.98) becomes

$$E_k c_k + e_k \sim \frac{1}{k}c^k + e^k, \qquad E_k K_k e_k \sim \frac{1}{k}e^k, \qquad k \to \infty, \tag{10.103}$$

we are again in regime 1. Hence

$$A_n \sim \frac{1}{2} \sum_{k=0}^{n-1} \frac{1}{kc^k}, \qquad n \to \infty, \tag{10.104}$$

and clustering occurs if and only if $c \leq 1$. In that case

$$\begin{aligned} c < 1\colon &\quad A_n \sim \frac{1}{2(1-c)}\,\frac{1}{n}\,c^{-(n-1)}, \\ c = 1\colon &\quad A_n \sim \frac{1}{2}\,\log n. \end{aligned} \tag{10.105}$$

□

In the above computations, only regimes 1 and 2 arise. Regime 3 arises, for instance, when

$$\lim_{k\to\infty} \frac{1}{k} \log K_k = \infty, \qquad \lim_{k\to\infty} K_k e_k / c_k = \infty. \tag{10.106}$$

Indeed, the first condition implies that $E_k \sim 1/K_{k-1}$ and $E_k K_k \gg 1$, while the second implies that $E_k K_k e_k \gg E_k c_k$. There are two subcases:

$$\begin{aligned} K_{k-1} e_k \ll c_k\colon &\quad A_n \sim \tfrac{1}{2} \sum_{k=0}^{n-1} \frac{1}{K_k K_{k-1} e_k}, \\ K_{k-1} e_k \gg c_k\colon &\quad A_n \sim \tfrac{1}{2} \sum_{k=0}^{n-1} \frac{1}{K_k c_k}. \end{aligned} \tag{10.107}$$

By picking, for instance, $e_k = 1/K_k K_{k-1}$, we find that $A_n \sim \frac{1}{2}n$ in the first subcase and $A_n \gg n$ in the second subcase. By picking, alternatively, $c_k = 1/K_k$, we find that $A_n \gg n$ in the first subcase and $A_n \sim \frac{1}{2}n$ in the second subcase.

Appendix A