Spatial populations with seed-bank
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CHAPTER 9

Proofs of the hierarchical multi-scale limit theorems

In this chapter we prove the hierarchical multi-scale limit theorems stated in Theorem 4.4.2 and Theorem 4.4.4. In Section 9.1 we first introduce the finite-level mean-field finite-systems scheme. In Section 9.2 we given an outline of how to prove the finite-level mean-field finite-systems scheme. In Section 9.3 we show how Theorems 4.4.2–4.4.4 can be obtained by a simple generalisation of the finite-level mean-field finite-systems scheme. The proof of the finite-level mean-field finite-systems scheme follows a similar line of argument as in Section 8.2 once we incorporate more levels. Since the proofs for the finite-level mean-field systems scheme are similar as the proofs in Section 8.3, we will not write out the full proof, but only give an outline and a sketch.

§9.1 Finite-level mean-field finite-systems scheme and interaction chain

In this section we extend the two-level three-colour system to a k-level $(k+1)$-colour system with an "outside world" for any $k \in \mathbb{N}$. This outside world allows also the highest level, the k-block, to start from equilibrium. It is also needed to generalize the results in this subsection to the infinite hierarchical group.

▶ **Definitions.** To set up the system, fix $k \in \mathbb{N}$ and consider the geographic space Ω_N^{k+1} obtained by truncating the hierarchical goup Ω_N (recall (4.2)) after hierarchical level $k+1$, i.e., $\Omega_N^{k+1} = B_{k+1}(0)$ the $(k+1)$-block centred at the origin (recall (4.2), (4.4) and Fig. 4.2). Note that the $k+1$-block consists of N k-blocks i.e., $B_{k+1}(0) = \bigcup_{i=0}^{N} B_k(i)$ and $B_{k+1}(0) = [N^{k+1}]$. The seed-bank in this model consists of the $k+2$ layers corresponding to colours $\{0, \cdots, k\} \cup \{k+1\}$. On this space we again consider a restricted version of the SSDE in (4.20) to the geographic space Ω_N^{k+1}. The migration kernel $a^{\Omega_N}(\cdot,\cdot)$ is restricted to Ω_N^{k+1} by setting all migration outside $B_{k+1}(0)$ equal to 0, i.e.,

$$a^{[\Omega_N^{k+1}]}(\xi,\eta) = \sum_{l=1}^{k+1} 1_{\left\{d_{\Omega_N^{k+1}}(\xi,\eta)\leq l\right\}} \frac{c_l}{N^{l-1}} \frac{1}{N^l}, \tag{9.1}$$

where $d_{\Omega_N^{k+1}}$ is the hierarchical distance d_{Ω_N} restricted to the space Ω_N^{k+1}. The colour-l dormant population exchanges individuals with the active population at rates $\frac{e_l}{N^l}$, $\frac{K_l e_l}{N^l}$ for all $0 \leq l \leq k$. We set the interaction of the active population with the colour $(k+1)$-dormant population equal to 0. This seed-bank is only needed later, namely for the "outside world".

The state space of the finite-level mean-field system is

$$S = (\mathfrak{s}^{k+1})^{\Omega_N^{k+1}}, \quad \mathfrak{s}^{k+1} = [0,1] \times [0,1]^{k+2}, \tag{9.2}$$

and the system is denoted by

$$\begin{aligned} &Z^{\Omega_N^{k+1}} = (Z^{\Omega_N^{k+1}}(t))_{t \geq 0}, \qquad Z^{\Omega_N^{k+1}}(t) = (z_\xi^{\Omega_N^{k+1}}(t))_{\xi \in \Omega_N^{k+1}}, \\ &z_\xi^{\Omega_N^{k+1}}(t) = (x_\xi^{\Omega_N^{k+1}}(t), (y_{\xi,m}^{\Omega_N^{k+1}}(t))_{m=0}^{k+1}). \end{aligned} \tag{9.3}$$

The components evolve according to the SSDE

$$\begin{aligned} \mathrm{d}x_\xi^{\Omega_N^{k+1}}(t) &= \sum_{l=1}^{k} \frac{c_{l-1}}{N^{l-1}} \frac{1}{N^l} \sum_{\eta \in B_l(\xi)} \left[x_\eta^{\Omega_N^{k+1}}(t) - x_\xi^{\Omega_N^{k+1}}(t) \right] \mathrm{d}t \\ &\quad + \sqrt{g(x_\xi^{\Omega_N^{k+1}}(t))}\, \mathrm{d}w_\xi(t) + \sum_{m=0}^{k} \frac{K_m e_m}{N^m} \left[y_{\xi,m}^{\Omega_N^{k+1}}(t) - x_\xi^{\Omega_N^{k+1}}(t) \right] \mathrm{d}t, \\ \mathrm{d}y_{\xi,m}^{\Omega_N^{k+1}}(t) &= \frac{e_m}{N^m} \left[x_\xi^{\Omega_N^{k+1}}(t) - y_{\xi,m}^{\Omega_N^{k+1}}(t) \right] \mathrm{d}t, \quad 0 \leq m \leq k, \\ \mathrm{d}y_{\xi,m}^{\Omega_N^{k+1}}(t) &= 0, \qquad \xi \in \Omega_N^{k+1}, \end{aligned} \tag{9.4}$$

with $B_l(\xi)$ the ball of radius l around $\xi \in \Omega_N^{k+1}$.

Note that this system is the hierarchical SSDE in (4.20) with all interactions at distance $> k$ switched off (i.e., $c_l = 0$ for $l > k+1$), and also the exchange with dormant populations of colour $m > k$ is switched off. As before, by [67] the martingale problem associated with (9.4) is well-posed, and for every initial state in S the SSDE has a unique strong solution. We will analyse (9.4) on time scales $1, N, N^2, \cdots, N^k$. If for $0 \leq l \leq k$ time runs on time scale N^l, then we write $N^l t_l$ with $t_l > 0$.

To study the k-level mean-field system, we analyse the equivalent of the block averages defined in (4.2.3). For the k-level mean-field system these are given by

$$\begin{aligned} x_l^{\Omega_N^{k+1}}(t) &= \frac{1}{N^l} \sum_{\eta \in B_l(0)} x_\eta^{\Omega_N^{k+1}}(N^l t), \\ y_{m,l}^{\Omega_N^{k+1}}(t) &= \frac{1}{N^l} \sum_{\eta \in B_l(0)} y_{\eta,m}^{\Omega_N^{k+1}}(N^l t), \qquad 0 \leq m \leq k+1, \qquad 0 \leq l \leq k. \end{aligned} \tag{9.5}$$

For $0 \leq l \leq k$ these block averages evolve according to the SSDE

$$\begin{aligned}
\mathrm{d}x_l^{\Omega_N^{k+1}}(t) &= \sum_{n=1}^{k-(l-1)} \frac{c_{l+n-1}}{N^{n-1}} \left[x_{l+n}^{\Omega_N^{k+1}}(N^{-n}t) - x_l^{\Omega_N^{k+1}}(t)\right] \mathrm{d}t \\
&\quad + \sqrt{\frac{1}{N^l} \sum_{i \in B_l(0)} g(x_i(N^l t))}\, \mathrm{d}w_l(t) \qquad (9.6)\\
&\quad + \sum_{m=0}^{k} N^l \frac{K_m e_m}{N^m} \left[y_{m,l}^{\Omega_N^{k+1}}(t) - x_l^{\Omega_N^{k+1}}(t)\right] \mathrm{d}t, \\
\mathrm{d}y_{m,l}^{\Omega_N^{k+1}}(t) &= N^l \frac{e_m}{N^m} \left[x_l^{\Omega_N^{k+1}}(t) - y_{m,l}^{\Omega_N^{k+1}}(t)\right] \mathrm{d}t, \qquad 0 \leq m \leq k, \qquad (9.7)\\
\mathrm{d}y_{k+1,l}^{\Omega_N^{k+1}}(t) &= 0, \qquad (9.8)
\end{aligned}$$

In the limit as $N \to \infty$, the active l-block average feels a drift towards the active $(l+1)$-block average, which is not moving on time scale N^l, at rate c_l. The diffusion term for the l-block average becomes the average diffusion over the l-block. The drift of the active l-block average towards the l-block average of m-dormant populations $y_{m,l}^{\Omega_N^{k+1}}$ with $m > l$ vanishes in the limit as $N \to \infty$. Therefore, the $m > l$ m-dormant populations are *slow seed-banks* on space-time scale l. The l-block average of the colour-l dormant population $y_{l,l}^{\Omega_N^{k+1}}$ has a non-trivial drift towards the active l-block average, written $x_l^{\Omega_N^{k+1}}$. Therefore the l-dormant population is the *effective seed-bank* on space-time scale l. For the colour m-dormant populations $y_{m,l}^{\Omega_N^{k+1}}$ with $m < l$, we see that infinite rates appear. Therefore the m-dormant populations with $m < l$ are *fast seed-banks* on space-time scale l. We again need the Meyer-Zheng topology to show that $\lim_{N\to\infty} y_{m,l}^{\Omega_N^{k+1}} = \lim_{N\to\infty} x_l^{\Omega_N^{k+1}}$. On space-time scale l, the colour-l dormant population is the effective seed-bank. To get rid of the infinite rates we again look at combinations. From the above discussion and the SSDE in (9.6)–(9.6), we see that if we consider the quantity

$$\frac{x_l^{\Omega_N^{k+1}}(t) + \sum_{m=0}^{l-1} K_m y_{l,m}^{\Omega_N^{k+1}}(t)}{1 + \sum_{m=0}^{l-1} K_m}, \tag{9.9}$$

then all infinite rates cancel out. Therefore

$$\left(\frac{x_l^{\Omega_N^{k+1}}(t) + \sum_{m=0}^{l-1} K_m y_{l,m}^{\Omega_N^{k+1}}(t)}{1 + \sum_{m=0}^{l-1} K_m}, y_{l,l}^{\Omega_N^{k+1}}(t)\right)_{t>0} \tag{9.10}$$

is called the *effective process* on space-time scale l. Like in the simpler mean-field finite-systems scheme, the effective process allows us to analyse our system in path space.

An important difference between the finite-level mean-field system in (9.6)–(9.7) and the two-level three-colour mean-field system in Section (8.1) is that in the finite-level mean-field system also the highest level k has a drift towards the outside world.

This outside world is the active $k+1$-block average, which does not evolve on time scale N^k. This drift allows the finite-level mean-field system to equilibrate to a non-trivial equilibrium. In the two-level mean-field system, the highest level, i.e., the active 2-block average, does not feel a drift due to migration. Consequently, the 2-block averages will eventually cluster.

► **Scaling limit.** To state and prove the finite-level multi-scale limit, we need the following three limiting processes. Recall (4.64) and (4.62). For $0 \le l \le k$, let

$$(z_{l,(\theta,(y_{m,l})_{m=0}^{k+1})}(t))_{t\ge0}) = (x_l(t),(y_{m,l}(t))_{m=0}^{k+1})_{t\ge0}) \tag{9.11}$$

be the process evolving according to

$$\begin{aligned} \mathrm{d}x_l(t) &= E_l\left[c_l[\theta - x_l(t)]\,\mathrm{d}t + \sqrt{\mathcal{F}^{(l)}g(x_1(t))}\,\mathrm{d}w(t) + K_l e_l\,[y_{l,l}(t)-x_l(t)]\,\mathrm{d}t\right],\\ y_{m,l}(t) &= x_l(t), \qquad \text{for } 0\le m<l\\ \mathrm{d}y_{l,l}(t) &= e_l\,[x_l(t)-y_{l,l}(t)]\,\mathrm{d}t,\\ y_{m,l}(t) &= y_{m,l}, \qquad \text{for } l<m\le k+1, \end{aligned} \tag{9.12}$$

where $\theta\in[0,1]$ and $y_{m,l}\in[0,1]$ for $l<m\le k+1$.

For $0\le l\le k$, let

$$(z^{\mathrm{aux}}_{l,(\theta,(y_{m,l})_{m=l+1}^{k+1})}(t))_{t\ge0} = (x_l^{\mathrm{aux}}(t),(y^{\mathrm{aux}}_{m,l}(t))_{m=l+1}^{k+1})_{t\ge0} \tag{9.13}$$

be the process evolving according to

$$\begin{aligned} \mathrm{d}x_l^{\mathrm{aux}}(t) &= E_l\Bigg[c_l[\theta - x_l^{\mathrm{aux}}(t)]\,\mathrm{d}t + \sqrt{\mathcal{F}^{(l)}g(x_1^{\mathrm{aux}}(t))}\,\mathrm{d}w(t)\\ &\qquad\qquad + K_l e_l\,[y_{l,l}^{\mathrm{aux}}(t)-x_l^{\mathrm{aux}}(t)]\,\mathrm{d}t\Bigg], \end{aligned} \tag{9.14}$$

$$\begin{aligned} \mathrm{d}y_{l,l}^{\mathrm{aux}}(t) &= e_l\,[x_l^{\mathrm{aux}}(t)-y_{l,l}^{\mathrm{aux}}(t)]\,\mathrm{d}t,\\ y_{m,l}^{\mathrm{aux}}(t) &= y_{m,l}, \qquad \text{for } l<m\le k+1, \end{aligned}$$

where $\theta\in[0,1]$ and $y^{\mathrm{aux}}_{m,l}\in[0,1]$, for $l<m\le k+1$.

For $0\le l\le k$, let

$$(z^{\mathrm{eff}}_{l,\theta}(t))_{t\ge0} = \left(x_l^{\mathrm{eff}}(t), y_{l,l}^{\mathrm{eff}}(t)\right)_{t\ge0} \tag{9.15}$$

be the *effective process* evolving according to

$$\begin{aligned} \mathrm{d}x_l^{\mathrm{eff}}(t) &= E_l\left[c_l\,[\theta - x_l^{\mathrm{eff}}(t)]\,\mathrm{d}t + \sqrt{(\mathcal{F}^{(l)}g)(x_l^{\mathrm{eff}}(t))}\,\mathrm{d}w(t) + K_l e_l\,[y_{l,l}^{\mathrm{eff}}(t)-x_l^{\mathrm{eff}}(t)]\,\mathrm{d}t\right],\\ \mathrm{d}y_{l,l}^{\mathrm{eff}}(t) &= e_l\,[x_l^{\mathrm{eff}}(t)-y_{l,l}^{\mathrm{eff}}(t)]\,\mathrm{d}t. \end{aligned} \tag{9.16}$$

Comparing (9.12) with (9.16), we see that the effective process looks at the non-trivial components of the full process. The auxiliary process in (9.14) looks at the active population, the effective seed-bank and the slow seed-banks.

To state and prove the finite-level multi-scale limit, we need the following list of ingredients:

(a) For $t > 0$ and for $0 \leq l \leq k$, define the *l-block estimators*

$$
\begin{aligned}
\bar{\Theta}^{(l),\Omega_N^{k+1}}(t) &= \frac{1}{N^l}\sum_{i\in B_l}\frac{x_i^{\Omega_N^{k+1}}(t)+\sum_{m=0}^{l-1}K_m y_{i,0}^{\Omega_N^{k+1}}(t)}{1+K_0},\\
\Theta_x^{(l),\Omega_N^{k+1}}(t) &= \frac{1}{N^l}\sum_{i\in B_l} x_i^{\Omega_N^{k+1}}(t),\\
\Theta_{y_m}^{(l),\Omega_N^{k+1}}(t) &= \frac{1}{N^l}\sum_{i\in B_l} y_{i,m}^{\Omega_N^{k+1}}(t), \qquad 0\leq m\leq k+1,
\end{aligned}
\tag{9.17}
$$

and put

$$
\begin{aligned}
\mathbf{\Theta}^{(l),\Omega_N^{k+1}}(t), &= \left(\Theta_x^{(l),\Omega_N^{k+1}}(t), \left(\Theta_{y_m}^{(l),\Omega_N^{k+1}}(t)\right)_{m=0}^{k+1}\right),\\
\mathbf{\Theta}^{\mathrm{aux},(l),\Omega_N^{k+1}}(t) &= \left(\bar{\Theta}^{(l),\Omega_N^{k+1}}(t), \left(\Theta_{y_l}^{(l),\Omega_N^{k+1}}(t)\right)_{m=l}^{k+1}\right),\\
\mathbf{\Theta}^{\mathrm{eff},(l),\Omega_N^{k+1}}(t) &= \left(\bar{\Theta}^{(l),\Omega_N^{k+1}}(t), \Theta_{y_l}^{(l),\Omega_N^{k+1}}(t)\right).
\end{aligned}
\tag{9.18}
$$

We call $(\mathbf{\Theta}^{(l),\Omega_N^{k+1}}(t))_{t>0}$ the *l-block estimator process*, $(\mathbf{\Theta}^{\mathrm{aux},(l),\Omega_N^{k+1}}(t))_{t>0}$ the *auxiliary l-block estimator process* and $(\mathbf{\Theta}^{\mathrm{eff},(l),\Omega_N^{k+1}}(t))_{t>0}$ the *effective l-block estimator process.*

(b) For $0 \leq l \leq k$, define the *time scales* N^l such that

$$
\mathcal{L}[\bar{\Theta}^{(l),\Omega_N^{k+1}}(N^l t_l - L(N)N^{l-1}) - \bar{\Theta}^{(l),\Omega_N^{k+1}}(N^l t_l)] = \delta_0 \tag{9.19}
$$

for all $L(N)$ such that $\lim_{N\to\infty} L(N) = \infty$ and $\lim_{N\to\infty} L(N)/N = 0$, but not for $L(N) = N$. In words, N is the time scale on which $\bar{\Theta}^{(l),\Omega_N^{k+1}}(\cdot)$ starts evolving, i.e., $\left(\bar{\Theta}^{(l),\Omega_N^{k+1}}(N^l t_l)\right)_{t_l>0}$, is no longer a fixed process.

(c) The *invariant measure* for the evolution of the l-block average in (9.12), denoted by

$$
\Gamma_{\theta,y_l}^{(l)}, \qquad y_l = (y_{m,l})_{m=0}^{k+1}. \tag{9.20}
$$

The *invariant measures* of the auxiliary l-block process in (9.15) and the effective l-block process in (9.16), denoted by, respectively,

$$
\Gamma_{\theta,y_l}^{(l),\mathrm{aux}}, \qquad y_l = (y_{m,l})_{m=l+1}^{k+1} \tag{9.21}
$$

and

$$
\Gamma_{\theta}^{(l),\mathrm{eff}}. \tag{9.22}
$$

(d) For $0 \leq l \leq k$, let $\mathcal{F}^{E_l,c_l,K_l,e_l}$ denote the *renormalisation transformation* acting on $\mathcal{G}$ defined by

$$
(\mathcal{F}^{E_l,c_l,K_l,e_l} g)(\theta) = \int_{[0,1]^2} g(x)\,\Gamma_\theta^{(l)}(\mathrm{d}x, (\mathrm{d}y)_{m=0}^{k+1}), \qquad \theta\in[0,1], \tag{9.23}
$$

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and define the *iterates* $\mathcal{F}^{(n)}$, $0 \leq n \leq k$, of the renormalisation transformation as the compositions

$$\mathcal{F}^{(l)} = \mathcal{F}^{E_{n-1},c_{n-1},K_{n-1},e_{n-1}} \circ \cdots \circ \mathcal{F}^{E_0,c_0,K_0,e_0}, \qquad 0 \leq l \leq k. \tag{9.24}$$

(Recall (4.76).)

(e) To give a detailed description of the multi-scale behaviour of the SSDE in (4.20), define the *interaction chain*

$$(M^k_{-l})_{-l=-(k+1),-k,\dots,0} \tag{9.25}$$

as the *time-inhomogeneous* Markov chain on $[0,1] \times [0,1]^{k+1}$ with initial state

$$M^k_{-(k+1)} = (\vartheta_k, \overbrace{\vartheta_k, \cdots, \vartheta_k}^{k+1 \text{ times}}, \theta_{y,k+1}) \tag{9.26}$$

that evolves according to the transition kernel $Q^{[l]}$ from time $-(l+1)$ to time $-l$ given by

$$Q^{[l]}(u, \mathrm{d}v) = \Gamma^{(l)}_u(\mathrm{d}v), \qquad 0 \leq l \leq k. \tag{9.27}$$

(Recall (4.77).)

We are now ready to state the scaling limit for the evolution of the averages in (7.7).

Proposition 9.1.1. *[Finite-level mean-field: finite-systems scheme] Suppose that the initial state of the system in* (9.4) *is given by* $\mu(0) = \mu^{\otimes[\Omega^{k+1}_N]}$ *for some* $\mu \in \mathcal{P}([0,1] \times [0,1]^{k+2})$. *Let* $L(N)$ *be such that* $\lim_{N\to\infty} L(N) = \infty$ *and* $\lim_{N\to\infty} L(N)/N = 0$, *and for* $t_k, \dots, t_0 \in (0,\infty)$ *set* $\bar{t} = L(N)N^k + \sum_{n=0}^{k} t_n N^n$.

(a) For every $t_k, \dots, t_0 \in (0,\infty)$,

$$\lim_{N\to\infty} \mathcal{L}\left[\left(\mathbf{\Theta}^{(l),\Omega^{k+1}_N}(\bar{t})\right)_{l=k+1,k,\dots,0}\right] = \mathcal{L}\left[(M^k_{-l})_{-l=-(k+1),-k,\dots,0}\right], \tag{9.28}$$

where $(M^k_{-l})_{-l=-(k+1),-k,\dots,0}$ *is the interaction chain in* (9.25) *starting from*

$$M^k_{-(k+1)} = (\vartheta_k, \overbrace{\vartheta_k, \cdots, \vartheta_k}^{k+1 \textit{ times}}, \theta_{y,k+1}). \tag{9.29}$$

(b) For all $0 \leq l \leq k$,

$$\lim_{N\to\infty} \mathcal{L}\left[\left(\mathbf{\Theta}^{(l),\Omega^{k+1}_N}(\bar{t} + t_l N^l)\right)_{t_l>0}\right] = \mathcal{L}\left[\left(z_{l,M^k_{-(l+1)}}(t_l)\right)_{t_l>0}\right], \tag{9.30}$$

in the Meyer-Zheng topology,

where

$$(z_{l,M^k_{-(l+1)}}(t_l))_{t_l>0} \tag{9.31}$$

is the processes defined (9.12) *with* θ, $(y_m, l)_{m=l+1}^{k+1}$ *given by the corresponding components in* $M^k_{-(l+1)}$ *and with initial measure*

$$\mathcal{L}\left[z_{l,M^k_{-(l+1)}}(0)\right] = \Gamma^{(l)}_{M^k_{-(l+1)}}$$
$$\Gamma^{(l)}_{M^k_{-(l+1)}} = \int_{\mathfrak{s}^{k+1}} \cdots \int_{\mathfrak{s}^{k+1}} \int_{\mathfrak{s}^{k+1}} \Gamma^{(k)}_{M^k_{-(k+1)}}(\mathrm{d}u_k)\Gamma^{(k-1)}_{u_k}(\mathrm{d}u_{k-1}) \cdots \Gamma^{(l+1)}_{u_2}(\mathrm{d}u_{l+2})\Gamma^{(l)}_{u_{l+1}}. \tag{9.32}$$

In Part (a), the limit does not depend on the choice of the times $t_k, \dots, t_0$, since we let time start from a time larger than $L(N)N^k$, so that in the limit as $N \to \infty$ all the l-block averages with $l \le k$ are already in quasi-equilibrium. In Part (b), for $l < k$ the center of the drift for the active population is *random* and is determined by the first component of the interaction chain. Also the states of the m-dormant populations with $l < m \le k+1$ are determined by the interaction chain.

Remark 9.1.2. In contrast to Propositions 7.1.2–8.1.1, there are no assumptions on the seed-bank behaviour in Proposition 9.1.1. This is because all the block-averages that we consider are in equilibrium at time $\bar{t}$. Consequently on space-time scales $l < m$ the m-dormant l-block average will equal the state of the m-dormant m-block average at time $\bar{t}$. Therefore we say that the state of the *slow seed-banks* is determined by the space-time scaleon which this seed-bank is effective. Hence the state of the slow seed-banks is determined by the interaction chain. ■

The proof of Proposition 9.1.1 will be given in Section 9.2.

§9.2 Proof of the mean-field finite-systems scheme: finite-level

We give a sketch of the proof Proposition 9.1.1. The proof uses a similar scheme as the proof of Proposition 8.1.1. We state the scheme and indicate at each step how it can be proved.

1 Tightness of the auxiliary l-block estimator processes, for $0 \le l \le k$,

$$\left(\left(\boldsymbol{\Theta}^{\mathrm{aux},(l),\Omega_N^{k+1}}(N^l t_l)\right)_{t_l>0}\right)_{N\in\mathbb{N}}. \tag{9.33}$$

Proof. For each $0 \le l \le k$ we use the tightness criterion in [49, Proposition 3.2.3.]. □

2 Stability property of the 2-block estimators, i.e., for $L(N)$ such that $\lim_{N\to\infty} L(N) = \infty$ and $\lim_{N\to\infty} L(N)/N = 0$,

$$\lim_{N\to\infty} \sup_{0\le t\le L(N)} \left|\bar{\Theta}^{(l),\Omega_N^{k+1}}(N^l t_l) - \bar{\Theta}^{(l),\Omega_N^{k+1}}(N^l t_l - N^{l-1}t)\right| = 0 \text{ in probability} \tag{9.34}$$

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and, for all $l \leq m \leq k+1$,

$$\lim_{N\to\infty} \sup_{0\leq t\leq L(N)} \left| \Theta_{y_m}^{(l),\Omega_N^{k+1}}(N^l t_l) - \Theta_{y_m}^{(l),\Omega_N^{k+1}}(N^l t_l - N^{l-1} t_l) \right| = 0 \text{ in probability.} \tag{9.35}$$

Proof. Use a similar computation as in the proof of Lemma 8.3.4. □

3 We analyse the behaviour of the slow seed-banks by proving the following lemma.

Lemma 9.2.1. *[Slow seed-banks in the multi-level system] Let $\Theta_{y_m,i}^{(l)}$ denote the m-dormant l-block average containing colony $i \in \Omega_N^{k+1}$. Then for all $i \in \Omega_N^{k+1}$, $m < k+1$, $l < m$ and $t_l > 0$,*

$$\lim_{N\to\infty} \left[y_{i,m}^{\Omega_N^{k+1}}(\bar{t} + N^l t_l) - \Theta_{y_m,i}^{(m),\Omega_N^{k+1}}(\bar{t} + N^l t_l) \right] = 0 \quad a.s. \tag{9.36}$$

and hence

$$\lim_{N\to\infty} \left[\Theta_{y_m,i}^{(l),\Omega_N^{k+1}}(\bar{t} + N^l t_l) - \Theta_{y_m,i}^{(m),\Omega_N^{k+1}}(\bar{t} + N^l t_l) \right] = 0 \quad a.s. \tag{9.37}$$

Proof. We can proceed as in the proof of Lemma 8.3.20, after adapting the kernel $b^{[N^2]}(\cdot,\cdot)$ to the kernel $b^{\Omega_N^{k+1}}(\cdot,\cdot)$. Then we can use that, from each of the $m < k+1$ m-dormant populations, individuals wake up before time $\bar{t}$ with probability 1. For individuals starting from an m-dormant state, we define the coupling event

$$H_t^{m,\Omega_N^{k+1}} = \{RW^{\Omega_N^{k+1}} \text{ has migrated over distance } m \text{ at least once up to time } t\}. \tag{9.38}$$

The migration over distance m is needed because we need m-dormant individuals to be uniformly distributed over the m-block in order to almost surely equal the state of the m-block. □

4 We prove the convergence of the single components. Recall that there are N^{k+1-l} l-blocks in Ω_N^{k+1}. Since tightness of components implies tightness of the process, step 1 implies that for $0 \leq l \leq k$ the full l-block processes

$$\left(\left(\boldsymbol{\Theta}_i^{\mathrm{aux},(l),\Omega_N^{k+1}}(\bar{t} + N^l t_l) \right)_{t_l>0,\, i\in[N^{k+1-l}]} \right)_{N\in\mathbb{N}} \tag{9.39}$$

are tight. From the tightness in steps 1 we can construct a subsequence $(N_n)_{n\in\mathbb{N}}$ along which, for all $0 \leq l \leq k$,

$$\lim_{n\to\infty} \mathcal{L}\left[\left(\boldsymbol{\Theta}_i^{\mathrm{aux},(1),\Omega_{N_n}^{k+1}}(\bar{t} + N_n^l t_l) \right)_{t_l>0,\, i\in[N_n^{k+1-l}]} \right] \tag{9.40}$$

exists. Note that $\bar{t}$ depends on the subsequence. For example, along the subsequence $(N_{\tilde{n}})_{\tilde{n}\in\mathbb{N}}$,

$$\bar{t} = L(N)N^k + \sum_{n=0}^{k} t_n N_{\tilde{n}}^n. \tag{9.41}$$

We define the measure

$$\nu_{\mathbf{\Theta}}^{(0)} = \prod_{i\in\mathbb{N}_0} \Gamma_{\Theta_i}^{(0)}(\bar{t}), \tag{9.42}$$

where

$$\Theta_i \in \mathfrak{s}^{k+1}. \tag{9.43}$$

In this step we show that along the same subsequence the single components converge to the infinite system. We show that if

$$\lim_{n\to\infty} \mathcal{L}[(\mathbf{\Theta}^{\mathrm{aux},(1),\Omega_{N_n}^{k+1}}(\bar{t}))_{i\in[N_n^k]}] = P^{(1)}, \tag{9.44}$$

then

$$\lim_{n\to\infty} \mathcal{L}\Big[\Big(Z^{\Omega_{N_n}^{k+1}}(\bar{t}+t_0)\Big)_{t_0\geq 0}\Big] = \mathcal{L}\left[(Z^{\nu^{(0)}(\bar{t})}(t_0))_{t_0\geq 0}\right], \tag{9.45}$$

where

$$\nu^{(0)}(\bar{t}) = \int \nu_u^{(0)} P^{(1)}(\mathrm{d}u). \tag{9.46}$$

Here, $(Z^{\nu^{(0)}(\bar{t})}(t_0))_{t_0\geq 0}$ is the process starting from $\nu^{(0)}(\bar{t})$ with components evolving according to (8.18), where θ is now a random variable that inherits its law from

$$\lim_{n\to\infty} \mathcal{L}[(\mathbf{\Theta}^{\mathrm{aux},(1),\Omega_{N_n}^{k+1}}(\bar{t}))_{i\in[N_n^k]}] \tag{9.47}$$

and, similarly, the laws of $y_{m,0}$, $1\leq l\leq k+1$ in the limiting process $(Z^{\nu^{(0)}(t_2)}(t_0))_{t_0\geq 0}$ are determined by

$$\lim_{n\to\infty} \mathcal{L}[(\mathbf{\Theta}^{\mathrm{aux},(1),\Omega_{N_n}^{k+1}}(\bar{t}))_{i\in[N_n^k]}]. \tag{9.48}$$

Note that we choose the subsequence $(N_n)_{n\in\mathbb{N}}$ in such a way that we know that the law $P^{(1)}$ in (9.44) exists.

Proof. Proceed as in the proof of Proposition 8.3.5. Note that the assumptions on the seed-banks in Proposition 8.3.5 follow from the choice of the subsequence and Lemma 9.2.1. □

5 Using the limiting evolution of the single colonies obtained in step 4, we can identify the limiting l-block process along the same subsequence. For $1\leq l\leq k$, we show that if

$$\lim_{n\to\infty} \mathcal{L}[(\mathbf{\Theta}^{\mathrm{aux},(l+1),\Omega_{N_n}^{k+1}}(\bar{t}))_{i\in[N_n^k]}] = P^{(l+1)}, \tag{9.49}$$

then

$$\lim_{n\to\infty} \mathcal{L}\left[\left(\mathbf{\Theta}^{\mathrm{aux},(l),\Omega_{N_n}^{k+1}}(\bar{t}+N_n^l t_l)\right)_{t_l>0,\, i\in[N_n^{k+1-l}]}\right] = \mathcal{L}\left[(z_{l,\mathbf{\Theta}^{(l+1)}}^{\mathrm{aux}}(t))_{t\geq 0}\right], \tag{9.50}$$

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where $\mathbf{\Theta}^{(l+1)} = (\mathbf{\Theta}_x^{(l+1)}, (\mathbf{\Theta}_{y_m,l}^{(l+1)})_{m=l+1}^{k+1}) \in [0,1] \times [0,1]^{k+2-(l+1)}$,

$$\begin{aligned} \mathcal{L}\left[z^{\text{aux}}_{l,\mathbf{\Theta}^{(l)}}(0)\right] &= \Gamma^{(l),\text{aux}}_{\mathbf{\Theta}^{(l+1)}}, \\ \Gamma^{(l),\text{aux}}_{\mathbf{\Theta}^{(l+1)}} &= \int_{[0,1]\times[0,1]^{k+2-(l+1)}} \Gamma^{(l),\text{aux}}_u P^{(l+1)}(\mathrm{d}u) \end{aligned} \tag{9.51}$$

and $(z^{\text{aux}}_{l,\mathbf{\Theta}^{(l+1)}}(t))_{t\geq 0}$ is the process evolving according to (9.14) with $\theta, (y_{m,l})_{m=l}^{k+1}$ replaced by the random variables $\mathbf{\Theta}_x^{(l+1)}, (\mathbf{\Theta}_{y_m,l}^{(l+1)})_{m=l}^{k+1}$. Note that by the choice of the subsequence $(N_n)_{n\in\mathbb{N}}$ we know that for $1 \leq l \leq k$ the limiting laws in (9.49) exist.

Proof. The proof goes by induction. Using the convergence of the single components, we can proceed as in the proof of Proposition 8.3.10 to prove the convergence of the 1-blocks averages

$$\lim_{n\to\infty} \mathcal{L}\left[\left(\mathbf{\Theta}_i^{\text{aux},(1),\Omega_{N_n}^{k+1}}(\bar{t} + N_n t_1)\right)_{t_1>0,\, i\in[N_n^{k+1-l}]}\right]. \tag{9.52}$$

Then, assuming that we have the convergence for all $0 \leq l \leq L$, we get

$$\lim_{n\to\infty} \mathcal{L}\left[\left(\mathbf{\Theta}_i^{\text{aux},(l),\Omega_{N_n}^{k+1}}(\bar{t} + N_n^l t_l)\right)_{t_l>0,\, i\in[N_n^{k+1-l}]}\right], \tag{9.53}$$

and we prove the convergence of

$$\lim_{n\to\infty} \mathcal{L}\left[\left(\mathbf{\Theta}_i^{\text{aux},(L+1),\Omega_{N_n}^{k+1}}(\bar{t} + N_n^{(L+1)} t_{(L+1)})\right)_{t_{(L+1)}>0,\, i\in[N_n^{k+1-(L+1)}]}\right]. \tag{9.54}$$

This is done using a similar proof strategy as in the proof of Proposition 8.3.10. In particular, we need to derive the l-level equivalent of Lemma 8.3.13. Since this lemma is also key to proving convergence in the Meyer-Zheng topology, we state it explicitly below. □

Lemma 9.2.2 (l-block averages). *Define*

$$\begin{aligned} &\Delta_{\Sigma}^{(l),\Omega_N^{k+1}}(N^{l-1}t_{l-1}) \\ &= \frac{\Theta_x^{(l),\Omega_N^{k+1}}(N^{l-1}t_{l-1}) + \sum_{m=0}^{l-2} K_m \Theta_{y_m}^{(l),\Omega_N^{k+1}}(N^{l-1}t_{l-1})}{1+\sum_{m=0}^{l-2} K_m} - \Theta_{y_{l-1}}^{(l),\Omega_N^{k+1}}(N^{l-1}t_{l-1}) \end{aligned} \tag{9.55}$$

and

$$R_l = \frac{1+\sum_{m=0}^{l-1} K_m}{1+\sum_{m=0}^{l-2} K_m}. \tag{9.56}$$

For $t \geq 0$ *set* $\bar{\Theta}^{(0)}(t) = \Theta_x^{(0)}(t) = x_0(t)$. *Then, for* $1 \leq l \leq k$,

$$
\begin{aligned}
&\mathbb{E}\left[\left|\Delta_\Sigma^{(l),\Omega_N^{k+1}}(N^{l-1}t_{l-1})\right|\right] \\
&\leq \sqrt{\mathbb{E}\left[\left(\Delta_\Sigma^{(l),\Omega_N^{k+1}}(0)\right)^2\right]\mathrm{e}^{-e_l R_l t_{l-1}}} \\
&\quad + \sqrt{\int_0^{t_1} \mathrm{d}s\, 2e_l R_l \mathrm{e}^{-2e_l R_l (t_1 - s)} \mathbb{E}\left[\left|\bar{\Theta}^{(l-1),\Omega_N^{k+1}}(N^{l-1}s) - \Theta_x^{(l-1),\Omega_N^{k+1}}(N^{l-1}s)\right|\right]} \\
&\quad + \sqrt{\frac{1}{N}\frac{1}{2e_l(1+\sum_{m=0}^{l-1}K_m)}\left[\sum_{n=l+1}^{k}\frac{c_{n-1}}{N^{n-(l+1)}} + \sum_{m=l}^{k}\frac{K_m e_m}{N^{m-l}} + E_{l-1}||g||\right]}.
\end{aligned}
\tag{9.57}
$$

Proof. Proceed like in the proof of Lemma 8.3.13, using the SSDE in (9.4) instead of the SSDE in (8.6). □

We obtain the following useful corollary from Lemma 9.2.2.

Corollary 9.2.3. *For all* $1 \leq l \leq k$, $s > 0$ *and* $\tilde{l} \geq l$,

$$
\lim_{N\to\infty} \mathbb{E}\left[\left|\bar{\Theta}^{(l-1),\Omega_N^{k+1}}(N^{\tilde{l}-1}s) - \Theta_x^{(l-1),\Omega_N^{k+1}}(N^{\tilde{l}-1}s)\right|\right] = 0. \tag{9.58}
$$

Proof. We proceed by induction. The result for $l = 1$ is trivial. Suppose that the result holds for $l = L$. Then for $l = L + 1$ we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\left|\bar{\Theta}^{(L),\Omega_N^{k+1}}(N^L s) - \Theta_x^{(L),\Omega_N^{k+1}}(N^L s)\right|\right] \\
&\leq \mathbb{E}\left[\left|\bar{\Theta}^{(L),\Omega_N^{k+1}}(N^L s) - \frac{1}{N}\sum_{i=0}^{N-1}\bar{\Theta}_i^{(L-1),\Omega_N^{k+1}}(N^L s)\right|\right] \\
&\qquad + \frac{1}{N}\sum_{i=0}^{N-1}\mathbb{E}\left[\left|\bar{\Theta}_i^{(L-1),\Omega_N^{k+1}}(N^L s) - \Theta_{x,i}^{(L-1),\Omega_N^{k+1}}(N^L s)\right|\right].
\end{aligned}
\tag{9.59}
$$

Note that the second term in the right-hand side of (9.59) tends to 0 as $N \to \infty$ by the induction hypothesis. For the first term in the right-hand side of (9.59), note that

$$\begin{aligned}
&\mathbb{E}\left[\left|\bar{\Theta}^{(L),\Omega_N^{k+1}}(N^L s) - \frac{1}{N}\sum_{i=0}^{N-1}\bar{\Theta}_i^{(L-1),\Omega_N^{k+1}}(N^L s)\right|\right] \\
&= \mathbb{E}\left[\left|\frac{\Theta_x^{(L),\Omega_N^{k+1}}(N^L s) + \sum_{m=0}^{L-1} K_m \Theta_{y_m}^{(L),\Omega_N^{k+1}}(N^L s)}{1+\sum_{m=0}^{L-1} K_m}\right.\right. \\
&\qquad \left.\left. - \frac{\Theta_x^{(L),\Omega_N^{k+1}}(N^L s) + \sum_{m=0}^{L-2} K_m \Theta_{y_m}^{(L),\Omega_N^{k+1}}(N^L s)}{1+\sum_{m=0}^{L-2} K_m}\right|\right] \\
&= \frac{K_{L-1}}{1+\sum_{m=0}^{L-1} K_m}\mathbb{E}\left[\left|\Theta_{y_{L-1}}^{(L),\Omega_N^{k+1}}(N^L s) - \bar{\Theta}^{(L),\Omega_N^{k+1}}(N^L s)\right|\right].
\end{aligned} \tag{9.60}$$

Invoking Lemma 9.2.2 and using the induction hypothesis, we see that for $s > 0$ and $\tilde{l} \geq L$ indeed

$$\lim_{N\to\infty} \mathbb{E}\left[\left|\bar{\Theta}^{(L),\Omega_N^{k+1}}(N^{\tilde{l}} s) - \Theta_x^{(L),\Omega_N^{k+1}}(N^{\tilde{l}} s)\right|\right] = 0. \tag{9.61}$$

□

6 Show that the convergence in step 4 and step 5 actually holds along each subsequence. Therefore we obtain the limiting evolution of the single colonies, the auxiliary 1-block process and the effective 2-block process. This follows from the fact that the auxiliary k-estimator process converges to the same limit along every subsequence. Consequently, the same holds for the auxiliary $k-1$-estimator process. In this way we can traverse back through the levels to obtain that all l-estimator process converges as $N \to \infty$.

Define, for $0 \leq l \leq k$,

$$\mathfrak{s}_l^{k+1} = [0,1] \times [0,1]^{k+2-l}. \tag{9.62}$$

We obtain, for $0 \leq l \leq k-1$,

$$\begin{aligned}
&\lim_{N\to\infty} \mathcal{L}[(\boldsymbol{\Theta}^{\mathrm{aux},(l+1),\Omega_N^{k+1}}(\bar{t}))]] = \Gamma_{\boldsymbol{\Theta}^{(l+2)}}^{(l+1),\mathrm{aux}}, \\
&\Gamma_{\boldsymbol{\Theta}^{(l+2)}}^{(l+1),\mathrm{aux}} = \int_{\mathfrak{s}_{l+2}^{k+1}} \cdots \int_{\mathfrak{s}_k^{k+1}} \Gamma_{\vartheta_k}^{(k),\mathrm{aux}}(\mathrm{d}u_k) \cdots \Gamma_{u_{l+3}}^{(l+2),\mathrm{aux}}(\mathrm{d}u_{l+2}) \Gamma_{u_{l+2}}^{(l+1),\mathrm{aux}}.
\end{aligned} \tag{9.63}$$

Therefore by step 5

$$\lim_{N\to\infty} \mathcal{L}\left[\left(\boldsymbol{\Theta}^{\mathrm{aux},(l),\Omega_N^{k+1}}(\bar{t} + N_n^l t_l)\right)_{t_l>0,\, i\in[N^{k+1-l}]}\right] = \mathcal{L}\left[(z_{l,\boldsymbol{\Theta}^{(l+1)}}^{\mathrm{aux}}(t))_{t\geq 0}\right], \tag{9.64}$$

where $\boldsymbol{\Theta}^{(l+1)} = (\boldsymbol{\Theta}_x^{(l+1)}, (\boldsymbol{\Theta}_{y_m,l}^{(l+1)})_{m=l}^{k+1}) \in \mathfrak{s}_l^{(k+1)}$ are random variables with law

$$\mathcal{L}\left[\boldsymbol{\Theta}^{(l+1)}\right] = \Gamma_{\boldsymbol{\Theta}^{(l+2)}}^{(l+1),\mathrm{aux}}. \tag{9.65}$$

The initial state of the limiting process in (9.64) is given by

$$\mathcal{L}\left[z^{\text{aux}}_{l,\boldsymbol{\Theta}^{(l)}}(0)\right] = \Gamma^{(l),\text{aux}}_{\boldsymbol{\Theta}^{(l+1)}},$$
$$\Gamma^{(l),\text{aux}}_{\boldsymbol{\Theta}^{(l+1)}} = \int_{\mathfrak{s}^{k+1}_{l+2}} \cdots \int_{\mathfrak{s}^{k+1}_{k}} \Gamma^{(k),\text{aux}}_{\vartheta+k}(\mathrm{d}u_k) \cdots \Gamma^{(l+2),\text{aux}}_{u_{l+2}}(\mathrm{d}u_{l+1}) \Gamma^{(l+1),\text{aux}}_{u_{l+1}} \tag{9.66}$$

and $(z^{\text{aux}}_{l,\boldsymbol{\Theta}^{(l+1)}}(t))_{t\geq 0}$ is the process evolving according to (9.14) with θ, $(y_{m,l})^{k+1}_{m=l+1}$ replaced by the random variables $\boldsymbol{\Theta}^{(l+1)}_x, (\boldsymbol{\Theta}^{(l+1)}_{y_m})^{k+1}_{m=l+1}$. Recall that, by Lemma 9.2.1, we have, for $l+1 \leq m \leq k+1$,

$$\boldsymbol{\Theta}^{(l+1)}_{y_m} = \boldsymbol{\Theta}^{(m)}_{y_m}. \tag{9.67}$$

7 Use the Meyer-Zheng topology to obtain Proposition 9.1.1(b).

Proof. Note that Lemma 9.2.2 and Corollary 9.2.3 together imply

$$\begin{aligned}
&\lim_{N\to\infty} \mathbb{E}\left[\left|\bar{\Theta}^{(l),\Omega^{k+1}_N}(N^l t_l) - \Theta^{(l),\Omega^{k+1}_N}_x(N^l t_l)\right|\right] = 0,\\
&\lim_{N\to\infty} \mathbb{E}\left[\left|\bar{\Theta}^{(l),\Omega^{k+1}_N}(N^l t_l) - \Theta^{(l),\Omega^{k+1}_N}_{y_m}(N^l t_l)\right|\right] = 0, \text{ for } 0 \leq m \leq l-1.
\end{aligned} \tag{9.68}$$

Combining the result obtained in step 6 with the proof strategy followed in Section 8.3.10, we get the claim. □

8 Finally, we prove Proposition 9.1.1(a).

Proof. Step 6 and step 7 yield the laws of the components $\mathcal{L}[M^k_l]$ of the interaction chain $(M^k_{-l})^0_{-l=-(k+1)}$. Note that the state space $([0,1]\times[0,1]^{k+2})^{k+2}$ is compact, and therefore the sequence of random variables

$$\left(\left(\boldsymbol{\Theta}^{(l),\Omega^{k+1}_N}(\bar{t})\right)_{l=k+1,k,\dots,0}\right)_{N\in\mathbb{N}} \tag{9.69}$$

is tight. For any

$$\begin{aligned}
&f\colon\ ([0,1]\times[0,1]^{k+2})^{n+2} \to \mathbb{R},\\
&f(x) = \prod_{i=1}^{n} f_i(x_i),\\
&f_i \in \mathcal{C}_b([0,1]\to\mathbb{R}),
\end{aligned} \tag{9.70}$$

we can use conditioning on the previous block average to obtain

$$\lim_{N\to\infty} \mathbb{E}\left[f\left(\left(\boldsymbol{\Theta}^{(l),\Omega^{k+1}_N}(\bar{t})\right)_{l=k+1,k,\dots,0}\right)\right] = \mathbb{E}\left[f\left((M^k_{-l})^0_{-l=-(k+1)}\right)\right]. \tag{9.71}$$

Using that the set of functions of the form (9.70) is separating, we obtain the claim. □

§9.3 Proof: of the hierarchical multi-scale limit theorems.

In this section we prove Theorems 4.4.2 and 4.4.4. We start by proving Theorem 4.4.4. Theorem 4.4.2 will follow from Theorem 4.4.4 by projection onto the effective components.

Proof of Theorem 4.4.2

Proof. Recall the estimators in (4.70). Like for the finite-level hierarchical mean-field system, we can define the auxiliary estimator process by

$$\mathbf{\Theta}^{(l),\mathrm{aux},\Omega_N}(t) = \big(\bar\Theta^{(l),\Omega_N}(t), \big(\Theta^{(l),\Omega_N}_{y_m}(t)\big)_{m=l}^{\infty}\big). \tag{9.72}$$

For $l,k \in \mathbb{N}$ the processes $(\mathbf{\Theta}^{(l),\mathrm{aux},\Omega_N}(\bar t{+}N^k t))_{t>0}$ evolve according to (recall, (4.114))

$$\begin{aligned}
\mathrm{d}\bar\Theta^{(l),\Omega_N}(N^k t) &= E_l \sum_{n=l+1}^{\infty} \frac{c_{n-1}}{N^{n-1-k}} \Big[\Theta^{(n),\Omega_N}_x(N^k t) - \Theta^{(l),\Omega_N}_x(N^k t)\Big]\,\mathrm{d}t \\
&\quad + E_l \sqrt{\frac{N^k}{N^{2l}} \sum_{\xi\in B_l} g\big(x_\xi(N^k t)\big)}\,\mathrm{d}w(t) \\
&\quad + E_l \sum_{m=l}^{\infty} \frac{K_m e_m}{N^{m-k}} \Big[\Theta^{(l),\Omega_N}_{y_m}(N^k t) - \Theta^{(l),\Omega_N}_x(N^k t)\Big]\,\mathrm{d}t, \\
\mathrm{d}\Theta^{(l),\Omega_N}_{y_m}(N^k t) &= \frac{e_m}{N^{m-k}} \Big[\Theta^{(l),\Omega_N}_x(N^k t) - \Theta^{(l),\Omega_N}_{y_m}(N^k t)\Big]\,\mathrm{d}t, \qquad l \le m \le \infty.
\end{aligned} \tag{9.73}$$

Therefore, for $l > k$ and all $\epsilon > 0$,

$$\begin{aligned}
&\mathbb{P}\Bigg[\sup_{0\le t\le L(N)} \Big|\bar\Theta^{(l),\Omega_N}(\bar t) - \bar\Theta^{(l),\Omega_N}(\bar t + N^k t)\Big| > \epsilon\Bigg] \\
&= \mathbb{P}\Bigg[\sup_{0\le t\le L(N)} E_l\Bigg|\int_{\bar t}^{\bar t+N^k t} \mathrm{d}r \sum_{n=l+1}^{\infty} \frac{c_{n-1}}{N^{n-1}} \Big[\Theta^{(n),\Omega_N}_x(r) - \Theta^{(l),\Omega_N}_x(r)\Big] \\
&\qquad + \int_{\bar t}^{\bar t+N^k t} \mathrm{d}r \sum_{m=l}^{\infty} \frac{K_m e_m}{N^m} \Big[\Theta^{(l),\Omega_N}_{y_m}(r) - \Theta^{(l),\Omega_N}_x(r)\Big] \\
&\qquad + \int_{\bar t}^{\bar t+N^k t} \mathrm{d}w_i(r) \sqrt{\frac{1}{N^{2l}} \sum_{\xi\in B_l} g\big(x_\xi(r)\big)}\,\Bigg| > \epsilon\Bigg] \\
&\le \mathbb{P}\Bigg[\sup_{0\le t\le L(N)} E_l\Bigg|\int_{\bar t}^{\bar t+N^k t} \mathrm{d}w_i(r) \sqrt{\frac{1}{N^{2l}} \sum_{\xi\in B_l} g\big(x_\xi(r)\big)}\,\Bigg| \\
&\qquad > \epsilon - t\Bigg[\sum_{n=l+1}^{\infty} \frac{c_{n-1}}{N^{n-1-k}} + \sum_{m=l}^{\infty} \frac{K_m e_m}{N^{m-k}}\Bigg]\Bigg].
\end{aligned} \tag{9.74}$$

Note that, since $l > k$,

$$\lim_{N\to\infty} t\left[\sum_{n=l+1}^{\infty}\frac{c_{n-1}}{N^{n-1-k}}+\sum_{m=l}^{\infty}\frac{K_m e_m}{N^{m-k}}\right]=0. \tag{9.75}$$

Hence, like in the proof of Lemma 8.3.4, we can use an optional stopping argument to obtain

$$\lim_{N\to\infty}\sup_{0\leq t\leq L(N)}\left|\bar{\Theta}^{(l),\Omega_N}(\bar{t})-\bar{\Theta}^{(l),\Omega_N}(\bar{t}+N^k t)\right|=0 \qquad \text{in probability.} \tag{9.76}$$

Using a similar computation as in (9.74), we can show

$$\lim_{N\to\infty}\sup_{0\leq t\leq L(N)}\left|\bar{\Theta}^{(l),\Omega_N}_{y_m}(\bar{t})-\Theta^{(l),\Omega_N}_{y_m}(\bar{t}+N^k t)\right|=0 \qquad \text{in probability.} \tag{9.77}$$

Hence we obtain that, on time scale N^k as $N\to\infty$, the process $(\mathbf{\Theta}^{(l),\mathrm{aux},\Omega_N}(\bar{t}+N^k t))_{t>0}$ does not evolve and therefore is still in its initial state$(\mathbf{\Theta}^{(l),\mathrm{aux},\Omega_N}(\bar{t}))$.

Using that the l-auxiliary estimator processes do not move for $l > k$, they function like the "outside world" for the finite-level mean-field system in Section 9.1. Therefore we can proceed as in the proof of Proposition 9.1.1 to prove the second and third line in (4.88) in Theorem 4.4.2. The l-block estimator process $(\mathbf{\Theta}^{(l),\mathrm{aux},\Omega_N}(\bar{t}+N^l t))_{t>0}$ evolves according to (9.73) with $l = k$. Note that the extra interactions due to migration over larger blocks $l > k$ and exchange with deeper seed-banks $m > k$ in (4.126) are of order $\mathcal{O}(1/N)$. Therefore these terms vanish as $N\to\infty$, and we can just proceed as in the scheme of Section 9.2, to obtain the second and third line in (4.88) in Theorem 4.4.2. Using these results, we obtain that, for $l > k$,

$$\mathbf{\Theta}^{(l),\mathrm{aux},\Omega_N}(\bar{t})=\delta_{M^k_{-(k+1)}}. \tag{9.78}$$

□

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