Spatial populations with seed-bank

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CHAPTER 8

Two-level three-colour mean-field system

To get a proper understanding of how the migration comes into play on different space-time scales, we next look at a two-level mean-field system where the geographic space consists of two layers and the seed-bank consist of three layers, corresponding three colours 0, 1, 2. In Section 8.1 we give the set-up of the two-level three-colour mean-field model. In Section 8.2 we give a scheme to prove the analysis of the two-level three-colour mean-field model. Finally, in Section 8.3 we prove the steps of the scheme given in Section 8.2.

§8.1 Two-level three-colour mean-field finite-systems scheme

We consider a restricted version of the SSDE in (4.20) on the finite geographic space

$$[N^2] = \{0, 1, \ldots, N^2 - 1\}, \qquad N \in \mathbb{N}. \tag{8.1}$$

This space should be interpreted as grouping the N-blocks consisting of N colonies together, i.e.

$$[N^2] = \bigcup_{l=0}^{N-1} \{Nl, Nl+1, \cdots, Nl+N-1\}. \tag{8.2}$$

With this interpretation we can use the metric $d_{[N^2]}$ that is induced by the metric d_{Ω_N} on the hierarchical group Ω_N (recall (4.4)). The migration kernel $a^{\Omega_N}(\cdot, \cdot)$ is restricted to $[N^2]$ by setting all migration rates outside the 2-block equal to 0, i.e., $c_k = 0$ for all $k \geq 2$. Hence the migration kernel is given by

$$a^{[N^2]}(i,j) = 1_{\{d_{[N^2]}(i,j) \leq 1\}} \frac{c_0}{N} + \frac{c_1}{N^3}, \tag{8.3}$$

where $c_0, c_1 \in (0, \infty)$ are constants. The seed-bank of the restricted system consists of *three colours*, labeled 0 1 and 2, with exchange rates given by $K_0 e_0, e_0$, $\frac{K_1 e_1}{N}, \frac{e_1}{N}$ and $\frac{K_2 e_2}{N^2}, \frac{e_2}{N^2}$ respectively. The state space of the restricted system is

$$S = \mathfrak{s}^{[N^2]}, \qquad \mathfrak{s} = [0,1] \times [0,1]^3, \tag{8.4}$$

and the restricted system is denoted by

$$(Z^{[N^2]}(t))_{t\geq 0} = \Big(X^{[N^2]}(t), \Big(Y_0^{[N^2]}(t), Y_1^{[N^2]}(t), Y_2^{[N^2]}(t)\Big)\Big)_{t\geq 0},$$
$$\Big(X^{[N^2]}(t), \Big(Y_0^{[N^2]}(t), Y_1^{[N^2]}(t), Y_2^{[N^2]}(t)\Big)\Big) = \Big(x_i^{[N^2]}(t), \Big(y_{i,0}^{[N^2]}(t), y_{i,1}^{[N^2]}(t), y_{i,2}^{[N^2]}(t)\Big)\Big)_{i\in[N^2]}. \tag{8.5}$$

The components of the restricted system $(Z^{[N^2]}(t))_{t\geq 0}$ evolve according to the SSDE

$$\begin{aligned}
\mathrm{d}x_i^{[N^2]}(t) = {} & \frac{c_0}{N} \sum_{j\in[N^2]} 1_{\{d_{[N^2]}(i,j)\leq 1\}} [x_j^{[N^2]}(t) - x_i^{[N^2]}(t)]\,\mathrm{d}t \\
& + \frac{c_1}{N^3} \sum_{j\in[N^2]} [x_j^{[N^2]}(t) - x_i^{[N^2]}(t)]\,\mathrm{d}t + \sqrt{g(x_i^{[N^2]}(t))}\,\mathrm{d}w_i(t) \\
& + K_0 e_0\,[y_{i,0}^{[N^2]}(t) - x_i^{[N^2]}(t)]\,\mathrm{d}t \\
& + \frac{K_1 e_1}{N}\,[y_{i,1}^{[N^2]}(t) - x_i^{[N^2]}(t)]\,\mathrm{d}t \\
& + \frac{K_2 e_2}{N^2}\,[y_{i,2}^{[N^2]}(t) - x_i^{[N^2]}(t)]\,\mathrm{d}t, \\
\mathrm{d}y_{i,0}^{[N^2]}(t) = {} & e_0\,[x_i^{[N^2]}(t) - y_{i,0}^{[N^2]}(t)]\,\mathrm{d}t, \\
\mathrm{d}y_{i,1}^{[N^2]}(t) = {} & \frac{e_1}{N}\,[x_i^{[N^2]}(t) - y_{i,1}^{[N^2]}(t)]\,\mathrm{d}t, \\
\mathrm{d}y_{i,2}^{[N^2]}(t) = {} & \frac{e_2}{N^2}\,[x_i^{[N^2]}(t) - y_{i,2}^{[N^2]}(t)]\,\mathrm{d}t, \qquad i \in [N^2],
\end{aligned} \tag{8.6}$$

which is a special case of (4.20). By [67, Theorem 3.1], the SSDE in (8.6) is the unique solution. It is important to note that we can write the SSDE also

$$\begin{aligned}
\mathrm{d}x_i^{[N^2]}(t) = {} & c_0 \left[\frac{1}{N} \sum_{j\in[N]_i} x_j^{[N^2]}(t) - x_i^{[N^2]}(t)\right] \mathrm{d}t \\
& + \frac{c_1}{N} \left[\frac{1}{N^2} \sum_{j\in[N^2]} x_j^{[N^2]}(t) - x_i^{[N^2]}(t)\right] \mathrm{d}t + \sqrt{g(x_i^{[N^2]}(t))}\,\mathrm{d}w_i(t) \\
& + K_0 e_0\,[y_{i,0}^{[N^2]}(t) - x_i^{[N^2]}(t)]\,\mathrm{d}t + \frac{K_1 e_1}{N}\,[y_{i,1}^{[N^2]}(t) - x_i^{[N^2]}(t)]\,\mathrm{d}t \\
& + \frac{K_2 e_2}{N^2}\,[y_{i,2}^{[N^2]}(t) - x_i^{[N^2]}(t)]\,\mathrm{d}t, \\
\mathrm{d}y_{i,0}^{[N^2]}(t) = {} & e_0\,[x_i^{[N^2]}(t) - y_{i,0}^{[N^2]}(t)]\,\mathrm{d}t, \\
\mathrm{d}y_{i,1}^{[N^2]}(t) = {} & \frac{e_1}{N}\,[x_i^{[N^2]}(t) - y_{i,1}^{[N^2]}(t)]\,\mathrm{d}t, \\
\mathrm{d}y_{i,2}^{[N^2]}(t) = {} & \frac{e_2}{N^2}\,[x_i^{[N^2]}(t) - y_{i,2}^{[N^2]}(t)]\,\mathrm{d}t, \qquad i \in [N^2],
\end{aligned} \tag{8.7}$$

where $[N]_i$ denotes the set of colonies in the 1-block around site $i \in [N^2]$. Therefore the migration term for a single colony in the two-level mean-field system can be interpreted as a drift towards the 1-block average of the active population at rate c_0 and a drift towards the 2-block average of the active population at rate $\frac{c_1}{N}$. We are interested in (8.7) on time scales N^0, N and N^2. On time scale N^0 we will look at the single colonies, i.e., space-time scale 0. On time scale N we will look at the 1-block averages, i.e., space-time scale 1 and on time scale N^2 we will look at the 2-block averages, i.e., space-time scale 2. In the sequel we will focus on site 0, the 1-block around site 0 and the 2-block around site 0. We will suppress this site from the notation, but instead use subscripts 0, 1, 2 to indicate when we look at a single colony, a 1-block average or a 2-block average. We will use the convention that in the subscript of a dormant population the first subscript denotes the colour and the second subscript denotes the level of the block, so $y_{0,1}$ is the 1-block average around site 0 of the dormant population with colour 0, while $y_{1,0}$ is the 1-dormant single colony at site 0. Heuristically, we can read off the following results from the SSDE in (8.7).

• **On time scale** $1 = N^0$ (i.e., space-time scale 0) in the limit as $N \to \infty$, the colour-1 dormant population and the colour-2 dormant population do not yet move. Hence

$$\left(y_{1,0}^{[N^2]}(t_0), y_{2,0}^{[N^2]}(t_0)\right)_{t_0 \geq 0}, \tag{8.8}$$

converges as $N \to \infty$ to the constant processes on time scale t_0. Therefore the colour 1-dormant population and the colour 2-dormant population are both slow seed-banks on space-time scale 0. The components $((x_0^{[N^2]}(t_0), y_{0,0}^{[N^2]}(t_0)))_{t_0 \geq 0}$ converge to i.i.d. copies of the single-colony McKean-Vlasov process in (6.1), where in the corresponding SSDE the parameters e, K, c are replaced by c_0, e_0, K_0 and $E = 1$. So, on time scale 1 we only see the colour 0-dormant population evolve. Therefore the colour-0 dormant population is the *effective seed-bank* on time scale t_0. The process

$$(z_0^{[N^2]}(t_0))_{t_0 \geq 0} = (x_0^{[N^2]}(t_0), y_{0,0}^{[N^2]}(t_0))_{t_0 \geq 0}, \tag{8.9}$$

will be called *the single colony effective process.*

• **On time scale** N^1 (i.e., space-time scale 1), we look at the averages

$$\begin{aligned}(z_1^{[N^2]}(t_1))_{t_1>0} &= \left(x_1^{[N^2]}(t_1), \left(y_{0,1}^{[N^2]}(t_1), y_{1,1}^{[N^2]}(t_1), y_{2,1}^{[N^2]}(t_1)\right)\right)_{t_1>0} \\ &= \left(\frac{1}{N}\sum_{i\in[N]} x_i^{[N^2]}(Nt_1), \left(\frac{1}{N}\sum_{i\in[N]} y_{i,0}^{[N^2]}(Nt_1), \frac{1}{N}\sum_{i\in[N]} y_{i,1}^{[N^2]}(Nt_1), \frac{1}{N}\sum_{i\in[N]} y_{i,2}^{[N^2]}(Nt_1)\right)\right)_{t_1>0}.\end{aligned} \tag{8.10}$$

(Recall Remark 4.2.4 to appreciate the notation.) We use the lower index 1 to indicate that the average is the analogue of the 1-block average defined in (4.22). Using (8.6),

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we see that the dynamics of the system in (8.10) is given by the SSDE

$$
\begin{aligned}
\mathrm{d}x_1^{[N^2]}(t_1) &= c_1 \left[\frac{1}{N^2}\sum_{j\in[N^2]} x_j(Nt_1) - x_1(t_1)\right]\mathrm{d}t_1 + \sqrt{\frac{1}{N}\sum_{i\in[N]} g(x_i(Nt_1))}\,\mathrm{d}w(t_1)\\
&\qquad + NK_0e_0\left[y_{0,1}^{[N^2]}(t_1) - x_1^{[N^2]}(t_1)\right]\mathrm{d}t_1\\
&\qquad + K_1e_1\left[y_{1,1}^{[N^2]}(t_1) - x_1^{[N^2]}(t_1)\right]\mathrm{d}t_1\\
&\qquad + \frac{K_2e_2}{N}\left[y_{2,1}^{[N^2]}(t_1) - x_1^{[N^2]}(t_1)\right]\mathrm{d}t_1,\\
\mathrm{d}y_{0,1}^{[N^2]}(t_1) &= Ne_0\left[x_1^{[N^2]}(t_1) - y_{0,1}^{[N^2]}(t_1)\right]\mathrm{d}t_1,\\
\mathrm{d}y_{1,1}^{[N^2]}(t_1) &= e_1\left[x_1^{[N^2]}(t_1) - y_{1,1}^{[N^2]}(t_1)\right]\mathrm{d}t_1,\\
\mathrm{d}y_{2,1}^{[N^2]}(t_1) &= \frac{e_2}{N}\left[x_1^{[N^2]}(t_1) - y_{1,1}^{[N^2]}(t_1)\right]\mathrm{d}t_1.
\end{aligned}
\tag{8.11}
$$

In the limit $N\to\infty$ we expect that the colour 2-dormant population does not move, since it only interacts with the active population at rate $\frac{e_2}{N}$. Therefore we expect $(y_{2,1}^{[N^2]}(t))_{t>0}$ to converge to a constant process and hence we say that the colour 2-dormant population behaves like a *slow seed-bank.* The colour 1-dormant population, however, has a non-trivial interaction with the active population and therefore is the *effective seed-bank* on space-time scale 1. The colour 0-dormant population has, in the limit as $N\to\infty$, an infinitely strong interaction with the active population. Therefore we expect that, in the limit as $N\to\infty$, its path becomes rougher and rougher at rarer and rarer times. We will need to use the *Meyer-Zheng topology* to prove that

$$
\lim_{N\to\infty} y_{0,1}^{[N^2]}(t_1) = \lim_{N\to\infty} x_1^{[N^2]}(t_1) \text{ for most } t_1. \tag{8.12}
$$

Therefore the colour 0-dormant population equalizes with the active population, due to its infinitely strong interaction with the active population. Hence at space-time scale 1, the colour 0-dormant population behaves like a *fast seed-bank.* If we look at the active population, then we see that it feels a drift towards the 2-block average of the active population, and resamples at a rate that is the 1-block average of the resampling rates in the single colonies. Furthermore, in the limit as $N\to\infty$, it feels an infinitely fast drift towards the colour 0-dormant population, has a non-trivial interaction with the colour 1-dormant population, and its interaction with the colour 2-dormant population cancels out. As long as we focus on the combination

$$
\frac{x_1^{[N^2]}(t_1) + K_0 y_{0,1}^{[N^2]}(t_1)}{1+K_0},
$$

we see that the colour-0 terms with the factor N in front cancel out. This will allow us to do most of the analysis in the path space topology, without using the Meyer-Zheng topology. The process

$$
\left(\frac{x_1^{[N^2]}(t_1) + K_0 y_{0,1}^{[N^2]}(t_1)}{1+K_0},\, y_{1,1}^{[N^2]}(t_1)\right)_{t_1>0}
$$

will therefore be called the *effective process.*

• **On time scale** N^2 (i.e., space-time scale 2) we look at the equivalent of the 2-block averages in (4.22),

$$\begin{aligned}
&\left(x_2^{[N^2]}(t_2), (y_{0,2}^{[N^2]}(t_2), y_{1,2}^{[N^2]}(t_2), y_{2,2}^{[N^2]}(t_2))\right)_{t_2>0} \\
&= \Bigg(\frac{1}{N^2} \sum_{i\in[N^2]} x_i^{[N^2]}(N^2 t_2), \\
&\qquad \left(\frac{1}{N^2} \sum_{i\in[N^2]} y_{i,0}^{[N^2]}(N^2 t_2), \frac{1}{N^2} \sum_{i\in[N^2]} y_{i,1}^{[N^2]}(N^2 t_2), \frac{1}{N^2} \sum_{i\in[N^2]} y_{i,2}^{[N^2]}(N^2 t_2) \right)\Bigg)_{t_2>0},
\end{aligned} \tag{8.13}$$

which evolves according to the SSDE

$$\begin{aligned}
\mathrm{d}x_2^{[N^2]}(t_2) &= \sqrt{\frac{1}{N^2} \sum_{i\in[N^2]} g(x_i(N^2 t_2))}\, \mathrm{d}w(t_2) \\
&\qquad + N^2 K_0 e_0 \left[y_{0,2}^{[N^2]}(t_2) - x_2^{[N^2]}(t_2) \right] \mathrm{d}t_2 \\
&\qquad + N K_1 e_1 \left[y_{1,2}^{[N^2]}(t_2) - x_2^{[N^2]}(t_2) \right] \mathrm{d}t_2 \\
&\qquad + K_2 e_2 \left[y_{2,2}^{[N^2]}(t_2) - x_2^{[N^2]}(t_2) \right] \mathrm{d}t_2, \\
\mathrm{d}y_{0,2}^{[N^2]}(t_2) &= N^2 e_0 \left[x_2^{[N^2]}(t_2) - y_{0,2}^{[N^2]}(t_2) \right] \mathrm{d}t_2, \\
\mathrm{d}y_{1,2}^{[N^2]}(t_2) &= N e_1 \left[x_2^{[N^2]}(t_2) - y_{1,2}^{[N^2]}(t_2) \right] \mathrm{d}t_2, \\
\mathrm{d}y_{2,2}^{[N^2]}(t_2) &= e_2 \left[x_2^{[N^2]}(t_2) - y_{2,2}^{[N^2]}(t_2) \right] \mathrm{d}t_2.
\end{aligned} \tag{8.14}$$

In this case we see that migration in the active component cancels out and the resampling rate is given by the average over the complete population. In the limit as $N \to \infty$, we see that the active population interacts at an infinitely fast rate with the 0-dormant population as well as with the colour 1-dormant population. Hence both the colour 0 and the colour 1 seed-banks are fast seed-banks and we expect equalisation of the active population and the colour 0-dormant population and the colour 1-dormant population in Meyer-Zheng topology. The active population, in the limit as $N \to \infty$, has a non-trivial interaction with the colour 2-dormant population, and hence the colour 2-dormant population is the *effective seed-bank* on time scale N^2. Looking at the quantity

$$\frac{x_2^{[N^2]}(t_2) + K_0 y_{0,2}^{[N^2]}(t_2) + K_1 y_{1,2}^{[N^2]}(t_2)}{1 + K_0 + K_1}, \tag{8.15}$$

for which we find

$$
\begin{aligned}
&\mathrm{d}\left[\frac{x_2^{[N^2]}(t_2)+K_0 y_{0,2}^{[N^2]}(t_2)+K_1 y_{1,2}^{[N^2]}(t_2)}{1+K_0+K_1}\right]\\
&=\frac{1}{1+K_0+K_1}\sqrt{\frac{1}{N^2}\sum_{i\in[N^2]} g(x_i(N^2 t_2))}\,\mathrm{d}w(t_2)+K_2 e_2\left[y_{2,2}^{[N^2]}(t_2)-x_2^{[N^2]}(t_2)\right]\mathrm{d}t_2,
\end{aligned}
\tag{8.16}
$$

we see that the infinite rates cancel out. We will call

$$
\left(\frac{x_2^{[N^2]}(t_2)+K_0 y_{0,2}^{[N^2]}(t_2)+K_1 y_{1,2}^{[N^2]}(t_2)}{1+K_0+K_1}, y_{2,2}^{[N^2]}(t_2)\right)_{t_2>0}
\tag{8.17}
$$

the effective process. Using the effective process we can analyse our system in path space.

► **Scaling limit.** Let $(z_0(t))_{t\geq 0}=(x_0(t),(y_{0,0}(t),y_{1,0}(t),y_{2,0}(t)))_{t\geq 0}$ be the process evolving according to

$$
\begin{aligned}
\mathrm{d}x_0(t) &= c_0\left[\theta - x_0(t)\right]\mathrm{d}t+\sqrt{g(x_0(t))}\,\mathrm{d}w(t)\\
&\qquad + K_0 e_0\left[y_{0,0}(t)-x_0(t)\right]\mathrm{d}t,\\
\mathrm{d}y_{0,0}(t) &= e_0\left[x_0(t)-y_{0,0}(t)\right]\mathrm{d}t,\\
y_{1,0}(t) &= y_{1,0},\\
y_{2,0}(t) &= y_{2,0},
\end{aligned}
\tag{8.18}
$$

where $\theta\in[0,1]$, $y_{1,0}\in[0,1]$ and $y_{2,0}\in[0,1]$. The process $(z_0(t))_{t\geq 0})$ will be the limiting process for the single colonies. The corresponding single colony effective processes are given by

$$
\begin{aligned}
&\mathrm{d}x_0^{\mathrm{eff}}(t)=c_0[\theta-x_0^{\mathrm{eff}}(t)]\,\mathrm{d}t+\sqrt{g(x_0^{\mathrm{eff}}(t))}\,\mathrm{d}w(t)+K_0 e_0\left[y_{0,0}^{\mathrm{eff}}(t)-x_0^{\mathrm{eff}}(t)\right]\mathrm{d}t,\\
&\mathrm{d}y_{0,0}^{\mathrm{eff}}(t)=e_0\left[x_0^{\mathrm{eff}}(t)-y_{0,0}^{\mathrm{eff}}(t)\right]\mathrm{d}t,\qquad i\in\mathbb{N}_0,
\end{aligned}
\tag{8.19}
$$

where $\theta\in[0,1]$. By [72], (8.18) and (8.19) have a unique strong solution. Like for the one-colour mean-field finite-systems scheme, we need the following list of ingredients to formally state the multi-scale analysis:

(a) For positive times $t>0$, we define the following 1-*block estimators* for the finite system:

$$
\begin{aligned}
\bar\Theta^{(1),[N^2]}(t) &= \frac{1}{N}\sum_{i\in[N]} \frac{x_i^{[N^2]}(t)+K_0 y_{i,0}^{[N^2]}(t)}{1+K_0},\\
\Theta_x^{(1),[N^2]}(t) &= \frac{1}{N}\sum_{i\in[N]} x_i^{[N^2]}(t),\\
\Theta_{y_0}^{(1),[N^2]}(t) &= \frac{1}{N}\sum_{i\in[N]} y_{i,0}^{[N^2]}(t),\\
\Theta_{y_1}^{(1),[N^2]}(t) &= \frac{1}{N}\sum_{i\in[N]} y_{i,1}^{[N^2]}(t),\\
\Theta_{y_2}^{(1),[N^2]}(t) &= \frac{1}{N}\sum_{i\in[N]} y_{i,2}^{[N^2]}(t).
\end{aligned}
\tag{8.20}
$$

We abbreviate

$$
\begin{aligned}
\mathbf{\Theta}^{(1),[N^2]}(t) &= \left(\Theta_x^{(1),[N^2]}(t), \left(\Theta_{y_0}^{(1),[N^2]}(t), \Theta_{y_1}^{(1),[N^2]}(t), \Theta_{y_2}^{(1),[N^2]}(t)\right)\right),\\
\mathbf{\Theta}^{\mathrm{aux},(1),[N^2]}(t) &= \left(\bar\Theta^{(1),[N^2]}(t), \Theta_{y_1}^{(1),[N^2]}(t), \Theta_{y_2}^{(1),[N^2]}(t)\right),\\
\mathbf{\Theta}^{\mathrm{eff},(1),[N^2]}(t) &= \left(\bar\Theta^{(1),[N^2]}(t), \Theta_{y_1}^{(1),[N^2]}(t)\right).
\end{aligned}
\tag{8.21}
$$

We call $(\mathbf{\Theta}^{(1),[N^2]}(t))_{t>0}$ the 1-*block estimator process*, $(\mathbf{\Theta}^{\mathrm{aux},(1),[N^2]}(t))_{t>0}$ the *auxiliary* 1-*block estimator process* and $(\mathbf{\Theta}^{\mathrm{eff},(1),[N^2]}(t))_{t>0}$ the *effective* 1-*block estimator process*. The auxiliary 1-block estimator will be useful in the proofs. For $t>0$, we define the following 2-*block estimators* for the finite system:

$$
\begin{aligned}
\bar\Theta^{(2),[N^2]}(t) &= \frac{1}{N^2}\sum_{i\in[N^2]} \frac{x_i^{[N^2]}(t)+K_0 y_{i,0}^{[N^2]}(t)+K_1 y_{i,1}^{[N^2]}(t)}{1+K_0+K_1},\\
\Theta_x^{(2),[N^2]}(t) &= \frac{1}{N^2}\sum_{i\in[N^2]} x_i^{[N^2]}(t),\\
\Theta_{y_0}^{(2),[N^2]}(t) &= \frac{1}{N^2}\sum_{i\in[N^2]} y_{i,0}^{[N^2]}(t),\\
\Theta_{y_1}^{(2),[N^2]}(t) &= \frac{1}{N^2}\sum_{i\in[N^2]} y_{i,1}^{[N^2]}(t),\\
\Theta_{y_2}^{(2),[N^2]}(t) &= \frac{1}{N^2}\sum_{i\in[N^2]} y_{i,2}^{[N^2]}(t).
\end{aligned}
\tag{8.22}
$$

We abbreviate

$$
\begin{aligned}
\mathbf{\Theta}^{(2),[N^2]}(t) &= \left(\Theta_x^{(2),[N^2]}(t), \left(\Theta_{y_0}^{(2),[N^2]}(t), \Theta_{y_1}^{(2),[N^2]}(t), \Theta_{y_2}^{(2),[N^2]}(t)\right)\right),\\
\mathbf{\Theta}^{\mathrm{eff},(2),[N^2]}(t) &= \left(\bar\Theta^{(2),[N^2]}(t), \Theta_{y_1}^{(2),[N^2]}(t)\right).
\end{aligned}
\tag{8.23}
$$

We call $(\mathbf{\Theta}^{\mathrm{eff},(2),[N^2]}(t))_{t>0}$ the *effective* 2-*block estimator process* and $(\mathbf{\Theta}^{(2),[N^2]}(t))_{t>0}$ as the 2-*block estimator process.*

(b) The *time scale* N for which $\mathcal{L}[\bar{\Theta}^{(1),[N^2]}(Nt_1 - L(N)) - \bar{\Theta}^{(1),[N^2]}(Nt_1)] = \delta_0$ for all $L(N)$ such that $L(N) \to \infty$ and $\lim_{N\to\infty} L(N)/N = 0$, but not for $L(N) = N$. In words, N is the time scale on which $\bar{\Theta}^{(1),[N^2]}(\cdot)$ starts evolving, i.e., $(\bar{\Theta}^{(1),[N^2]}(Nt_1))_{t_1>0}$ is no longer a fixed process. When we use time scale N, we will use t_1 as a time index, which indicates the "faster time scale". For the "slow time scale" we use t_0 as time index.

The *time scale* N^2 for which $\mathcal{L}[\bar{\Theta}^{(2),[N^2]}(N^2t_2 - L(N)N) - \bar{\Theta}^{(2),[N^2]}(N^2t_2)] = \delta_0$ for all $L(N)$ such that $L(N) \to \infty$ and $\lim_{N\to\infty} L(N)/N = 0$, but not for $L(N) = N$. In words, N^2 is the time scale on which $\bar{\Theta}^{(2),[N^2]}(\cdot)$ starts evolving, i.e., $(\bar{\Theta}^{(2),[N^2]}(N^2t_2))_{t_2>0}$, is no longer a fixed process. When we use time scale N^2, we will use t_2 as a time index, which indicates the "fastest time scale".

(c) The *invariant measure* for the evolution of a single colony in (8.18), written

$$\Gamma^{(0)}_{\theta,\mathbf{y}_0}, \qquad \mathbf{y}_0 = (\theta, y_{1,0}, y_{2,0}), \tag{8.24}$$

and the invariant measure of the level-0 effective process evolving according to (8.19), written

$$\Gamma^{\mathrm{eff},(0)}_{\theta}. \tag{8.25}$$

(d) The renormalisation transformation $\mathcal{F}\colon \mathcal{G} \to \mathcal{G}$,

$$(\mathcal{F}g)(\theta) = \int_{[0,1]^2} g(x)\, \Gamma^{\mathrm{eff},(0)}_{\theta}(\mathrm{d}x, \mathrm{d}y_0), \quad \theta \in [0,1], \tag{8.26}$$

where $\Gamma^{\mathrm{eff},(0)}_{\theta}$ is the equilibrium measure in (8.25). Note that this is the same transformation as defined in (4.75), but defined for the truncated system. Later we will study iterates of the renormalisation transformation. Therefore we will write $\mathcal{F}^{(1)}g = \mathcal{F}g$, to indicate that we apply the renormalisation transformation only once.

(e) The limiting 1-block process is given by

$$(z_1(t))_{t>0}) = (x_1(t), (y_{0,1}(t), y_{1,1}(t), y_{2,1}(t)))_{t>0} \tag{8.27}$$

and evolves according to

$$\begin{aligned} \mathrm{d}x_1(t) &= \frac{1}{1+K_0}\Bigg[c_1[\theta - x_1(t)]\,\mathrm{d}t + \sqrt{(\mathcal{F}^{(1)}g)(x_1(t))}\,\mathrm{d}w(t) \\ &\qquad\qquad + K_1e_1\,[y_{1,1}(t) - x_1(t)]\,\mathrm{d}t\Bigg], \\ y_{0,1}(t) &= x_1(t), \\ \mathrm{d}y_{1,1}(t) &= e_1\,[x_1(t) - y_{1,1}(t)]\,\mathrm{d}t, \\ y_{2,1}(t) &= y_{2,1}, \end{aligned} \tag{8.28}$$

where $\theta \in [0,1]$, and $y_{2,1} \in [0,1]$, and $\mathcal{F}^{(1)}$ is the renormalisation transformation defined in (8.26). The limiting 1-block process for the auxiliary estimator process is given by $(z_1^{\mathrm{aux}}(t))_{t>0} = (x_1^{\mathrm{aux}}(t), y_{1,1}^{\mathrm{aux}}(t), y_{2,1}^{\mathrm{aux}}(t))_{t>0}$ and evolves according to

$$\begin{aligned}
\mathrm{d}x_1^{\mathrm{aux}}(t) &= \frac{1}{1+K_0}\Bigg[c_1[\theta - x_1^{\mathrm{aux}}(t)]\,\mathrm{d}t + \sqrt{(\mathcal{F}^{(1)}g)(x_1^{\mathrm{aux}}(t))}\,\mathrm{d}w(t) \\
&\qquad + K_1 e_1\,[y_{1,1}^{\mathrm{aux}}(t) - x_1^{\mathrm{aux}}(t)]\,\mathrm{d}t\Bigg], \\
\mathrm{d}y_{1,1}^{\mathrm{aux}}(t) &= e_1\,[x_1^{\mathrm{aux}}(t) - y_{1,1}^{\mathrm{aux}}(t)]\,\mathrm{d}t, \\
y_{2,1}^{\mathrm{aux}}(t) &= y_{2,1},
\end{aligned} \tag{8.29}$$

for $\theta \in [0,1]$. The auxiliary estimator process turns out to be important in the next section. The effective limiting 1-block process is given by $(z_1^{\mathrm{eff}}(t))_{t>0}) = (x_1^{\mathrm{eff}}(t), y_{1,1}^{\mathrm{eff}}(t))_{t>0}$ and evolves according to

$$\begin{aligned}
\mathrm{d}x_1^{\mathrm{eff}}(t) &= \frac{1}{1+K_0}\Bigg[c_1[\theta - x_1^{\mathrm{eff}}(t)]\,\mathrm{d}t + \sqrt{(\mathcal{F}^{(1)}g)(x_1^{\mathrm{eff}}(t))}\,\mathrm{d}w(t) \\
&\qquad + K_1 e_1\,[y_{1,1}^{\mathrm{eff}}(t) - x_1^{\mathrm{eff}}(t)]\,\mathrm{d}t\Bigg], \\
\mathrm{d}y_{1,1}^{\mathrm{eff}}(t) &= e_1\,[x_1^{\mathrm{eff}}(t) - y_{1,1}^{\mathrm{eff}}(t)]\,\mathrm{d}t,
\end{aligned} \tag{8.30}$$

for $\theta \in [0,1]$. By [72], (8.28), (8.29) and (8.30) have a unique strong solution.

(f) The *invariant measure* of the infinite system in (8.28), written

$$\Gamma^{(1)}_{\theta,y_1}, \qquad y_1 = (\theta,\theta,y_{2,1}), \tag{8.31}$$

and the invariant measures of the level-1 limiting estimator process evolving according to (8.29) and the level-1 effective process evolving according to (8.30),

$$\Gamma_\theta^{\mathrm{aux},(1)},\ \Gamma_\theta^{\mathrm{eff},(1)}. \tag{8.32}$$

(g) The first iteration of the renormalisation transformation,

$$(\mathcal{F}^{(2)}g)(\theta) = \int_{[0,1]^2} (\mathcal{F}g)(x)\,\Gamma_\theta^{\mathrm{eff},(1)}(\mathrm{d}x,\mathrm{d}y_1), \quad \theta\in[0,1]. \tag{8.33}$$

Hence

$$(\mathcal{F}^{(2)}g)(\theta) = \int_{[0,1]^2} \Gamma_\theta^{\mathrm{eff},(1)}(\mathrm{d}u,\mathrm{d}v) \int_{[0,1]^2} g(x)\,\Gamma_u^{\mathrm{eff},(0)}(\mathrm{d}x,\mathrm{d}y). \tag{8.34}$$

(h) The limiting 2-block process $(z_2(t))_{t>0} = (x_2(t),(y_{0,2}(t),y_{1,2}(t),y_{2,2}(t)))_{t>0}$

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evolves according to

$$
\begin{aligned}
\mathrm{d}x_2(t) &= \frac{1}{1+K_0+K_1}\left[\sqrt{(\mathcal{F}^{(2)}g)\big(x_2(t)\big)}\,\mathrm{d}w(t) + K_2e_2\left[y_{2,2}(t)-x_2(t)\right]\mathrm{d}t\right],\\
y_{0,2}(t) &= x_2(t),\\
\mathrm{d}y_{1,2}(t) &= x_2(t),\\
\mathrm{d}y_{2,2}(t) &= e_2\left[x_2(t)-y_{2,2}(t)\right]\,\mathrm{d}t,
\end{aligned}
\tag{8.35}
$$

where $\mathcal{F}^{(2)}g$ is defined as in (8.33). The limiting effective 2-block process on space-time scale 2 is $(z_2^{\mathrm{eff}}(t))_{t>0}) = (x_2^{\mathrm{eff}}(t), y_{2,2}^{\mathrm{eff}}(t))_{t>0}$ and evolves according to

$$
\begin{aligned}
\mathrm{d}x_2^{\mathrm{eff}}(t) &= \frac{1}{1+K_0+K_1}\left[\sqrt{(\mathcal{F}^{(2)}g)(x_2^{\mathrm{eff}}(t))}\,\mathrm{d}w(t) + K_2e_2\,[y_{2,2}^{\mathrm{eff}}(t)-x_2^{\mathrm{eff}}(t)]\,\mathrm{d}t\right],\\
\mathrm{d}y_{2,2}^{\mathrm{eff}}(t) &= e_2\,[x_2^{\mathrm{eff}}(t)-y_{2,2}^{\mathrm{eff}}(t)]\,\mathrm{d}t.
\end{aligned}
\tag{8.36}
$$

We are now ready to state the scaling limit for the evolution of the averages in (7.7).

Proposition 8.1.1 (Two-level three-colour finite-systems scheme). *Suppose that* $\mu(0) = \mu^{\otimes[N^2]}$ *for some* $\mu \in \mathcal{P}\left([0,1]\times[0,1]^2\right)$. *Let*

$$
\begin{aligned}
&\vartheta_0 = \mathbb{E}^{\mu}\left[\frac{x+K_0y_0}{1+K_0}\right], \quad \vartheta_1 = \mathbb{E}^{\mu}\left[\frac{x+K_0y_0+K_1y_1}{1+K_0+K_1}\right],\\
&\theta_{y_1} = \mathbb{E}^{\mu}\left[y_1\right], \quad \theta_{y_2} = \mathbb{E}^{\mu}\left[y_2\right].
\end{aligned}
\tag{8.37}
$$

and recall the limiting process $(z_2(t))_{t>0}$ *in* (8.35) *and the limiting process* $(z_1(t))_{t>0}$ *in* (8.28). *Assume for the* 2*-dormant* 1*-blocks that*

$$
\lim_{N\to\infty}\mathcal{L}\left[Y_{2,1}^{[N^2]}(Nt_2)\middle|\boldsymbol{\Theta}^{(2),[N^2]}(N^2t_2)\right] = P^{z_2(t_2)}, \tag{8.38}
$$

and for the 2*-dormant* 0*-blocks (= single colonies) that*

$$
\lim_{N\to\infty}\mathcal{L}\left[Y_{2,0}^{[N^2]}(Nt_2+Nt_1)\middle|\boldsymbol{\Theta}^{\mathrm{eff},(1),[N^2]}(N^2t_2+Nt_1)\right] = P^{z_1(t_1)}. \tag{8.39}
$$

Then the following hold:

(a) For the effective 2*-block estimator process defined in* (8.23),

$$
\lim_{N\to\infty}\mathcal{L}\left[\left(\boldsymbol{\Theta}^{\mathrm{eff},(2),[N^2]}(N^2t_2)\right)_{t_2>0}\right] = \mathcal{L}\left[\left(z_2^{\mathrm{eff}}(t_2)\right)_{t_2>0}\right], \tag{8.40}
$$

where the limit is determined by the unique solution of the SSDE (8.36) *with initial state*

$$
z_2^{\mathrm{eff}}(0) = \left(x_2^{\mathrm{eff}}(0), y_2^{\mathrm{eff}}(0)\right) = (\vartheta_1, \theta_{y_2}). \tag{8.41}
$$

(b) *For the effective* 1*-block estimator process defined in* (8.21),

$$\lim_{N\to\infty}\mathcal{L}\left[\left(\mathbf{\Theta}^{\mathrm{eff},(1),[N^2]}(N^2t_2+Nt_1)\right)_{t_1>0}\right]=\mathcal{L}\left[(z_1^{\mathrm{eff}}(t_1))_{t_1>0}\right], \tag{8.42}$$

where, conditional on $x_2^{\mathrm{eff}}(t_2)=u$*, the limit process is the unique solution of the SSDE in* (8.30) *with* θ *replaced by* u *and with initial measure* $\Gamma_u^{\mathrm{eff},(1)}$.

(c) *For the single colony effective process defined in* (8.9),

$$\lim_{N\to\infty}\mathcal{L}\left[\left(z_0^{\mathrm{eff},[N^2]}(N^2t_2+Nt_1+t_0)\right)_{t_0\geq 0}\right]=\mathcal{L}\left[(z_0^{\mathrm{eff}}(t_0))_{t_0\geq 0}\right], \tag{8.43}$$

where, conditional on $x_1^{\mathrm{eff}}(t_1)=v$*, the limit process is the unique solution of the SSDE in* (8.19) *with* θ *replaced by* v *and with initial measure* $\Gamma_v^{\mathrm{eff},(0)}$.

(d) *For the* 2*-block estimator process defined in* (8.23),

$$\lim_{N\to\infty}\mathcal{L}\left[\left(\mathbf{\Theta}^{(2),[N^2]}(N^2t_2)\right)_{t_2>0}\right]=\mathcal{L}\left[(z_2(t_2))_{t_2>0}\right] \tag{8.44}$$

in the Meyer-Zheng topology,

where the limit process is the unique solution of the SSDE in (8.35) *with initial state*

$$z_2(0)=(\vartheta_1,(\vartheta_1,\vartheta_1,\theta_{y_2}))\,. \tag{8.45}$$

(e) *Fix* $t_2>0$*. Assume* (8.38)*. Define*

$$\begin{aligned}\Gamma^{(1)}(t_2)=&\int_{[0,1]^4}S_{t_2}^{[2]}\big((\vartheta_1,(\vartheta_1,\vartheta_1,\theta_{y_2})),\mathrm{d}(u_x,u_x,u_x,u_{y_{2,2}})\big)\\&\int_{[0,1]}P^{(u_x,u_x,u_x,u_{y_{2,2}})}(\mathrm{d}y_{2,1})\,\Gamma^{(1)}_{(u_x,(u_x,u_x,y_{2,1}))}\in\mathcal{P}([0,1]^4),\end{aligned} \tag{8.46}$$

where $\Gamma^{(1)}_{(u_x,(u_x,u_x,y_{2,1}))}$ *is the equilibrium measure in* (8.31) *and* $S_{t_2}^{[2]}((\vartheta_1,(\vartheta_1,\vartheta_1,\theta_{y_2})),\cdot)$ *is the time-*t_2 *law of the limiting process* $(z_2(t_2))_{t_2>0}$ *in* (8.44) *starting from* $(\vartheta_1,(\vartheta_1,\vartheta_1,\theta_{y_2}))\in[0,1]\times[0,1]^3$.

Let $(z^{\Gamma^{(1)}(t_2)}(t_1))_{t_1\geq 0}$ *be the random process that conditioned on* $z_2(t_2)=(\theta,(\theta,\theta,y_{2,2}))$ *moves according to* (8.28) *with* $\theta=\theta$ *and* $y_{2,1}(0)=y_{2,1}$ *and with* $z^{\Gamma^{(1)}(t_2)}(0)$ *be drawn according to* $\Gamma^{(1)}(t_2)$ *(which is a mixture of random processes in equilibrium). Then for the* 1*-block estimator process defined in* (8.21),

$$\lim_{N\to\infty}\mathcal{L}\left[\left(\mathbf{\Theta}^{(1),[N^2]}(N^2t_2+Nt_1)\right)_{t_1>0}\right]=\mathcal{L}\left[(z^{\Gamma^{(1)}(t_2)}(t_1))_{t_1>0}\right] \tag{8.47}$$

in the Meyer-Zheng topology.

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(f) Let $z_1(t_1)$ be the limiting process obtained in (e). Assume (8.39)*. Define, for $t_2 \in (0,\infty)$,*

$$\Gamma^{(0)}(t_2) = \int_{[0,1]^4} \Gamma^{(1)}(t_2)(\mathrm{d}z_1) \int_{[0,1]} P^{z_1}(\mathrm{d}y_{2,0})\, \Gamma^{(0)}_{(x_1,(x_1,y_{1,1}.y_{2,0}))}, \tag{8.48}$$

where $\Gamma^{(1)}(t_2)$ is as defined in (8.46)*. Let $(z^{\Gamma^{(0)}(t_2)}(t_0))_{t_0\geq 0}$ be the random process in* (8.18) *with $z^{\Gamma^{(0)}(t_2)}(0)$ drawn according to $\Gamma^{(0)}(t_2)$ which is a mixture of random processes in equilibrium. Then*

$$\lim_{N\to\infty} \mathcal{L}\left[\left(z_0^{[N^2]}(N^2t_2+Nt_1+t_0)\right)_{t_0\geq 0}\right] = \mathcal{L}\left[(z^{\Gamma^{(0)}(t_2)}(t_0))_{t_0\geq 0}\right]. \tag{8.49}$$

Remark 8.1.2. Note that Proposition 8.1.1(f) does not depend on the choice of t_1, because $\Gamma^{(1)}(t_2)$ is already a mixture of equilibrium measures of the 1-block process. ■

Remark 8.1.3. Note that in Propostion 8.1.1(f) $\Gamma^{(0)}_{(x_1,(x_1,y_{1,1},y_{2,0}))}$ is the equilibrium measure of (8.18) (see also (8.24)), where $y_{1,0} = y_{1,1}$. This means that all colour 1-dormant single colonies equal the current state of the colour 1-dormant 1-block. We say that given the state of the 1-dormant 1-block, the 1-dormant single colonies become deterministic. This effect occurs once a slow seed-bank, in this case the colour 1 seed-bank, is already in equilbrium on the space-time scale where it is effective, in this case space-time-scale 1. Since we start at times N^2t_2, the 1-dormant 1-blocks are already in equilibrium. This will turn out to be the reason that the single colour 1-dormant colonies are equal to the current value of the 1-dormant 1-block averages. Note that at time N^2t_2 the 2-dormant 2-blocks do not yet have reached equilibrium. Hence the colour 2-dormant 1-blocks and the colour 2-dormant single colonies do not equal the instantaneous value of the 2-dormant 2-block averages. In the Section 8.3.8 we will treat this effect in detail. ■

§8.2 Scheme for the two-level three-colour mean-field analysis.

In this section we give a scheme to prove Proposition 8.1.1. The proof of the steps in the scheme will be written in Section 8.3. To analyse the two-level hierarchical mean-field system we use the results obtained in Sections 6.2.2, 6.3 and 7.2.

The scheme for the two-level three-colour hierarchical mean-field system comes in 11 steps. Recall the estimators defined in (8.20) and (8.22).

1 Tightness of the effective 2-block estimator processes

$$\left(\left(\boldsymbol{\Theta}^{\mathrm{eff},(2),[N^2]}(N^2t_2)\right)_{t_2>0}\right)_{N\in\mathbb{N}}. \tag{8.50}$$

2 Stability property of the 2-block estimators, i.e., for $L(N)$ such that $\lim_{N\to\infty} L(N)=\infty$ and $\lim_{N\to\infty} L(N)/N = 0$,

$$\lim_{N\to\infty} \sup_{0\leq t\leq L(N)} \left|\bar{\Theta}^{(2),[N^2]}(N^2t_2) - \bar{\Theta}^{(2),[N^2]}(N^2t_2-Nt)\right| = 0 \text{ in probability} \tag{8.51}$$

and

$$\lim_{N\to\infty} \sup_{0\le t\le L(N)} \left|\Theta_{y_2}^{(2),[N^2]}(N^2t_2) - \Theta_{y_2}^{(2),[N^2]}(N^2t_2 - Nt)\right| = 0 \text{ in probability.} \tag{8.52}$$

3 Tightness of the effective 1-block estimator process (recall (8.21)),

$$\left(\left(\boldsymbol{\Theta}^{\mathrm{aux},(1),[N^2]}(N^2t_2 + Nt_1)\right)_{t_1>0}\right)_{N\in\mathbb{N}}. \tag{8.53}$$

4 Stability property of $(\boldsymbol{\Theta}^{\mathrm{aux},(1),[N^2]}(N^2t_2 + Nt_1))_{t_1>0}$, i.e., for $L(N)$ such that $\lim_{N\to\infty} L(N) = \infty$ and $\lim_{N\to\infty} L(N)/N = 0$, for all $\epsilon > 0$,

$$\lim_{N\to\infty} \sup_{0\le t\le L(N)} \left|\bar{\Theta}^{(1),[N^2]}(N^2t_2 + Nt_1) - \bar{\Theta}^{(1),[N^2]}(N^2t_2 + Nt_1 - t)\right| = 0 \text{ in probability,} \tag{8.54}$$

$$\lim_{N\to\infty} \sup_{0\le t\le L(N)} \left|\Theta_{y_1}^{(1),[N^2]}(N^2t_2 + Nt_1) - \Theta_{y_1}^{(1),[N]}(N^2t_2 + Nt_1 - t)\right| = 0 \text{ in probability,} \tag{8.55}$$

$$\lim_{N\to\infty} \sup_{0\le t\le L(N)} \left|\Theta_{y_2}^{(1),[N^2]}(N^2t_2 + Nt_1) - \Theta_{y_2}^{(1),[N]}(N^2t_2 + Nt_1 - t)\right| = 0 \text{ in probability.} \tag{8.56}$$

5 Recall that there are N 1-blocks in $[N^2]$. Since tightness of components implies tightness of the process, step 3 implies that the full 1-block process

$$\left(\left(\boldsymbol{\Theta}_i^{\mathrm{aux},(1),[N^2]}(N^2t_2 + Nt_1)\right)_{t_1>0,\, i\in[N]}\right)_{N\in\mathbb{N}} \tag{8.57}$$

is tight. From the tightness in steps 1 and 3 we can construct a subsequence $(N_k)_{k\in\mathbb{N}}$ along which

$$\begin{aligned} &\lim_{k\to\infty} \mathcal{L}\left[\left(\boldsymbol{\Theta}^{\mathrm{eff},(2),[N_k^2]}(N_k^2t_2)\right)_{t_2>0}\right], \\ &\lim_{k\to\infty} \mathcal{L}\left[\left(\boldsymbol{\Theta}_i^{\mathrm{aux},(1),[N_k^2]}(N_k^2t_2 + N_kt_1)\right)_{t_1>0,\, i\in[N_k]}\right] \end{aligned} \tag{8.58}$$

both exists. Define the measure

$$\nu^{(0)}(t_2) = \prod_{i\in\mathbb{N}_0} \Gamma_i^{(0)}(t_2). \tag{8.59}$$

Show that along the same subsequence the single components converge to the infinite system, i.e.,

$$\lim_{k\to\infty} \mathcal{L}\left[\left(Z^{[N_k^2]}(N_k^2t_2 + N_kt_1 + t_0)\right)_{t_0\ge 0}\right] = \mathcal{L}\left[(Z^{\nu^{(0)}(t_2)}(t_0))_{t_0\ge 0}\right]. \tag{8.60}$$

Here, $(Z^{\nu^{(0)}(t_2)}(t_0))_{t_0\geq 0}$ is the process starting from $\nu^{(0)}(t_2)$ with components evolving according to (8.18), where θ is now a random variable that inherits its law from

$$\lim_{k\to\infty} \mathcal{L}[(\mathbf{\Theta}^{\mathrm{aux},(1),[N_k^2]}(N_k^2 t_2 + N_k t_1))_{i\in[N_k^2]}], \tag{8.61}$$

and, similarly, the laws of $y_{1,0}$ and $y_{2,0}$ in the limiting process $(Z^{\nu^{(0)}(t_2)}(t_0))_{t_0\geq 0}$ are determined by

$$\lim_{k\to\infty} \mathcal{L}[(\mathbf{\Theta}^{\mathrm{aux},(1),[N_k^2]}(N_k^2 t_2 + N_k t_1))_{i\in[N_k^2]}]. \tag{8.62}$$

6 Use the limiting evolution of the single colonies obtained in step 5 to identify the limiting 1-block process along the same subsequence, i.e., identify the limit

$$\lim_{k\to\infty} \mathcal{L}\left[\left(\mathbf{\Theta}^{\mathrm{aux},(1),[N_k^2]}(N_k^2 t_2 + N_k t_1)\right)_{t_1>0,i\in[N_k]}\right]. \tag{8.63}$$

7 Identify the limit $\lim_{k\to\infty} \mathcal{L}[(\mathbf{\Theta}^{\mathrm{eff},(2),[N_k^2]}(N_k^2 t_2))_{t_2>0}]$ with the help of the limiting evolution of the single colonies obtained in step 5 and the limiting evolution of the full 1-block process obtained in step 6.

8 Prove that the 1-dormant single colonies at time $N^2t_2 + Nt_1$ equal, in the limit as $N\to\infty$, the 1-dormant 1-block averages. The proof of this step shows how the evolution of the slow seed-banks must be analysed.

9 Show that the convergence in step 8, step 7 and step 5 actually holds along each subsequence. Therefore we obtain the limiting evolution of the single colonies, the auxiliary 1-block process and the effective 2-block process.

10 Use the Meyer-Zheng topology to describe the limiting evolution of

$$\begin{aligned}&\Big(\Theta_x^{(1),[N^2]}(N^2t_2+Nt_1), \Theta_{y_0}^{(1),[N^2]}(N^2t_2+Nt_1), \Theta_{y_1}^{(1),[N^2]}(N^2t_2+Nt_1),\\ &\quad \Theta_{y_2}^{(1),[N^2]}(N^2t_2+Nt_1)\Big)_{t_1>0}\end{aligned} \tag{8.64}$$

and

$$\Big(\Theta_x^{(2),[N^2]}(N^2t_2), \Theta_{y_0}^{(2),[N^2]}(N^2t_2), \Theta_{y_1}^{(2),[N^2]}(N^2t_2), \Theta_{y_2}^{(2),[N^2]}(N^2t_2)\Big)_{t_2>0}. \tag{8.65}$$

11 Combine the above steps to complete the proof of Proposition 8.1.1.

§8.3 Proof of two-level three-colour mean-field finite-systems scheme

In this section we prove the steps in the scheme given in Section 8.2.

§8.3.1 Tightness of the 2-block estimators

In this section we prove step 1 of the scheme.

Lemma 8.3.1 (Tightness of the 2-block estimator). *Let*

$$\mathbf{\Theta}^{\mathrm{eff},(2),[N^2]}(N^2t_2) = (\bar{\Theta}^{(2),[N^2]}(N^2t_2), \Theta_{y_2}^{(2),[N^2]}(N^2t_2)) \tag{8.66}$$

be defined as in (8.22)*. Then* $(\mathcal{L}[(\mathbf{\Theta}^{\mathrm{eff},(2),[N^2]}(N^2t_2))_{t_2>0}])_{N\in\mathbb{N}}$ *is a tight sequence of probability measures on* $\mathcal{C}((0,\infty),[0,1]^2)$*.*

Proof. To prove the tightness of the 1-blocks, we use [49, Proposition 3.2.3]. From (8.6) we find that $(\mathbf{\Theta}^{\mathrm{eff},(2),[N^2]}(t))_{t>0}$ evolves according to

$$\begin{aligned}
\mathrm{d}\bar{\Theta}^{(2),[N^2]}(t) &= \frac{1}{1+K_0+K_1}\frac{1}{N^2}\sum_{i\in[N^2]}\sqrt{g(x_i^{[N^2]}(t))}\,\mathrm{d}w_i(t)\\
&\quad+\frac{1}{1+K_0+K_1}\frac{K_2e_2}{N^2}\left[\frac{1}{N^2}\sum_{i\in[N^2]}y_{i,2}^{[N^2]}(t)-\frac{1}{N^2}\sum_{i\in[N^2]}x_i^{[N^2]}(t)\right]\mathrm{d}t,\\
\mathrm{d}\Theta_{y_2}^{(2),[N^2]}(t) &= \frac{e_2}{N^2}\left[\frac{1}{N^2}\sum_{i\in[N^2]}x_i^{[N^2]}(t)-\frac{1}{N^2}\sum_{i\in[N^2]}y_{i,2}^{[N^2]}(t)\right]\mathrm{d}t.
\end{aligned}\tag{8.67}$$

Therefore the process $(\mathbf{\Theta}^{\mathrm{eff},(2),[N^2]}(N^2t_2))_{t_2>0}$ evolves according to

$$\begin{aligned}
\mathrm{d}\bar{\Theta}^{(2),[N^2]}(N^2t_2) &= \frac{1}{1+K_0+K_1}\sqrt{\frac{1}{N^2}\sum_{i\in[N^2]}g(x_i^{[N^2]}(N^2t_2))}\,\mathrm{d}w_i(t_2)\\
&\quad+\frac{1}{1+K_0+K_1}K_2e_2\left[\frac{1}{N^2}\sum_{i\in[N^2]}y_{i,2}^{[N^2]}(N^2t_2)-\frac{1}{N^2}\sum_{i\in[N^2]}x_i^{[N^2]}(N^2t_2)\right]\mathrm{d}t_2,\\
\mathrm{d}\Theta_{y_2}^{(2),[N^2]}(N^2t_2) &= e_2\left[\frac{1}{N^2}\sum_{i\in[N^2]}x_i^{[N^2]}(N^2t_2)-\frac{1}{N^2}\sum_{i\in[N^2]}y_{i,2}^{[N^2]}(N^2t_2)\right]\mathrm{d}t_2.
\end{aligned}\tag{8.68}$$

To use [49, Proposition 3.2.3], we define $\mathcal{C}^*$ as the set of polynomials on $([0,1]^2)$. Since $(\mathbf{\Theta}^{\mathrm{eff},(2),[N^2]}(N^2t_2))_{t_2>0}$ is a semi-martingale, by applying Itô's formula we obtain that $(\mathbf{\Theta}^{\mathrm{eff},(2),[N^2]}(N^2t_2))_{t_2>0}$ is a $\mathcal{D}$-semi-martingale with corresponding operator

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$$
\begin{aligned}
G_{\dagger}^{(2),[N^2]}\colon\ & (\mathcal{C}^*, [0,1]^2, (0,\infty), \Omega) \to \mathbb{R}, \\
G_{\dagger}^{(2),[N^2]}(f,(x,y),t,\omega) = {} & \frac{K_2 e_2}{1+K_0+K_1}\left[y - \frac{1}{N^2}\sum_{i\in[N^2]} x_i^{[N^2]}(N^2t,\omega)\right]\frac{\partial f}{\partial x} \\
& + e_2\left[\frac{1}{N^2}\sum_{i\in[N^2]} x_i^{[N^2]}(N^2t,\omega) - y\right]\frac{\partial f}{\partial y} \\
& + \frac{1}{2(1+K_0+K_1)^2}\frac{1}{N^2}\sum_{i\in[N^2]} g(x_i^{[N^2]}(N^2t,\omega))\frac{\partial^2 f}{\partial x^2}.
\end{aligned}
\tag{8.69}
$$

The conditions $H_1,\ H_2,\ H_3$ in [49, Proposition 3.2.3] are satisfied. Hence tightness follows from [49, Proposition 3.2.3]. □

§8.3.2 Stability of the 2-block estimators

Lemma 8.3.2 (Stability property of the 2-block estimator). *Let* $(\mathbf{\Theta}^{\mathrm{eff},(2),[N^2]}(t))_{t>0}$ *be defined as in* (8.23). *For any* $L(N)$ *such that* $\lim_{N\to\infty} L(N) = \infty$ *and* $\lim_{N\to\infty} L(N)/N = 0$,

$$
\lim_{N\to\infty}\sup_{0\le t\le L(N)}\left|\bar{\Theta}^{(2),[N^2]}(N^2t_2) - \bar{\Theta}^{(2),[N^2]}(N^2t_2 - Nt)\right| = 0 \text{ in probability} \tag{8.70}
$$

and

$$
\lim_{N\to\infty}\sup_{0\le t\le L(N)}\left|\Theta_{y_2}^{(2),[N^2]}(N^2t_2) - \Theta_{y_2}^{(2),[N^2]}(N^2t_2 - Nt)\right| = 0 \text{ in probability.} \tag{8.71}
$$

Proof. Fix $\epsilon > 0$. From the SSDE in (8.67) we obtain that, for N large enough,

$$\begin{aligned}
&\mathbb{P}\left[\sup_{0\le t\le L(N)}\left|\bar\Theta^{(2),[N^2]}(N^2t_2)-\bar\Theta^{(2),[N^2]}(N^2t_2-Nt)\right|>\epsilon\right]\\
&=\mathbb{P}\left[\sup_{0\le t\le L(N)}\frac{1}{1+K_0+K_1}\left|\int_{N^2t_2-Nt}^{N^2t_2}\mathrm{d}w_i(r)\,\frac{1}{N^2}\sum_{i\in[N^2]}\sqrt{g(x_i^{[N^2]}(r))}\right.\right.\\
&\qquad\left.\left.+\int_{N^2t_2-Nt}^{N^2t_2}\mathrm{d}r\,\frac{K_2e_2}{N^2}\left[\Theta_{y_2}^{(2),[N^2]}(r)-\frac{1}{N^2}\sum_{i\in[N^2]}x_i^{[N^2]}(r)\right]\right|>\epsilon\right]\\
&\le\mathbb{P}\left[\sup_{0\le t\le L(N)}\frac{1}{1+K_0+K_1}\left|\int_{N^2t_2-Nt}^{N^2t_2}\mathrm{d}w_i(r)\,\frac{1}{N^2}\sum_{i\in[N^2]}\sqrt{g(x_i^{[N^2]}(r))}\right|\right.\\
&\qquad\left.>\epsilon-\frac{K_2e_2}{1+K_0+K_1}\frac{L(N)N}{N^2}\right]\\
&\le\mathbb{P}\left[\sup_{0\le t\le L(N)}\frac{1}{1+K_0+K_1}\left|\int_{N^2t_2-Nt}^{N^2t_2}\mathrm{d}w_i(r)\frac{1}{N^2}\sum_{i\in[N^2]}\sqrt{g(x_i^{[N^2]}(r))}\right|>\frac{\epsilon}{2}\right].
\end{aligned}\tag{8.72}$$

By a similar optional stopping time argument as in the proof of Lemma 6.2.15, the above computation shows that (8.70) holds. Equation (8.71) holds by a similar argument as given in the proof of Lemma 7.2.2. □

§8.3.3 Tightness of the 1-block estimators

Lemma 8.3.3 (Tightness of the 1-block estimator). *Let*

$$\begin{aligned}
&\boldsymbol{\Theta}^{\mathrm{aux},(1),[N^2]}(N^2t_2+Nt_1)\\
&=(\bar\Theta^{(1),[N^2]}(N^2t_2+Nt_1),\Theta_{y_1}^{(1),[N^2]}(N^2t_2+Nt_1),\Theta_{y_2}^{(1),[N^2]}(N^2t_2+Nt_1))
\end{aligned}\tag{8.73}$$

be defined as in (8.20). *Then* $(\mathcal{L}[(\boldsymbol{\Theta}^{\mathrm{aux},(1),[N]}(N^2t_2+Nt_1))_{t_1>0}])_{N\in\mathbb{N}}$ *is a tight sequence of probability measures on* $\mathcal{C}((0,\infty),[0,1]^3)$.

Proof. To prove the tightness of the 1-blocks, we again use [49, Proposition 3.2.3]. From (8.11) we find that the effective process $(\boldsymbol{\Theta}^{\mathrm{aux},(1),[N^2]}(N^2t_2+Nt_1))_{t_1>0}$ evolves

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according to

$$
\begin{aligned}
\mathrm{d}\bar{\Theta}^{(1),[N^2]}(Nt_1) &= \frac{1}{1+K_0} c_1 \left[\frac{1}{N^2} \sum_{j\in[N^2]} x_j^{[N^2]}(Nt_1) - \frac{1}{N} \sum_{i\in[N]} x_i^{[N^2]}(Nt_1) \right] \mathrm{d}t_1 \\
&\quad + \frac{1}{1+K_0} \sqrt{\frac{1}{N} \sum_{i\in[N]} g(x_i^{[N^2]}(Nt_1))}\, \mathrm{d}w_i(t_1) \\
&\quad + \frac{K_1 e_1}{1+K_0} \left[\Theta_{y_1}^{(1),[N^2]}(Nt_1) - \frac{1}{N} \sum_{i\in[N]} x_i^{[N^2]}(Nt_1) \right] \mathrm{d}t_1 \\
&\quad + \frac{K_2 e_2}{N(1+K_0)} \left[\Theta_{y_2}^{(1),[N^2]}(Nt_1) - \frac{1}{N} \sum_{i\in[N]} x_i^{[N^2]}(Nt_1) \right] \mathrm{d}t_1, \\
\mathrm{d}\Theta_{y_1}^{(1),[N^2]}(Nt_1) &= e_1 \left[\frac{1}{N} \sum_{i\in[N]} x_i^{[N^2]}(Nt_1) - \Theta_{y_1}^{(1),[N^2]}(Nt_1) \right] \mathrm{d}t_1, \\
\mathrm{d}\Theta_{y_2}^{(1),[N^2]}(Nt_1) &= \frac{e_2}{N} \left[\frac{1}{N} \sum_{i\in[N]} x_i^{[N^2]}(Nt_1) - \Theta_{y_2}^{(1),[N^2]}(Nt_1) \right] \mathrm{d}t_1.
\end{aligned}
\tag{8.74}
$$

To use [49, Proposition 3.2.3], we define $\mathcal{C}^*$ as the set of polynomials on $([0,1]^2)$. Since $(\boldsymbol{\Theta}^{\mathrm{aux},(1),[N^2]}(N^2t_2 + Nt_1))_{t_1>0}$ is a semi-martingale, by applying Itô's formula we obtain that $(\boldsymbol{\Theta}^{\mathrm{aux},(1),[N^2]}(N^2t_2+Nt_1))_{t_1>0}$ is a $\mathcal{D}$-semi-martingale with corresponding operator

$$
\begin{aligned}
G_{\dagger}^{(1),[N^2]}&\colon\ (\mathcal{C}^*, [0,1]^3, (0,\infty), \Omega) \to \mathbb{R}, \\
G_{\dagger}^{(1),[N^2]}(f,(x,y_1,y_2),t,\omega) &= \frac{c_1}{1+K_0} \left[\frac{1}{N^2} \sum_{j\in[N^2]} x_j^{[N^2]}(Nt,\omega) - \frac{1}{N} \sum_{i\in[N]} x_i^{[N^2]}(Nt,\omega) \right] \frac{\partial f}{\partial x} \\
&\quad + \frac{K_1 e_1}{1+K_0} \left[y_1 - \frac{1}{N} \sum_{i\in[N]} x_i^{[N^2]}(Nt,\omega) \right] \frac{\partial f}{\partial x} \\
&\quad + \frac{K_2 e_2}{N(1+K_0)} \left[y_2(Nt,\omega) - \frac{1}{N} \sum_{i\in[N]} x_i^{[N^2]}(Nt) \right] \frac{\partial f}{\partial x} \\
&\quad + e_1 \left[\frac{1}{N} \sum_{i\in[N]} x_i^{[N^2]}(Nt,\omega) - y_1 \right] \frac{\partial f}{\partial y_1} \\
&\quad + \frac{e_2}{N} \left[\frac{1}{N} \sum_{i\in[N]} x_i^{[N^2]}(Nt,\omega) - y_2(Nt,\omega) \right] \frac{\partial f}{\partial y_2} \\
&\quad + \frac{1}{2(1+K_0)^2} \frac{1}{N} \sum_{i\in[N]} g(x_i^{[N^2]}(Nt,\omega)) \frac{\partial^2 f}{\partial x^2}.
\end{aligned}
\tag{8.75}
$$

The conditions $H_1,\ H_2,\ H_3$ in [49, Proposition 3.2.3] are satisfied as before. Hence we conclude that the sequence $(\mathcal{L}[(\boldsymbol{\Theta}^{\mathrm{aux},(1),[N^2]}(N^2t_2+Nt_1))_{t_1>0}])_{N\in\mathbb{N}}$ is tight. $\square$

§8.3.4 Stability of the 1-block estimators

Lemma 8.3.4 (Stability property of the 1-block estimator). *Let $\boldsymbol{\Theta}^{\mathrm{aux},(1),[N^2]}(t)$ be defined as in (7.14). For any $L(N)$ such that $\lim_{N\to\infty} L(N)=\infty$ and $\lim_{N\to\infty} L(N)/N=0$,*

$$\lim_{N\to\infty}\sup_{0\le t\le L(N)}\left|\bar\Theta^{(1),[N^2]}(N^2t_2+Nt_1)-\bar\Theta^{(1),[N^2]}(N^2t_2+Nt_1-t)\right|=0 \text{ in probability,} \tag{8.76}$$

$$\lim_{N\to\infty}\sup_{0\le t\le L(N)}\left|\Theta^{(1),[N^2]}_{y_1}(N^2t_2+Nt_1)-\Theta^{(1),[N^2]}_{y_1}(N^2t_2+Nt_1-t)\right|=0 \text{ in probability,} \tag{8.77}$$

$$\lim_{N\to\infty}\sup_{0\le t\le L(N)}\left|\Theta^{(1),[N^2]}_{y_2}(N^2t_2+Nt_1)-\Theta^{(1),[N^2]}_{y_2}(N^2t_2+Nt_1-t)\right|=0 \text{ in probability.} \tag{8.78}$$

Proof. Define
$$u=N^2t_2+Nt_1. \tag{8.79}$$
From the SSDE in (8.7) we obtain that

$$\begin{aligned}
&\mathbb{P}\left[\sup_{0\le t\le L(N)}\left|\bar\Theta^{(1),[N^2]}(u)-\bar\Theta^{(1),[N^2]}(u-t)\right|>\epsilon\right]\\
&=\mathbb{P}\Bigg[\sup_{0\le t\le L(N)}\frac{1}{1+K_0}\Bigg|\int_{u-t}^{u}\mathrm{d}r\,\frac{c_1}{N}\left[\frac{1}{N^2}\sum_{j\in[N^2]}x_j^{[N^2]}(r)-\frac{1}{N}\sum_{i\in[N]}x_i^{[N^2]}(r)\right]\\
&\qquad+\int_{u-t}^{u}\mathrm{d}r\,\frac{K_1e_1}{N}\left[\Theta^{(1),[N^2]}_{y_1}(r)-\frac{1}{N}\sum_{i\in[N]}x_i^{[N^2]}(r)\right]\\
&\qquad+\frac{K_2e_2}{N^2}\left[\frac{1}{N}\sum_{i\in[N]}y_{i,2}^{[N^2]}(r)-\frac{1}{N}\sum_{i\in[N]}x_i^{[N^2]}(r)\right]\\
&\qquad+\int_{u-t}^{u}\mathrm{d}w_i(r)\,\frac{1}{N}\sum_{i\in[N]}\sqrt{g(x_i^{[N^2]}(r))}\Bigg|>\epsilon\Bigg]\\
&\le\mathbb{P}\Bigg[\sup_{0\le t\le L(N)}\frac{1}{1+K_0}\left|\int_{u-t}^{u}\mathrm{d}w_i(r)\,\frac{1}{N}\sum_{i\in[N]}\sqrt{g(x_i^{[N^2]}(r))}\right|\\
&\qquad>\epsilon-\frac{L(N)2(c_1+K_1e_1+\frac{K_2e_2}{N})}{N(1+K_0)}\Bigg]\\
&\le\mathbb{P}\left[\sup_{0\le t\le L(N)}\frac{1}{1+K_0}\left|\int_{u-t}^{u}\mathrm{d}w_i(r)\frac{1}{N}\sum_{i\in[N]}\sqrt{g(x_i^{[N^2]}(r))}\right|>\frac{\epsilon}{2}\right].
\end{aligned} \tag{8.80}$$

Via the same optional stopping time argument as in the proof of Lemma 6.2.15, the above computation shows that (8.76) holds. Note that the extra drift term $\frac{K_2 e_2}{N}$ does not have any influence. Equations (8.77)–(8.78) hold by a similar argument as given in the proof of Lemma 7.2.2. □

§8.3.5 Limiting evolution for the single components

Proposition 8.3.5 (Equilibrium for the infinite system). *Fix $t_2, t_1 > 0$. Let $(N_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ and let $L(N)$ be any sequence satisfying $\lim_{N\to\infty} L(N) = \infty$ and $\lim_{N\to\infty} L(N)/N = 0$ such that*

$$\begin{aligned}
&\lim_{k\to\infty} \mathcal{L}\left[\mathbf{\Theta}^{\mathrm{aux},(1),[N_k^2]}(N_k^2 t_2 + N_k t_1)\right] = P_{t_1,t_2},\\
&\lim_{k\to\infty} \mathcal{L}\left[\left(Y_{1,0}^{[N_k^2]}(N_k^2 t_2 + N_k t_1), Y_{2,0}^{[N_k^2]}(N_k^2 t_2)\right) \middle| \mathbf{\Theta}^{\mathrm{aux},(1),[N_k^2]}(N_k^2 t_2 + N_k t_1)\right]\\
&\qquad = P^{z_1^{\mathrm{eff}}(t_1)},\\
&\lim_{k\to\infty} \mathcal{L}\Big[\sup_{0\le t\le L(N_k)} \left|\bar\Theta^{[N_k^2]}(N_k^2 t_2 + N_k t_1) - \bar\Theta^{[N_k^2]}(N_k^2 t_2 + N_k t_1 - t)\right| \\
&\qquad\qquad + \left|\Theta_{y_1}^{[N_k^2]}(N_k^2 t_2 + N_k t_1) - \Theta_{y_1}{}^{[N_k^2]}(N_k^2 t_2 + N_k t_1 - t)\right| \\
&\qquad\qquad + \left|\Theta_{y_2}^{[N_k^2]}(N_k^2 t_2 + N_k t_1) - \Theta_{y_2}{}^{[N_k^2]}(N_k^2 t_2 + N_k t_1 - t)\right|\Big] = \delta_0,\\
&\lim_{k\to\infty} \mathcal{L}\Big(Z^{[N_k^2]}(N_k^2 t_2 + N_k t_1),\Big) = \nu(t_1, t_2).
\end{aligned} \tag{8.81}$$

Then $\nu(t_1, t_2)$ is of the form

$$\nu(t_1, t_2) = \int_{[0,1]^2} P_{t_1,t_2}(\mathrm{d}\theta^{(1)}, \mathrm{d}\theta_y^{(1)}) \int_{[0,1]^{\mathbb{N}_0}} P^{(\theta^{(1)},\theta_y^{(1)})}(\mathrm{d}\boldsymbol{y})\, \nu_{\theta,\boldsymbol{y}}, \tag{8.82}$$

where

$$\nu_{\theta,\boldsymbol{y}_0} = \prod_{i\in\mathbb{N}_0} \Gamma_{(\theta,\boldsymbol{y}_{0,i})} \tag{8.83}$$

with $\Gamma_{(\theta,\boldsymbol{y}_{0,i})}$ the equilibrium measure for the i'th single colony defined in (8.24).

Note that by step 1 and step 3 we can find a subsequence $(N_k)_{k\in\mathbb{N}}$ such that the first and third line in (8.81) hold. The second line in (8.81) follows from assumptions (8.38) and (8.39). To prove Proposition 8.3.5 we proceed as in the proof of Proposition 7.2.3, but with the finite system in (7.4) replaced by the system in (8.6) and the infinite system in (7.11) replaced by the system in (8.18). Note that Lemma 7.2.4 holds also for the system in (8.18), after adding the non-interacting component $y_{2,0}$ to the equilibrium. The equivalent of Lemma 7.2.5 will again follow from the equivalent of Lemma 7.2.9. We will derive the analogue of Lemmas 7.2.6 and 7.2.7 (see Lemma's 8.3.6 and 8.3.7 below). Lemma 7.2.8 can be extended with an extra colour-2 seed-bank estimator by using the same proof. Since the infinite system for the single colonies in the two-layer three-colour mean-field system (see (8.89)) equals the one for the one-layer two-colour mean-field system, up to a non-interacting component, we obtain an equivalent of Lemma 7.2.9. Finally, also the equivalent of Lemma 7.2.10

holds under an additional assumption, see Lemma 8.3.8. Finally Corollary 8.3.9 states the equivalent of Corollary 7.2.11. With the help of the lemma's and the corollary, the proof of Proposition 8.3.5 follows from the same argument as used in the proof of Proposition 7.2.3.

Lemma 8.3.6 (Comparison of empirical averages).
Let $(\Theta_x^{(1),[N^2]}(t_0))_{t_0\geq 0}$ *and* $(\Theta_{y_0}^{(1),[N^2]}(t_0))_{t_0\geq 0}$ *be defined as in* (8.20). *Then*

$$\mathbb{E}\left[\left|\Theta_x^{(1),[N]}(t)-\Theta_{y_0}^{(1),[N]}(t)\right|\right]\leq\sqrt{\mathbb{E}\left[\left(\Theta_x^{(1),[N]}(0)-\Theta_{y_0}^{(1),[N]}(0)\right)^2\right]}\mathrm{e}^{-(K_0e_0+e_0)t}$$
$$+\sqrt{\frac{1}{K_0e_0+e_0}\left[\frac{c_1}{N}+\frac{||g||}{N}+\frac{K_1e_1}{N}+\frac{K_2e_2}{N^2}\right]}. \tag{8.84}$$

Proof. The result follows by Itô-calculus on the SSDE in (8.6) and the same type of argument as used in the proof of Lemma 7.2.6. □

Like for the mean-field system with one colour, we need to compare the finite system in (8.6) with an infinite system. To derive the analogue of Lemma 7.2.7, let $L(N)$ satisfy $\lim_{N\to\infty}L(N)=\infty$ and $\lim_{N\to\infty}L(N)/N=0$. Define $[N]_i$ to be the 1-block that contains site $i\in[N^2]$. Since we start our system in an exchangeable measure and the dynamics are exchangeable, we will only consider the single colonies in $[N]_0$, the 1-block containing the site $0\in[N^2]$. In the rest of the prove, we will suppress the 0 from the notation i.e., $[N]_0=[N]$ and $\bar{\Theta}_0^{(1),[N^2]}=\bar{\Theta}^{(1),[N^2]}$. Define

$$u=N^2t_2+Nt_1 \tag{8.85}$$

and let μ_N be the measure on $([0,1]^3)^{\mathbb{N}_0}$ by continuing the configuration of

$$\begin{aligned}&\left(Z^{[N^2]}(u-L(N))\right)\\&=\left(X^{[N^2]}(u-L(N)),\left(Y_0^{[N^2]}(u-L(N)),Y_1^{[N^2]}(u-L(N)),Y_2^{[N^2]}(u-L(N))\right)\right)\end{aligned} \tag{8.86}$$

periodically to $([0,1]^4)^{\mathbb{N}_0}$, i.e., we continue the configuration of the single colonies in the first block to $([0,1]^4)^{\mathbb{N}_0}$. Let

$$\bar{\Theta}^{(1),[N^2]}=\frac{1}{N}\sum_{i\in[N]}\frac{x_i^{[N^2]}(u-L(N))+K_0y_{i,0}^{[N^2]}(u-L(N))}{1+K_0}. \tag{8.87}$$

Let

$$(Z^{\mu_N}(t))_{t\geq 0}=\left(X^{\mu_N}(t),(Y_0^{\mu_N}(t),Y_1^{\mu_N}(t),Y_2^{\mu_N}(t))\right)_{t\geq 0} \tag{8.88}$$

be the infinite system evolving according to

$$\begin{aligned}&\mathrm{d}x_i^{\mu_N}(t)=c_0\,[\bar{\Theta}^{(1),[N^2]}-x_i^{\mu_N}(t)]\,\mathrm{d}t+\sqrt{g\big(x_i^{\mu_N}(t)\big)}\,\mathrm{d}w_i(t)+K_0e_0\,[y_{i,0}^{\mu_N}(t)-x_i^{\mu_N}(t)]\,\mathrm{d}t,\\&\mathrm{d}y_{i,0}^{\mu_N}(t)=e_0\,[x_i^{\mu_N}(t)-y_{i,0}^{\mu_N}(t)]\,\mathrm{d}t,\\&\quad y_{i,1}^{\mu_N}(t)=y_{i,1}^{\mu_N}(0),\\&\quad y_{i,2}^{\mu_N}(t)=y_{i,2}^{\mu_N}(0),\qquad i\in\mathbb{N}_0,\end{aligned} \tag{8.89}$$

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starting from initial distribution μ_N. Then the following Lemma 8.3.7 is the equivalent of Lemma 7.2.7 for the three-colour two-layer mean-field system. In particular, the infinite system considered in Lemma 8.3.7 is similar to the infinite system in Lemma 7.2.7. The only difference is that there is one more non-interacting component added in (8.89).

Lemma 8.3.7. *[Comparison of finite and infinite systems] Fix* $t_1, t_2 > 0$, *and let* $u = N^2 t_2 + N t_1$. *Let* $L(N)$ *satisfy* $\lim_{N\to\infty} L(N) = \infty$ *and* $\lim_{N\to\infty} L(N)/N$. *Suppose that*

$$\lim_{N\to\infty} \sup_{0\leq t\leq L(N)} \left|\bar{\Theta}^{(1),[N]}(u) - \bar{\Theta}^{(1),[N]}(u-t)\right| = 0 \quad \textit{in probability.} \tag{8.90}$$

Then, for all $t \geq 0$,

$$\lim_{k\to\infty} \left|\mathbb{E}\left[f\big(Z^{\mu_N}(t)\big) - f\big(Z^{[N^2]}(u-L(N)+t)\big)\right]\right| = 0 \qquad \forall\, f \in \mathcal{C}\big(([0,1]^3)^{\mathbb{N}_0}, \mathbb{R}\big). \tag{8.91}$$

Proof. We proceed as in the proof of Lemma 7.2.7 and couple the finite and infinite systems by their Brownian motion, exactly as was done there. The single components in the block around site 0 of the finite process $(Z^{[N^2]}(t))$ are evolving according to

$$\begin{aligned}
\mathrm{d}x_i^{[N^2]}(t) &= c_0 \left[\Theta^{(1),[N^2]} - x_i^{[N^2]}(t)\right] \mathrm{d}t + c_0 \left[\bar{\Theta}^{(1),[N^2]}(t) - \Theta^{(1),[N^2]}\right] \mathrm{d}t \\
&\quad + c_0 \left[\Theta_x^{(1),[N^2]}(t) - \bar{\Theta}^{(1),[N^2]}(t)\right] \mathrm{d}t + \frac{c_1}{N}\left[\frac{1}{N^2}\sum_{i\in[N^2]} x_j^{[N^2]}(t) - x_i^{[N^2]}(t)\right] \mathrm{d}t \\
&\quad + \sqrt{g(x_i^{[N^2]}(t))}\, \mathrm{d}w_i(t) + K_0 e_0 \, [y_{i,0}^{[N^2]}(t) - x_i^{[N^2]}(t)]\, \mathrm{d}t \\
&\quad + \frac{K_1 e_1}{N}\left[y_{i,1}^{[N^2]}(t) - x_i^{[N^2]}(t)\right] \mathrm{d}t + \frac{K_2 e_2}{N^2}\, [y_{i,0}^{[N^2]}(t) - x_i^{[N^2]}(t)]\, \mathrm{d}t, \\
\mathrm{d}y_{i,0}^{[N^2]}(t) &= e_0\, [x_i^{[N^2]}(t) - y_{i,0}^{[N^2]}(t)]\, \mathrm{d}t, \\
\mathrm{d}y_{i,1}^{[N^2]}(t) &= \frac{e_1}{N}\, [x_i^{[N^2]}(t) - y_{i,1}^{[N^2]}(t)]\, \mathrm{d}t, \\
\mathrm{d}y_{i,2}^{[N^2]}(t) &= \frac{e_2}{N^2}\, [x_i^{[N^2]}(t) - y_{i,2}^{[N^2]}(t)]\, \mathrm{d}t, \qquad i \in [N].
\end{aligned} \tag{8.92}$$

Using this SSDE we can exactly proceed as in the proof of Lemma 7.2.7 to obtain the result. Note that the colour-2 seed-bank can be treated just in the same way as the colour-1 seed-bank in the proof of Lemma 7.2.7, since its rate of interaction with the active population is even slower than the rate of interaction of the colour-1 seed-bank. □

Finally, we state the equivalent of Lemma 7.2.10 for the three-colour two-layer mean-field system.

Lemma 8.3.8 (Coupling of finite systems). *Let*

$$Z^{[N^2],1} = (X^{[N^2],1}, Y_0^{[N^2],1}, Y_1^{[N^2],1}, Y_2^{[N^2],1}) \tag{8.93}$$

be the finite system evolving according to (8.6) *starting from an exchangeable initial measure. Let* $\mu^{[N],1}$ *be the measure obtain by periodic continuation of the configuration of* $Z^{[N^2],1}(0)$ *in the* 1*-block around* 0*. Similarly, let*

$$Z^{[N^2],2} = (X^{[N^2],2}, Y_0^{[N^2],2}, Y_1^{[N^2],2}, Y_2^{[N^2],2}) \tag{8.94}$$

be the finite system evolving according to (8.6) *starting from an exchangeable initial measure. Let* $\mu^{[N],2}$ *be the measure obtained by periodic continuation of the configuration of* $Z^{[N^2],1}(0)$ *in the* 1*-block around* 0*. Let* $\tilde{\mu}$ *be any weak limit point of the sequence of measures* $(\mu^{[N],1} \times \mu^{[N],2})_{N\in\mathbb{N}}$*. Define the variables* $\bar{\Theta}^{[N],1}$ *on* $(([0,1]^4, \mu^{[N],1})^{\mathbb{N}_0})$*,* $\bar{\Theta}^{[N],2}$ *on* $(([0,1]^4)^{\mathbb{N}_0}, \mu^{[N],2})$ *and* $\bar{\Theta}_1$ *and* $\bar{\Theta}_2$ *on* $(([0,1]^4)^{\mathbb{N}_0} \times ([0,1]^4)^{\mathbb{N}_0}, \mu)$ *by*

$$\begin{aligned}
\bar{\Theta}^{[N],1} &= \frac{1}{N}\sum_{i\in[N]} \frac{x_i^{[N^2],1} + K_0 y_{i,0}^{[N^2],1}}{1+K_0}, \qquad \bar{\Theta}^{[N],2} = \frac{1}{N}\sum_{i\in[N]} \frac{x_i^{[N^2],2} + K_0 y_{i,0}^{[N^2],2}}{1+K_0},\\
\bar{\Theta}^1 &= \lim_{n\to\infty}\frac{1}{n}\sum_{i\in[n]} \frac{x_i^1 + K_0 y_{i,0}^1}{1+K_0}, \qquad \bar{\Theta}^2 = \lim_{n\to\infty}\frac{1}{n}\sum_{i\in[n]} \frac{x_i^2 + K_0 y_{i,0}^2}{1+K_0},
\end{aligned} \tag{8.95}$$

and let $(\bar{\Theta}^{(1),[N],1}(t))_{t\geq 0}$ *and* $(\bar{\Theta}^{(1),[N],2}(t))_{t\geq 0}$ *be defined as in* (7.14) *for* $Z^{[N^2],1}$*, respectively,* $Z^{[N^2],2}$*. Suppose that*

$$\lim_{N\to\infty} \sup_{0\leq t\leq L(N)} \left(\left|\bar{\Theta}^{[N],k}(0) - \bar{\Theta}^{[N],k}(t)\right|\right) = 0 \quad \textit{in probability}, \quad k\in\{1,2\}, \tag{8.96}$$

and suppose that $\tilde{\mu}(\{\bar{\Theta}_1 = \bar{\Theta}_2,\, Y_1^1 = Y_1^2, Y_2^1 = Y_2^2\}) = 1$*. Then, for any sequence* $(t(N))_{N\in\mathbb{N}}$ *with* $\lim_{N\to\infty} t(N) = \infty$*,*

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\big[&|x_i^{[N],1}(t(N)) - x_i^{[N],2}(t(N))| + K_0|y_{i,0}^{[N],1}(t(N)) - y_{i,0}^{[N],2}(t(N))|\\
&+ K_1|y_{i,1}^{[N],1}(t(N)) - y_{i,1}^{[N],2}(t(N))| + K_2|y_{i,2}^{[N],1}(t(N)) - y_{i,2}^{[N],2}(t(N))|\big] = 0.
\end{aligned} \tag{8.97}$$

Proof. Like in the proof of Lemma 7.2.10, we can show with Itô calculus that the function

$$\begin{aligned}
t \to \mathbb{E}\big[&|x_i^{[N],1}(t(N)) - x_i^{[N],2}(t(N))| + K_0|y_{i,0}^{[N],1}(t(N)) - y_{i,0}^{[N],2}(t(N))|\\
&+ K_1|y_{i,1}^{[N],1}(t(N)) - y_{i,1}^{[N],2}(t(N))| + K_2|y_{i,2}^{[N],1}(t(N)) - y_{i,2}^{[N],2}(t(N))|\big]
\end{aligned} \tag{8.98}$$

is monotonically decreasing. Hence we can proceed as in the proof of Lemma 6.2.13 to show that (8.97) is true. □

From the above couplings we can derive the following corollary, which is the analogue of Corollary 7.2.11 for the two-level three-colour mean-field system.

Corollary 8.3.9. *Fix* $t_1, t_2 > 0$ *and set* $u = N^2 t_2 + N t_1$*. Let* μ_N *be the measure obtained by periodic continuation of*

$$Z^{[N^2]}(u - L(N)) = \big(X^{[N^2]}(u-L(N)), Y_0^{[N^2]}(u-L(N)), Y_1^{[N^2]}(u-L(N)), Y_2^{[N^2]}(u-L(N))\big), \tag{8.99}$$

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and let μ be a weak limit point of the sequence $(\mu_N)_{N\in\mathbb{N}}$. Let

$$\Theta = \lim_{N\to\infty} \frac{1}{N}\sum_{i\in[N]} \frac{x_i^\mu + K y_i^\mu}{1+K} \qquad \text{in } L^2(\mu), \tag{8.100}$$

and let $(Z^{\nu_\Theta}(t))_{t>0} = (X^{\nu_\Theta}(t), Y_0^{\nu_\Theta}(t), Y_1^{\nu_\Theta}(t), Y_2^{\nu_\Theta}(t))_{t>0}$ be the infinite system with components evolving according to (8.18) *with $\theta = \Theta$ and $y_{i,1,0}$ and $y_{i,2,0}$ determined by assumption* (8.81) *and starting from its equilibrium measure. Extend the finite system $Z^{[N^2]}$ as a system on $([0,1]^4)^{\mathbb{N}_0}$ by periodic continuation. Construct $(Z^{[N^2]}(t))_{t>0}$ and $(Z^{\nu_\Theta}(t))_{t>0}$ on one probability space. Then there exists a sequence $(\bar{L}(N))_{N\in\mathbb{N}}$ such that $\lim_{N\to\infty}\bar{L}(N)=\infty$, $\lim_{N\to\infty}\frac{L(\bar{N})}{N}=0$ and*

$$\begin{aligned}\lim_{N\to\infty} \mathbb{E}\left[\left|x_i^{[N^2]}(Ns) - x_i^{\nu_\Theta}(\bar{L}(N))\right|\right] + K_0\,\mathbb{E}\left[\left|y_{i,0}^{[N^2]}(Ns) - y_{i,0}^{\nu_\Theta}(\bar{L}(N))\right|\right]\\ + K_1\,\mathbb{E}\left[\left|y_{i,1}^{[N^2]}(Ns) - y_{i,1}^{\nu_\Theta}(\bar{L}(N))\right|\right] + K_2\,\mathbb{E}\left[\left|y_{i,2}^{[N^2]}(Ns) - y_{i,2}^{\nu_\Theta}(\bar{L}(N))\right|\right] = 0,\ i\in[N].\end{aligned} \tag{8.101}$$

Note that Lemmas 8.3.7, 8.3.8 and Corollary 8.3.9 do not only hold for sites i in the 1-block around 0, but hold for for all sites $i \in [N^2]$, after we replace $\Theta_0^{[N]_0}$ by $\Theta_i^{[N]_i}$.

§8.3.6 Limiting evolution of the 1-block estimator process

Proposition 8.3.10 (Limiting evolution of the 1-blocks). *Fix $t_2 > 0$. Let $(L(N))_{N\in\mathbb{N}}$ satisfy $\lim_{N\to\infty} L(N)=\infty$ and $\lim_{N\to\infty} L(N)/N = 0$. Let $(N_k)_{k\in\mathbb{N}}$ be a subsequence such that*

$$\begin{aligned}
&\lim_{k\to\infty}\mathcal{L}\left[\left(\mathbf{\Theta}^{\mathrm{eff},(2),[N_k^2]}(N_k^2 t_2)\right)\right] = P_{t_2}(\cdot),\\
&\lim_{k\to\infty}\mathcal{L}\left[y_{2,1}^{[N_k^2]}(N_k t_2)\,\middle|\,\mathbf{\Theta}^{(2),[N_k^2]}(N_k^2 t_2)\right] = P^{z_2(t_2)},\\
&\lim_{k\to\infty}\mathcal{L}\left[\left(Y_{1,0}^{[N_k^2]}(N_k^2 t_2 + N_k t_1), Y_{2,0}^{[N_k^2]}(N_k^2 t_2)\right)\,\middle|\,\mathbf{\Theta}^{\mathrm{aux},(1),[N_k^2]}(N_k^2 t_2 + N_k t_1)\right] = P^{z_1^{\mathrm{eff}}(t_1)},\\
&\lim_{k\to\infty}\mathcal{L}\Bigg[\sup_{0\le t\le L(N_k)}\left|\bar{\Theta}^{(2),[N_k^2]}(N_k^2 t_2) - \bar{\Theta}^{(2),[N_k^2]}(N_k^2 t_2 - N_k t)\right|\\
&\qquad\qquad + \left|\Theta_{y_2}^{(2),[N_k^2]}(N_k^2 t_2) - \Theta_{y_2}{}^{(2),[N_k^2]}(N_k^2 t_2 - N t)\right|\Bigg] = \delta_0.
\end{aligned} \tag{8.102}$$

Then, for the 1-block around 0,

$$\lim_{k\to\infty}\mathcal{L}\left[\mathbf{\Theta}^{\mathrm{aux},(1),[N_k^2]}(N_k^2 t_2)\right] = \int_{[0,1]^2}\int_{[0,1]} \Gamma^{\mathrm{aux},(1)}_{u,y_{2,1}}\, P^{(u,v)}(\mathrm{d}y_{2,1})\, P_{t_2}(\mathrm{d}u,\mathrm{d}v), \tag{8.103}$$

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where $\Gamma^{\mathrm{aux},(1)}_{u,y_{2,1}}$ *is the equilibrium measure of* (8.29) *with* θ *replaced by* u*, and*

$$\lim_{k\to\infty}\mathcal{L}\left[\left(\mathbf{\Theta}^{\mathrm{aux},(1),[N_k^2]}(N_k^2 t_2+N_k t_1)\right)_{t_1>0}\right]=\mathcal{L}\left[(z_1^{\mathrm{aux}}(t_1))_{t_1>0})\right], \tag{8.104}$$

where $(z_1^{\mathrm{aux}}(t_1))_{t_1>0}$ *is the process evolving according to* (8.29) *with* θ *replaced by the random variable* $\bar{\Theta}^{(2)}(t_2)$ *and with initial measure* $\int_{[0,1]^2}\int_{[0,1]}\Gamma^{\mathrm{aux},(1)}_{u,y_{2,1}}\,P^{(u,v)}(\mathrm{d}y_{2,1})\,P_{t_2}(\mathrm{d}u,\mathrm{d}v)$*, and* $y_{2,1}$ *is a random variable.*

Note that by tightness of the 2-blocks and the assumptions in Proposition 8.1.1, we can always find a subsequence $(N_k)_{k\in\mathbb{N}}$ such that (8.102) holds and also (8.81) holds. To prepare for the proof of Proposition 8.3.10, we prove four lemmas: Lemma 8.3.11 shows that the limiting 1-block system has a unique equilibrium, Lemma 8.3.13 implies convergence of the active 2-block estimator and the combined 2-block estimator, Lemma 8.3.14 gives a regularity property for the 2-block estimator, and Lemma 8.3.15 shows the limiting evolution of the auxiliary 1-block estimator process. Lemma 8.3.17 proves equation (8.103). After we derive these lemmas we prove Proposition 8.3.10.

Lemma 8.3.11 (1-block equilibrium). *For any initial distribution* $\mu\in([0,1]^3)$*, the process* $(z_1^{\mathrm{aux}}(t_1))_{t_1>0}$ *evolving according to* (8.29) *is well defined and converges to a unique equilibrium measure*

$$\lim_{t_1\to\infty}\mathcal{L}[z_1^{\mathrm{aux}}(t_1)]=\Gamma^{\mathrm{aux},(1)}_{\theta,y_{2,1}}. \tag{8.105}$$

Proof. By [72], the SSDE in (8.29) has a unique strong solution. By a similar argument as in the proof of Lemma 7.2.4, the SSDE in (8.29) converges to a unique equilibrium measure $\Gamma^{\mathrm{aux},(1)}_{\theta,y_{2,1}}$. □

Remark 8.3.12 (Equilibrium measure). Note that Lemma 8.3.11 still holds when we allow θ and $y_{2,1}$ to be the random variables $\bar{\Theta}(t_2)$ and $y_{2,1}$. Assuming (8.102), we can derive the distributions of $\bar{\Theta}(t_2)$ and $y_{2,1}$, and we can write the equilibrium as $\int_{[0,1]^2}\int_{[0,1]}\Gamma^{\mathrm{aux},(1)}_{u,y_{2,1}}\,P^{(u,v)}(\mathrm{d}y_{2,1})\,P_{t_2}(\mathrm{d}u,\mathrm{d}v)$. In what follows we abbreviate

$$\Gamma^{(1)}_{\bar{\Theta}(t_2),y_{2,1,i}}=\int_{[0,1]^2}\int_{[0,1]}\Gamma^{\mathrm{aux},(1)}_{u,y_{2,1,i}}\,P^{(u,v)}(\mathrm{d}y_{2,1,i})\,P_{t_2}(\mathrm{d}u,\mathrm{d}v). \tag{8.106}$$

■

Lemma 8.3.13 (2-block averages). *Define*

$$\Delta_{\Sigma}^{(2),[N^2]}(Nt_1)=\frac{\Theta_x^{(2),[N^2]}(Nt_1)+K_0\Theta_{y_0}^{(2),[N^2]}(Nt_1)}{1+K_0}-\Theta_{y_1}^{(2),[N^2]}(Nt_1). \tag{8.107}$$

Then

$$
\begin{aligned}
&\mathbb{E}\left[\left|\Delta_{\Sigma}^{(2),[N^2]}(Nt_1)\right|\right] \\
&\leq \sqrt{\mathbb{E}\left[\left(\Delta_{\Sigma}^{(2),[N^2]}(0)\right)^2\right]}\mathrm{e}^{-e_1\left(\frac{1+K_0+K_1}{1+K_0}\right)t_1} \\
&+\sqrt{\int_0^{t_1}\mathrm{d}s\, 2e_1\left(\frac{1+K_0+K_1}{1+K_0}\right)\mathrm{e}^{-2e_1\left(\frac{1+K_0+K_1}{1+K_0}\right)(t_1-s)}\mathbb{E}\left[\left|\bar{\Theta}^{(1),[N^2]}(Ns)-\Theta_x^{(1),[N^2]}(Ns)\right|\right]} \\
&+\sqrt{\frac{1}{e_1}\left[\frac{K_2e_2}{N(1+K_0+K_1)}+\frac{||g||}{2N(1+K_0+K_1)}\right]}.
\end{aligned}
\tag{8.108}
$$

Proof. For the two-level mean-field system we have the following SSDE for the 2-block averages:

$$
\begin{aligned}
\mathrm{d}\Theta_x^{(2),[N^2]}(Nt_1) &= \sqrt{\frac{1}{N^3}\sum_{i\in[N^2]} g(x_i^{[N^2]}(Nt_1))}\,\mathrm{d}\tilde{w}(t_1) \\
&\quad + NK_0e_0\left[\Theta_{y_0}^{(2),[N^2]}(Nt_1)-\Theta_x^{(2),[N^2]}(Nt_1)\right]\mathrm{d}t_1 \\
&\quad + K_1e_1\left[\Theta_{y_1}^{(2),[N^2]}(Nt_1)-\Theta_x^{(2),[N^2]}(Nt_1)\right]\mathrm{d}t_1 \\
&\quad + \frac{K_2e_2}{N}\left[\Theta_{y_2}^{(2),[N^2]}(Nt_1)-\Theta_x^{(2),[N^2]}(Nt_1)\right]\mathrm{d}t_1, \\
\mathrm{d}\Theta_{y_0}^{(2),[N^2]}(Nt_1) &= Ne_0\left[\Theta_x^{(2),[N^2]}(Nt_1)-\Theta_{y_0}^{(2),[N^2]}(Nt_1)\right]\mathrm{d}t_1, \\
\mathrm{d}\Theta_{y_1}^{(2),[N^2]}(Nt_1) &= e_1\left[\Theta_x^{(2),[N^2]}(Nt_1)-\Theta_{y_1}^{(2),[N^2]}(Nt_1)\right]\mathrm{d}t_1, \\
\mathrm{d}\Theta_{y_2}^{(2),[N^2]}(Nt_1) &= \frac{e_2}{N}\left[\Theta_x^{(2),[N^2]}(Nt_1)-\Theta_{y_2}^{(2),[N^2]}(Nt_1)\right]\mathrm{d}t_1.
\end{aligned}
\tag{8.109}
$$

Therefore

$$
\begin{aligned}
&\mathrm{d}\left(\Delta_{\Sigma}^{(2),[N^2]}(Nt_1)\right)^2 = 2\Delta_{\Sigma}^{(2),[N^2]}(Nt_1)\,\mathrm{d}\Delta_{\Sigma}^{(2),[N^2]}(Nt_1)+\mathrm{d}\left\langle\Delta_{\Sigma}^{(2),[N^2]}(Nt_1)\right\rangle \\
&= 2\Delta_{\Sigma}^{(2),[N^2]}(Nt_1)\,\frac{1}{1+K_0}\sqrt{\frac{1}{N^3}\sum_{i\in[N^2]} g(x_i^{[N^2]}(Nt_1))}\,\mathrm{d}\tilde{w}(t_1) \\
&\quad + 2\Delta_{\Sigma}^{(2),[N^2]}(Nt_1)\,\frac{K_1e_1}{(1+K_0)}\left[\Theta_{y_1}^{(2),[N^2]}(Nt_1)-\Theta_x^{(2),[N^2]}(Nt_1)\right]\mathrm{d}t_1, \\
&\quad + 2\Delta_{\Sigma}^{(2),[N^2]}(Nt_1)\,\frac{K_2e_2}{N(1+K_0)}\left[\Theta_{y_2}^{(2),[N^2]}(Nt_1)-\Theta_x^{(2),[N^2]}(Nt_1)\right]\mathrm{d}t_1, \\
&\quad - 2\Delta_{\Sigma}^{(2),[N^2]}(Nt_1)\,e_1\left[\Theta_x^{(2),[N^2]}(Nt_1)-\Theta_{y_1}^{(2),[N^2]}(Nt_1)\right]\mathrm{d}t_1 \\
&\quad + \frac{1}{(1+K_0)^2}\frac{1}{N^3}\sum_{i\in[N^2]} g(x_i^{[N^2]}(Nt_1))\,\mathrm{d}t_1.
\end{aligned}
\tag{8.110}
$$

Hence

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[\left(\Delta_{\Sigma}^{(2),[N^2]}(Nt_1)\right)^2\right]\\
&= -2e_1\left(\frac{1+K_0+K_1}{1+K_0}\right)\mathbb{E}\left[\left(\Delta_{\Sigma}^{(2),[N^2]}(Nt_1)\right)^2\right]\\
&\quad + 2e_1\left(\frac{1+K_0+K_1}{1+K_0}\right)\\
&\qquad \times \mathbb{E}\left[\Delta_{\Sigma}^{(2),[N^2]}(Nt_1)\left(\frac{\Theta_x^{(2),[N^2]}(Nt_1)+K_0\Theta_{y_0}^{(2),[N^2]}(Nt_1)}{1+K_0}-\Theta_x^{(2),[N^2]}(Nt_1)\right)\right]\\
&\quad + \frac{K_2e_2}{N(1+K_0)}\,2\mathbb{E}\left[\Delta_{\Sigma}^{(2),[N^2]}(Nt_1)\,\left[\Theta_{y_2}^{(2),[N^2]}(Nt_1)-\Theta_x^{(2),[N^2]}(Nt_1)\right]\right]\\
&\quad + \frac{1}{(1+K_0)^2}\mathbb{E}\left[\frac{1}{N^3}\sum_{i\in[N^2]} g(x_i^{[N^2]}(Nt_1))\right],
\end{aligned}
\tag{8.111}
$$

and therefore

$$
\begin{aligned}
&\mathbb{E}\left[\left(\Delta_{\Sigma}^{(2),[N^2]}(Nt_1)\right)^2\right]\\
&= \mathbb{E}\left[\left(\Delta_{\Sigma}^{(2),[N^2]}(0)\right)^2\right]\mathrm{e}^{-2e_1\left(\frac{1+K_0+K_1}{1+K_0}\right)t_1} + \int_0^{t_1}\mathrm{d}s\,\mathrm{e}^{-2e_1\left(\frac{1+K_0+K_1}{1+K_0}\right)(t_1-s)}h^{[N]}(s),
\end{aligned}
\tag{8.112}
$$

where

$$
\begin{aligned}
h^{[N]}(s) &= 2e_1\left(\frac{1+K_0+K_1}{1+K_0}\right)\\
&\quad \times \mathbb{E}\left[\Delta_{\Sigma}^{(2),[N^2]}(Ns)\left(\frac{\Theta_x^{(2),[N^2]}(Ns)+K_0\Theta_{y_0}^{(2),[N^2]}(Ns)}{1+K_0}-\Theta_x^{(2),[N^2]}(Ns)\right)\right]\\
&\quad + \frac{2K_2e_2}{N(1+K_0)}\mathbb{E}\left[\Delta_{\Sigma}^{(2),[N^2]}(Ns)\,\left[\Theta_{y_2}^{(2),[N^2]}(Ns)-\Theta_x^{(2),[N^2]}(Ns)\right]\right]\\
&\quad + \frac{1}{(1+K_0)^2}\mathbb{E}\left[\frac{1}{N^3}\sum_{i\in[N^2]} g(x_i^{[N^2]}(Ns))\right].
\end{aligned}
\tag{8.113}
$$

Therefore

$$
\begin{aligned}
&\mathbb{E}\left[\left|\Delta_{\Sigma}^{(2),[N^2]}(Nt_1)\right|\right]\\
&\leq \sqrt{\mathbb{E}\left[\left(\Delta_{\Sigma}^{(2),[N^2]}(0)\right)^2\right]}\mathrm{e}^{-e_1\left(\frac{1+K_0+K_1}{1+K_0}\right)t_1}\\
&+ \sqrt{\int_0^{t_1}\mathrm{d}s\,2e_1\left(\frac{1+K_0+K_1}{1+K_0}\right)\mathrm{e}^{-2e_1\left(\frac{1+K_0+K_1}{1+K_0}\right)(t_1-s)}\mathbb{E}\left[\left|\bar{\Theta}^{(1),[N^2]}(Ns)-\Theta_x^{(1),[N^2]}(Ns)\right|\right]}\\
&+ \sqrt{\frac{1}{e_1}\left[\frac{K_2e_2}{N(1+K_0+K_1)}+\frac{||g||}{2N(1+K_0+K_1)}\right]}.
\end{aligned}
\tag{8.114}
$$

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□

Let μ_{N_k} be the measure obtained by periodic continuation of the configuration

$$Z^{[N^2]}(N_k^2 t_2). \tag{8.115}$$

Since the state space $([0,1] \times [0,1]^3)^{\mathbb{N}_0}$ is compact, we can pass to a further subsequence, to obtain

$$\mu = \lim_{k\to\infty} \mu_{N_k}. \tag{8.116}$$

Lemma 8.3.14 (Regularity for 2-block estimator). *Let μ and μ_N be as defined above. Let $(x_i, y_{1,i}, y_{2,i})_{i\in\mathbb{N}_0}$ be distributed according to μ. Define the random variable*

$$\begin{aligned}
&\phi = (\phi_1, \phi_2),\\
&\phi_1 = \lim_{n\to\infty} \frac{1}{n^2} \sum_{i\in[n^2]} \frac{x_i + K_0 y_{i,0} + K_1 y_{i,1}}{1+K_0+K_1}, \qquad \phi_2 = \lim_{n\to\infty} \frac{1}{n^2} \sum_{i\in[n^2]} y_{i,2},
\end{aligned} \tag{8.117}$$

and the random variable $\phi^{[N]}$ on $(\mu_N, ([0,1]^3)^{\mathbb{N}_0})$ by putting

$$\begin{aligned}
&\phi^{[N^2]} = (\phi_1^{[N^2]}, \phi_2^{[N^2]}),\\
&\phi_1^{[N^2]} = \frac{1}{N^2} \sum_{i\in[N^2]} \frac{x_i^{[N^2]} + K_0 y_{i,0}^{[N^2]} + K_1 y_{i,1}^{[N^2]}}{1+K_0+K_1}, \qquad \phi_2 = \lim_{N\to\infty} \frac{1}{N^2} \sum_{i\in[N^2]} y_{i,2}^{[N^2]}.
\end{aligned} \tag{8.118}$$

Then

$$\lim_{N\to\infty} \mathcal{L}[\phi^{[N^2]}] = \mathcal{L}[\phi]. \tag{8.119}$$

Proof. Use a similar argument as in the proof of Lemma 7.2.8. □

We will first determine the limiting evolution of $(\mathbf{\Theta}^{\mathrm{aux},(1),[N_k^2]}(N_k^2 t_2 + N t_1))_{t_1>0}$. To do so we consider all the N_k 1-blocks in $[N_k^2]$. After that we show that

$$\lim_{k\to\infty} \mathcal{L}\left[\left(\mathbf{\Theta}_i^{\mathrm{aux},(1),[N_k^2]}(N_k^2 t_2)\right)_{i\in[N_k]}\right] = \prod_{i\in\mathbb{N}_0} \Gamma^{(1)}_{\bar{\Theta}(t_2),y_{2,1,i}}, \tag{8.120}$$

The limiting 1-block process for the auxiliary estimator process (recall (8.29)) is given by

$$\begin{aligned}
&(\mathbf{z}_1^{\mathrm{aux}}(t))_{t>0} = (\mathbf{x}_1^{\mathrm{aux}}(t), \mathbf{y}_{1,1}^{\mathrm{aux}}(t), \mathbf{y}_{2,1}^{\mathrm{aux}}(t))_{t>0},\\
&\mathbf{z}_1^{\mathrm{aux}}(t) = (z_{1,i}^{\mathrm{aux}}(t))_{i\in\mathbb{N}_0}, \qquad \mathbf{x}_1^{\mathrm{aux}}(t) = (x_{1,i}^{\mathrm{aux}}(t))_{i\in\mathbb{N}_0},\\
&\mathbf{y}_{1,1}^{\mathrm{aux}}(t) = (y_{1,1,i}^{\mathrm{aux}}(t))_{i\in\mathbb{N}_0} \qquad \mathbf{y}_{2,1}^{\mathrm{aux}}(t) = (y_{2,i}^{\mathrm{aux}}(t))_{i\in\mathbb{N}_0}
\end{aligned} \tag{8.121}$$

and its components evolve according to

$$\begin{aligned}
\mathrm{d}x_{1,i}^{\mathrm{aux}}(t) &= \frac{1}{1+K_0}\Bigg[c_1[\bar{\Theta}^{(2)}(t_2) - x_{1,i}^{\mathrm{aux}}(t)]\,\mathrm{d}t + \sqrt{(\mathcal{F}^{(1)}g)(x_{1,i}^{\mathrm{aux}}(t))}\,\mathrm{d}w(t)\\
&\qquad + K_1 e_1\,[y_{1,1,i}^{\mathrm{aux}}(t) - x_{1,i}^{\mathrm{aux}}(t)]\,\mathrm{d}t\Bigg], \qquad (8.122)\\
\mathrm{d}y_{1,1,i}^{\mathrm{aux}}(t) &= e_1\,[x_{1,i}^{\mathrm{aux}}(t) - y_{1,1,i}^{\mathrm{aux}}(t)]\,\mathrm{d}t,\\
y_{2,1,i}^{\mathrm{aux}}(t) &= y_{2,1,i}, \qquad i\in\mathbb{N}_0,
\end{aligned}$$

where

$$\bar{\Theta}^{(2)}(t_2) = \lim_{N\to\infty} \sum_{i\in[N^2]} \frac{x_i^{[N^2]} + K_0 y_{i,0}^{[N^2]} + K_1 y_{i,1}^{[N^2]}}{1+K_0+K_1} \text{ in } L_2(\mu). \tag{8.123}$$

Let $\mu_{N_k}^{(1)}$ be the law obtained by periodic continuation of $(\boldsymbol{\Theta}_i^{\mathrm{aux},(1),[N_k^2]}(N_k^2 t_2))_{i\in[N_k]}$, and let $\mu^{(1)} = \lim_{k\to\infty} \mu_{N_k}^{(1)}$ be any weak limit point of the sequence $(\mu_{N_k}^{(1)})_{k\in\mathbb{N}}$.

Lemma 8.3.15 (Limiting evolution of auxiliary 1-block estimator). *Let $\mathcal{L}[(\boldsymbol{z}_1^{\mathrm{aux}}(0))] = \mu^{(1)}$. Then the following hold.*

(a) *For all $t_1 > 0$ and $i \in [N_k]$,*

$$\begin{aligned}\lim_{k\to\infty} \mathbb{E}\Bigg[&(1+K_0)\left(x_{1,i}^{\mathrm{aux}}(t_1) - \bar{\Theta}_i^{\mathrm{aux},(1),[N_k^2]}(N_k^2 t_2 + N_k t_1)\right)^2\\ &+ K_1\left(y_{1,1,i}(t_1) - \Theta_{y_1,i}^{\mathrm{aux},(1),[N_k^2]}(N_k^2 t_2 + N_k t_1)\right)^2\\ &+ K_2\left(y_{2,1,i}^{\mathrm{aux},(1),[N_k^2]}(t_1) - \Theta_{y_2,i}^{\mathrm{aux},(1),[N_k]}(N_k^2 t_2 + N_k t_1)\right)^2\Bigg] = 0.\end{aligned} \tag{8.124}$$

(b) *For all $t_2 > 0$,*

$$\lim_{k\to\infty} \mathcal{L}\left[(\boldsymbol{\Theta}^{\mathrm{aux},(1),[N_k^2]}(N_k^2 t_2 + N t_1))_{t_1>0}\right] = \mathcal{L}[(\boldsymbol{z}_1^{\mathrm{aux}}(t_1))_{t_1>0}]. \tag{8.125}$$

Proof. Abbreviate

$$\begin{aligned}\Delta_i^{(1),[N_k^2]}(N_k t_1) &= x_{1,i}^{\mathrm{aux}}(t_1) - \Theta_i^{\mathrm{aux},(1),[N_k^2]}(N_k^2 t_2 + N_k t_1),\\ \delta_{y_1,i}^{(1),[N_k^2]}(N_k t_1) &= y_{1,1,i}(t_1) - \Theta_{y_1,i}^{\mathrm{aux},(1),[N_k^2]}(N_k^2 t_2 + N_k t_1),\\ \delta_{y_2,i}^{(1),[N_k^2]}(N_k t_1) &= y_{2,1,i}^{\mathrm{aux}}(t_1) - \Theta_{y_2,i}^{\mathrm{aux},(1),[N_k^2]}(N_k^2 t_2 + N_k t_1).\end{aligned} \tag{8.126}$$

Extending $(\boldsymbol{\Theta}^{\mathrm{aux},(1),[N_k^2]}(N_k t_1))_{t_1>0}$ as a process on $\mathbb{N}_0$ by periodic continuation, we can construct $(\mathbf{z}_1^{\mathrm{aux}}(t_1))_{t_1>0}$ and $(\boldsymbol{\Theta}^{\mathrm{aux},(1)}(N_k t_1))_{t_1>0}$ on one probability space such that

$$\lim_{k\to\infty} \boldsymbol{\Theta}^{\mathrm{aux},(1),[N_k^2]}(N_k^2 t_2) = \mathbf{z}_1^{\mathrm{aux},(1)}(0) \quad a.s. \tag{8.127}$$

We couple the processes $(\mathbf{z}_1^{\mathrm{aux}}(t_1))_{t_1>0}$ and $(\boldsymbol{\Theta}^{\mathrm{aux},(1),[N_k^2]}(N_k^2 t_2 + N_k t_1))_{t_1>0}$ by using the same Brownian motions for both processes. By Itô-calculus we obtain for the coupled process (recall (8.74))

$$
\begin{aligned}
&\mathbb{E}\left[(1+K_0)\left(\Delta_i^{(1),[N_k^2]}(N_k t_1)\right)^2 + K_1\left(\delta_{y_1,i}^{(1),[N_k^2]}(N_k t_1)\right)^2 + K_2\left(\delta_{y_2,i}^{(1),[N_k^2]}(N_k t_1)\right)^2\right]\\
=&\mathbb{E}\left[(1+K_0)\left(\Delta_i^{(1),[N_k^2]}(0)\right)^2 + K_1\left(\delta_{y_1,i}^{(1),[N_k^2]}(0)\right)^2 + K_2\left(\delta_{y_2,i}^{(1),[N_k^2]}(0)\right)^2\right]\\
&-2c_1\int_0^{t_1}\mathbb{E}\left[\left(\Delta_i^{(1),[N_k^2]}(N_k s)\right)^2\right]\mathrm{d}s\\
&-2K_1e_1\int_0^{t_1}\mathbb{E}\left[\left(\Delta_i^{(1),[N_k^2]}(N_k s)-\delta_{y_1,i}^{(1),[N_k^2]}(N_k s)\right)^2\right]\mathrm{d}s\\
&+2c_1\int_0^{t_1}\mathbb{E}\left[\Delta_i^{(1),[N_k^2]}(N_k s)\left(\Theta^{(2)}(t_2)-\frac{1}{N_k^2}\sum_{j\in[N_k^2]}x_j^{[N_k^2]}(N_k^2t_2+N_k s)\right)\right]\mathrm{d}s\\
&+(K_1e_1+c_1)\int_0^{t_1}\mathbb{E}\Big[\Delta_i^{(1),[N_k^2]}(N_k s)\\
&\qquad\times\left[\frac{1}{N_k}\sum_{j\in[N_k]_i}x_j^{[N_k^2]}(N_k^2t_2+N_k s)-\bar{\Theta}_i^{\mathrm{aux},(1),[N_k^2]}(N_k^2t_2+N_k s)\right]\Big]\mathrm{d}s\\
&+2K_1e_1\int_0^{t_1}\mathbb{E}\Big[\delta_{y_1,i}^{(1),[N_k^2]}(N_k s)\\
&\qquad\times\left[\bar{\Theta}_i^{\mathrm{aux},(1),[N_k^2]}(N_k^2t_2+N_k s)-\frac{1}{N_k}\sum_{j\in[N_k]_i}x_j^{[N_k^2]}(N_k^2t_2+N_k s)\right]\Big]\mathrm{d}s\\
&+2\frac{K_2e_2}{N_k}\int_0^{t_1}\mathbb{E}\Bigg[\left[\delta_{y_1,i}^{(1),[N_k^2]}(N_k s)-\Delta_i^{(1),[N_k^2]}(N_k^2t_2+N_k s)\right]\\
&\qquad\times\left[\frac{1}{N_k}\sum_{j\in[N_k]_i}x_j^{[N_k^2]}(N_k^2 s)-\Theta_{y_2,i}^{\mathrm{aux},(1),[N_k^2]}(N_k^2t_2+Nt_1)\right]\Bigg]\mathrm{d}s\\
&+(1+K_0)^2\int_0^{t_1}\mathbb{E}\left[\left(\sqrt{(\mathcal{F}g)\left(x_1^{\mathrm{aux}}(s)\right)}-\sqrt{\frac{1}{N}\sum_{i\in[N]}g(x_i^{[N_k^2]}(N_k^2t_2+N_k s))}\right)^2\right]\mathrm{d}s.
\end{aligned}
\tag{8.128}
$$

Note that $|\Delta_i^{(1),[N_k^2]}|\leq 1$ and $|\delta_{y_1,i}^{(1),[N_k^2]}|\leq 1$. Note that the first term tends to 0 by (8.127). We show by dominated convergence that also all other positive terms in the right-hand side of (8.128) tend to 0 as $k\to\infty$.

For the third term, we can estimate

$$\begin{aligned}
&\lim_{k\to\infty}\mathbb{E}\left[\frac{c_1}{1+K_0}\left|\frac{1}{N_k^2}\sum_{i\in[N_k^2]}x_j^{[N_k^2]}(N_k^2t_2+N_ks)-\bar{\Theta}^{(2)}(t_2)\right|\right]\\
&\leq\lim_{k\to\infty}\mathbb{E}\left[\frac{c_1}{1+K_0}\left|\Theta_x^{(2),[N_k^2]}(N_k^2t_2+N_ks)\right.\right.\\
&\qquad\left.\left.-\frac{\Theta_x^{(2),[N_k^2]}(N_k^2t_2+N_ks)+K_0\Theta_{y_0}^{(2),[N_k^2]}(N_k^2t_2+N_ks)}{1+K_0}\right|\right]\\
&\quad+\mathbb{E}\left[\frac{c_1}{1+K_0}\left|\frac{\Theta_x^{(2),[N_k^2]}(N_k^2t_2+N_ks)+K_0\Theta_{y_0}^{(2),[N_k^2]}(N_k^2t_2+N_ks)}{1+K_0}\right.\right.\\
&\quad\left.\left.-\frac{\Theta_x^{(2),[N_k^2]}(N_k^2t_2+N_ks)+K_0\Theta_{y_0}^{(2),[N_k^2]}(N_k^2t_2+N_ks)+K_1\Theta_{y_1}^{(2),[N_k^2]}(N_k^2t_2+N_ks)}{1+K_0+K_1}\right|\right]\\
&\quad+\mathbb{E}\left[\frac{c_1}{1+K_0}\left|\frac{\Theta_x^{(2),[N_k^2]}(N_k^2t_2+N_ks)+K_0\Theta_{y_0}^{(2),[N_k^2]}(N_k^2t_2+N_ks)+K_1\Theta_{y_1}^{(2),[N_k^2]}(N_k^2t_2+N_ks)}{1+K_0+K_1}\right.\right.\\
&\qquad\left.\left.-\bar{\Theta}^{(2)}(t_2)\right|\right].
\end{aligned}\tag{8.129}$$

The first term in (8.129) tends to zero by Lemma 8.3.6, the second term tends to zero by Lemma 8.3.13, while the third term tends to zero by Lemma 8.3.14 and Lemma 8.3.2, which is the third assumption in (8.102). Hence the third term in (8.128) tends to zero by dominated convergence as $k\to\infty$.

The fourth and fifth term in (8.128) tend to zero by Lemma 8.3.6 and dominated convergence. The sixth term in (8.128) tends to zero because the integral is bounded by t_1 and there is a factor $\frac{1}{N_k}$ in front. To see that the last term in the right-hand side in (8.128) tends to zero, recall that the subsequence N_k is chosen such that

$$\lim_{k\to\infty}\mathcal{L}\left[(\boldsymbol{\Theta}^{\mathrm{aux},(1),[N_k^2]}(N_k^2t_2+Nt_1))_{t_1>0}\right]\tag{8.130}$$

exists. Note that

$$\begin{aligned}
&\mathbb{E}\left[\left(\sqrt{(\mathcal{F}g)(x_1^{\mathrm{aux}}(s))}-\sqrt{\frac{1}{N}\sum_{i\in[N]}g(x_i^{[N_k^2]}(N_k^2t_2+N_ks))}\right)^2\right]\\
&\leq\mathbb{E}\left[\left|(\mathcal{F}g)(x_1^{\mathrm{aux}}(s))-\frac{1}{N}\sum_{i\in[N]}g(x_i^{[N_k^2]}(N_k^2t_2+N_ks))\right|\right],
\end{aligned}\tag{8.131}$$

and hence we can apply a similar reasoning as in (6.198) to see that (8.131) tends to zero as $k\to\infty$. Therefore we obtain

$$\begin{aligned}
&\lim_{k\to\infty}\mathbb{E}\left[(1+K_0)(\Delta_i^{(1),[N_k^2]}(N_kt_1))^2+K_1(\delta_{y_1,i}^{(1),[N_k^2]}(N_kt_1))^2+K_2(\delta_{y_2,i}^{(1),[N_k^2]}(N_kt_1))^2\right]\\
&\qquad=0.
\end{aligned}\tag{8.132}$$

CHAPTER 8

To prove (8.125), note that (8.132) implies convergence of the finite-dimensional distributions of $(\boldsymbol{\Theta}^{\mathrm{aux},(1),[N_k^2]}(N_k^2 t_2 + N_k t_1))_{t_1>0}$ by a similar argument as given below (6.137). By Lemma 8.3.3 we see that the laws of the processes

$$\left(\mathcal{L}\left[(\boldsymbol{\Theta}^{\mathrm{aux},(1),[N_k^2]}(N_k^2 t_2 + N_k t_1))_{t_1>0}\right]\right)_{k\in\mathbb{N}_0} \tag{8.133}$$

are tight. Therefore (8.125) indeed holds. □

Remark 8.3.16. Note that in the proof of Lemma 7.2.12 we could have proceeded as in the proof of Lemma 8.3.15, instead of using the criterion in [49, Theorem 3.3.1]. ■

Lemma 8.3.17 (Proof of (8.103)**).** *Under the assumptions in Proposition 8.3.10,*

$$\lim_{k\to\infty} \mathcal{L}\left[\boldsymbol{\Theta}^{\mathrm{aux},(1),[N_k^2]}(N_k^2 t_2)\right] = \int_{[0,1]^2}\int_{[0,1]} \Gamma^{\mathrm{aux},(1)}_{u,y_{2,1}}\, P^{(u,v)}(\mathrm{d}y_{2,1})\, P_{t_2}(\mathrm{d}u,\mathrm{d}v). \tag{8.134}$$

Proof. For ease of notation, we drop the subsequence notation in this proof. Let $(t_n)_{n\in\mathbb{N}_0}$ be any sequence satisfying $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} t_n/n = 0$. For each t_n, let $\mu^{(1)}_{N,t_n}$ be the law obtained by periodic continuation of the configuration of $(\boldsymbol{\Theta}_i^{\mathrm{aux},(1),[N^2]}(N^2 t_2 - N t_n))_{i\in[N]}$. Recall that, since our state space is compact, the sequence $(\mu^{(1)}_{N,t_n})_{N\in\mathbb{N}}$ is tight. Let $\mu^{(1)}_{t_n}$ be any weak limit point of the sequence $(\mu^{(1)}_{N,t_n})_{N\in\mathbb{N}}$.

Let $\mathcal{L}[\mathbf{z}_1^{\mathrm{aux},n}(0)]$ be the law obtained by periodic continuation of $\boldsymbol{\Theta}^{\mathrm{aux},(1),[N^2]}(N^2 t_2 - N t_n)$. By Lemma 8.3.3 we know that the sequence

$$\left(\mathcal{L}\left[\left(\boldsymbol{\Theta}_i^{\mathrm{aux},(1),[N^2]}(N^2 t_2 - N t_n + N t_1)\right)_{t_1>0,i\in[N]}\right]\right)_{N\in\mathbb{N}} \tag{8.135}$$

is tight and hence for each t_n we can pass to a subsequence such that

$$\lim_{k\to\infty} \mathcal{L}\left[\left(\boldsymbol{\Theta}_i^{\mathrm{aux},(1),[N_k^2]}(N_k^2 t_2 - N_k t_n + N_k t_1)\right)_{t_1>0,i\in[N]}\right] \tag{8.136}$$

exists. By Lemmas 8.3.2–8.3.14, we obtain for all t_n that

$$\lim_{N\to\infty} \bar{\Theta}^{(2),[N_k^2]}(N_k^2 t_2 - N t_n) = \bar{\Theta}^{(2)}(t_2) \text{ in probability.} \tag{8.137}$$

Then, by (8.128) in the proof of Lemma 8.3.15, for fixed t_n and all $i\in[N]$,

$$\begin{aligned}\lim_{k\to\infty}\mathbb{E}\Bigg[&(1+K_0)\left(x^{\mathrm{aux},n}_{1,i}(t_n) - \bar{\Theta}_i^{\mathrm{aux},(1),[N_k^2]}(N_k^2 t_2 - N_k t_n + N_k t_n)\right)^2\\ &+ K_1\left(y^{\mathrm{aux},n}_{1,1,i}(t_n) - \Theta_{y_1,i}^{\mathrm{aux},(1),[N_k^2]}(N_k^2 t_2 - N_k t_n + N_k t_n)\right)^2\\ &+ K_2\left(y^{\mathrm{aux},n}_{2,1,i}(t_n) - \Theta_{y_2,i}^{\mathrm{aux},(1),[N_k^2]}(N_k^2 t_2 - N_k t_n + N_k t_n)\right)^2\Bigg] = 0.\end{aligned} \tag{8.138}$$

By contradiction we can argue that

$$\begin{aligned}\lim_{N\to\infty}\mathbb{E}\Bigg[&(1+K_0)\left(x^{\mathrm{aux},n}_{1,i}(t_n)-\bar\Theta^{\mathrm{aux},(1),[N^2]}_i(N^2t_2-Nt_n+Nt_n)\right)^2\\&+K_1\left(y^{\mathrm{aux},n}_{1,1,i}(t_n)-\Theta^{\mathrm{aux},(1),[N^2]}_{y_1,i}(N^2t_2-Nt_n+Nt_n)\right)^2\\&+K_2\left(y^{\mathrm{aux},n}_{2,1,i}(t_n)-\Theta^{\mathrm{aux},(1),[N^2]}_{y_2,i}(N^2t_2-Nt_n+Nt_n)\right)^2\Bigg]=0.\end{aligned}\tag{8.139}$$

To see why, suppose that (8.139) does not hold. Then for any $\delta>0$ we can construct a sequence $(N_l)_{l>0}$ such that, for $l\in\mathbb{N}$,

$$\begin{aligned}\mathbb{E}\Bigg[&(1+K_0)\left(x^{\mathrm{aux},n}_{1,i}(t_n)-\bar\Theta^{\mathrm{aux},(1),[N_l^2]}_i(N_l^2t_2-N_lt_n+N_lt_n)\right)^2\\&+K_1\left(y^{\mathrm{aux},n}_{1,1,i}(t_n)-\Theta^{\mathrm{aux},(1),[N_l^2]}_{y_1,i}(N_l^2t_2-N_lt_n+N_lt_n)\right)^2\\&+K_2\left(y^{\mathrm{aux},n}_{2,1,i}(t_n)-\Theta^{\mathrm{aux},(1),[N_l^2]}_{y_2,i}(N_l^2t_2-N_lt_n+N_lt_n)\right)^2\Bigg]>\delta.\end{aligned}\tag{8.140}$$

However, also the sequence

$$\left(\mathcal{L}\left[\left(\mathbf{\Theta}^{\mathrm{aux},(1),[N_l^2]}_i(N_l^2t_2-N_lt_n+N_lt_1)\right)_{t_1>0,i\in[N]}\right]\right)_{l\in\mathbb{N}}\tag{8.141}$$

is tight. Hence we can pass to a further subsequence $(N_{\tilde l})_{\tilde l\in\mathbb{N}}$ for which (8.138) holds. But this contradicts (8.140). We conclude that (8.139) indeed holds. Moreover the argument holds for all t_n, so that (8.139) holds for all t_n.

Hence for every t_n there exists a N_n such that, for all $N\geq N_n$,

$$\begin{aligned}\mathbb{E}\Bigg[&(1+K_0)\left(x^{\mathrm{aux},n}_{1,i}(t_n)-\bar\Theta^{\mathrm{aux},(1),[N_k^2]}_i(N^2t_2)\right)^2\\&+K_1\left(y^{\mathrm{aux},n}_{1,1,i}(t_n)-\Theta^{\mathrm{aux},(1),[N_k^2]}_{y_1,i}(N^2t_2)\right)^2\\&+K_2\left(y^{\mathrm{aux},n}_{2,1,i}(t_n)-\Theta^{\mathrm{aux},(1),[N_k^2]}_{y_2,i}(N^2t_2)\right)^2\Bigg]<\frac{1}{n}.\end{aligned}\tag{8.142}$$

In particular, we may require that $N_n>N_{n-1}$. Setting $N=N_n$, we obtain

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\Bigg[&(1+K_0)\left(x^{\mathrm{aux},n}_{1,i}(t_n)-\bar\Theta^{\mathrm{aux},(1),[N_n^2]}_i(N_n^2t_2)\right)^2\\&+K_1\left(y^{\mathrm{aux},n}_{1,1,i}(t_n)-\Theta^{\mathrm{aux},(1),[N_n^2]}_{y_1,i}(N_n^2t_2)\right)^2\\&+K_2\left(y^{\mathrm{aux},n}_{2,1,i}(t_n)-\Theta^{\mathrm{aux},(1),[N_n^2]}_{y_2,i}(N_n^2t_2)\right)^2\Bigg]=0.\end{aligned}\tag{8.143}$$

If we can prove that

$$\lim_{n\to\infty} \mathcal{L}[z_i^{\mathrm{aux},n}(t_n)] = \Gamma^{(1)}_{\bar\Theta(t_2)}, \tag{8.144}$$

then we are done. To see why, note that, for all $f \in \mathcal{C}_b([0,1]\times[0,1]^2)$, f Lipschitz continuous

$$\begin{aligned}
&\left|\mathbb{E}[f(\mathbf{\Theta}_i^{\mathrm{aux},(1),[N^2]}(N^2t_2))] - \mathbb{E}^{\Gamma^{(1)}_{\bar\Theta(t_2)}}[f]\right| \\
&\leq \left|\mathbb{E}[f(\mathbf{\Theta}_i^{\mathrm{aux},(1),[N^2]}(N^2t_2)) - f(z_{1,i}{}^{\mathrm{aux},n}(t_n))]\right| + \left|\mathbb{E}[f(z_{1,i}{}^{\mathrm{aux},n}(t_n))] - \mathbb{E}^{\Gamma^{(1)}_{\bar\Theta(t_2)}}[f]\right|.
\end{aligned} \tag{8.145}$$

Therefore if (8.144) holds, then for all $\epsilon > 0$ we can choose $\bar n$ such that, for all $n > \bar n$,

$$\left|\mathbb{E}[f(z_{1,i}{}^{\mathrm{aux},n}(t_n))] - \mathbb{E}^{\Gamma^{(1)}_{\bar\Theta(t_2)}}[f]\right| < \frac{\epsilon}{2}. \tag{8.146}$$

By (8.143) we can find a $\hat n > \bar n$ such that for all $n > \hat n$

$$\left|\mathbb{E}[f(\mathbf{\Theta}_i^{\mathrm{aux}}(N_n^2 t_2)) - f(z_{1,i}{}^{\mathrm{aux},n}(t_{\hat n}))]\right| < \frac{\epsilon}{2}. \tag{8.147}$$

Using (8.144) and the fact that the Lipschitz functions are dense in $\mathcal{C}_b([0,1]\times[0,1]^2)$, we obtain (8.134).

Proof of (8.144)**.** We use that any two systems $(\mathbf{z}^{\mathrm{aux},1}(t_1))_{t_1>0}$ and $(\mathbf{z}^{\mathrm{aux},2}(t_1))_{t_1>0}$ evolving according to (8.29), and having the same $y_{2,1}$-components and the same $\bar\Theta(t_2)$, can be constructed on one probability space and can coupled by their Brownian motions. We obtain, for a component $i \in \mathbb{N}_0$,

$$\begin{aligned}
&\mathbb{E}[|x_{1,i}^{\mathrm{aux},1}(t_n) - x_{1,i}^{\mathrm{aux},2}(t_n)| + K_1|y_{1,1,i}^{\mathrm{aux},1}(t_n) - y_{1,1,i}^{\mathrm{aux},2}(t_n)| + K_2|y_{2,1,i}^{\mathrm{aux},1}(t_n) - y_{2,1,i}^{\mathrm{aux},2}(t_n)|] \\
&= \mathbb{E}[|x_{1,i}^{\mathrm{aux},1}(0) - x_{1,i}^{\mathrm{aux},2}(0)| + K_1|y_{1,1,i}^{\mathrm{aux},1}(0) - y_{1,1,i}^{\mathrm{aux},2}(0)| + K_2|y_{2,1,i}^{\mathrm{aux},1}(0) - y_{2,1,i}^{\mathrm{aux},2}(0)|] \\
&- c\int_0^{t_n} \mathbb{E}[|x_{1,i}^{\mathrm{aux},1}(s) - x_{1,i}^{\mathrm{aux},2}(s)|]\mathrm{d}s \\
&- 2K_1e_1 \int_0^{t_n} \mathbb{E}\big[[|x_{1,i}^{\mathrm{aux},1}(s) - x_{1,i}^{\mathrm{aux},2}(s)| + K_1|y_{1,1,i}^{\mathrm{aux},1}(s) - y_{1,1,i}^{\mathrm{aux},2}(s)|] \\
&\qquad\qquad \times 1_{\{\mathrm{sgn}\,(x_{1,i}^{\mathrm{aux},1}(s) - x_{1,i}^{\mathrm{aux},2}(s)) \neq \mathrm{sgn}\,(y_{1,1,i}^{\mathrm{aux},1}(s) - y_{1,1,i}^{\mathrm{aux},2}(s))\}}\big]\mathrm{d}s.
\end{aligned} \tag{8.148}$$

Therefore the difference between these two systems monotonically decreases.

Since the state space $[0,1]\times[0,1]^2$ is compact, the sequence of laws

$$(\mathcal{L}[z_i^{\mathrm{aux},n}(0)])_{n\in\mathbb{N}} \tag{8.149}$$

is tight. Therefore we can find converging subsequences such that

$$\lim_{k\to\infty} \mathcal{L}[z_i{}^{\mathrm{aux},n_k}(0)] = \mu \tag{8.150}$$

for some probability measure μ on $[0,1]\times[0,1]^2$.

Let $(z^{\mathrm{aux},0}(t_1))_{t_1>0}$ be the limiting system evolving according to (8.29) and starting from initial distribution μ. By Skorohod's theorem, we can construct the sequence of limiting systems $((z^{\mathrm{aux},n_k}(t_1))_{t_1>0})_{k\in\mathbb{N}}$ and $(z^{\mathrm{aux},0}(t_1))_{t_1>0}$ on one probability space such that

$$\lim_{k\to\infty} z^{\mathrm{aux},n_k}(0) = z^{\mathrm{aux},0}(0) \quad a.s. \tag{8.151}$$

Use the coupling of Brownian motions to obtain

$$\begin{aligned}
&\mathbb{E}[|x^{\mathrm{aux},n_k}_{1,i}(t_{n_k}) - x^{\mathrm{aux},0}_{1,i}(t_{n_k})| + K_1|y^{\mathrm{aux},n_k}_{1,1,i}(t_{n_k}) - y^{\mathrm{aux},0}_{1,1,i}(t_{n_k})| \\
&\qquad + K_2|y^{\mathrm{aux},n_k}_{2,1,i}(t_{n_k}) - y^{\mathrm{aux},0}_{2,1,i}(t_{n_k})|] \\
&= \mathbb{E}[|x^{\mathrm{aux},n_k}_{1,i}(0) - x^{\mathrm{aux},0}_{1,i}(0)| + K_1|y^{\mathrm{aux},n_k}_{1,1,i}(0) - y^{\mathrm{aux},0}_{1,1,i}(0)| + K_2|y^{\mathrm{aux},n_k}_{2,1,i}(0) - y^{\mathrm{aux},0}_{2,1,i}(0)|] \\
&- c\int_0^{t_{n_k}} \mathbb{E}[|x^{\mathrm{aux},n_k}_{1,i}(s) - x^{\mathrm{aux},0}_{1,i}(s)|]\mathrm{d}s \\
&- 2K_1e_1\int_0^{t_{n_k}} \mathbb{E}\big[[|x^{\mathrm{aux},n_k}_{1,i}(s) - x^{\mathrm{aux},0}_{1,i}(s)| + K_1|y^{\mathrm{aux},n_k}_{1,1,i}(s) - y^{\mathrm{aux},0}_{1,1,i}(s)|] \\
&\qquad\qquad \times 1_{\{\mathrm{sgn}\,(x^{\mathrm{aux},n_k}_{1,i}(s) - x^{\mathrm{aux},0}_{1,i}(s)) \neq \mathrm{sgn}\,(y^{\mathrm{aux},n_k}_{1,1,i}(s) - y^{\mathrm{aux},0}_{1,1,i}(s))\}}\big]\mathrm{d}s.
\end{aligned} \tag{8.152}$$

Taking the limit $k\to\infty$ on both sides of (8.152), we obtain

$$\begin{aligned}
\lim_{k\to\infty} &\mathbb{E}[|x^{\mathrm{aux},n_k}_{1,i}(t_{n_k}) - x^{\mathrm{aux},0}_{1,i}(t_{n_k})| + K_1|y^{\mathrm{aux},n_k}_{1,1,i}(t_{n_k}) - y^{\mathrm{aux},0}_{1,1,i}(t_{n_k})| \\
&+ K_2|y^{\mathrm{aux},n_k}_{2,1,i}(t_{n_k}) - y^{\mathrm{aux},0}_{2,1,i}(t_{n_k})|] = 0.
\end{aligned} \tag{8.153}$$

Note that $\lim_{n\to\infty} t_n = \infty$ implies that $\lim_{k\to\infty} t_{n_k} = \infty$, so $z^{\mathrm{aux},0}_i$ is the limiting system in (8.29) with θ replaced by the random variable $\bar\Theta(t_2)$ and $y_{2,1,i}$. Therefore we can condition on $\bar\Theta(t_2)$ and $y_{2,1,i}$, and use the assumption in (8.81), to obtain

$$\lim_{k\to\infty} \mathcal{L}\left[z^{\mathrm{aux},0}_i(t_{n_k})\right] = \int_{[0,1]^2}\int_{[0,1]} \Gamma^{\mathrm{aux},(1)}_{u,y_{2,1}}\, P^{(u,v)}(\mathrm{d}y_{2,1})\, P_{t_2}(\mathrm{d}u,\mathrm{d}v). \tag{8.154}$$

Hence we conclude that

$$\lim_{k\to\infty} \mathcal{L}[z^{\mathrm{aux},n_k}_i(t_{n_k})] = \int_{[0,1]^2}\int_{[0,1]} \Gamma^{\mathrm{aux},(1)}_{u,y_{2,1}}\, P^{(u,v)}(\mathrm{d}y_{2,1})\, P_{t_2}(\mathrm{d}u,\mathrm{d}v). \tag{8.155}$$

Equation (8.155) holds for all subsequences along which the initial distribution converges,

$$\lim_{k\to\infty} z^{\mathrm{aux},n_k}_i(0) = z^{\mathrm{aux},0}_i(0) \quad a.s. \tag{8.156}$$

We will show that this implies (8.144).

Suppose that

$$\lim_{n\to\infty} \mathcal{L}[z^{\mathrm{aux},n}_i(t_n)] \neq \int_{[0,1]^2}\int_{[0,1]} \Gamma^{\mathrm{aux},(1)}_{u,y_{2,1}}\, P^{(u,v)}(\mathrm{d}y_{2,1})\, P_{t_2}(\mathrm{d}u,\mathrm{d}v). \tag{8.157}$$

Then there exist $f\in\mathcal{C}_b([0,1]\times[0,1]^3)$ and $\delta>0$ such that for all $N\in\mathbb{N}$ there exists an $n\in\mathbb{N}$, $n>N$ such that

$$\left|\mathbb{E}[f(z^{\mathrm{aux},n}_i(t_n))] - \mathbb{E}^{\Gamma^{(1)}_{\bar\Theta(t_2)}}[f]\right| > \delta. \tag{8.158}$$

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Hence we can construct a subsequence $(z_i^{\mathrm{aux},n_k}(t_1))_{t_1>0,\,k\in\mathbb{N}}$ such that (8.158) holds for each $k\in\mathbb{N}$. However, also for this sequence $(\mathcal{L}[z_i^{\mathrm{aux},n_k}(0)])_{k\in\mathbb{N}}$ is tight. Passing to a possibly further subsequence of converging initial distributions, we argue like before to obtain that along this subsequence

$$\lim_{k\to\infty}\left|\mathbb{E}[f(z_i^{\mathrm{aux},n_k}(t_{n_k}))]-\mathbb{E}^{\Gamma^{(1)}_{\Theta(t_2)}}[f]\right|=0. \tag{8.159}$$

This contradicts (8.158) and so (8.144) is indeed true. □

Proof of Proposition 8.3.10

Proof. Lemma 8.3.17 implies (8.103). Therefore Lemma 8.3.15 implies (8.104). □

§8.3.7 Convergence of 2-block process

In this section we derive the limiting evolution of the effective 2-block process.

Lemma 8.3.18 (Convergence of the 2-block averages). *Assume that $(N_k)_{k\in\mathbb{N}}\subset\mathbb{N}$ is a subsequence satisfying*

$$\begin{aligned}&\lim_{k\to\infty}\mathcal{L}\left[y_{2,1}^{[N_k^2]}(N_kt_2)\middle|\mathbf{\Theta}^{(2),[N_k^2]}(N_k^2t_2)\right]=P^{z_2(t_2)},\\&\lim_{k\to\infty}\mathcal{L}\left[\left(Y_{1,0}^{[N_k^2]}(N_k^2t_2+N_kt_1),Y_{2,0}^{[N_k^2]}(N_k^2t_2)\right)\middle|\mathbf{\Theta}^{\mathrm{aux},(1),[N_k^2]}(N_k^2t_2+N_kt_1)\right]\\&\qquad=P^{z_1^{\mathrm{eff}}(t_1)}.\end{aligned} \tag{8.160}$$

Then, for the effective 2-block estimator process defined in (8.23),

$$\lim_{k\to\infty}\mathcal{L}\left[\left(\mathbf{\Theta}^{\mathrm{eff},(2),[N_k^2]}(N_k^2t_2)\right)_{t_2>0}\right]=\mathcal{L}\left[\left(z_2^{\mathrm{eff}}(t_2)\right)_{t_2>0}\right], \tag{8.161}$$

where the limit is determined by the unique solution of the SSDE (8.36) *with initial state*

$$z_2^{\mathrm{eff}}(0)=\left(x_2^{\mathrm{eff}}(0),y_2^{\mathrm{eff}}(0)\right)=(\vartheta_1,\theta_{y_2}). \tag{8.162}$$

Proof. Again we use [49, Theorem 3.3.1]. By a similar argument as used in the proof of Lemma 7.2.12 we can show that

$$\lim_{t_2\downarrow 0}\mathcal{L}\left[\mathbf{\Theta}^{\mathrm{eff},(2),[N_k^2]}(N_k^2t_2)\right]=\delta_{(\vartheta_1,\theta_{y_2})}. \tag{8.163}$$

Note that by steps 1-4 of the scheme in Section 8.2 we can choose the subsequence $(N_k)_{k\in\mathbb{N}}$ such that both (8.81) and (8.102) hold. Since we already established the tightness of the 2-block in Lemma 8.3.1, we are left to show that, for all $t_2>0$,

$$\begin{aligned}\lim_{N\to\infty}\mathbb{E}\Big[\Big|&G_\dagger^{(2),[N_k^2]}(f,\mathbf{\Theta}^{\mathrm{eff},(2),[N_k^2]}(N_k^2t_2),t_2,\omega)\\&-G^{(2)}f\left(\mathbf{\Theta}^{\mathrm{eff},(2),[N_k^2]}(N_k^2t_2)\right)\Big|\Big]=0,\end{aligned} \tag{8.164}$$

where $G_\dagger^{(2),[N_k^2]}$ is the $\mathcal{D}$-semi-martingale operator defined in (8.69), $G^{(2)}$ is the generator of the process $(z_2^{\mathrm{eff}}(t_2))_{t_2>0}$ defined in (8.36), and both generators work on a probability space driven by one set of Brownian motions. Note that, for all $t_2>0$,

$$\begin{aligned}
&\mathbb{E}\Big[\Big|G_\dagger^{(2),[N_k^2]}\left(f,\left(\bar\Theta^{(2),[N_k^2]}(N_k^2t_2),\Theta_{y_2}^{(2),[N_k^2]}(N_k^2t_2)\right),t_2,\omega\right)\\
&\qquad -G^{(2)}f\left(\bar\Theta^{(2),[N_k^2]}(N_k^2t_2),\Theta_{y_2}^{(2),[N_k^2]}(N_k^2t_2)\right)\Big|\Big]\\
&\leq \frac{K_2e_2}{1+K_0+K_1}\mathbb{E}\left[\left|\bar\Theta^{(2),[N_k^2]}(N_k^2t_2)-\Theta_x^{(2),[N_k^2]}(N_k^2t_2,\omega)\right|\left|\frac{\partial f}{\partial x}\right|\right]\\
&\quad + e_2\mathbb{E}\left[\left|\Theta_x^{(2),[N_k^2]}(N_k^2t_2,\omega)-\bar\Theta^{(2),[N_k^2]}(N_k^2t_2)\right|\left|\frac{\partial f}{\partial y}\right|\right]\\
&\quad +\frac{1}{(1+K_0+K_1)^2}\\
&\qquad\times\mathbb{E}\left[\left|\frac{1}{N_k^2}\sum_{i\in[N_k^2]}g(x_i^{[N_k^2]}(N_k^2t_2,\omega))-(\mathcal{F}^{(2)}g)(\bar\Theta^{(2),[N_k^2]}(N_k^2t_2))\right|\left|\frac{\partial^2 f}{\partial x^2}\right|\right].
\end{aligned}\tag{8.165}$$

The first and second term on the right-hand side tend to 0 as $k\to\infty$ by a similar argument as used in (8.129) and below. For the third let $[N]_i$ denote the 1-block that contains site i and let $(z^{\nu_{\bar\Theta_i^{(1)}}}(t))_{t>0}$ be the limiting single colony system, with drift towards the random variable $\bar\Theta_i$ and starting from the equilibrium measure $\nu_{\bar\Theta_i^{(1)}}$. We construct the single colony system $Z^{[N_k^2]}(N_k^2t_2-L(N)+t)_{t\geq 0}$ and the limiting system $(z^{\nu_{\bar\Theta_i^{(1)}}}(t))_{t>0}$ on one probability space, such that by Skohorod's theorem, we can assume that the convergence is almost surely. Note that $\bar\Theta_i$ is the limiting one block. Then we can write

$$\begin{aligned}
&\mathbb{E}\left[\left|\frac{1}{N_k^2}\sum_{i\in[N_k^2]}g(x_i^{[N_k^2]}(N_k^2t_2,\omega))-(\mathcal{F}^{(2)}g)(\bar\Theta^{(2),[N_k^2]}(N_k^2t_2))\right|\left|\frac{\partial^2 f}{\partial x^2}\right|\right]\\
&\quad\leq\frac{1}{N_k}\sum_{i\in[N]}\mathbb{E}\left[\left|\frac{1}{N_k}\sum_{j\in[N]_i}g(x_j^{[N_k^2]}(N_k^2t_2,\omega))-\frac{1}{N_k}\sum_{j\in[N]_i}g(x_j^{\nu_{\bar\Theta_i^{(1)}}}(L(N))\right|\left|\frac{\partial^2 f}{\partial x^2}\right|\right]\\
&\quad+\frac{1}{N_k}\sum_{i\in[N]}\mathbb{E}\left[\left|\frac{1}{N_k}\sum_{j\in[N]_i}g(x_j^{\nu_{\bar\Theta_i^{(1)}}}(L(N))-(\mathcal{F}^{(1)}g)(\bar\Theta_i^{(1)})\right|\left|\frac{\partial^2 f}{\partial x^2}\right|\right]\\
&\quad+\mathbb{E}\left[\left|\frac{1}{N_k}\sum_{i\in[N]}(\mathcal{F}^{(1)}g)(\bar\Theta_i^{(1)})-(\mathcal{F}^{(2)}g)(\bar\Theta_i^{(2)})\right|\left|\frac{\partial^2 f}{\partial x^2}\right|\right]\\
&\quad+\mathbb{E}\left[\left|(\mathcal{F}^{(2)}g)(\bar\Theta_i^{(2)})-(\mathcal{F}^{(2)}g)(\bar\Theta_i^{(2)}(N_k^2t_2))\right|\left|\frac{\partial^2 f}{\partial x^2}\right|\right].
\end{aligned}\tag{8.166}$$

The first term on the right-hand side tends to zero by Lipschitz continuity for g and Corollary 8.3.9. The second term tends to zero by the law of large numbers, since the limiting single colonies are i.i.d. given the value of the random variable $\bar\Theta_i^{(1)}$.

The third term tends to zero since by Proposition 8.3.10 also the limiting 1-blocks become independent given the value of the 2-block. Hence we can again apply the law of large numbers. Finally, for the last term, note that since we construct the single components and the limiting process on one probability space, we can argue like in the proof of Lemma 7.2.8 that

$$\lim_{k\to\infty} \mathbb{E}\left[\left|\bar{\Theta}_i^{(2)} - \bar{\Theta}_i^{(2)}(N_k^2 t_2)\right|\right] = 0. \tag{8.167}$$

Hence the last term tends to zero by the Lipschitz property of $\mathcal{F}^{(2)}g$. □

Remark 8.3.19. Instead of [49, Theorem 3.3.1] we could have used a similar strategy as in the proof of Lemma 8.3.15 to obtain Lemma 8.3.18. ■

§8.3.8 State of the slow seed-banks

On time scale t_0, i.e., space-time scale 0, the colour-1 seed-bank is a "slow seed-bank," since it does not move on this time scale. Because we study the two-layer three-colour mean-field system from time $N^2 t_2$ onwards, the 1-block averages of the colour 1-dormant population are already in equilibrium. As a consequence we can exactly describe the single 1-dormant colonies, which turn out to be in a state that equals the current 1-block average of the dormant population of colour 1. To obtain the formal result we will first prove the following lemma.

Lemma 8.3.20 (Slow seed-banks). *Fix $t_2, t_1 > 0$, for $i \in [N^2]$ and all $t_0 \geq 0$,*

$$\lim_{N\to\infty} \left[y_{i,1}^{[N^2]}(N^2 t_2 + N t_1 + t_0) - \Theta_{y_1,i}^{(1),[N^2]}(N^2 t_2 + N t_1 + t_0)\right] = 0 \qquad a.s., \tag{8.168}$$

where $\Theta_{y_1,i}^{(1),[N^2]}$ is the 1-block average to which $y_{i,1}^{[N^2]}$ contributes.

To prove Lemma 8.3.20, we need the kernel $b^{[N^2]}(\cdot,\cdot)$ defined in 4.31, which becomes in the current setting

$$b^{[N^2]}((i,R_i),(j,R_j)) = \begin{cases} 1_{\{d_{[N^2]}(i,j)\leq 1\}}\frac{c_0}{N} + \frac{c_1}{N^3}, & \text{if } R_i = R_j = A,\\ K_m \frac{e_m}{N^m}, & \text{if } i=j,\ R_i = A,\ R_j = D_m,\ m\in\{0,1,2\},\\ \frac{e_m}{N^m}, & \text{if } i=j,\ R_i = D_m,\ R_j = A,\ m\in\{0,1,2\},\\ 0, & \text{otherwise.}\end{cases} \tag{8.169}$$

The corresponding semigroup of the kernel $b^{[N^2]}(\cdot,\cdot)$ is denoted by $b_t^{[N^2]}(\cdot,\cdot)$.

To prove Lemma 8.3.20 we will use the following lemma, which was proved in [43][Lemma 6.1] and for our setting reads as follows.

Lemma 8.3.21 (First and second moment).
Let $\mathbb{E}_{z^{[N^2]}}$ the expectation if the process start from some state $z^{[N^2]} \in ([0,1]\times[0,1]^3)^{[N^2]}$. For $z^{[N^2]} \in ([0,1]\times[0,1]^2)^{[N^2]}$, $t \geq 0$ and $(i,R_i),(j,R_j) \in [N^2]\times\{A,D_0,D_1,D_2\}$,

$$\mathbb{E}_{z^{[N^2]}}[z_{(i,R_i)}^{[N^2]}(t)] = \sum_{\substack{(k,R_k)\in\\ \Omega_N\times\{A,D_0,D_1,D_2\}}} b_t^{[N^2]}\big((i,R_i),(k,R_k)\big)\, z_{(k,R_k)}^{[N^2]} \tag{8.170}$$

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and

$$\begin{aligned}
&\mathbb{E}_{z^{[N^2]}}[z^{[N^2]}_{(i,R_i)}(t) z^{[N^2]}_{(j,R_j)}(t)] \\
&= \sum_{\substack{(k,R_k),(l,R_l)\in \\ \Omega_N\times\{A,D_0,D_1,D_2\}}} b^{[N^2]}_t\big((i,R_i),(k,R_k)\big)\, b^{[N^2]}_t\big((j,R_j),(l,R_l)\big)\, z^{[N^2]}_{(k,R_k)} z^{[N^2]}_{(l,R_l)} \\
&\quad + 2\int_0^t \mathrm{d}s \sum_{k\in\Omega_N} b^{[N^2]}_{(t-s)}((i,R_i),(k,A))\, b^{[N^2]}_{(t-s)}((j,R_j),(k,A))\, \mathbb{E}^{[N^2]}_z[g(x^{[N^2]}_k(s))].
\end{aligned} \tag{8.171}$$

Proof of Lemma 8.3.20. The argument is given in such a way that it can easily be generalised to more complicated systems, which we treat later. Let $\bar{t}(N) = N^2 t_2 + N t_1 + t_0$. We will show that if $i, j \in [N]_i$, i.e., i and j belong to the same 1-block, then

$$\lim_{N\to\infty} \mathbb{E}\left[\left(y^{[N^2]}_{i,1}(\bar{t}(N)) - y^{[N^2]}_{j,1}(\bar{t}(N))\right)^2\right] = 0. \tag{8.172}$$

This implies (8.168). By Lemma 8.3.21, we can write

$$\begin{aligned}
&\mathbb{E}\left[\left(y^{[N^2]}_{i,1}(\bar{t}(N)) - y^{[N^2]}_{j,1}(\bar{t}(N))\right)^2\right] \\
&= \sum_{\substack{(k,R_k),(l,R_l)\in \\ [N^2]\times\{A,D_0,D_1,D_2\}}} \left(b^{[N^2]}_{\bar{t}(N)}\left((i,D_1),(k,R_k)\right) - b^{[N^2]}_{\bar{t}(N)}\left((j,D_1),(k,R_k)\right)\right) \\
&\times \left(b^{[N^2]}_{\bar{t}(N)}\left((i,D_1),(l,R_l)\right) - b^{[N^2]}_{\bar{t}(N)}\left((j,D_1),(l,R_l)\right)\right) \mathbb{E}[z^{[N^2]}_{(k,R_k)} z^{[N^2]}_{(l,R_l)}] \\
&+ 2\int_0^{\bar{t}(N)} \mathrm{d}s \sum_{k\in[N^2]} \left(b^{[N^2]}_{(\bar{t}(N)-s)}((i,D_1),(k,A)) - b^{[N^2]}_{(\bar{t}(N)-s)}((j,D_1),(k,A))\right)^2 \\
&\times \mathbb{E}[g(x^{[N^2]}_k(s))].
\end{aligned} \tag{8.173}$$

Using a coupling argument, we show that both terms in (8.173) tend to 0 as $N\to\infty$. To prove that the first term tends to 0, we will show that

$$\lim_{N\to\infty} \sum_{\substack{(k,R_k)\in \\ [N^2]\times\{A,D_0,D_1,D_2\}}} \left| b^{[N^2]}_{\bar{t}(N)}\left((i,D_1),(k,R_k)\right) - b^{[N^2]}_{\bar{t}(N)}\left((j,D_1),(k,R_k)\right)\right| = 0. \tag{8.174}$$

To do so, let $(RW^{[N^2]}(t))_{t\geq 0}$ and $(RW'^{[N^2]}(t))_{t\geq 0}$ be two independent random walks, starting from $RW^{[N^2]}(0) = (i, D_1)$ and $RW'^{[N^2]}(0) = (j, D_1)$, where i and j are in the same 1-block. Let $RW^{[N^2]}$ and $RW'^{[N^2]}$ both evolve according to the kernel $b^{[N^2]}(\cdot,\cdot)$, so $b^{[N^2]}_t(\cdot,\cdot)$ is their corresponding semigroup. Since $RW^{[N^2]}$ and $RW'^{[N^2]}$ both start from the colour 1-seed-bank, we can perfectly couple their switches between A, D_0, D_1 and D_2. Since this implies that both $RW^{[N^2]}$ and $RW'^{[N^2]}$ are always simultaneously active, we can also couple the times when they jump due to migration and the distance over which they migrate. However, we do not couple their migrations, i.e. $RW^{[N^2]}$ and $RW'^{[N^2]}$ jump at the same time and over the same distance, but they can jump to different sites. This implies that the coupled process $(RW^{[N^2]}(t), RW'^{[N^2]}(t))_{t\geq 0}$ has transition rates

$$((i,R_i),(j,R_i)) \to \begin{cases} ((k,A),(l,A)) & \text{if } R_i = R_j = A \text{ and } d_{[N^2]}(i,k) = d_{[N^2]}(j,l) \\ & \text{at rate } 1_{\{d_{[N^2]}(i,j)\leq 1\}}\frac{c_0}{N} + \frac{c_1}{N^3}, \\ ((i,D_m),(j,D_m)) & \text{if } R_i = R_j = A \text{ at rate } \frac{K_m e_m}{N^m},\ m\in\{0,1,2\}, \\ ((i,A),(j,A)) & \text{if } R_i = R_j = D_m \text{ at rate } \frac{e_m}{N^m},\ m\in\{0,1,2\}. \end{cases} \tag{8.175}$$

Define the event

$$H_t^{[N^2]} = \{RW^{[N^2]} \text{ has migrated at least once up to time } t\}. \tag{8.176}$$

Note that if $H_t^{[N^2]}$ has happened, then also $RW'^{[N^2]}$ has migrated. Hence

$$\begin{aligned} & b^{[N^2]}_{\bar t(N)}((i,D_1),(k,R_k)) - b^{[N^2]}_{\bar t(N)}((j,D_1),(k,R_k)) \\ &\quad = \mathbb{P}_{(i,D_1)}\left(RW^{[N^2]}(\bar t(N)) = (k,R_k)\right) - \mathbb{P}_{(j,D_1)}\left(RW'^{[N^2]}(\bar t(N)) = (k,R_k)\right) \\ &\quad = \tilde{\mathbb{P}}_{(i,D_1),(j,D_1)}\left(RW^{[N^2]}(\bar t(N)) = (k,R_k),\ H^{[N^2]}_{\bar t(N)}\right) \\ &\qquad + \tilde{\mathbb{P}}_{(i,D_1),(j,D_1)}\left(RW^{[N^2]}(\bar t(N)) = (k,R_k),\ (H^{[N^2]}_{\bar t(N)})^c\right) \\ &\qquad - \tilde{\mathbb{P}}_{(i,D_1),(j,D_1)}\left(RW'^{[N^2]}(\bar t(N)) = (k,R_k),\ H^{[N^2]}_{\bar t(N)}\right) \\ &\qquad - \tilde{\mathbb{P}}_{(i,D_1),(j,D_1)}\left(RW'^{[N^2]}(\bar t(N)) = (k,R_k),\ (H^{[N^2]}_{\bar t(N)})^c\right) \\ &\quad = \tilde{\mathbb{P}}_{(i,D_1),(j,D_1)}\left(RW^{[N^2]}(\bar t(N)) = (k,R_k),\ (H^{[N^2]}_{\bar t(N)})^c\right) \\ &\qquad - \tilde{\mathbb{P}}_{(i,D_1),(j,D_1)}\left(RW'^{[N^2]}(\bar t(N)) = (k,R_k),\ (H^{[N^2]}_{\bar t(N)})^c\right), \end{aligned} \tag{8.177}$$

where the last equality follows because, once the random walks have just jumped once, $RW^{[N^2]}$ and $RW'^{[N^2]}$ are uniformly distributed over $[N]\times A$ if their jump horizon was 1 and they are uniformly distributed of $[N^2]\times A$ if they jumped over distance 2. Hence if $H_t^{[N^2]}$ has occurred, then $RW^{[N^2]}$ and $RW'^{[N^2]}$ have the same distribution. Therefore

$$\sum_{\substack{(k,R_k)\in\\ [N^2]\times\{A,D_0,D_1,D_2\}}} \left| b^{[N^2]}_{\bar t(N)}((i,D_1),(k,R_k)) - b^{[N^2]}_{\bar t(N)}((j,D_1),(k,R_k))\right| \leq 2\tilde{\mathbb{P}}((H^{[N^2]}_{\bar t(N)})^c) \tag{8.178}$$

and we are left to show that

$$\lim_{N\to\infty} \tilde{\mathbb{P}}((H^{[N^2]}_{\bar t(N)})^c) = 0. \tag{8.179}$$

The event $(H^{[N^2]}_{\bar t(N)})^c$ occurs either when the random walks do not wake up before time $\bar t(N)$ or when the random walks wake up before time $\bar t(N)$ but do not migrate. By the coupling we only have to consider one of the random walks. Therefore the probability that $RW^{[N^2]}$ and $RW'^{[N^2]}$ do not wake up before time $\bar{t}(N)$ is given by

$$\tilde{\mathbb{P}}_{(i,D_1),(j,D_1)}\left(RW^{[N^2]} \text{ does not wake up before } \bar{t}(N)\right) = \mathrm{e}^{-\frac{e_1}{N}\bar{t}(N)} = \mathrm{e}^{-\frac{e_1(N^2t_2+Nt_1+t_0)}{N}} \tag{8.180}$$

and hence

$$\lim_{N\to\infty} \mathbb{P}_{(i,D_1),(j,D_1)}\left(RW^{[N^2]} \text{ does not wake up before } \bar{t}(N)\right) = 0. \tag{8.181}$$

The probability that the random walks do wake up, but do not migrate is a little more complicated, since each time they wake up with positive probability they go to sleep before they migrate. Define

$$\begin{aligned} C^{[N^2]}(t) &= \{\#\text{ times } RW^{[N^2]} \text{ gets active before time } t\},\\ T_A^{[N^2]}(t) &= \{\text{total time } RW^{[N^2]} \text{ is active up to time } t\},\\ T_D^{[N^2]}(t) &= \{\text{total time } RW^{[N^2]} \text{ is dormant up to time } t\}. \end{aligned} \tag{8.182}$$

Thus, $C^{[N^2]}(t)$ counts the number of active/dormant cycles. Define $T_{A,n}^{[N^2]}$, $T_{D,n}^{[N^2]}$ as the active respectively, dormant time during the nth cycle. Define

$$\chi = K_0e_0 + \frac{K_1e_1}{N} + \frac{K_2e_2}{N^2}, \tag{8.183}$$

so χ is the total rate at which RW and RW' become dormant when they are active. Define

$$c = c_0 + \frac{c_1}{N}, \tag{8.184}$$

so c is the total rate at which RW and RW' migrate when they are active. Then

$$T_A^{[N^2]}(t) = \sum_{n=1}^{C^{[N^2]}(t)} T_{A,n}^{[N^2]}, \qquad T_D^{[N^2]}(t) = \sum_{n=1}^{C^{[N^2]}(t)} T_{D,n}^{[N^2]}, \tag{8.185}$$

where $T_{A,n}^{[N^2]} \stackrel{d}{=} \exp(\chi)$ and $T_{D,n}^{[N^2]} \stackrel{d}{=} \frac{1}{\chi}K_0e_0 \exp(e_0) + \frac{1}{\chi}\frac{K_1e_1}{N} \exp(\frac{e_1}{N}) + \frac{1}{\chi}\frac{K_2e_2}{N^2} \exp(\frac{e_2}{N^2})$. Once awake, $RW^{[N^2]}$ migrates at rate c and hence the probability to migrate before time $\bar{t}(N)$ is given by $1-\mathrm{e}^{-cT_A^{[N^2]}(\bar{t}(N))}$. Therefore we are left to show that

$$\lim_{N\to\infty} cT_A^{[N^2]}(\bar{t}(N)) = \lim_{N\to\infty} c \sum_{n=1}^{C^{[N^2]}(\bar{t}(N))} T_{A,n}^{[N^2]} = \infty, \qquad a.s. \tag{8.186}$$

Since $T_{A,n}^{[N^2]} \stackrel{d}{=} \exp(\chi)$, it is enough to show that

$$\lim_{N\to\infty} C^{[N^2]}(\bar{t}(N)) = \infty \qquad a.s. \tag{8.187}$$

To do so, we assume the contrary, i.e., there exists an $R \in \mathbb{N}$ such that for all $\bar{N} \in \mathbb{N}$ there exists an $N > \bar{N}$ such that

$$\mathbb{P}_{(i,D_1)}(C^{[N^2]}(\bar{t}(N)) \le R) > 0. \tag{8.188}$$

Let $L(N)$ be such that $\lim_{N\to\infty} L(N) = \infty$ and $\lim_{N\to\infty} L(N)/N = 0$. Note that, by (8.181), we can condition on the first wake-up time and estimate

$$\begin{aligned}
\mathbb{P}_{(i,D_1)}(C^{[N^2]}(\bar{t}(N)) \le R) &= \int_0^{\bar{t}(N)} \mathrm{d}s\, \mathbb{P}_{(i,A)}(C^{[N^2]}(\bar{t}(N)-s) \le R)\ \frac{e_1}{N}\mathrm{e}^{-\frac{e_1}{N}s} \\
&= \int_0^{\bar{t}(N)-L(N)} \mathrm{d}s\, \mathbb{P}_{(i,A)}(C^{[N^2]}(\bar{t}(N)-s) \le R)\ \frac{e_1}{N}\mathrm{e}^{-\frac{e_1}{N}s} \\
&\quad + \int_{\bar{t}(N)-L(N)}^{\bar{t}(N)} \mathrm{d}s\, \mathbb{P}_{(i,A)}(C^{[N^2]}(\bar{t}(N)-s) \le R)\ \frac{e_1}{N}\mathrm{e}^{-\frac{e_1}{N}s} \\
&\le \mathbb{P}_{(i,A)}(C^{[N^2]}(L(N)) \le R) + \mathrm{e}^{-\frac{e_1}{N}\bar{t}(N)}\left[\mathrm{e}^{\frac{e_1}{N}L(N)} - 1\right].
\end{aligned} \tag{8.189}$$

Note that the second term in the last inequality tends to 0 as $N \to \infty$. For the first term, note that we are now looking at time $L(N)$, i.e., time scale N^0. Since

$$\begin{aligned}
&\lim_{N\to\infty} \mathbb{P}_{i,A}\left(RW^{[N^2]} \text{ jumps to } D_1 \text{ or } D_2 \text{ before time } L(N)\right) \\
&\quad = \lim_{N\to\infty} 1 - \mathrm{e}^{-(\frac{K_1 e_1}{N} + \frac{K_2 e_2}{N^2})L(N)} = 0.
\end{aligned} \tag{8.190}$$

we have $\lim_{N\to\infty} \mathbb{P}_{i,A}\left(\{RW^{[N^2]}(s) \in \{A, D_0\} \text{ for } s \in [0, L(N)]\}\right) = 1$. Hence, conditioned on the event $\{RW^{[N^2]} \in \{A, D_0\}\}$, $T^{[N^2]}_{A,n} \stackrel{d}{=} \exp(K_0 e_0)$ and $T^{[N^2]}_{D,n} \stackrel{d}{=} \exp(e_0)$. We therefore obtain

$$\begin{aligned}
\mathbb{P}_{(i,A)}(C^{[N^2]}(L(N)) \le R) &= \mathbb{P}_{(i,A)}\left(\sum_{n=1}^{R} (T^{[N^2]}_{A,n} + T^{[N^2]}_{D,n}) \ge L(N)\right) \\
&\le \frac{R}{L(N)} \mathbb{E}_{(i,A)}\left[T^{[N^2]}_{A,n} + T^{[N^2]}_{D,n}\right] \\
&= \frac{R}{L(N)}\left[\frac{1}{K_0 e_0} + \frac{1}{e_0}\right].
\end{aligned} \tag{8.191}$$

Taking the limit $N \to \infty$ in (8.191) and combining this with (8.189), we conclude that (8.187) indeed holds. Hence also (8.179) and (8.174) hold.

We are left to show that

$$\begin{aligned}
\lim_{N\to\infty} 2\int_0^{\bar{t}(N)} \mathrm{d}s \sum_{k\in[N^2]} &\left(b^{[N^2]}_{(\bar{t}(N)-s)}((i,D_1),(k,A)) - b^{[N^2]}_{(\bar{t}(N)-s)}((j,D_1),(k,A))\right)^2 \\
&\times \mathbb{E}[g(x_k(s))] = 0.
\end{aligned} \tag{8.192}$$

Also here the idea is to make a similar coupling. As soon as the random walks migrate, they are equally distributed. On time scale N, after waking up from the colour 1 seed-bank they will almost immediately migrate, since migration happens on time scale 1, i.e., by time $L(N)$ they have migrated with probability tending to 1. This will again be the key to show that (8.192) tends to 0 as $N \to \infty$.

Note that, by (8.173),

$$\left|2\int_0^{\bar{t}(N)} \mathrm{d}s \sum_{k\in[N^2]} \left(b^{[N^2]}_{(\bar{t}(N)-s)}((i,D_1),(k,A)) - b^{[N^2]}_{(\bar{t}(N)-s)}((j,D_1),(k,A))\right)^2 \times \mathbb{E}[g(x_k^{[N^2]}(s))]\right| \leq 2. \tag{8.193}$$

We will again use the coupling in (8.176). Define

$$\tau^{[N^2]} = \inf\{t \geq 0 : RW^{[N^2]}(t) = (k,A) \text{ for some } k \in [N^2]\}. \tag{8.194}$$

Then, for all $l \in [N^2]$,

$$\mathbb{P}_{(l,D_1)}(\tau^{[N^2]} \leq t) = 1 - \mathrm{e}^{-\frac{e_1}{N}t}. \tag{8.195}$$

Setting $s = \bar{t}(N) - s$, we can rewrite the integral in (8.192) as

$$\begin{aligned}
&2\int_0^{\bar{t}(N)} \mathrm{d}s \sum_{k\in[N^2]} \left(\tilde{\mathbb{P}}^{[N^2]}_{(i,D_1),(j,D_1)}\left(RW^{[N^2]}(s) = (k,A)\right)\right.\\
&\qquad \left.- \tilde{\mathbb{P}}^{[N^2]}_{(i,D_1),(j,D_1)}\left(RW'^{[N^2]}(s) = (k,A),\right)\right)^2 \mathbb{E}[g(x_k^{[N^2]}(\bar{t}(N)-s))]\\
&= 2\int_0^{\bar{t}(N)} \mathrm{d}s \sum_{k\in[N^2]} \left[\int_0^s \mathrm{d}r\, \tilde{\mathbb{P}}(\tau^{[N^2]} = r)\right.\\
&\qquad \left.\left(\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\left(RW^{[N^2]}(s-r) = (k,A)\right) - \tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\left(RW'^{[N^2]}(s-r) = (k,A)\right)\right)\right]\\
&\qquad \times \left(\tilde{\mathbb{P}}^{[N^2]}_{(i,D_1),(j,D_1)}\left(RW^{[N^2]}(s) = (k,A)\right) - \tilde{\mathbb{P}}^{[N^2]}_{(i,D_1),(j,D_1)}\left(RW'^{[N^2]}(s) = (k,A)\right)\right)\\
&\qquad \times \mathbb{E}[g(x_k^{[N^2]}(\bar{t}(N)-s))].
\end{aligned} \tag{8.196}$$

In what follows we will abbreviate

$$\begin{aligned}
&P_{\Delta^{[N^2]}_{(i,A),(j,A)}}(t,((l,R_l),(j,R_j)))\\
&= \tilde{\mathbb{P}}_{(i,A),(j,A)}\left(RW^{[N^2]}(t) = (l,R_l)\right) - \tilde{\mathbb{P}}_{(i,A),(j,A)}\left(RW'^{[N^2]}(t) = (j,R_j)\right),
\end{aligned} \tag{8.197}$$

and similarly, for $m \in \{0,1,2\}$,

$$\begin{aligned}
&P_{\Delta^{[N^2]}_{(i,D_m),(j,D_m)}}(t,((l,R_l),(j,R_j)))\\
&= \tilde{\mathbb{P}}_{(i,D_m),(j,D_m)}\left(RW^{[N^2]}(t) = (l,R_l)\right) - \tilde{\mathbb{P}}_{(i,D_m),(j,D_m)}\left(RW'^{[N^2]}(t) = (j,R_j)\right).
\end{aligned} \tag{8.198}$$

By (8.193), we can use Fubini to swap the order of integration and subsequently

CHAPTER 8

substitute $v = s - r$, to obtain

$$
\begin{aligned}
&2\int_0^{\bar t(N)} \mathrm{d}r\, \tilde{\mathbb{P}}(\tau^{[N^2]} = r) \int_0^{\bar t(N)-r} \mathrm{d}v \sum_{k\in[N^2]} P_{\Delta^{[N^2]}_{(i,A),(j,A)}}(v,((k,A),(k,A)))\\
&\qquad \times P_{\Delta^{[N^2]}_{(i,D_1),(j,D_1)}}(v+r,((k,A),(k,A)))\mathbb{E}[g(x_k^{[N^2]}(\bar t(N)-r-v))]\\
&= 2\int_0^{\bar t(N)} \mathrm{d}r\, \tilde{\mathbb{P}}(\tau^{[N^2]} = r)\\
&\sum_{\substack{(l,R_l)\in\\ [N^2]\times\{A,D_0,D_1,D_2\}}} \left(\mathbb{P}_{(i,D_1)}\left(RW^{[N^2]}(r) = (l,R_l)\right) - \mathbb{P}_{(j,D_1)}\left(RW'^{[N^2]}(r) = (l,R_l)\right)\right)\\
&\quad \times \int_0^{\bar t(N)-r} \mathrm{d}v \sum_{k\in[N^2]} P_{\Delta^{[N^2]}_{(i,A),(j,A)}}(v,((k,A),(k,A)))\\
&\quad \times \mathbb{P}_{(l,R_l)}\left(RW^{[N^2]}(v) = (k,A)\right)\mathbb{E}[g(x_k^{[N^2]}(\bar t(N)-r-v))],
\end{aligned}
\tag{8.199}
$$

where in the last equality we use that the random walks move according to the same kernel $b(\cdot,\cdot)$.

We can continue by writing

$$
\begin{aligned}
&2\int_0^{\bar{t}(N)} \mathrm{d}r\, \tilde{\mathbb{P}}(\tau^{[N^2]}=r) \\
&\quad\sum_{\substack{(l,R_l)\in\\ [N^2]\times\{A,D_0,D_1,D_2\}}} \Big(\mathbb{P}^{[N^2]}_{(i,D_1)}\Big(RW^{[N^2]}(r)=(l,R_l)\Big)-\mathbb{P}^{[N^2]}_{(j,D_1)}\Big(RW'^{[N^2]}(r)=(l,R_l)\Big)\Big)\\
&\quad\times\int_0^{\bar{t}(N)-r}\mathrm{d}v\sum_{k\in[N^2]}\Big(\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\Big(RW^{[N^2]}(v)=(k,A)\Big)\\
&\qquad\qquad -\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\Big(RW'^{[N^2]}(v)=(k,A)\Big)\Big)\\
&\quad\times\mathbb{P}^{[N^2]}_{(l,R_l)}\Big(RW^{[N^2]}(v)=(k,A)\Big)\,\mathbb{E}[g(x_k^{[N^2]}(\bar{t}(N)-r-v))]\\
&=2\int_0^{\bar{t}(N)}\mathrm{d}r\,\tilde{\mathbb{P}}(\tau^{[N^2]}=r)\int_0^r\mathrm{d}u\,\tilde{\mathbb{P}}(\tau^{[N^2]}=u)\sum_{\substack{(l,R_l)\in\\ [N^2]\times\{A,D_0,D_1,D_2\}}}\\
&\quad\Big(\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\Big(RW^{[N^2]}(r-u)=(l,R_l)\Big)-\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\Big(RW'^{[N^2]}(r-u)=(l,R_l)\Big)\Big)\\
&\quad\times\int_0^{\bar{t}(N)-r}\mathrm{d}v\sum_{k\in[N^2]}\Big(\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\Big(RW^{[N^2]}(v)=(k,A)\Big)\\
&\qquad\qquad -\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\Big(RW'^{[N^2]}(v)=(k,A)\Big)\Big)\\
&\quad\times\mathbb{P}^{[N^2]}_{(l,R_l)}\Big(RW^{[N^2]}(v)=(k,A)\Big)\,\mathbb{E}[g(x_k^{[N^2]}(\bar{t}(N)-r-v))]\\
&+2\int_0^{\bar{t}(N)}\mathrm{d}r\,\tilde{\mathbb{P}}(\tau^{[N^2]}=r)\tilde{\mathbb{P}}(\tau^{[N^2]}\geq r)\\
&\quad\times\int_0^{\bar{t}(N)-r}\mathrm{d}v\sum_{k\in[N^2]}\Big(\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\Big(RW^{[N^2]}(v)=(k,A)\Big)\\
&\qquad\qquad -\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\Big(RW'^{[N^2]}(v)=(k,A)\Big)\Big)\\
&\quad\times\Big(\mathbb{P}^{[N^2]}_{(i,D_1)}\Big(RW^{[N^2]}(v)=(k,A)\Big)-\mathbb{P}^{[N^2]}_{(j,D_1)}\Big(RW'^{[N^2]}(v)=(k,A)\Big)\Big)\\
&\quad\times\mathbb{E}[g(x_k^{[N^2]}(\bar{t}(N)-r-v))].
\end{aligned}
\tag{8.200}
$$

We will show that both terms in the last equality of (8.200) tends to 0 as $N\to\infty$.

For the first term note that, by (8.171), (8.173) and (8.195), we have

$$
\begin{aligned}
&2\int_0^{\bar t(N)} \mathrm{d}r\, \tilde{\mathbb{P}}(\tau^{[N^2]}=r)\int_0^r \mathrm{d}u\, \tilde{\mathbb{P}}(\tau^{[N^2]}=u)\\
&\quad\times\Bigg[\sum_{\substack{(l,R_l)\in\\ [N^2]\times\{A,D_0,D_1,D_2\}}} \tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\left(RW^{[N^2]}(r-u)=(l,R_l)\right)\\
&\qquad\qquad - \tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\left(RW'^{[N^2]}(r-u)=(l,R_l)\right)\Bigg]\\
&\quad\times\int_0^{\bar t(N)-r}\mathrm{d}v\sum_{k\in[N^2]}\Bigg(\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\left(RW^{[N^2]}(v)=(k,A)\right)\\
&\qquad\qquad - \tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\left(RW'^{[N^2]}(v)=(k,A)\right)\Bigg)\\
&\quad\times \mathbb{P}^{[N^2]}_{(l,R_l)}\left(RW^{[N^2]}(v)=(k,A)\right)\mathbb{E}[g(x_k(\bar t(N)-r-v))]\\
&\leq 4\int_0^{\bar t(N)} \mathrm{d}r\, \tilde{\mathbb{P}}(\tau^{[N^2]}=r)\int_0^r \mathrm{d}u\, \tilde{\mathbb{P}}(\tau^{[N^2]}=u)\\
&\quad\times\Bigg|\sum_{\substack{(l,R_l)\in\\ [N^2]\times\{A,D_0,D_1,D_2\}}} \tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\left(RW^{[N^2]}(r-u)=(l,R_l)\right)\\
&\qquad\qquad - \tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\left(RW'^{[N^2]}(r-u)=(l,R_l)\right)\Bigg| \qquad (8.201)\\
&\leq 4\int_0^{\bar t(N)} \mathrm{d}r\, \tilde{\mathbb{P}}(\tau^{[N^2]}=r)\int_0^{r-L(N)} \mathrm{d}u\, \tilde{\mathbb{P}}(\tau^{[N^2]}=u)\\
&\quad\times\Bigg|\sum_{\substack{(l,R_l)\in\\ [N^2]\times\{A,D_0,D_1,D_2\}}} \tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\left(RW^{[N^2]}(r-u)=(l,R_l)\right)\\
&\qquad\qquad - \tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\left(RW'^{[N^2]}(r-u)=(l,R_l)\right)\Bigg|\\
&+8\int_{L(N)}^{\bar t(N)} \mathrm{d}r\, \tilde{\mathbb{P}}(\tau^{[N^2]}=r)\tilde{\mathbb{P}}(\tau^{[N^2]}\in[r-L(N),r]) + 8\mathbb{P}[\tau^{[N^2]}\in[0,L(N)]]\\
&\leq 4\int_0^{\bar t(N)} \mathrm{d}r\, \tilde{\mathbb{P}}(\tau^{[N^2]}=r)\int_0^{r-L(N)} \mathrm{d}u\, \tilde{\mathbb{P}}(\tau^{[N^2]}=u)\\
&\quad\times\Bigg|\sum_{\substack{(l,R_l)\in\\ [N^2]\times\{A,D_0,D_1,D_2\}}} \tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\left(RW^{[N^2]}(r-u)=(l,R_l)\right)\\
&\qquad\qquad - \tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\left(RW'^{[N^2]}(r-u)=(l,R_l)\right)\Bigg|\\
&+16[1-\mathrm{e}^{-\frac{e_1}{N}L(N)}].
\end{aligned}
$$

Hence the last term in the last inequality tends to 0 as $N \to \infty$.

To show that the first term in the last inequality tends to 0 we use the coupling again. Recall the definition of $H_t^{[N^2]}$ in (8.176). Note that we can rewrite the sum as

$$\begin{aligned}
&\sum_{\substack{(l,R_l)\in\\ [N^2]\times\{A,D_0,D_1,D_2\}}} \tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\left(RW^{[N^2]}(r-u)=(l,R_l)\right)-\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\left(RW'^{[N^2]}(r-u)=(l,R_l)\right)\\
&\leq \sum_{\substack{(l,R_l)\in\\ [N^2]\times\{A,D_0,D_1,D_2\}}} \sum_{\substack{(l',R_{l'})\in\\ [N^2]\times\{A,D_0,D_1,D_2\}}} \left(\mathbb{P}^{[N^2]}_{(i,A)}\left(RW^{[N^2]}(L(N))=(l',R_l')\right)\right.\\
&\qquad \times \mathbb{P}^{[N^2]}_{(l',R_l')}\left(RW^{[N^2]}(r-u-L(N))=(l,R_l)\right)\\
&\leq \sum_{\substack{(l',R_{l'})\in\\ [N^2]\times\{A,D_0,D_1,D_2\}}} \left(\mathbb{P}^{[N^2]}_{(i,A)}\left(RW^{[N^2]}(L(N))=(l',R_l')\right)-\mathbb{P}^{[N^2]}_{(j,A)}\left(RW'^{[N^2]}(L(N))=(l',R_l')\right)\right)\\
&= \sum_{\substack{(l',R_{l'})\in\\ [N^2]\times\{A,D_0,D_1,D_2\}}} \left(\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\left(RW^{[N^2]}(L(N))=(l',R_l'),(H_t^{[N^2]})^c\right)\right.\\
&\qquad\qquad -\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\left(RW'^{[N^2]}(L(N))=(l',R_l'),(H_t^{[N^2]})^c\right)\Big)\\
&\leq 2\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\left((H_t^{[N^2]})^c\right).
\end{aligned} \tag{8.202}$$

To show that

$$\lim_{n\to\infty}\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\left((H_t^{[N^2]})^c\right)=0, \tag{8.203}$$

we can use a similar strategy as between (8.189) and (8.191), but note that we now start from two active sites instead of two 1-dormant sites. Therefore (8.187) directly follows from (8.190) and (8.191).

To show the second term in (8.200) tends to 0, we write it as

$$\begin{aligned}
&2\int_0^{\bar{t}(N)} \mathrm{d}r\, \tilde{\mathbb{P}}(\tau^{[N^2]}=r)\tilde{\mathbb{P}}(\tau^{[N^2]}\geq r)\\
&\times\int_0^{\bar{t}(N)-r}\mathrm{d}v \sum_{k\in[N^2]}\left(\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\left(RW^{[N^2]}(v)=(k,A)\right)-\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\left(RW'^{[N^2]}(v)=(k,A)\right)\right)\\
&\times\left[\mathbb{P}^{[N^2]}_{(i,D_1)}\left(RW^{[N^2]}(v)=(k,A)\right)-\mathbb{P}^{[N^2]}_{(j,D_1)}\left(RW'^{[N^2]}(v)=(k,A)\right)\right]\mathbb{E}[g(x_k^{[N^2]}(\bar{t}(N)-r-v))]\\
&=2\int_0^{\bar{t}(N)} \mathrm{d}r\, \tilde{\mathbb{P}}(\tau^{[N^2]}=r)\tilde{\mathbb{P}}(\tau^{[N^2]}\geq r)\\
&\times\int_0^{\bar{t}(N)-r}\mathrm{d}v \sum_{k\in[N^2]}\left(\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\left(RW^{[N^2]}(v)=(k,A)\right)-\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\left(RW'^{[N^2]}(v)=(k,A)\right)\right)\\
&\times\int_0^v \mathrm{d}u\, \tilde{\mathbb{P}}^{[N^2]}(\tau^{[N^2]}=u)\\
&\left(\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\left(RW^{[N^2]}(v-u)=(k,A)\right)-\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\left(RW'^{[N^2]}(v-u)=(k,A)\right)\right)\\
&\times\mathbb{E}[g(x_k^{[N^2]}(\bar{t}(N)-r-v))].
\end{aligned} \tag{8.204}$$

Changing the order of integration and setting $w = v - u$, we obtain

$$
\begin{aligned}
&2\int_0^{\bar{t}(N)} \mathrm{d}r\, \tilde{\mathbb{P}}(\tau^{[N^2]} = r)\tilde{\mathbb{P}}(\tau^{[N^2]} \geq r) \\
&\times \int_0^{\bar{t}(N)-r} \mathrm{d}u\, \tilde{\mathbb{P}}^{[N^2]}(\tau^{[N^2]} = u) \int_u^{\bar{t}(N)-r} \mathrm{d}v \\
&\sum_{k\in[N^2]} \Big[\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\Big(RW^{[N^2]}(v) = (k,A)\Big) - \tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\Big(RW'^{[N^2]}(v) = (k,A)\Big)\Big] \\
&\times \Big[\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\Big(RW^{[N^2]}(v-u) = (k,A)\Big) - \tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\Big(RW'^{[N^2]}(v-u) = (k,A)\Big)\Big] \\
&\times \mathbb{E}[g(x_k^{[N^2]}(\bar{t}(N) - r - v))] \\
&= 2\int_0^{\bar{t}(N)} \mathrm{d}r\, \tilde{\mathbb{P}}(\tau^{[N^2]} = r)\tilde{\mathbb{P}}(\tau^{[N^2]} \geq r) \\
&\times \int_0^{\bar{t}(N)-r} \mathrm{d}u\, \tilde{\mathbb{P}}^{[N^2]}(\tau^{[N^2]} = u) \int_0^{\bar{t}(N)-r-u} \mathrm{d}w \\
&\sum_{k\in[N^2]} \Big[\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\Big(RW^{[N^2]}(w+u) = (k,A)\Big) - \tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\Big(RW'^{[N^2]}(w+u) = (k,A)\Big)\Big] \\
&\times \Big[\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\Big(RW^{[N^2]}(w) = (k,A)\Big) - \tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\Big(RW'^{[N^2]}(w) = (k,A)\Big)\Big] \\
&\times \mathbb{E}[g(x_k^{[N^2]}(\bar{t}(N) - r - u - w))].
\end{aligned}
\tag{8.205}
$$

This can be rewritten as

$$
\begin{aligned}
&2\int_0^{\bar t(N)} \mathrm{d}r\, \tilde{\mathbb{P}}(\tau^{[N^2]}=r)\tilde{\mathbb{P}}(\tau^{[N^2]}\geq r)\\
&\int_0^{\bar t(N)-r} \mathrm{d}u\, \tilde{\mathbb{P}}^{[N^2]}(\tau^{[N^2]}=u)\\
&\quad\times \sum_{(l,R_l)\in[N^2]} \Big[\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\Big(RW^{[N^2]}(u)=(l,R_l)\Big) - \tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\Big(RW'^{[N^2]}(u)=(l,R_l)\Big)\Big]\\
&\quad\times \int_0^{\bar t(N)-r-u} \mathrm{d}w \sum_{k\in[N^2]} \Big[\mathbb{P}^{[N^2]}_{(l,R_l)}\Big(RW^{[N^2]}(w)=(k,A)\Big)\Big]\\
&\quad\times \Big[\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\Big(RW^{[N^2]}(w)=(k,A)\Big) - \tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\Big(RW'^{[N^2]}(w)=(k,A)\Big)\Big]\\
&\quad\times \mathbb{E}[g(x_k^{[N^2]}(\bar t(N)-r-u-w))]\\
&\leq 8\int_{L(N)}^{\bar t(N)} \mathrm{d}r\, \tilde{\mathbb{P}}(\tau^{[N^2]}=r)\tilde{\mathbb{P}}(\tau^{[N^2]}\geq r)\\
&\int_{L(N)}^{\bar t(N)-r} \mathrm{d}u\, \tilde{\mathbb{P}}^{[N^2]}(\tau^{[N^2]}=u)\\
&\quad\times \sum_{(l,R_l)\in[N^2]} \Big[\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\Big(RW^{[N^2]}(u)=(l,R_l)\Big) - \tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\Big(RW'^{[N^2]}(u)=(l,R_l)\Big)\Big]\\
&+8\int_{L(N)}^{\bar t(N)} \mathrm{d}r\, \tilde{\mathbb{P}}(\tau^{[N^2]}=r)\tilde{\mathbb{P}}(\tau^{[N^2]}\geq r)\\
&\int_0^{L(N)} \mathrm{d}u\, \tilde{\mathbb{P}}^{[N^2]}(\tau^{[N^2]}=u)\\
&\quad\times \sum_{(l,R_l)\in[N^2]} \Big[\tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\Big(RW^{[N^2]}(u)=(l,R_l)\Big) - \tilde{\mathbb{P}}^{[N^2]}_{(i,A),(j,A)}\Big(RW'^{[N^2]}(u)=(l,R_l)\Big)\Big]\\
&+16\int_0^{L(N)} \mathrm{d}r\, \tilde{\mathbb{P}}(\tau^{[N^2]}=r)\tilde{\mathbb{P}}(\tau^{[N^2]}\geq r).
\end{aligned}
\tag{8.206}
$$

This tends to 0 by (8.202) and the reasoning below (8.203). □

§8.3.9 Limiting evolution of the estimator processes

In this section we show that the results along the subsequences used in steps 5-8 of the scheme for the two-level three-colour mean-field system actually hold for all subsequences. Therefore the limiting evolution holds for $N\to\infty$. Recall that Lemma 8.3.20 tells us that all single 1-dormant colonies equal the value of the 1-dormant 1-block average. Therefore the second assumption in (8.81) in Proposition (8.3.5) can be replaced by

$$
\lim_{k\to\infty}\mathcal{L}\left[\left(Y_{2,0}^{[N_k^2]}(N_k^2t_2)\right)\Big|\boldsymbol{\Theta}^{\mathrm{aux},(1),[N_k^2]}(N_k^2t_2+N_kt_1)\right] = P^{z_1^{\mathrm{eff}}(t_1)}, \tag{8.207}
$$

since, by Lemma 8.3.20, the limiting law

$$
\lim_{k\to\infty}\mathcal{L}\left[\left(Y_{1,0}^{[N_k^2]}(N_k^2t_2)\right)\right] \tag{8.208}
$$

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is completely determined by the first line in (8.81). Hence, the assumptions in Proposition 8.3.10 and Lemma 8.3.18 can be weakened in the same way. Using that in Proposition 8.1.1 we assume (8.38) and (8.39), we find that the 2-block convergence stated in Lemma 8.3.18 holds along all subsequences we choose in Step 5. We conclude that Proposition 8.1.1(a) is indeed true. Combining Proposition 8.1.1(a) with steps 1-4 of the scheme and Lemma 8.3.20, we find that the assumptions in Proposition 8.3.10 are true for all subsequences, and we obtain the limiting evolution of the 1-block estimator process. Projecting this limiting evolution onto the active 1-block average and the 1-dormant 1-block average, we obtain Proposition 8.1.1(b). Finally, combining Proposition 8.1.1(a), steps1-4, and the fact that Proposition 8.3.10 is true along all subsequences, we obtain Proposition 8.1.1(c) and (f).

§8.3.10 Convergence in the Meyer-Zheng topology

In this section we show how the results on the effective and estimator processes can be use to show convergence of the full 1- and 2-block processes.

Lemma 8.3.22 ([Convergence of 1-process in the Meyer-Zheng topology]). *Assume that for the 1-block estimator process defined in* (8.21)

$$\lim_{N\to\infty} \mathcal{L}\left[\left(\boldsymbol{\Theta}^{\mathrm{aux},(1),[N^2]}(N^2t_2+Nt_1)\right)_{t_1>0}\right] = \mathcal{L}\left[(z_1^{\mathrm{aux}}(t_1))_{t_1>0}\right], \tag{8.209}$$

where, conditional on $x_2^{\mathrm{eff}}(t_2)=u$, the limit process is the unique solution of the SSDE in (8.30) *with θ replaced by u and with initial measure $\Gamma_u^{\mathrm{eff},(1)}$. Then*

$$\lim_{N\to\infty} \mathcal{L}\left[\left(\boldsymbol{\Theta}^{(1),[N^2]}(N^2t_2+Nt_1)\right)_{t_1>0}\right] = \mathcal{L}\left[(z_1^{\Gamma^{(1)}(t_2)}(t_1))_{t_1>0}\right] \quad \text{in the Meyer-Zheng topology,} \tag{8.210}$$

where $\Gamma^{(1)}(t_2)$ is defined as in (8.46) *and $(z_1^{\Gamma^{(1)}(t_2)}(t_1))_{t_1>0}$ is the process moving according to* (8.28) *with initial measure $\Gamma^{(1)}(t_2)$.*

Proof. By assumption 8.209 and Lemma 8.3.6, we can proceed as in the proof of Proposition 7.2.13 to find (8.210). □

Lemma 8.3.23 ([Convergence of 2-process in the Meyer-Zheng topology]).

Assume that for the effective 2-block process defined in (8.23)

$$\lim_{N\to\infty} \mathcal{L}\left[\left(\boldsymbol{\Theta}^{\mathrm{eff},(2),[N^2]}(N^2t_2)\right)_{t_2>0}\right] = \mathcal{L}\left[(z_2^{\mathrm{aux}}(t_2))_{t_2>0}\right], \tag{8.211}$$

where $(z_2^{\mathrm{eff}}(t_2))_{t_2>0}$ is the process evolving according to (8.36) *and starting from $(\vartheta_1,\theta_{y_2})$. Then for the 2-block estimator process defined in* (8.23)

$$\lim_{N\to\infty} \mathcal{L}\left[\left(\boldsymbol{\Theta}^{(2),[N^2]}(N^2t_2)\right)_{t_2>0}\right] = \mathcal{L}\left[(z_2(t_2))_{t_2>0}\right] \quad \text{in the Meyer-Zheng topology,} \tag{8.212}$$

where $(z_2(t_2))_{t_2>0}$ *is the process evolving according to* (8.35) *and starting in state* $(\vartheta_1, \vartheta_1, \vartheta_1, \theta_{y_2})$.

Proof. Combining Lemmas 8.3.6 and 8.3.13, we find for $t_2 > 0$

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\left[\left|\bar{\Theta}^{(2),[N^2]}(N^2t_2) - \Theta_x^{(2),[N^2]}(N^2t_2)\right|\right] &= 0,\\
\lim_{N\to\infty} \mathbb{E}\left[\left|\bar{\Theta}^{(2),[N^2]}(N^2t_2) - \Theta_{y_0}^{(2),[N^2]}(N^2t_2)\right|\right] &= 0, \qquad (8.213)\\
\lim_{N\to\infty} \mathbb{E}\left[\left|\bar{\Theta}^{(2),[N^2]}(N^2t_2) - \Theta_{y_1}^{(2),[N^2]}(N^2t_2)\right|\right] &= 0.
\end{aligned}$$

Therefore we can again proceed as in the proof of Proposition 7.2.13 to find (8.212).
□

§8.3.11 Proof of the two-level three-colour mean-field finite-systems scheme

In Section 8.3.9 we already proved Proposition 8.1.1(a),(b),(c) and (f). The proof of Proposition 8.1.1(d) follows from Proposition 8.1.1(a) by applying Lemma 8.3.23. The proof of Proposition 8.1.1(e) follows from Proposition 8.1.1(b) by applying Lemma 8.3.22. This completes the proof of Proposition 8.1.1.