Spatial populations with seed-bank
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CHAPTER 7

Two-colour mean-field system

In this chapter we extend the results obtained in Section 6.2.1 to a mean-field system where the seed-bank consists of two colours, one colour that interacts on the slow time scale and one colour that interacts on the fast time scale. To do so we follow the set-up used in Chapter 4.4. In particular, we highlight the role of the second colour. Section 7.1 builds up the setting and states the main scaling result: Proposition 7.1.2. Section 7.2 provides the proof of this proposition based on a series of lemmas, which are stated and proved first.

§7.1 Two-colour mean-field finite-systems scheme

Setup. In this section we consider a simplified version of our SSDE in (4.20) on the finite geographic space

$$[N] = \{0, 1, \dots, N-1\}, \qquad N \in \mathbb{N}. \tag{7.1}$$

The migration kernel $a^{\Omega_N}(\cdot,\cdot)$ is replaced by the migration kernel $a^{[N]}(i,j) = c_0 N^{-1}$ for all $i,j \in [N]$, where $c_0 \in (0,\infty)$ is a constant. The seed-bank consists of *two colours*, labeled 0 and 1. The exchange rates between the active and the colour-0 dormant population are given by $K_0 e_0, e_0$. The exchange rates between active and the colour-1 dormant population are given by $\frac{K_1 e_1}{N}, \frac{e_1}{N}$. The state space is

$$S = \mathfrak{s}^{[N]}, \qquad \mathfrak{s} = [0,1] \times [0,1]^2, \tag{7.2}$$

and the system, consisting of three components, is denoted by

$$\begin{aligned} Z^{[N]}(t) &= \big(X^{[N]}(t), (Y_0^{[N]}(t), Y_1^{[N]}(t))\big)_{t \geq 0}, \\ \big(X^{[N]}(t), (Y_0^{[N]}(t), Y_1^{[N]}(t))\big) &= \big(x_i(t), (y_{i,0}(t), y_{i,1}(t))\big)_{i \in [N]}. \end{aligned} \tag{7.3}$$

The components of $(Z^{[N]}(t))_{t\geq 0}$ evolve according to the SSDE

$$\begin{aligned} \mathrm{d}x_i^{[N]}(t) &= \frac{c_0}{N} \sum_{j \in [N]} [x_j^{[N]}(t) - x_i^{[N]}(t)]\,\mathrm{d}t + \sqrt{g(x_i^{[N]}(t))}\,\mathrm{d}w_i(t) \\ &\quad + K_0 e_0\,[y_{i,0}^{[N]}(t) - x_i^{[N]}(t)]\,\mathrm{d}t + \frac{K_1 e_1}{N}\,[y_{i,1}^{[N]}(t) - x_i^{[N]}(t)]\,\mathrm{d}t, \\ \mathrm{d}y_{i,0}^{[N]}(t) &= e_0\,[x_i^{[N]}(t) - y_{i,0}^{[N]}(t)]\,\mathrm{d}t, \\ \mathrm{d}y_{i,1}^{[N]}(t) &= \frac{e_1}{N}\,[x_i^{[N]}(t) - y_{i,1}^{[N]}(t)]\,\mathrm{d}t, \qquad i \in [N], \end{aligned} \tag{7.4}$$

which is a special case of (4.20). The initial state is $\mu(0) = \mu^{\otimes[N]}$ for some $\mu \in \mathcal{P}([0,1]^3)$. The SSDE in (7.4) has a unique weak solution coming from a well-posed martingale problem [67, Theorem 3.1]. By [67, Theorem 3.2], (7.4) has a unique strong solution for every deterministic initial state $Z^{[N]}(0)$. Therefore the solutino of (7.4) is Feller and Markov for any initial law. The SSDE in (7.4) can alternatively be written as

$$\begin{aligned}
\mathrm{d}x_i^{[N]}(t) &= c_0 \left[\frac{1}{N} \sum_{j\in[N]} x_j^{[N]}(t) - x_i^{[N]}(t)\right] \mathrm{d}t + \sqrt{g\big(x_i^{[N]}(t)\big)}\, \mathrm{d}w_i(t) \\
&\quad + K_0 e_0 \,[y_{i,0}^{[N]}(t) - x_i^{[N]}(t)]\,\mathrm{d}t + \frac{K_1 e_1}{N}\,[y_{i,1}^{[N]}(t) - x_i^{[N]}(t)]\,\mathrm{d}t, \\
\mathrm{d}y_{i,0}^{[N]}(t) &= e_0\,[x_i^{[N]}(t) - y_{i,0}^{[N]}(t)]\,\mathrm{d}t, \\
\mathrm{d}y_{i,1}^{[N]}(t) &= \frac{e_1}{N}\,[x_i^{[N]}(t) - y_{i,1}^{[N]}(t)]\,\mathrm{d}t, \qquad i \in [N].
\end{aligned} \tag{7.5}$$

So the migration term for a single colony can be interpreted as a drift towards the average of the active population. We are interested in $\mathcal{L}[(Z^{[N]}(t)))_{t\geq 0}]$ in the limit as $N \to \infty$, on time scales t and Ns. Heuristically, analysing the SSDE in (7.5), we can foresee the following results, which are made precise in Proposition 7.1.2.

• **On time scale** $1 = N^0$ (space-time scale 0), in the limit as $N \to \infty$ the colour-1 dormant population $(Y_1^{[N]}(t))_{t\geq 0}$ in (7.4) converges to a constant process, since the single components $y_{i,1}$ do not move on time scale t. The components of $(X^{[N]}(t), Y_0^{[N]}(t))_{t\geq 0}$ converge to i.i.d. copies of the single-colony McKean-Vlasov process in (6.1), where in the corresponding SSDE the parameters c, e, K are replaced by c_0, e_0, K_0 and $E = 1$. So on time scale t we only see the colour-0 dormant population interacting with the active population, and the colour-1 dormant population is not yet coming into play. Therefore the colour-0 dormant population is the *effective seed-bank* on time scale 1, and the process

$$z_0^{\mathrm{eff},[N]}(t) = (x_0^{[N]}(t), y_{0,0}^{[N]}(t))_{t\geq 0} \tag{7.6}$$

is called the effective process on level 0. Note that the active population has a drift towards $\frac{1}{N}\sum_{j\in[N]} x_j(t)$, which in the McKean-Vlasov limit is replaced by $\mathbb{E}[x(t)]$ given by (4.111).

• **On time scale** N (space-time scale 1), we look at the averages

$$\begin{aligned}
(z_1^{[N]}(s))_{s>0} &= \Big(x_1^{[N]}(s), (y_{0,1}^{[N]}(s), y_{1,1}^{[N]}(s))\Big)_{s>0} \\
&= \left(\frac{1}{N}\sum_{i\in[N]} x_i^{[N]}(Ns), \left(\frac{1}{N}\sum_{i\in[N]} y_{i,0}^{[N]}(Ns), \frac{1}{N}\sum_{i\in[N]} y_{i,1}^{[N]}(Ns)\right)\right)_{s>0}.
\end{aligned} \tag{7.7}$$

Again the lower index 1 indicates that the average is the analogue of the 1-block average defined in (4.22). Using (7.4), we see that the dynamics of the system in (7.7) is given by the SSDE

$$
\begin{aligned}
\mathrm{d}x_1^{[N]}(s) = {} & \sqrt{\frac{1}{N}\sum_{i\in[N]} g(x_i^{[N]}(Ns))}\,\mathrm{d}w(s) + NK_0e_0\left[y_{0,1}^{[N]}(s) - x_1^{[N]}(s)\right]\mathrm{d}s \\
& + K_1e_1\left[y_{1,1}^{[N]}(s) - x_1^{[N]}(s)\right]\mathrm{d}s, \\
\mathrm{d}y_{0,1}^{[N]}(s) = {} & Ne_0\left[x_1^{[N]}(s) - y_{0,1}^{[N]}(s)\right]\mathrm{d}s, \\
\mathrm{d}y_{1,1}^{[N]}(s) = {} & e_1\left[x_1^{[N]}(s) - y_{1,1}^{[N]}(s)\right]\mathrm{d}s.
\end{aligned}
\tag{7.8}
$$

Thus, as in the mean-field system with one-colour, on time scale N infinite rates appear in the interaction of the active population with the colour-0 dormant population. Therefore in the limit as $N \to \infty$ the path becomes rougher and rougher at rarer and rarer times. Using the *Meyer-Zheng topology* we can prove that $\lim_{N\to\infty} y_{0,1}^{[N]}(s) = \lim_{N\to\infty} x_1^{[N]}(s)$ most of the time. On the other hand, on time scale N, $x_1^{[N]}(s)$ has a non-trivial interaction with $y_{1,1}^{[N]}(s)$, and therefore we say that on time scale N the colour-1 dormant population is the *effective seed-bank*. Note that for the evolution of the average $\frac{x_1^{[N]}(s)+K_0y_{0,1}^{[N]}(s)}{1+K_0}$ the rates with a factor N in front cancel out. We will use the quantity $\frac{x_1^{[N]}(s)+K_0y_{0,1}^{[N]}(s)}{1+K_0}$ to obtain results in the classical path-space topology. We call

$$
(z_1^{[N],\mathrm{eff}}(s))_{s>0} = \left(\frac{x_1^{[N]}(s)+K_0y_{0,1}^{[N]}(s)}{1+K_0}, y_{1,1}(s)\right)_{s>0}
\tag{7.9}
$$

the effective process on space-time scale 1. We will call space-time scale 1 also level 1.

Scaling limit. To describe the limiting dynamics of the system in (7.4), we need the infinite-dimensional process

$$
(Z(t))_{t\geq 0} = \big((z_i(t))_{t\geq 0}\big)_{i\in\mathbb{N}_0} = \big((x_i(t),(y_{i,0}(t),y_{i,1}(t)))_{t\geq 0}\big)_{i\in\mathbb{N}_0}
\tag{7.10}
$$

with state space $([0,1]^3)^{\mathbb{N}_0}$ that evolves according to

$$
\begin{aligned}
\mathrm{d}x_i(t) &= c_0[\theta - x_i(t)]\,\mathrm{d}t + \sqrt{g(x_i(t))}\,\mathrm{d}w_i(t) + K_0e_0\,[y_{i,0}(t) - x_i(t)]\,\mathrm{d}t, \\
\mathrm{d}y_{i,0}(t) &= e_0\,[x_i(t) - y_{i,0}(t)]\,\mathrm{d}t, \\
y_{i,1}(t) &= y_{i,1}, \qquad i\in\mathbb{N}_0.
\end{aligned}
\tag{7.11}
$$

Here, $\theta\in[0,1]$ and $y_{i,1}\in[0,1]$ for all $i\in\mathbb{N}_0$. We will also need the limiting effective process

$$
(Z^{\mathrm{eff}}(t))_{t\geq 0} = \big((z_i^{\mathrm{eff}}(t))_{t\geq 0}\big)_{i\in\mathbb{N}_0} = \big((x_i^{\mathrm{eff}}(t), y_{i,0}^{\mathrm{eff}}(t))_{t\geq 0}\big)_{i\in\mathbb{N}_0}
\tag{7.12}
$$

with state space $([0,1]^2)^{\mathbb{N}_0}$ that evolves according to

$$
\begin{aligned}
\mathrm{d}x_i^{\mathrm{eff}}(t) &= c_0[\theta - x_i^{\mathrm{eff}}(t)]\,\mathrm{d}t + \sqrt{g\big(x_i^{\mathrm{eff}}(t)\big)}\,\mathrm{d}w(t) + K_0e_0\,[y_{i,0}^{\mathrm{eff}}(t) - x_i^{\mathrm{eff}}(t)]\,\mathrm{d}t, \\
\mathrm{d}y_{i,0}^{\mathrm{eff}}(t) &= e_0\,[x_i^{\mathrm{eff}}(t) - y_{i,0}^{\mathrm{eff}}(t)]\,\mathrm{d}t, \qquad i\in\mathbb{N}_0.
\end{aligned}
\tag{7.13}
$$

Like for the one-colour mean-field finite-systems scheme, we need the following list of ingredients to formally state our multi-scaling properties:

(a) For positive times $t > 0$, we define the so-called *estimators* for the finite system by:

$$\begin{aligned}\bar{\Theta}^{(1),[N]}(t) &= \frac{1}{N}\sum_{i\in[N]}\frac{x_i^{[N]}(t)+K_0 y_{i,0}^{[N]}(t)}{1+K_0},\\ \Theta_x^{(1),[N]}(t) &= \frac{1}{N}\sum_{i\in[N]} x_i^{[N]}(t),\\ \Theta_{y_0}^{(1),[N]}(t) &= \frac{1}{N}\sum_{i\in[N]} y_{i,0}^{[N]}(t),\\ \Theta_{y_1}^{(1),[N]}(t) &= \frac{1}{N}\sum_{i\in[N]} y_{i,1}^{[N]}(t).\end{aligned} \tag{7.14}$$

We abbreviate

$$\begin{aligned}\mathbf{\Theta}^{(1),[N]}(t) &= \left(\Theta_x^{(1),[N]}(t), \Theta_{y_0}^{(1),[N]}(t), \Theta_{y_1}^{(1),[N]}(t)\right),\\ \mathbf{\Theta}^{\mathrm{eff},(1),[N]}(t) &= \left(\bar{\Theta}^{(1),[N]}(t), \Theta_{y_1}^{(1),[N]}(t)\right).\end{aligned} \tag{7.15}$$

We refer to $(\mathbf{\Theta}^{\mathrm{eff},(1),[N]}(t))_{t\geq 0}$ as the *effective estimator process* and to $(\mathbf{\Theta}^{(1),[N]}(t))_{t\geq 0}$ as the *estimator process*.

(b) The *time scale* Ns is such that $\mathcal{L}[\bar{\Theta}^{[N]}(Ns - L(N)) - \bar{\Theta}^{[N]}(Ns)] = \delta_0$ for all $L(N)$ satisfying $\lim_{N\to\infty} L(N) = \infty$ and $\lim_{N\to\infty} L(N)/N = 0$, but not for $L(N) = N$. In words, Ns is the time scale on which $\bar{\Theta}^{[N]}(\cdot)$ starts evolving, i.e., $(\bar{\Theta}^{[N]}(Ns))_{s>0}$ is no longer a fixed process. When we scale time by Ns, we will use s as a time index, which indicates the "fast time scale". The "slow time scale" will be indicated by t. Thus, the time scales for the two-colour mean-field system are the same as the time scales for the one-colour mean-field system.

Remark 7.1.1 (Notation). The upper index 1 in $\bar{\Theta}^{(1)}$ and $\Theta_{y_1}^{(1)}$ is used to indicate that we are working with a system of level 1, so the system that lives on space-time scale 1. This can later be easily generalized to levels 2 and k. ■

(c) The *invariant measure* (i.e., the equilibrium measure) for the evolution of a single colony in (7.11), written

$$\Gamma_{\theta,\theta,y_1}, \tag{7.16}$$

and the *invariant measure* of the infinite system in (7.11), written $\nu_{\theta,\theta,\mathbf{y}_1} = \Gamma_{\theta,\theta,y_1}^{\otimes\mathbb{N}_0}$ with $\theta \in [0,1]$ and $\mathbf{y}_1 \in [0,1]^{\mathbb{N}_0}$ a random variable. The existence of the invariant measure ν_θ and the convergence of $\mathcal{L}[Z(t)_{t\geq 0}]$ towards ν_θ will be shown in the proof of Proposition 7.1.2.

(d) The invariant measure of the effective process in (7.13),

$$\Gamma_\theta^{\mathrm{eff}}, \tag{7.17}$$

and the invariant measure for the full process, $\nu_\theta^{\mathrm{eff}} = (\Gamma_\theta^{\mathrm{eff}})^{\otimes\mathbb{N}_0}$.

(e) The renormalisation transformation $\mathcal{F}\colon \mathcal{G} \to \mathcal{G}$,

$$(\mathcal{F}g)(\theta) = \int_{([0,1]^2)^{\mathbb{N}_0}} g(x_0)\, \nu_\theta^{\mathrm{eff}}(\mathrm{d}x_0, \mathrm{d}y_{0,0}), \quad \theta \in [0,1], \tag{7.18}$$

where $\Gamma_\theta^{\mathrm{eff}}$ is the equilibrium measure of (7.16). Note that this is the same transformation as defined in (4.75), but for the truncated system. Since $\nu_\theta^{\mathrm{eff}}$ is a product measure, we can write

$$(\mathcal{F}g)(\theta) = \int_{[0,1]^2} g(x)\, \Gamma_\theta^{\mathrm{eff}}(\mathrm{d}x, \mathrm{d}y_0), \quad \theta \in [0,1], \tag{7.19}$$

(f) The limiting 1-block process $(z_1(s))_{s>0} = (x_1(s), (y_{0,1}(s), y_{1,1}(s)))_{s>0}$ evolving according to

$$\begin{aligned} \mathrm{d}x_1(s) &= \frac{1}{1+K_0}\left[\sqrt{(\mathcal{F}g)\big(x_1(s)\big)}\,\mathrm{d}w(s) + K_1 e_1\,[y_{1,1}(s) - x_1(s)]\,\mathrm{d}s\right], \\ y_{0,1}(s) &= x_1(s), \\ \mathrm{d}y_{1,1}(s) &= e_1\,[x_1(s) - y_{1,1}(s)]\,\mathrm{d}s, \end{aligned} \tag{7.20}$$

where $\mathcal{F}g$ is defined in (7.20). The effective process $(z_1^{\mathrm{eff}}(s))_{s>0} = (x_1^{\mathrm{eff}}(s), y_{1,1}^{\mathrm{eff}}(s))_{s>0}$ on space-time scale 1,

$$\begin{aligned} \mathrm{d}x_1^{\mathrm{eff}}(s) &= \frac{1}{1+K_0}\left[\sqrt{(\mathcal{F}g)(x_1^{\mathrm{eff}}(s))}\,\mathrm{d}w(s) + K_1 e_1\,[y_{1,1}^{\mathrm{eff}}(s) - x_1^{\mathrm{eff}}(s)]\,\mathrm{d}s\right], \\ \mathrm{d}y_{1,1}^{\mathrm{eff}}(s) &= e_1\,[x_1^{\mathrm{eff}}(s) - y_{1,1}^{\mathrm{eff}}(s)]\,\mathrm{d}s. \end{aligned} \tag{7.21}$$

We are now ready to state the scaling limit for the evolution of the averages in (7.7), which we refer to as the *mean-field finite-systems scheme with two colours.*

Proposition 7.1.2 (Mean-field: two-colour finite-systems scheme). *Suppose that $\mathcal{L}[Z^{[N]}(0)] = \mu^{\otimes[N]}$ for some $\mu \in \mathcal{P}\left([0,1]\times[0,1]^2\right)$. Let*

$$\vartheta_0 = \mathbb{E}^\mu\left[\frac{x + K_0 y_0}{1+K_0}\right], \qquad \theta_{y_1} = \mathbb{E}^\mu\,[y_1]. \tag{7.22}$$

(a) For the effective estimator process defined in (7.15),

$$\lim_{N\to\infty} \mathcal{L}\left[\left(\mathbf{\Theta}^{\mathrm{eff},(1),[N]}(Ns)\right)_{s>0}\right] = \mathcal{L}\left[\left(z_1^{\mathrm{eff}}(s)\right)_{s>0}\right], \tag{7.23}$$

where the limit is determined by the unique solution of the SSDE (7.21), *with initial state*

$$z_1^{\mathrm{eff}}(0) = \left(x_1^{\mathrm{eff}}(0), y_1^{\mathrm{eff}}(0)\right) = (\vartheta_0, \theta_{y_1}). \tag{7.24}$$

(b) Assume for the 1*-dormant single components that*

$$\lim_{N\to\infty} \mathcal{L}\left[Y_1^{[N]}(Ns)\,\middle|\,\mathbf{\Theta}^{(1),[N]}(Ns)\right] = P^{z_1(s)}_{Y_1(s)}. \tag{7.25}$$

Define

$$\Gamma^{\mathrm{eff}}_{(\vartheta_0,\theta_{y_1})}(s) = \int_{[0,1]^2} S_s\big((\vartheta_0,\theta_{y_1}),\mathrm{d}(u_x,u_y)\big)\,\Gamma^{\mathrm{eff}}_{u_x} \in \mathcal{P}([0,1]^2), \tag{7.26}$$

where $S_s((\vartheta_0,\theta_{y_1}),\cdot)$ is the time-s marginal law of the process $(z_1^{\mathrm{eff}}(s))_{s>0}$ starting from $(\theta_0,\theta_{y_1}) \in [0,1]^2$ and $\Gamma^{\mathrm{eff}}_{u_x}$ is the equilibrium distribution of the system in (7.13) *with $\theta = u_x$ (note that $\Gamma^{\mathrm{eff}}_{\vartheta_0,\theta_{y_1}}(0) = \Gamma^{\mathrm{eff}}_{\vartheta_0}$). Let $(z^{\mathrm{eff},\Gamma_{(\vartheta_0,\theta_{y_1})}(s)}(t))_{t\geq 0}$ be the process with initial law $z^{\mathrm{eff},\Gamma_{(\vartheta_0,\theta_{y_1})}(s)}(0)$ drawn according to $\Gamma^{\mathrm{eff}}_{(\vartheta_0,\theta_{y_1})}(s)$ (which is a mixture of random processes in equilibrium) that, conditional on $x_1^{\mathrm{eff}}(s) = \theta$, evolves according to* (7.13). *Then, for every $s \in (0,\infty)$,*

$$\lim_{N\to\infty} \mathcal{L}\left[\left(z_0^{\mathrm{eff},[N]}(Ns+t)\right)_{t\geq 0}\right] = \mathcal{L}\left[(z^{\Gamma^{\mathrm{eff}}_{(\vartheta_0,\theta_{y_1})}(s)}(t))_{t\geq 0}\right]. \tag{7.27}$$

(c) *For the averages in* (7.7),

$$\lim_{N\to\infty} \mathcal{L}\left[\left(z_1^{[N]}(s)\right)_{s>0}\right] = \mathcal{L}\left[(z_1(s))_{s>0}\right] \tag{7.28}$$

in the Meyer-Zheng topology,

where the limit process is the unique solution of the SSDE in (7.20) *with initial state*

$$z_1(0) = (x_1(0), y_{0,1}(0), y_{1,1}(0)) = (\vartheta_0,\vartheta_0,\theta_{y_1}). \tag{7.29}$$

(d) *Assume 7.25 and define*

$$\nu(s) = \int_{[0,1]^3} S_s\big((\vartheta_0,\vartheta_0,\theta_{y_1}),\mathrm{d}(u_x,u_x,u_{y_1})\big) \int_{[0,1]^{\mathbb{N}_0}} P^{(u_x,u_x,u_{y_1})}_{Y_1(s)}(\mathrm{d}\boldsymbol{y}_1)\,\nu_{u_x,\boldsymbol{y}_1}, \tag{7.30}$$

where $S_s((\vartheta_0,\vartheta_0,\theta_{y_1}),\cdot)$ is the time-s marginal law of the process $(z_1(s))_{s>0}$ in (7.20), *starting from $(\vartheta_0,\vartheta_0,\theta_{y_1}) \in [0,1]^3$, and $\nu_{u_x,\boldsymbol{y}_1}$ is the equilibrium distribution of the system in* (7.11) *with $\theta = u_x$ and $(y_{i,1})_{i\in\mathbb{N}_0} = \boldsymbol{y_1}$, (note that $\nu(0) = \nu_{\vartheta_0,(y_{i,1}(0))_{i\in\mathbb{N}_0}}$). Let $(z^{\nu(s)}(t))_{t\geq 0}$ be the process on $([0,1]^3)^{\mathbb{N}_0}$ with initial measure $z^{\nu(s)}(0)$ drawn according to $\nu(s)$ (which is a mixture of random processes in equilibrium) that conditional on $x_1(s) = \theta$ and $Y_1(s) = \boldsymbol{y}_1$ evolves according to* (7.11) *with $\theta = u_x$ and $(y_{i,1})_{i\in\mathbb{N}_0} = \boldsymbol{y}_1$. Then, for every $s \in (0,\infty)$,*

$$\lim_{N\to\infty} \mathcal{L}\left[\left(Z^{[N]}(Ns+t)\right)_{t\geq 0}\right] = \mathcal{L}\left[(z^{\nu(s)}(t))_{t\geq 0}\right]. \tag{7.31}$$

Remark 7.1.3 (Law of 1-dormant single components). Note that

$$\left(\mathcal{L}\left[Y_1^{[N]}(Ns)\,\middle|\,\left(\bar{\Theta}^{[N]}(Ns),\Theta^{[N]}_{y_1}(Ns)\right)\right]\right)_{N\in\mathbb{N}_0} \tag{7.32}$$

is a tight sequence of measures. Hence there exist weak limit points. In Section 8 we will see that if there is a higher layer in the hierarchy, then we can show that all weak limit points of (7.32) are the same and we can identify the limit. For Theorems 4.4.2 and 4.4.4 we do not need this assumption, since there will alwyas be multiple higher levels. ■

§7.2 Proof of the two-colour mean-field finite-systems scheme

The proof of Proposition 7.1.2, the finite-systems scheme with one level and two colours, follows the strategy used in Section 6.3 for the proof of Proposition 6.2.1. Like for the one-colour finite-systems scheme, we denote the slow time scale by t and the fast time scale by s. The proof consists of the following 6 steps:

1 Tightness of the effective estimator processes defined in (7.15).

$$\left((\mathbf{\Theta}^{\mathrm{eff},(1),[N]}(Ns))_{s>0}\right)_{N\in\mathbb{N}} \tag{7.33}$$

2 Stability property of $(\mathbf{\Theta}^{\mathrm{eff},(1),[N]}(Ns+t))_{t>0}$, i.e., for $L(N)$ satisfying $\lim_{N\to\infty} L(N) = \infty$ and $\lim_{N\to\infty} L(N)/N = 0$, and all $\epsilon > 0$,

$$\lim_{N\to\infty} \mathbb{P}\left[\sup_{0\le t\le L(N)} \left|\bar{\Theta}^{(1),[N]}(Ns) - \bar{\Theta}^{(1),[N]}(Ns-t)\right| > \epsilon\right] = 0. \tag{7.34}$$

and

$$\lim_{N\to\infty} \mathbb{P}\left[\sup_{0\le t\le L(N)} \left|\Theta_{y_1}^{(1),[N]}(Ns) - \Theta_{y_1}^{(1),[N]}(Ns-t)\right| > \epsilon\right] = 0. \tag{7.35}$$

3 Equilibrium of the infinite system and the one-dimensional distribution of the effective single components $(Z(Ns+t))_{t>0}$, analogous to Proposition 6.2.4.

4 Limiting evolution of the effective processes $((\mathbf{\Theta}^{\mathrm{eff},(1),[N]}(Ns))_{s>0})_{N\in\mathbb{N}}$.

5 Evolution of the 1-blocks in the Meyer-Zheng topology.

6 Proof of Proposition 7.1.2.

Step 1: Tightness of the 1-block estimators.

Lemma 7.2.1 (Tightness of the 1-block estimator). *Let*

$$(\mathbf{\Theta}^{\mathrm{eff},(1),[N]}(Ns))_{s>0} \tag{7.36}$$

be defined as in (7.14)*. Then* $(\mathcal{L}[(\mathbf{\Theta}^{\mathrm{eff},(1),[N]}(Ns))_{s>0}])_{N\in\mathbb{N}}$ *is a tight sequence of probability measures on* $\mathcal{C}((0,\infty),[0,1]^2)$*.*

Proof. To prove tightness of $((\mathbf{\Theta}^{\mathrm{eff},(1),[N]}(Ns))_{s>0})_{N\in\mathbb{N}}$, we will prove for all $\epsilon > 0$ that the set of measures $((\mathbf{\Theta}^{\mathrm{eff},(1),[N]}(Ns))_{s\ge\epsilon})_{N\in\mathbb{N}}$ is tight. To do so, fix $\epsilon > 0$. We will again use [49, Proposition 3.2.3]. From (7.4) we find that the 1-block averages

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$(\boldsymbol{\Theta}^{\mathrm{eff},(1),[N]}(Ns))_{s>0}$ evolve according to

$$\begin{aligned}
\mathrm{d}\bar{\Theta}^{(1),[N]}(Ns) &= \frac{1}{1+K_0}\left[\sqrt{\frac{1}{N}\sum_{i\in[N]} g\big(x_i^{[N]}(Ns)\big)}\,\mathrm{d}w_i(s)\right.\\
&\qquad \left. + K_1 e_1 \left[\Theta_{y_1}^{(1),[N]} - \frac{1}{N}\sum_{i\in[N]} x_i^{[N]}(Ns)\right]\,\mathrm{d}s\right], \qquad (7.37)\\
\mathrm{d}\Theta_{y_1}^{(1),[N]}(Ns) &= e_1\left[\frac{1}{N}\sum_{i\in[N]} x_i^{[N]}(Ns) - \Theta_{y_1}^{(1),[N]}(Ns)\right]\,\mathrm{d}s.
\end{aligned}$$

To use [49, Proposition 3.2.3], we define $\mathcal{C}^*$ as the set of polynomials on $([0,1]^2)$. Note that $(\boldsymbol{\Theta}^{\mathrm{eff},(1),[N]}(Ns))_{s\geq\epsilon}$ is a semi-martingale. Applying Itô's formula, we get

$$\begin{aligned}
&f\left(\boldsymbol{\Theta}^{\mathrm{eff},(1),[N]}(Ns)\right)\\
&= f\left(\boldsymbol{\Theta}^{\mathrm{eff},(1),[N]}(N\epsilon)\right)\\
&\quad + \int_\epsilon^s \mathrm{d}w_i(r)\,\frac{1}{1+K_0}\sqrt{\frac{1}{N}\sum_{i\in[N]} g\big(x_i^{[N]}(Nr)\big)}\,\frac{\partial f}{\partial x}\left(\boldsymbol{\Theta}^{\mathrm{eff},(1),[N]}(Nr)\right)\\
&\quad + \int_\epsilon^s \mathrm{d}r\,\frac{K_1e_1}{1+K_0}\left[\Theta_{y_1}^{(1),[N]}(Nr) - \frac{1}{N}\sum_{i\in[N]} x_i^{[N]}(Nr)\right]\frac{\partial f}{\partial x}\left(\boldsymbol{\Theta}^{\mathrm{eff},(1),[N]}(Nr)\right)\\
&\quad + \int_\epsilon^s \mathrm{d}r\,e_1\left[\frac{1}{N}\sum_{i\in[N]} x_i^{[N]}(Nr) - \Theta_{y_1}^{(1),[N]}(Nr)\right]\frac{\partial f}{\partial y}\left(\boldsymbol{\Theta}^{\mathrm{eff},(1),[N]}(Nr)\right)\\
&\quad + \int_\epsilon^s \mathrm{d}r\,\frac{1}{2(1+K_0)^2}\frac{1}{N}\sum_{i\in[N]} g\big(x_i^{[N]}(Nr)\big)\,\frac{\partial^2 f}{\partial x^2}\left(\boldsymbol{\Theta}^{\mathrm{eff},(1),[N]}(Nr)\right)
\end{aligned} \qquad (7.38)$$

for all $f\in\mathcal{C}^*$. Hence, if we define the operator

$$\begin{aligned}
&G_\dagger^{(1),[N]}\colon\ (\mathcal{C}^*,[0,1]^2,[\epsilon,\infty),\Omega)\to\mathbb{R},\\
&G_\dagger^{(1),[N]}(f,(x,y),s,\omega) = \frac{K_1e_1}{1+K_0}\left[y - \frac{1}{N}\sum_{i\in[N]} x_i^{[N]}(Ns,\omega)\right]\frac{\partial f}{\partial x}\\
&\qquad + e_1\left[\frac{1}{N}\sum_{i\in[N]} x_i^{[N]}(Ns,\omega) - y\right]\frac{\partial f}{\partial y}\\
&\qquad + \frac{1}{2(1+K_0)^2}\frac{1}{N}\sum_{i\in[N]} g(x_i^{[N]}(Ns,\omega))\,\frac{\partial^2 f}{\partial x^2},
\end{aligned} \qquad (7.39)$$

then we see that the process $(\boldsymbol{\Theta}^{\mathrm{eff},(1),[N]}(Ns))_{s\geq\epsilon}$ is a $\mathcal{D}$-semi-martingale for all $\epsilon>0$. For all $\epsilon>0$ the conditions H_1, H_2, H_3 are satisfied as before. Therefore we conclude from [49, Proposition 3.2.3] that the sequence $((\boldsymbol{\Theta}^{\mathrm{eff},(1),[N]}(Ns))_{s\geq\epsilon})_{N\in\mathbb{N}}$ is tight. Since this is true for all $\epsilon>0$, we conclude that $(\mathcal{L}[(\boldsymbol{\Theta}^{\mathrm{eff},(1),[N]}(Ns))_{s>0}])_{N\in\mathbb{N}}$ is tight. □

Step 2: Stability of the 1-block estimators.

Lemma 7.2.2 (Stability property of the 1-block estimator). *Let* $\boldsymbol{\Theta}^{\mathrm{eff},(1),[N]}(t)$ *be defined as in* (7.14). *For any* $L(N)$ *satisfying* $\lim_{N\to\infty} L(N)=\infty$ *and* $\lim_{N\to\infty} L(N)/N=0$,

$$\lim_{N\to\infty}\sup_{0\leq t\leq L(N)}\left|\bar\Theta^{(1),[N]}(Ns)-\bar\Theta^{(1),[N]}(Ns-t)\right|=0 \text{ in probability} \tag{7.40}$$

and

$$\lim_{N\to\infty}\sup_{0\leq t\leq L(N)}\left|\Theta_{y_1}^{(1),[N]}(Ns)-\Theta_{y_1}^{(1),[N]}(Ns-t)\right|=0 \text{ in probability.} \tag{7.41}$$

Proof. Fix $\epsilon>0$. From the SSDE (7.4) we obtain that, for N large enough,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\leq t\leq L(N)}\left|\bar\Theta^{(1),[N]}(Ns)-\bar\Theta^{(1),[N]}(Ns-t)\right|>\epsilon\right)\\
&=\mathbb{P}\Bigg(\sup_{0\leq t\leq L(N)}\frac{1}{1+K_0}\Bigg|\int_{Ns-t}^{Ns}\mathrm{d}r\,\frac{K_1e_1}{N}\left[\Theta_{y_1}^{(1),[N]}(r)-\frac{1}{N}\sum_{i\in[N]}x_i^{[N]}(r)\right]\\
&\qquad\qquad+\int_{Ns-t}^{Ns}\mathrm{d}w_i(r)\,\frac{1}{N}\sum_{i\in[N]}\sqrt{g(x_i^{[N]}(r))}\Bigg|>\epsilon\Bigg]\\
&\leq\mathbb{P}\left(\left|\frac{L(N)2K_1e_1}{N(1+K_0)}\right|+\sup_{0\leq t\leq L(N)}\left|\frac{1}{1+K_0}\int_{Ns-t}^{Ns}\mathrm{d}w_i(r)\,\frac{1}{N}\sum_{i\in[N]}\sqrt{g(x_i^{[N]}(r))}\right|>\epsilon\right)\\
&=\mathbb{P}\left(\sup_{0\leq t\leq L(N)}\left|\frac{1}{1+K_0}\int_{Ns-t}^{Ns}\mathrm{d}w_i(r)\,\frac{1}{N}\sum_{i\in[N]}\sqrt{g(x_i^{[N]}(r))}\right|>\epsilon-\frac{L(N)2K_1e_1}{N(1+K_0)}\right)\\
&\leq\mathbb{P}\left(\sup_{0\leq t\leq L(N)}\left|\frac{1}{1+K_0}\int_{Ns-t}^{Ns}\mathrm{d}w_i(r)\,\frac{1}{N}\sum_{i\in[N]}\sqrt{g(x_i^{[N]}(r))}\right|>\frac{\epsilon}{2}\right).
\end{aligned}\tag{7.42}$$

Applying the same optional stopping argument as used in the proof of Lemma 6.2.15, we find (7.40). For (7.41), note that

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\leq t\leq L(N)}\left|\Theta_{y_1}^{(1),[N]}(Ns)-\Theta_{y_2}^{(1),[N]}(Ns-t)\right|>\epsilon\right)\\
&=\mathbb{P}\left(\sup_{0\leq t\leq L(N)}\frac{1}{1+K_0}\left|\int_{Ns-t}^{Ns}\mathrm{d}r\,\frac{e_1}{N}\left[\Theta_{y_1}^{(1),[N]}(r)-\frac{1}{N}\sum_{i\in[N]}x_i^{[N]}(r)\right]\right|>\epsilon\right) \qquad (7.43)\\
&\leq\mathbb{P}\left(\frac{2e_1L(N)}{(1+K_0)N}>\epsilon\right).
\end{aligned}$$

Let $N \to \infty$ to obtain (7.41). $\square$

Step 3: Equilibrium for the infinite system. To derive the equilibrium of the single components in the infinite system, we derive the following analoque of Proposition 6.2.4. Recall that the finite system is denoted by $Z^{[N_k]}$ in (7.3), and recall the list of ingredients in Section 7.1.

Proposition 7.2.3 (Equilibrium for the infinite 2-colour system). *Let $(N_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{N}$. Fix $s > 0$. Let $L(N)$ satisfy $\lim_{N\to\infty} L(N) = \infty$ and $\lim_{N\to\infty} L(N)/N = 0$, and suppose that*

$$\begin{aligned}
&\lim_{k\to\infty} \mathcal{L}\left[\boldsymbol{\Theta}^{\mathrm{eff},(1),[N_k]}(N_k s)\right] = P_{\boldsymbol{\Theta}^{\mathrm{eff}}(s)},\\
&\lim_{k\to\infty} \mathcal{L}\left[Y_1^{[N_k]}(N_k s)\middle|\boldsymbol{\Theta}^{\mathrm{eff},(1),[N_k]}(N_k s)\right] = P_{Y_1(s)}^{\boldsymbol{\Theta}^{\mathrm{eff},(1)}(s)},\\
&\lim_{k\to\infty} \mathcal{L}\left[\sup_{0\le t\le L(N_k)} \left|\bar{\Theta}^{[N_k]}(N_k s) - \bar{\Theta}^{[N_k]}(N_k s - t)\right| + \left|\Theta_{y_1}^{[N_k]}(N_k s) - \Theta_{y_1}{}^{[N_k]}(N_k s - t)\right|\right]\\
&\qquad\qquad\qquad = \delta_0,\\
&\lim_{k\to\infty} \mathcal{L}\big[Z^{[N_k]}(N_k s)\big] = \nu(s).
\end{aligned} \tag{7.44}$$

Then $\nu(s)$ is of the form

$$\nu(s) = \int_{[0,1]^2} P_{\boldsymbol{\Theta}^{\mathrm{eff}}(s)}(\mathrm{d}\theta, \mathrm{d}\theta_y) \int_{[0,1]^{\mathbb{N}_0}} P_{Y_1(s)}^{(\theta,\theta_y)}(\mathrm{d}\boldsymbol{y}_1)\, \nu_{\theta,\boldsymbol{y}_1}, \tag{7.45}$$

where $\boldsymbol{y}_1 = (y_{i,1})_{i\in\mathbb{N}_0}$ is a sequence with elements in $[0,1]$, and $\nu_{\theta,\boldsymbol{y}_1}$ is the equilibrium measure of the process in (7.10) evolving according to (7.11) with $(y_{i,1})_{i\in\mathbb{N}_0}$ given by the sequence $\boldsymbol{y}_1 =$.

Preparation for the proof of Proposition 7.2.3. The proof of Proposition 7.2.3 follows the same line of argument as used in the proof of Proposition 6.2.4. We need lemmas that are similar to Lemmas 6.2.5-6.2.11, but this time in the setting of the two-colour hierarchical mean-field finite-systems scheme. Afterwards we prove Proposition 7.2.3.

Lemma 7.2.4 (Convergence for the infinite system). *Let μ be an exchangeable probability measure on $([0,1]^3)^{\mathbb{N}_0}$. Then for the system $(Z(t))_{t\ge 0}$ given by (7.10) with $\mathcal{L}[Z(0)] = \mu$,*

$$\lim_{t\to\infty} \mathcal{L}[Z(t)] = \nu_{\theta,\boldsymbol{y}_1}, \tag{7.46}$$

where $\nu_{\theta,\boldsymbol{y}_1}$ is of the form

$$\nu_{\theta,\boldsymbol{y}_1} = \prod_{i\in\mathbb{N}_0} \Gamma_{\theta,y_{i,1}} \tag{7.47}$$

with $\Gamma_{\theta,y_{i,1}}$ the equilibrium of the ith single-component process in (7.11).

Proof. For each component of the infinite system in (7.10) the 1-dormant single component process $(y_{i,1}(t))_{t\ge 0}$ does not move on time scale t. Hence, given the states of 1-dormant single components, we can use a similar argument as in the proof of Proposition 6.1.2 (see Section 6.1.3) to show that the single components converge to an equilibrium measure $\Gamma_{\theta,y_{i,1}}$. Since the single components do not interact, the claim in Lemma 7.2.4 follows. □

The second lemma establishes the continuity of the equilibrium with respect to θ, its center of drift.

Lemma 7.2.5 (Continuity of the equilibrium). *Let $\mathcal{P}(([0,1]^3)^{\mathbb{N}_0})$ denote the space of probability measures on $([0,1]^3)^{\mathbb{N}_0}$. The mapping*

$$\begin{gathered}[0,1]\times[0,1]^{\mathbb{N}_0}\to\mathcal{P}(([0,1]^3)^{\mathbb{N}_0})\\(\theta,\boldsymbol{y}_1)\mapsto\nu_{\theta,\boldsymbol{y}_1}\end{gathered}\tag{7.48}$$

is continuous. Furthermore, if h is a Lipschitz function on $[0,1]$, then also $\mathcal{F}h$ defined by

$$(\mathcal{F}h)(\theta)=\mathbb{E}^{\nu_{\theta,\boldsymbol{y}_1}}[h(\cdot)]=\int_{([0,1]^3)^{\mathbb{N}_0}}\nu_{\theta,\boldsymbol{y}_1}(\mathrm{d}z)\,h(x_0)\tag{7.49}$$

is a Lipschitz function on $[0,1]$, whose values are independent of $\boldsymbol{y}_1$.

Proof. Lemma 7.2.5 follows from the proof of Lemma 7.2.9. □

The third lemma characterises the speed at which the estimators $(\Theta_x^{[N]}(t))_{t\geq 0}$ and $(\Theta_y^{[N]}(t))_{t\geq 0}$ converge to each other when $N\to\infty$ and $t\to\infty$.

Lemma 7.2.6 (Comparison of empirical averages). *Let $(\Theta_x^{(1),[N]}(t))_{t\geq 0}$ and $(\Theta_{y_0}^{(1),[N]}(t))_{t\geq 0}$ be defined as in (7.14). Then*

$$\begin{aligned}\mathbb{E}\left[\left|\Theta_x^{(1),[N]}(t)-\Theta_{y_0}^{(1),[N]}(t)\right|\right]\leq&\sqrt{\mathbb{E}\left[\left(\Theta_x^{(1),[N]}(0)-\Theta_{y_0}^{(1),[N]}(0)\right)^2\right]}\,\mathrm{e}^{-(K_0e_0+e_0)t}\\&+\sqrt{\frac{2}{K_0e_0+e_0}\left[\frac{||g||}{N}+\frac{4K_1e_1}{N}\right]}.\end{aligned}\tag{7.50}$$

Proof. From (7.4) it follows via Itô-calculus that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[\left(\Theta_x^{(1),[N]}(t)-\Theta_{y_0}^{(1),[N]}(t)\right)^2\right]=&-2(K_0e_0+e_0)\,\mathbb{E}\left[\left(\Theta_x^{(1),[N]}(t)-\Theta_{y_0}^{(1),[N]}(t)\right)^2\right]\\&+h^{[N]}(t),\end{aligned}\tag{7.51}$$

where

$$\begin{aligned}h^{[N]}(t)=\mathbb{E}&\left[\frac{2K_1e_1}{N}\left(\Theta_x^{(1),[N]}(t)-\Theta_{y_0}^{(1),[N]}(t)\right)\left[\Theta_{y_1}^{(1),[N]}(t)-\Theta_x^{(1),[N]}(t)\right]\right]\\&+\frac{2}{N^2}\sum_{i\in[N]}\mathbb{E}\big[g\big(x_i^{[N]}(r)\big)\big].\end{aligned}\tag{7.52}$$

Hence

$$\mathbb{E}\left[\left(\Theta_x^{(1),[N]}(t) - \Theta_{y_0}^{(1),[N]}(t)\right)^2\right] = \mathbb{E}\left[\left(\Theta_x^{(1),[N]}(0) - \Theta_{y_0}^{(1),[N]}(0)\right)^2\right] \mathrm{e}^{-2(K_0e_0+e_0)t} + \int_0^t \mathrm{d}r\, \mathrm{e}^{-2(K_0e_0+e_0)(t-r)} h^{[N]}(r). \tag{7.53}$$

Take the square root on both sides and use Jensen's inequality to get (7.50). □

Like for the mean-field system with one colour, we need to compare the finite system in (7.3) with an infinite system. To derive the analogue of Lemma 6.2.9, let $L(N)$ satisfy $\lim_{N\to\infty} L(N) = \infty$ and $\lim_{N\to\infty} L(N)/N = 0$. Define the measure μ_N on $([0,1]^3)^{\mathbb{N}_0}$ by continuing the configuration of

$$Z^{[N]}(Ns - L(N)) = \Big(X^{[N]}(Ns - L(N)), \Big(Y_0^{[N]}(Ns - L(N)), Y_1^{[N]}(Ns - L(N))\Big)\Big) \tag{7.54}$$

periodically to $([0,1]^3)^{\mathbb{N}_0}$. Let

$$\bar{\Theta}^{(1),[N]} = \frac{1}{N} \sum_{i\in[N]} \frac{x_i^{[N]}(Ns - L(N)) + K_0 y_{i,0}^{[N]}(Ns - L(N))}{1+K_0}. \tag{7.55}$$

Let

$$(Z^{\mu_N}(t))_{t\geq 0} = \big(X^{\mu_N}(t), (Y_0^{\mu_N}(t), Y_1^{\mu_N}(t))\big)_{t\geq 0} \tag{7.56}$$

be the infinite system evolving according to

$$\begin{aligned} \mathrm{d}x_i^{\mu_N}(t) &= c_0\,[\bar{\Theta}^{(1),[N]} - x_i^{\mu_N}(t)]\,\mathrm{d}t + \sqrt{g\big(x_i^{\mu_N}(t)\big)}\,\mathrm{d}w_i(t) + K_0e_0\,[y_{i,0}^{\mu_N}(t) - x_i^{\mu_N}(t)]\,\mathrm{d}t,\\ \mathrm{d}y_{i,0}^{\mu_N}(t) &= e_0\,[x_i^{\mu_N}(t) - y_{i,0}^{\mu_N}(t)]\,\mathrm{d}t,\\ y_{i,1}^{\mu_N}(t) &= y_{i,1}^{\mu_N}(0), \qquad i\in\mathbb{N}_0, \end{aligned} \tag{7.57}$$

starting from initial distribution μ_N. Then the following lemma is the equivalent of Lemma 6.2.9 for the two-colour mean-field system.

Lemma 7.2.7 (Comparison of finite and infinite systems). *Fix $s>0$, and let $L(N)$ satisfy $\lim_{N\to\infty} L(N) = \infty$ and $\lim_{N\to\infty} L(N)/N$. Suppose that*

$$\lim_{N\to\infty} \sup_{0\leq t\leq L(N)} \left|\bar{\Theta}^{(1),[N]}(Ns) - \bar{\Theta}^{(1),[N]}(Ns-t)\right| = 0 \quad \text{in probability.} \tag{7.58}$$

Then, for all $t\geq 0$,

$$\lim_{k\to\infty} \left|\mathbb{E}\left[f\big(Z^{\mu_N}(t)\big) - f\big(Z^{[N]}(Ns - L(N) + t)\big)\right]\right| = 0 \qquad \forall\, f\in\mathcal{C}\big(([0,1]^3)^{\mathbb{N}_0},\mathbb{R}\big). \tag{7.59}$$

Proof. We proceed as in the proof of Lemma 6.2.9. We rewrite the SSDE in (7.4) as

$$\begin{aligned}
\mathrm{d}x_i^{[N]}(t) &= c_0 \left[\Theta^{(1),[N]} - x_i^{[N]}(t)\right] \mathrm{d}t \\
&\quad + c_0 \left[\bar{\Theta}^{(1),[N]}(t) - \Theta^{(1),[N]}\right] \mathrm{d}t + c_0 \left[\Theta_x^{(1),[N]}(t) - \bar{\Theta}^{(1),[N]}(t)\right] \mathrm{d}t \\
&\quad + \sqrt{g\big(x_i^{[N]}(t)\big)}\, \mathrm{d}w_i(t) \\
&\quad + K_0 e_0 \left[y_{i,0}^{[N]}(t) - x_i^{[N]}(t)\right]\mathrm{d}t + \frac{K_1 e_1}{N}\left[y_{i,1}^{[N]}(t) - x_i^{[N]}(t)\right]\mathrm{d}t, \\
\mathrm{d}y_{i,0}^{[N]}(t) &= e_0 \left[x_i^{[N]}(t) - y_{i,0}^{[N]}(t)\right] \mathrm{d}t, \\
\mathrm{d}y_{i,1}^{[N]}(t) &= \frac{e_1}{N} \left[x_i^{[N]}(t) - y_{i,1}^{[N]}(t)\right] \mathrm{d}t, \qquad i \in [N].
\end{aligned} \tag{7.60}$$

As before, we consider the finite system in (7.60) as a system on $([0,1]^3)^{\mathbb{N}_0}$ by periodic continuation, and we couple the finite system in (7.60) and the infinite system in (7.59) via there Brownian motions. We denote the coupled process by $\tilde{z}(t) = (\tilde{z}_i(t))_{i\in\mathbb{N}_0} = (\tilde{z}_i^{[N]}(t), \tilde{z}_i^{\mu_N}(t))_{i\in\mathbb{N}_0}$, where $\tilde{z}_i^{[N]}(t) = (\tilde{x}_i^{[N]}(t), \tilde{y}_{i,0}^{[N]}(t), \tilde{y}_{i,1}^{[N]}(t))$ and $\tilde{z}_i^{\mu_N}(t) = (\tilde{x}_i^{\mu_N}(t), \tilde{y}_{i,0}^{\mu_N}(t), \tilde{y}_{i,1}^{\mu_N}(t))$. We define

$$\begin{aligned}
\Delta_{i,0}^{[N]}(t) &= \tilde{x}_i^{[N]}(t) - \tilde{x}_i^{\mu_N}(t), \\
\delta_{i,0}^{[N]}(t) &= \tilde{y}_{i,0}^{[N]}(t) - \tilde{y}_{i,0}^{\mu_N}(t), \\
\delta_{i,1}^{[N]}(t) &= \tilde{y}_{i,1}^{[N]}(t) - \tilde{y}_{i,1}^{\mu_N}(t).
\end{aligned} \tag{7.61}$$

As in the proof of Lemma 6.2.9, we have to show that, for all $t \geq 0$,

$$\lim_{N\to\infty} \mathbb{E}[|\Delta_i^{[N]}(t)|] = 0, \qquad \lim_{N\to\infty} \mathbb{E}[|\delta_{i,0}^{[N]}(t)|] = 0, \qquad \lim_{N\to\infty} \mathbb{E}[|\delta_{i,1}^{[N]}(t)|] = 0. \tag{7.62}$$

To prove the third limit in (7.63), note that, by (7.57), (7.60) and the choice of the initial measure in the coupling,

$$y_{i,1}^{[N]}(t) = y_{i,1}^{[N]}(0) + \frac{e_1}{N}\int_0^t \mathrm{d}r\, \left[x_i^{[N]}(r) - y_{i,1}^{[N]}(r)\right] = y_{i,1}^{\mu_N}(t) + \frac{e_1}{N}\int_0^t \mathrm{d}r\, \left[x_i^{[N]}(r) - y_{i,1}^{[N]}(r)\right]. \tag{7.63}$$

Hence

$$\lim_{N\to\infty} \mathbb{E}[|\delta_{i,1}^{[N]}(L(N))|] = 0. \tag{7.64}$$

To prove the first two limits in (7.63), we argue as in the proof of Lemma 6.2.9, but we need to add extra drift terms towards the first seed-bank. Using Itô-calculus, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[|\Delta_i^{[N]}(t)| + K|\delta_{i,0}^{[N]}(t)|] \\
&= -c\,\mathbb{E}[\Delta_i^{[N]}(t)] \\
&\quad - 2K_0 e_0\, \mathbb{E}\left[[|\Delta_i^{[N]}(t)| + |\delta_i^{[N]}(t)|]\, 1_{\{\operatorname{sgn} \Delta_i^{[N]}(t) \neq \operatorname{sgn} \delta_{i,0}^{[N]}(t)\}}\right] \\
&\quad + c \operatorname{sgn} \Delta_i^{[N]}(t)\left[\bar{\Theta}^{(1),[N]}(t) - \bar{\Theta}^{(1),[N]}\right] \\
&\quad + c \operatorname{sgn} \Delta_i^{[N]}(t)\left[\bar{\Theta}_x^{(1),[N]}(t) - \bar{\Theta}^{(1),[N]}(t)\right] \\
&\quad + \frac{K_1 e_1}{N} \operatorname{sgn} \Delta_i^{[N]}(t)\left[\delta_{i,1}^{[N]}(t) - \Delta_i^{[N]}(t)\right].
\end{aligned} \tag{7.65}$$

CHAPTER 7

This can be rewritten as

$$
\begin{aligned}
0 &\le \mathbb{E}[|\Delta_i^{[N]}(t)| + K_0|\delta_{i,0}^{[N]}(t)|] \\
&\le \mathbb{E}[|\Delta_i^{[N]}(0)| + K|\delta_{i,0}^{[N]}(0)|] - c\int_0^t \mathrm{d}r\, \mathbb{E}[\Delta_i^{[N]}(r)] \\
&\quad - 2K_0 e_0 \int_0^t \mathrm{d}r\, \mathbb{E}\left[[|\Delta_i^{[N]}(r)| + |\delta_{i,0}^{[N]}(r)|]\, 1_{\{\operatorname{sgn}\Delta_i^{[N]}(t) \neq \operatorname{sgn}\delta_{i,0}^{[N]}(t)\}}\right] \\
&\quad + c\int_0^t \mathrm{d}r\, |\bar{\Theta}^{(1),[N]}(r) - \Theta^{(1),[N]}| \\
&\quad + c\int_0^t \mathrm{d}r\, |\bar{\Theta}_x^{(1),[N]}(r) - \bar{\Theta}^{(1),[N]}(r)| \\
&\quad + \frac{K_1 e_1}{N}\int_0^t \mathrm{d}r\, \left|\delta_{i,1}^{[N]}(r) - \Delta_i^{[N]}(r)\right|.
\end{aligned}
\tag{7.66}
$$

By the construction of the measure μ_N, we have

$$
\lim_{N\to\infty} \mathbb{E}[|\Delta_i^{[N]}(0)| + K_0|\delta_i^{[N]}(0)|] = 0. \tag{7.67}
$$

Therefore, for all $t \ge 0$,

$$
\lim_{N\to\infty} \mathbb{E}[|\Delta_i^{[N]}(t)| + K_0|\delta_{i,0}^{[N]}(t)|] = 0. \tag{7.68}
$$

Combine this with (7.64) and use that Lipschitz functions are dense in the set of bounded continuous functions. Then, as in the proof of Lemma 6.2.9, we get the claim in (7.59). □

Before we can prove that the infinite system $(X^{\mu_N}(t), Y_0^{\mu_N}(t), Y_1^{\mu_N}(t))_{t\ge 0}$ converges to a limiting system as $N \to \infty$, we need the following regularity property for the estimators $(\bar{\Theta}^{[N]}, \Theta_{y_1}^{[N]})$.

Lemma 7.2.8 (Stability of the estimator for the conserved quantity). *Define μ_N as in Lemma 7.2.7. Let $(x_i^N, y_{i,0}^N, y_{i,1}^N)_{i\in[N]}$ be distributed according to the exchangeable probability measure μ_N on $([0,1]^3)^{\mathbb{N}_0}$ restricted to $([0,1]^3)^{[N]}$. Suppose that $\lim_{N\to\infty}\mu_N = \mu$ for some exchangeable probability measure μ on $([0,1]^3)^{\mathbb{N}_0}$. Define the random variable ϕ on $(\mu, ([0,1]^3)^{\mathbb{N}_0})$ by putting*

$$
\begin{aligned}
&\phi = (\phi_1, \phi_2), \\
&\phi_1 = \lim_{n\to\infty} \frac{1}{n}\sum_{i\in[n]} \frac{x_i + K y_{i,0}}{1+K}, \qquad \phi_2 = \lim_{n\to\infty}\frac{1}{n}\sum_{i\in[n]} y_{i,1},
\end{aligned}
\tag{7.69}
$$

and the random variable $\phi^{[N]}$ on $(\mu_N, ([0,1]^3)^{\mathbb{N}_0})$ by putting

$$
\begin{aligned}
&\phi^{[N]} = (\phi_1^{[N]}, \phi_2^{[N]}) \\
&\phi_1^{[N]} = \frac{1}{N}\sum_{i\in[N]} \frac{x_i^N + K y_{i,0}^N}{1+K}, \qquad \phi_2^{[N]} = \frac{1}{N}\sum_{i\in[N]} y_{i,1}^N.
\end{aligned}
\tag{7.70}
$$

Then

$$
\lim_{N\to\infty} \mathcal{L}[\phi^{[N]}] = \mathcal{L}[\phi]. \tag{7.71}
$$

Proof. We can use a similar argument as in the proof of Lemma 6.2.10. Define

$$D^{[N]}(Z) = \left(\frac{1}{N} \sum_{j \in [N]} \frac{x_j + K_0 y_{j,0}}{1+K_0}, \frac{1}{N} \sum_{j \in [N]} y_{i,1} \right). \tag{7.72}$$

Then we can proceed as in the proof of Lemma 6.2.10, using Fourier analysis for both components of $D^{[N]}(Z)$ separately. □

In the fifth and final lemma we state the convergence of $\mathcal{L}[(X^{\mu_N}(t), Y_0^{\mu_N}(t), Y_1^{\mu_N}(t))]$ to the law of a limiting system as $N \to \infty$.

Lemma 7.2.9 (Uniformity of the ergodic theorem for the infinite system). *Let μ_N be defined as in (7.56). Since $(\mu_N)_{N\in\mathbb{N}}$ is tight, it has convergent subsequences. Let $(N_k)_{k\in\mathbb{N}}$ be a subsequence such that $\mu = \lim_{k\to\infty} \mu_{N_k}$. Define*

$$\Theta = \lim_{N\to\infty} \frac{1}{N} \sum_{i\in[N]} \frac{x_i^\mu + K y_{i,0}^\mu}{1+K} \qquad \text{in } L_2(\mu). \tag{7.73}$$

Let $Z^\mu(t) = \big(X^\mu(t), Y_0^\mu(t), Y_1^\mu(t)\big)_{t\geq 0}$ be the infinite system evolving according to

$$\begin{aligned}
\mathrm{d}x_i^\mu(t) &= c\,[\Theta - x_i^\mu(t)]\,\mathrm{d}t + \sqrt{g(x_i^\mu(t))}\,\mathrm{d}w_i(t) + Ke\,[y_{i,1}^\mu(t) - x_i^\mu(t)]\,\mathrm{d}t,\\
\mathrm{d}y_{i,0}^\mu(t) &= e\,[x_i^\mu(t) - y_{i,1}^\mu(t)]\,\mathrm{d}t,\\
\mathrm{d}y_{i,1}^\mu(t) &= y_{i,1}^\mu(0), \qquad i \in \mathbb{N}_0.
\end{aligned} \tag{7.74}$$

and let $Z^{\mu_{N_k}}(t) = (X^{\mu_{N_k}}(t), Y_0^{\mu_{N_k}}(t), Y_1^{\mu_{N_k}}(t))_{t\geq 0}$ be the infinite system defined in (7.56). Then

(a) *For all $t \geq 0$,*

$$\lim_{k\to\infty} \big|\mathbb{E}\big[f\big(Z^{\mu_{N_k}}(t)\big)\big] - \mathbb{E}\big[f\big(Z^\mu(t)\big)\big]\big| = 0 \qquad \forall f \in \mathcal{C}\big(([0,1]^2)^{\mathbb{N}_0}, \mathbb{R}\big). \tag{7.75}$$

(b) *There exists a sequence $\bar{L}(N)$ satisfying $\lim_{N\to\infty} \bar{L}(N) = \infty$ and $\lim_{N\to\infty} \bar{L}(N)/N = 0$ such that*

$$\begin{aligned}
\lim_{k\to\infty} &\big|\mathbb{E}\big[f\big(Z^{[N_k]}(N_k s - L(N_k) + \bar{L}(N_k))\big) - f\big(Z^{\mu_{N_k}}(\bar{L}(N_k))\big)\big|\big]\\
&+ \big|\mathbb{E}\big[f\big(Z^{\mu_{N_k}}(\bar{L}(N_k))\big)\big] - \mathbb{E}\big[f\big(Z^\mu(\bar{L}(N_k))\big)\big]\big| = 0 \qquad \forall f \in \mathcal{C}\big(([0,1]^2)^{\mathbb{N}_0}, \mathbb{R}\big).
\end{aligned} \tag{7.76}$$

Proof. As in the proof of Lemma 6.2.11, we can construct $(z_i^{\mu_N})_{i\in\mathbb{N}_0}$ and $(z_i^\mu)_{i\in\mathbb{N}_0}$ on one probability space. Then

$$\lim_{N\to\infty} y_{i,1}^{\mu_N}(0) = y_{i,1}^\mu(0) \text{ a.s.} \tag{7.77}$$

and

$$\lim_{N\to\infty} \mathbb{E}[|\bar{\Theta}^{[N]} - \Theta|] = 0. \tag{7.78}$$

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Via a similar coupling as in Lemma (7.2.7), it follows via Itô-calculus that (7.75) holds. Combining (7.64), (7.68), (7.77) and (7.78), we obtain, via a similar construction as in the proof of Lemma 6.2.11, a sequence $\bar{L}(N)$ such that

$$\begin{aligned}&\lim_{N\to\infty} \mathbb{E}[|\Delta_i^N(\bar{L}(N))| + K_0|\delta_{i,0}^N(\bar{L}(N))|] + K_1|\delta_{i,1}^N(\bar{L}(N))|] \\ &+ \mathbb{E}[|\Delta_i^{\mu_N}(\bar{L}(N))| + K_0|\delta_{i,0}^{\mu_N}(\bar{L}(N))|] + K_1|\delta_{i,1}^{\mu_N}(\bar{L}(N))|] = 0.\end{aligned} \tag{7.79}$$

As in the proof of Lemma 6.2.11, we can again use Lipschitz functions to conclude (7.76). □

Lemma 7.2.10 (Coupling of finite systems). *Let*

$$Z^{[N],1} = (X^{[N],1}, Y_0^{[N],1}, Y_1^{[N],1}) \tag{7.80}$$

be the finite system evolving according to (7.4) *starting from an exchangeable initial measure. Let* $\mu^{[N],1}$ *be the measure obtained by periodic continuation of the configuration of* $Z^{[N],1}(0)$*. Similarly, let*

$$Z^{[N],2} = (X^{[N],2}, Y_0^{[N],2}, Y_1^{[N],2}) \tag{7.81}$$

be the finite system evolving according to (7.4) *starting from an exchangeable initial measure. Let* $\mu^{[N],2}$ *be the measure obtained by periodic continuation of the configuration of* $Z^{[N],2}(0)$*. Let* $\tilde{\mu}$ *be any weak limit point of the sequence of measures* $(\mu^{[N],1} \times \mu^{[N],2})_{N\in\mathbb{N}}$*. Define the random variables* $\bar{\Theta}^{[N],1}$ *and* $\bar{\Theta}^{[N],2}$ *on* $(([0,1]^3)^{\mathbb{N}_0} \times ([0,1]^3)^{\mathbb{N}_0}, \mu^{[N],1} \times \mu^{[N],2})$ *and* $\bar{\Theta}_1$ *and* $\bar{\Theta}_2$ *on* $(([0,1]^3)^{\mathbb{N}_0} \times ([0,1]^3)^{\mathbb{N}_0}, \bar{\mu})$ *by*

$$\begin{aligned}&\bar{\Theta}^{[N],1} = \frac{1}{N}\sum_{i\in[N]} \frac{x_i^{[N],1} + K_0 y_{i,0}^{[N],1}}{1+K_0}, \qquad \bar{\Theta}^{[N],2} = \frac{1}{N}\sum_{i\in[N]} \frac{x_i^{[N],2} + K_0 y_{i,0}^{[N],2}}{1+K_0}, \\ &\bar{\Theta}^1 = \lim_{n\to\infty}\frac{1}{n}\sum_{i\in[n]} \frac{x_i^1 + K_0 y_{i,0}^1}{1+K_0}, \qquad \bar{\Theta}^2 = \lim_{n\to\infty}\frac{1}{n}\sum_{i\in[n]} \frac{x_i^2 + K_0 y_{i,0}^2}{1+K_0},\end{aligned} \tag{7.82}$$

and let $(\bar{\Theta}^{(1),[N],1}(t))_{t\geq 0}$ *and* $(\bar{\Theta}^{(1),[N],2}(t))_{t\geq 0}$ *be defined as in* (7.14) *for* $Z^{[N],1}$*, respectively,* $Z^{[N],2}$*. Suppose that*

$$\lim_{N\to\infty}\sup_{0\leq t\leq L(N)} \left(\left|\bar{\Theta}^{[N],k}(0) - \bar{\Theta}^{[N],k}(t)\right|\right) = 0 \quad \textit{in probability}, \quad k\in\{1,2\}, \tag{7.83}$$

and suppose that $\tilde{\mu}(\{\bar{\Theta}_1 = \bar{\Theta}_2,\, Y_1^1 = Y_1^2\}) = 1$*. Then, for any* $t(N)\to\infty$*,*

$$\begin{aligned}&\lim_{N\to\infty} \mathbb{E}\big[|x_i^{[N],1}(t(N)) - x_i^{[N],2}(t(N))| + K_0|y_{i,0}^{[N],1}(t(N)) - y_{i,0}^{[N],2}(t(N))| \\ &\quad + K_1|y_{i,1}^{[N],1}(t(N)) - y_{i,1}^{[N],2}(t(N))|\big] = 0.\end{aligned} \tag{7.84}$$

Proof. Via standard Itô-calculus we obtain from (7.4) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[|x_i^{[N],1}(t)-x_i^{[N],2}(t)|+K_0|y_{i,0}^{[N],1}(t)-y_{i,0}^{[N],2}(t)|+K_1|y_{i,1}^{[N],1}(t)-y_{i,1}^{[N],2}(t)|\right]\\
&=-\frac{2c}{N}\sum_{j\in[N]}\mathbb{E}\left[|x_j^{[N],1}(t)-x_j^{[N],2}(t)|1_{\{\mathrm{sgn}\,(x_j^{[N],1}(t)-x_j^{[N],2}(t))\neq\mathrm{sgn}\,(x_i^{[N],1}(t)-x_i^{[N],2}(t))\}}\right]\\
&\qquad-2K_0e_0\,\mathbb{E}\Bigg[|x_i^{[N],1}(t)-x_i^{[N],2}(t)|+K|y_{i,0}^{[N],1}(t)-y_{i,0}^{[N],2}(t)|\\
&\qquad\qquad\times 1_{\{\mathrm{sgn}\,(x_i^{[N],1}(t)-x_i^{[N],2}(t))\neq\mathrm{sgn}\,(y_{i,0}^{[N],1}(t)-y_{i,0}^{[N],2}(t))\}}\Bigg]\\
&\qquad-2\frac{K_1e_1}{N}\,\mathbb{E}\Bigg[|x_i^{[N],1}(t)-x_i^{[N],2}(t)|+K_1|y_{i,0}^{[N],1}(t)-y_{i,0}^{[N],2}(t)|\\
&\qquad\qquad\times 1_{\{\mathrm{sgn}\,(x_i^{[N],1}(t)-x_i^{[N],2}(t))\neq\mathrm{sgn}\,(y_{i,1}^{[N],1}(t)-y_{i,1}^{[N],2}(t))\}}\Bigg].
\end{aligned}\tag{7.85}$$

Therefore, for all $N\in\mathbb{N}$,

$$t\mapsto\mathbb{E}\left[|x_i^{[N],1}(t)-x_i^{[N],2}(t)|+K_0|y_{i,0}^{[N],1}(t)-y_{i,0}^{[N],2}(t)|+K_1|y_{i,1}^{[N],1}(t)-y_{i,1}^{[N],2}(t)|\right]\tag{7.86}$$

is a decreasing function. Hence we can use the same strategy as in the proof of Lemma 6.2.13 to finish the proof. □

- **Proof of Proposition 7.2.3**

Proof. We follow a similar argument as in the proof of Proposition 6.2.4. Let $L(N)$ satisfy $\lim_{N\to\infty} L(N)=\infty$ and $\lim_{N\to\infty} L(N)/N=0$. Let μ_N be the measure on $([0,1]^3)^{\mathbb{N}_0}$ obtained by periodic continuation of $\mathcal{L}[Z^{[N]}(Ns-L(N))]$. Note that $([0,1]^3)^{\mathbb{N}_0}$ is compact. Hence, letting $(N_k)_{k\in\mathbb{N}}$ be the subsequence in Proposition 7.2.3, we can pass to a possibly further subsequence and obtain

$$\lim_{k\to\infty}\mu_{N_k}=\mu.\tag{7.87}$$

Since we assumed that $\mathcal{L}[Z^{[N]}(0)]$ is exchangeable and the dynamics preserve exchangeability, the measures μ_{N_k} are translation invariant and also the limiting law μ is translation invariant.

Let $\phi=(\phi_1,\phi_2)$ be defined as in (7.69) in Lemma 7.2.8. Then we can condition on $\phi=(\phi_1,\phi_2)$ and write

$$\mu=\int_{[0,1]^2}\mu_\rho\,\mathrm{d}\Lambda(\rho),\tag{7.88}$$

where $\Lambda(\cdot)=\mathcal{L}[\phi]=\mathcal{L}[(\phi_1,\phi_2)]$ and $\rho=(\rho_1,\rho_2)$. By assumption we know that

$$\lim_{k\to\infty}\mathcal{L}\left[\mathbf{\Theta}^{\mathrm{eff},(1),[N_k]}(N_ks)\right]=P_{\mathbf{\Theta}^{\mathrm{eff}}(s)}(\cdot)\tag{7.89}$$

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and

$$\lim_{k\to\infty}\mathcal{L}\left[\sup_{0\le t\le L(N_k)}\left|\bar{\Theta}^{[N_k]}(N_k s)-\bar{\Theta}^{[N_k]}(N_k s-t)\right|+\left|\Theta_{y_1}^{[N_k]}(N_k s)-\Theta_{y_1}{}^{[N_k]}(N_k s-t)\right|\right]$$
$$=\delta_0. \tag{7.90}$$

Hence

$$\lim_{k\to\infty}\mathcal{L}\left[\boldsymbol{\Theta}^{\mathrm{eff},(1),[N_k]}(N_k s-L(N_k))\right]=P_{\boldsymbol{\Theta}^{\mathrm{eff}}(s)}(\cdot). \tag{7.91}$$

Recall that

$$\Lambda(\cdot)=\mathcal{L}\left[\lim_{n\to\infty}\left(\frac{1}{n}\sum_{i\in[n]}\frac{x_i+Ky_{i,0}}{1+K},\frac{1}{n}\sum_{i\in[n]}y_{i,1}\right)\right]\quad\text{on }(\mu,([0,1]^2)^{\mathbb{N}_0}). \tag{7.92}$$

By Lemma 6.2.10, if

$$\phi^{N_k}=(\phi_1^{N_k},\phi_2^{N_k})=\left(\frac{1}{N_k}\sum_{i\in[N_k]}\frac{x_i+Ky_{i,0}}{1+K},\frac{1}{N_k}\sum_{i\in[N_k]}y_{i,1}^{[N_k]}\right)\text{ on }(\mu_{N_k},([0,1]^3)^{\mathbb{N}_0}), \tag{7.93}$$

then $\lim_{k\to\infty}\mathcal{L}[\phi^{N_k}]=\mathcal{L}[\phi]$. Taking the subsequence $(\mu_{N_k})_{k\in\mathbb{N}}$, we get $\Lambda(\cdot)=P_{\boldsymbol{\Theta}^{\mathrm{eff}}(s)}(\cdot)$, and hence

$$\mu=\int_{[0,1]}\mu_\rho\,\mathrm{d}P_s(\rho). \tag{7.94}$$

Let $\bar{L}(N)$ be the sequence constructed in Lemma 7.2.9[b]. By construction we can require that $\bar{L}(N)\le L(N)$ for all $N\in\mathbb{N}$. Write

$$\begin{aligned}&\mathcal{L}\big[Z^{[N_k]}(N_k s-L(N_k)+\bar{L}(N_k))\big]\\&=\mathcal{L}\big[Z^{[N_k]}(N_k s-L(N_k)+\bar{L}(N_k))\big]-\mathcal{L}\big[Z^{\mu_{N_k}}(\bar{L}(N_k))\big],\\&\quad+\mathcal{L}\big[Z^{\mu_{N_k}}(\bar{L}(N_k))\big]-\mathcal{L}\big[Z^{\mu}(\bar{L}(N_k))\big]\\&\quad+\mathcal{L}\big[Z^{\mu}(\bar{L}(N_k))\big].\end{aligned} \tag{7.95}$$

By Lemma 7.2.9 the first and second differences tend to zero as $k\to\infty$. Hence

$$\lim_{k\to\infty}\mathcal{L}\big[Z^{[N_k]}(N_k s-L(N_k)+\bar{L}(N_k))\big]=\mathcal{L}\big[Z^{\mu}(\bar{L}(N_k))\big]. \tag{7.96}$$

By (7.88),

$$\mathcal{L}\big[Z^{\mu}(\bar{L}(N_k))\big]=\int_{[0,1]^2}\mathcal{L}\big[Z^{\mu_\rho}(\bar{L}(N_k))\big]\,P_{\boldsymbol{\Theta}^{\mathrm{eff}}(s)}(\mathrm{d}\rho). \tag{7.97}$$

For the infinite system $(Z^{\mu_\rho}(t))_{t\ge0}=\big(X^{\mu_\rho}(t),Y_0^{\mu_\rho}(t),Y_1^{\mu_\rho}(t)\big)_{t\ge0}$ we have

$$Y_1^{\mu}(t)=Y_1^{\mu}(0)\;a.s. \tag{7.98}$$

and hence, since $\lim_{k\to\infty}\bar{L}(N_k)/N_k=0$ by (7.44),

$$\lim_{k\to\infty}\mathcal{L}[Y_1^{\mu_\rho}(\bar{L}(N_k))]=\mathcal{L}[Y_1^{\mu_\rho}(0)]\;\forall\,\rho\in[0,1]. \tag{7.99}$$

Therefore

$$\lim_{k\to\infty} \mathcal{L}[Y_1^{\mu_\rho}(\bar{L}(N_k))] = P^{\rho}_{Y_1(s)}(\cdot) \tag{7.100}$$

and

$$\begin{aligned} &\mathcal{L}\big[X^{\mu_\rho}(\bar{L}(N_k)), Y_0^{\mu_\rho}(\bar{L}(N_k)), Y_1^{\mu_\rho}(\bar{L}(N_k))\big] \\ &\quad = \int \mathcal{L}\big[X_1^{\mu_\rho}(\bar{L}(N_k)), Y_0^{\mu_\rho}(\bar{L}(N_k)), \mathbf{y}_1\big] \mathrm{d}P^{\rho}_{Y_1(s)}(\mathrm{d}\mathbf{y}_1). \end{aligned} \tag{7.101}$$

Hence, since $\lim_{k\to\infty} \bar{L}(N_k) = \infty$, by Lemma 6.2.5 we have

$$\begin{aligned} \lim_{k\to\infty} \mathcal{L}\big[Z^{\mu_\rho}(\bar{L}(N_k))\big] &= \lim_{k\to\infty} \mathcal{L}\big[X_1^{\mu_\rho}(\bar{L}(N_k)), Y_0^{\mu_\rho}(\bar{L}(N_k)), Y_1^{\mu_\rho}(\bar{L}(N_k))\big] \\ &= \int \nu_{\rho,\mathbf{y}_1} P^{\rho}_{Y_1(s)}(\mathrm{d}\mathbf{y}_1). \end{aligned} \tag{7.102}$$

Therefore, by (6.109), (7.97) and Lemma 6.2.6,

$$\lim_{k\to\infty} \mathcal{L}\big[Z^{[N_k]}(N_k s - L(N_k) + \bar{L}(N_k))\big] = \int_{[0,1]} P_{\mathbf{\Theta}^{\mathrm{eff}}(s)}(\mathrm{d}\rho) \int \nu_{\rho,\mathbf{y}_1} P^{\rho}_{Y_1(s)}(\mathrm{d}\mathbf{y}_1). \tag{7.103}$$

To finish the proof, we proceed as in the proof of Proposition 6.2.4 and invoke Lemma 7.2.10. Let $Z^{[N],1} = (X^{[N],1}, Y_0^{[N],1}, Y_1^{[N],1})$ be the finite system starting from

$$\mathcal{L}\big[Z^{[N]}(Ns - L(N))\big] = \mathcal{L}\big[X^{[N]}(Ns - L(N)), Y_0^{[N]}(Ns - L(N)), Y_1^{[N]}(Ns - L(N))\big]. \tag{7.104}$$

Let $(\bar{L}(N))_{N\in\mathbb{N}}$ be the sequence constructed in Lemma 7.2.9. Let $Z^{[N],2} = (X^{[N],2}, Y_0^{[N],2}, Y_1^{[N],2})$ be the finite system starting from

$$\mathcal{L}\big[X^{[N]}(Ns - \bar{L}(N))\big] = \mathcal{L}\left[X^{[N]}(Ns - \bar{L}(N)), Y_0^{[N]}(Ns - \bar{L}(N)), Y_1^{[N]}(Ns - \bar{L}(N))\right]. \tag{7.105}$$

Choose for $t(N)$ in Lemma 7.2.10 the sequence $\bar{L}(N)$. Let $\mu^{[N],1}$ be defined by the periodic continuation of the configuration of $Z^{[N]}(Ns - L(N))$ and $\mu^{[N],2}$ be defined by periodic continuation of the configuration of $Z^{[N]}(Ns - \bar{L}(N))$. Define Θ_1 and Θ_2 according to (6.85), where under $\mu^{[N],2}$ we replace $L(N)$ by $\bar{L}(N)$. Then, by the assumptions in (7.44),

$$\begin{aligned} \lim_{k\to\infty} |\Theta^{(1),[N_k],1} - \Theta^{(1),[N_k],2}| &= \lim_{k\to\infty} |\Theta^{N_k}(N_k s - L(N_k)) - \Theta^{N_k}(N_k s - \bar{L}(N_k))| \\ &= 0 \text{ in probability.} \end{aligned} \tag{7.106}$$

Using 7.63 we see that also, for all $i \in [N]$,

$$\begin{aligned} \lim_{k\to\infty} |y_{i,1}^{[N_k],1}(0) - y_{i,1}^{[N_k],2}(0)| &= \lim_{k\to\infty} |y_{i,1}^{[N_k]}(N_k s - L(N_k)) - y_{i,1}^{[N_k]}(N_k s - \bar{L}(N_k))| \\ &= 0 \text{ in probability.} \end{aligned} \tag{7.107}$$

Therefore, if μ is any weak limit point of the sequence $\big(\mu^{[N_k],1} \times \mu^{[N_k],2}\big)_{k\in\mathbb{N}}$, then

$$\mu(\{\Theta_1 = \Theta_2,\, Y_1^1 = Y_1^2\}) = 1. \tag{7.108}$$

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Hence, by possibly passing to a further subsequence, we can now apply Lemma 7.2.10 to obtain, for all i,

$$\begin{aligned}\lim_{k\to\infty} \mathbb{E}\Big[|x_i^{[N_k],1}(\bar L(N_k)) - x_i^{[N_k],2}(\bar L(N_k))| \\ + K_0\,|y_{i,0}^{[N_k],1}(\bar L(N_k)) - y_{i,0}^{[N_k],2}(\bar L(N_k))| \\ + K_1\,|y_{i,1}^{[N_k],1}(\bar L(N_k)) - y_{i,1}^{[N_k],2}(\bar L(N_k))|\Big] = 0.\end{aligned} \tag{7.109}$$

Hence

$$\lim_{N\to\infty} \Big(\mathcal{L}[Z^{[N],1}(\bar L(N_k))] - \mathcal{L}[Z^{[N],2}(\bar L(N_k))]\Big) = \delta_0 \tag{7.110}$$

and therefore

$$\lim_{k\to\infty} \mathcal{L}\big(Z^{[N_k]}(N_k s)\big) = \int_{[0,1]} P_{\mathbf{\Theta}^{\mathrm{eff}}(s)}(\mathrm{d}\rho) \int \nu_{\rho,\mathbf{y}_1} P^{\rho}_{Y_1(s)}(\mathrm{d}\mathbf{y}_1). \tag{7.111}$$

This concludes the proof of Proposition 7.2.3. □

Like for the one-colour mean-field system, Proposition 7.2.3 and Lemmas 7.2.4–7.2.10 give rise to the following corollary, which will be important to derive the evolution of the 1-blocks on time scale Ns.

Corollary 7.2.11. *Fix $s > 0$. Let μ_N be the measure obtained by periodic continuation of*

$$Z^{[N]}(Ns - L(N)) = (X^{[N]}(Ns - L(N)), Y_0^{[N]}(Ns - L(N)), Y_1^{[N]}(Ns - L(N))), \tag{7.112}$$

and let μ be a weak limit point of the sequence $(\mu_N)_{N\in\mathbb{N}}$. Let

$$\Theta = \lim_{N\to\infty} \frac{1}{N} \sum_{i\in[N]} \frac{x_i^{\mu} + K y_i^{\mu}}{1+K} \qquad \text{in } L_2(\mu), \tag{7.113}$$

and let $(Z^{\nu_\Theta}(t))_{t>0} = (X^{\nu_\Theta}(t), Y_0^{\nu_\Theta}(t), Y_1^{\nu_\Theta}(t))_{t>0}$ be the infinite system evolving according to (7.74) starting from its equilibrium measure. Consider the finite system $Z^{[N]}$ as a system on $([0,1]^3)^{\mathbb{N}_0}$ by periodic continuation. Construct $(Z^{[N]}(t))_{t>0}$ and $(Z^{\nu_\Theta}(t))_{t>0}$ on one probability space. Then, for all $t \geq 0$,

$$\begin{aligned}\lim_{N\to\infty} \mathbb{E}\left[\left|x_i^{[N]}(Ns+t) - x_i^{\nu_\Theta}(t)\right|\right] + K_0\,\mathbb{E}\left[\left|y_{i,0}^{[N]}(Ns+t) - y_{i,0}^{\nu_\Theta}(t)\right|\right] \\ + K_1\,\mathbb{E}\left[\left|y_{i,1}^{[N]}(Ns+t) - y_{i,1}^{\nu_\Theta}(t)\right|\right] = 0 \qquad \forall\, i \in [N].\end{aligned} \tag{7.114}$$

Proof. Proceed as in the proof of Corollary 6.3.1, but use the setup of the two-colour mean-field system and therefore replace Proposition 6.2.4, Lemma 6.2.11 and Lemma 6.2.13 by, respectively Proposition 7.2.3, Lemma 7.2.9 and Lemma 7.2.10. □

Step 4: Limiting evolution of the 1-blocks.

Lemma 7.2.12 (Limiting evolution of the 1-blocks). *Let $(z_1^{\rm eff}(s))_{s>0}$ be the process defined in* (7.21) *with initial state*

$$z_1^{\rm eff}(0) = (\vartheta_0, \theta_{y_1}). \tag{7.115}$$

Then

$$\lim_{N\to\infty} \mathcal{L}\left[\left(\boldsymbol{\Theta}^{{\rm eff},(1),[N]}(Ns)\right)_{s>0}\right] = \mathcal{L}\left[(z_1^{\rm eff}(s))_{s>0}\right]. \tag{7.116}$$

Proof. By [72], the SSDE in (7.21) has a unique strong solution. Therefore the process $(z_1^{\rm eff}(s))_{s>0}$ is Markov. Its generator G is given by

$$G = \frac{K_1 e_1}{1+K_0}\,(y-x)\,\frac{\partial}{\partial x} + e_1(x-y)\,\frac{\partial}{\partial y} + \frac{1}{(1+K_0)^2}\,(\mathcal{F}g)(x)\,\frac{\partial^2}{\partial x^2}, \tag{7.117}$$

and hence $(z_1^{\rm eff}(s))_{s\geq 0}$ solves the martingale problem for G. We will use [49, Theorem 3.3.1], to prove that (7.116) holds.

Define

$$(\vartheta_0^N, \vartheta_{y_1}^N) = \left(\bar{\Theta}^{(1),[N]}(0), \Theta_{y_1}^{(1),[N]}(0)\right). \tag{7.118}$$

Since we start from an i.i.d. law, by the law of large numbers we have that

$$\lim_{N\to\infty} \boldsymbol{\Theta}^{{\rm eff},(1),[N]}(0) = \lim_{N\to\infty} (\vartheta_0^N, \vartheta_{y_1}^N) = (\vartheta_0, \theta_{y_1}) \qquad a.s. \tag{7.119}$$

By the SSDE in (7.37) and an optional sampling argument, we have, for all $N \in \mathbb{N}$,

$$\lim_{s\downarrow 0} \left(\bar{\Theta}^{(1),[N]}(Ns), \Theta^{(1),[N]}(Ns)\right) = (\vartheta_0^N, \vartheta_{y_1}^N) \quad \text{a.s.} \tag{7.120}$$

Therefore we can continuously extend the process $(\boldsymbol{\Theta}^{{\rm eff},(1),[N]}(Ns))_{s>0}$ to 0 and, in particular,

$$\lim_{N\to\infty} \mathcal{L}\left[\boldsymbol{\Theta}^{{\rm eff},(1),[N]}(0)\right] = \mathcal{L}\left[z_1^{\rm eff}(0)\right]. \tag{7.121}$$

Since we already showed that the processes

$$\left(\boldsymbol{\Theta}^{{\rm eff},(1),[N]}(Ns)\right)_{s>0} \tag{7.122}$$

are $\mathcal{D}$-semimartingales, and are trivially bounded, we are left to show that

$$\lim_{N\to\infty} \int_0^s {\rm d}r\, \mathbb{E}\left[\left|G_\dagger^{(1),[N]}\big(f, \boldsymbol{\Theta}^{{\rm eff},(1),[N]}(Nr), r, \cdot\big) - (Gf)\big(\boldsymbol{\Theta}^{{\rm eff},(1),[N]}(Nr)\big)\right|\right] = 0. \tag{7.123}$$

Here, $G_\dagger^{(1),[N]}$ is the operator defined in (7.39). Since we are working on the space $\mathcal{C}^*$ of polynomials on $[0,1]^2$, all derivatives of $f \in \mathcal{C}^*$ are bounded. Hence, by dominated convergence, it is enough to prove that, for all $s > 0$,

$$\lim_{N\to\infty} \mathbb{E}^{[N]}\left[\left|G_\dagger^{(1),[N]}\big(f, \boldsymbol{\Theta}^{{\rm eff},(1),[N]}(Ns), s, \cdot\big) - (Gf)\big(\boldsymbol{\Theta}^{{\rm eff},(1),[N]}(Ns)\big)\right|\right] = 0. \tag{7.124}$$

Note that

$$\begin{aligned}
&\mathbb{E}\Big[\Big|G_{\dagger}^{(1),[N]}\big(f,\boldsymbol{\Theta}^{\mathrm{eff},(1),[N]}(Ns),s,\cdot\big)-(Gf)\big(\boldsymbol{\Theta}^{\mathrm{eff},(1),[N]}(Ns)\big)\Big|\Big]\\
&=\mathbb{E}\Bigg[\Bigg|\frac{K_1e_1}{1+K_0}\Big[\Theta_{y_1}^{(1),[N]}(Ns)-\frac{1}{N}\sum_{i\in[N]}x_i(Ns,\omega)\Big]\frac{\partial f}{\partial x}\big(\boldsymbol{\Theta}^{\mathrm{eff},(1),[N]}(Ns)\big)\\
&\qquad+e_1\Big[\frac{1}{N}\sum_{i\in[N]}x_i(Ns,\omega)-\Theta_{y_1}^{(1),[N]}(Ns)\Big]\frac{\partial f}{\partial y}\big(\boldsymbol{\Theta}^{\mathrm{eff},(1),[N]}(Ns)\big)\\
&\qquad+\frac{1}{(1+K_0)^2}\frac{1}{N}\sum_{i\in[N]}g(x_i(Ns,\omega))\frac{\partial^2 f}{\partial x^2}\big(\boldsymbol{\Theta}^{\mathrm{eff},(1),[N]}(Ns)\big)\\
&\qquad-\frac{K_1e_1}{1+K_0}\big[\Theta_{y_1}^{(1),[N]}(Ns)-\bar{\Theta}^{(1),[N]}(Ns)\big]\frac{\partial f}{\partial x}\big(\boldsymbol{\Theta}^{\mathrm{eff},(1),[N]}(Ns)\big)\\
&\qquad-e_1\big[\bar{\Theta}^{(1),[N]}(Ns)-\Theta_{y_1}^{(1),[N]}(Ns)\big]\frac{\partial f}{\partial y}\big(\boldsymbol{\Theta}^{\mathrm{eff},(1),[N]}(Ns)\big)\\
&\qquad-\frac{1}{(1+K_0)^2}(\mathcal{F}g)(\bar{\Theta}^{(1),[N]}(Ns))\frac{\partial^2 f}{\partial x^2}\big(\boldsymbol{\Theta}^{\mathrm{eff},(1),[N]}(Ns)\big)\Bigg|\Bigg].
\end{aligned} \tag{7.125}$$

Hence

$$\begin{aligned}
&\lim_{N\to\infty}\mathbb{E}\Big[\Big|G_{\dagger}^{(1),[N]}\big(f,\boldsymbol{\Theta}^{\mathrm{eff},(1),[N]}(Ns),s,\cdot\big)-(Gf)\big(\boldsymbol{\Theta}^{\mathrm{eff},(1),[N]}(Ns)\big)\Big|\Big]\\
&\le\lim_{N\to\infty}\mathbb{E}\left[\frac{K_1e_1}{1+K_0}\left|\bar{\Theta}^{(1),[N]}(Ns)-\frac{1}{N}\sum_{i\in[N]}x_i(Ns,\omega)\right|\left|\frac{\partial f}{\partial x}\big(\boldsymbol{\Theta}^{\mathrm{eff},(1),[N]}(Ns)\big)\right|\right]\\
&+\lim_{N\to\infty}\mathbb{E}\left[e_1\left|\frac{1}{N}\sum_{i\in[N]}x_i(Ns,\omega)-\bar{\Theta}^{(1),[N]}(Ns)\right|\left|\frac{\partial f}{\partial y}\big(\boldsymbol{\Theta}^{\mathrm{eff},(1),[N]}(Ns)\big)\right|\right]\\
&+\lim_{N\to\infty}\mathbb{E}\left[\frac{1}{(1+K_0)^2}\left|\frac{1}{N}\sum_{i\in[N]}g(x_i(Ns,\omega))-(\mathcal{F}g)\big(\bar{\Theta}^{(1),[N]}(Ns)\big)\right|\left|\frac{\partial^2 f}{\partial x^2}\big(\boldsymbol{\Theta}^{\mathrm{eff},(1),[N]}(Ns)\big)\right|\right].
\end{aligned} \tag{7.126}$$

Note that each of the derivatives is bounded by aconstant because we work on $\mathcal{C}^*$. The first and the second term tend to zero by Lemma 7.2.6. For the third term we can use a similar argument as used in (6.198), since we showed Lemmas 7.2.4–7.2.10 for the single components in the mean-field system with two colours. □

Step 5: Evolution of the averages in the Meyer-Zheng topology. In this section we prove the following proposition

Proposition 7.2.13 (Convergence in the Meyer-Zheng topology). *Suppose that the effective estimator process defined in* (7.15) *satisfies*

$$\lim_{N\to\infty}\mathcal{L}\left[\left(\boldsymbol{\Theta}^{\mathrm{eff},(1),[N]}(Ns)\right)_{s>0}\right]=\mathcal{L}\left[\left(z_1^{\mathrm{eff}}(s)\right)_{s>0}\right]. \tag{7.127}$$

Then for the averages in (7.7),

$$\lim_{N\to\infty} \mathcal{L}\left[\left(z_1^{[N]}(s)\right)_{s>0}\right] = \mathcal{L}\left[(z_1(s))_{s>0}\right] \quad \text{in the Meyer-Zheng topology,} \tag{7.128}$$

where the limiting process $(z_1(s))_{s>0}$ *is defined as in* (7.20).

To prove Proposition 7.2.13 we need the following characterisation of continuous functions in the Meyer-Zheng topology

Lemma 7.2.14 (Convergence of marginals in the Meyer-Zheng topology). *Let* (E,d) *be a Polish space with metric* d. *Suppose that* $(X_n(s),Y_n(s))_{s>0}$ *is a stochastic process with state space* E^2. *If*

$$\lim_{n\to\infty} \mathcal{L}\left[(X_n(s),Y_n(s))_{s>0}\right] = \mathcal{L}\left[(X(s),Y(s))_{s>0}\right] \ \text{in the Meyer-Zheng topology,} \tag{7.129}$$

then the marginals also converge in the Meyer-Zheng topology, i.e.,

$$\begin{aligned}&\lim_{n\to\infty} \mathcal{L}\left[(X_n(s))_{s>0}\right] = \mathcal{L}\left[(X(s))_{s>0}\right] \ \text{in the Meyer-Zheng topology,}\\ &\lim_{n\to\infty} \mathcal{L}\left[(Y_n(s))_{s>0}\right] = \mathcal{L}\left[(Y(s))_{s>0}\right] \ \text{in the Meyer-Zheng topology.}\end{aligned} \tag{7.130}$$

The proof of Lemma 7.2.14 is given in Appendix B.2.3.

Proof of Proposition 7.2.13. By Lemma 7.2.6, we have that, for all $s>0$,

$$\lim_{n\to\infty} \mathbb{E}\left[\left|\bar{\Theta}^{[N]}(Ns) - x_1^{[N]}(s)\right|\right] = 0 \tag{7.131}$$

and

$$\lim_{n\to\infty} \mathbb{E}\left[\left|\bar{\Theta}^{[N]}(Ns) - y_{0,1}^{[N]}(s)\right|\right] = 0. \tag{7.132}$$

Applying Lemmas 6.2.19, 6.2.20 and 6.2.21, like in the proof of Proposition 6.2.18, we obtain

$$\begin{aligned}&\lim_{N\to\infty} \mathcal{L}\left[\left(x_1^{[N]}(s), y_{0,1}^{[N]}(s), \bar{\Theta}^{[N]}(Ns), \Theta_{y_{1,1}}^{[N]}(Ns)\right)_{s>0}\right]\\ &= \mathcal{L}\left[\left(x_1^{\rm eff}(s), x_1^{\rm eff}(s), x_1^{\rm eff}(s), y_1^{\rm eff}(s)\right)_{s>0}\right] \ \text{in the Meyer-Zheng topology.}\end{aligned} \tag{7.133}$$

Applying Lemma 7.2.14, we get the claim. □

Step 6: Proof of the two-colour mean-field finite-systems scheme.

Proof. The proof of Proposition 7.1.2(a) follows directly from Lemma 7.2.12. The proof of Proposition 7.1.2(b) is a consequence of Proposition 7.1.2(d). The proof of Prosition 7.1.2(c) follows from Proposition 7.1.2(a) by applying Proposition 7.2.13. The proof of Proposition (7.1.2)(d) follows by the same argument as used in the proof Proposition 6.2.1(c) in Section 6.3.4. In this argument we have to replace the two-component system $Z^{[N]}(Ns+t) = (X^{[N]}(Ns+t), Y^{[N]}(Ns+t))_{t\geq 0}$ by the three-component system $Z^{[N]}(Ns+t) = (X^{[N]}(Ns+t), Y_0^{[N]}(Ns+t), Y_1^{[N]}(Ns+t))_{t\geq 0}$

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and use the infinite system defined in 7.11 instead of the infinite system defined in (6.42). We now use the two-dimensional transition kernel in (7.30), which controls the transition probabilities of the two-dimensional process $(\bar{\Theta}^{(1)}(s), \Theta_{y_1}^{(1)}(s))_{s>0}$, instead of the one-dimensional transition kernel in (6.59). $\square$