# Spatial populations with seed-bank

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# CHAPTER 6
# Mean-field system

## §6.1 Preparation: $N \to \infty$, McKean-Vlasov process and mean-field system

To analyse the scaling of our hierarchical system in the hierarchichal mean-field limit $N \to \infty$, we first need to understand simpler systems. In this section we consider the mean-field system consisting of a *single hierarchy*, and introduce the following:

(a) McKean-Vlasov process (Section 6.1.1).

(b) Mean-field system and McKean-Vlasov limit (Section 6.1.2).

For each we derive a key proposition that will play a crucial role in our analysis of the truncated system with *finitely many hierarchies* in Sections 7–9 and the full system with *infinitely many hierarchies* in Section 9. The proofs of the propositions stated in this section will be given in Sections 6.1.3 and 6.1.4.

### §6.1.1 McKean-Vlasov process

In this section we introduce the McKean-Vlasov process, which will play an important role in our analysis of the mean-field system to be introduced in Sections 6.1.2–6.2.1. (In the full system the effective process introduced in (4.68) will be seen to be an example of a McKean-Vlasov process.)

For $g \in \mathcal{G}$ and $c, K, e \in (0, \infty)$, consider the single-colony process

$$z(t) = (x(t), y(t))_{t \geq 0}, \tag{6.1}$$

taking values in $[0,1]^2$, with initial law $\mathcal{L}[(x(0), y(0))] = \mu$ and with components evolving according to

$$\begin{aligned} \mathrm{d}x(t) &= c\,[\mathbb{E}[x(t)] - x(t)]\,\mathrm{d}t + \sqrt{g(x(t))}\,\mathrm{d}w(t) + Ke\,[y(t) - x(t)]\,\mathrm{d}t, \\ \mathrm{d}y(t) &= e\,[x(t) - y(t)]\,\mathrm{d}t, \end{aligned} \tag{6.2}$$

where $\mathbb{E}$ denotes expectation with respect to $\mu$. With the help of Itô-calculus we can compute the expectation $\mathbb{E}[x(t)]$. Indeed, from (6.2) we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[x(t)] &= Ke\,\big[\mathbb{E}[y(t)] - \mathbb{E}[x(t)]\big], \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[y(t)] &= e\,\big[\mathbb{E}[x(t)] - \mathbb{E}[y(t)]\big]. \end{aligned} \tag{6.3}$$

Define

$$\theta_x = \mathbb{E}^\mu[x(0)], \qquad \theta_y = \mathbb{E}^\mu[y(0)], \qquad \theta = \mathbb{E}^\mu\left[\frac{x(0)+Ky(0)}{1+K}\right]. \tag{6.4}$$

Note that (6.3) implies that $\theta$ is a preserved quantity, i.e.,

$$\mathbb{E}^\mu\left[\frac{x(0)+Ky(0)}{1+K}\right] = \mathbb{E}^\mu\left[\frac{x(t)+Ky(t)}{1+K}\right] = \theta, \qquad t \geq 0. \tag{6.5}$$

Solving (6.3), we find

$$\begin{aligned} \mathbb{E}[x(t)] &= \theta + \frac{K}{1+K}(\theta_x - \theta_y)\,\mathrm{e}^{-(K+1)et}, \\ \mathbb{E}[y(t)] &= \theta - \frac{1}{1+K}(\theta_x - \theta_y)\,\mathrm{e}^{-(K+1)et}. \end{aligned} \tag{6.6}$$

In particular, from (4.111) we see that

$$\lim_{t\to\infty}(\mathbb{E}[x(t)], \mathbb{E}[y(t)]) = (\theta, \theta). \tag{6.7}$$

Hence, in equilibrium we can replace $\mathbb{E}[x(t)]$ in (6.2) by $\theta$. After inserting (6.6) into (6.2), we can use [72, Theorem 1, Remark on p.156] to show that for every deterministic initial state $(x(0), y(0)) \in [0,1]^2$ the SSDE in (6.2) has a unique strong solution. We will refer to this solution as the *McKean-Vlasov process.*

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**Remark 6.1.1 (Self-consistency).** To prove uniqueness of the solution to (6.2) we can also use [38], where self-consistent mean-field dynamics are treated in detail. The solution has the Feller property. ■

**Proposition 6.1.2 (McKean-Vlasov process: equilibrium).** *For every initial law $\mu \in \mathcal{P}([0,1]^2)$ satisfying*

$$\mathbb{E}^\mu\left[\frac{x(0)+Ky(0)}{1+K}\right] = \theta, \qquad \theta \in [0,1], \tag{6.8}$$

*the process in* (6.1) *converges to a unique equilibrium,*

$$\lim_{t\to\infty} \mathcal{L}[(x(t), y(t))] = \Gamma_\theta, \tag{6.9}$$

*and*

$$\Gamma_\theta \in \mathcal{P}([0,1]^2), \tag{6.10}$$

*satisfies*

$$\theta = \int_{[0,1]^2} x\,\Gamma_\theta(\mathrm{d}x, \mathrm{d}y) = \int_{[0,1]^2} y\,\Gamma_\theta(\mathrm{d}x, \mathrm{d}y). \tag{6.11}$$

The proof of Proposition 6.1.2 is given in Section 6.1.3. Note that $\Gamma_\theta = \Gamma_\theta^{g,c,K,e}$ depends on all the parameters appearing in (6.2). In Section 6.2 we will see that $\Gamma_\theta$ is continuous as a function of $\theta$.

**Remark 6.1.3 (Non-linear Markov process).** Note that (6.1) is a *non-linear* Markov process: the evolution not only depends on the current state $z(t)$, but also on the current law $\mathcal{L}[z(t)]$ via the expectation $\mathbb{E}[x(t)]$ appearing in the SSDE (6.2). This is different from the model without seed-bank, where the non-linearity is replaced by a drift towards $\theta$ that is constant in time. In equilibrium we can replace $\mathbb{E}[x(t)]$ by $\theta$ in (6.2), but before equilibrium is reached we cannot, because $t \mapsto \mathbb{E}[x(t)]$ is not constant, as is clear from (4.111). Note that $\mathbb{E}[x(t)]$ is a linear functional of $z(0)$. This fact will play an important role in the renormalisation analysis in Section 10. ■

## §6.1.2 Mean-field system and McKean-Vlasov limit

In this section we consider a simplified version of the SSDE in (4.20), namely, we restrict to the finite geographic space

$$[N] = \{0, 1, \dots, N-1\}, \qquad N \in \mathbb{N}. \tag{6.12}$$

In this simplified version, the migration kernel $a^{\Omega_N}(\cdot,\cdot)$ is replaced by $a^{[N]}(\xi,\eta) = cN^{-1}$ for all $(\xi,\eta) \in [N]$, where $c \in (0,\infty)$ is a constant. The seed-bank consists of only *one colour* and the exchange rates between active and dormant are given by $Ke, e$. The state space is

$$S = \mathfrak{s}^{[N]}, \qquad \mathfrak{s} = [0,1]^2, \tag{6.13}$$

the system is denoted by

$$Z^{[N]}(t) = \big(X^{[N]}(t), Y^{[N]}(t)\big)_{t \geq 0}, \qquad \big(X^{[N]}(t), Y^{[N]}(t)\big) = \big(x_i^{[N]}(t), y_i^{[N]}(t)\big)_{i \in [N]}, \tag{6.14}$$

and its components evolve according to the SSDE

$$\begin{aligned} \mathrm{d}x_i^{[N]}(t) &= \frac{c}{N} \sum_{j \in [N]} \big[x_j^{[N]}(t) - x_i^{[N]}(t)\big]\,\mathrm{d}t + \sqrt{g\big(x_i^{[N]}(t)\big)}\,\mathrm{d}w_i(t) \\ &\quad + Ke\,\big[y_i^{[N]}(t) - x_i^{[N]}(t)\big]\,\mathrm{d}t, \\ \mathrm{d}y_i^{[N]}(t) &= e\,\big[x_i^{[N]}(t) - y_i^{[N]}(t)\big]\,\mathrm{d}t, \qquad i \in [N], \end{aligned} \tag{6.15}$$

which is the special case of (4.20) obtained by setting $a^{\Omega_N}(\eta,\xi) = 0$ if $d(\eta,\xi) > 1$ and $K_m = e_m = 0$ for $m \geq 1$. It is natural to take an *exchangeable random initial state*, because the evolution preserves exchangeability. According to De Finetti's theorem, there is no loss of generality in taking an i.i.d. initial state, i.e.,

$$\mathcal{L}\big[X^{[N]}(0), Y^{[N]}(0)\big] = \mu^{\otimes [N]}, \qquad \mu \in \mathcal{P}\left([0,1]^2\right). \tag{6.16}$$

By [67, Theorem 3.1], the SSDE in (6.15) is the unique weak solution of a well-posed martingale problem. By [67, Theorem 3.2], for every deterministic initial state $(X^{[N]}(0), Y^{[N]}(0))$, (6.15) has a unique strong solution. We are interested in the limit $N \to \infty$. For the limiting process we define

$$(Z(t))_{t \geq 0} = (X(t), Y(t))_{t \geq 0} = \big((x_i(t), y_i(t))_{i \in \mathbb{N}_0}\big)_{t \geq 0} \tag{6.17}$$

with components evolving according to (6.2), i.e.,

$$\begin{aligned}\mathrm{d}x_i(t) &= c\,[\mathbb{E}[x_i(t)] - x_i(t)]\,\mathrm{d}t + \sqrt{g(x_i(t))}\,\mathrm{d}w(t)\\ &\qquad\qquad\qquad\qquad + Ke\,[y_i(t) - x_i(t)]\,\mathrm{d}t,\\ \mathrm{d}y_i(t) &= e\,[x_i(t) - y_i(t)]\,\mathrm{d}t, \qquad i \in \mathbb{N}_0,\end{aligned} \tag{6.18}$$

with $\mathcal{L}\left[(X(0), Y(0))\right] = \mu$ for some exchangeable $\mu \in \mathcal{P}(([0,1]^2)^{\otimes[\mathbb{N}_0]}$. Note that (6.18) consists of i.i.d. copies of the single-colony McKean-Vlasov process in (6.1), labelled by $i \in \mathbb{N}_0$.

**Proposition 6.1.4 (Infinite-system McKean-Vlasov limit: convergence).** *Suppose that* $\mathcal{L}[(X^{[N]}(0), Y^{[N]}(0))] = \mu^{[N]}$ *is exchangeable and*

$$\theta = \mathbb{E}^{\mu^{[N]}}\left[\frac{x(0) + Ky(0)}{1+K}\right]. \tag{6.19}$$

*Then*

$$\lim_{N\to\infty} \mathcal{L}\Big[\big(X^{[N]}(t), Y^{[N]}(t)\big)_{t\geq 0}\Big] = \mathcal{L}\big[(X(t), Y(t))_{t\geq 0}\big] \tag{6.20}$$

*with*

$$\mathcal{L}\big[(X(0), Y(0))_{t\geq 0}\big] = \mu, \qquad \mu = \lim_{N\to\infty} \mu^{[N]}, \tag{6.21}$$

*where the limit is the McKean-Vlasov process in* (6.1)–(6.2).

The proof of Proposition 6.1.4 is given in Section 6.1.4. For the system without seed-bank the McKean-Vlasov limit was proved in [38]. The fact that the components decouple is a property referred to as *propagation of chaos.*



## §6.1.3 Proof of equilibrium and ergodicity

In this section we prove Proposition 6.1.2.

*Proof.* Note that, by (4.111), we can rewrite (6.2) as

$$\begin{aligned}\mathrm{d}x(t) &= c\,\left[\theta + \frac{K}{1+K}(\theta_x - \theta_y)\,\mathrm{e}^{-(K+1)et} - x(t)\right]\,\mathrm{d}t + \sqrt{g(x(t))}\,\mathrm{d}w(t)\\ &\qquad\qquad + Ke\,[y(t) - x(t)]\,\mathrm{d}t,\\ \mathrm{d}y(t) &= e\,[x(t) - y(t)]\,\mathrm{d}t.\end{aligned} \tag{6.22}$$

Existence and uniqueness of a strong solution is again standard (see e.g. [72, Theorem 1] and recall Remark 6.1.1). We start by proving existence and uniqueness of the equilibrium. Afterwards we show that the solution converges to this equilibrium.

Consider two copies $(x_1, y_1)$ and $(x_2, y_2)$ of the system defined in (6.22), with $\mathcal{L}[(x_1(0), y_1(0))] = \mu_1$ and $\mathcal{L}[(x_2(0), y_2(0))] = \mu_2$, where $\mu_1$ and $\mu_2$ satisfy

$$\mathbb{E}^{\mu_1}\left[\frac{x_1(0) + Ky_1(0)}{1+K}\right] = \theta = \mathbb{E}^{\mu_2}\left[\frac{x_2(0) + Ky_2(0)}{1+K}\right] \tag{6.23}$$

for some $\theta \in [0,1]$. Write

$$\theta_{x_1} = \mathbb{E}^{\mu_1}[x_1(0)], \quad \theta_{y_1} = \mathbb{E}^{\mu_1}[y_1(0)], \quad \theta_{x_2} = \mathbb{E}^{\mu_2}[x_2(0)], \quad \theta_{y_2} = \mathbb{E}^{\mu_2}[y_2(0)]. \tag{6.24}$$

Couple the two systems by coupling their Brownian motions. Denote the coupled process by

$$\begin{aligned}&(\bar{z}(t))_{t\geq 0} = (z_1(t), z_2(t))_{t\geq 0}, \quad z_1(t) = (x_1(t), y_1(t)), \quad z_2(t) = (x_2(t), y_2(t)),\\ &\mathcal{L}(\bar{z}(0)) = \mu_1 \times \mu_2,\end{aligned} \tag{6.25}$$

which has a unique strong solution. Put

$$\Delta(t) = x_1(t) - x_2(t), \qquad \delta(t) = y_1(t) - y_2(t). \tag{6.26}$$

To show that the equilibrium is unique, it is enough to show that

$$\lim_{t\to\infty} \mathbb{E}\left[|\Delta(t)| + EK|\delta(t)|\right] = 0. \tag{6.27}$$

Using a generalised form of Itô's formula, we find

$$\begin{aligned}\mathrm{d}|\Delta(t)| &= (\mathrm{sgn}\ \Delta(t))\,\mathrm{d}\Delta(t) + \mathrm{d}L_t^0\\ &= (\mathrm{sgn}\ \Delta(t))\, c\left[\frac{K}{1+K}\big((\theta_{x_1}-\theta_{x_2}) - (\theta_{y_1}-\theta_{y_2})\big)\,\mathrm{e}^{-(K+1)et} - \Delta(t)\right]\mathrm{d}t\\ &\qquad + (\mathrm{sgn}\ \Delta(t))\left(\sqrt{g(x_1(t))} - \sqrt{g(x_2(t))}\right)\mathrm{d}w(t)\\ &\qquad + (\mathrm{sgn}\ \Delta(t))\, Ke\left[\delta(t) - \Delta(t)\right]\mathrm{d}t,\end{aligned} \tag{6.28}$$

where we use that the local time $L_t^0$ (see [63, Section IV.43]) of $\Delta(t)$ at 0 equals 0, since $g$ is Lipschitz (see [63, Proposition V.39.3]). Again using Itô's formula, we also find

$$\mathrm{d}|\delta(t)| = (\mathrm{sgn}\ \delta(t))\,\mathrm{d}\delta(t) = (\mathrm{sgn}\ \delta(t))\, e\left[\Delta(t) - \delta(t)\right]\mathrm{d}t. \tag{6.29}$$

Taking expectations in (6.28)–(6.29), we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[|\Delta(t)| + K|\delta(t)|]\\ &= \mathbb{E}\left[c\left[(\mathrm{sgn}\ \Delta(t))\frac{K}{1+K}\big((\theta_{x_1}-\theta_{x_2}) - (\theta_{y_1}-\theta_{y_2})\big)\,\mathrm{e}^{-(K+1)et} - |\Delta(t)|\right]\right]\\ &\quad + Ke\,\mathbb{E}\Big[(\mathrm{sgn}\ \Delta(t) - \mathrm{sgn}\ \delta(t))\,(\delta(t) - \Delta(t))\Big]\\ &= \mathbb{E}\left[c\,(\mathrm{sgn}\ \Delta(t))\frac{K}{1+K}\big((\theta_{x_1}-\theta_{x_2}) - (\theta_{y_1}-\theta_{y_2})\big)\,\mathrm{e}^{-(K+1)et}\right]\\ &\quad - c\,\mathbb{E}[|\Delta(t)|]\\ &\quad - 2Ke\,\mathbb{E}\left[1_{\{\mathrm{sgn}\ \delta(t)\neq \mathrm{sgn}\,\Delta(t)\}}\,(|\delta(t)| + |\Delta(t)|)\right].\end{aligned} \tag{6.30}$$

Define

$$h(t) = c\,\mathbb{E}[|\Delta(t)|] + 2Ke\,\mathbb{E}\left[1_{\{\mathrm{sgn}\ \delta(t))\neq \mathrm{sgn}\ \Delta(t)\}}\,(|\delta(t)| + |\Delta(t)|)\right]. \tag{6.31}$$

Then $h(t)$ satisfies

(a) $h(t) > 0$.

(b) $0 \leq \int_0^\infty \mathrm{d}t\, h(t) \leq 1+K+c\,|(\theta_{x_1}-\theta_{x_2}) - (\theta_{y_1}-\theta_{y_2})|\,\frac{K}{K+1}\frac{1}{e(K+1)}\left[1 - \mathrm{e}^{-(K+1)et}\right]$.

(c) $h$ is differentiable with $h'$ bounded (see [43, Appendix D]).

Hence it follows that $\lim_{t\to\infty} h(t) = 0$, which implies that

$$\lim_{t\to\infty} \mathbb{E}\left[|\Delta(t)|\right] = 0. \tag{6.32}$$

We are left to prove that $\lim_{t\to\infty} \mathbb{E}[|\delta(t)|] = 0$. To do so, we define

$$f(t) = \mathbb{E}[|\delta(t)|], \qquad G(t) = e\,\mathbb{E}[(\mathrm{sgn}\ \delta(t))\Delta(t)]. \tag{6.33}$$

Note that $G$ is bounded and continuous. Taking expectations in (6.29), we find

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) = -ef(t) + G(t), \tag{6.34}$$

Solving (6.34) explicitly, we find that

$$f(t) = f(r)\,\mathrm{e}^{-e(t-r)} + \int_r^t \mathrm{d}s\,\mathrm{e}^{-e(t-s)} G(s), \qquad r,t \in \mathbb{R},\ t > r \geq 0. \tag{6.35}$$

By (6.32), for each $\epsilon > 0$ we can find an $r \in \mathbb{R}$ such that $\mathbb{E}[|\Delta(s)|] < \epsilon$ for all $s > r$, and hence $\sup_{t>r} |G(t)| < \epsilon$. Therefore

$$f(t) \leq f(r)\,\mathrm{e}^{-e(t-r)} + \epsilon \tag{6.36}$$

and, since $|f| < 1$, we find, for each $\epsilon > 0$,

$$\lim_{t\to\infty} f(t) < \epsilon. \tag{6.37}$$

Therefore $\lim_{t\to\infty} \mathbb{E}[|\delta(t)|] = 0$, which completes the proof of uniqueness of the equilibrium for given $\theta$.

To prove existence of the equilibrium, let $(t_n)_{n\in\mathbb{N}}$ be any increasing sequenc eof times such that $\lim_{n\to\infty} t_n = \infty$. Let $\mu = \mathcal{L}[(x(0), y(0))]$ be any initial measure of the system in (6.22) with $\mathbb{E}^{\mu}[\frac{x(0)+Ky(0)}{1+K}] = \theta$, and let $\mu(t_n) = \mathcal{L}[(x(t_n), y(t_n))]$. Since the state space is compact, the sequence $(\mu(t_n))_{n\in\mathbb{N}}$ is tight, and by Prohorov's theorem we can find a converging subsequence $(\mu(t_{n_k}))_{k\in\mathbb{N}}$. Put $\nu = \lim_{k\to\infty} \mu(t_{n_k})$. We will show that $\nu$ is invariant. To that end, recall from Section 6.1.1 that

$$\mathbb{E}^{\mu}\left[\frac{x(t) + Ky(t)}{1+K}\right] = \theta, \qquad t \geq 0. \tag{6.38}$$

Hence we can use the coupling in (6.25) to show that the system starting in $\mu$ and the system starting $\mu(t)$ converge to the same law as $t \to \infty$, from which it follows that $\lim_{k\to\infty} \mu(t + t_{n_k}) = \nu$. Let $(S_t)_{t\geq 0}$ denote the semigroup of the system in (6.22). By the Feller property for semigroups,

$$S_t\nu = \lim_{k\to\infty} S_t\mu(t_{n_k}) = \lim_{k\to\infty} S_{t_{n_k}}\left(S_t\mu\right) = \nu, \tag{6.39}$$

where in the last equality we use the uniqueness of the equilibrium given $\theta$. Thus, $\nu$ is an invariant measure. To exhibit its dependence on $\theta$ we write $\nu_\theta$. Using the same coupling as in (6.25), and starting from $\mu \times \nu_\theta$ with $\nu_\theta$ the invariant measure just obtained, we see that for every $\theta$ the system in (6.2) converges to a unique equilibrium measure $\nu_\theta$, and so (6.11) is immediate from (6.38). $\square$

## §6.1.4 Proof of McKean-Vlasov limit

In this section we give a sketch of the proof of Proposition 6.1.4. In Chapters 6.2-9 we encounter more difficult versions of Proposition 6.1.4. There we will give the proofs in full detail.

*Proof.* Since we start from a distribution $\mu(0)$ that is exchangeable, Aldous's ergodic theorem gives

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j\in[N]} x_j(0) = \mathbb{E}^{\mu(0)}[x_0] \quad \mathbb{P}\text{-a.s.} \tag{6.40}$$

By Ioffe's theorem [25, Eqs. (1.1)–(1.2)], tightness of the associated sequence of processes (uniformly on the state space) follows from boundedness of the generator as an operator. To apply the generator criterion in [49] we must show propagation of chaos and prove the weak law of large numbers

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j\in[N]} x_j(t) = \mathbb{E}[x_0(t)]. \tag{6.41}$$

The propagation of chaos and the weak law of large numbers for $t > 0$ therefore follows from [38, Section 4]. Since the martingale problem is well-posed [38, Section 2], the limiting process exists and is unique. □

## §6.2 Proofs: $N \to \infty$, mean-field finite-systems scheme

Chapter 6

In Sections 6.2.1 we introduce the so called mean-field finite-systems scheme for the mean-field system introduced in Section 6.1.2. In Section 6.2.2 we outline the *abstract scheme* behind the proof behind the mean-field finite-systems scheme. The computations in the proof of the abstract scheme are long and technical, and are deferred to Section 6.3.

## §6.2.1 Mean-field finite-systems scheme

In this section we describe the limiting dynamics of the finite system in (6.14) from a *multiple space-time scale* viewpoint. To do so, we need the following limiting SSDE for the infinite system $Z(t) = (z_i(t))_{i\in\mathbb{N}_0} = (x_i(t), y_i(t))_{i\in\mathbb{N}_0}$, with initial law $\mathcal{L}[Z(0)] = \mu^{\otimes\mathbb{N}_0}$, evolving according to

$$\begin{aligned} \mathrm{d}x_i(t) &= c\,[\theta - x_i(t)]\,\mathrm{d}t + \sqrt{g(x_i(t))}\,\mathrm{d}w_i(t) + Ke\,[y_i(t) - x_i(t)]\,\mathrm{d}t, \\ \mathrm{d}y_i(t) &= e\,[x_i(t) - y_i(t)]\,\mathrm{d}t,\ i \in \mathbb{N}_0, \end{aligned} \tag{6.42}$$

where $\theta$ is defined in (6.11). Note that each component of (6.42) is an autonomous copy of the McKean-Vlasov process in (6.2) in equilibrium.

For the multiscale analysis we will need the following ingredients:

(a) The *estimator* for the finite system is defined by

$$\bar{\Theta}^{[N]}(t) = \bar{\Theta}^{[N]}\big(Z^{[N]}(t)\big) = \frac{1}{N} \sum_{i\in[N]} \frac{x_i^{[N]}(t) + Ky_i^{[N]}(t)}{1+K} \tag{6.43}$$

and its active and dormant counterparts

$$\begin{aligned}\bar{\Theta}_x^{[N]}(t) &= \frac{1}{N}\sum_{i\in[N]} x_i^{[N]}(t),\\ \bar{\Theta}_y^{[N]}(t) &= \frac{1}{N}\sum_{i\in[N]} y_i^{[N]}(t).\end{aligned} \tag{6.44}$$

(b) The *time scale* $N$, on which $\lim_{N\to\infty}\mathcal{L}[\bar{\Theta}^{[N]}(L(N)) - \bar{\Theta}^{[N]}(0)] = \delta_0$ for all $L(N)$ such that $\lim_{N\to\infty} L(N) = \infty$ and $\lim_{N\to\infty} L(N)/N = 0$, but not for $L(N) = N$. In words, $N$ is the time scale on which $\bar{\Theta}^{[N]}(\cdot)$ starts evolving, i.e., $\big(\bar{\Theta}^{[N]}(Ns)\big)_{s>0}$ is not a fixed process. When we scale time by $N$, putting $t = Ns$, we view $s$ as the "fast time scale" and $t$ as the "slow time scale".

(c) The *invariant measure*, i.e., the equilibrium measure of a single component in (6.42) written

$$\Gamma_\theta, \tag{6.45}$$

and the *invariant measure* of the infinite system in (6.42), written $\nu_\theta = \Gamma_\theta^{\otimes\mathbb{N}_0}$, with $\theta \in [0,1]$ controlled by the initial measure (recall (6.4)–(4.111)).

(d) The *renormalisation transformation* $\mathcal{F}\colon\, \mathcal{G}\to\mathcal{G}$,

$$(\mathcal{F}g)(\theta) = \int_{[0,1]^2} g(x)\,\nu_\theta(\mathrm{d}x,\mathrm{d}y_0), \quad \theta\in[0,1], \tag{6.46}$$

where $\nu_\theta$ is the equilibrium measure of 6.42. Note that $\mathcal{F}$ is the same transformation as defined in (4.75), but for the truncated system. Note that we can also write

$$(\mathcal{F}g)(\theta) = \int_{[0,1]^2} g(x)\,\Gamma_\theta(\mathrm{d}x,\mathrm{d}y_0), \quad \theta\in[0,1], \tag{6.47}$$

where $\Gamma_\theta$ is as defined in (6.45).

(e) The *macroscopic observable* $\big(\bar{\Theta}(s)\big)_{s>0}$ satisfying the SSDE

$$\mathrm{d}\bar{\Theta}(s) = \frac{1}{1+K}\sqrt{\mathbb{E}^{\Gamma_{\bar{\Theta}(s)}}[g(u)]}\,\mathrm{d}w(s) = \frac{1}{1+K}\sqrt{(\mathcal{F}g)(\bar{\Theta}(s))}\,\mathrm{d}w(s), \tag{6.48}$$

To obtain the multi-scale limit dynamics for the system in (6.14), we speed up time by a factor $N$ and define the process

$$\Big(x_1^{[N]}(s), y_1^{[N]}(s)\Big)_{s>0} = \Big(\Theta_x^{[N]}(Ns), \Theta_y^{[N]}(Ns)\Big)_{s>0}, \tag{6.49}$$

which is the analogue of the 1-block average in (4.22). We use the lower index 1 to indicate that the average is taken over $[N]$ components. Using (6.15), we see that the dynamics of (6.49) is given by the SSDE

$$\begin{aligned}\mathrm{d}x_1^{[N]}(s) &= \sqrt{\frac{1}{N}\sum_{i\in[N]} g\big(x_i(Ns)\big)}\,\mathrm{d}w(s) + NKe\left[y_1^{[N]}(s) - x_1^{[N]}(s)\right]\mathrm{d}s,\\ \mathrm{d}y_1^{[N]}(s) &= Ne\left[x_1^{[N]}(s) - y_1^{[N]}(s)\right]\mathrm{d}s.\end{aligned} \tag{6.50}$$

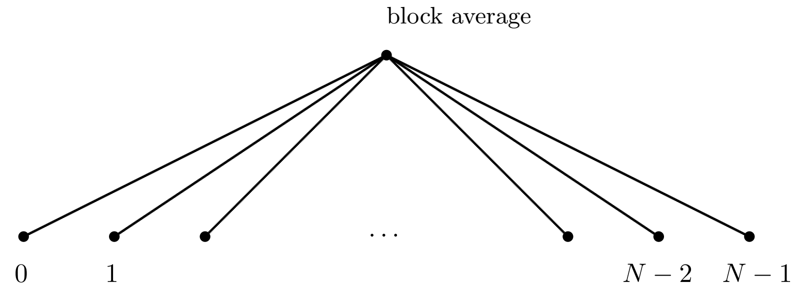


*Figure 6.1: Given the value of the block average, the $N \gg 1$ constituent components equilibrate on a time scale that is fast with respect to the time scale on which the block average fluctuates. Consequently, the volatility of the block average is the expectation of the volatility of the constituent components under the conditional quasi-equilibrium.*

In (6.50), in the limit as $N \to \infty$ infinite rates appear in the exchange between the active and the dormant population. However, looking at the process

$$\left(\frac{x_1^{[N]}(s) + K y_1^{[N]}(s)}{1+K}\right)_{s>0} = \left(\bar{\Theta}^{[N]}(Ns)\right)_{s>0} \tag{6.51}$$

we see that the terms carrying a factor $N$ in front cancel out. Consequently, for the process in (6.51) we can use ideas from [20] to prove tightness as $N \to \infty$ in the *classical topology* of continuum path processes. We will show in Section 6.3.3 that

$$\lim_{N\to\infty} \mathcal{L}\left[\left(\left[x_1^{[N]}(s) - y_1^{[N]}(s)\right]\right)_{s\geq 0}\right] = \mathcal{L}\left[(0)_{s\geq 0}\right] \tag{6.52}$$

in the *Meyer-Zheng topology*.

Combining (6.51) and (6.52), we obtain the multiple space-time scaling behaviour of the system in (6.14).

**Proposition 6.2.1 (Mean-field: finite-systems scheme).** *Suppose that the SSDE in* (6.15) *has initial measure* $\mathcal{L}[Z^{[N]}(0)] = \mu^{\otimes[N]}$ *for some* $\mu \in \mathcal{P}\left([0,1]^2\right)$. *Let*

$$\theta = \mathbb{E}^{\mu}\left[\frac{x + K y_0}{1+K}\right]. \tag{6.53}$$

*(a) For the averages in* (6.49),

$$\lim_{N\to\infty} \mathcal{L}\left[\left(x_1^{[N]}(s), y_{0,1}^{[N]}(s)\right)_{s>0}\right] = \mathcal{L}\left[\left(x_1^{\mathbb{N}_0}(s), y_{0,1}^{\mathbb{N}_0}(s)\right)_{s>0}\right] \tag{6.54}$$

*in the Meyer-Zheng topology,*

*where the limit process is the unique solution of the SSDE*

$$\begin{aligned} \mathrm{d}x_1^{\mathbb{N}_0}(s) &= \frac{1}{1+K}\sqrt{(\mathcal{F}g)\big(x_1^{\mathbb{N}_0}(s)\big)}\,\mathrm{d}w(s), \\ y_{0,1}^{\mathbb{N}_0}(s) &= x_1^{\mathbb{N}_0}(s), \end{aligned} \tag{6.55}$$

*with initial state*

$$\left(x_1^{\mathbb{N}_0}(0), y_{0,1}^{\mathbb{N}_0}(0)\right) = (\theta, \theta)\,. \tag{6.56}$$

*(b) For the weighted sum of the averages in* (6.51),

$$\lim_{N\to\infty} \mathcal{L}\left[\left(\bar{\Theta}^{[N]}(Ns)\right)_{s>0}\right] = \mathcal{L}\left[\left(\bar{\Theta}(s)\right)_{s>0}\right], \tag{6.57}$$

*where the limit is the macroscopic observable in* (6.48) *with initial state*

$$\bar{\Theta}(0) = \theta. \tag{6.58}$$

*(c) Define*

$$\nu_\theta(s) = \int_{[0,1]} Q_s\big(\theta, \mathrm{d}\theta'\big)\, \nu_{\theta'} \in \mathcal{P}([0,1]^2), \tag{6.59}$$

*where $Q_s(\theta,\cdot)$ is the time-s marginal law of the process $(\bar{\Theta}(s))_{s>0}$ starting from $\theta \in [0,1]$ (note that $\nu_\theta(0) = \nu_\theta$). Then, for every $s \in (0,\infty)$,*

$$\lim_{N\to\infty} \mathcal{L}\left[\left(X^{[N]}(Ns+t), Y^{[N]}(Ns+t)\right)_{t>0}\right] = \mathcal{L}\left[(Z^{\nu_\theta(s)}(t))_{t>0}\right] \tag{6.60}$$

*where, conditional on $\bar{\Theta}(s) = \theta$, $(z^{\nu_\theta(s)}(t))_{t\geq 0}$ is the random process in* (6.42) *and $z^{\nu_\theta(s)}(0)$ is drawn according to $\nu_\theta(s)$ (which is a mixture of random processes in equilibrium).*

CHAPTER 6

The proof of Proposition 6.2.1 is given in Section 6.2.2.

The result in Part (a) shows that the limit dynamics of the averages follows a similar type of diffusion as a single colony, but with four important changes:

- For the limit of the time-scaled average in (6.51) the diffusion function $g$ is replaced by a *renormalised diffusion function* $\mathcal{F}g$, defined by (6.46) (recall Fig. 6.1). In section 6.2.2 we will show that $\mathcal{F}\mathcal{G} \subset \mathcal{G}$, i.e., $\mathcal{F}$ preserves the class of diffusion functions defined in (4.15).
- The average of the dormant population is the same as the average of the active population, and hence the term that accounts for the exchange between the active and the dormant population *vanishes.* This happens because when time is speeded up by a factor $N$ also the rates of exchange between active and dormant are speeded up by a factor $N$ (see (6.50)). Hence the exchange rates become infinitely large, which implies that the active and the dormant population equilibrate instantly in the Meyer-Zheng topology.
- Since we take the average over all the components, the migration terms in (6.15) cancel out against each other.
- Comparing the system in (6.14) with the system of interacting Fisher-Wright diffusions in the mean-field limit studied in [21], we see from (6.55) that the single-colour seed-bank *slows down the average* by a factor $1/(1+K)$, but does not change the system qualitatively. This is a direct consequence of the fact that the averages of the active and the dormant population equilibrate (due to the infinite rates), while only individuals in the active part of the population resample.

The result in Part (b) shows that the limit dynamics of the averages in (6.51) follows an autonomous SDE, with convergence in the classical topology, i.e., in $C_b([0,\infty),[0,1])$. The Brownian motion in (6.1) is taken to be independent of the initial state. The result in Part (c) says that, on time scale 1 and starting from time $Ns$ with $N \to \infty$, the system has a *McKean-Vlasov limit*, i.e., exhibits propagation of chaos, with components that are versions of a McKean-Vlasov process with a random initial state whose law depends on $s$. So, in particular, the components become independent, and we see *decoupling* The proof of Part (c) will use Part (b). The proof of Part (a) will follow from Part (b) after we use the Meyer-Zheng topology.

**Remark 6.2.2 (Basic multi-scale).** Note that Proposition 6.2.1 already reveals several phenomena that we encountered in Theorems 4.4.2 and 4.4.4, capturing the hierarchical multiscale behaviour. Even for the one-layer mean-field system we find decoupling of components, the occurrence of a renormalisation transformation, equalisation of the seed-bank with the active population, and the need for the Meyer-Zheng topology. Later we will see that the role of the macroscopic observable $\bar{\Theta}$ is the same as that of the effective process. ■

**Remark 6.2.3 (Interchange of limits).** The notation $x_1^{\mathbb{N}_0}, y_{0,1}^{\mathbb{N}_0}$ indicates that the limit arises from taking averages over $[N]$ and letting $N \to \infty$. Note that, for i.i.d. initial states,

$$x_1^{\mathbb{N}_0}(0) = \lim_{N\to\infty} x_1^{[N]}(0) = \lim_{N\to\infty} \frac{1}{N} \sum_{i\in[N]} x_i(0) = \theta_x \qquad \mathbb{P}\text{-a.s.} \tag{6.61}$$

On the other hand, picking any sequence of times $L(N)$ such that $\lim_{N\to\infty} L(N) = \infty$ and $\lim_{N\to\infty} L(N)/N = 0$, we get

$$x_1^{\mathbb{N}_0}(0+) = \lim_{N\to\infty} x_1^{[N]}\big(\tfrac{L(N)}{N}t\big) = \lim_{N\to\infty} \frac{1}{N} \sum_{i\in[N]} x_i(L(N)t) = \theta \qquad \mathbb{P}\text{-a.s.} \tag{6.62}$$

The mismatch between (6.61) and (6.62) indicates that we must be careful with interchanging the limits $N \to \infty$ and $s \downarrow 0$. This is why (6.54), which lives on the fast time scale, is restricted to $s > 0$. ■

## §6.2.2 Abstract scheme behind finite-systems scheme

To prove Proposition 6.2.1, we follow the abstract scheme outlined in [25, p. 2314–2315] and based on [21], [20]. Below we state the abstract scheme for our model. The scheme consists of 4 steps, each of the steps consists of a series of propositions and lemmas. The proofs of these are given in Section 6.3.

**Step 1. Equilibrium of the single components.** This step fixes the one-dimensional distributions of the single components when $t, N \to \infty$ in a combined way, and is the equivalent of [21, Proposition 1]. Recall that $\bar{\Theta}^{[N]}$ is defined in (6.43).

**Proposition 6.2.4 (Equilibrium for the infinite system).** *Let $(N_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{N}$. Fix $s>0$. Let $L(N)$ be such that $\lim_{N\to\infty} L(N)=\infty$ and $\lim_{N\to\infty} L(N)/N=0$, and suppose that*

$$\begin{aligned}
&\lim_{k\to\infty}\mathcal{L}\left[\bar\Theta^{[N_k]}(N_k s)\right]=P_s,\\
&\lim_{k\to\infty}\mathcal{L}\left[\sup_{0\le t\le L(N_k)}\left|\bar\Theta^{[N_k]}(N_k s)-\bar\Theta^{[N_k]}(N_k s-t)\right|\right]=\delta_0,\\
&\lim_{k\to\infty}\mathcal{L}\big(X^{[N_k]}(N_k s),Y^{[N_k]}(N_k s)\big)=\nu(s).
\end{aligned}\tag{6.63}$$

*Then $\nu(s)$ is of the form*

$$\nu(s)=\int_{[0,1]}P_s(\mathrm{d}\theta)\,\nu_\theta,\tag{6.64}$$

*where $\nu_\theta$ is the equilibrium measure of the process defined in* (6.42).

Proposition 6.2.4 follows from the following seven lemmas, which are the analogues of the five lemmas used in [21, p. 477–478] for the system without seed-bank.

The first lemma establishes convergence of the infinite system in (6.42) to its equilibrium.

**Lemma 6.2.5 (Convergence for the infinite system).** *Let $\mu$ be an exchangeable probability measure on $([0,1]^2)^{\mathbb{N}_0}$. Then for the system $(Z(t))_{t\ge0}$ given by* (6.42) *with $\mathcal{L}(Z(0))=\mu$,*

$$\lim_{t\to\infty}\mathcal{L}[Z(t)]=\nu_\theta,\tag{6.65}$$

*where $\nu_\theta$ is of the form*

$$\nu_\theta=\Gamma_\theta^{\otimes\mathbb{N}_0},\tag{6.66}$$

*with $\Gamma_\theta$ the equilibrium of the single-colony process defined in* (6.45). *Moreover, $\nu_\theta$ is ergodic.*

The second lemma establishes the continuity of the equilibrium with respect to its center of drift $\theta$.

**Lemma 6.2.6 (Continuity of the equilibrium).** *Let $\mathcal{P}([0,1]^{\mathbb{N}_0})$ denote the space of probability measures on $[0,1]^{\mathbb{N}_0}$. The mapping $[0,1]\to\mathcal{P}([0,1]^{\mathbb{N}_0})$ given by*

$$\theta\mapsto\nu_\theta\tag{6.67}$$

*is continuous. Furthermore, if $h$ is a Lipschitz function on $[0,1]$, then also $\mathcal{F}h$ defined by*

$$(\mathcal{F}h)(\theta)=\mathbb{E}^{\nu_\theta}[h(\cdot)]=\int_{([0,1]^2)^{\mathbb{N}_0}}\nu_\theta(\mathrm{d}z)\,h(x_0)\tag{6.68}$$

*is a Lipschitz function on $[0,1]$.*

The third lemma characterises the speed at which the estimators $\Theta_x^{[N]}$ and $\Theta_y^{[N]}$ converge to each other.

**Lemma 6.2.7 (Comparison of empirical averages).** *Let* $(\Theta_x^{[N]}(t))_{t\geq 0}$ *and* $(\bar{\Theta}_y^{[N]}(t))_{t\geq 0}$ *be defined as in* (6.44)*, and define*

$$\Delta_{\bar{\Theta}}^{[N]}(t) = \Theta_x^{[N]}(t) - \Theta_y^{[N]}(t). \tag{6.69}$$

*Then*

$$\mathbb{E}\left[\left|\Delta_{\bar{\Theta}}^{[N]}(t)\right|\right] \leq \sqrt{\mathbb{E}\left[\left(\Delta_{\bar{\Theta}}^{[N]}(0)\right)^2\right]}\, \mathrm{e}^{-(Ke+e)t} + \sqrt{\frac{\|g\|}{N(Ke+e)}}. \tag{6.70}$$

**Remark 6.2.8 (Key estimate for Meyer-Zheng convergence).** The estimate in (6.70) in Lemma 6.2.7 will be the key estimate to show convergence of the active and dormant 1-block in Meyer-Zheng topology. Note that if we look at times $Ns$ for $s > 0$, then (6.70) shows that $\mathbb{E}[|\Delta_{\bar{\Theta}}^{[N]}(Ns)|]$ is $\mathcal{O}(\sqrt{1/N})$. ■

The fourth lemma compares the finite system with an infinite system. To that end we construct both the finite and the infinite system on the same state-space by considering the finite system $(X^{[N]}(t), Y^{[N]}(t))$ as an element of $([0,1]^2)^{\mathbb{N}_0}$ via periodic continuation. Let $L(N)$ be such that $\lim_{N\to\infty} L(N) = \infty$ and $\lim_{N\to\infty} L(N)/N = 0$, and define the distribution $\mu_N$ by continuing the configuration of $(X^{[N]}(Ns - L(N)), Y^{[N]}(Ns - L(N)))$ periodically to $([0,1]^2)^{\mathbb{N}_0}$. Define

$$\bar{\Theta}^{[N]} = \bar{\Theta}^{[N]}(Ns - L(N)). \tag{6.71}$$

Note that

$$\begin{aligned}\bar{\Theta}^{[N]} &= \frac{\frac{1}{N}\sum_{j\in[N]} x_j^{[N]}(Ns - L(N)) + \frac{K}{N}\sum_{j\in[N]} y_j^{[N]}(Ns - L(N))}{1+K}\\ &= \frac{1}{N}\sum_{j\in[N]} \frac{x_j^{\mu_N}(0) + K y_j^{\mu_N}(0)}{1+K}.\end{aligned} \tag{6.72}$$

Thus, $\bar{\Theta}^{[N]}$ is a random variable whose law depends on $\mathcal{L}\left[X^{[N]}(Ns - L(N)), Y^{[N]}(Ns - L(N))\right] = \mu_N$. The infinite system with initial law $\mu_N$ is denoted by

$$\left(X^{\mu_N}(t), Y^{\mu_N}(t)\right)_{i\in\mathbb{N}_0,\ t\geq 0} = \left(x_i^{\mu_N}(t), y_i^{\mu_N}(t)\right)_{t\geq 0} \tag{6.73}$$

and evolves according to

$$\begin{aligned}\mathrm{d}x_i^{\mu_N}(t) &= c\left[\bar{\Theta}^{[N]} - x_i^{\mu_N}(t)\right]\mathrm{d}t + \sqrt{g(x_i^{\mu_N}(t))}\,\mathrm{d}w_i(t) + Ke\left[y_i^{\mu_N}(t) - x_i^{\mu_N}(t)\right]\mathrm{d}t,\\ \mathrm{d}y_i^{\mu_N}(t) &= e\left[x_i^{\mu_N}(t) - y_i^{\mu_N}(t)\right]\mathrm{d}t, \qquad i\in\mathbb{N}_0,\end{aligned} \tag{6.74}$$

where $\{w_i\}_{i\in\mathbb{N}_0}$ is a collection of independent Brownian motions.

**Lemma 6.2.9.** *[Comparison of finite and infinite systems] Fix* $s > 0$ *and assume that, for any* $L(N)$ *satisfying* $\lim_{N\to\infty} L(N) = \infty$ *and* $\lim_{N\to\infty} L(N)/N = 0$*,*

$$\lim_{N\to\infty} \sup_{0\leq t\leq L(N)} \left|\bar{\Theta}^{[N]}(Ns) - \bar{\Theta}^{[N]}(Ns - t)\right| = 0 \quad \textit{in probability.} \tag{6.75}$$

*Let*

$$\big(X^{\mu_N}(t), Y^{\mu_N}(t)\big)_{t\geq 0} \tag{6.76}$$

*be the infinite system defined in* (6.74) *starting in the distribution* $\mu_N$*, where* $\mu_N$ *is defined by continuing the configuration of* $(X^{[N]}(Ns-L(N)), Y^{[N]}(Ns-L(N)))$ *periodically to* $([0,1]^2)^{\mathbb{N}_0}$*. Similarly, view* $(X^{[N]}(t), Y^{[N]}(t))$ *as an element of* $([0,1]^2)^{\mathbb{N}_0}$ *by periodic continuation. Then, for all* $t \geq 0$*,*

$$\begin{aligned}&\lim_{N\to\infty}\big|\mathbb{E}\big[f\big(X^{\mu_N}(t), Y^{\mu_N}(t)\big) - f\big(X^{[N]}(Ns-L(N)+t), Y^{[N]}(Ns-L(N)+t)\big)\big]\big| = 0\\ &\qquad \forall\, f \in \mathcal{C}\big(([0,1]^2)^{\mathbb{N}_0}, \mathbb{R}\big).\end{aligned} \tag{6.77}$$

Before we can prove that the infinite system $(X^{\mu_N}(t), Y^{\mu_N}(t))_{t\geq 0}$ converges to some limiting system as $N \to \infty$, we need the following regularity property for the estimator $\bar{\Theta}^{[N]}$. This is stated in our fifth lemma.

**Lemma 6.2.10 (Stability of the estimator for the conserved quantity).** *Define* $\mu_N$ *as in Lemma 6.2.9. Let* $(x_i, y_i)_{i\in[N]}$ *be distributed according to the exchangeable probability measure* $\mu_N$ *on* $([0,1]^2)^{\mathbb{N}_0}$ *restricted to* $([0,1]^2)^{[N]}$*. Suppose that* $\lim_{N\to\infty}\mu_N = \mu$ *for some exchangeable probability measure* $\mu$ *on* $([0,1]^2)^{\mathbb{N}_0}$*. Define a random variable* $\phi$ *on* $(\mu, ([0,1]^2)^{\mathbb{N}_0})$ *by putting*

$$\phi = \lim_{n\to\infty}\frac{1}{n}\sum_{i\in[n]}\frac{x_i + Ky_i}{1+K}, \tag{6.78}$$

*and a random variable* $\phi_N$ *on* $(\mu_N, ([0,1]^2)^{\mathbb{N}_0})$ *by putting*

$$\phi_N = \frac{1}{N}\sum_{i\in[N]}\frac{x_i + Ky_i}{1+K}. \tag{6.79}$$

*Then*

$$\lim_{N\to\infty}\mathcal{L}[\phi_N] = \mathcal{L}[\phi]. \tag{6.80}$$

In the sixth lemma we state the convergence of the law $\mathcal{L}[(X^{\mu_N}(t), Y^{\mu_N}(t))]$ to the law of a limiting system as $N \to \infty$.

**Lemma 6.2.11 (Uniformity of the ergodic theorem for the infinite system).** *Let* $\mu_N$ *be defined as in Lemma 6.2.9. Since* $(\mu_N)_{N\in\mathbb{N}}$ *is tight, it has convergent subsequences. Let* $(N_k)_{k\in\mathbb{N}}$ *be a subsequence such that* $\mu = \lim_{k\to\infty}\mu_{N_k}$*. Define*

$$\bar{\Theta} = \lim_{N\to\infty}\frac{1}{N}\sum_{i\in[N]}\frac{x_i^\mu + Ky_i^\mu}{1+K} \qquad \text{in } L_2(\mu), \tag{6.81}$$

*and let* $(X^\mu(t), Y^\mu(t))_{t\geq 0}$ *be the infinite system evolving according to*

$$\begin{aligned}\mathrm{d}x_i^\mu(t) &= c\,\big[\bar{\Theta} - x_i^\mu(t)\big]\,\mathrm{d}t + \sqrt{g(x_i^\mu(t))}\,\mathrm{d}w_i(t) + Ke\,[y_i^\mu(t) - x_i^\mu(t)]\,\mathrm{d}t,\\ \mathrm{d}y_i^\mu(t) &= e\,[x_i^\mu(t) - y_i^\mu(t)]\,\mathrm{d}t, \qquad i \in \mathbb{N}_0.\end{aligned} \tag{6.82}$$

*Then*

(a) *For all* $t \geq 0$,

$$\begin{aligned}&\lim_{k\to\infty}\big|\mathbb{E}\big[f\big(X^{\mu_{N_k}}(t),Y^{\mu_{N_k}}(t)\big)\big]-\mathbb{E}\big[f\big(X^{\mu}(t),Y^{\mu}(t)\big)\big]\big|=0,\\&\qquad\forall\, f\in\mathcal{C}\big(([0,1]^2)^{\mathbb{N}_0},\mathbb{R}\big).\end{aligned}\tag{6.83}$$

(b) *There exists a sequence* $(\bar{L}(N))_{N\in\mathbb{N}}$ *satisfying* $\lim_{N\to\infty}\bar{L}(N)=\infty$ *and* $\lim_{N\to\infty}\bar{L}(N)/N=0$ *such that*

$$\begin{aligned}&\lim_{k\to\infty}\big|\mathbb{E}\big[f\big(X^{[N_k]}(N_ks-L(N_k)+\bar{L}(N_k)),Y^{[N_k]}(N_ks-L(N_k)+\bar{L}(N_k))\big)\\&\qquad-f\big(X^{\mu_{N_k}}(\bar{L}(N_k)),Y^{\mu_{N_k}}(\bar{L}(N_k))\big)\big|\big]\\&\quad+\big|\mathbb{E}\big[f\big(X^{\mu_{N_k}}(\bar{L}(N_k)),Y^{\mu_{N_k}}(\bar{L}(N_k))\big)\big]-\mathbb{E}\big[f\big(X^{\mu}(\bar{L}(N_k)),Y^{\mu}(\bar{L}(N_k))\big)\big]\big|=0\\&\qquad\forall\, f\in\mathcal{C}\big(([0,1]^2)^{\mathbb{N}_0},\mathbb{R}\big).\end{aligned}\tag{6.84}$$

**Remark 6.2.12 (Existence of $\bar{\Theta}$).** Note that the limit in (6.81) is well-defined by the ergodic theorem in $L_2$, since $\mu$ is the limit of translation invariant measures and hence is itself translation invariant. ■

In the seventh lemma we provide a coupling of two copies of the finite system starting from different measures.

**Lemma 6.2.13 (Coupling of finite systems).** *Let* $(X^{[N],1},Y^{[N],1})$ *be a finite system evolving according to* (6.15) *and starting from some exchangeable measure. Let* $\mu^{[N],1}$ *be the measure obtain by periodic continuation of the configuration of* $(X^{[N],1}(0),Y^{[N],1}(0))$. *Similarly, let* $(X^{[N],2},Y^{[N],2})$ *be a finite system evolving according to* (6.15) *and starting from some exchangeable measure. Let* $\mu^{[N],2}$ *be the measure obtain by periodic continuation of the configuration of* $(X^{[N],2}(0),Y^{[N],2}(0))$. *Let* $\tilde{\mu}$ *be any weak limit point of the sequence of measures* $\{\mu^{[N],1}\times\mu^{[N],2}\}_{N\in\mathbb{N}}$. *Define random variables* $\bar{\Theta}^{[N],1}$ *on* $(\mu^{[N],1},([0,1]^2)^{\mathbb{N}_0})$, $\bar{\Theta}^{[N],2}$ *on* $(\mu^{[N],1},([0,1]^2)^{\mathbb{N}_0})$ *and* $\bar{\Theta}_1$ *and* $\bar{\Theta}_2$ *on* $(\mu,([0,1]^2)^{\mathbb{N}_0})$ *by*

$$\begin{aligned}&\bar{\Theta}^{[N],1}=\frac{1}{N}\sum_{i\in[N]}\frac{x_i^{[N],1}+Ky_i^{[N],1}}{1+K},\qquad \bar{\Theta}^{[N],2}=\frac{1}{N}\sum_{i\in[N]}\frac{x_i^{[N],2}+Ky_i^{[N],2}}{1+K},\\&\bar{\Theta}_1=\lim_{n\to\infty}\frac{1}{n}\sum_{i\in[n]}\frac{x_i^1+Ky_i^1}{1+K},\qquad \bar{\Theta}_2=\lim_{n\to\infty}\frac{1}{n}\sum_{i\in[n]}\frac{x_i^2+Ky_i^2}{1+K},\end{aligned}\tag{6.85}$$

*and let* $(\bar{\Theta}^{[N],1}(t))_{t\geq0}$ *and* $(\bar{\Theta}^{[N],2}(t))_{t\geq0}$ *be defined according to* (6.43) *for* $(X_1^{[N]},Y_1^{[N]})$, *respectively,* $(X_2^{[N]},Y_2^{[N]})$. *Assume that*

$$\lim_{N\to\infty}\sup_{0\leq t\leq L(N)}\big|\bar{\Theta}^{[N],k}(0)-\bar{\Theta}^{[N],k}(t)\big|=0\quad\text{in probability},\quad k\in\{1,2\},\tag{6.86}$$

*and suppose that* $\tilde{\mu}(\{\bar{\Theta}_1=\bar{\Theta}_2\})=1$. *Then, for any sequence* $t(N)\to\infty$,

$$\lim_{N\to\infty}\mathbb{E}\Big[\big|x_i^{[N],1}(t(N))-x_i^{[N],2}(t(N))\big|+K\big|y_i^{[N],1}(t(N))-y_i^{[N],2}(t(N))\big|\Big]=0.\tag{6.87}$$

**Step 2. Convergence of the estimator.** This step is the equivalent of [21, Proposition 2]. We first prove the tightness of the estimator $\bar{\Theta}^{[N]}$ in path space. After that we settle convergence of the finite-dimensional distributions and identify the limit.

**Proposition 6.2.14 (Convergence of average sum process).**

$$\lim_{N\to\infty} \mathcal{L}\left[\left(\bar{\Theta}^{[N]}(Ns)\right)_{s>0}\right] = \mathcal{L}[(\bar{\Theta}(s))_{s>0}], \tag{6.88}$$

*where* $(\bar{\Theta}(s))_{s>0}$ *evolves according to*

$$\mathrm{d}\bar{\Theta}(s) = \frac{1}{(1+K)}\sqrt{(\mathcal{F}g)(\bar{\Theta}(s))}\,\mathrm{d}w(s). \tag{6.89}$$

Proposition 6.2.14 follows from the following three lemmas, which are the equivalent of the three lemmas used in [21, p. 488–493] for the system without seed-bank.

**Lemma 6.2.15 (Martingale property of average sum process).**

(1) *The process* $(\bar{\Theta}^{[N]}(Ns))_{s>0}$ *is a square-integrable martingale with continuous paths and increasing process*

$$\left\langle \bar{\Theta}^{[N]}(Ns)\right\rangle_{s>0} = \frac{1}{(1+K)^2}\int_0^s \mathrm{d}r\,\frac{1}{N}\sum_{i\in[N]} g\big(x_i^{[N]}(Nr)\big). \tag{6.90}$$

(2) *Let* $L(N)$ *be such that* $\lim_{N\to\infty} L(N)=\infty$ *and* $\lim_{N\to\infty} L(N)/N=0$. *Then*

$$\lim_{N\to\infty}\sup_{0\le t\le L(N)} \left|\bar{\Theta}^{[N]}(Ns)-\bar{\Theta}^{[N]}(Ns-t)\right| = 0 \text{ in probability.} \tag{6.91}$$

(3) $\big(\mathcal{L}[(\bar{\Theta}^{[N]}(Ns))_{s>0}]\big)_{N\in\mathbb{N}}$ *is tight as a sequence of probability measures on* $\mathcal{C}([0,\infty),[0,1])$.

**Lemma 6.2.16 (Martingale property of limit process).** *Let* $(N_k)_{k\in\mathbb{N}}$ *be any subsequence such that*

$$\lim_{k\to\infty} \mathcal{L}\left[\big(\bar{\Theta}^{[N_k]}(N_k s)\big)_{s>0}\right] = \mathcal{L}\big[(\bar{\Theta}(s))_{s>0}\big]. \tag{6.92}$$

*Then* $(\bar{\Theta}(s))_{s>0}$ *is a square-integrable martingale with continuous paths, and*

$$\left(\bar{\Theta}^2(s) - \int_0^s \mathrm{d}r\,\frac{1}{(1+K)^2}\,\mathbb{E}^{\nu_{\bar{\Theta}(r)}}[g(x_0)]\right)_{s>0} \tag{6.93}$$

*is a martingale.*

**Lemma 6.2.17 (Uniqueness).** *The following martingale problem has a unique solution:*

$$\begin{gathered}(\bar{\Theta}_s)_{s>0} \text{ is a continuous martingale with values in } [0,1],\\ \left(\bar{\Theta}^2(s) - \frac{1}{(1+K)^2}\int_0^s \mathrm{d}r\,\mathbb{E}^{\nu_{\bar{\Theta}(r)}}[g(x_0)]\right)_{s>0} \text{ is a martingale.}\end{gathered} \tag{6.94}$$

*The solution of* (6.94) *is given by the diffusion generated by* $\mathbb{E}^{\nu_u}[g(\cdot)]\frac{\partial^2}{\partial u^2}$.

**Step 3. Convergence of the averages in the Meyer-Zheng topology.** Recall the definition of the Meyer-Zheng topology in Section 4.4.1. We have to prove the following proposition.

**Proposition 6.2.18 (Convergence in Meyer-Zheng topology).** *If*

$$\lim_{N\to\infty} \mathcal{L}\left[\left(\bar{\Theta}(Ns)\right)_{s>0}\right] = \mathcal{L}\left[\left(\bar{\Theta}(s)\right)_{s>0}\right], \tag{6.95}$$

*then*

$$\begin{aligned}\lim_{N\to\infty} \mathcal{L}\Big[\big(x_1^{[N]}(t), y_1^{[N]}(t)\big)_{t\geq 0}\Big] &= \mathcal{L}\Big[\big(x_1^{\mathbb{N}_0}(t), y_1^{\mathbb{N}_0}(t)\big)_{t\geq 0}\Big] \\ &\text{in the Meyer-Zheng topology,}\end{aligned} \tag{6.96}$$

*where* $(x_1^{\mathbb{N}_0}(t), y_1^{\mathbb{N}_0}(t))_{t\geq 0})$ *evolves according to* (6.55).

To prove Proposition 6.2.18 we will use Lemma 6.2.7 in combination with the following three general lemmas about the Meyer-Zheng topology, which are proven in Appendix B.2.3.

**Lemma 6.2.19 (Convergence in probability in the Meyer-Zheng topology).** *Let* $((Z_n(t))_{t\geq 0})_{n\in\mathbb{N}}$ *and* $(Z(t))_{t\geq 0}$ *be stochastic processes on the Polish space* $(E,d)$*. If, for all* $t \geq 0$,

$$\lim_{n\to\infty} \mathbb{E}\left[d(Z_n(t), Z(t))\right] = 0, \tag{6.97}$$

*then,*

$$\lim_{n\to\infty} (Z_n(t))_{t\geq 0} = (Z(t))_{t\geq 0} \text{ in probability in the Meyer-Zheng topology.} \tag{6.98}$$

**Lemma 6.2.20 (Convergence of the joint law).** *Let* $((X_n(t))t \geq 0)_{n\in\mathbb{N}}$, $((Y_n(t))_{t\geq 0})_{n\in\mathbb{N}}, (X(t))_[t \geq 0]$ *be stochastic processes on a metric space* $(E,d)$ *and let* $c \in E$ *be a constant. If* $\lim_{n\to\infty} \mathcal{L}[X_n] = \mathcal{L}[X]$ *in the Meyer-Zheng topology and for all* $t \geq 0$, $\lim_{n\to\infty} \mathbb{E}[d(Y_n(t), c)] = 0$*, then* $\lim_{n\to\infty} \mathcal{L}[(X_n, Y_n)] = \mathcal{L}[(X, c)]$ *in the Meyer-Zheng topology.*

**Lemma 6.2.21 (Continuous mapping theorem).** *Let* $f\colon E \to E$ *be a continuous function and* $x \in M_E[0,\infty)$.

*(a) The function*

$$h\colon \Psi \to \Psi, \qquad \psi_x \to \psi_{f(x)}, \tag{6.99}$$

*is continuous.*

*(b) If the stochastic processes* $(X_n)_{n\in\mathbb{N}}, X$ *on state space* $(E,d)$ *satisfy*

$$\lim_{n\to\infty} \mathcal{L}[X_n] = \mathcal{L}[X] \text{ in the Meyer-Zheng topology,} \tag{6.100}$$

*then*

$$\lim_{n\to\infty} \mathcal{L}[f(X_n)] = \mathcal{L}[f(X)] \text{ in the Meyer-Zheng topology.} \tag{6.101}$$

Note that Lemma 6.2.21 allows us to use the continuous mapping theorem in the Meyer-Zheng topology.

**Step 4. Mean-field finite-systems scheme.** Use Steps 1– 4 to prove Proposition 6.2.1.

Having completed the abstract scheme of steps 1–4, we set out to prove the constituent propositions and lemmas.

# §6.3 Proofs: $N \to \infty$, mean-field, proof of abstract scheme

In Sections 6.3.1–6.3.4 we prove the propositions and the lemmas stated in Steps 1–4 in Section 6.2.2.

## §6.3.1 Proof of step 1. Equilibrium of the single components

We start by proving Proposition 6.2.4 with the help of the seven lemmas stated in Step 1 of Section 6.2.2. Afterwards we prove each of the lemmas.



• **Proof of Proposition 6.2.4**

*Proof.* We use an argument similar to the one used in [21, Section 2 (i)]. Let $(L(N))_{N\in\mathbb{N}}$ be any sequence satisfying $\lim_{N\to\infty} L(N) = \infty$ and $\lim_{N\to\infty} L(N)/N = 0$. Let $\mu_N$ be the measure on $([0,1]^2)^{\mathbb{N}_0}$ obtained by periodic continuation of $\mathcal{L}[X^{[N]}(Ns - L(N)), Y^{[N]}(Ns - L(N))]$. Note that $([0,1]^2)^{\mathbb{N}_0}$ is compact. Hence, letting $(N_k)_{k\in\mathbb{N}}$ be the subsequence in Proposition 6.2.4, we can pass to a further subsequence and obtain

$$\lim_{k\to\infty} \mu_{N_k} = \mu. \tag{6.102}$$

Since we assumed that $\mathcal{L}[X^{[N]}(0), Y^{[N]}(0)]$ is exchangeable and the dynamics preserves exchangeability, the measures $\mu_{N_k}$ are exchangeable and also the limiting law $\mu$ is exchangeable. Define $\phi$ as in (6.78) in Lemma 6.2.10. Then we can condition on $\phi$ and write

$$\mu = \int_{[0,1]} \mu_\rho \, \mathrm{d}\Lambda(\rho), \tag{6.103}$$

where $\Lambda(\cdot) = \mathcal{L}[\phi]$. By assumption we know that

$$\lim_{k\to\infty} \mathcal{L}\big[\bar\Theta^{[N_k]}(N_k s)\big] = P_s \tag{6.104}$$

and

$$\lim_{k\to\infty} \mathcal{L}\left[\sup_{0\le t\le L(N_k)} \left|\bar\Theta^{[N_k]}(N_k s) - \bar\Theta^{[N_k]}(N_k s - t)\right|\right] = \delta_0. \tag{6.105}$$

Hence

$$\lim_{k\to\infty} \mathcal{L}\left[\bar\Theta^{[N_k]}(N_k s - L(N_k))\right] = P_s. \tag{6.106}$$

Recall that

$$\Lambda = \mathcal{L}[\phi] = \mathcal{L}\left[\lim_{n\to\infty}\frac{1}{n}\sum_{i\in[n]}\frac{x_i+Ky_i}{1+K}\right] \qquad \text{on } (\mu,([0,1]^2)^{\mathbb{N}_0}). \tag{6.107}$$

By Lemma 6.2.10, if $\phi_{N_k} = \frac{1}{N_k}\sum_{i\in[N_k]}\frac{x_i+Ky_i}{1+K}$ on $(\mu_{N_k},([0,1]^2)^{\mathbb{N}_0})$, then $\lim_{k\to\infty}\mathcal{L}[\phi_{N_k}] = \mathcal{L}[\phi]$. Taking the subsequence $(\mu_{N_k})_{k\in\mathbb{N}}$, we get $\Lambda(\cdot) = P_s(\cdot)$, and hence

$$\mu = \int_{[0,1]} \mu_\rho \,\mathrm{d}P_s(\rho). \tag{6.108}$$

Let $\bar{L}(N)$ be the sequence constructed in Lemma 6.2.11[b]. We can require that $\bar{L}(N) \le L(N)$ for all $N \in \mathbb{N}$. Write

$$\begin{aligned}
&\mathcal{L}\big[X^{[N_k]}(N_k s - L(N_k)+\bar{L}(N_k)), Y^{[N_k]}(N_k s - L(N_k)+\bar{L}(N_k))\big]\\
&= \mathcal{L}\big[X^{[N_k]}(N_k s - L(N_k)+\bar{L}(N_k)), Y^{N_k}(N_k s - L(N_k)+\bar{L}(N_k))\big]\\
&\quad - \mathcal{L}\big[X^{\mu_{N_k}}(\bar{L}(N_k)), Y^{\mu_{N_k}}(\bar{L}(N_k))\big],\\
&\quad + \mathcal{L}\big[X^{\mu_{N_k}}(\bar{L}(N_k)), Y^{\mu_{N_k}}(\bar{L}(N_k))\big] - \mathcal{L}\big[X^{\mu}(\bar{L}(N_k)), Y^{\mu}(\bar{L}(N_k))\big]\\
&\quad + \mathcal{L}\big[X^{\mu}(\bar{L}(N_k)), Y^{\mu}(\bar{L}(N_k))\big].
\end{aligned} \tag{6.109}$$

By Lemma 6.2.11, the first and the second term tend to zero as $k\to\infty$. Hence

$$\mathcal{L}\big[X^{[N_k]}(N_k s - L(N_k)+\bar{L}(N_k)), Y^{[N_k]}(N_k s - L(N_k)+\bar{L}(N_k))\big] \tag{6.110}$$

tends to $\mathcal{L}\big[X^{\mu}(L(N_k)), Y^{\mu}(L(N_k))\big]$ as $k\to\infty$. By (6.108),

$$\mathcal{L}\big[X^{\mu}(\bar{L}(N_k)), Y^{\mu}(\bar{L}(N_k))\big] = \int_{[0,1]} \mathcal{L}\big[X^{\mu_\rho}(\bar{L}(N_k)), Y^{\mu_\rho}(\bar{L}(N_k))\big]\,\mathrm{d}P_s(\rho). \tag{6.111}$$

Since $\lim_{k\to\infty}\bar{L}(N_k) = \infty$, by Lemma 6.2.5 we have

$$\lim_{k\to\infty}\mathcal{L}\big[X^{\mu_\rho}(\bar{L}(N_k)), Y^{\mu_\rho}(\bar{L}(N_k))\big] = \nu_\rho. \tag{6.112}$$

Therefore, by (6.109) and Lemma 6.2.6,

$$\begin{aligned}
&\lim_{k\to\infty}\mathcal{L}\big[X^{[N_k]}(N_k s - L(N_k)+\bar{L}(N_k)), Y^{[N_k]}(N_k s - L(N_k)+\bar{L}(N_k))\big]\\
&\quad = \int_{[0,1]} \nu_\rho \,\mathrm{d}P_s(\rho).
\end{aligned} \tag{6.113}$$

To show that

$$\lim_{k\to\infty}\mathcal{L}\big[X^{[N_k]}(N_k s), Y^{[N_k]}(N_k s)\big] = \int_{[0,1]} \nu_\rho\,\mathrm{d}P_s(\rho). \tag{6.114}$$

we invoke Lemma 6.2.13. Let $(X^{[N],1}, Y^{[N],1})$ be the finite system starting from

$$\mathcal{L}\big[X^{[N]}(Ns - L(N)), Y^{[N]}(Ns - L(N))\big], \tag{6.115}$$

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and let $(\bar{L}(N))_{N\in\mathbb{N}}$ be the sequence such that (6.113) holds. Let $(X^{[N],2}.Y^{[N],2})$ be the finite system starting from

$$\mathcal{L}\big[X^{[N]}(Ns-\bar{L}(N)),Y^{[N]}(Ns-\bar{L}(N))\big]. \tag{6.116}$$

Choose for the sequence $t(N)$ in Lemma 6.2.13 the sequence $\bar{L}(N)$. Let $\mu^{[N],1}$ be defined by periodic continuation of $(X^{[N]}(Ns-L(N)),Y^{[N]}(Ns-L(N)))$, and $\mu^{[N],2}$ by periodic continuation of $(X^{[N]}(Ns-\bar{L}(N)),Y^{[N]}(Ns-\bar{L}(N)))$. Defining $\bar{\Theta}_1$ and $\bar{\Theta}_2$ according to (6.85), where for $\mu^{[N],2}$ we replace $L(N)$ by $\bar{L}(N)$, we get

$$\lim_{k\to\infty}|\bar{\Theta}_1^{N_k}-\bar{\Theta}_2^{N_k}|=\lim_{k\to\infty}\big|\bar{\Theta}^{N_k}(N_ks-L(N_k))-\bar{\Theta}^{N_k}(N_ks-\bar{L}(N_k))\big|=0 \text{ in probability} \tag{6.117}$$

by the assumptions in (6.63). Hence, if $\mu$ is any weak limit point of the sequence $(\mu^{[N_k],1}\times\mu^{[N_k],2})_{k\in\mathbb{N}}$, then

$$\mu(\bar{\Theta}_1=\bar{\Theta}_2)=1. \tag{6.118}$$

By passing to a further subsequence, we can now apply Lemma 6.2.13, to obtain

$$\lim_{k\to\infty}\mathbb{E}\left[|x_{i,1}^{N_k}(\bar{L}(N_k))-x_{i,2}^{N_k}(\bar{L}(N_k))|+K|y_{i,1}^{N_k}(\bar{L}(N_k))-y_{i,2}^{N_k}(\bar{L}(N_k))|\right]=0. \tag{6.119}$$

Note that

$$\begin{aligned}&\mathcal{L}\big[X_1(\bar{L}(N_k),Y_1(\bar{L}(N_k)))\big]\\&=\mathcal{L}\big[X^{[N_k]}(N_ks-L(N_k)+\bar{L}(N_k)),Y^{[N_k]}(N_ks-L(N_k)+\bar{L}(N_k))\big],\\&\mathcal{L}\big[X_2(\bar{L}(N_k),Y_2(\bar{L}(N_k)))\big]=\mathcal{L}\big[X^{[N_k]}(N_ks),Y^{[N_k]}(N_ks)\big].\end{aligned} \tag{6.120}$$

Moreover, we know from (6.113) that

$$\begin{aligned}&\lim_{k\to\infty}\mathcal{L}\left[X^{[N_k]}(N_ks-L(N_k)+\bar{L}(N_k)),Y^{[N_k]}(N_ks-L(N_k)+\bar{L}(N_k))\right]\\&=\int_{[0,1]}\nu_\rho P_s(\mathrm{d}\rho).\end{aligned} \tag{6.121}$$

Combining (6.119)–(6.121), we find that

$$\mathcal{L}\big[X^{[N_k]}(N_ks),Y^{[N_k]}(N_ks)\big]=\int_{[0,1]}\nu_\rho P_s(\mathrm{d}\rho). \tag{6.122}$$

□

In the remainder of this section we prove Lemmas 6.2.5–6.2.11 and 6.2.13.

- **Proof of Lemma 6.2.5**

*Proof.* Since the components of the infinite system in (6.42) evolve independently, it is enough to show that each component converges to $\Gamma_\theta$. This convergence follows from the proof of Proposition 6.1.2 (see Section 6.1.3). Hence the infinite system defined by (6.18) converges to $\nu_\theta=\Gamma_\theta^{\otimes\mathbb{N}_0}$. Ergodicity of $\nu_\theta$ with respect to translations follows from Kolmogorov's zero-one law. □

• **Proof of Lemma 6.2.7**

*Proof.* Using the definition of $\Theta_x^{[N]}(t)$, $\Theta_y^{[N]}(t)$ in (6.44) and the SSDE in (6.15), we find the following evolution for the averages:

$$\begin{aligned} \mathrm{d}\Theta_x^{[N]}(t) &= \frac{1}{N}\sum_{i\in[N]} \sqrt{g(x_i(t))}\,\mathrm{d}w_i(t) + Ke\,[\Theta_y^{[N]}(t) - \Theta_x^{[N]}(t)]\,\mathrm{d}t, \\ \mathrm{d}\Theta_y^{[N]}(t) &= e\,[\Theta_x^{[N]}(t) - \Theta_y^{[N]}(t)]\,\mathrm{d}t. \end{aligned} \tag{6.123}$$

Consequently,

$$\begin{aligned} \mathrm{d}\big(\Delta_{\bar\Theta}^{[N]}(t)\big)^2 &= 2\Delta_{\bar\Theta}^{[N]}(t)\,\mathrm{d}\Delta_{\bar\Theta}^{[N]}(t) + 2\mathrm{d}\langle \Delta_{\bar\Theta}^{[N]}\rangle(t) \\ &= \Delta_{\bar\Theta}^{[N]}(t)\frac{1}{N}\sum_{i\in[N]} \sqrt{g(x_i(t))}\,\mathrm{d}w_i(t) - (Ke+e)\,\big(\Delta_{\bar\Theta}^{[N]}(t)\big)^2\,\mathrm{d}t \\ &\qquad + 2\frac{1}{N^2}\sum_{i\in[N]} g(x_i(t))\;\mathrm{d}t, \end{aligned} \tag{6.124}$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}\left[\big(\Delta_{\bar\Theta}^{[N]}(t)\big)^2\right] = -2(Ke+e)\,\mathbb{E}\left[\big(\Delta_{\bar\Theta}^{[N]}(t)\big)^2\right] + \frac{2}{N^2}\sum_{i\in[N]} g(x_i(t)) \tag{6.125}$$

and

$$\mathbb{E}\left[\big(\Delta_{\bar\Theta}^{[N]}(t)\big)^2\right] = \mathbb{E}\left[\big(\Delta_{\bar\Theta}^{[N]}(0)\big)^2\right]\mathrm{e}^{-2(Ke+e)t} + \int_0^t \mathrm{d}r\,\mathrm{e}^{-2(Ke+e)(t-r)}\,\frac{2}{N^2}\sum_{i\in[N]} g(x_i(r)). \tag{6.126}$$

Therefore we get the bound

$$\mathbb{E}\left[\big|\Delta_{\bar\Theta}^{[N]}(t)\big|\right] \leq \sqrt{\mathbb{E}\left[\big(\Delta_{\bar\Theta}^{[N]}(0)\big)^2\right]}\mathrm{e}^{-(Ke+e)t} + \sqrt{\frac{2\|g\|}{N(Ke+e)}}. \tag{6.127}$$

□

• **Proof of Lemma 6.2.9**

*Proof.* To compare the systems in (6.15) and (6.74), we couple them via their Brownian motions. Therefore for all $i \in [N]$ we assume that the evolution in (6.15) and (6.74) is driven by the same Brownian motion, $\tilde w_i = w_i$. If $i \notin [N]$, then we set $w_i = w_j$ for $j = i \mod N$. We denote the coupled process by $\tilde z(t) = (\tilde z_i(t))_{i\in\mathbb{N}_0} = (\tilde z_i^{[N]}(t), \tilde z_i^{\mu_N}(t))_{i\in\mathbb{N}_0}$, where $\tilde z_i^{[N]}(t) = (\tilde x_i^{[N]}(t), \tilde y_i^{[N]}(t))$ and $\tilde z_i^{\mu_N}(t) = (\tilde x_i^{\mu_N}(t), \tilde y_i^{\mu_N}(t))$. The tilde indicates that we are considering the coupled process, and

$$\begin{aligned} \mathcal{L}[\tilde z(0)] &= \mathcal{L}\big[X^{[N]}(Ns - L(N)), Y^{[N]}(Ns - L(N))\big] \times \mu_N \\ &= \mathcal{L}\big[X^{[N]}(Ns - L(N)), Y^{[N]}(Ns - L(N))\big]^2. \end{aligned} \tag{6.128}$$

Define

$$\Delta_i^N(t) = \tilde x_i^{[N]}(t) - \tilde x_i^{\mu_N}(t), \qquad \delta_i^N(t) = \tilde y_i^{[N]}(t) - \tilde y_i^{\mu_N}(t). \tag{6.129}$$

To prove that the coupling is successful, we show that, for all $t \geq 0$,

$$\lim_{N\to\infty} \mathbb{E}\left[|\Delta_i^N(t)| + K|\delta_i^N(t)|\right] = 0 \quad \forall\, i \in \mathbb{N}_0. \tag{6.130}$$

From now on we will only consider sites $i \in [0,N]$ for which both infinite systems have the same Brownian motion.

From (6.15) and (6.74) it follows that

$$\begin{aligned}
&\mathrm{d}\left[|\Delta_i^N(t)| + K|\delta_i^N(t)|\right]\\
&\quad = (\mathrm{sgn}\ \Delta_i^N(t))\ \mathrm{d}\Delta_i^N(t) + \mathrm{d}L_t^0 + K\,\mathrm{sgn}\ \delta_i^N(t)\ \mathrm{d}\delta_i^N(t)\\
&\quad = -c\ (\mathrm{sgn}\ \Delta_i^N(t))\ \Delta_i^N(t)\,\mathrm{d}t + c\,(\mathrm{sgn}\ \Delta_i^N(t))\ \left[\bar\Theta^{[N]}(t) - \bar\Theta^{[N]}\right]\ \mathrm{d}t\\
&\qquad + c\,(\mathrm{sgn}\ \Delta_i^N(t))\ \left[\Theta_x^{[N]}(t) - \bar\Theta^{[N]}(t)\right]\ \mathrm{d}t\\
&\qquad + (\mathrm{sgn}\ \Delta_i^N(t))\ \left(\sqrt{g(x_i^{[N]}(t))} - \sqrt{g(x_i^{\mu_N}(t))}\right)\mathrm{d}w_i(t)\\
&\qquad + (\mathrm{sgn}\ \Delta_i^N(t))\,Ke\,[\delta_i^N(t) - \Delta_i^N(t)]\,\mathrm{d}t\\
&\qquad + (\mathrm{sgn}\ \delta_i^N(t))\,Ke\,[\Delta_i^N(t) - \delta_i^N(t)]\,\mathrm{d}t,
\end{aligned} \tag{6.131}$$

where we use that the local time $L_t^0$ is zero, since $g$ is Lipschitz (see [63, Proposition V.39.3]).

Taking expectations in (6.131), we find

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[|\Delta_i^N(t)| + K|\delta_i^N(t)|] = &-c\,\mathbb{E}\left[|\Delta_i^N(t)|\right]\\
&+ c\ \mathbb{E}\left[(\mathrm{sgn}\ \Delta_i^N(t))\ \left[\bar\Theta^{[N]}(t) - \bar\Theta^{[N]}\right]\right]\\
&+ c\,\mathbb{E}\left[(\mathrm{sgn}\ \Delta_i^N(t))\ \left[\Theta_x^{[N]}(t) - \bar\Theta^{[N]}(t)\right]\right]\\
&+ Ke\,\mathbb{E}\left[\left(\mathrm{sgn}\ \Delta_i^N(t) - \mathrm{sgn}\ \delta_i^N(t)\right)\left[\delta_i^N(t) - \Delta_i^N(t)\right]\right].
\end{aligned} \tag{6.132}$$

Note that we can rewrite (6.132) as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[|\Delta_i^N(t)| + K|\delta_i^N(t)|] = &-c\,\mathbb{E}\left[|\Delta_i^N(t)|\right]\\
&- 2Ke\,\mathbb{E}\left[1_{\mathrm{sgn}\ \Delta_i^N(t)\neq \mathrm{sgn}\ \delta_i^N(t)}\ \left[|\delta_i^N(t)| + |\Delta_i^N(t)|\right]\right]\\
&+ c\ \mathbb{E}\left[(\mathrm{sgn}\ \Delta_i^N(t))\ \left[\bar\Theta^{[N]}(t) - \bar\Theta^{[N]}\right]\right]\\
&+ c\,\mathbb{E}\left[(\mathrm{sgn}\ \Delta_i^N(t))\ \left[\Theta_x^{[N]}(t) - \bar\Theta^{[N]}(t)\right]\right].
\end{aligned} \tag{6.133}$$

It therefore follows that

$$\begin{aligned}\mathbb{E}[|\Delta_i^N(t)| + K|\delta_i^N(t)|] = \mathbb{E}[|\Delta_i^N(0)| &+ K|\delta_i^N(0)|] \\ &- c\int_0^t \mathrm{d}r\, \mathbb{E}\left[|\Delta_i^N(r)|\right] \\ &- 2Ke\int_0^t \mathrm{d}r\, \mathbb{E}\left[1_{\mathrm{sgn}\ \Delta_i^N(r) \neq \mathrm{sgn}\ \delta_i^N(r)}\ [|\delta_i^N(r)| + |\Delta_i^N(r)|]\right] \\ &+ \int_0^t \mathrm{d}r\, c\,\mathbb{E}\left[(\mathrm{sgn}\ \Delta_i^N(r))\ \left[\bar{\Theta}^{[N]}(r) - \bar{\Theta}^{[N]}\right]\right] \\ &+ \int_0^t \mathrm{d}r\, c\,\mathbb{E}\left[(\mathrm{sgn}\ \Delta_i^N(r))\ \left[\Theta_x^{[N]}(r) - \bar{\Theta}^{[N]}(r)\right]\right].\end{aligned} \tag{6.134}$$

Note that, by the choice of initial distribution for the coupling, we have

$$\mathbb{E}[|\Delta_i^N(0)| + K|\delta_i^N(0)|] = 0. \tag{6.135}$$

Therefore we get

$$\begin{aligned}0 &\leq \mathbb{E}[|\Delta_i^N(t)| + K|\delta_i^N(t)|] \\ &\leq -c\int_0^t \mathrm{d}r\, \mathbb{E}\left[|\Delta_i^N(r)|\right] \\ &\quad - 2Ke\int_0^t \mathrm{d}r\, \mathbb{E}\left[1_{\mathrm{sgn}\ \Delta_i^N(r) \neq \mathrm{sgn}\ \delta_i^N(r)}\ [|\delta_i^N(r)| + |\Delta_i^N(r)|]\right] \\ &\quad + \int_0^t \mathrm{d}r\, c\,\mathbb{E}\left[\left|\bar{\Theta}^{[N]}(r) - \bar{\Theta}^{[N]}\right|\right] \\ &\quad + \int_0^t \mathrm{d}r\, c\,\mathbb{E}\left[\left|\Theta_x^{[N]}(r) - \bar{\Theta}^{[N]}(r)\right|\right] \\ &\leq t\left(\sup_{0 \leq r \leq t} c\ \mathbb{E}\left[\left|\bar{\Theta}^{[N]}(r) - \bar{\Theta}^{[N]}\right|\right] + c\,\mathbb{E}\left[\left|\Theta_x^{[N]}(r) - \bar{\Theta}^{[N]}(r)\right|\right]\right).\end{aligned} \tag{6.136}$$

Hence, by the assumption in (6.75) and Lemma 6.2.7 (recall (6.128)) , we see that, for all $t > 0$,

$$\lim_{N \to \infty} \mathbb{E}[|\Delta_i^N(t)| + K|\delta_i^N(t)|] = 0. \tag{6.137}$$

Therefore, for every Lipschitz function $f \in \mathcal{C}\big(([0,1]),\mathbb{R}\big)$ of $x_i(t)$,

$$\lim_{n \to \infty} \left|\mathbb{E}[f(x_i^{[N]}(t)) - f(x_i^{\mu_N}(t))]\right| \leq \lim_{n \to \infty} \mathrm{Lip} f\, \mathbb{E}[||\Delta_i^N(L(N))||] = 0, \tag{6.138}$$

and the same holds for Lipschitz functions of $y_i$. Using that the Lipschitz functions are dense in $\mathcal{C}\big(([0,1]),\mathbb{R}\big)$, we obtain that the result actually holds for all $f \in \mathcal{C}\big(([0,1]^2)^{\mathbb{N}_0},\mathbb{R}\big)$ depending on finitely many components. This in turn implies that the result holds for all $f \in \mathcal{C}\big(([0,1]^2)^{\mathbb{N}_0},\mathbb{R}\big)$. □

- **Proof of Lemma 6.2.10**

*Proof.* Define

$$D^N(Z) = \frac{1}{N}\sum_{j\in[N]} \frac{x_j + Ky_j}{1+K}, \qquad D(Z) = \lim_{N\to\infty} D^N(Z) \text{ in } L_2(\mu). \tag{6.139}$$

Since $\mu$ is translation invariant with $\int_{[0,1]^2} \frac{x_0+Ky_0}{1+K}\,\mathrm{d}\mu < 1$, the $L_2(\mu)$-limit $D(Z)$ exists by the ergodic theorem. Since, by assumption, $\mu_N \to \mu$ as $N \to \infty$ for all fixed $M \in \mathbb{N}_0$, we have

$$\lim_{N\to\infty} \mathcal{L}_{\mu_N}[D^M(Z)] = \mathcal{L}_\mu[D^M(Z)]. \tag{6.140}$$

Therefore, in order to prove Lemma 6.2.10, we are left to show

$$\lim_{M\to\infty} \sup_{N\geq M} \|D^M(Z) - D^N(Z)\|_{L_2(\mu_N)} = 0. \tag{6.141}$$

This can be done by using Fourier transforms and spectral densities, and to do so we follow the same strategy as in [21, Lemma 2.5].

Define

$$\bar{\theta}^N = \mathbb{E}^{\mu_N}\left[\frac{x_0+Ky_0}{1+K}\right]. \tag{6.142}$$

Since $\mu_N$ is translation invariant on $\mathbb{N}_0$, by Herglotz's theorem there exists a unique measure $\lambda_N$ such that, for all $j,k \in \mathbb{N}_0$,

$$\mathbb{E}^{\mu_N}\left[\left(\frac{x_j+Ky_j}{1+K} - \bar{\theta}^N\right)\left(\frac{x_k+Ky_k}{1+K} - \bar{\theta}^N\right)\right] = \int_{(-\pi,\pi]} \lambda_N(\mathrm{d}u)\, \mathrm{e}^{\mathrm{i}(j-k)u}. \tag{6.143}$$

For $N \in \mathbb{N}_0$, define

$$D^N(u) = \frac{1}{N}\sum_{j\in[N]} \mathrm{e}^{\mathrm{i}ju}. \tag{6.144}$$

By (6.143), it follows that

$$\left\|D^M(Z) - D^N(Z)\right\|_{L_2(\mu_N)} = \left\|D^M(u) - D^N(u)\right\|_{L_2(\lambda_N)}. \tag{6.145}$$

Polynomials of the type $D^N(u)$ are called trigonometric polynomials and satisfy:

(a) $\lim_{N\to\infty} D^N(u) = 1_{\{0\}}(u)$.

(b) For $\delta > 0$ and $M < \infty$ there exists an $\epsilon(M,\delta)$ such that, for all $N \geq M$,

$$|D^N(u) - 1_{\{0\}}(u)| \leq 1_{(-\delta,\delta)\setminus\{0\}} + \epsilon(M,\delta) \text{ with } \epsilon(M,\delta) \to 0 \text{ as } M \to \infty. \tag{6.146}$$

Hence it follows that

$$\left\|D^M(u) - D^N(u)\right\|^2_{L_2(\lambda_N)} \leq 2\lambda_N((-\delta,\delta)\setminus\{0\}) + 2\epsilon(M,\delta). \tag{6.147}$$

Now let $M \to \infty$, to obtain

$$\sup_{N\geq M} \|D^M(u) - D^N(u)\|_{L_2(\lambda_N)} \leq 2\lambda_N((-\delta,\delta)\setminus\{0\}). \tag{6.148}$$

Subsequently let $\delta \to 0$, so that $(-\delta,\delta)\setminus\{0\} \to \emptyset$ and

$$\lim_{M\to\infty} \sup_{N\geq M} \|D^M(u) - D^N(u)\|^2_{L_2(\lambda_N)} = 0. \tag{6.149}$$

□

• **Proof of Lemma 6.2.11**

*Proof.* We first prove Lemma 6.2.11[1]. Afterwards we construct $(\bar{L}(N))_{N\in\mathbb{N}}$ to prove Lemma 6.2.11[2].

Since $\lim_{k\to\infty} \mu_{N_k} = \mu$, Lemma 6.2.10 implies that $\lim_{k\to\infty} \mathcal{L}[\bar{\Theta}^{[N_k]}] = \mathcal{L}[\bar{\Theta}]$. For ease of notation we drop the subsequence notation in the remainder of this proof. By Skohorod's theorem we can construct the random variables $(z_i^{\mu_N})_{N\in\mathbb{N}}$ and $z_i^{\mu}$ on one probability space such that $\lim_{N\to\infty} z_i^{\mu_N} = z_i$ a.s. Then, as in the proof of Lemma 6.2.10, we obtain

$$\lim_{N\to\infty} \mathbb{E}[|\bar{\Theta}^{[N]} - \bar{\Theta}|] = 0. \tag{6.150}$$

To prove the claim we couple the two infinite systems, namely,

$$\begin{aligned} \mathrm{d}x_i^{\mu_N}(t) &= c\left[\bar{\Theta}^{[N]} - x_i^{\mu_N}(t)\right]\mathrm{d}t + \sqrt{g(x_i^{\mu_N}(t))}\,\mathrm{d}w_i(t) + Ke\,[y_i^{\mu_N}(t) - x_i^{\mu_N}(t)]\,\mathrm{d}t,\\ \mathrm{d}y_i^{\mu_N}(t) &= e\,[x_i^{\mu_N}(t) - y_i^{\mu_N}(t)]\,\mathrm{d}t, \qquad i\in\mathbb{N}_0, \end{aligned} \tag{6.151}$$

and

$$\begin{aligned} \mathrm{d}x_i^{\mu}(t) &= c\left[\bar{\Theta} - x_i^{\mu}(t)\right]\mathrm{d}t + \sqrt{g(x_i^{\mu}(t))}\,\mathrm{d}w_i(t) + Ke\,[y_i^{\mu}(t) - x_i^{\mu}(t)]\,\mathrm{d}t,\\ \mathrm{d}y_i^{\mu}(t) &= e\,[x_i^{\mu}(t) - y_i^{\mu}(t)]\,\mathrm{d}t, \qquad i\in\mathbb{N}_0, \end{aligned} \tag{6.152}$$

are coupled by using the same Brownian motions in (6.151) and (6.152). Like before, define $\Delta_i^{\mu_N} = x_i^{\mu_N} - x_i^{\mu}$ and $\delta_i^{\mu_N} = y_i^{\mu_N} - y_i^{\mu}$. By the construction with Skohorod's theorem, we have that

$$\lim_{N\to\infty} \mathbb{E}\big[|\Delta_i^{\mu_N}(0)| + K|\delta_i^{\mu_N}(0)|\big] = 0. \tag{6.153}$$

To prove that, for all $t \geq 0$,

$$\lim_{N\to\infty} \mathbb{E}\big[|\Delta_i^{\mu_N}(t)| + K|\delta_i^{\mu_N}(t)|\big] = 0, \tag{6.154}$$

we proceed as in the proof of Lemma 6.2.9. By Itô-calculus, we find that

$$\begin{aligned} &\mathbb{E}\big[|\Delta_i^{\mu_N}(t)| + K|\delta_i^{\mu_N}(t)|\big]\\ &= \mathbb{E}\big[|\Delta_i^{\mu_N}(0)| + K|\delta_i^{\mu_N}(0)|\big]\\ &\quad - c\int_0^t \mathrm{d}r\,\mathbb{E}\left[|\Delta_i^{\mu_N}(r)|\right]\\ &\quad - 2Ke\int_0^t \mathrm{d}r\,\mathbb{E}\left[1_{\mathrm{sgn}\,\Delta_i^{\mu_N}(r)\neq \mathrm{sgn}\,\delta_i^{\mu_N}(r)}\,[|\delta_i^{\mu_N}(r)| + |\Delta_i^{\mu_N}(r)|]\right]\\ &\quad + \int_0^t \mathrm{d}r\,c\,\mathbb{E}\left[\left|\bar{\Theta}^{[N]} - \bar{\Theta}\right|\right]\\ &\leq \mathbb{E}\big[|\Delta_i^{\mu_N}(0)| + K|\delta_i^{\mu_N}(0)|\big] + tc\,\mathbb{E}\big[\big|\bar{\Theta}^{[N]} - \bar{\Theta}\big|\big]. \end{aligned} \tag{6.155}$$

From (6.155) it the follows that, for every $t \geq 0$,

$$\lim_{N\to\infty} \mathbb{E}\big[|\Delta_i^{\mu_N}(t)| + K|\delta_i^{\mu_N}(t)|\big] = 0. \tag{6.156}$$

We next construct the sequence $(\bar{L}(N))_{N\in\mathbb{N}}$. From (6.137) and (6.156),we have

$$\lim_{N\to\infty} \mathbb{E}\big[|\Delta_i^N(t)| + K|\delta_i^N(t)|\big] + \mathbb{E}\big[|\Delta_i^{\mu_N}(t)| + K|\delta_i^{\mu_N}(t)|\big] = 0. \tag{6.157}$$

Let $(t_k)_{k\in\mathbb{N}}$ be an increasing sequence such that $\lim_{k\to\infty} t_k = \infty$ and $\lim_{k\to\infty} t_k/k = 0$. By (6.157), we can for each $k$ find an $N_k \in \mathbb{N}$ such that, for all $N \geq N_k$,

$$\mathbb{E}\big[|\Delta_i^N(t_k)| + K|\delta_i^N(t_k)|\big] + \mathbb{E}\big[|\Delta_i^{\mu_N}(t_k)| + K|\delta_i^{\mu_N}(t_k)|\big] < \frac{1}{k}. \tag{6.158}$$

Requiring that $N_{k+1} > N_k$, we obtain a strictly increasing sequence $(N_k)_{k\in\mathbb{N}}$ that partitions $\mathbb{N}$. Set

$$\bar{L}(N) = \sum_{k\in\mathbb{N}} t_k 1_{\{N_k,\cdots,N_{k+1}-1\}}(N). \tag{6.159}$$

We show that $\bar{L}(N)$ satisfies the required properties:

- $\lim_{N\to\infty} \mathbb{E}[|\Delta_i^N(\bar{L}(N))|+K|\delta_i^N(\bar{L}(N))|]+\mathbb{E}[|\Delta_i^{\mu_N}(\bar{L}(N))|+K|\delta_i^{\mu_N}(\bar{L}(N))|] = 0$. To proof this, we fix $\epsilon > 0$ and let $K$ be such that $\frac{1}{K} < \epsilon$. Then, for all $N \geq N_K$,

$$\begin{aligned}
&\mathbb{E}\big[|\Delta_i^N(\bar{L}(N))| + K|\delta_i^N(\bar{L}(N))|\big] + \mathbb{E}\big[|\Delta_i^{\mu_N}(\bar{L}(N))| + K|\delta_i^{\mu_N}(\bar{L}(N))|\big] \\
&\quad= \sum_{k\in\mathbb{N}} \mathbb{E}\big[|\Delta_i^N(t_k)| + K|\delta_i^N(t_k)|\big] + \mathbb{E}\big[|\Delta_i^{\mu_N}(t_k)| + K|\delta_i^{\mu_N}(t_k)|\big]\, 1_{\{N_k,\cdots,N_{k-1}\}}(N) \\
&\quad< \frac{1}{K} < \epsilon.
\end{aligned} \tag{6.160}$$

  We conclude that

$$\lim_{N\to\infty} \mathbb{E}\big[|\Delta_i^N(\bar{L}(N))|+K|\delta_i^N(\bar{L}(N))|\big]+\mathbb{E}\big[|\Delta_i^{\mu_N}(\bar{L}(N))|v+K|\delta_i^{\mu_N}(\bar{L}(N))|\big] = 0. \tag{6.161}$$

- $\lim_{N\to\infty} \bar{L}(N) = \infty$. By (6.159), for each $k \in \mathbb{N}$ there exists an $N_k \in \mathbb{N}$ such that, for all $N \geq N_k$, $\bar{L}(N) \geq t_k$ and $t_k \to \infty$. We conclude that $\lim_{N\to\infty} \bar{L}(N) = \infty$.

- $\lim_{N\to\infty} \bar{L}(N)/N = 0$. Recall that $\lim_{k\to\infty} t_k/k = 0$ and $N_k \geq k$ by construction. Hence $\lim_{N\to\infty} \bar{L}(N)/N \leq \lim_{N\to\infty} \sum_{k\in\mathbb{N}}(t_k/k)]\, 1_{\{N_k,\cdots,N_{k-1}\}}(N) = 0$.

Choosing $(t_k)_{k\in\mathbb{N}} = (L(N))_{N\in\mathbb{N}}$, we complete the proof of Lemma 6.2.11. □

### • Proof of Lemma 6.2.6

*Proof.* The goal is to prove that $\nu_\theta$ is continuous in $\theta$. To do so, let $(\theta_n)_{n\in\mathbb{N}}$ be a sequence in $[0, 1]$ (note that $\theta_n$ is not a random variable) such that $\lim_{n\to\infty} \theta_n = \theta$. Couple the two infinite systems

$$\begin{aligned}
\mathrm{d}x_i^n(t) &= c\,[\theta_n - x_i^n(t)]\,\mathrm{d}t + \sqrt{g(x_i^n(t))}\,\mathrm{d}w_i(t) + Ke\,[y_i^n(t) - x_i^n(t)]\,\mathrm{d}t, \\
\mathrm{d}y_i^n(t) &= e\,[x_i^n(t) - y_i^n(t)]\,\mathrm{d}t, \qquad i \in \mathbb{N}_0,
\end{aligned} \tag{6.162}$$

and

$$\begin{aligned} \mathrm{d}x_i(t) &= c\,[\theta - x_i(t)]\,\mathrm{d}t + \sqrt{g(x_i(t))}\,\mathrm{d}w_i(t) + Ke\,[y_i(t) - x_i(t)]\,\mathrm{d}t, \\ \mathrm{d}y_i(t) &= e\,[x_i(t) - y_i(t)]\,\mathrm{d}t, \qquad i \in \mathbb{N}_0, \end{aligned} \tag{6.163}$$

via their Brownian motions, like in the proof of Lemma 6.2.11. Let $\mathcal{L}[(x_i^n(0), y_i^n(0))] = \delta_{(\theta_n,\theta_n)}$ and $\mathcal{L}[(x_i(0), y_i(0))] = \delta_{(\theta,\theta)}$. As before, define $\Delta_i^n = x_i^n - x_i$ and $\delta_i^n = y_i^n - y_i$. Note that

$$\lim_{n\to\infty} \mathbb{E}\big[|\Delta_i^n(0)| + K|\delta_i^n(0)|\big] = 0. \tag{6.164}$$

By a similar argument as in the proof of Lemma 6.2.11, we obtain that, for all $t \geq 0$,

$$\lim_{n\to\infty} \mathbb{E}\big[|\Delta_i^n(t)| + K|\delta_i^n(t)|\big] = 0. \tag{6.165}$$

Hence we can construct a sequence $(L(n))_{n\in\mathbb{N}}$ satisfying $\lim_{n\to\infty} L(n) = \infty$ and $\lim_{n\to\infty} L(n)/n = 0$ such that

$$\lim_{n\to\infty} \mathbb{E}\big[|\Delta_i^n(L(n))| + K|\delta_i^n(L(n))|\big] = 0. \tag{6.166}$$

To prove the continuity of the equilibrium $\nu_\theta$ in $\theta$, we reason as follows. First note that we can couple the system in (6.162) starting from $\delta_{(\theta_n,\theta_n)}$ with the system in (6.162) starting from $\nu_{\theta_n}$. By the uniqueness and convergence to equilibrium (see Lemma 6.2.5), we see that this coupling is successful. Similarly, we can couple the system in (6.163) starting from $\delta_{(\theta,\theta)}$ with the system in (6.163) starting from $\nu_{\theta_n}$, and see the coupling succesful. Finally, we use (6.166) to obtain

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{E}^{\nu_{\theta_n}\times\nu_\theta}\big[|\Delta_i^n(L(n))| + K|\delta_i^n(L(n))|\big] \\ &\leq \lim_{n\to\infty} \mathbb{E}^{\nu_{\theta_n}\times\delta_{(\theta_n,\theta_n)}}\big[|\Delta_i^n(L(n))| + K|\delta_i^n(L(n))|\big] \\ &\quad + \lim_{n\to\infty} \mathbb{E}^{\delta_{(\theta,\theta)}\times\delta_{(\theta_n,\theta_n)}}\big[|\Delta_i^n(L(n))| + K|\delta_i^n(L(n))|\big] \\ &\quad + \lim_{n\to\infty} \mathbb{E}^{\nu_\theta\times\delta_{(\theta,\theta)}}\big[|\Delta_i^n(L(n))| + K|\delta_i^n(L(n))|\big] = 0. \end{aligned} \tag{6.167}$$

Let $f$ be a Lipschitz function. Then, by the equilibrium property of $\nu_{\theta_n}$ and $\nu_\theta$,

$$\begin{aligned} &\lim_{n\to\infty} \big|\mathbb{E}^{\nu_{\theta_n}}\big[f(x^n(0))\big] - \mathbb{E}^{\nu_\theta}\big[f(x(0))\big]\big| \\ &= \lim_{n\to\infty} \big|\mathbb{E}^{\nu_{\theta_n}}\big[f(x^n(L(n)))\big] - \mathbb{E}^{\nu_\theta}\big[f(x(L(n)))\big]\big| \\ &= \lim_{n\to\infty} \big|\mathbb{E}^{\nu_{\theta_n}\times\nu_\theta}\big[f(x^n(L(n))) - f(x(L(n)))\big]\big| \\ &\leq \lim_{n\to\infty} (\mathrm{Lip}f)\,\mathbb{E}^{\nu_{\theta_n}\times\nu_\theta}\big[|(x^n(L(n))) - (x(L(n)))|\big] = 0. \end{aligned} \tag{6.168}$$

We can also show this if $f$ is a Lipschitz function of the $y$ component. Hence $\nu_\theta$ is indeed continuous as a function of $\theta$. □

- **Proof of Lemma 6.2.13**

*Proof.* Note that for all $N \in \mathbb{N}$ fixed, by Itô-calculus we find from (6.15) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[|x_i^{[N],1}(t) - x_i^{[N],2}(t)| + K|y_i^{[N],1}(t) - y_i^{[N],2}(t)|\right] \\
&= -\frac{2c}{N}\sum_{j\in[N]}\mathbb{E}\left[|x_{j,1}^{[N]}(t) - x_{j,2}^{[N]}(t)|\,1_{\left\{\mathrm{sgn}\left(x_{j,1}^{[N]}(t)-x_{j,2}^{[N]}(t)\right)\neq \mathrm{sgn}\left(x_i^{[N],1}(t)-x_i^{[N],2}(t)\right)\right\}}\right] \\
&\quad - 2Ke\,\mathbb{E}\Big[|x_i^{[N],1}(t) - x_i^{[N],2}(t)| \\
&\qquad + |y_i^{[N],1}(t) - y_i^{[N],2}(t)|\,1_{\left\{\mathrm{sgn}\left(x_i^{[N],1}(t)-x_i^{[N],2}(t)\right)\neq \mathrm{sgn}\left(y_i^{[N],1}(t)-y_i^{[N],2}(t)\right)\right\}}\Big].
\end{aligned} \tag{6.169}$$

Therefore, for each $N \in \mathbb{N}$,

$$t \mapsto \mathbb{E}\left[|x_i^{[N],1}(t) - x_i^{[N],2}(t)| + K|y_i^{[N],1}(t) - y_i^{[N],2}(t)|\right] \text{ is decreasing.} \tag{6.170}$$

Fix any $t(N) \to \infty$. By the assumption in Lemma 6.2.13, the proofs of Lemma 6.2.9 and Lemma 6.2.11 imply that (6.157) holds for both $(X^{[N],1}, Y^{[N],1})$ and $(X^{[N],2}, Y^{[N],2})$. Using the construction in the proof of Lemma 6.2.11, we can construct *one* sequence $(l(N))_{N\in\mathbb{N}}$, satisfying $l(N) \leq t(N)$, $\lim_{N\to\infty} l(N) = \infty$ and $\lim_{N\to\infty} l(N)/N = 0$, such that (6.161) with $\bar{L}(N)$ replaced by $l(N)$ holds for both the systems arising from $(X^{[N],1}, Y^{[N],1})$ and $(X^{[N],2}, Y^{[N],2})$.

Write

$$\begin{aligned}
&\mathbb{E}\left[|x_i^{[N],1}(l(N)) - x_i^{[N],2}(l(N))| + K|y_i^{[N],1}(l(N)) - y_i^{[N],2}(l(N))|\right] \\
\leq&\mathbb{E}\left[|x_i^{[N],1}(l(N)) - x_i^{\mu,1}(l(N))| + K|y_i^{[N],1}(l(N)) - y_i^{\mu,1}(l(N))|\right] \\
&+\mathbb{E}\left[|x_i^{\mu,1}(l(N)) - x_i^{\mu,2}(l(N))| + K|y_i^{\mu,1}(l(N)) - y_i^{\mu,2}(l(N))|\right] \\
&+\mathbb{E}\left[|x_i^{\mu,2}(l(N)) - x_i^{[N],2}(l(N))| + K|y_i^{\mu,2}(l(N)) - y_i^{[N],2}(l(N))|\right].
\end{aligned} \tag{6.171}$$

Note that in the right-hand side of the inequality the first and the third term tend to zero by (6.161). The second term tends to zero because $\mu\{\bar{\Theta}_1 = \bar{\Theta}_2\} = 1$, and hence Lemma 6.2.5 can be applied. Therefore

$$\lim_{N\to\infty}\mathbb{E}\left[|x_i^{[N],1}(l(N)) - x_i^{[N],2}(l(N))| + K|y_i^{[N],1}(l(N)) - y_i^{[N],2}(l(N))|\right] = 0. \tag{6.172}$$

Using the monotonicity in (6.170), we get

$$\begin{aligned}
&\lim_{N\to\infty}\mathbb{E}\left[|x_i^{[N],1}(t(N)) - x_i^{[N],2}(t(N))| + K|y_i^{[N],1}(t(N)) - y_i^{[N],2}(t(N))|\right] \\
&\leq \lim_{N\to\infty}\mathbb{E}\left[|x_i^{[N],1}(l(N)) - x_i^{[N],2}(l(N))| + K|y_i^{[N],1}(l(N)) - y_i^{[N],2}(l(N))|\right] = 0.
\end{aligned} \tag{6.173}$$

□

Combining the proofs of Proposition 6.2.4, Lemma 6.2.11 and Lemma 6.2.13, we obtain the following corollary. This corollary turns out to be important in Section 6.3.2 in the proof of Lemma 6.2.16 to obtain the limiting evolution of the 1-blocks.

**Corollary 6.3.1.** *Fix $s > 0$. Let $\mu_N$ be the measure obtained by periodic configuration of*

$$\mathcal{L}\big[X^{[N]}(Ns - L(N)), Y^{[N]}(Ns - L(N))\big], \tag{6.174}$$

*and let $\mu$ be a weak limit point of the sequence $(\mu_N)_{N\in\mathbb{N}}$. Let*

$$\bar{\Theta} = \lim_{N\to\infty} \frac{1}{N} \sum_{i\in[N]} \frac{x_i^\mu + K y_i^\mu}{1+K} \text{ in } L^2(\mu). \tag{6.175}$$

*and let $(X^{\nu_{\bar{\Theta}}}, Y^{\nu_{\bar{\Theta}}})$ be the infinite system evolving according to (6.82) and starting from its equilibrium measure. Consider the finite system $(X^{[N]}, Y^{[N]})$ as a system on $([0,1]\times[0,1))^{\mathbb{N}_0}$ obtained by periodic continuation. Construct $(X^{[N]}, Y^{[N]})$ and $(X^{\nu_{\bar{\Theta}}}, Y^{\nu_{\bar{\Theta}}})$ on one probability space. Then, for all $t \geq 0$,*

$$\lim_{N\to\infty} \mathbb{E}\left[\left|x_i^{[N]}(Ns+t) - x_i^{\nu_{\bar{\Theta}}}(t)\right|\right] + K\,\mathbb{E}\left[\left|y_i^{[N]}(Ns+t) - y_i^{\nu_{\bar{\Theta}}}(t)\right|\right] = 0, \qquad \forall\, i \in [N]. \tag{6.176}$$

*Proof.* By Proposition 6.2.4 we have that $\lim_{k\to\infty} \mathcal{L}[X^{[N_k]}(N_k s) + Y^{[N_k]}(N_k s)] = \nu(s) = \nu_{\bar{\Theta}}$. Let $\nu_{N_k}$ be defined by periodic continuation of the configuration of $(X^{[N_k]}(N_k s), Y^{[N_k]}(N_k s))$ and let $\nu = \lim_{k\to\infty} \nu_{N_k}$, so $\nu = \nu_{\bar{\Theta}}$. Construct the process $(X^{[N]}(t), Y^{[N]}(t))_{t\geq 0}$, $(X^{\nu_{N_k}}(t), Y^{\nu_{N_k}}(t))_{t\geq 0}$ and $(X^{\nu}(t), Y^{\nu}(t))_{t\geq 0}$ on one probability space and use for all processes the same Brownian motions. Then the couplings in the proofs of Lemma 6.2.9 and Lemma 6.2.11 imply (6.176). $\square$

## §6.3.2 Proof of step 2. Convergence of the estimator

In this section we prove the three lemmas stated in Step 2 of Section 6.2.2. Afterwards we prove Proposition 6.2.14 with the help of these lemmas.

- **Proof of Lemma 6.2.15**

*Proof.* Recall the definition of $\bar{\Theta}^{[N]}(t)$ in (6.43). It follows from the SSDE in (6.15) that

$$\mathrm{d}\bar{\Theta}^{[N]}(t) = \frac{1}{1+K}\frac{1}{N} \sum_{i\in[N]} \sqrt{g\big(x_i^{[N]}(t)\big)}\,\mathrm{d}w_i(t). \tag{6.177}$$

Hence we see that $t \mapsto \bar{\Theta}^{[N]}(t)$ is a continuous martingale. By Itô's formula we have

$$\begin{aligned}\mathbb{E}\big[\big(\bar{\Theta}^{[N]}(t)\big)^2\big] &= \mathbb{E}\big[\big(\bar{\Theta}^{[N]}(0)\big)^2\big] + \frac{1}{(1+K)^2}\int_0^t \mathrm{d}r\, \frac{1}{N^2}\sum_{i\in[N]} g\big(x_i^{[N]}(r)\big) \\ &\leq 1 + \frac{1}{N}\frac{\|g\|}{(1+K)^2}\,t.\end{aligned} \tag{6.178}$$

Since $g$ is a bounded function, we get that $t \mapsto \bar{\Theta}^{[N]}(t)$ is square integrable. It follows that,

$$\Big(\big(\bar{\Theta}^{[N]}(Ns+t) - \bar{\Theta}^{[N]}(Ns)\big)^2\Big)_{t\geq 0} \tag{6.179}$$

is a sub-martingale. Therefore, defining the stopping time

$$S_\epsilon^N = \inf\left\{t \geq 0 : \left(\bar{\Theta}^{[N]}(Ns+t) - \bar{\Theta}^{[N]}(Ns)\right)^2 \geq \epsilon\right\} \wedge L(N), \tag{6.180}$$

we find, by the continuity of $t \mapsto \bar{\Theta}^{[N]}(Ns+t)$ and the optional sampling theorem, that

$$\mathbb{P}\left(S_\epsilon^N \in [Ns, Ns+L(N))\right) \leq \frac{1}{\epsilon^2}\mathbb{E}\left[\left(\bar{\Theta}^{[N]}(Ns+L(N)) - \bar{\Theta}^{[N]}(Ns)\right)^2\right]. \tag{6.181}$$

Combining (6.178) and (6.181), we find

$$\lim_{N\to\infty} \sup_{0\leq t\leq L(N)} \left|\bar{\Theta}^{[N]}(Ns+t) - \bar{\Theta}^{[N]}(Ns)\right| = 0 \text{ in probability.} \tag{6.182}$$

To obtain the increasing process, note that by Itô-calculus it follows from (6.177) that

$$\left\langle \bar{\Theta}^{[N]}(Ns)\right\rangle_{s\geq 0} = \frac{1}{(1+K)^2}\int_0^s \mathrm{d}r\, \frac{1}{N}\sum_{i\in[N]} g\left(x_i^{[N]}(Nr)\right). \tag{6.183}$$

We are left to show that the sequence of processes $((\bar{\Theta}^{[N]}(Ns))_{s\geq 0})_{N\in\mathbb{N}}$ is tight. Note that (6.183) implies that, for all $N\in\mathbb{N}$ and $s\geq 0$,

$$\left\langle \bar{\Theta}^{[N]}(Ns)\right\rangle \leq \frac{\|g\|}{(1+K)^2}\, s \tag{6.184}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\langle \bar{\Theta}^{[N]}(Ns)\right\rangle \leq \frac{\|g\|}{(1+K)^2}. \tag{6.185}$$

Hence the sequence $(\langle\bar{\Theta}^{[N]}(Ns)\rangle_{s\geq 0})_{N\in\mathbb{N}}$ is equicontinuous. Therefore, by the Arzela-Ascoli theorem (see e.g. [7, Theorem 7.2]), for each $T\geq 0$ the set $(\langle\bar{\Theta}^{[N]}(Ns)\rangle_{0\leq s\leq T})_{N\in\mathbb{N}}$ is relatively compact in $C([0,T],\mathbb{R})$, the space of continuous functions from $[0,T]\to\mathbb{R}$. Therefore the set of laws $(\mathcal{L}\langle\bar{\Theta}^{[N]}(Ns)\rangle_{0\leq s\leq T})_{N\in\mathbb{N}}$ is tight in $\mathcal{P}(C([0,T],\mathbb{R}))$. Hence it follows that $(\mathcal{L}\langle\bar{\Theta}^{[N]}(Ns)\rangle_{s\geq 0})_{N\in\mathbb{N}}$ is tight in $\mathcal{P}(C([0,\infty),\mathbb{R}))$, the set of probability laws on $C([0,\infty),\mathbb{R})$.

Since $(\bar{\Theta}^{[N]}(Ns) - \bar{\Theta}^{[N]}(0))_{s\geq 0}$ is a stochastic integral, we can represent it as a time-transformed Brownian motion (see e.g. [62, Chapter 5]):

$$\left(\bar{\Theta}^{[N]}(Ns) - \bar{\Theta}^{[N]}(0)\right)_{s\geq 0} = w\left(\langle\bar{\Theta}^{[N]}(Ns)\rangle\right)_{s\geq 0}. \tag{6.186}$$

Let $\chi$ be a standard normal random variable. Then

$$\begin{aligned}\mathbb{E}\left[\left(w\left(\langle\bar{\Theta}^{[N]}(Ns)\rangle\right) - w\left(\langle\bar{\Theta}^{[N]}(Nr)\rangle\right)\right)^2\right] &\leq \mathbb{E}\left[\left(\langle\bar{\Theta}^{[N]}(Ns)\rangle - \langle\bar{\Theta}^{[N]}(Nr)\rangle\right)^2\right]\mathbb{E}\left[\chi^4\right]\\ &\leq (s-r)^2\,\frac{\|g\|^2}{(1+K)^4}\,\mathbb{E}\left[\chi^4\right].\end{aligned} \tag{6.187}$$

Hence it follows from Kolmogorov's criterion for weak compactness (see e.g. [62, Chapter XIII, Theorem 1.8]) that the sequence $(\mathcal{L}[(\bar{\Theta}^{[N]}(Ns))_{s\geq 0}])_{N\in\mathbb{N}}$ is tight in $\mathcal{P}(C([0,\infty),\mathbb{R}))$. □

- **Proof of Lemma 6.2.16**

*Proof.* For ease of notation we will suppress the subsequence notation and assume that

$$\lim_{N\to\infty} \mathcal{L}\left[\left(\bar{\Theta}^{[N]}(Ns)\right)_{s>0}\right] = \mathcal{L}\big[(\bar{\Theta}(s))_{s>0}\big]. \tag{6.188}$$

The processes $(\bar{\Theta}^{[N]}(Ns))_{s\geq 0}$ are martingales, see (6.177), measurable w.r.t. the canonical filtration $(\mathcal{F}_s)_{s\geq 0}$ and so are there weak limit points. Therefore also the weak limit point $(\bar{\Theta}(s))_{s>0}$ is a martingale, (see [21, Section 3]). To obtain (6.93), we use the following strategy. Recall from the proof of Lemma 6.2.15 that the sequence $\{\langle \bar{\Theta}^{[N]}(Ns)\rangle_{s>0}\}_{N\in\mathbb{N}}$ is tight. Hence, in order to prove that

$$\lim_{N\to\infty} \mathcal{L}\left[\langle \bar{\Theta}^{[N]}(Ns)\rangle_{s>0}\right] = \mathcal{L}\left[\left(\int_0^s \mathrm{d}r \frac{1}{(1+K)^2}\, \mathbb{E}^{\nu_{\bar{\Theta}(r)}}[g(x_0)]\right)_{s>0}\right], \tag{6.189}$$

it is enough to prove that the finite-dimensional distributions of $(\langle \bar{\Theta}^{[N]}(Ns)\rangle_{s>0})_{N\in\mathbb{N}}$ converge to the finite-dimensional distribution of $(\int_0^s \mathrm{d}r\, \frac{1}{(1+K)^2}\mathbb{E}^{\nu_{\bar{\Theta}(r)}}[g(x_0)])_{s>0}$. We will prove a slightly stronger result, namely,

$$\lim_{N\to\infty} \mathbb{E}\left[\left|\langle \bar{\Theta}^{[N]}(Ns)\rangle - \int_0^s \mathrm{d}r\, \frac{1}{(1+K)^2}\, \mathbb{E}^{\nu_{\bar{\Theta}(r)}}[g(x_0)]\right|\right] = 0. \tag{6.190}$$

Once we have obtained (6.190) and hence (6.189), by Skohorod's theorem we can construct the processes $(\langle \bar{\Theta}^{[N]}(Ns)\rangle_{s>0})_{N\in\mathbb{N}}$ on a single probability space, to obtain

$$\lim_{N\to\infty} \langle \bar{\Theta}^{[N]}(Ns)\rangle_{s\geq 0} = \left(\int_0^s \mathrm{d}r\, \frac{1}{(1+K)^2}\mathbb{E}^{\nu_{\bar{\Theta}(r)}}[g(x_0)]\right)_{s\geq 0} \quad a.s. \tag{6.191}$$

Using the continuity of Brownian motion, we get that (recall (6.186))

$$\begin{aligned}\lim_{N\to\infty} \left(\bar{\Theta}^{[N]}(Ns)\right)_{s>0} &= \lim_{N\to\infty}\left[w\big(\langle \bar{\Theta}^{[N]}(Ns)\rangle\big)_{s>0} + \bar{\Theta}^{[N]}(0)\right]\\ &= w\left(\int_0^s \mathrm{d}r\, \frac{1}{(1+K)^2}\, \mathbb{E}^{\nu_{\bar{\Theta}(r)}}[g(x_0)]\right)_{s>0} + \vartheta_0 \quad a.s.\end{aligned} \tag{6.192}$$

Therefore we can choose a version of $(\bar{\Theta}(s))_{s>0}$ such that

$$\lim_{N\to\infty} \left(\bar{\Theta}^{[N]}(Ns)\right)_{s>0} = \lim_{N\to\infty} \left(\bar{\Theta}(s)\right)_{s>0} \quad a.s. \tag{6.193}$$

and

$$\lim_{N\to\infty} \left(\bar{\Theta}^{[N]}(Ns), \langle \bar{\Theta}^{[N]}(Ns)\rangle\right)_{s>0} = \left(\bar{\Theta}(s), \langle \bar{\Theta}(s)\rangle\right)_{s>0} \quad a.s. \tag{6.194}$$

By the continuous mapping theorem, (6.93) follows. The martingale property follows from the fact that $\left(\bar{\Theta}^{[N]}(Ns)^2 - \langle \bar{\Theta}^{[N]}(Ns)\rangle\right)_{s>0}$ are martingales. Therefore, to finish the proof of Lemma 6.2.16 we are left to prove (6.190).

To prove (6.190), define the empirical measures on $[0,1]$ by

$$U^{[N]}(Ns) = \frac{1}{N}\sum_{i\in[N]} \delta_{x_i(Ns)}. \tag{6.195}$$

Note that we can write

$$
\begin{aligned}
&\mathbb{E}\left[\left|\langle\bar{\Theta}^{[N]}(Ns)\rangle - \int_0^s \mathrm{d}r\,\frac{1}{(1+K)^2}\,\mathbb{E}^{\nu_{\bar{\Theta}(r)}}[g(x_0)]\right|\right] \\
&= \frac{1}{(1+K)^2}\,\mathbb{E}\left[\left|\int_0^s \mathrm{d}r\;\mathbb{E}^{U^{[N]}(Nr)}[g(x_0)] - \int_0^s \mathrm{d}r\;\mathbb{E}^{\nu_{\bar{\Theta}(r)}}[g(x_0)]\right|\right] \\
&\leq \frac{1}{(1+K)^2}\int_0^s \mathrm{d}r\,\mathbb{E}\left[\left|\mathbb{E}^{U^{[N]}(Nr)}[g(x_0)] - \mathbb{E}^{\nu_{\bar{\Theta}(r)}}[g(x_0)]\right|\right].
\end{aligned}
\tag{6.196}
$$

Hence, to prove (6.190) it is enough to prove that, for all $r > 0$,

$$
\lim_{N\to\infty}\mathbb{E}\left[\left|\mathbb{E}^{U^{[N]}(Nr)}[g(x_0)] - \mathbb{E}^{\nu_{\bar{\Theta}(r)}}[g(x_0)]\right|\right] = 0 \tag{6.197}
$$

and apply the dominated convergence theorem.

To prove (6.197), we will use the coupling arguments from Section 6.3.1, as well as ergodicity and invariance under the evolution of $\nu_{\bar{\Theta}}$. As before, let $z^{[N]}(t)$ denote the $[N]$-component system $(x_i^{[N]}(t), y_i^{[N]}(t))_{t\geq 0}$ evolving according to (6.15), viewed as a system on $\mathbb{N}_0$ obtained by periodic continuation and with initial law $\mathcal{L}[z^{[N]}(0)] = \mathcal{L}[x_i^{[N]}(Nr - L(N)), y_i^{[N]}(Nr - L(N))]$. Let $(z^{\mu_N}(t))_{t\geq 0}$ denote the infinite system $(x_i^{\mu_N}(t), y_i^{\mu_N}(t))_{t\geq 0}$ evolving according to (6.74) with initial law $\mu_N$ obtained by periodic continuation of the configuration of $(x_i^{[N]}(Nr-L(N)), y_i^{[N]}(Nr-L(N)))$, and let $\mu$ be a weak limit point of the sequence $\mu_N$. Note that, for all $r > 0$, by Lemma 6.2.10 we have that $\lim_{N\to\infty}\bar{\Theta}^{[N]}(Nr) = \bar{\Theta}(r)$ for $\bar{\Theta}(r) = \lim_{n\to\infty}\frac{1}{n}\sum_{i\in[n]}\frac{x_i+Ky_i}{1+K}$ in $L_2(\mu)$. Let $\bar{L}(N)$ be the sequence constructed in Corollary 6.3.1. Then we can write

$$
\begin{aligned}
&\mathbb{E}\left[\left|\mathbb{E}^{U^{[N]}(Nr)}[g(x_0)] - \mathbb{E}^{\nu_{\bar{\Theta}}(r)}[g(x_0)]\right|\right] \\
&\leq \mathbb{E}\left[\left|\frac{1}{N}\sum_{i\in[N]} g(x_i^{[N]}(Nr)) - \frac{1}{N}\sum_{i\in[N]} g\big(x_i^{\nu_{\bar{\Theta}}(r)}(\bar{L}(N))\big)\right|\right] \\
&\qquad + \mathbb{E}\left[\left|\frac{1}{N}\sum_{i\in[N]} g\big(x_i^{\nu_{\bar{\Theta}(r)}}(\bar{L}(N))\big) - \mathbb{E}^{\nu_{\bar{\Theta}(r)}}[g(x_0)]\right|\right] \\
&\leq (\mathrm{Lip}\, g)\,\mathbb{E}\left[\left|x_0^{[N]}(Nr) - x_0^{\nu_{\bar{\Theta}}(r)}(\bar{L}(N))\right|\right] \\
&\qquad + \mathbb{E}\left[\left|\frac{1}{N}\sum_{i\in[N]} g\big(x_i^{\nu_{\bar{\Theta}(r)}}(\bar{L}(N))\big) - \mathbb{E}^{\nu_{\bar{\Theta}(r)}}[g(x_0)]\right|\right],
\end{aligned}
\tag{6.198}
$$

where in the second inequality we use the Lipschitz property of $g$ and the translation invariance of the system. Note that the first term tends to 0 as $N \to \infty$ by Corollary 6.3.1. Finally, note that by Lemma 6.2.5 $(x_i)_{i\in\mathbb{N}_0}$ is a sequence of bounded i.i.d. random variables under $\nu_{\bar{\Theta}(r)}$. Hence the last term tends to zero by the law of large numbers. □

- **Proof of Lemma 6.2.17**

*Proof.* Note that, since $g$ is Lipschitz, the function

$$\theta \mapsto \mathbb{E}^{\nu_\theta}[g], \tag{6.199}$$

is Lipschitz by Lemma 6.2.6. Hence, by [72, Theorem 1], the SDE

$$\mathrm{d}\Phi(s) = \frac{1}{(1+K)} \sqrt{\mathbb{E}^{\nu_{\Phi(s)}}[g]}\,\mathrm{d}w(s) \tag{6.200}$$

has a pathwise unique solution (see ţte[p315–317]KS91) and a unique solution in law (see [72, Proposition 1]). This implies that the martingale problem with generator

$$\frac{1}{(1+K)^2}\,\mathbb{E}^{\nu_\Phi}[g]\frac{\mathrm{d}}{\mathrm{d}\Phi^2} \tag{6.201}$$

has a unique solution. In particular, choosing $f(\Phi) = \Phi^2$, we see that the martingale problem implies that

$$\left(\Phi^2(s) - \frac{1}{(1+K)^2}\int_0^s \mathrm{d}u\,\mathbb{E}^{\nu_{\Phi(u)}}[g(x_0)]\right)_{s>0} \tag{6.202}$$

is a martingale.

Since $(\bar{\Theta}(s))_{s>0}$ is a continuous bounded martingale, it could be written as a time transformed Brownian motion. The uniqueness of the martingale problem in (6.94) now follows from the fact that the quadratic variation of a martingale is unique. □

Chapter 6

- **Proof of Proposition 6.2.14**

*Proof.* Combining Lemma 6.2.16–6.2.17, all converging subsequences of $(\mathcal{L}[(\bar{\Theta}^{[N]}(Ns))_{s\geq 0}])_{N\in\mathbb{N}}$ converge to the same limit, which is the unique process satisfying the martingale problem in (6.94). □

## §6.3.3 Proof of step 3. Convergence of the 1-blocks in the Meyer-Zheng topology

In this section we prove Proposition 6.2.18 stated in Step 3 of Section 6.2.2. The Lemmas 6.2.19, 6.2.20 and 6.2.21 are proven in Appendix B.2.

- **Proof of Proposition 6.2.18**

*Proof.* By Proposition 6.2.14 we have that

$$\lim_{N\to\infty} \mathcal{L}[\bar{\Theta}^{[N]}(Ns)_{s>0}] = \mathcal{L}[(\bar{\Theta}(s))_{s>0}] \tag{6.203}$$

in the normal topology and therefore (see B.2 Lemma B.2.1)

$$\lim_{N\to\infty} \mathcal{L}[\bar{\Theta}^{[N]}(Ns)_{s>0}] = \mathcal{L}[(\bar{\Theta}(s))_{s>0}] \text{ in Meyer-Zheng topology.} \tag{6.204}$$

By Lemma 6.2.7, for $s > 0$,

$$\lim_{N\to\infty} \mathbb{E}\left[\left|\bar{\Theta}^{[N]}(Ns) - x_1^{[N]}(s)\right|\right] = 0 \tag{6.205}$$

and therefore, by Lemma 6.2.19,

$$\lim_{N\to\infty} d_P(\psi_{\bar{\Theta}^{[N]}}, \psi_{x_1^{[N]}}) = 0 \text{ in probability.} \tag{6.206}$$

To apply the above results to our model, we recall the following basic result (see [7, Chapter 1]), which also holds for the Meyer-Zheng topology.

**Lemma 6.3.2.** *Let $X_n$, $Y_n$ be random variables. If*

$$\lim_{n\to\infty} \mathcal{L}[X_n] = \mathcal{L}[X] \tag{6.207}$$

*and*

$$\lim_{n\to\infty} d(X_n, Y_n) = 0 \textit{ in probability,} \tag{6.208}$$

*then*

$$\lim_{n\to\infty} \mathcal{L}[Y_n] = \mathcal{L}(X). \tag{6.209}$$

Applying Lemma (6.3.2) to our case, we obtain

$$\lim_{N\to\infty} \mathcal{L}[(x_1^{[N]}(s))_{s>0}] = \mathcal{L}[(\bar{\Theta}(s))_{s>0}] \text{ in the Meyer-Zheng topology.} \tag{6.210}$$

The argument for

$$\lim_{N\to\infty} \mathcal{L}[(y_1^{[N]}(s))_{s>0}] = \mathcal{L}[(\bar{\Theta}(s))_{s>0}] \text{ in the Meyer-Zheng topology} \tag{6.211}$$

follows in the same way. By Lemma 6.2.20, we obtain

$$\lim_{N\to\infty} \mathcal{L}[(x_1^{[N]}(s), y_1^{[N]}(s) - x_1^{[N]}(s))_{s>0}] = \mathcal{L}[(\bar{\Theta}(s), 0)_{s>0}] \text{ in the Meyer-Zheng topology.} \tag{6.212}$$

Applying Lemma 6.2.21 with $f(x, y) = f(x, y + x)$ and the continuous mapping theorem, we obtain

$$\lim_{N\to\infty} \mathcal{L}\left[\left(x_1^{[N]}(s), y_1^{[N]}(s)\right)_{s>0}\right] = \mathcal{L}[(\bar{\Theta}(s), \bar{\Theta}(s))_{s>0}] \text{ in the Meyer-Zheng topology.} \tag{6.213}$$

□

## §6.3.4 Proof of step 4. Mean-field finite-systems scheme

- **Proof of Proposition 6.2.1**

*Proof.* Proposition 6.2.1(b) follows directly from Propostion 6.2.14. The proof of Proposition 6.2.1(a) follows from Proposition 6.2.1(b) and Proposition 6.2.18.

To prove Proposition 6.2.1(c) fix $t > 0$. Consider the processes $(X^{[N]}(sN+t), Y^{[N]}(sN+t))_{t\geq 0}$ as processes on $([0,1]^2)^{\mathbb{N}_0}$ by periodic continuation. Since $([0,1]^2)^{\mathbb{N}_0}$ is compact, the sequence $(X^{[N]}(sN+t), Y^{[N]}(sN+t))_{N\in\mathbb{N}}$ is tight and hence has a converging subsequence. Let $(\bar{\Theta}(s))_{s>0}$ be the limiting process obtained in Proposition (6.2.14). This has continuous paths and is the unique solution of a well-posed martingale problem, and hence is a Markov process. Denote by $Q_s$ the time-$s$ semigroup corresponding to $(\bar{\Theta}(s))_{s\geq 0}$. Combining Proposition 6.2.4, Proposition 6.2.14 and Lemma 6.2.15, we get that, for each converging subsequence,

$$\lim_{k\to\infty} \mathcal{L}\left[X^{[N_k]}(sN_k+t), Y^{[N_k]}(sN_k+t)\right] = \int Q_s\left(\theta, \mathrm{d}\theta'\right)\nu_{\theta'} = \nu(s), \tag{6.214}$$

and hence it follows that, for all $t \geq 0$,

$$\lim_{N\to\infty} \mathcal{L}\left[X^{[N]}(sN+t), Y^{[N]}(sN+t)\right] = \int Q_s\left(\theta, \mathrm{d}\theta'\right)\nu_{\theta'} = \nu(s). \tag{6.215}$$

Let $(X^{\nu(s)}(t), Y^{\nu(s)}(t))_{t\geq 0}$ be the infinite system defined in (6.18), starting from initial measure $\nu(s)$. Then it follows from Corollary 6.3.1 that we can construct the processes $(X^{[N]}(sN+t), Y^{[N]}(sN+t))_{t\geq 0}$ and $(X^{\nu(s)}(t), Y^{\nu(s)}(t))_{t\geq 0}$ on one probability space such that, for all $t \geq 0$,

$$\lim_{N\to\infty} \mathbb{E}\left[\left|z^{\nu(s)}_{(i,R_i)}(t) - z^{[N]}_{(i,R_i)}(sN+t)\right|\right] = 0 \qquad \forall\,(i,R_i) \in \mathbb{Z}\times\{A,D\}. \tag{6.216}$$

Hence we see that the finite-dimensional distributions of the process $(X^{[N]}(sN+t), Y^{[N]}(sN+t))_{t\geq 0}$ converge to the finite-dimensional distributions of te process $(X^{\nu(s)}(t), Y^{\nu(s)}(t))_{t\geq 0}$.

Since we want that

$$\lim_{N\to\infty} \mathcal{L}\left[\left(X^{[N]}(sN+t), Y^{[N]}(sN+t)\right)_{t\geq 0}\right] = \mathcal{L}\left[\left(X^{\nu(s)}(t), Y^{\nu(s)}(t)\right)_{t\geq 0}\right], \tag{6.217}$$

we are left to show the tightness of $\mathcal{L}[(X^{[N]}(sN+t), Y^{[N]}(sN+t))_{t\geq 0}]_{N\in\mathbb{N}}$ in the path space $\mathcal{C}([0,\infty), ([0,1]\times[0,1))^{\mathbb{N}_0})$. Since $([0,1]^2)^{\mathbb{N}_0}$ is endowed with the product topology, it is enough to show for all sequence components $(x^{[N]}_i(t))_{t\geq 0}$ and $(y^{[N]}_i(t))_{t\geq 0}$ that they are tight in path space (see [7, Theorem 7.3]).

To prove that the components are tight, we use a tightness criterion for semimartingales by Joffe and Metivier, [49, Proposition 3.2.3]. To use this criterion, we have to show that for all $i \in [N]$ the components $(x^{[N]}_i(t), y^{[N]}_i(t))$ are $\mathcal{D}$-semimartingales as defined in [49, Definition 3.1.1]. To do this, let $\mathcal{C}^* \subset \mathcal{C}_b\left([0,1]^2\right)$ be the set of polynomials on $[0,1]^2$, and define the operator

$$G^{[N]}_{\dagger}\colon\ (\mathcal{C}^*\times[0,1]^2\times[0,\infty), \Omega) \to \mathbb{R} \tag{6.218}$$

by

$$\begin{aligned} G^{[N]}_{\dagger}(f,(x,y),t,\omega) = &\left[\frac{c}{N}\sum_{j\in[N]}[x^{[N]}_j(t,\omega) - x] + K\,[y-x]\right]\frac{\partial f}{\partial x} \\ &+ \frac{1}{2}g(x)\frac{\partial^2 f}{\partial x^2} + e\,[x-y]\frac{\partial f}{\partial y}. \end{aligned} \tag{6.219}$$

We use the subscript † to emphasize that $G_\dagger^{[N]}$ is the operator of a $\mathcal{D}$-semi-martingale and not a generator. Below we check in 4 steps that the component processes $(x_i^{[N]}(t), y_i^{[N]}(t))_{t\geq 0}$ are indeed $\mathcal{D}$-semi-martingales.

(a) The functions

$$f_1(x_i, y_i) = x_i, \quad f_2(x_i, y_i) = y_i, \tag{6.220}$$

are in $\mathcal{C}^*$, and so are $f_1^2$, $f_1 f_2$, $f_2^2$.

(b) For every $((x,y),t,\omega) \in \big([0,1]^2 \times [0,\infty) \times \Omega\big)$, the mapping $f \mapsto G_\dagger^{[N]}(f,(x,y),t,\omega)$ is linear on $\mathcal{C}^*$ and $G_\dagger^{[N]}(f,\cdot,t,\omega) \in \mathcal{C}^*$.

(c) Let $(\mathcal{F}_s)_{s\geq 0}$ be the filtration generated by the Brownian motions $((w_i(s))_{s\geq 0})_{i\in[N]}$, and let $\mathcal{P}$ be the $\sigma$-algebra generated by the predictable sets, i.e., sets of the form $(s,t] \times F$ for $F \in \mathcal{F}_s$. Since the component processes $(x_j^{[N]}(t))_{t\geq 0}$ are continuous, $((x,y),t,\omega) \mapsto G_\dagger^{[N]}(f,(x,y),t,\omega)$ is $\mathcal{B}([0,1]^2) \otimes \mathcal{P}$ measurable for every $f \in \mathcal{C}^*$, where $\mathcal{P}$ is the $\sigma$-algebra generated by the sets of the form $(s,t] \times F$ for $F \in \mathcal{F}_s$.

(d) Applying Itô's formula to the SSDE in (6.15), we obtain, for every $f \in \mathcal{C}^*$,

$$\begin{aligned} f\big(x_i^{[N]}(t), y_i^{[N]}(t)\big) &= f\big(x_i^{[N]}(0), y_i^{[N]}(0)\big) \\ &+ \int_0^t \mathrm{d}s\, \frac{c}{N} \sum_{j\in[N]} \big[x_j^{[N]}(s,\omega) - x_i^{[N]}(s)\big]\, \frac{\partial f}{\partial x}\big(x_i^{[N]}(t), y_i^{[N]}(t)\big) \\ &+ \frac{1}{2}\int_0^t \mathrm{d}w_i(s)\, \sqrt{g(x_i^{[N]}(s))}\, \frac{\partial f}{\partial x}\big(x_i^{[N]}(t), y_i^{[N]}(t)\big) \\ &+ \int_0^t \mathrm{d}s\, Ke\big[y_i^{[N]}(s) - x_i^{[N]}(s)\big]\, \frac{\partial f}{\partial x}\big(x_i^{[N]}(t), y_i^{[N]}(t)\big) \\ &+ \int_0^t \mathrm{d}s\, e\big[x_i^{[N]}(s) - y_i^{[N]}(s)\big]\, \frac{\partial f}{\partial y}\big(x_i^{[N]}(t), y_i^{[N]}(t)\big) \\ &+ \int_0^t \mathrm{d}s\, g(x_i^{[N]}(s))\, \frac{\partial^2 f}{\partial x^2}\big(x_i^{[N]}(t), y_i^{[N]}(t)\big). \end{aligned} \tag{6.221}$$

Therefore

$$\begin{aligned} M^{[N],f}(t,\omega) &= f\big(x_i^{[N]}(t,\omega), y_i^{[N]}(t,\omega)\big) - f\big(x_i^{[N]}(0,\omega), y_i^{[N]}(0,\omega)\big) \\ &\quad - \int_0^t \mathrm{d}s\, G_\dagger^{[N]}\big(f(x_i^{[N]}(s,\omega), y_i^{[N]}(s,\omega)), s, \omega\big) \end{aligned} \tag{6.222}$$

is a square-integrable martingale on $(\Omega, (\mathcal{F}_s)_{s\geq 0}, \mathbb{P})$.

To check that the sequence of component processes $((x_i^{[N]}(t), y_i^{[N]}(t)))_{N\in\mathbb{N}}$ is tight, we need the local characteristics of the $\mathcal{D}$-semi-martingale, which are defined in [49, Definition 3.1.2] as (recall (6.220))

$$
\begin{aligned}
b_1^{[N]}((x,y),t,\omega) &= G_\dagger^{[N]}(f_1,(x,y),t,\omega),\\
b_2^{[N]}((x,y),t,\omega) &= G_\dagger^{[N]}(f_2,(x,y),t,\omega),\\
a_{(1,1)}^{[N]}((x,y),t,\omega) &= G_\dagger^{[N]}(f_1f_1,(x,y),t,\omega) - 2x\, b_1((x,y),t,\omega),\\
a_{(2,1)}^{[N]}((x,y),t,\omega) &= G_\dagger^{[N]}(f_1f_2,(x,y),t,\omega) - x\, b_2((x,y),t,\omega) - y b_1((x,y),t,\omega),\\
a_{(1,2)}^{[N]}((x,y),t,\omega) &= a_{(2,1)}((x,y),t,\omega),\\
a_{(2,2)}^{[N]}((x,y),t,\omega) &= G_\dagger^{[N]}(f_2f_2,(x,y),t,\omega) - 2y\, b_2((x,y),t,\omega).
\end{aligned}
\tag{6.223}
$$

Hence

$$
\begin{aligned}
b_1^{[N]}((x,y),t,\omega) &= \frac{c}{N}\sum_{j\in[N]}[x_j^{[N]}(t,\omega)-x] + Ke[y-x],\\
b_2^{[N]}((x,y),t,\omega) &= e[x-y],\\
a_{(1,1)}^{[N]}((x,y),t,\omega) &= 2g(x),\\
a_{(1,2)}^{[N]}((x,y),t,\omega) &= a_{(2,1)}((x,y),t,\omega) = 0,\\
a_{(2,2)}^{[N]}((x,y),t,\omega) &= 0.
\end{aligned}
\tag{6.224}
$$

Here, $b_i^{[N]}$ and $a_{i,j}^{[N]}$, $i,j \in \{1,2\}$, are called the local coefficients of first and second order. We check that the hypotheses [49, H1, H2, H3 in Section 3.2.1] are satisfied.

$H_1$: For all $N \in \mathbb{N}$,

$$
\sum_{i\in\{1,2\}} |b_i^{[N]}((x,y),t,\omega)|^2 + \sum_{i,j\in\{1,2\}} |a_{(i,j)}^{[N]}((x,y),t,\omega)|^2 \le 4(c+Ke+e)^2 + 2||g||^2.
\tag{6.225}
$$

Hence, choosing as positive adapted process the constant process 1 and letting the constant be equal to $4(c+Ke+e)^2+2||g||^2$, we see that $H_1$ is satisfied.

$H_2$: Since the component processes are bounded by 1, also $H_2$ is satisfied.

$H_3$: Since the increasing càdlàg function $(A^{[N]}(t))_{t\ge 0}$ in [49, Definition 3.1.1] is in our case $A^{[N]}(t) = t$, also $H_3$ is satisfied.

Since $H_1$, $H_2$, $H_3$ are met, [49, Proposition 3.2.3] implies that $((x_i^{[N]}(t), y_i^{[N]}(t))_{t>0})_{N\in\mathbb{N}}$ are tight in the space of càdlàg paths $\mathcal{D}((0,\infty],[0,1]^2)$. Hence (6.217) indeed holds. □