Spatial populations with seed-bank
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APPENDIX **A**

Appendix Part I

§A.1 Derivation of continuum frequency equations

Model 1. We give the derivation of (2.4)–(2.5) as the continuum limit of an individual-based model when the size of the colonies tends to infinity. We start with the continuum limit of the Fisher-Wright model with (strong) seed-bank for a *single-colony* model as defined in [12]. Subsequently we show how the limit extends to a *multi-colony* model with seed-bank.

Single-colony model. The Fisher-Wright model with (strong) seed-bank defined in [12] consists of a *single colony* with $N \in \mathbb{N}$ active individuals and $M \in \mathbb{N}$ dormant individuals. Each individual can carry one of two types: $\heartsuit$ or $\diamondsuit$. Let $\epsilon \in [0,1]$ be such that ϵN is integer and $\epsilon N \leq M$. Put $\delta = \frac{\epsilon N}{M}$. The evolution of the population is described by a discrete-time Markov chain that undergoes four transitions per step:

(1) From the N active individuals, $(1-\epsilon)N$ are selected uniformly at random without replacement. Each of these individuals resamples, i.e. it adopts the type of an active individual selected uniformly at random with replacement, and remains active.

(2) Each of the ϵN active individuals not selected first resamples, it adopts the type of an active individual selected uniformly at random with replacement, and subsequently becomes dormant.

(3) From the M dormant individuals, $\delta M = \epsilon N$ are selected uniformly at random without replacement, and each of these becomes active. Since these individuals come from the dormant population they do not resample.

(4) Each of $(1-\delta)M$ dormant individuals not selected remains dormant and retains its type.

Note that the total sizes of the active and the dormant population remain fixed. During the evolution the dormant and active population *exchange* individuals. We are interested in the fractions of individuals of type $\heartsuit$ in the active and the dormant population. For an example of the evolution see Fig. 1.3.

Let $c = \epsilon N = \delta M$, i.e., c is the number of pairs of individuals that change state. Label the N active individuals from 1 to N and the M dormant individuals from 1 up to M. We denote by $[N] = \{1, \ldots, N\}$ and by $[M] = \{1, \ldots, M\}$. Let

$\xi(k) = (\xi_j(k))_{j\in[N]} \in \{0,1\}^{[N]}$ be the random vector where $\xi_j(k) = 1$ if the j'th individual is of type ♡ at time k and $\xi_j(k) = 0$ if the j'th individual is of type ♢ at time k. Similarly, we let $\eta(k) = (\eta_j(k))_{j\in[M]} \in \{0,1\}^{[M]}$ be the random vector where $\eta_j(k) = 1$ if the j'th individual is of type ♡ at time k and $\eta_j(k) = 0$ if the j'th individual is of type ♢ at time k. Let $I^N = \{0, \frac{1}{N}, \frac{2}{N}, \frac{3}{N} \dots, 1\}$ and $I^M = \{0, \frac{1}{M}, \frac{2}{M}, \frac{3}{M} \dots, 1\}$. Define the variables

$$\begin{aligned} X^N(k) &= \frac{1}{N}\sum_{j\in[N]} \mathbf{1}_{\{\xi_j(k)=\heartsuit\}} \quad \text{on } I^N, \\ Y^N(k) &= \frac{1}{N}\sum_{j\in[N]} \mathbf{1}_{\{\eta_j(k)=\heartsuit\}} \quad \text{on } I^M. \end{aligned} \tag{A.1}$$

Let $\mathbb{P}_{x,y}$ denote the law of

$$(X^N, Y^N) = (X^N(k), Y^N(k))_{k\in\mathbb{N}_0} \tag{A.2}$$

given that $(X^N(0), Y^N(0)) = (x,y) \in I^N \times I^M$. Then, as shown in [12],

$$\begin{aligned} p_{x,y}(\bar{x},\bar{y}) &= \mathbb{P}_{x,y}(X_1^N = \bar{x}, Y_1^N = \bar{y}) \\ &= \sum_{c'=0}^{c} \mathbb{P}_{x,y}(Z = c')\,\mathbb{P}_{x,y}(U = \bar{x}N - c')\,\mathbb{P}_{x,y}(V = \bar{y}M - yM + c'). \end{aligned} \tag{A.3}$$

Here, Z denotes the number of dormant ♡-individuals in generation 0 that become active in generation 1 ($\mathcal{L}_{x,y}(Z) = \mathrm{Hyp}_{M,c,yM}$), U denotes the number of active individuals in generation 1 that are offspring of active ♡-individuals in generation 0 ($\mathcal{L}_{x,y}(U) = \mathrm{Bin}_{N-c,x}$), and V denotes the number of active individuals in generation 0 that become dormant ♡-individuals in generation 1 ($\mathcal{L}_{x,y}(V) = \mathrm{Bin}_{c,x}$).

Speed up time by a factor N. The generator G^N for the process

$$((X^N(\lfloor Nk\rfloor), Y^N(\lfloor Nk\rfloor))_{k\in\mathbb{N}_0} \tag{A.4}$$

equals

$$\begin{aligned} &(G^N f)(x,y) = N\,\mathbb{E}_{x,y}\big[f(X^N(1), Y^N(1)) - f(x,y)\big], \\ &(x,y) \in I^N \times I^M, \end{aligned} \tag{A.5}$$

where the prefactor N appears because one step of the Markov chain takes time $\frac{1}{N}$. Inserting the Taylor expansion for f (which we assume to be smooth), using that $X^N(1) = \frac{U+Z}{N}$ and $Y^N(1) = \frac{yM+V-U}{M}$ and letting $N \to \infty$, we end up with the limiting generator G given by

$$\begin{aligned} &(Gf)(x,y) = c(y-x)\frac{\partial f}{\partial x}(x,y) + \frac{c}{K}(x-y)\frac{\partial f}{\partial y}(x,y) + \tfrac{1}{2}x(1-x)\frac{\partial^2 f}{\partial x^2}(x,y), \\ &(x,y) \in [0,1]\times[0,1], \end{aligned} \tag{A.6}$$

where $K = \frac{M}{N}$ is the relative size of the dormant population compared to the active population. This is the generator of the Markov process in the continuum limit [32, Section 7.8]. It follows from the form of G that this limit is described by the system of coupled stochastic differential equations

$$\begin{aligned} \mathrm{d}x(t) &= c\,[y(t) - x(t)]\,\mathrm{d}t + \sqrt{x(t)(1-x(t))}\,\mathrm{d}w(t), \\ \mathrm{d}y(t) &= \frac{c}{K}\,[x(t) - y(t)]\,\mathrm{d}t. \end{aligned} \tag{A.7}$$

This is the version of (2.4)–(2.5) for a single colony (no migration) and exchange rate

$$e = \frac{c}{K}. \tag{A.8}$$

Multi-colony model. First fix a number $L \in \mathbb{N}$ and consider $|\mathbb{G}| = L$ colonies. The *multi-colony* version with migration is obtained by allowing the $(1-\epsilon)N$ selected active individuals to undergo a migration in step (1):

(1) Each active individual at colony $i \in \mathbb{G}$ chooses colony $j \in \mathbb{G}$ with probability $\frac{1}{N}a(i,j)$ and adopts the type of a parent chosen from colony j. If an active individual does not migrate, it adopts the type of a parent chosen from its own population.

Using the same strategy as in the single-colony model, this results in (2.4)–(2.5), for $|\mathbb{G}| = L$. Subsequently we can let $L \to \infty$ and use convergence of generators to obtain (2.4)–(2.5) for countable $\mathbb{G}$.

Model 2. The same argument works for (2.12)–(2.13). Steps (1)-(4) are extended by considering a seed-bank with colours labelled by $\mathbb{N}_0$. First we consider the truncation where only finitely many colours are allowed, for which the argument carries through with minor adaptations. Afterwards, we pass to the limit of infinitely many colours, which is straightforward for a finite time horizon because large colours are only seen after large times. See also [60].

Model 3. To get (2.18)–(2.19), also extend Step (3) by adding a displacement via the kernel $a^{\dagger}(\cdot,\cdot)$ for each transition into the seed-bank.

§A.2 Alternative models

In this appendix we consider the Moran versions of models 1 and 2. What is written below is based on [60]. In the Moran version each active individual resamples at rate 1 and becomes dormant at a certain rate, while each dormant individual does not resample and becomes active at a certain rate. Since switches between active and dormant are done independently, the sizes of the active and the dormant population are *no longer* fixed and individuals *change* state without the necessity to *exchange* state. In model 1 there are two Poisson clocks, in model 2 there are two sequences of Poisson clocks, namely, two for each colour. In Appendices A.2.1–A.2.2 we compute the scaling limit for the case where the number of colours is $\mathfrak{m} = 1$ and $\mathfrak{m} = 2$, respectively. The extension to $\mathfrak{m} \geq 3$ is given in Appendix A.2.3. Migration can be added in the same way as is done in Appendix A.1.

§A.2.1 Alternative for Model 1

To describe the Moran version of Model 1 we need the following variables.

- Total number of individuals: $N \in \mathbb{N}$.
- Two types: $\heartsuit$ and $\diamondsuit$.
- $X(t)$ is the number of $\heartsuit$-individuals in the active population at time t.
- $Y(t)$ is the number of $\heartsuit$-individuals in the dormant population at time t.
- $Z(t)$ is the number of individuals in the active population at time t (either $\heartsuit$ or $\diamondsuit$).

In the Moran model with seed-bank each active individual resamples at rate 1, each active individual becomes dormant at rate ϵ and each dormant individual becomes active at rate δ. Hence the transition rates for $(X(t), Y(t), Z(t))$ are:

- $(i,j,k) \to (i+1,j,k)$ at rate $(k-i)\frac{i}{k}$.
- $(i,j,k) \to (i-1,j,k)$ at rate $i\frac{(k-i)}{k}$.
- $(i,j,k) \to (i-1,j+1,k-1)$ at rate ϵi.
- $(i,j,k) \to (i+1,j-1,k+1)$ at rate δj.
- $(i,j,k) \to (i,j,k-1)$ at rate $\epsilon\frac{k-i}{N}$.
- $(i,j,k) \to (i,j,k+1)$ at rate $\delta\frac{N-k-j}{N}$.

For the scaling limit we consider the variables

$$\bar{X}(t) = \frac{1}{N}X(Nt), \quad \bar{Y}(t) = \frac{1}{N}Y(Nt), \quad \bar{Z}(t) = \frac{1}{N}Z(Nt). \tag{A.9}$$

Hence

$$(\bar{X}(t), \bar{Y}(t), \bar{Z}(t)) \in I^N \times I^N \times I^N, \qquad I^N = \left\{0, \tfrac{1}{N}, \tfrac{2}{N}, \dots, \tfrac{N-1}{N}, 1\right\}. \tag{A.10}$$

Since in (A.9) we speed up time by a factor N, we must also speed up the transition rates by a factor N. To get a meaningful scaling limit, we assume that there exist $c^A, c^D \in (0,\infty)$ such that (see [12, p. 8])

$$N\epsilon = c^A, \qquad N\delta = c^D, \qquad N \in \mathbb{N}. \tag{A.11}$$

We can then write down the generator G^N:

$$\begin{aligned}(G^N f)\left(\frac{i}{N},\frac{j}{N},\frac{k}{N}\right) &= N(k-i)\frac{i}{k}\left[f\left(\frac{i+1}{N},\frac{j}{N},\frac{k}{N}\right)-f\left(\frac{i}{N},\frac{j}{N},\frac{k}{N}\right)\right]\\ &+ Ni\frac{k-i}{k}\left[f\left(\frac{i-1}{N},\frac{j}{N},\frac{k}{N}\right)-f\left(\frac{i}{N},\frac{j}{N},\frac{k}{N}\right)\right]\\ &+ c^A i\left[f\left(\frac{i-1}{N},\frac{j+1}{N},\frac{k-1}{N}\right)-f\left(\frac{i}{N},\frac{j}{N},\frac{k}{N}\right)\right]\\ &+ c^D j\left[f\left(\frac{i+1}{N},\frac{j-1}{N},\frac{k+1}{N}\right)-f\left(\frac{i}{N},\frac{j}{N},\frac{k}{N}\right)\right]\\ &+ c^A (k-i)\left[f\left(\frac{i}{N},\frac{j}{N},\frac{k-1}{N}\right)-f\left(\frac{i}{N},\frac{j}{N},\frac{k}{N}\right)\right]\\ &+ c^D (N-k-j)\left[f\left(\frac{i}{N},\frac{j}{N},\frac{k+1}{N}\right)-f\left(\frac{i}{N},\frac{j}{N},\frac{k}{N}\right)\right]\end{aligned} \tag{A.12}$$

Assuming that f is smooth and Taylor expanding f around $(\frac{i}{N},\frac{j}{N},\frac{k}{N})$, we get

$$\begin{aligned}(G^N f)\left(\frac{i}{N},\frac{j}{N},\frac{k}{N}\right) &= \frac{i(k-i)}{k}\left[\left(\frac{1}{N}\right)\frac{\partial^2 f}{\partial x^2}+\mathcal{O}\left(\left(\frac{1}{N}\right)^2\right)\right]\\ &+ c^A i\left[\left(\frac{-1}{N}\right)\frac{\partial f}{\partial x}+\left(\frac{1}{N}\right)\frac{\partial f}{\partial y}+\left(\frac{-1}{N}\right)\frac{\partial f}{\partial z}+\mathcal{O}\left(\left(\frac{1}{N}\right)^2\right)\right]\\ &+ c^D j\left[\left(\frac{1}{N}\right)\frac{\partial f}{\partial x}+\left(\frac{-1}{N}\right)\frac{\partial f}{\partial y}+\left(\frac{1}{N}\right)\frac{\partial f}{\partial z}+\mathcal{O}\left(\left(\frac{1}{N}\right)^2\right)\right]\\ &+ c^A (k-i)\left[\left(\frac{-1}{N}\right)\frac{\partial f}{\partial z}+\mathcal{O}\left(\left(\frac{1}{N}\right)^2\right)\right]\\ &+ c^D (N-k-j)\left[\left(\frac{1}{N}\right)\frac{\partial f}{\partial z}+\mathcal{O}\left(\left(\frac{1}{N}\right)^2\right)\right].\end{aligned} \tag{A.13}$$

Next, suppose that

$$\lim_{N\to\infty}\frac{i}{N}=x, \qquad \lim_{N\to\infty}\frac{j}{N}=y, \qquad \lim_{N\to\infty}\frac{k}{N}=z. \tag{A.14}$$

Letting $N\to\infty$ in (A.13), we obtain the limiting generator G:

$$\begin{aligned}(Gf)(x,y,z) = \ & z\frac{x}{z}\left(1-\frac{x}{z}\right)\left(\frac{\partial^2 f}{\partial x^2}\right)+[c^D\,y-c^A\,x]\frac{\partial f}{\partial x}\\ &+[c^A\,x-c^D\,y]\frac{\partial f}{\partial y}+\left[c^D\,(1-z)-c^A\,z\right]\frac{\partial f}{\partial z}.\end{aligned} \tag{A.15}$$

Therefore the continuum limit equals

$$\begin{aligned}\mathrm{d}x(t) &= \sqrt{z(t)\,\frac{x(t)}{z(t)}\left(1-\frac{x(t)}{z(t)}\right)}\,\mathrm{d}w(t)+\left[c^D\,y(t)-c^A\,x(t)\right]\mathrm{d}t,\\ \mathrm{d}y(t) &= \left[c^A\,x(t)-c^D\,y(t)\right]\mathrm{d}t,\\ \mathrm{d}z(t) &= \left[c^D\,(1-z(t))-c^A\,z(t)\right]\mathrm{d}t.\end{aligned} \tag{A.16}$$

Since $z(t)$ is the fraction of active individuals in the population, $1-z(t)$ is the fraction of dormant individuals in the population. Therefore the equivalent of the parameter K in Appendix A.1 is $K(t) = (1-z(t))/z(t)$. Moreover, $x(t)/z(t)$ is the fraction of $\heartsuit$-individuals in the active population at time t and $y(t)/(1-z(t))$ is the fraction of $\heartsuit$-individuals in the dormant population at time t. The last line of (A.16) is an autonomous differential equation whose solution converges to

$$z^* = \frac{1}{1+\frac{c^A}{c^D}} \tag{A.17}$$

exponentially fast. After this transition period we can replace $z(t)$ by z^*, and we see that $K^* = c^A/c^D$.

Time is to be scaled by the total number of active *and* dormant individuals, instead of the total number of active individuals only:

$$\begin{aligned} x(t) &= \frac{\text{number of active individuals of type } \heartsuit}{\text{total number of individuals}}, \\ y(t) &= \frac{\text{number of dormant individuals of type } \heartsuit}{\text{total number of individuals}}. \end{aligned} \tag{A.18}$$

To compare the Moran model with a 1-colour seed-bank with the Fisher-Wright model with a 1-colour seed-bank, we look at the variables

$$\bar{x}(t) = \left(1+\frac{c^A}{c^D}\right) x\left(\frac{t}{1+\frac{c^A}{c^D}}\right), \qquad \bar{y}(t) = \left(1+\frac{c^A}{c^D}\right)\left(\frac{c^D}{c^A}\right) y\left(\frac{t}{1+\frac{c^A}{c^D}}\right). \tag{A.19}$$

After a short transition period in which $z(t)$ tends to z^*, we see that by setting

$$K = K^* = \frac{c^A}{c^D}, \qquad e = \frac{c^D}{c^A}\frac{c^A c^D}{c^A+c^D}, \tag{A.20}$$

we obtain

$$\begin{aligned} \mathrm{d}\bar{x}(t) &= \sqrt{\bar{x}(t)(1-\bar{x}(t))}\,\mathrm{d}w(t) + Ke\,[\bar{y}(t)-\bar{x}(t)]\,\mathrm{d}t, \\ \mathrm{d}\bar{y}(t) &= e\,[\bar{x}(t)-\bar{y}(t)]\,\mathrm{d}t, \end{aligned} \tag{A.21}$$

which is the single-colony version of (2.4)–(2.5) but without migration. Migration can be added in the same way as was done in Appendix A.1.

§A.2.2 Alternative for Model 2: Two colours

We consider the following system:

- Total number of individuals: $N \in \mathbb{N}$.
- Two types: $\heartsuit$ and $\diamondsuit$.
- $X(t)$ is the number of $\heartsuit$-individuals in the active population at time t.
- $Y_1(t)$ is the number of $\heartsuit$-individuals of colour 1 in the dormant population at time t.

- $Y_2(t)$ is the number of $\heartsuit$-individuals of colour 2 in the dormant population at time t.

- $Z_{D_1}(t)$ is the number of dormant individuals of colour 1 at time t (either $\heartsuit$ or $\diamondsuit$).

- $Z_{D_2}(t)$ is the number of dormant individuals of colour 2 at time t. (either $\heartsuit$ or $\diamondsuit$).

Note that the number of active individuals at time t (either $\heartsuit$ or $\diamondsuit$) is given by $Z_A(t) = N - Z_{D_1}(t) - Z_{D_2}(t)$. Since the number of individuals N is constant during the evolution, $Z_A(t)$ can be derived from $Z_{D_1}(t)$ and $Z_{D_2}(t)$. Each active individual resamples at rate 1, and becomes dormant at rate ϵ. When an individual becomes dormant, it gets either colour 1 with probability p_1 or colour 2 with probability p_2, where $p_1, p_2 \in (0,1)$ and $p_1 + p_2 = 1$. For ease of notation, we denote the rate to become dormant with colour 1 by $\epsilon_1 = \epsilon \cdot p_1$ and the rate to become dormant with colour 2 by $\epsilon_2 = \epsilon \cdot p_2$. A dormant individual with colour 1 becomes active at rate δ_1, a dormant individual with colour 2 becomes active at rate δ_2. Thus, the transition rates for $(X(t), Y_1(t), Y_2(t), Z_{D_1}(t), Z_{D_2}(t))$ are:

- $(i,j,k,l,m) \to (i+1,j,k,l,m)$ at rate $(N-l-m-i)\frac{i}{N-l-m}$.
- $(i,j,k,l,m) \to (i-1,j,k,l,m)$ at rate $i\frac{(N-l-m-i)}{N-l-m}$.
- $(i,j,k,l,m) \to (i-1,j+1,k,l+1,m)$ at rate $\epsilon_1 i$.
- $(i,j,k,l,m) \to (i+1,j-1,k,l-1,m)$ at rate $\delta_1 j$.
- $(i,j,k,l,m) \to (i-1,j,k+1,l,m+1)$ at rate $\epsilon_2 i$.
- $(i,j,k,l,m) \to (i+1,j,k-1,l,m-1)$ at rate $\delta_2 k$.
- $(i,j,k,l,m) \to (i,j,k,l+1,m)$ at rate $\epsilon_1(N-l-m-i)$.
- $(i,j,k,l,m) \to (i,j,k,l,m+1)$ at rate $\epsilon_2(N-l-m-i)$.
- $(i,j,k,l,m) \to (i,j,k,l-1,m)$ at rate $\delta_1(l-j)$.
- $(i,j,k,l,m) \to (i,j,k,l,m-1)$ at rate $\delta_2(m-k)$.

Proceeding in the same way as for the 1-colour seed-bank, we define the scaled variables

$$\begin{aligned} \bar{X}(t) &= \frac{1}{N}X(Nt), \quad \bar{Y}_1(t) = \frac{1}{N}Y_1(Nt), \quad \bar{Y}_2(t) = \frac{1}{N}Y_2(Nt), \\ \bar{Z}_{D_1}(t) &= \frac{1}{N}Z_{D_1}(Nt), \quad \bar{Z}_{D_2}(t) = \frac{1}{N}Z_{D_1}(Nt). \end{aligned} \tag{A.22}$$

We assume that there exist $c_1^A, c_2^A, c_1^D, c_2^D \in (0,\infty)$ such that

$$N\epsilon_1 = c_1^A, \qquad N\epsilon_2 = c_2^A, \qquad N\delta_1 = c_1^D, \qquad N\delta_2 = c_2^D, \qquad N \in \mathbb{N}, \tag{A.23}$$

and further assume that

$$\begin{aligned}&\lim_{N\to\infty}\frac{i}{N}=x,\quad \lim_{N\to\infty}\frac{j}{N}=y_1,\quad \lim_{N\to\infty}\frac{k}{N}=y_2,\\ &\lim_{N\to\infty}\frac{N-l-m}{N}=z_A\quad \lim_{N\to\infty}\frac{N-l-m}{N}=z_{D_2},\quad \lim_{N\to\infty}\frac{N-l-m}{N}=z_{D_1}.\end{aligned}\tag{A.24}$$

Using the same method of converging generators as for model 1, we obtain the following continuum limit:

$$\begin{aligned}\mathrm{d}x(t)&=\sqrt{z_A(t)\frac{z_A-x(t)}{z_A(t)}\frac{x(t)}{z_A(t)}}\,\mathrm{d}w(t)\\&\quad+\left[c_1^Dy_1(t)-c_1^Ax(t)\right]\mathrm{d}t+\left[c_2^Dy_2(t)-c_2^Ax(t)\right]\mathrm{d}t,\\ \mathrm{d}y_1(t)&=\left[c_1^Ax(t)-c_1^Dy_1(t)\right]\mathrm{d}t,\\ \mathrm{d}y_2(t)&=\left[c_2^Ax(t)-c_2^Dy_2(t)\right]\mathrm{d}t,\\ \mathrm{d}z_A(t)&=\left[c_1^Dz_{D_1}(t)-c_1^Az_A(t)+c_2^Dz_{D_2}(t)-c_2^Az_A(t)\right]\mathrm{d}t,\\ \mathrm{d}z_{D_1}(t)&=\left[c_1^Az_A(t)-c_1^Dz_{D_1}(t)\right]\mathrm{d}t,\\ \mathrm{d}z_{D_2}(t)&=\left[c_2^Az_A(t)-c_2^Dz_{D_2}(t)\right]\mathrm{d}t.\end{aligned}\tag{A.25}$$

Note that the equation for $z_A(t)=1-z_{D_1}(t)-z_{D_2}(t)$ follows directly from the equations from $z_{D_1}(t)$ and $z_{D_2}(t)$. It is therefore redundant, but we use it for notational reasons. Again, we see that $z(t)=(z_A(t),z_{D_1}(t),z_{D_2}(t))$ is governed by an autonomous system of differential equations. Solving this system, we see that

$$\lim_{t\to\infty}z_A(t)=\frac{1}{1+\frac{c_1^A}{c_1^D}+\frac{c_2^A}{c_2^D}},\quad \lim_{t\to\infty}z_{D_1}(t)=\frac{\frac{c_1^A}{c_1^D}}{1+\frac{c_1^A}{c_1^D}+\frac{c_2^A}{c_2^D}},\quad \lim_{t\to\infty}z_{D_2}(t))=\frac{\frac{c_2^A}{c_2^D}}{1+\frac{c_1^A}{c_1^D}+\frac{c_2^A}{c_2^D}}.\tag{A.26}$$

To compare the Moran model with a 2-colour seed-bank with the Fisher-Wright model with a 2-colour seed-bank, we look at the variables

$$\begin{aligned}\bar{x}(t)&=\left(1+\frac{c_1^A}{c_1^D}+\frac{c_2^A}{c_2^D}\right)x\left(\frac{t}{1+\frac{c_1^A}{c_1^D}+\frac{c_2^A}{c_2^D}}\right),\\ \bar{y}_1(t)&=\left(1+\frac{c_1^A}{c_1^D}+\frac{c_2^A}{c_2^D}\right)\left(\frac{c_1^D}{c_1^A}\right)y_1\left(\frac{t}{1+\frac{c_1^A}{c_1^D}+\frac{c_2^A}{c_2^D}}\right),\\ \bar{y}_2(t)&=\left(1+\frac{c_1^A}{c_1^D}+\frac{c_2^A}{c_2^D}\right)\left(\frac{c_2^D}{c_2^A}\right)y_2\left(\frac{t}{1+\frac{c_1^A}{c_1^D}+\frac{c_2^A}{c_2^D}}\right).\end{aligned}\tag{A.27}$$

Defining

$$K_m=\frac{c_m^A}{c_m^D},\quad e_m=\frac{c_m^D}{1+\frac{c_1^A}{c_1^D}+\frac{c_2^A}{c_2^D}},\qquad m\in\{1,2\},\tag{A.28}$$

we see that, after a short transition period, the system becomes

$$\begin{aligned}
\mathrm{d}\bar{x}(t) &= \sqrt{\bar{x}(t)(1-\bar{x}(t))}\,\mathrm{d}w(t) + K_1 e_1\,[\bar{y}_2(t)-\bar{x}(t)]\,\mathrm{d}t + K_2 e_2\,[\bar{y}_1(t)-\bar{x}(t)]\,\mathrm{d}t,\\
\mathrm{d}\bar{y}_1(t) &= e_1\,[\bar{x}(t)-\bar{y}_1(t)]\,\mathrm{d}t,\\
\mathrm{d}\bar{y}_2(t) &= e_2\,[\bar{x}(t)-\bar{y}_2(t)]\,\mathrm{d}t,
\end{aligned} \tag{A.29}$$

which is the single-colony version of (2.12)–(2.13) with 2 colours and without migration. Note, in particular, that after $z(t)$ reaches the equilibrium point in (A.26), we have

$$K_m = \frac{\text{number of dormant individuals with colour } m}{\text{number of active individuals}}, \quad m \in \{1,2\}. \tag{A.30}$$

It is instructive to show how the above result can also be derived with the help of *duality*. The argument that follows easily extends to an n-coloured seed-bank for any $n \in \mathbb{N}$ finite, to be considered in Appendix A.2.3. Recall from (A.25) that

$$\begin{aligned}
\mathrm{d}z_A(t) &= \left[c_1^D z_{D_1}(t) - c_1^A z_A(t) + c_2^D z_{D_2}(t) - c_2^A z_A(t)\right]\mathrm{d}t,\\
\mathrm{d}z_{D_1}(t) &= \left[c_1^A z_A(t) - c_1^D z_{D_1}(t)\right]\mathrm{d}t,\\
\mathrm{d}z_{D_2}(t) &= \left[c_2^A z_A(t) - c_2^D z_{D_2}(t)\right]\mathrm{d}t.
\end{aligned} \tag{A.31}$$

Let

$$\begin{aligned}
\bar{z}_A(t) &= \left(1+\frac{c_1^A}{c_1^D}+\frac{c_2^A}{c_2^D}\right) z_A(t),\\
\bar{z}_{D_1}(t) &= \left(1+\frac{c_1^A}{c_1^D}+\frac{c_2^A}{c_2^D}\right)\left(\frac{c_1^D}{c_1^A}\right) z_{D_1}(t),\\
\bar{z}_{D_2}(t) &= \left(1+\frac{c_1^A}{c_1^D}+\frac{c_2^A}{c_2^D}\right)\left(\frac{c_2^D}{c_2^A}\right) z_{D_2}(t).
\end{aligned} \tag{A.32}$$

Substitute (A.32) into (A.31), to obtain

$$\begin{aligned}
\mathrm{d}\bar{z}_A(t) &= c_1^A\,[\bar{z}_{D_1}(t)-\bar{z}_A(t)] + c_2^A\,[\bar{z}_{D_2}(t)-\bar{z}_A(t)]\,\mathrm{d}t,\\
\mathrm{d}\bar{z}_{D_1}(t) &= c_1^D\,[\bar{z}_A(t)-\bar{z}_{D_1}(t)]\,\mathrm{d}t,\\
\mathrm{d}\bar{z}_{D_2}(t) &= c_2^D\,[\bar{z}_A(t)-\bar{z}_{D_2}(t)]\,\mathrm{d}t.
\end{aligned} \tag{A.33}$$

To define a dual for the process $(\bar{z}_A(t), \bar{z}_{D_1}(t), \bar{z}_{D_2}(t))_{t\geq 0}$, let $(M(t))_{t\geq 0}$ be the continuous-time Markov chain on $\{A, D_1, D_2\}$ with transition rates

$$\begin{aligned}
A \to D_m &\text{ at rate } c_m^A, \quad m\in\{1,2\},\\
D_m \to A &\text{ at rate } c_m^D, \quad m\in\{1,2\}.
\end{aligned} \tag{A.34}$$

Consider l independent copies of $(M(t))_{t\geq 0}$, evolving on the same state space $\{A, D_1, D_2\}$. Let $(L(t))_{t\geq 0} = (L_A(t), L_{D_1}(t), L_{D_2}(t))_{t\geq 0}$ be the process that counts how many copies of $M(t)$ are on site $\{A\}$, $\{D_1\}$ and $\{D_2\}$ at time t. Let $l = m + n_1 + n_2$. Then $(L(t))_{t\geq 0}$ is the Markov process on $\mathbb{N}_0^3$ with transition rates

$$(m,n_1,n_2) \to \begin{cases}
(m-1,n_1+1,n_2) & \text{at rate } mc_1^A,\\
(m-1,n_1,n_2+1) & \text{at rate } mc_2^A,\\
(m+1,n_1-1,n_2) & \text{at rate } n_1c_1^D,\\
(m+1,n_1,n_2-1) & \text{at rate } n_2c_2^D.
\end{cases}. \tag{A.35}$$

Note that $L_A(t) + L_{D_1}(t) + L_{D_2}(t) = L_A(0) + L_{D_1}(0) + L_{D_2}(0) = m + n_1 + n_2 = l$. Define $H\colon \mathbb{R}^3 \times \mathbb{N}_0^3 \to \mathbb{R}$ by

$$H((\bar{z}_A, \bar{z}_{D_1}, \bar{z}_{D_2}), (m, n_1, n_2)) := \bar{z}_A^m \bar{z}_{D_1}^{n_1} \bar{z}_{D_2}^{n_2} \tag{A.36}$$

Using the generator criterion [48, Proposition 1.2], we see that, for all $t \geq 0$,

$$\begin{aligned} &\mathbb{E}\left[H((\bar{z}_A(t), \bar{z}_{D_1}(t), \bar{z}_{D_2}(t)), (m(0), n_1(0), n_2(0)))\right] \\ &\quad = \mathbb{E}\left[H((\bar{z}_A(0), \bar{z}_{D_1}(0), \bar{z}_{D_2}(0)), (m(t), n_1(t), n_2(t)))\right]. \end{aligned} \tag{A.37}$$

Therefore $(L(t))_{t\geq 0}$ and $(\bar{z}(t))_{t\geq 0}$ are dual to each other with duality function H.

Since $(M(t))_{t\geq 0}$ is a irreducible and recurrent, we can define

$$\begin{aligned} \pi_A &= \lim_{t\to\infty} \mathbb{P}(M(t) = A) = \frac{1}{1 + \frac{c_1^A}{c_1^D} + \frac{c_2^A}{c_2^D}}, \\ \pi_{D_1} &= \lim_{t\to\infty} \mathbb{P}(M(t) = D_1) = \frac{\frac{c_1^A}{c_1^D}}{1 + \frac{c_1^A}{c_1^D} + \frac{c_2^A}{c_2^D}}, \\ \pi_{D_2} &= \lim_{t\to\infty} \mathbb{P}(M(t) = D_2) = \frac{\frac{c_2^A}{c_2^D}}{1 + \frac{c_1^A}{c_1^D} + \frac{c_2^A}{c_2^D}}. \end{aligned} \tag{A.38}$$

Using the duality relation in (A.37) together with (A.38) and (A.32), we find

$$\begin{aligned} \lim_{t\to\infty} \mathbb{E}[\bar{z}_A(t)] &= \pi_A \bar{z}_A(0) + \pi_{D_1} \bar{z}_{D_1}(0) + \pi_{D_2} \bar{z}_{D_2}(0) \\ &= \frac{1}{1 + \frac{c_1^A}{c_1^D} + \frac{c_2^A}{c_2^D}} \bar{z}_A(0) + \frac{\frac{c_1^A}{c_1^D}}{1 + \frac{c_1^A}{c_1^D} + \frac{c_2^A}{c_2^D}} \bar{z}_{D_1}(0) + \frac{\frac{c_2^A}{c_2^D}}{1 + \frac{c_1^A}{c_1^D} + \frac{c_2^A}{c_2^D}} \bar{z}_{D_2}(0) \\ &= z_A(0) + z_{D_1}(0) + z_{D_2}(0) = 1. \end{aligned} \tag{A.39}$$

Using the duality relation in(A.37) once more, we get

$$\lim_{t\to\infty} \mathbb{E}[\bar{z}_A(t)] = \lim_{t\to\infty} \mathbb{E}[\bar{z}_{D_1}(t)] = \lim_{t\to\infty} \mathbb{E}[\bar{z}_{D_2}(t)] = 1. \tag{A.40}$$

Computing the limiting second moment $\lim_{t\to\infty} \mathbb{E}[\bar{z}_A(t)^2]$ by duality, we obtain

$$\begin{aligned} \lim_{t\to\infty} \mathbb{E}[\bar{z}_A(t)^2] &= \lim_{t\to\infty} \sum_{\substack{i,j\in \\ \{A,D_1,D_2\}}} \mathbb{P}(M_t^1 = i)\, \bar{z}_i(0)\, \mathbb{P}(M_t^2 = j)\, \bar{z}_j(0) \\ &= \sum_{i\in\{A,D_1,D_2\}} \pi_i \bar{z}_i(0) \sum_{j\in\{A,D_1,D_2\}} \pi_j \bar{z}_j(0) = 1. \end{aligned} \tag{A.41}$$

Similarly, we find $\lim_{t\to\infty} \mathbb{E}[\bar{z}_{D_1}(t)^2] = 1$ and $\lim_{t\to\infty} \mathbb{E}[\bar{z}_{D_2}(t)^2] = 1$. Combining (A.40) and (A.41), we find

$$\lim_{t\to\infty} \bar{z}_A(t) = \lim_{t\to\infty} \bar{z}_{D_1}(t) = \lim_{t\to\infty} \bar{z}_{D_2}(t) = 1. \tag{A.42}$$

Hence we conclude that

$$\lim_{t\to\infty} z_A(t) = \frac{1}{1+\frac{c_1^A}{c_1^D}+\frac{c_2^A}{c_2^D}}, \quad \lim_{t\to\infty} z_{D_1}(t) = \frac{\frac{c_1^A}{c_1^D}}{1+\frac{c_1^A}{c_1^D}+\frac{c_2^A}{c_2^D}}, \quad \lim_{t\to\infty} z_{D_2}(t) = \frac{\frac{c_2^A}{c_2^D}}{1+\frac{c_1^A}{c_1^D}+\frac{c_2^A}{c_2^D}}. \tag{A.43}$$

Continuing as in (A.27), we again find the single-colony version of (2.12)-(2.13) with 2 colours and no migration.

§A.2.3 Alternative for Model 2: Three or more colours

The argument in Appendix A.2.2 can be extended to an $\mathfrak{m} \in \mathbb{N}$-colour seed-bank, by introducing sequences of variables $(Y_m(t))_{m=0}^{\mathfrak{m}}$ and $(Z_m(t))_{m=0}^{\mathfrak{m}}$ that count the number of $\heartsuit$-individuals in the colour-m seed-bank at time t, respectively, the total number of individuals in the colour-m seed-bank at time t. Let $\epsilon > 0$ be the total rate at which an active individual becomes dormant, and define a probability vector $(p_m)_{m=0}^{\mathfrak{m}}$ such that $\epsilon_m = \epsilon p_m$ is the rate at which an active individual becomes dormant with colour m. Let δ_m be the rate at which m-dormant individuals become active. Via the same line of argument as in Appendix A.2.2, we see that the equivalent of (A.25) reads

$$\begin{aligned} \mathrm{d}x(t) &= \sqrt{z_A(t)\frac{z_A - x(t)}{z_A(t)}\frac{x(t)}{z_A(t)}}\,\mathrm{d}w(t) + \sum_{m=0}^{\mathfrak{m}} \left[c_m^D y_m(t) - c_m^A x(t)\right]\mathrm{d}t, \\ \mathrm{d}y_m(t) &= \left[c_m^A x(t) - c_m^D y_m(t)\right]\mathrm{d}t, \\ \mathrm{d}z_A(t) &= \sum_{m=0}^{\mathfrak{m}} \left[c_m^D z_{D_m}(t) - c_m^A z_A(t)\right]\mathrm{d}t, \\ \mathrm{d}z_{D_m}(t) &= \left[c_m^A z_A(t) - c_m^D z_{D_m}(t)\right]\mathrm{d}t, \qquad 0 \le m \le N. \end{aligned} \tag{A.44}$$

Solving the autonomous system describing $z(t) = (z_A(t), (z_{D_m}(t))_{m=0}^N)$ via duality, and subsequently substituting into (A.44) the variables

$$\begin{aligned} \bar{x}(t) &= \left(1+\sum_{n=0}^{\mathfrak{m}}\frac{c_n^A}{c_n^D}\right) x\left(\frac{t}{1+\sum_{n=0}^{\mathfrak{m}}\frac{c_n^A}{c_n^D}}\right), \\ \bar{y}_m(t) &= \left(1+\sum_{n=0}^{\mathfrak{m}}\frac{c_n^A}{c_n^D}\right)\left(\frac{c_m^D}{c_m^A}\right) y_m\left(\frac{t}{1+\sum_{n=0}^{\mathfrak{m}}\frac{c_n^A}{c_n^D}}\right), \qquad 0 \le m \le N, \end{aligned} \tag{A.45}$$

we find the single-colony version of (2.12)–(2.13) with N-colours and no migration. Migration can be added as in Appendix A.1.

It is straightforward to derive the version (2.12)–(2.13) with N-colours and M colonies. Afterwards we can let $N, M \to \infty$ and use convergence of generators, to find (2.12)–(2.13). The limit is unproblematic because we are interested in finite time horizons only.

§A.3 Successful coupling

To prove Lemma 3.2.11 we proceed as in [14], with minor adaptations. The notation used in this appendix is the same as in Section 3.2.3. For model 1 we write down the full proof. The proof holds works for model 2 and 3 by invoking the colours $m \in \mathbb{N}_0$ and the SSDE in (2.12)–(2.13), respectively, (2.18)–(2.19).

Proof of Lemma 3.2.11. The proof consists of 5 steps.

Step 1. If $z \in E$ with $x_i = 0$ and $x_k > 0$ for some $k \neq i$, then

$$\mathbb{P}_z\left(\exists\, t^* > 0 \text{ such that } x_i(t) = 0\ \forall\, t \in [0, t^*]\right) = 0. \tag{A.46}$$

Proof. Suppose that z is such that $x_i = 0$, but $x_k > 0$ for some $i, k \in \mathbb{G}$. By (2.4),

$$x_i(t) = \int_0^t \sum_{j \in \mathbb{G}} a(i,j)[x_j(s) - x_i(s)]\,\mathrm{d}s + \int_0^t Ke[y_i(s) - x_i(s)]\,\mathrm{d}s + \int_0^t \sqrt{g(x_i(s))}\,\mathrm{d}w_i(s). \tag{A.47}$$

Suppose that there exists a $T > 0$ such that $x_i(t) = 0$ for all $t \in [0, T]$, and therefore $g(x_i(t)) = 0$. Then we obtain for all $t \in [0, T]$ that

$$\int_0^t \sum_{j \in \mathbb{G}} a(i,j) x_j(s)\,\mathrm{d}s + \int_0^t Key_i(s)\,\mathrm{d}s = 0. \tag{A.48}$$

Hence, by path continuity of $(Z(t))_{t \geq 0}$, we see that $y_i(t) = 0$ for all $t \in [0, T]$, as well as $x_j(t) = 0$ for all $j \in \mathbb{G}$ such that $a(i,j) > 0$. Repeating this argument, we obtain by irreducibility of $a(\cdot,\cdot)$ that $x_k(t) = 0$ for all $k \in \mathbb{G}$ and hence $y_k(t) = 0$ for all $k \in \mathbb{G}$. By path continuity, this contradicts the assumption that $x_k(0) > 0$. We conclude that (A.46) holds. □

Step 2. If $\bar{z} \in E \times E$ and $g(x_i^1) \neq g(x_i^2)$, then for all j,

$$\hat{\mathbb{P}}_{\bar{z}}\left(\exists\, t^* > 0 \text{ such that } \Delta_j(t) = 0\ \forall\, t \in [0, t^*]\right) = 0. \tag{A.49}$$

Proof. Note that the SSDE in (2.4)–(2.5) can be rewritten as

$$\begin{aligned} \mathrm{d}z_{(i,R_i)}(t) = &\sum_{(j,R_j) \in \mathbb{G} \times \{A,D\}} b^{(1)}((i,R_i),(j,R_j))[z_{(j,R_j)}(t) - z_{(i,R_i)}(t)]\,\mathrm{d}t \\ &\qquad + \sqrt{g(z_{(i,R_i)}(t))}\; 1_{\{R_i = A\}}\,\mathrm{d}w_i(t), \\ &\forall\,(i,R_i) \in \mathbb{G} \times \{A,D\}, \end{aligned} \tag{A.50}$$

with $b^{(1)}(\cdot,\cdot)$ defined as in (2.31).

Suppose that $\bar{z}$ is such that $g(x_i^1) \neq g(x_i^2)$. Suppose there exist a $T > 0$ such that $\Delta_j(t) = 0$ for all $t \in [0, T]$. Then also $\sqrt{g(x_j^1(t))} - \sqrt{g(x_j^2(t))} = 0$ for all $t \in [0, T]$.

Using (A.50) on $\Delta_j(t) = z^1_{(j,A)}(t) - z^2_{(j,A)}(t)$, we obtain

$$0 = \int_0^t \sum_{(k,R_k)\in\mathbb{G}\times\{A,D\}} b^{(1)}((j,A),(k,R_k)) \times \Big[\Big(z^1_{(k,R_k)}(s) - z^2_{(k,R_k)}(s)\Big) - \Big(z^1_{(j,R_j)}(s) - z^2_{(j,R_j)}(s)\Big)\Big]\,\mathrm{d}s. \tag{A.51}$$

Hence

$$\sum_{(k,R_k)\in\mathbb{G}\times\{A,D\}} b^{(1)}((j,A),(k,R_k)) \Big[\Big(z^1_{(k,R_k)}(t) - z^2_{(k,R_k)}(t)\Big) - \Big(z^1_{(j,R_j)}(t) - z^2_{(j,R_j)}(t)\Big)\Big] = 0 \qquad \forall\, t\in[0,T]. \tag{A.52}$$

Using (A.50), we can write the SDE for

$$\sum_{(k,R_k)\in\mathbb{G}\times\{A,D\}} b^{(1)}((j,A),(k,R_k)) \Big[\Big(z^1_{(j,R_j)}(t) - z^2_{(j,R_j)}(t)\Big) - \Big(z^1_{(i,R_i)}(t) - z^2_{(i,R_i)}(t)\Big)\Big], \tag{A.53}$$

which yields that, for all $t\in[0,T]$,

$$\begin{aligned} &-\int_0^t \sum_{(k,R_k)\in\mathbb{G}\times\{A,D\}} b^{(1)}((j,A),(k,R_k)) \Big(\sqrt{g(z^1_{k,R_k}(s))} - \sqrt{g(z^2_{k,R_k}(s))}\Big) 1_{\{R_k=A\}}\,\mathrm{d}w_k(s) \\ &= \int_0^t \sum_{(k,R_k)\in\mathbb{G}\times\{A,D\}} b^{(1),2}((j,A),(l,R_l)) \\ &\qquad \times \Big[\Big(z^1_{(j,R_j)}(s) - z^2_{(j,R_j)}(s)\Big) - \Big(z^1_{(i,R_i)}(s) - z^2_{(i,R_i)}(s)\Big)\Big]\,\mathrm{d}s, \end{aligned} \tag{A.54}$$

where $b^{(1),2}(\cdot,\cdot)$ is the 2-step kernel of $b^{(1)}(\cdot,\cdot)$.

The two process in the right-hand side form a process of bounded variation, while the process in the left-hand side is a continuous square-integrable martingale, whose quadratic variation is given by

$$\int_0^t \sum_{k\in\mathbb{G}} a(j,k)^2 \Big(\sqrt{g(x^1_k(s))} - \sqrt{g(x^2_k(s))}\Big)^2\,\mathrm{d}s. \tag{A.55}$$

Since a square-integrable martingale of bounded variation is constant, it follows that (A.55) equals 0. Hence, for all k such that $a(j,k) > 0$, it follows that $g(x^1_k(t)) = g(x^2_k(t))$ for all $t\in[0,T]$. Moreover, the right-hand side of (A.54) is equal to 0. Iterating the right-hand side of (A.54) further, we find by the irreducibility of $a(\cdot,\cdot)$ that $g(x^1_i(t)) = g(x^2_i(t))$ for all $t\in[0,T]$, which contradicts the assumption on $\bar z$ that $g(x^1_i(0)) \neq g(x^2_i(0))$. Hence we find that there does not exist a $T > 0$ such that $\Delta_j(t) = 0$ for all $t\in[0,T]$. □

Step 3. If $\bar z\in E\times E,\ i,k\in\mathbb{G}$ and $g(x^1_i) = g(x^2_i)$ with $\Delta_i < 0$ and $\Delta_k > 0$ for some $k\neq i$, then

$$\hat{\mathbb{P}}_{\bar z}\left(\exists t^*\in[0,\tfrac12]\colon\ \Delta_i(t^*) < 0,\ \Delta_k(t^*) > 0,\ g(x^1_i(t^*)) \neq g(x^2_i(t^*))\right) > 0. \tag{A.56}$$

Proof. Note that by assumption we have $x_i^1 < 1$ and $x_k^1 > 0$. Let $t_0 \in [0, \frac{1}{4}]$. If $x_i^1 > 0$, then set $t_0 = 0$. Otherwise, by Step 1 and path continuity, we find with probability 1 a $t_0 \in [0, \frac{1}{4}]$ such that $x_i^1(t_0) > 0$, $\Delta_i(t_0) < 0$ and $\Delta_k(t_0) > 0$. Let $\tilde{z} = \bar{z}(t_0)$. By the existence of t_0 and the Markov property, it is enough to prove that

$$\hat{\mathbb{P}}_{\tilde{z}}\left(\exists\, t^* \in [0, \tfrac{1}{4}]\colon\ \Delta_i(t^*) < 0,\ \Delta_k(t^*) > 0,\ g(x_i^1(t^*)) \neq g(x_i^2(t^*))\right) > 0 \tag{A.57}$$

in order to prove (A.56). Define the following two martingales:

$$M_i(t) = \int_0^t \sqrt{g(x_i^1(s))}\,\mathrm{d}w_i(s), \tag{A.58}$$

$$M_k(t) = \int_0^t \left(\sqrt{g(x_k^1(s))} - \sqrt{2g(x_k^2(s))}\right)\mathrm{d}w_k(s). \tag{A.59}$$

Their corresponding quadratic variation processes are given by

$$\langle M_i(t)\rangle = \int_0^t g(x_i(s))\,\mathrm{d}s, \tag{A.60}$$

$$\langle M_k(t)\rangle = \int_0^t \left(\sqrt{g(x_k^1(s))} - \sqrt{2g(x_k^2(s))}\right)^2 \mathrm{d}s. \tag{A.61}$$

By Knight's theorem (see [62, Theorem V.1.9 p.183]), we can write $M_i(t)$ and $M_k(t)$ as time-transformed Brownian motions:

$$M_i(t) = w_i\left(\langle M_i(t)\rangle\right), \tag{A.62}$$

$$M_k(t) = w_k\left(\langle M_k(t)\rangle\right). \tag{A.63}$$

We may assume that $g(\tilde{x}_i^1) = g(\tilde{x}_i^2)$, otherwise we can set $t^* = 0$. Recall that $0 < \tilde{x}_i^1 < 1$, $\tilde{\Delta}_i < 0$ and $\tilde{\Delta}_k > 0$, and, since $0 < g(\tilde{x}_i^1) = g(\tilde{x}_i^2)$, also $\tilde{x}_i^2 < 1$. Choose an $\epsilon \in (0, \frac{1}{15})$ such that $\tilde{x}_i^1, \tilde{x}_i^2 \in [5\epsilon, 1-5\epsilon]$, $-\tilde{\Delta}_i > 5\epsilon$ and $\tilde{\Delta}_k > 5\epsilon$. Let $\xi \in (0, \epsilon)$ be such that $g(\xi) < \min\{g(u)\colon \epsilon \le u \le 1-\epsilon\}$, and set $c_1 = \min\{g(u)\colon \xi \le u \le 1-\xi\}$ and $c_2 = \|g\|$. Then we can make the following estimates:

$$\langle M_i(t)\rangle \le c_2 t \qquad \langle M_k(t)\rangle \le c_2 t,\ t \ge 0, \tag{A.64}$$

$$\langle M_i(t)\rangle \ge c_1 t \qquad \text{for } t \ge 0 \text{ such that } x_i(s) \in [\xi, 1-\xi]\ \forall\, s \in [0,t]. \tag{A.65}$$

Define $c_3 = \min\{\frac{\xi}{2Ke}, \frac{\xi}{2}\}$. Fix $T \in [0, c_3]$ and define

$$\begin{aligned}
\Omega_0 &= \left\{\min_{t\in[0,c_1T]} w_i(t) < -1,\ \max_{t\in[0,c_2T]} w_i(t) < \epsilon,\ \max_{t\in[0,c_2T]} |w_k(t)| < \epsilon\right\},\\
\Omega_1 &= \left\{\exists t^* \in [0,1] \text{ such that } \Delta_i(t^*) < 0,\ \Delta_k(t^*) > 0,\ g(x_i^1(t^*)) = g(x_i^2(t^*))\right\}.
\end{aligned} \tag{A.66}$$

Note that $\mathbb{P}(\Omega_0) > 0$. Therefore it suffices that $\Omega_0 \subset \Omega_1$.

We start by checking the conditions Δ_k. Using (2.4), we can write

$$\begin{aligned}
\Delta_k(t) = \Delta_k(0) &+ \int_0^t \sum_{l\in\mathbb{G}} a(k,l)(\Delta_l(s) - \Delta_k(s))\,\mathrm{d}s + \int_0^t Ke\left[\delta_k(s) - \Delta_k(s)\,\mathrm{d}s\right]\\
&+ \int_0^t \left(\sqrt{g(x_k^1(s))} - \sqrt{2g(x_k^2(s))}\right)^2 \mathrm{d}w_k(s).
\end{aligned} \tag{A.67}$$

Since $|\Delta_l(t)| \leq 1$, $|\delta_k(t)| \leq 1$ for all $t \geq 0$, and $M_k(t) = w_k(\langle M_k(t)\rangle)$ for $t \in [0, T]$, we may estimate

$$\Delta_k(t) > 5\epsilon - 2c_3 - 2Kec_3 - \epsilon = 2\epsilon. \tag{A.68}$$

So, on Ω_0, $\Delta_k(t) > 0$ for all $t \in [0, T]$. By expanding $x_i^1(t)$, we find

$$x_i^1(t) = x_i^1(0) + \int_0^t \sum_{l\in\mathbb{G}} a(i,l)(x_l^1(s) - x_i^1(s))\,\mathrm{d}s + \int_0^t Ke(y_i^1(s) - x_i^1(s))\,\mathrm{d}s + M_i(t), \tag{A.69}$$

so that on Ω_0 we have, for $t \in [0, T]$,

$$x_i^1(t) < 1 - 10\epsilon + c_3 + Kec_3 + \epsilon = 1 - 8\epsilon. \tag{A.70}$$

To check the conditions on $x_i^1(t)$ and $\Delta_i(t)$, we define the following random times:

$$\begin{aligned} \sigma &= \inf\{t \geq 0 : x_i^1(t) = \xi\}, \\ \tau &= \inf\left\{t > 0\colon\ g(x_i^1(t)) \neq g(x_i^2(t))\right\}. \end{aligned} \tag{A.71}$$

We will prove that, on Ω_0, we have $\sigma < \tau$ and $x_i^2(\tau) \geq x_i^1(\tau) + 3\epsilon$. To do so, we first prove that $\sigma < T$. Assume the contrary $\sigma \geq T$. Then by (A.70) we have $x_i^1(t) \in [\xi, 1-\xi]$ for all $t \in [0, T]$, which implies that $\min_{[0,T]} M_i(t) < -1$. Hence there exists a κ such that, by (A.69),

$$x_i^1(\kappa) < 1 - 10\epsilon + \epsilon - 1 < 0. \tag{A.72}$$

However, this contradicts the fact that $x_i^1 > 0$ for all $t \geq 0$. We conclude that $\sigma < T$. Now suppose that $\tau > \sigma$. Expanding Δ_i, we get, for $t < \tau$,

$$\Delta_i(t) = \Delta_i(0) + \int_0^t \sum_{l\in\mathbb{G}} a(i,l)(\Delta_l(s) - \Delta_i(s))\,\mathrm{d}s + \int_0^t Ke[\delta_i(s) - \Delta_i(s)]\,\mathrm{d}s, \tag{A.73}$$

which can be rewritten as

$$x_i^2(t) = x_i^1(t) - x_i^1(0) + x_i^2(0) - \int_0^t \sum_{l\in\mathbb{G}} a(i,l)[\Delta_l(s) - \Delta_i(s)]\,\mathrm{d}s - \int_0^t Ke[\delta_i(s) - \Delta_i(s)]\,\mathrm{d}s. \tag{A.74}$$

By (A.74), we obtain, for $t \in [0, \sigma]$,

$$\begin{aligned} x_i^2(t) &\leq 1 - 5\epsilon + 2\epsilon + 2\epsilon = 1 - \epsilon, \\ x_i^2(t) &\geq x_i^1(t) + 5\epsilon - 2\epsilon \geq 3\epsilon, \end{aligned} \tag{A.75}$$

so $x_i^2(t) \in [\epsilon, 1-\epsilon]$ for $t \in [0, \sigma]$. But then $g(x_i^1(\sigma)) = g(\xi) < g(x_i^2(t))$ by the definition of ξ. Hence we obtain a contradiction and conclude that $\tau \leq \sigma$. From (A.75) we obtain that $\Delta_i(t) < 0$ for all $t \in [0, \tau]$, which concludes the proof that $\Omega_0 \subset \Omega_1$. □

Step 4. If $\bar{z} \in E \times E$ and $\Delta_i < 0, \Delta_j = 0,\ \Delta_k > 0$ for some i, j, k, then

$$\hat{\mathbb{P}}_{\bar{z}} \left(\exists\, t^* \in [0,1]\colon\ \Delta_i(t^*) < 0, \Delta_j(t^*) \neq 0, \Delta_k(t^*) > 0\right) > 0. \tag{A.76}$$

Proof. Suppose that $\bar{z}$ satisfies $\Delta_i < 0,\ \Delta_j = 0,\ \Delta_k > 0$. Define

$$\begin{aligned} \Gamma_0 &= \{\bar{z} \in E \times E : \Delta_i < 0, \Delta_j \neq 0, \Delta_k > 0\}, \\ \Gamma_1 &= \{\bar{z} \in E \times E : \Delta_i < 0,\ g(x_i^1) \neq g(x_i^2),\ \Delta_k > 0\}. \end{aligned} \tag{A.77}$$

By Step 3 and path continuity, there exists a $T \in [0, \frac{1}{2}]$ such that $\mathbb{P}^{\bar{z}} \left(\bar{z}(T) \in \Gamma_1\right) > 0$. By the Markov property,

$$\hat{\mathbb{P}}_{\bar{z}}\big(\exists\, t^* \in [0,1]\colon\ \bar{z}(t^*) \in \Gamma_0\big) \geq \int_{\Gamma_1} \hat{\mathbb{P}}_{\bar{z}}(\bar{z}(T) \in d\tilde{z})\, \hat{\mathbb{P}}_{\tilde{z}}\big(\exists t^* \in [0, \tfrac{1}{2}]\colon\ \bar{z}(t^*) \in \Gamma_0\big). \tag{A.78}$$

By path continuity, we can find for $\bar{z} \in \Gamma_1$ a t' such that, for all $t \leq t'$, $\Delta_i(t) < 0$, $\Delta_k(t) > 0$ and $g(x_i^1(t)) \neq g(x_i^2(t))$. By Step 2 there exists a $t^* < t'$ such that $\bar{z}(t^*) \in \Gamma_0$. Hence both probabilities in the integral on the right-hand side of (A.78) are positive. □

Step 5. Proof of Lemma 3.2.11.

Proof. Suppose that (3.147) holds for the pair i, j, and $a(j,k) > 0$, but (3.147) fails for the pair i, k. This implies that there exist $\epsilon_0 > 0$, $\delta_0 > 0$ and a positive increasing sequence $(t_n)_{n\in\mathbb{N}}$ of times with $t_n \to \infty$, such that

$$\lim_{t\to\infty} \hat{\mathbb{P}}_{\bar{z}} \left(\{\Delta_i(t) < \epsilon_0, \Delta_k(t) > \epsilon_0\} \cup \{\Delta_i(t) > \epsilon_0, \Delta_k(t) < \epsilon_0\}\right) > \delta_0. \tag{A.79}$$

By compactness of $E\times E$, there exists a subsequence t_{n_k} such that $\mathcal{L}(\bar{z}(t_{n_k}))$ converges and (A.79) holds. Let $\bar{\nu} = \lim_{k\to\infty} \mathcal{L}(\bar{z}(t_{n_k}))$. Then

$$\begin{aligned} \bar{\nu}\left(\{\Delta_i < \epsilon_0, \Delta_j > \epsilon_0\} \cup \{\Delta_i > \epsilon_0, \Delta_j < \epsilon_0\}\right) &= 0, \\ \bar{\nu}\left(\{\Delta_j < \epsilon_0, \Delta_k > \epsilon_0\} \cup \{\Delta_j > \epsilon_0, \Delta_k < \epsilon_0\}\right) &= 0, \\ \bar{\nu}\left(\{\Delta_i < \epsilon_0, \Delta_k > \epsilon_0\} \cup \{\Delta_i > \epsilon_0, \Delta_k < \epsilon_0\}\right) &> \delta_0. \end{aligned} \tag{A.80}$$

Assume without loss of generality that $\bar{\nu}\left(\{\Delta_i < \epsilon_0, \Delta_k > \epsilon_0\}\right) > 0$. Hence, by (A.80),

$$\bar{\nu}\left(\{\Delta_i < \epsilon_0, \Delta_k > \epsilon_0\}\right) = \bar{\nu}\left(\{\Delta_i < \epsilon_0, \Delta_j \in (-\epsilon_0, \epsilon_0), \Delta_k > \epsilon_0\}\right) > 0. \tag{A.81}$$

For each $\bar{z} \in \{\Delta_i < \epsilon_0, \Delta_j \in (-\epsilon_0, \epsilon_0), \Delta_k > \epsilon_0\}$, Step 4 implies that

$$\hat{\mathbb{P}}_{\bar{z}} \left(\exists\, t^* \in [0,1]\colon\ \Delta_i(t^*) < 0, \Delta_j(t^*) \neq 0, \Delta_k(t^*) > 0\right) > 0, \tag{A.82}$$

and therefore, by (A.81),

$$\hat{\mathbb{P}}_{\bar{\nu}} \left(\exists\, t^* \in [0,1]\colon\ \Delta_i(t^*) < 0, \Delta_j(t^*) \neq 0, \Delta_k(t^*) > 0\right) > 0. \tag{A.83}$$

By path continuity, we can find $T \in [0,1]$ and $\epsilon > 0$ such that

$$\hat{\mathbb{P}}_{\bar{\nu}} \left(\Delta_i(T) < -\epsilon,\ |\Delta_j(T)|,\ \Delta_k(T) > \epsilon\right) > 0. \tag{A.84}$$

Let $\bar{\mu}(t_n) = \mathcal{L}(\bar{z}(t_n))$. Then, by the Markov property and (A.84),

$$\begin{aligned} &\liminf_{n\to\infty} \hat{\mathbb{P}}_{\bar{\mu}(t_n)} \left(\Delta_i(T) < -\epsilon,\ |\Delta_j(T)| > \epsilon,\ \Delta_k(T) > \epsilon\right) \\ &= \liminf_{n\to\infty} \hat{\mathbb{P}}_{\bar{\mu}(0)} \left(\Delta_i(T+t_n) < -\epsilon,\ |\Delta_j(T+t_n)| > \epsilon,\ \Delta_k(T+t_n) > \epsilon\right) > 0. \end{aligned} \tag{A.85}$$

However, this violates (3.147) for either i, j or j, k. We conclude that (A.79) fails and that (A.79) holds for i, k. By irreducibility, (A.79) holds for all $i, k \in \mathbb{G}$. □

§A.4 Bounded derivative of Lyapunov function

Recall from Section 3.2.3 that

$$\begin{aligned} h(t) = 2\sum_{j\in\mathbb{G}} a(i,j)\,\hat{\mathbb{E}}\left[|\Delta_j(t)|\, 1_{\{\operatorname{sgn}\Delta_i(t)\neq\operatorname{sgn}\Delta_j(t)\}}\right] \\ + 2Ke\,\hat{\mathbb{E}}\left[\left(|\Delta_i(t)| + |\delta_i(t)|\right) 1_{\{\operatorname{sgn}\Delta_i(t)\neq\operatorname{sgn}\delta_i(t)\}}\right]. \end{aligned} \tag{A.86}$$

In this section we show that $h'(t)$ exists for all $t > 0$ and is bounded. To do so, we need to get rid of the indicator in the expectations.

Let

$$h_{1,j}(t) = \hat{\mathbb{E}}\left[|\Delta_j(t)|\, 1_{\{\operatorname{sgn}\Delta_i(t)\neq\operatorname{sgn}\Delta_j(t)\}}\right] \tag{A.87}$$

and

$$h_2(t) = 2Ke\,\hat{\mathbb{E}}\left[\left(|\Delta_i(t)| + |\delta_i(t)|\right) 1_{\{\operatorname{sgn}\Delta_i(t)\neq\operatorname{sgn}\delta_i(t)\}}\right]. \tag{A.88}$$

Then $h(t) = 2\sum_{j\in\mathbb{G}} a(i,j) h_{1,j}(t) + h_2(t)$. We show that $h_{1,j}(t)$ is differentiable with bounded derivative for $j \in \mathbb{G}$. The proof of the differentiability of $h_2(t)$ is similar. Fix $t \geq 0$. Note that

$$\begin{aligned} &\hat{\mathbb{E}}\left[|\Delta_j(t)|\, 1_{\{\operatorname{sgn}\Delta_i(t)\neq\operatorname{sgn}\Delta_j(t)\}}\right] \\ &= \hat{\mathbb{E}}\left[|\Delta_j(t)|\, 1_{\{\operatorname{sgn}\Delta_i(t)\neq\operatorname{sgn}\Delta_j(t)\}} \mid |\Delta_i(t)| \neq 0, |\Delta_i(t)| \neq 0\right] \mathbb{P}\left(|\Delta_i(t)| \neq 0, |\Delta_j(t)| \neq 0\right) \\ &\quad + \hat{\mathbb{E}}\left[|\Delta_j(t)|\, 1_{\{\operatorname{sgn}\Delta_i(t)\neq\operatorname{sgn}\Delta_j(t)\}} \mid |\Delta_i(t)| = 0 \text{ or } |\Delta_j(t)| = 0\right] \\ &\qquad \times \mathbb{P}\left(|\Delta_i(t)| = 0 \text{ or } |\Delta_j(t)| = 0\right). \end{aligned} \tag{A.89}$$

Since $\Delta_i(t)$ and $\Delta_j(t)$ have zero local time, the second term vanishes and $\mathbb{P}(|\Delta_i(t)| \neq 0, |\Delta_j(t)| \neq 0) = 1$. By continuity of $\Delta_i(t)$ and $\Delta_j(t)$, we can define sets

$$B_n = \left\{|\Delta_i(r)| > 0 \text{ and } |\Delta_j(r)| > 0, \forall r \in \mathcal{B}(t, \tfrac{1}{n})\right\}. \tag{A.90}$$

Then

$$\cdots \subset B_n \subset B_{n+1} \subset B_{n+2} \subset \cdots, \tag{A.91}$$

so

$$B_n = \bigcup_{i=0}^{n} B_i \tag{A.92}$$

and we define

$$B := \bigcup_{i=0}^{\infty} B_n = \lim_{n\to\infty} B_n. \tag{A.93}$$

Since $\mathbb{P}(|\Delta_i(t)| \neq 0, |\Delta_j(t)| \neq 0) = 1$, it follows that $\mathbb{P}(B) = 1$.

For each B_n, we have

$$B_n = C_n \cup C_n^c, \qquad C_n = \left\{\omega \in B_n \colon 1_{\{\operatorname{sgn}\Delta_i(r) \neq \operatorname{sgn}\Delta_j(r)\}} = 1,\ \forall r \in \mathcal{B}(t, \tfrac{1}{n})\right\}, \tag{A.94}$$

and, by the definition of B_n,

$$\cdots \subset C_n \subset C_{n+1} \subset C_{n+2} \subset \cdots \qquad \qquad \cdots \subset C_n^c \subset C_{n+1}^c \subset C_{n+2}^c \subset \cdots \tag{A.95}$$

Let $C = \bigcup_{i=0}^{\infty} C_i$ and $C^c = \bigcup_{i=0}^{\infty} C_i^c$ be such that $B = C \cup C^c$. Using (3.136), we obtain

$$\begin{aligned}
&\frac{1}{s}\left(h_{1,j}(t+s) - h_{1,j}(t)\right) \\
&= \frac{1}{s}\left[\hat{\mathbb{E}}\left[|\Delta_j(t)|\, 1_{\{\operatorname{sgn}\Delta_i(t+s) \neq \operatorname{sgn}\Delta_j(t+s)\}}\right] - \hat{\mathbb{E}}\left[|\Delta_j(t)|\, 1_{\{\operatorname{sgn}\Delta_i(t) \neq \operatorname{sgn}\Delta_j(t)\}}\right]\right] \\
&= \frac{1}{s}\left[\hat{\mathbb{E}}\left[|\Delta_j(t+s)|\, 1_{\{\operatorname{sgn}\Delta_i(t+s) \neq \operatorname{sgn}\Delta_j(t+s)\}} - |\Delta_j(t)|\, 1_{\{\operatorname{sgn}\Delta_i(t) \neq \operatorname{sgn}\Delta_j(t)\}} \middle| B\right]\right] \\
&= \frac{1}{s}\left[\hat{\mathbb{E}}\left[|\Delta_j(t+s)| - |\Delta_j(t)| \middle| C\right]\right] \mathbb{P}(C) \\
&= \frac{1}{s}\hat{\mathbb{E}}\left[\sum_{j\in\mathbb{G}} a(i,j) \int_t^{t+s} \operatorname{sgn}\left(\Delta_i(r)\right)[\Delta_j(r) - \Delta_i(r)]\,\mathrm{d}r \middle| C\right] \mathbb{P}(C) \\
&\quad + \frac{1}{s}\hat{\mathbb{E}}\left[\int_t^{t+s} \operatorname{sgn}\left(\Delta_i(r)\right)\left[\sqrt{g(x_i^1(r))} - \sqrt{g(x_i^2(r))}\right]\mathrm{d}w_i(r) \middle| C\right] \mathbb{P}(C) \\
&\quad + \frac{1}{s}\hat{\mathbb{E}}\left[Ke \int_t^{t+s} \operatorname{sgn}\left(\Delta_i(r)\right)\left[\delta_i(r) - \Delta_i(r)\right]\mathrm{d}r \middle| C\right] \mathbb{P}(C) \\
&= \sum_{j\in\mathbb{G}} a(i,j)\hat{\mathbb{E}}\left[\frac{1}{s}\int_t^{t+s} \operatorname{sgn}\left(\Delta_i(r)\right)[\Delta_j(r) - \Delta_i(r)]\,\mathrm{d}r \middle| C\right] \mathbb{P}(C) \\
&\quad + \frac{1}{s}\hat{\mathbb{E}}\left[\int_t^{t+s} \operatorname{sgn}\left(\Delta_i(r)\right)\left[\sqrt{g(x_i^1(r))} - \sqrt{g(x_i^2(r))}\right]\mathrm{d}w_i(r) \middle| C\right] \mathbb{P}(C) \\
&\quad + \hat{\mathbb{E}}\left[Ke\, \frac{1}{s}\int_t^{t+s} \operatorname{sgn}\left(\Delta_i(r)\right)\left[\delta_i(r) - \Delta_i(r)\right]\mathrm{d}r \middle| C\right] \mathbb{P}(C).
\end{aligned} \tag{A.96}$$

In the last equality, the first and third term are bounded, because $\Delta_i(t), \delta_i(t)$ and $\Delta_j(t)$ are continuous functions of t, and $\operatorname{sgn}(\Delta_i)$ is constant since we conditioned on the set C. Therefore, letting $s \to 0$, it follows from the fundamental theorem of calculus that these terms are bounded. The second term is more involved. Since, on the set C,

$$\operatorname{sgn}\left(\Delta_i(r)\right)\left[\sqrt{g(x_i^1(r))} - \sqrt{g(x_i^2(r))}\right] \tag{A.97}$$

is a continuous function, we can rewrite the stochastic integral as a time-transformed

Brownian motion:

$$\frac{1}{s}\hat{\mathbb{E}}\left[\int_t^{t+s} \operatorname{sgn}\left(\Delta_i(r)\right)\left[\sqrt{g(x_i^1(r))} - \sqrt{g(x_i^2(r))}\right]\,\mathrm{d}w_i(r)\middle| C\right]$$
$$= \frac{1}{s}\hat{\mathbb{E}}\left[W\left(\int_0^{t+s}\left[\sqrt{g(x_i^1(r))} - \sqrt{g(x_i^2(r))}\right]^2 \mathrm{d}r\right)\right.$$
$$\left. - W\left(\int_0^{t}\left[\sqrt{g(x_i^1(r))} - \sqrt{g(x_i^2(r))}\right]^2 \mathrm{d}r\right)\middle| C\right]. \tag{A.98}$$

Since the normal distribution is differentiable with respect to its variance, we are done.