Spatial populations with seed-bank
Oomen, M.

CHAPTER 3

Spatial populations with seed-bank, proofs

§3.1 Proofs: Well-posedness and duality

In Section 3.1.1 we prove Theorem 2.2.4, in Section 3.1.2 Theorems 2.2.5, 2.2.8 and 2.2.10, and in Section 3.1.3 Theorems 2.2.11 and 2.2.13.

§3.1.1 Well-posedness

In this section we prove Theorem 2.2.4.

Proof. (a) We first prove Theorem 2.2.4(a): existence and uniqueness of solutions to the SSDE. We do this for each of the three models separately.

Model 1. Existence of the process defined in (2.4)–(2.5) for model 1 is a consequence of the assumptions in (2.1), (2.17) and (2.20), in combination with [67, Theorem 3.2], which reads as follows:

Theorem 3.1.1 (Unique strong solution). *Let $\mathbb{S}$ be a countable set, and let $Z = \{z_u\}_{u\in\mathbb{S}} \in [0,1]^{\mathbb{S}}$. Consider the stochastic differential equation*

$$\mathrm{d}z_u(t) = \alpha_u(z_u(t))\,\mathrm{d}B_u(t) + f_u(Z(t))\,\mathrm{d}t, \qquad u \in \mathbb{S}, \tag{3.1}$$

where $\alpha_u\colon [0,1] \to \mathbb{R}$ for all $u \in \mathbb{S}$, $f_u\colon [0,1]^{\mathbb{S}} \to [0,1]$ for all $u \in \mathbb{S}$, and $B = \{B_u\}_{u\in\mathbb{S}}$ is a collection of independent standard Brownian motions. Suppose that:

1. *The functions α_u, $u \in \mathbb{S}$, are real-valued, $\frac{1}{2}$-Hölder continuous (i.e., there are $C_u \in (0,\infty)$ such that $|\alpha_u(x) - \alpha_u(y)| \leq C_u|x-y|^{\frac{1}{2}}$ for all $x, y \in [0,1]$) and uniformly bounded, with $\alpha_u(0) = \alpha_u(1) = 0$, $u \in \mathbb{S}$.*
2. *The functions f_u, $u \in \mathbb{S}$, are continuous and satisfy:*
   - *There exists a matrix $Q = \{Q_{u,v}\}_{u,v\in\mathbb{S}}$ such that $Q_{u,v} \geq 0$ for all $u, v \in \mathbb{S}$, $\sup_{u\in\mathbb{S}} \sum_{v\in\mathbb{S}} Q_{u,v} < \infty$, and*

$$\begin{aligned} &|f_u(Z^1) - f_u(Z^2)| \leq \sum_{v\in\mathbb{S}} Q_{u,v}|z^1_v - z^2_v|, \\ &\textit{for} \quad Z^1 = \{z^1_v\}_{v\in\mathbb{S}} \in [0,1]^{\mathbb{S}},\ Z^2 = \{z^2_v\}_{v\in\mathbb{S}} \in [0,1]^{\mathbb{S}}. \end{aligned} \tag{3.2}$$

- *For* $Z \in [0,1]^{\mathbb{S}}$ *and* $z_u = 0$,
$$f_u(Z) \geq 0. \tag{3.3}$$
- *For* $Z \in [0,1]^{\mathbb{S}}$ *and* $z_u = 1$,
$$f_u(Z) \leq 0. \tag{3.4}$$

Then (3.1) *has a unique* $[0,1]^{\mathbb{S}}$*-valued strong solution with a continuous path.*

To apply Theorem 3.1.1 to model 1, recall that
$$\mathbb{S} = \mathbb{G} \times \{A, D\}, \tag{3.5}$$
where A denotes the active part of a colony and D the dormant part of a colony. Since $\mathbb{G}$ is countable and $\{A, D\}$ is finite, $\mathbb{S}$ is countable. As before, we denote the fraction of active individuals of type $\heartsuit$ at colony $i \in \mathbb{G}$ by x_i and the fraction of dormant individuals of type $\heartsuit$ at colony $i \in \mathbb{G}$ by y_i. Note that for every $u \in \mathbb{S}$ we have either $u = (i, A)$ or $u = (i, D)$ for some $i \in \mathbb{G}$. Therefore
$$Z = \{z_u\}_{u \in \mathbb{S}} = \{x_i \colon\ i \in \mathbb{G}\} \cup \{y_i \colon\ i \in \mathbb{G}\}, \tag{3.6}$$
and $z_u = x_i$ when $u = (i, A)$ and $z_u = y_i$ when $u = (i, D)$. We can rewrite (2.4)–(2.5) in the form of (3.1) by picking
$$\alpha_u(z_u) = \begin{cases} \sqrt{g(x_i)}, & u = (i, A), \\ 0, & u = (i, D), \end{cases} \tag{3.7}$$
and
$$f_u(Z) = \begin{cases} \sum_{j \in \mathbb{G}} a(i,j)\,(x_j - x_i) + Ke\,(y_i - x_i), & u = (i, A), \\ e\,(x_i - y_i), & u = (i, D). \end{cases} \tag{3.8}$$
Since $g \in \mathcal{G}$ (recall (2.23)), the conditions in (1) are satisfied. To check the conditions in (2), define the matrix $Q = \{Q_{u,v}\}_{u,v \in \mathbb{S}}$ by
$$Q_{u,v} = \begin{cases} \sum_{j \in \mathbb{G}} a(i,j) + Ke, & u = (i, A),\ v = (i, A), \\ a(i,j), & u = (i, A),\ v = (j, A), \\ Ke, & u = (i, A),\ v = (i, D), \\ e, & u = (i, D),\ v = (i, D) \text{ or } u = (i, D),\ v = (i, A), \\ 0, & \text{otherwise}. \end{cases} \tag{3.9}$$
Then
$$\sum_{v \in S} Q_{u,v} = \begin{cases} 2\sum_{j \in \mathbb{G}} a(i,j) + 2Ke, & u = (i, A), \\ 2e, & u = (i, D). \end{cases} \tag{3.10}$$
Since we have assumed that $\sum_{j \in \mathbb{G}} a(i,j) = \sum_{j \in \mathbb{G}} a(0, j - i) < \infty$, it follows that $\sup_{u \in \mathbb{S}} \sum_{v \in \mathbb{S}} Q_{u,v} < \infty$. Since $x_i \in [0,1]$ and $y_i \in [0,1]$, the requirements on f_u are immediate. Hence we have a unique strong solution with a continuous path.

By Itô's formula, the law of the strong solution solves the martingale problem. Uniqueness of that solution follows from [62, Theorem IX 1.7(i)]. This in turn implies the Markov property.

Model 2. To apply Theorem 3.1.1 to model 2, recall that

$$\mathbb{S} = \mathbb{G} \times \{A, (D_m)_{m\in\mathbb{N}_0}\}. \tag{3.11}$$

Pick

$$\alpha_u(z_u) = \begin{cases} \sqrt{g(x_i)}, & u = (i, A), \\ 0, & u = (i, D_m),\ m \in \mathbb{N}_0, \end{cases} \tag{3.12}$$

and

$$f_u(Z) = \begin{cases} \sum_{j\in\mathbb{G}} a(i,j)\,(x_j - x_i) + \sum_{m\in\mathbb{N}_0} K_m e_m\,(y_{i,m} - x_i), & u = (i, A), \\ e_m\,(x_i - y_{i,m}), & u = (i, D_m). \end{cases} \tag{3.13}$$

Set

$$Q_{u,v} = \begin{cases} \sum_{j\in\mathbb{G}} a(i,j) + \sum_{m\in\mathbb{N}_0} K_m e_m, & u = (i,A),\ v = (i,A), \\ a(i,j), & u = (i,A),\ v = (j,A),\ j \neq i, \\ K_m e_m, & u = (i,A),\ v = (i,D_m), \\ e_m, & u = (i,D_m),\ v = (i,D_m) \\ & \text{ or } u = (i,D_m),\ v = (i,A), \\ 0, & \text{otherwise.} \end{cases} \tag{3.14}$$

Then, by assumptions (2.1) and (2.20), Q, f and α satisfy the conditions of Theorem 3.1.1.

Model 3. The state space $\mathbb{S}$ and the function α are the same as in model 2. When $u \in \mathbb{S}$ is of the form (i, A), we must adapt the function f_u such that it takes the displacement of seeds into account. The matrix Q must be adapted accordingly and, by assumption (2.17), the conditions of Theorem 3.1.1 are again satisfied.

(b) The proof of Theorem 2.2.4(b) is the same for models 1–3. The Feller property can be proved by using duality if $g = dg_{\mathrm{FW}}$, $d \in (0,\infty)$. For general g we use [67, Remark 3.2] (see also [56, Theorem 5.8]). The Feller property in turn implies the strong Markov property. □

§3.1.2 Duality

In this section we prove Theorems 2.2.5, 2.2.8 and 2.2.10.

Model 1: Proof of Theorem 2.2.5.

Proof. We use the generator criterion (see [32, p.190–193] or [48, Proposition 1.2]) to prove the duality relation given in (2.35). Let F be the generator of the spatial block-counting process defined in (2.33), and let $H((m_j, n_j)_{j\in\mathbb{G}})$ be defined as in (2.34), but read as a function of the second sequence only. Then

$$
\begin{aligned}
&(FH)\big((m_j,n_j)_{j\in\mathbb{G}}\big)\\
&=\sum_{i\in\mathbb{G}}\Bigg[\sum_{k\in\mathbb{G}} m_i a(i,k)\,\big[H\big((m_j,n_j)_{j\in\mathbb{G}}-\delta_{(i,A)}+\delta_{(k,A)}\big)-H\big((m_j,n_j)_{j\in\mathbb{G}}\big)\big]\\
&\qquad+d\binom{m_i}{2}\big[H\big((m_j,n_j)_{j\in\mathbb{G}}-\delta_{(i,A)}\big)-H\big((m_j,n_j)_{j\in\mathbb{G}}\big)\big]\\
&\qquad+m_iKe\big[H\big((m_j,n_j)_{j\in\mathbb{G}}-\delta_{(i,A)}+\delta_{(i,D)}\big)-H\big((m_j,n_j)_{j\in\mathbb{G}}\big)\big]\\
&\qquad+n_ie\big[H\big((m_j,n_j)_{j\in\mathbb{G}}+\delta_{(i,A)}-\delta_{(i,D)}\big)-H\big((m_j,n_j)_{j\in\mathbb{G}}\big)\big]\Bigg].
\end{aligned}
\tag{3.15}
$$

Recall that G is the generator of the SSDE (recall (2.24)–(2.25)). Let $\mathcal{D}_G$ denote the domain of G and $\mathcal{D}_F$ the domain of F. Let $(S_t)_{t\geq 0}$ denote the semigroup of the process $(Z(t))_{t\geq 0}$ in (2.2) and $(R_t)_{t\geq 0}$ the semigroup of the process $(L(t))_{t\geq 0}$ in (2.32). Since

$$
\frac{d^2}{dt^2}(R_tH)((x_j,y_j,n_j,m_j)_{j\in\mathbb{G}})=(F^2R_tH)((x_j,y_j,n_j,m_j)_{j\in\mathbb{G}}),
\tag{3.16}
$$

we see that $H((x_j,y_j,n_j,m_j)_{j\in\mathbb{G}})\in\mathcal{D}_G$ and $(R_tH)((x_j,y_j,n_j,m_j)_{j\in\mathbb{G}})\in\mathcal{D}_G$. It is also immediate that $H((x_j,y_j,n_j,m_j)_{j\in\mathbb{G}})\in\mathcal{D}_F$ and
$(S_tH)((x_j,y_j,n_j,m_j)_{j\in\mathbb{G}})\in\mathcal{D}_F$. Applying the generator G in (2.25) with $g=\frac{d}{2}g_{\mathrm{FW}}$

to (2.34), we find

$$
\begin{aligned}
&(GH)\big((x_j,y_j)_{j\in\mathbb{G}}\big)\\
&=\sum_{i\in\mathbb{G}}\Bigg\{\Bigg[\sum_{k\in\mathbb{G}}a(i,k)\,(x_k-x_i)\Bigg]\frac{\partial}{\partial x_i}\Big(\prod_{j\in\mathbb{G}}x_j^{m_j}y_j^{n_j}\Big)\\
&\qquad+\tfrac{d}{2}\,x_i(1-x_i)\frac{\partial^2}{\partial x_i^2}\Big(\prod_{j\in\mathbb{G}}x_j^{m_j}y_j^{n_j}\Big)+Ke\,(y_i-x_i)\frac{\partial}{\partial x_i}\Big(\prod_{j\in\mathbb{G}}x_j^{m_j}y_j^{n_j}\Big)\\
&\qquad\qquad+e\,(x_i-y_i)\frac{\partial}{\partial y_i}\Big(\prod_{j\in\mathbb{G}}x_j^{m_j}y_j^{n_j}\Big)\Bigg\}\\
&=\sum_{i\in\mathbb{G}}\Bigg\{\Bigg[\sum_{k\in\mathbb{G}}m_i a(i,k)\prod_{\substack{j\in\mathbb{G}\\ j\neq i\\ j\neq k}}x_j^{m_j}y_j^{n_j}\,\big(x_i^{m_i-1}y_i^{n_i}x_k^{m_k+1}y_k^{n_k}-x_i^{m_i}y_i^{n_i}x_k^{m_l}y_k^{n_k}\big)\Bigg]\\
&\qquad+\prod_{\substack{j\in\mathbb{G}\\ j\neq i}}x_j^{m_j}y_j^{n_j}\,\tfrac{d}{2}\,m_i(m_i-1)\,\big(x_i^{m_i-1}y_i^{n_i}-x_i^{m_i}y_i^{n_i}\big)\,1_{\{m_i\geq 2\}}\\
&\qquad\qquad+m_iKe\prod_{\substack{j\in\mathbb{G}\\ j\neq i}}x_j^{m_j}y_j^{n_j}\,\big(x_i^{m_i-1}y_i^{n_i+1}-x_i^{m_i}y_i^{n_i}\big)\\
&\qquad\qquad+n_ie\prod_{\substack{j\in\mathbb{G}\\ j\neq i}}x_j^{m_j}y_j^{n_j}\,\big(x_i^{m_i+1}y_i^{n_i-1}-x_i^{m_i}y_i^{n_i}\big)\Bigg\}\\
&=(FH)\big((m_j,n_j)_{j\in\mathbb{G}}\big).
\end{aligned}
\tag{3.17}
$$

Consequently, it follows from the generator criterion that

$$
\mathbb{E}\Big[H\Big(\big(X_i(t),Y_i(t),m_i,n_i\big)_{i\in\mathbb{G}}\Big)\Big]=\mathbb{E}\Big[H\Big(\big(x_i,y_i,M_i(t),N_i(t)\big)_{i\in\mathbb{G}}\Big)\Big]. \tag{3.18}
$$

This settles Theorem 2.2.5. □

Model 2: Proof of Theorem 2.2.8.

Proof. Theorem 2.2.8 follows after replacing in the above proof the block-counting process in (2.33) by the one in (2.43), the duality function by the one in (2.44), and checking the generator criterion. □

Model 3: Proof of Theorem 2.2.10.

Proof. Theorem 2.2.10 follows after replacing the block-counting process in (2.33) by the one in (2.54), the duality function is by the one in (2.44), and checking the generator criterion. □

§3.1.3 Dichotomy criterion

In this section we prove Theorems 2.2.11 and 2.2.13.

Model 1: Proof of Theorem 2.2.11.

Proof.

"$\Longleftarrow$" The proof uses the duality relation in Theorem 2.2.5. Define $\theta_x = \mathbb{E}_{\mu(0)}[x_0]$ and $\theta_y = \mathbb{E}_{\mu(0)}[y_0]$. Note that, since $\mu(0)$ is invariant under translations, we have $\mathbb{E}_{\mu(0)}[x_i] = \theta_x$ and $\mathbb{E}_{\mu(0)}[y_i] = \theta_y$ for all $i \in \mathbb{G}$. We proceed as in [12, Proposition 2.9]. Let $(m_i, n_i)_{i\in\mathbb{G}} \in E'$ be such that $\sum_{i\in\mathbb{G}}[m_i(0) + n_i(0)] < \infty$, and put

$$T = \inf\left\{ t \geq 0\colon \sum_{i\in\mathbb{G}}[m_i(t) + n_i(t)] = 1 \right\}. \tag{3.19}$$

By assumption, each pair of partition elements coalesces with probability 1, and hence $\mathbb{P}(T < \infty) = 1$. By duality

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}\left[\prod_{i\in\mathbb{G}} x_i(t)^{m_i} y_i(t)^{n_i}\right] \\
&= \lim_{t\to\infty} \mathbb{E}\left[\prod_{i\in\mathbb{G}} x_i^{m_i(t)} y_i^{n_i(t)}\right] \\
&= \lim_{t\to\infty} \mathbb{E}\left[\prod_{i\in\mathbb{G}} x_i^{m_i(t)} y_i^{n_i(t)} \;\Big|\; T<\infty\right] \mathbb{P}(T<\infty) \\
&\qquad + \mathbb{E}\left[\prod_{i\in\mathbb{G}} x_i^{m_i(t)} y_i^{n_i(t)} \;\Big|\; T=\infty\right] \mathbb{P}(T=\infty) \\
&= \lim_{t\to\infty} \mathbb{E}\left[\prod_{i\in\mathbb{G}} x_i^{m_i(t)} y_i^{n_i(t)} \;\Big|\; T<\infty,\ m(t)=1,\ n(t)=0\right] \mathbb{P}(m(t)=1,\ n(t)=0) \\
&\qquad + \lim_{t\to\infty} \mathbb{E}\left[\prod_{i\in\mathbb{G}} x_i^{m_i(t)} y_i^{n_i(t)} \;\Big|\; T<\infty,\ m(t)=0,\ n(t)=1\right] \mathbb{P}(m(t)=0,\ n(t)=1) \\
&= \theta_x \frac{1}{1+K} + \theta_y \frac{K}{1+K},
\end{aligned} \tag{3.20}$$

where in the last step we use that a single lineage in the dual behaves like the Markov chain with transition kernel $b^{(1)}(\cdot,\cdot)$ defined in (2.31). It follows from (3.20) that, for all $i,j \in \mathbb{G}$,

$$\lim_{t\to\infty} \mathbb{E}\left[\frac{x_i(t) + Ky_i(t)}{1+K}\left(1 - \frac{x_j(t) + Ky_j(t)}{1+K}\right)\right] = 0. \tag{3.21}$$

Hence, either $\lim_{t\to\infty}(x(t), y(t)) = (0,0)^{\mathbb{G}}$ or $\lim_{t\to\infty}(x(t), y(t)) = (1,1)^{\mathbb{G}}$. Computing $\lim_{t\to\infty} \mathbb{E}[x_i(t)]$ with the help of (3.20), we find

$$\lim_{t\to\infty} \mu(t) = (1-\theta)\,[\delta_{(0,0)}]^{\bigotimes \mathbb{G}} + \theta\,[\delta_{(1,1)}]^{\bigotimes \mathbb{G}} \tag{3.22}$$

with $\theta = \mathbb{E}_{\mu(0)}\left[\frac{x_0 + Ky_0}{1+K}\right] = \frac{\theta_x + K\theta_y}{1+K}$, which means that the system clusters.

"$\Longrightarrow$" Suppose that the systems clusters. Then (3.21) holds for all $i, j \in \mathbb{G}$, which means that

$$\lim_{t\to\infty} \mathbb{E}\left[z_u(t)\,(1 - z_v(t))\right] = 0 \qquad \forall\, u, v \in \mathbb{S}. \tag{3.23}$$

Let

$$|L(t)| = \sum_{u\in\mathbb{S}} L_u(t), \tag{3.24}$$

be the total number of lineages left at time t. Applying the duality relation in (2.38) to (3.23), we find

$$\begin{aligned}
0 &= \lim_{t\to\infty} \mathbb{E}\left[z_u(t)(1 - z_v(t))\right] \\
&= \lim_{t\to\infty} \mathbb{E}_{\mu(0)}\left[\mathbb{E}_{\delta_u}\left[\prod_{u\in\mathbb{S}} z_u^{L_u(t)}\right]\right] - \mathbb{E}_{\mu(0)}\left[\mathbb{E}_{\delta_u+\delta_v}\left[\prod_{u\in\mathbb{S}} z_u^{L_u(t)}\right]\right] \\
&= \lim_{t\to\infty}\left[\frac{\theta_x + K\theta_y}{1+K}\left[1 - \mathbb{P}_{\delta_u+\delta_v}\left(|L(t)| = 1\right)\right]\right. \\
&\qquad \left. - \mathbb{E}_{\mu(0)}\left[\mathbb{E}_{\delta_u+\delta_v}\left[\prod_{u\in\mathbb{S}} z_u^{L_u(t)} \;\Big|\; |L(t)| = 2\right]\right]\mathbb{P}_{\delta_u+\delta_v}(|L(t)| = 2)\right].
\end{aligned} \tag{3.25}$$

As to the last term in the right-hand side of (3.25), we note that

$$\begin{aligned}
&\limsup_{t\to\infty} \mathbb{E}_{\mu(0)}\left[\mathbb{E}_{\delta_u+\delta_v}\left[\prod_{u\in\mathbb{S}} z_u^{L_u(t)} \;\Big|\; |L(t)| = 2\right]\right] \\
&= \limsup_{t\to\infty} \frac{1}{(1+K)^2}\mathbb{E}\left[\prod_{u\in\mathbb{S}} z_u^{L_u(t)} \;\Big|\; L(t) = \delta_{(i,A)} + \delta_{(j,A)},\ i, j \in \mathbb{G}\right] \\
&\quad + \limsup_{t\to\infty} \frac{2K}{(1+K)^2}\mathbb{E}\left[\prod_{u\in\mathbb{S}} z_u^{L_u(t)} \;\Big|\; L(t) = \delta_{(i,A)} + \delta_{(j,D)},\ i, j \in \mathbb{G}\right] \\
&\quad + \limsup_{t\to\infty} \frac{K^2}{(1+K)^2}\mathbb{E}\left[\prod_{u\in\mathbb{S}} z_u^{L_u(t)} \;\Big|\; L(t) = \delta_{(i,D)} + \delta_{(j,D)}, i, j \in \mathbb{G}\right] \\
&< \frac{\theta_x}{(1+K)^2} + \frac{K\theta_x + K\theta_y}{(1+K)^2} + \frac{K^2\theta_y}{(1+K)^2} = \frac{\theta_x + K\theta_y}{1+K} = \theta.
\end{aligned} \tag{3.26}$$

Here, the strict inequality follows from the non-trivial invariant initial distribution (ruling out $z \equiv 0$ and $z \equiv 1$), together with the fact that the swapping between active and dormant is driven by a positive recurrent Markov chain on $\{A, D\}$. Hence (3.23) holds if and only if $\lim_{t\to\infty} \mathbb{P}_{\delta_u+\delta_v}(|L(t)| = 2 \,|\, |L(0)| = 2) = 0$ for every $u, v \in \mathbb{S}$. Therefore every pair of lineages coalesces with probability 1. Thus, we have proved Theorem 2.2.11.

Model 2: Proof of Theorem 2.2.13.

Case $\rho < \infty$. Like for model 1, we define

$$\theta_x = \mathbb{E}_{\mu(0)}[x_0], \qquad \theta_{y,m} = \mathbb{E}_{\mu(0)}[y_{0,m}], \qquad \theta = \frac{\theta_x + \sum_{m=0}^{\infty} K_m \theta_{y,m}}{1+\rho}. \tag{3.27}$$

For $\rho < \infty$, a lineage in the dual moves as a positive recurrent Markov chain on $\{A, (D_m)_{m\in\mathbb{N}_0}\}$. Therefore the argument for "$\Longleftarrow$" given for model 1 goes through via the duality relation, which gives

$$\lim_{t\to\infty} \mathbb{E}\left[\prod_{u\in\mathbb{S}} z_u(t)^{l_u}\right] = \lim_{t\to\infty} \mathbb{E}\left[\prod_{u\in\mathbb{S}} z_u^{L_u(t)}\right] = \frac{\theta_x + \sum_{m\in\mathbb{N}_0} K_m \theta_{y,m}}{1+\sum_{m\in\mathbb{N}_0} K_m}. \tag{3.28}$$

With the duality relation in (2.47), the argument for "$\Longrightarrow$" given for model 1 also goes through directly.

Case $\rho = \infty$. For $\rho = \infty$, a lineage in the dual moves as a null-recurrent Markov chain, which has no stationary distribution, and so (3.28) does not carry over. However, from [58, Section 3] it follows that, for all $u_1, u_2 \in \mathbb{S}$,

$$\lim_{t\to\infty} \left\|\mathbb{P}_{u_1}(L(t) = \delta_{(\cdot)} \mid L(t) = 1) - \mathbb{P}_{u_2}(L(t) = \delta_{(\cdot)} \mid L(t) = 1)\right\|_{tv} = 0. \tag{3.29}$$

Moreover, by null-recurrence,

$$\begin{aligned} \lim_{t\to\infty} \mathbb{P}(L(t) = \delta_{(\cdot,A)}) &= 0, \\ \lim_{t\to\infty} \mathbb{P}(L(t) = \delta_{(\cdot,D_m)}) &= 0 \qquad \forall m \in \mathbb{N}_0, \\ \lim_{t\to\infty} \sum_{m=M}^{\infty} \mathbb{P}(L(t) = \delta_{(\cdot,D_m)}) &= 1 \qquad \forall M \in \mathbb{N}_0. \end{aligned} \tag{3.30}$$

"$\Longleftarrow$" By duality, we have

$$\begin{aligned} \lim_{t\to\infty} \mathbb{E}\left[\prod_{u\in\mathbb{S}} z_u(t)^{l_u}\right] &= \lim_{t\to\infty} \mathbb{E}\left[\prod_{u\in\mathbb{S}} z_u^{L_u(t)}\right] \\ &= \lim_{t\to\infty} \left[\theta_x \mathbb{P}(L(t) = \delta_{(\cdot,A)}) + \sum_{m\in\mathbb{N}_0} \theta_{y,m} \mathbb{P}(L(t) = \delta_{(\cdot,D_m)})\right], \end{aligned} \tag{3.31}$$

where we follow an argument similar as in (3.20) and use that $\mathbb{P}(T < \infty) = 1$. Because the initial measure is colour regular, we know that $\lim_{m\to\infty} \theta_{y,m} = \theta$ (recall Definition 2.2.12). But (3.30)–(3.31) imply that all moments tend to θ. In particular,

$$\lim_{t\to\infty} \mathbb{E}[x_i(t)] = \theta = \lim_{t\to\infty} \mathbb{E}[y_{i,m}(t)], \qquad i \in \mathbb{G},\ m \in \mathbb{N}_0. \tag{3.32}$$

"$\Longrightarrow$" By the duality relation in (2.47) and the assumption of clustering, we find

$$\lim_{t\to\infty} \mathbb{E}\left[z_u(t)(1-z_v(t))\right] = 0 \qquad \forall u,v \in \mathbb{S}. \tag{3.33}$$

Therefore

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}\left[z_u(t)(1-z_v(t))\right] \\
&= \lim_{t\to\infty} \mathbb{E}_{\mu(0)}\left[\mathbb{E}_{\delta_u}\left[\prod_{u\in\mathbb{S}} z_u^{L_u(t)}\right]\right] - \mathbb{E}_{\mu(0)}\left[\mathbb{E}_{\delta_u+\delta_v}\left[\prod_{u\in\mathbb{S}} z_u^{L_u(t)}\right]\right] \\
&= \theta \lim_{t\to\infty}\Bigg[\left[1-\mathbb{P}_{\delta_u+\delta_v}\left(|L(t)|=1\right)\right] \\
&\qquad - \mathbb{E}_{\mu(0)}\left[\mathbb{E}_{\delta_u+\delta_v}\left[\prod_{u\in\mathbb{S}} z_u^{L_u(t)} \,\middle|\, |L(t)|=2\right]\right]\mathbb{P}_{\delta_u+\delta_v}(|L(t)|=2)\Bigg] = 0.
\end{aligned} \tag{3.34}$$

Suppose that $\lim_{t\to\infty}\mathbb{P}_{\delta_u+\delta_v}(|L(t)|=2)\neq 0$. Then

$$\lim_{t\to\infty} \mathbb{E}_{\delta_u+\delta_v}\left[\prod_{u\in\mathbb{S}} z_u^{L_u(t)} \mid |L(t)|=2\right] = \theta. \tag{3.35}$$

However,

$$\limsup_{t\to\infty} \quad \mathbb{E}_{\mu(0)} \quad \left[\mathbb{E}_{\delta_u+\delta_v}\left[\prod_{u\in\mathbb{S}} z_u^{L_u(t)} \,\middle|\, |L(t)|=2\right]\right] \tag{3.36}$$

$$< \quad \mathbb{E}_{\mu(0)}\left[\mathbb{E}_{\delta_u+\delta_v}\left[\prod_{u\in\mathbb{S}} z_u^{L_u(t)} \,\middle|\, |L(t)|=1\right]\right] = \theta, \tag{3.37}$$

because we start from a nontrivial stationary distribution. Thus, we have proved Theorem 2.2.13.

Model 3: Proof of Theorem 2.2.13. Since the duality relation for model 3 is exactly the same as for model 2, the same results hold by translation invariance and the extra displacement does not affect the dichotomy criterion. □

§3.1.4 Outline remainder of paper

In Sections 3.2–3.4 we prove Theorems 2.3.1, 2.3.3 and 2.3.6, respectively. For each of the three models we split the proof into four parts:

(a) Moment relations.

(b) The clustering case.

(c) The coexistence case.

(d) Proof of the dichotomy.

§3.2 Proofs: Long-time behaviour for Model 1

In Section 3.2.1 we relate the first and second moments of the process $(Z(t))_{t\geq 0}$ in (2.4)–(2.5) to the random walk with internal states $\{A, D\}$ that evolves according to the transition kernel $b^{(1)}(\cdot,\cdot)$ given in (2.31) (Lemma 3.2.1 below). These moment relations hold for all $g \in \mathcal{G}$. In Section 3.2.2 we deal with the clustering case (Lemmas 3.2.4–3.2.5 below), in Section 3.2.3 with the coexistence case (Lemmas 3.2.7–3.2.13 below). In Section 3.2.4 we prove Theorem 2.3.1. In Sections 3.2.2 and 3.2.3 we will see that the moment relations are crucial when no duality is available.

Below we write $\mathbb{E}_z$ for $\mathbb{E}_{\delta_z}$, the expectation when the process starts from the initial distribution δ_z, $z \in E$.

§3.2.1 Moment relations

Lemma 3.2.1 (First and second moment). *For $z \in E$, $t \geq 0$ and $(i,R_i),(j,R_j) \in \mathbb{G}\times\{A,D\}$,*

$$\mathbb{E}_z[z_{(i,R_i)}(t)] = \sum_{(k,R_k)\in\mathbb{G}\times\{A,D\}} b_t^{(1)}\big((i,R_i),(k,R_k)\big)\, z_{(k,R_k)} \tag{3.38}$$

and

$$\begin{aligned}
&\mathbb{E}_z[z_{(i,R_i)}(t)z_{(j,R_j)}(t)]\\
&= \sum_{(k,R_k),(l,R_l)\in\mathbb{G}\times\{A,D\}} b_t^{(1)}\big((i,R_i),(k,R_k)\big)\, b_t^{(1)}\big((j,R_j),(l,R_l)\big)\, z_{(k,R_k)}z_{(l,R_l)}\\
&\qquad + \int_0^t \mathrm{d}s \sum_{k\in\mathbb{G}} b_{(t-s)}^{(1)}((i,R_i),(k,A))\, b_{(t-s)}^{(1)}((j,R_j),(k,A))\, \mathbb{E}_z[g(x_k(s))].
\end{aligned} \tag{3.39}$$

Proof. We derive systems of differential equations for the moments and solve these in terms of the random walk. Let $(RW_t)_{t\geq 0}$ denote the semigroup of the random walk with transition kernel $b^{(1)}(\cdot,\cdot)$, and recall that the corresponding generator is given by

$$(G_{RW}f)(i,R_i) = \sum_{(j,R_j)\in\mathbb{G}\times\{A,D\}} b^{(1)}\big((i,R_i),(j,R_j)\big)\,[f(j,R_j)-f(i,R_i)]\,. \tag{3.40}$$

Applying the generator (2.25) of the system in (2.4)–(2.5) to the function

$$f_{(i,R_i)}\colon\ E\to\mathbb{R},\ f_{(i,R_i)}(z) = z_{(i,R_i)}, \tag{3.41}$$

we obtain by standard stochastic calculus

$$\begin{aligned}
&\frac{\mathrm{d}\mathbb{E}_z[z_{(i,R_i)}(t)]}{\mathrm{d}t}\\
&= \left[\sum_{j\in\mathbb{G}} a(i,j)\,\big(\mathbb{E}_z[x_j(t)]-\mathbb{E}_z[x_i(t)]\big) + Ke\,(\mathbb{E}_z[y_i(t)]-\mathbb{E}_z[x_i(t)])\right] 1_{(R_i=A)}\\
&\qquad + e\,\big(\mathbb{E}_z[x_i(t)]-\mathbb{E}_z[y_i(t)]\big)\, 1_{(R_i=D)}.
\end{aligned} \tag{3.42}$$

Hence, denoting by $(S_t)_{t\geq 0}$ the semigroup of the system in (2.4)–(2.5), we see from (3.42) and the definition of $b^{(1)}(\cdot,\cdot)$ in (2.31) that $(S_t f_{(i,R_i)})$ solves the differential equation

$$F'(t) = (G_{RW}F)(t). \tag{3.43}$$

On the other hand, for each $f \in \mathcal{C}_b(\mathbb{G} \times \{A, D\})$, $RW_t f$ also solves (3.43). In particular, for $z \in \mathbb{E}$ define $f_z \colon \mathbb{G} \times \{A, D\} \to \mathbb{R}$ by $f_z(i, R_i) = z(i, R_i)$ for $z \in E$, then $RW_t f_z$ is a solution to (3.43). Since

$$(RW_0 f_z)(i, R_i) = z(i, R_i) = (S_0 f_{(i,R_i)})(z), \tag{3.44}$$

we see that (3.38) holds. To prove (3.39), we derive a similar system of differential equations and again solve this in terms of the random walk moving according to the kernel $b(\cdot, \cdot)$. Let $f \colon E \to \mathbb{R}$ be given by $f(z) = z_{(i,R_i)} z_{(j,R_j)}$. Using the generator (2.25), we obtain via Itô-calculus that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E}_z[z_{(i,R_i)}(t) z_{(j,R_j)}(t)] \\
&= \sum_{k \in \mathbb{G}} a(i,k) \left(\mathbb{E}_z[x_k(t) z_{(j,R_j)}(t)] - \mathbb{E}_z[x_i(t) z_{(j,R_j)}(t)]\right) 1_{\{R_i = A\}} \\
&\quad + Ke \left(\mathbb{E}_z[y_i(t) z_{(j,R_j)}(t)] - \mathbb{E}_z[x_i(t) z_{(j,R_j)}(t)]\right) 1_{\{R_i = A\}} \\
&\quad + e \left(\mathbb{E}_z[x_i(t) z_{(j,R_j)}(t)] - \mathbb{E}_z[y_i(t) z_{(j,R_j)}(t)]\right) 1_{\{R_i = D\}} \\
&\quad + \sum_{l \in \mathbb{G}} a(j,l) \left(\mathbb{E}_z[x_l(t) z_{(i,R_i)}(t)] - \mathbb{E}_z[x_j(t) z_{(i,R_i)}(t)]\right) 1_{\{R_j = A\}} \\
&\quad + Ke \left(\mathbb{E}_z[y_j(t) z_{(i,R_i)}(t)] - \mathbb{E}_z[x_j(t) z_{(i,R_i)}(t)]\right) 1_{\{R_j = A\}} \\
&\quad + e \left(\mathbb{E}_z[x_j(t) z_{(i,R_i)}(t)] - \mathbb{E}_z[y_j(t) z_{(i,R_i)}(t)]\right) 1_{\{R_j = D\}} \\
&\quad + \mathbb{E}_z[g(x_i(t))] \, 1_{\{i=j\}} \, 1_{\{R_i = R_j = A\}}.
\end{aligned} \tag{3.45}$$

Let U be the generator of two independent random walks each moving with transition kernel $b^{(1)}(\cdot, \cdot)$, i.e., for all $h \in \mathcal{C}_b((\mathbb{G} \times \{A, D\})^2)$,

$$\begin{aligned}
&(Uh)((i, R_i), (j, R_j)) \\
&= \sum_{k \in \mathbb{G}} a(i,k) \left[h((k, A), (j, R_j)) - h((i, R_i), (j, R_j))\right] 1_{\{i, R_i = A\}} \\
&\quad + Ke \left[h((i, D), (j, R_j)) - h((i, R_i), (j, R_j))\right] 1_{\{i, R_i = A\}} \\
&\quad + e \left[h((i, A), (j, R_j)) - h((i, R_i), (j, R_j))\right] 1_{\{i, R_i = D\}} \\
&\quad + \sum_{l \in \mathbb{G}} a(j,l) \left[h((i, R_i), (l, A)) - h((i, R_i), (j, R_j))\right] 1_{\{R_j = A\}} \\
&\quad + Ke \left[h((i, R_i), (j, D)) - h((i, R_i), (j, R_j))\right] 1_{\{R_j = A\}} \\
&\quad + e \left[h((i, R_i), (j, A)) - h((i, R_i), (j, D))\right] 1_{\{R_j = D\}}.
\end{aligned} \tag{3.46}$$

Let $F(t) = \mathbb{E}_z[z_{(i,R_i)}(t) z_{(j,R_j)}(t)]$ and $H(t) = 2\mathbb{E}_z[g(x_i(t))] 1_{\{i=j\}} 1_{\{R_i = R_j = A\}}$. Then we can rewrite (3.45) as

$$\frac{\mathrm{d}}{\mathrm{d}t} F(t) = (UF)(t) + H(t). \tag{3.47}$$

Denote by $(RW_t^{(2)})_{t \geq 0}$ the semigroup corresponding to U. Applying [56, Theorem I.2.15], we obtain

$$F(t) = RW_t^{(2)} F(0) + \int_0^t \mathrm{d}s \, RW_{t-s}^{(2)} H(s). \tag{3.48}$$

Hence

$$\begin{aligned}
&\mathbb{E}_z[z_{(i,R_i)}(t)z_{(j,R_j)}(t)]\\
&= \sum_{(k,R_k),(l,R_l)\in\mathbb{G}\times\{A,D\}} b_t^{(1)}\big((i,R_i),(k,R_k)\big)\, b_t^{(1)}\big((j,R_j),(l,R_l)\big)\, \mathbb{E}_z[z_{(k,R_k)}z_{(l,R_l)}]\\
&\qquad + \int_0^t \mathrm{d}s \sum_{k\in\mathbb{G}} b_{t-s}^{(1)}\big((i,R_i),(k,A)\big)\, b_{t-s}^{(1)}\big((j,R_j),(k,A)\big)\, \mathbb{E}_z[g(x_k(s))].
\end{aligned} \tag{3.49}$$

□

Remark 3.2.2 (Density). From Lemma 3.2.1 we obtain that if μ is a translation invariant measure such that $\mathbb{E}_\mu[x_0(0)] = \theta_x$ and $\mathbb{E}_\mu[y_0(0)] = \theta_y$, then

$$\begin{aligned}
\mathbb{E}_\mu[z_{(i,R_i)}(t)] = \theta_x &\sum_{(k,R_k)\in\mathbb{G}\times\{A\}} b_t^{(1)}\big((i,R_i),(k,R_k)\big)\\
&+ \theta_y \sum_{(k,R_k)\in\mathbb{G}\times\{D\}} b_t^{(1)}\big((i,R_i),(k,R_k)\big),
\end{aligned} \tag{3.50}$$

in particular, $\lim_{t\to\infty} \mathbb{E}_\mu[z_{(i,R_i)}(t)] = \frac{\theta_x + K\theta_y}{1+K} = \theta$, recall (2.62), and

$$\begin{aligned}
&\mathbb{E}_\mu[z_{(i,R_i)}(t)z_{(j,R_j)}(t)]\\
&= \sum_{(k,R_k),(l,R_l)\in\mathbb{G}\times\{A,D\}} b_t^{(1)}\big((i,R_i),(k,R_k)\big)\, b_t^{(1)}\big((j,R_j),(l,R_l)\big)\, \mathbb{E}_\mu[z_{(k,R_k)}z_{(l,R_l)}]\\
&\qquad + 2\int_0^t \mathrm{d}s \sum_{k\in\mathbb{G}} b_{t-s}^{(1)}\big((i,R_i),(k,A)\big)\, b_{t-s}^{(1)}\big((j,R_j),(k,A)\big)\, \mathbb{E}_\mu[g(x_i(s))].
\end{aligned} \tag{3.51}$$

□

Remark 3.2.3 (First moment duality). Note that (3.38) shows that even for general $g \in \mathcal{G}$ there is a *first moment duality* between the process $Z(t)$ and the random walk $RW(t)$, that moves according to the kernel $b^{(1)}(\cdot,\cdot)$. The duality function is given by

$$H : E \times \mathbb{G} \times \{A, D\} \to \mathbb{R}, \qquad H(z,(i,R_i)) = z_{(i,R_i)}. \tag{3.52}$$

Equation (3.38) in Lemma 3.2.1 tells us that $\mathbb{E}[H(Z(t), RW(0))] = \mathbb{E}[H(Z(0), RW(t))]$.

§3.2.2 The clustering case

The proof that the system in (2.4)–(2.5) converges to a unique trivial equilibrium when $\hat{a}(\cdot,\cdot)$ is recurrent goes as follows. We first consider the case where $g = dg_{\mathrm{FW}}$, for which *duality* is available (Lemma 3.2.4). Afterwards we use a *duality comparison argument* to show that the dichotomy between coexistence and clustering does not depend on the choice of $g \in \mathcal{G}$ (Lemma 3.2.5).

• **Case $g = dg_{\mathbf{FW}}$.**

Lemma 3.2.4 (Clustering). *Suppose that $\mu(0) \in \mathcal{T}_\theta^{\mathrm{erg}}$ and $g = dg_{\mathrm{FW}}$. Moreover, suppose that $\hat{a}(\cdot,\cdot)$ defined in* (2.59) *is recurrent, i.e., $I_{\hat{a}} = \infty$. Let $\mu(t)$ be the law at time t of the system defined in* (2.4)*–*(2.5)*. Then*

$$\lim_{t\to\infty} \mu(t) = \theta\,[\delta_{(1,1)}]^{\otimes\mathbb{G}} + (1-\theta)\,[\delta_{(0,0)}]^{\otimes\mathbb{G}}. \tag{3.53}$$

Proof. Since $g = dg_{\mathrm{FW}}$, we can use duality. By the dichotomy criterion in Theorem 2.2.11, it is enough to show that in the dual two partition elements coalesce with probability 1. Recall from Section 2.2.4 that each of the partition elements in the dual moves according to the transition kernel $b^{(1)}(\cdot,\cdot)$ on $\mathbb{G}\times\{A,D\}$ defined by (2.31) (see Fig. 2.3). Recall from Section (2.2.4) that $b^{(1)}(\cdot,\cdot)$ describes a random walk on $\mathbb{G}$ with migration rate kernel $a(\cdot,\cdot)$ that becomes dormant (state D) at rate Ke (after which it stops moving), and becomes active (state A) at rate e (after which it can move again). When two partition elements in the dual are active and are at the same site, they coalesce at rate d, i.e., each time they are active and meet at the same site they coalesce with probability $d/[\sum_{j\in\mathbb{Z}^d} a(i,j) + Ke + d] > 0$. Hence, in order to show that two partition elements coalesce with probability 1, we have to show that *with probability* 1 *two partition elements meet infinitely often while being active.* The latter holds if and only if the expected total time the random walks spend together at the same colony while being active is infinite. We will show that this occurs if and only the random walk with *symmetrised* transition rate kernel $\hat{a}(\cdot,\cdot)$ is recurrent. The proof comes in 4 Steps.

1. Active and dormant time lapses. Consider two copies of the random walk with kernel $b^{(1)}(\cdot,\cdot)$, both starting at 0 and in the active state. Let

$$(\sigma_k)_{k\in\mathbb{N}}, \quad (\sigma'_k)_{k\in\mathbb{N}}, \tag{3.54}$$

denote the successive time lapses during which they are active and let

$$(\tau_k)_{k\in\mathbb{N}}, \quad (\tau'_k)_{k\in\mathbb{N}}, \tag{3.55}$$

denote the successive time lapses during which they are dormant (see Fig. 3.1). These are mutually independent sequences of i.i.d. random variables with marginal laws

$$\begin{array}{lllll} \mathbb{P}(\sigma_1 > t) &=& \mathbb{P}(\sigma'_1 > t) &=& \mathrm{e}^{-Ke\,t}, \quad t \geq 0,\\ \mathbb{P}(\tau_1 > t) &=& \mathbb{P}(\tau'_1 > t) &=& \mathrm{e}^{-e\,t} \qquad t \geq 0, \end{array} \tag{3.56}$$

where we use the symbol $\mathbb{P}$ to denote the joint law of the two sequences.

Let $a_t(\cdot,\cdot)$ denote the time-t transition kernel of the random walk with migration kernel $a(\cdot,\cdot)$. Let

$$\begin{aligned} \mathcal{E}(k,t) &= \left\{ \sum_{\ell=1}^{k} (\sigma_\ell + \tau_\ell) \leq t < \sum_{\ell=1}^{k} (\sigma_\ell + \tau_\ell) + \sigma_{k+1} \right\},\\ \mathcal{E}'(k',t) &= \left\{ \sum_{\ell=1}^{k'} (\sigma'_\ell + \tau'_\ell) \leq t < \sum_{\ell=1}^{k'} (\sigma'_\ell + \tau'_\ell) + \sigma'_{k+1} \right\}, \end{aligned} \tag{3.57}$$

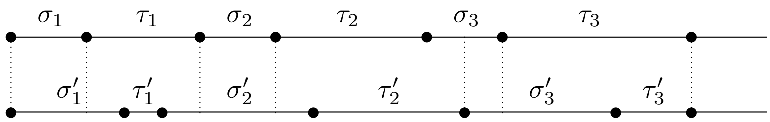

Figure 3.1: Successive periods during which the two random walks are active and dormant. The time lapses between the dotted lines represent periods of joint activity.

be the events that the random walks are active at time t after having become dormant and active exactly k, k' times, and let

$$\begin{aligned} T(k,t) &= \sum_{\ell=1}^{k} \sigma_\ell + \left(\left(t - \sum_{\ell=1}^{k} (\sigma_\ell + \tau_\ell) \right) \wedge \sigma_{k+1} \right), \\ T'(k',t) &= \sum_{\ell=1}^{k'} \sigma'_\ell + \left(\left(t - \sum_{\ell=1}^{k'} (\sigma'_\ell + \tau'_\ell) \right) \wedge \sigma_{k+1} \right), \end{aligned} \tag{3.58}$$

be the total accumulated activity times of the random walks on the events in (3.57). Note that the terms between brackets in (3.58) are at most σ_{k+1}, respectively, $\sigma'_{k'+1}$, and therefore are negligible as $k, k' \to \infty$.

Given the outcome of the sequences in (3.54)–(3.55), the probability that at time t both random walks are active and are at the same colony equals

$$\sum_{k,k' \in \mathbb{N}} \left(\sum_{i \in \mathbb{G}} a_{T(k,t)}(0,i)\, a_{T'(k',t)}(0,i) \right) 1_{\mathcal{E}(k,t)}\, 1_{\mathcal{E}(k',t)}, \tag{3.59}$$

Therefore the expected total time the random walks are active and are at the same colony equals

$$I = \int_0^\infty \mathrm{d}t \sum_{k,k' \in \mathbb{N}} \mathbb{E}_{(0,A),(0,A)} \left[\left(\sum_{i \in \mathbb{G}} a_{T(k,t)}(0,i)\, a_{T'(k',t)}(0,i) \right) 1_{\mathcal{E}(k,t)}\, 1_{\mathcal{E}'(k',t)} \right], \tag{3.60}$$

where $\mathbb{E}$ is the expectation over the sequences in (3.54). Let

$$\begin{aligned} N(t) &= \max \left\{ k \in \mathbb{N}\colon\ \sum_{\ell=1}^{k} (\sigma_\ell + \tau_\ell) \leq t \right\}, \\ N'(t) &= \max \left\{ k' \in \mathbb{N}\colon\ \sum_{\ell=1}^{k'} (\sigma_\ell + \tau_\ell) \leq t \right\}, \end{aligned} \tag{3.61}$$

be the number of times the random walks have become dormant and active up to time t. Let

$$T(t) = T(N(t),t), \quad T'(t) = T'(N'(t),t), \quad \mathcal{E}(t) = \mathcal{E}(N(t),t), \quad \mathcal{E}'(t) = \mathcal{E}'(N'(t),t), \tag{3.62}$$

be the total accumulated activity times of the random walks up to time t, respectively, the events that the random walks are active at time t. Then we may write

$$I = \int_0^\infty \mathrm{d}t \; \mathbb{E}_{(0,A),(0,A)} \left[\left(\sum_{i\in\mathbb{G}} a_{T(t)}(0,i)\, a_{T'(t)}(0,i) \right) 1_{\mathcal{E}(t)}\, 1_{\mathcal{E}'(t)} \right]. \tag{3.63}$$

We know that coalescence occurs with probability 1 if and only if $I = \infty$.

2. Fourier analysis. Define

$$M(t) = T(t) \wedge T'(t), \qquad \Delta(t) = [T(t) \vee T'(t)] - [T(t) \wedge T'(t)]. \tag{3.64}$$

Then

$$\sum_{i\in\mathbb{G}} a_{T(t)}(0,i)\, a_{T'(t)}(0,i) = \sum_{j\in\mathbb{G}} \hat{a}_{2M(t)}(0,j)\, a_{\Delta(t)}(j,0). \tag{3.65}$$

Indeed, the difference of the two random walks at time $M(t)$ has distribution $\hat{a}_{2M(t)}(0,\cdot)$, and in order for the random walk with the largest activity time to meet the random walk with the smallest activity time at time $2M(t)+\Delta(t)$, it must bridge this difference in time $\Delta(t)$. To work out (3.65), we assume without loss of generality that $\sum_{j\in\mathbb{G}} a(0,j) = 1$, and use *Fourier analysis.* For ease of exposition we focus on the special case where $\mathbb{G} = \mathbb{Z}^d$, but the argument below extends to *any* countable Abelian group endowed with the discrete topology, because these properties ensure that there is a version of Fourier analysis on $\mathbb{G}$ [64, Section 1.2]. For $\phi \in [-\pi,\pi]^d$, define

$$a(\phi) = \sum_{j\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}(\phi,j)} a(0,j), \qquad \hat{a}(\phi) = \mathrm{Re}\, a(\phi), \qquad \tilde{a}(\phi) = \mathrm{Im}\, a(\phi). \tag{3.66}$$

Then

$$\begin{aligned} \hat{a}_t(0,j) &= \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \mathrm{d}\phi\, \mathrm{e}^{-\mathrm{i}(\phi,j)}\, \mathrm{e}^{-t[1-\hat{a}(\phi)]}, \\ a_t(j,0) &= \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \mathrm{d}\phi'\, \mathrm{e}^{\mathrm{i}(\phi',j)}\, \mathrm{e}^{-t[1-\hat{a}(\phi')-\mathrm{i}\tilde{a}(\phi')]}, \end{aligned} \tag{3.67}$$

where we use that $a(\phi) = \hat{a}(\phi) + \mathrm{i}\tilde{a}(\phi)$. Inserting these representations into (3.65), we get

$$\sum_{i\in\mathbb{Z}^d} a_{T(t)}(0,i)\, a_{T'(t)}(0,i) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \mathrm{d}\phi\, \mathrm{e}^{-[2M(t)+\Delta(t)]\,[1-\hat{a}(\phi)]} \cos(\Delta(t)\tilde{a}(\phi)), \tag{3.68}$$

where we use that $\sum_{j\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}(\phi'-\phi,j)} = (2\pi)^d \delta(\phi'-\phi)$, with $\delta(\cdot)$ the Dirac distribution (Folland [37, Chapter 7]).

3. Limit theorems. By the strong law of large numbers, we have

$$\lim_{k\to\infty} \frac{1}{k} \sum_{\ell=1}^{k} \sigma_\ell = \frac{1}{Ke} \quad \mathbb{P}\text{-a.s.}, \qquad \lim_{k\to\infty} \frac{1}{k} \sum_{\ell=1}^{k} \tau_\ell = \frac{1}{e} \quad \mathbb{P}\text{-a.s.} \tag{3.69}$$

Therefore, by the standard renewal theorem (Asmussen [3, Chapter I, Theorem 2.2]),

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t}\, N(t) = \lim_{t\to\infty} \frac{1}{t}\, N'(t) = A \quad \mathbb{P}\text{-a.s.},\\
&\lim_{t\to\infty} \frac{1}{t}\, T(t) = \lim_{t\to\infty} \frac{1}{t}\, T'(t) = B \quad \mathbb{P}\text{-a.s.},\\
&\lim_{t\to\infty} \mathbb{P}\big(\mathcal{E}(t)\big) = \lim_{t\to\infty} \mathbb{P}\big(\mathcal{E}'(t)\big) = B,
\end{aligned} \tag{3.70}$$

with

$$A = \frac{1}{\frac{1}{Ke}+\frac{1}{e}} = \frac{K}{1+K}\, e, \qquad B = \frac{\frac{1}{Ke}}{\frac{1}{Ke}+\frac{1}{e}} = \frac{1}{1+K}. \tag{3.71}$$

Moreover, by the central limit theorem, we have

$$\left(\frac{T(t)-Bt}{c\sqrt{t}}, \frac{T'(t)-Bt}{c\sqrt{t}}\right) \quad \Longrightarrow \quad (Z,Z') \quad \text{in } \mathbb{P}\text{-distribution as } t\to\infty \tag{3.72}$$

with (Z,Z') independent standard normal random variables and

$$c^2 = A\left[(1-B)^2\,\mathbb{V}\mathrm{ar}(\sigma_1) + B^2\,\mathbb{V}\mathrm{ar}(\tau_1)\right] \tag{3.73}$$

(see [68] or [3, Theorem VI.3.2]). Since $T(t),\mathcal{E}(t)$ and $T'(t),\mathcal{E}'(t)$ are independent, and each pair is asymptotically independent as well, we find that

$$\mathbb{E}_{(0,A),(0,A)}\left[\left(\sum_{i\in\mathbb{Z}^d} a_{T(t)}(0,i)\, a_{T'(t)}(0,i)\right) 1_{\mathcal{E}(t)}\, 1_{\mathcal{E}'(t)}\right] \sim B^2 f(t), \qquad t\to\infty, \tag{3.74}$$

with

$$\begin{aligned}
f(t) &= \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \mathrm{d}\phi\, \mathrm{e}^{-[1+o(1)]\,2Bt\,[1-\hat{a}(\phi)]}\, \mathbb{E}\left[\cos\left([1+o(1)]\,c(Z-Z')\sqrt{t}\,\tilde{a}(\phi)\right)\right]\\
&= \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \mathrm{d}\phi\, \mathrm{e}^{-[1+o(1)]\,2Bt\,[1-\hat{a}(\phi)]}\, \mathrm{e}^{-[1+o(1)]\,c^2 t\,\tilde{a}(\phi)^2},
\end{aligned} \tag{3.75}$$

where we use that cos is symmetric, $Z-Z' = \sqrt{2}\,Z''$ in $\mathbb{P}$-distribution with Z'' standard normal, and $\mathbb{E}(\mathrm{e}^{\mathrm{i}\mu Z''}) = \mathrm{e}^{-\mu^2/2}$, $\mu\in\mathbb{R}$. From (3.63) and (3.74) we have that $I<\infty$ if and only if $t\mapsto f(t)$ is integrable. By Cramér's theorem, deviations of $T(t)/t$ and $T'(t)/t$ away from B are exponentially costly in t. Hence the error terms in (3.75), arising from (3.70) and (3.72), do *not* affect the integrability of $t\mapsto f(t)$. Note that, because $a(\cdot,\cdot)$ is assumed to be *irreducible* (recall (2.1)), $\hat{a}(\phi)=1$ if and only if $\phi=0$. Hence the integrability of $t\mapsto f(t)$ is determined by the behaviour of $\hat{a}(\phi)$ and $\tilde{a}(\phi)$ as $\phi\to 0$.

4. Irrelevance of asymmetric part of migration. We next observe that

$$\tilde{a}(\phi)^2 \le 1-\hat{a}(\phi)^2 \le 2[1-\hat{a}(\phi)]. \tag{3.76}$$

Hence, $t\,\tilde{a}(\phi)^2 \le 2t\,[1-\hat{a}(\phi)]$. Therefore we see from (3.75) that for sufficiently large $T\in\mathbb{R}$ we can bound $t\mapsto f(t)$ on $[T,\infty)$ from above and below by functions of the form $t \mapsto g_C(t)$ with

$$g_C(t) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \mathrm{d}\phi \, \mathrm{e}^{-Ct\,[1-\hat{a}(\phi)]}, \qquad C \in (0,\infty). \tag{3.77}$$

From (3.67) we have

$$g_C(t) = \hat{a}_{Ct}(0,0) \asymp \hat{a}_t(0,0), \tag{3.78}$$

where the last asymptotics uses that $t \mapsto \hat{a}_t(0,0)$ is regularly varying at infinity (recall (2.60)). Combining (3.63), (3.74) and (3.77)–(3.78), we get

$$I = \infty \quad \Longleftrightarrow \quad I_{\hat{a}} = \infty \tag{3.79}$$

with $I_{\hat{a}} = \int_1^\infty \mathrm{d}t \, \hat{a}_t(0,0)$. Thus, if $\hat{a}(\cdot,\cdot)$ is recurrent, then $I = \infty$ and the system clusters. Moreover, we see from the bounds on $f(t)$ (recall (3.75)) that *the asymmetric part of the migration kernel has no effect on the integrability.*

This settles the dichotomy between clustering and coexistence when $g = g_{\mathrm{FW}}$. □

• **Case $g \neq dg_{\mathbf{FW}}$.** For $g \neq dg_{\mathrm{FW}}$ the proof of Lemma 3.2.4 does not go through. However, the *moments relations* in Lemma 3.2.1 hold for general $g \in \mathcal{G}$. Using these moment relations and a technique called duality comparison (see [14]), we prove Lemma 3.2.4 for general $g \in \mathcal{G}$.

Lemma 3.2.5 (Duality comparison). *Suppose that $\mu(0) \in \mathcal{T}_\theta^{\mathrm{erg}}$ and $g \in \mathcal{G}$. Moreover, suppose that $\hat{a}(\cdot,\cdot)$ defined in (2.59) is recurrent, i.e., $I_{\hat{a}} = \infty$. Let $\mu(t)$ be the law at time t of the system defined in (2.4)–(2.5). Then*

$$\lim_{t\to\infty} \mu(t) = \theta\,[\delta_{(1,1)}]^{\otimes \mathbb{G}} + (1-\theta)\,[\delta_{(0,0)}]^{\otimes \mathbb{G}}. \tag{3.80}$$

Proof. We proceed as in the proof of [14, Theorem]. First assume that $\mu(0) = \delta_z$ for some $z \in E$, that satisfies

$$\lim_{t\to\infty} \sum_{(k,R_k)\in\mathbb{G}\times\{A,D\}} b_t^{(1)}((i,R_i),(k,R_k))\, z_{(k,R_k)} = \theta. \tag{3.81}$$

By Lemma 3.2.1, we have

$$\mathbb{E}_z\left[z_{(i,R_i)}(t)\right] = \sum_{(k,R_k)\in\mathbb{G}\times\{A,D\}} b_t^{(1)}((i,R_i),(k,R_k))\, z_{(k,R_k)}. \tag{3.82}$$

Hence, by assumption, for all $(i,R_i) \in \mathbb{G}\times\{A,D\}$ we have

$$\lim_{t\to\infty} \mathbb{E}_z\left[z_{(i,R_i)}(t)\right] = \theta. \tag{3.83}$$

Since we have clustering if, for all $(i,R_i),(j,R_j) \in \mathbb{G}\times\{A,D\}$,

$$\lim_{t\to\infty} \mathbb{E}_z\left[z_{(i,R_i)}(t)(1-z_{(j,R_j)}(t))\right] = 0, \tag{3.84}$$

we are left to prove that

$$\lim_{t\to\infty} \mathbb{E}_z\left[z_{(i,R_i)} z_{(j,R_j)}\right] = \theta. \tag{3.85}$$

Since (3.83) implies that $\limsup_{t\to\infty} \mathbb{E}_z[z_{(i,R_i)}z_{(j,R_j)}] \leq \theta$, we are left to prove that

$$\liminf_{t\to\infty} \mathbb{E}_z[z_{(i,R_i)}z_{(j,R_j)}] \geq \theta. \tag{3.86}$$

Like in [14], we will prove (3.86) by *comparison duality.*

Fix $\epsilon > 0$. Since $g \in \mathcal{G}$ we can choose a $c = c(\epsilon) > 0$ such that

$$g(x) \geq \tilde{g}(x) = c(x-\epsilon)(1-(x+\epsilon)),\ x \in [0,1]. \tag{3.87}$$

Note that $\tilde{g}(x) < 0$ for $x \in [0,\epsilon) \cup (1-\epsilon, 1]$, so we cannot replace g by $\tilde{g}$ in the SSDE. Instead we use $\tilde{g}$ as an auxiliary function.

Consider the Markov chain $(B(t))_{t\geq 0}$, with state space

$$\{1,2\} \times (\mathbb{G} \times \{A,D\}) \times (\mathbb{G} \times \{A,D\}) \tag{3.88}$$

and $B(t) = (B_0(t), B_1(t), B_2(t))$, evolving according to

$$\begin{aligned}
(1,(i,R_i),(i,R_i)) &\to \ (1,(k,R_k),(k,R_k)), \qquad \text{at rate } b^{(1)}((i,R_i),(k,R_k)),\\
(2,(i,R_i),(j,R_j)) &\to \begin{cases} (2,(k,R_k),(j,R_j)), & \text{at rate } b^{(1)}((i,R_i),(k,R_k)),\\ (2,(i,R_i),(l,R_l)), & \text{at rate } b^{(1)}((j,R_j),(l,R_l)),\\ (1,(i,R_i),(i,R_i)), & \text{at rate } c1_{\{i=j\}}1_{\{R_i=R_j=A\}}.\end{cases}
\end{aligned} \tag{3.89}$$

This describes two random walks, evolving independently according to the transition kernel $b^{(1)}(\cdot,\cdot)$, that coalesce at rate $c > 0$ when they are at the same site and are active. We put $B_0(t) = 1$ when the two random walks have already coalesced by time t, and $B_0(t) = 2$ otherwise. Let $\mathbb{P}_{(2,(i,R_i),(j,R_j))}$ denote the law of the Markov Chain $B(t)$ that starts in $(2,(i,R_i),(j,R_j))$. Note that

$$\mathbb{P}_{(2,(i,R_i),(j,R_j))}\left(B_1(t) = (k,R_k)\right) = b_t^{(1)}((i,R_i),(k,R_k)), \tag{3.90}$$

and similarly

$$\mathbb{P}_{(2,(i,R_i),(j,R_j))}\left(B_2(t) = (l,R_l)\right) = b_t^{(1)}((j,R_j),(l,R_l)). \tag{3.91}$$

Since we have assumed that $\hat{a}(\cdot,\cdot)$ is recurrent, i.e., $I_{\hat{a}} = \infty$, the two random walks meet infinitely often at the same site while being active and hence coalesce with probability 1. Therefore

$$\lim_{t\to\infty} \mathbb{P}_{(2,(i,R_i),(j,R_j))}\left(B_0(t) = 2\right) = 0. \tag{3.92}$$

We can rewrite the SSDE in (2.4)–(2.5) in terms of $b^{(1)}(\cdot,\cdot)$, namely, for all $(i,R_i) \in \mathbb{G} \times \{A,D\}$,

$$\begin{aligned}
\mathrm{d}z_{(i,R_i)}(t) = &\sum_{(k,R_k)\in\mathbb{G}\times\{A,D\}} b^{(1)}((i,R_i),(j,R_j))[z_{(j,R_j)}(t) - z_{(i,R_i)}(t)]\,\mathrm{d}t\\
&+ \sqrt{g(z_{i,R_i}(t))}\,1_{\{R_i=A\}}\,\mathrm{d}w_i(t).
\end{aligned} \tag{3.93}$$

Using (3.93) and Itô-calculus, we obtain

$$\begin{aligned} &\frac{\mathrm{d}\mathbb{E}_z[z_{(i,R_i)}(t)-\epsilon]}{\mathrm{d}t} \\ &= \sum_{(k,R_k)\in\mathbb{G}\times\{A,D\}} b^{(1)}((i,R_i),(k,R_k))\,\mathbb{E}\left[(z_{(k,R_k)}(t)-\epsilon)-(z_{(i,R_i)}(t)-\epsilon)\right] \end{aligned} \tag{3.94}$$

and

$$\begin{aligned} &\frac{\mathrm{d}\mathbb{E}_z[(z_{(i,R_i)}(t)-\epsilon)(z_{(j,R_j)}(t)+\epsilon)]}{\mathrm{d}t} \\ &= \sum_{(k,R_k)\in\mathbb{G}\times\{A,D\}} b^{(1)}((i,R_i),(k,R_k)) \\ &\qquad \times \mathbb{E}_z\left[(z_{(j,R_j)}(t)+\epsilon)(z_{(k,R_k)}(t)-\epsilon)-(z_{(j,R_j)}(t)+\epsilon)(z_{(i,R_i)}(t)-\epsilon)\right] \\ &\quad + \sum_{(l,R_l)\in\mathbb{G}\times\{A,D\}} b^{(1)}((j,R_j),(k,R_k)) \\ &\qquad \times \mathbb{E}_z\left[(z_{(i,R_i)}(t)-\epsilon)(z_{(l,R_l)}(t)+\epsilon)-(z_{(i,R_i)}(t)-\epsilon)(z_{(j,R_j)}(t)+\epsilon)\right] \\ &\quad + \mathbb{E}_z\left[c(z_{(i,R_i)}(t)-\epsilon)(1-(z_{(j,R_j)}(t)+\epsilon))1_{\{i=j\}}1_{\{R_i=R_j=A\}}\right] \\ &\quad + \mathbb{E}_z\left[\left(g(z_{(i,R_i)}(t))-\tilde{g}(z_{(i,R_i)}(t))\right)1_{\{i=j\}}1_{\{R_i=R_j=A\}}\right]. \end{aligned} \tag{3.95}$$

For $t \geq 0$, define $F_t\colon\ \{0,1\}\times(\mathbb{G}\times\{A,D\})\times(\mathbb{G}\times\{A,D\})\to\mathbb{R}$ by

$$\begin{aligned} F_t(1,(i,R_i),(i,R_i)) &= \mathbb{E}_z\left[z_{(i,R_i)}(t)-\epsilon\right] \\ F_t(2,(i,R_i),(j,R_j)), &= \mathbb{E}_z\left[(z_{(i,R_i)}(t)-\epsilon)(z_{(j,R_j)}(t)+\epsilon)\right], \end{aligned} \tag{3.96}$$

and $H_t\colon\ \{0,1\}\times(\mathbb{G}\times\{A,D\})\times(\mathbb{G}\times\{A,D\})\to\mathbb{R}$ by

$$\begin{aligned} &H_t(1,(i,R_i),(i,R_i)) = 0, \\ &H_t(2,(i,R_i),(j,R_j)) = \mathbb{E}_z\left[\left(g(z_{(i,R_i)}(t))-\tilde{g}(z_{(i,R_i)}(t))\right)1_{\{i=j\}}\,1_{\{R_i=R_j=A\}}\right]. \end{aligned} \tag{3.97}$$

Let $\mathfrak{B}$ denote the generator of $(B(t))_{t\geq 0}$, and let $(V_t)_{t\geq 0}$ the associated semigroup. Then

$$\frac{\mathrm{d}F_t}{\mathrm{d}t} = \mathfrak{B}F_t + H_t. \tag{3.98}$$

Hence, by [56, Theorem I.2.15], it follows that

$$F_t = V_tF_0 + \int_0^t \mathrm{d}s\, V_{(t-s)}H_s. \tag{3.99}$$

Since $H_t > 0$ for all $t \geq 0$, we obtain

$$
\begin{aligned}
&F_t(2,(i,R_i),(j,R_j)) \geq V_t F_0(2,(i,R_i),(j,R_j))\\
&= \mathbb{E}_{(2,(i,R_i),(j,R_j))}\left[F_0(B(t))\right]\\
&= \mathbb{E}_{(2,(i,R_i),(j,R_j))}\left[F_0(B(t))1_{\{B_0(t)=1\}} + F_0(B(t))1_{\{B_0(t)=2\}}\right]\\
&= \sum_{(k,R_k),(l,R_l)\in\mathbb{G}\times\{A,D\}} \mathbb{P}_{(2,(i,R_i),(j,R_j))}\left[B_0(t)=1, B_1(t)=(k,R_k)\right](z_{(k,R_k)}-\epsilon)\\
&\qquad + \mathbb{E}_{(2,(i,R_i),(j,R_j))}\left[F_0(B(t))1_{\{B_0(t)=2\}}\right]\\
&= \sum_{(k,R_k),(l,R_l)\in\mathbb{G}\times\{A,D\}} \mathbb{P}_{(2,(i,R_i),(j,R_j))}\left[B_1(t)=(k,R_k)\right](z_{(k,R_k)}-\epsilon)\\
&\qquad - \sum_{(k,R_k),(l,R_l)\in\mathbb{G}\times\{A,D\}} \mathbb{P}_{(2,(i,R_i),(j,R_j))}\left[B_0(t)=2, B_1(t)=(k,R_k)\right](z_{(k,R_k)}-\epsilon)\\
&\qquad + \mathbb{E}_{(2,(i,R_i),(j,R_j))}\left[F_0(B(t))1_{\{B_0(t)=2\}}\right]\\
&\geq \sum_{(k,R_k)\in\mathbb{G}\times\{A,D\}} b_t^{(1)}((i,R_i),(k,R_k))\,(z_{(k,R_k)}-\epsilon)\\
&\qquad - (1+\epsilon^2)\,\mathbb{P}_{(2,(i,R_i),(j,R_j))}\left[B_1(t)=2\right].
\end{aligned}
\tag{3.100}
$$

Hence, by (3.92), we obtain

$$
\begin{aligned}
\liminf_{t\to\infty} F_t(2,(i,R_i),(j,R_j)) &\geq \liminf_{t\to\infty} \mathbb{E}_z\left[(z_{(i,R_i)}(t)-\epsilon)(z_{(j,R_j)}(t)+\epsilon)\right]\\
&\geq \theta - \epsilon^2.
\end{aligned}
\tag{3.101}
$$

Letting $\epsilon \downarrow 0$, we get (3.85).

To get rid of the assumption $\mu(0) = \delta_z$, note that for $\mu(0) \in \mathcal{T}_\theta^{\text{erg}}$ we have (recall Remark 3.2.2)

$$
\lim_{t\to\infty} \sum_{(k,R_k)\in\mathbb{G}\times\{A,D\}} b_t((i,R_i),(k,R_k))\mathbb{E}_\mu[z_{(k,R_k)}] = \theta. \tag{3.102}
$$

Hence, by the above argument,

$$
\begin{aligned}
&\mathbb{E}_\mu\left[(z_{(i,R_i)}(t)-\epsilon)(z_{j,R_j}(t)+\epsilon)\right]\\
&= \int \mathbb{E}_z\left[(z_{(i,R_i)}(t)-\epsilon)(z_{j,R_j}(t)+\epsilon)\right]\mathrm{d}\mu(z)\\
&\geq \int \sum_{(k,R_k)\in\mathbb{G}\times\{A,D\}} b_t^{(1)}((i,R_i),(k,R_k))(z_{(k,R_k)}-\epsilon)\\
&\qquad - (1+\epsilon^2)\mathbb{P}_{(2,(i,R_i),(j,R_j))}\left[B_1(t)=2\right]\mathrm{d}\mu(z)
\end{aligned}
\tag{3.103}
$$

Letting first $t \to \infty$ and then $\epsilon \downarrow 0$, we find that

$$
\lim_{t\to\infty} \mathbb{E}_\mu\left[(z_{(i,R_i)}(t)-\epsilon)(z_{j,R_j}(t)+\epsilon)\right] = \theta, \tag{3.104}
$$

and, for all $(i,R_i),(j,R_j) \in \mathbb{G}\times\{A,D\}$,

$$
\lim_{t\to\infty} \mathbb{E}_\mu\left[z_{(i,R_i)}(t)(1-z_{j,R_j}(t))\right] = 0. \tag{3.105}
$$

□

§3.2.3 The coexistence case

For the coexistence case we proceed as in [14] with small adaptations. For the convenience of the reader we have written out the full proof. The proof relies on the moment relations in Lemma 3.2.1 and no distinction between $g = dg_{\mathrm{FW}}$ and general $g \in \mathcal{G}$ is needed. The proof consist of several lemmas (Lemmas 3.2.7–3.2.13 below), organised into 4 Steps. In Step 1 we use the moment relations in Lemma 3.2.1 to define a set of measures that are preserved under the evolution. In Step 2 we use coupling to prove that, for each given θ, the system converges to a unique equilibrium. In Step 3 we show that, for each given θ, each initial measure under the evolution converges to an invariant measure. In Step 4 we show that the limiting measure is invariant, ergodic and mixing under translations, and is associated.

1. Properties of measures preserved under the evolution. Let θ be defined as in (2.62) such that $\theta = \mathbb{E}_{\mu(0)}\left[\frac{x_0+Ky_0}{1+K}\right] = \frac{\theta_x+K\theta_y}{1+K}$.

Definition 3.2.6 (Preserved class of measure). Let $\mathcal{R}_\theta^{(1)}$ denote the set of measures $\mu \in \mathcal{T}$ satisfying:

(1) For all $(i,R_i) \in \mathbb{G} \times \{A,D\}$,

$$\lim_{t\to\infty} \mathbb{E}_\mu[z_{(i,R_i)}(t)] = \theta. \tag{3.106}$$

(2) for all (i,R_i), $(j,R_j) \in \mathbb{G} \times \{A,D\}$,

$$\begin{aligned}\lim_{t\to\infty} \sum_{(k,R_k),(l,R_l)\in\mathbb{G}\times\{A,D\}} & b_t^{(1)}\big((i,R_i),(k,R_k)\big)\, b_t^{(1)}\big((j,R_j),(l,R_l)\big)\\ & \times \mathbb{E}_\mu[z_{(k,R_k)} z_{(l,R_l)}] = \theta^2.\end{aligned} \tag{3.107}$$

□

Clearly, if $\mu \in \mathcal{R}_\theta^{(1)}$, then (1) and (2) together with Lemma 3.2.1 imply

$$\lim_{t\to\infty} \mathbb{E}_\mu\left[\left(\sum_{(k,R_k)\in\mathbb{G}\times\{A,D\}} b_t^{(1)}((i,R_i),(k,R_k))\, z_{(k,R_k)} - \theta\right)^2\right] = 0, \tag{3.108}$$

and so $\lim_{t\to\infty} z_{i,R_i}(t) = \theta$ in $L^2(\mu)$.

On the other hand, suppose that (3.108) holds for *some* $(i,R_i) \in \mathbb{G} \times \{A,D\}$. Then, by Lemma (3.2.1), we can rewrite (3.108) as

$$\lim_{t\to\infty} \mathbb{E}_\mu\left[\left(\mathbb{E}_z[z_{(i,R_i)}(t)] - \theta\right)^2\right] = 0. \tag{3.109}$$

This implies

$$\lim_{t\to\infty} \mathbb{E}_\mu[z_{(i,R_i)}(t)] = \theta, \tag{3.110}$$

and hence, by translation invariance,

$$\lim_{t\to\infty} \mathbb{E}_\mu[z_{(k,R_i)}(t)] = \theta \qquad \forall\, k \in \mathbb{G}. \tag{3.111}$$

Using that switches between the active state at the dormant state occur at a positive rate, we can use the strong Markov property to obtain that (3.111) holds both for $R_i = A$ and for $R_i = D$. Hence (3.106) holds. Combining (3.106) and (3.108), we see that also (3.107) holds.

Lemma 3.2.7. *$\mu \in \mathcal{R}_\theta^{(1)}$ for all $\mu \in \mathcal{T}_\theta^{\mathrm{erg}}$.*

Proof. The proof relies on *Fourier analysis* and the existence of spectral measures. As in Section 3.2.2, for ease of exposition we focus on the special case where $\mathbb{G} = \mathbb{Z}^d$, but the argument below extends to *any* countable Abelian group endowed with the discrete topology.

By translation invariance and the Herglotz theorem, there exist spectral measures λ_A and λ_D such that, for all $j,k \in \mathbb{Z}^d$,

$$\begin{aligned}
\mathbb{E}_\mu\left[(x_j-\theta_x)(x_k-\theta_x)\right] &= \int_{(-\pi,\pi]^d} \mathrm{e}^{\mathrm{i}(j-k,\phi)}\mathrm{d}\lambda_A(\phi),\\
\mathbb{E}_\mu\left[(y_j-\theta_y)(y_k-\theta_y)\right] &= \int_{(-\pi,\pi]^d} \mathrm{e}^{\mathrm{i}(j-k,\phi)}\mathrm{d}\lambda_D(\phi).
\end{aligned} \tag{3.112}$$

Let $a(\phi) = \sum_{k\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}(\phi,j)} a(0,k)$ be the characteristic function of the kernel $a(\cdot,\cdot)$ (recall (3.66)), and $T(t)$ the activity time of the random walk up to time t (recall (3.58)). Then

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^d} a_{T(t)}(0,k)\,\mathrm{e}^{\mathrm{i}(k,\phi)} &= \sum_{n\in\mathbb{N}_0} \frac{\mathrm{e}^{-T(t)}[T(t)]^n}{n!} \sum_{k\in\mathbb{Z}^d} a^n(0,k)\,\mathrm{e}^{\mathrm{i}(k,\phi)}\\
&= \sum_{n\in\mathbb{N}_0} \frac{\mathrm{e}^{-T(t)}[T(t)\,a(\phi)]^n}{n!}\\
&= \mathrm{e}^{-T(t)(1-a(\phi))}.
\end{aligned} \tag{3.113}$$

Let $\mathcal{E}(t)$ be defined as in (3.62). Then, for fixed $t>0$,

$$\mathbb{P}_{(0,A)}(\mathcal{E}(t)) = \sum_{k\in\mathbb{Z}^d} b_t^{(1)}((0,A),(k,A)) > 0. \tag{3.114}$$

and hence

$$
\begin{aligned}
&\mathbb{E}_\mu\left[\left(\frac{1}{\mathbb{P}_{(0,A)}(\mathcal{E}(t))}\sum_{k\in\mathbb{Z}^d} b_t^{(1)}((0,A),(k,A))x_k-\theta_x\right)^2\right]\\
&=\frac{1}{\mathbb{P}_{(0,A)}(\mathcal{E}(t))^2}\sum_{k,l\in\mathbb{Z}^d} b_t^{(1)}((0,A),(k,A))\,b_t^{(1)}((0,A),(l,A))\,\mathbb{E}_\mu\left[(x_k-\theta_x)(x_l-\theta_x)\right]\\
&=\frac{1}{\mathbb{P}_{(0,A)}(\mathcal{E}(t))^2}\sum_{k,l\in\mathbb{Z}^d} b_t^{(1)}((0,A),(k,A))\,b_t^{(1)}((0,A),(l,A))\\
&\qquad\qquad\times\int_{(-\pi,\pi]^d}\mathrm{e}^{\mathrm{i}(k-l,\phi)}\mathrm{d}\lambda_A(\phi)\\
&=\frac{1}{\mathbb{P}_{(0,A)}(\mathcal{E}(t))^2}\sum_{k,l\in\mathbb{Z}^d}\mathbb{E}_{(0,A),(0,A)}\left[a_{T(t)}(0,k)\,a_{T'(t)}(0,l)\,1_{\mathcal{E}(t)}\,1_{\mathcal{E}'(t)}\right]\\
&\qquad\qquad\times\int_{(-\pi,\pi]^d}\mathrm{e}^{\mathrm{i}(k-l,\phi)}\mathrm{d}\lambda_A(\phi)\\
&=\frac{1}{\mathbb{P}_{(0,A)}(\mathcal{E}(t))^2}\\
&\times\int_{(-\pi,\pi]^d}\mathbb{E}_{(0,A),(0,A)}\left[\sum_{k\in\mathbb{Z}^d}a_{T(t)}\,\mathrm{e}^{\mathrm{i}(k,\phi)}(0,k)1_{\mathcal{E}(t)}\sum_{l\in\mathbb{Z}^d}a_{T'(t)}\,\mathrm{e}^{-\mathrm{i}(l,\phi)}(0,l)1_{\mathcal{E}'(t)}\right]\mathrm{d}\lambda_A(\phi)\\
&=\frac{1}{\mathbb{P}_{(0,A)}(\mathcal{E}(t))^2}\int_{(-\pi,\pi]^d}\mathbb{E}_{(0,A),(0,A)}\left[\mathrm{e}^{-T(t)(1-a(\phi))}1_{\mathcal{E}(t)}\,\mathrm{e}^{-T'(t)(1-\bar{a}(\phi))}1_{\mathcal{E}'(t)}\right]\mathrm{d}\lambda_A(\phi).
\end{aligned}
\tag{3.115}
$$

Since $a(\cdot,\cdot)$ is irreducible, $a(\phi)\neq 1$ for all $\phi\in(-\pi,\pi]^d\backslash\{0\}$. Taking the limit $t\to\infty$, we find

$$
\lim_{t\to\infty}\mathbb{E}_\mu\left[\left(\frac{1}{\mathbb{P}_{(0,A)}(\mathcal{E}(t))}\sum_{k\in\mathbb{Z}^d} b_t^{(1)}((0,A),(k,A))x_k-\theta_x\right)^2\right]=\lambda_A(\{0\}). \tag{3.116}
$$

Similarly,

$$
\lim_{t\to\infty}\mathbb{E}_\mu\left[\left(\frac{1}{\mathbb{P}_{(0,A)}(\mathcal{E}^c(t))}\sum_{k\in\mathbb{Z}^d} b_t^{(1)}((0,A),(k,D))y_k-\theta_y\right)^2\right]=\lambda_D(\{0\}). \tag{3.117}
$$

Hence

$$
\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}_\mu \left[\left(\sum_{(k,R_k)\in\mathbb{Z}^d\times\{A,D\}} b_t^{(1)}((0,A),(k,R_k)\, z_{(k,R_k)} - \theta \right)^2 \right] \\
&= \lim_{t\to\infty} \mathbb{E}_\mu \Bigg[\Bigg(\mathbb{P}_{(0,A)}(\mathcal{E}(t)) \sum_{k\in\mathbb{Z}^d} \frac{b_t^{(1)}((0,A),(k,A))}{\mathbb{P}_{(0,A)}(\mathcal{E}(t))}\, x_k - \frac{\theta_x}{1+K} \\
&\qquad + \mathbb{P}_{(0,A)}(\mathcal{E}^c(t)) \sum_{k\in\mathbb{Z}^d} \frac{b_t^{(1)}((0,A),(k,D))}{\mathbb{P}_{(0,A)}(\mathcal{E}^c(t))}\, y_k - \frac{K\theta_y}{1+K} \Bigg)^2 \Bigg] \\
&\le \lim_{t\to\infty} \mathbb{P}_{(0,A)}(\mathcal{E}(t))\, \mathbb{E}_\mu \left[\left(\sum_{k\in\mathbb{Z}^d} \frac{b_t^{(1)}((0,A),(k,A))}{\mathbb{P}_{(0,A)}(\mathcal{E}(t))}\, x_k - \frac{\theta_x}{(1+K)} \frac{1}{\mathbb{P}_{(0,A)}(\mathcal{E}(t))} \right)^2 \right] \\
&\qquad + \mathbb{P}_{(0,A)}(\mathcal{E}^c(t))\, \mathbb{E}_\mu \left[\left(\sum_{k\in\mathbb{Z}^d} \frac{b_t^{(1)}((0,A),(k,D))}{\mathbb{P}_{(0,A)}(\mathcal{E}^c(t))}\, y_k - \frac{K\theta_y}{1+K} \frac{1}{\mathbb{P}_{(0,A)}(\mathcal{E}^c(t))} \right)^2 \right] \\
&= \frac{1}{1+K}\lambda_A(\{0\}) + \frac{K}{1+K}\lambda_D(\{0\}).
\end{aligned}
\tag{3.118}
$$

Hence, if $\lambda_A(\{0\}) = 0$ and $\lambda_D(\{0\}) = 0$, then $\mu \in \mathcal{R}_\theta^{(1)}$. We will show that $\lambda_A(\{0\}) = 0$ and $\lambda_D(\{0\}) = 0$ for $\mu \in \mathcal{T}_\theta^{\mathrm{erg}}$.

Let $\Lambda_N = [0,N)^d \cap \mathbb{Z}^d$. By the L^1-ergodic theorem, we have, for $\mu \in \mathcal{T}_\theta^{\mathrm{erg}}$,

$$
\lim_{N\to\infty} \mathbb{E}_\mu \left[\left(\frac{1}{\Lambda_N} \sum_{j\in\Lambda_N} x_j - \theta_x \right)^2 \right] = 0. \tag{3.119}
$$

(For general $\mathbb{G}$ not that countable groups endowed with the discrete topology are amenable. For amenable groups $\mathbb{G}$, $(\Lambda_N)_{N\in\mathbb{N}}$ must be replaced by a so-called Følner sequence, i.e., a sequence of finite subsets of $\mathbb{G}$ that exhaust $\mathbb{G}$ and satisfy

$$
\lim_{N\to\infty} |\mathfrak{g}\Lambda_N \triangle \Lambda_N| / |\Lambda_N| = 0 \tag{3.120}
$$

for any $\mathfrak{g} \in \mathbb{G}$ [57].) Using the spectral measure, we can write

$$
\begin{aligned}
&\lim_{N\to\infty} \mathbb{E}_\mu \left[\left(\frac{1}{\Lambda_N} \sum_{j\in\Lambda_N} x_j - \theta_x \right)^2 \right] \\
&= \lim_{N\to\infty} \frac{1}{\Lambda_N^2} \sum_{j,k\in\Lambda_N} \int_{(-\pi,\pi]^d} \mathrm{e}^{\mathrm{i}(j-k,\phi)} \mathrm{d}\lambda_A \\
&= \lim_{N\to\infty} \int_{(-\pi,\pi]^d} \left(\frac{1}{\Lambda_N} \sum_{j\in\Lambda_N} \mathrm{e}^{\mathrm{i}(j,\phi)} \right) \left(\frac{1}{\Lambda_N} \sum_{k\in\Lambda_N} \mathrm{e}^{-\mathrm{i}(k,\phi)} \right) \mathrm{d}\lambda_A \\
&= \lambda_A\{0\}.
\end{aligned}
\tag{3.121}
$$

In the last equality we use dominated convergence and

(a) For all $\phi \in (-\pi, \pi]^d$,

$$\lim_{N\to\infty} \frac{1}{\Lambda_N} \sum_{j,k\in\Lambda_N} \mathrm{e}^{-\mathrm{i}(k,\phi)} = 1_{\{0\}}(\phi). \tag{3.122}$$

(b) For all $\delta > 0$ there exist $\epsilon(N,\delta) > 0$ such that if $J_\delta = (-\delta, \delta)$, then

$$\left| \frac{1}{\Lambda_N} \sum_{j,k\in\Lambda_N} \mathrm{e}^{-\mathrm{i}(k,\phi)} - 1_{\{0\}}(\phi) \right| \leq 1_{J_\delta}(\phi) + \epsilon(N,\delta), \tag{3.123}$$

where $\epsilon(N,\delta) \downarrow 0$ as $N \to \infty$.

We conclude that $\lambda_A(\{0\}) = 0$. Similarly we can show that $\lambda_D(\{0\}) = 0$, and hence $\mu \in \mathcal{R}_\theta^{(1)}$. □

Recall that $(S_t)_{t\geq 0}$ is the semigroup associated with (2.4)–(2.5).

Lemma 3.2.8 (Preservation). *If $b(\cdot,\cdot)$ is transient and $\mu \in \mathcal{R}_\theta^{(1)}$, then the following hold:*

(a) *$\mu S_t \in \mathcal{R}_\theta^{(1)}$ for each $t \geq 0$.*

(b) *If $t_n \to \infty$ and $\mu S_{t_n} \to \mu(\infty)$, then $\mu(\infty) \in \mathcal{R}_\theta^{(1)}$.*

Proof. Our dynamics preserve translation invariance. To check property (1) of $\mathcal{R}_\theta^{(1)}$ (see (3.106)), set $f(z) = z_{(i,R_i)}$. Since $\mu \in \mathcal{R}_\theta^{(1)}$, applying Lemma 3.2.1 multiple times, we obtain

$$\begin{aligned}
\lim_{s\to\infty} \mathbb{E}_{\mu S_t}[z_{(i,R_i)}(s)] &= \lim_{s\to\infty} \sum_{(k,R_k)\in\mathbb{G}\times\{A,D\}} b_s^{(1)}\big((i,R_i),(k,R_k)\big)\, \mathbb{E}_{\mu S_t}[z_{(k,R_k)}] \\
&= \lim_{s\to\infty} \sum_{(k,R_k)\in\mathbb{G}\times\{A,D\}} b_s^{(1)}\big((i,R_i),(k,R_k)\big)\, \mathbb{E}_{\mu}[z_{(k,R_k)}(t)] \\
&= \lim_{s\to\infty} \sum_{(k',R'_{k'})\in\mathbb{G}\times\{A,D\}} b_{s+t}^{(1)}\big((i,R_i),(k',R_{k'})\big)\, \mathbb{E}_{\mu}[z_{(k',R_{k'})}] \\
&= \lim_{s\to\infty} \mathbb{E}_{\mu}[z_{(i,R_i)}(t+s)] = \theta.
\end{aligned} \tag{3.124}$$

To check property (2) of $\mathcal{R}_\theta^{(1)}$ (see (3.107)), we set $f(z) = z_{(i,R_i)} z_{(j,R_j)}$. Then, again by applying Lemma 3.2.1, we find

$$
\begin{aligned}
&\lim_{s\to\infty} \sum_{\substack{(k,R_k),(l,R_l)\\ \in\mathbb{G}\times\{A,D\}}} b_s^{(1)}\big((i,R_i),(k,R_k)\big)\, b_s^{(1)}\big((j,R_j),(l,R_l)\big)\,\mathbb{E}_{\mu S_t}[z_{(k,R_k)}z_{(l,R_l)}]\\
&= \lim_{s\to\infty} \sum_{\substack{(k,R_k),(l,R_l)\\ \in\mathbb{G}\times\{A,D\}}} b_s^{(1)}\big((i,R_i),(k,R_k)\big)\, b_s^{(1)}\big((j,R_j),(l,R_l)\big)\,\mathbb{E}_{\mu}[z_{(k,R_k)}(t)z_{(l,R_l)}(t)]\\
&= \lim_{s\to\infty}\Bigg[\sum_{\substack{(k',R_{k'}),(l',R_{l'})\\ \in\mathbb{G}\times\{A,D\}}} b_{t+s}^{(1)}\big((i,R_i),(k',R_{k'})\big)\, b_{t+s}^{(1)}\big((j,R_j),(l',R_{l'})\big)\\
&\qquad\qquad \times \mathbb{E}_\mu[z_{(k',R_{k'})}z_{(l',R_{l'})}]\\
&\quad + \int_0^t \mathrm{d}r \sum_{k'\in\mathbb{G}} b_{t-r+s}^{(1)}\big((i,R_i),(k',A)\big)\, b_{t-r+s}^{(1)}\big((j,R_j),(k',A)\big)\,\mathbb{E}_\mu[g(x_{k'}(r))]\Bigg].
\end{aligned}
\tag{3.125}
$$

Since $\mu\in\mathcal{R}_\theta^{(1)}$, we are left to show that

$$
\lim_{s\to\infty}\int_s^{t+s}\mathrm{d}u \sum_{k'\in\mathbb{G}} b_u^{(1)}\big((i,R_i),(k',A)\big)\, b_u^{(1)}\big((j,R_j),(k',A)\big)\,\mathbb{E}_\mu[g(x_{k'}(t+s-u))] = 0.
\tag{3.126}
$$

Using the notation of Section 3.2.2, we get

$$
\begin{aligned}
&\lim_{s\to\infty}\int_s^{t+s}\mathrm{d}u \sum_{k'\in\mathbb{G}} b_u^{(1)}\big((i,R_i),(k',A)\big)\, b_u^{(1)}\big((j,R_j),(k',A)\big)\,\mathbb{E}_\mu[g(x_{k'}(t+s-u))]\\
&\leq \|g\|\lim_{s\to\infty}\int_s^{t+s}\mathrm{d}u \sum_{k'\in\mathbb{G}} b_u^{(1)}\big((i,R_i),(k',A)\big)\, b_u^{(1)}\big((j,R_j),(k',A)\big)\\
&= \|g\|\lim_{s\to\infty}\int_s^{t+s}\mathrm{d}u\; \mathbb{E}_{(i,R_i),(j,R_j)}\Bigg[\sum_{k'\in\mathbb{G}} a_{T(u)}(i,k')\,1_{\mathcal{E}(u)}\,a_{T'(u)}(j,k')\,1_{\mathcal{E}'(u)}\Bigg]\\
&\leq \|g\|\lim_{s\to\infty}\int_s^{t+s}\mathrm{d}u\; \mathbb{E}_{(0,A),(0,A)}\Bigg[\sum_{k'\in\mathbb{G}} a_{T(u)}(i,k')\,1_{\mathcal{E}(u)}\,a_{T'(u)}(j,k')\,1_{\mathcal{E}'(u)}\Bigg] = 0,
\end{aligned}
\tag{3.127}
$$

where the last equality follows from the assumption $I_{\hat{a}}<\infty$ in Theorem 2.3.1, (3.60) and (3.79). The last inequality follows from the Markov property and the observation that, in order to get a contribution to the integral, the two random walks first have to meet at the same site and both be active. We conclude that $\mu S_t\in\mathcal{R}_\theta^{(1)}$ for all $t\geq 0$.

To show that $\mu(\infty)\in\mathcal{R}_\theta^{(1)}$, we proceed like in (3.124), to obtain

$$
\lim_{s\to\infty}\mathbb{E}_{\mu(\infty)}[z_{(i,R_i)}(s)] = \lim_{s\to\infty}\lim_{n\to\infty}\mathbb{E}_{\mu S_{t_n}}[z_{(i,R_i)}(s)] = \lim_{s\to\infty}\lim_{n\to\infty}\mathbb{E}_{\mu}[z_{(i,R_i)}(t_n+s)] = \theta,
\tag{3.128}
$$

and so (3.106) is satisfied. To get (3.107), we note that, by Lemma 3.2.1,

$$\begin{aligned}
&\sum_{(k,R_k),(l,R_l)\in\mathbb{G}\times\{A,D\}} b^{(1)}_{t_n}\big((i,R_i),(k,R_k)\big)\, b^{(1)}_{t_n}\big((j,R_j),(l,R_l)\big)\, \mathbb{E}_\mu[z_{(k,R_k)}z_{(l,R_l)}] \\
&\le E_\mu[z_{(i,R_i)}(t_n)z_{(j,R_j)}(t_n)] \\
&\le \sum_{(k,R_k),(l,R_l)\in\mathbb{G}\times\{A,D\}} b^{(1)}_{t_n}\big((i,R_i),(k,R_k)\big)\, b^{(1)}_{t_n}\big((j,R_j),(l,R_l)\big)\, \mathbb{E}_\mu[z_{(k,R_k)}z_{(l,R_l)}] \\
&\qquad + \|g\| \int_0^{t_n} \mathrm{d}s \sum_{k\in\mathbb{G}} b^{(1)}_{t_n-s}\big((i,R_i),(k,A)\big)\, b^{(1)}_{t_n-s}\big((j,R_j),(k,A)\big).
\end{aligned} \tag{3.129}$$

Letting $n\to\infty$, we see that, since $\mu\in\mathcal{R}^{(1)}_\theta$,

$$\begin{aligned}
\theta^2 &\le \mathbb{E}_{\mu(\infty)}[z_{(i,R_i)}z_{(j,R_j)}] \\
&\le \theta^2 + \|g\| \int_0^\infty \mathrm{d}s \sum_{k\in\mathbb{G}} b^{(1)}_r\big((i,R_i),(k,A)\big)\, b^{(1)}_r\big((j,R_j),(k,A)\big).
\end{aligned} \tag{3.130}$$

Inserting (3.130) into (3.107), we see that it is enough to show that

$$\begin{aligned}
&\lim_{s\to\infty} \sum_{(k,R_k),(l,R_l)\in\mathbb{G}\times\{A,D\}} b^{(1)}_s\big((i,R_i),(k,R_k)\big)\, b^{(1)}_s\big((j,R_j),(l,R_l)\big) \\
&\qquad\times\ \|g\| \int_0^\infty \mathrm{d}r \sum_{k'\in\mathbb{G}} b^{(1)}_r((k,R_k),(k',A))\, b^{(1)}_r((l,R_l),(k',A)) \\
&= \lim_{s\to\infty} \|g\| \int_0^\infty \mathrm{d}r \sum_{k'\in\mathbb{G}} b^{(1)}_{r+s}\big((i,R_i),(k',A)\big)\, b^{(1)}_{r+s}\big((j,R_j),(k',A)\big) = 0.
\end{aligned} \tag{3.131}$$

However, from the assumption $I_{\hat a}<\infty$ in Theorem 2.3.1, (3.60) and (3.79), we have

$$\begin{aligned}
&\lim_{s\to\infty} \|g\| \int_0^\infty \mathrm{d}r \sum_{k'\in\mathbb{G}} b^{(1)}_{r+s}\big((i,R_i),(k',A)\big)\, b^{(1)}_{r+s}\big((j,R_j),(k',A)\big) \\
&= \lim_{s\to\infty} \|g\| \int_s^\infty \mathrm{d}r\ \mathbb{E}_{(i,R_i),(j,R_j)}\left[\sum_{k'\in\mathbb{G}} a_{T(r)}(i,k')1_{\mathcal{E}(r)} a_{T'(r)}(j,k')1_{\mathcal{E}'(r)}\right] = 0.
\end{aligned} \tag{3.132}$$

□

2. Uniqueness of the equilibrium. In this section we show that, for given θ, the equilibrium when it exists is unique. To prove this we extend the coupling argument in [14]. Consider two copies of the system (2.4)–(2.5) *coupled via their Brownian motions*:

$$\begin{aligned}
\mathrm{d}x^k_i(t) &= \sum_{j\in\mathbb{G}} a(i,j)\,[x^k_j(t)-x^k_i(t)]\,\mathrm{d}t + \sqrt{g(x^k_i(t))}\,\mathrm{d}w_i(t) +\ Ke\,[y^k_i(t)-x^k_i(t)]\,\mathrm{d}t, \\
\mathrm{d}y^k_i(t) &= e\,[x^k_i(t)-y^k_i(t)]\,\mathrm{d}t, \qquad k\in\{1,2\}.
\end{aligned} \tag{3.133}$$

Here, k labels the copy, and the two copies are driven by the same set of Brownian motions $(w_i(t))_{t\geq 0}$, $i \in \mathbb{G}$. As initial probability distributions we choose $\mu^1(0)$ and $\mu^2(0)$ that are both invariant and ergodic under translations.

Let

$$\bar{z}_i(t) = (z_i^1(t), z_i^2(t)), \qquad z_i^k(t) = (x_i^k(t), y_i^k(t)), \quad k \in \{1, 2\}. \tag{3.134}$$

The coupled system $(\bar{z}_i(t))_{i\in\mathbb{G}}$ has a unique strong solution [67, Theorem 3.2] whose marginals are the single-component systems. Write $\hat{\mathbb{P}}$ to denote the law of the coupled system, and let $\Delta_i(t) = x_i^1(t) - x_i^2(t)$ and $\delta_i(t) = y_i^1(t) - y_i^2(t)$.

Lemma 3.2.9 (Coupling dynamics). *For every $t \geq 0$,*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,\hat{\mathbb{E}}\big[|\Delta_i(t)| + K|\delta_i(t)|\big] = &-2\sum_{j\in\mathbb{G}} a(i,j)\,\hat{\mathbb{E}}\left[|\Delta_j(t)|\,1_{\{\mathrm{sgn}\,\Delta_i(t)\neq \mathrm{sgn}\,\Delta_j(t)\}}\right]\\ &- 2Ke\,\hat{\mathbb{E}}\left[\big(|\Delta_i(t)| + |\delta_i(t)|\big)\,1_{\{\mathrm{sgn}\,\Delta_i(t)\neq \mathrm{sgn}\,\delta_i(t)\}}\right].\end{aligned} \tag{3.135}$$

Proof. Let $f(x) = |x|$, $x \in \mathbb{R}$. Then $f'(x) = \mathrm{sgn}\,x$ and $f''(x) = 0$ for $x \neq 0$, but f is not differentiable at $x = 0$, a point the path hits. Therefore, by a generalization of Itô's formula, we have

$$\begin{aligned}\mathrm{d}|\Delta_i(t)| &= \mathrm{sgn}\,\Delta_i(t)\,\mathrm{d}\Delta_i(t) + \mathrm{d}L_t^0,\\ \mathrm{d}\Delta_i(t) &= \sum_{j\in\mathbb{G}} a(i,j)[\Delta_j(t) - \Delta_i(t)]\,\mathrm{d}t + \left[\sqrt{g(x_i^1(t))} - \sqrt{g(x_i^2(t))}\right]\mathrm{d}w_i(t)\\ &\qquad + Ke\left[\delta_i(t) - \Delta_i(t)\right]\mathrm{d}t,\end{aligned} \tag{3.136}$$

where L_t^0 is the local time of $\Delta_i(t)$ at 0 (see [63, Section IV.43]). Next, we use that $\Delta_i(t)$ has zero local time at $x = 0$ because g is Lipschitz (see [63, Proposition V.39.3]). Taking expectation, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\hat{\mathbb{E}}\big[|\Delta_i(t)|\big] = \sum_{j\in\mathbb{G}} a(i,j)\,\hat{\mathbb{E}}\Big[\mathrm{sgn}\,\Delta_i(t)\,[\Delta_j(t)-\Delta_i(t)]\Big] + Ke\,\hat{\mathbb{E}}\Big[\mathrm{sgn}\,\Delta_i(t)\,[\delta_i(t)-\Delta_i(t)]\Big]. \tag{3.137}$$

Similarly, we have

$$\begin{aligned}\mathrm{d}|\delta_i(t)| &= \mathrm{sgn}\,\delta_i(t)\,\mathrm{d}\delta_i(t),\\ \mathrm{d}\delta_i(t) &= e\left[\Delta_i(t) - \delta_i(t)\right]\mathrm{d}t.\end{aligned} \tag{3.138}$$

Taking expectation, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\hat{\mathbb{E}}\big[|\delta_i(t)|\big] = e\,\hat{\mathbb{E}}\Big[\mathrm{sgn}\,\delta_i(t)\,[\Delta_i(t) - \delta_i(t)]\Big]. \tag{3.139}$$

Combining (3.137) and (3.139), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,\hat{\mathbb{E}}\big[|\Delta_i(t)| + K|\delta_i(t)|\big] = &\sum_{j\in\mathbb{G}} a(i,j)\,\hat{\mathbb{E}}\Big[\mathrm{sgn}\,\Delta_i(t)\,[\Delta_j(t) - \Delta_i(t)]\Big]\\ &+ K\,e\,\hat{\mathbb{E}}\Big[[\mathrm{sgn}\,\Delta_i(t) - \mathrm{sgn}\,\delta_i(t)]\,[\delta_i(t) - \Delta_i(t)]\Big].\end{aligned} \tag{3.140}$$

Note that

$$\operatorname{sgn}\Delta_i(t)\,[\Delta_j(t)-\Delta_i(t)] = |\Delta_j(t)| - |\Delta_i(t)| - 2\,|\Delta_j(t)|\,1_{\{\operatorname{sgn}\Delta_i(t)\neq \operatorname{sgn}\Delta_j(t)\}}. \tag{3.141}$$

By translation invariance, $\mathbb{E}[|\Delta_i(t)|]$ is independent of i. Hence the first sum in the right-hand side can be rewritten as

$$-2\sum_{j\in\mathbb{G}} a(i,j)\,\hat{\mathbb{E}}\Big[|\Delta_j(t)|\,1_{\{\operatorname{sgn}\Delta_i(t)\neq \operatorname{sgn}\Delta_j(t)\}}\Big]. \tag{3.142}$$

Similarly, the second sum in the right-hand side can be rewritten as

$$-2Ke\,\hat{\mathbb{E}}\left[\big(|\Delta_i(t)| + |\delta_i(t)|\big)\,1_{\{\operatorname{sgn}\Delta_i(t)\,\neq\,\operatorname{sgn}\delta_i(t)\}}\right]. \tag{3.143}$$

Combining (3.140) and (3.142)–(3.143), we get the claim. □

Lemma 3.2.9 tells us that $t\mapsto \hat{\mathbb{E}}[|\Delta_i(t)| + K|\delta_i(t)|]$ is a non-increasing Lyapunov function. Therefore $\lim_{t\to\infty}\hat{\mathbb{E}}[|\Delta_i(t)| + K|\delta_i(t)|] = c_i \in [0, 1+K]$ exists. To show that the coupling is successful we need the following lemma.

Lemma 3.2.10 (Uniqueness of equilibrium). *If $a(\cdot,\cdot)$ is transient, then $c_i = 0$ for all $i\in\mathbb{G}$, and so the coupling is successful, i.e.,*

$$\lim_{t\to\infty}\hat{\mathbb{E}}\big[|\Delta_i(t)| + K|\delta_i(t)|\big] = 0. \tag{3.144}$$

Proof. Write $-h_i(t)$ to denote the right-hand side of (3.135). We begin with the observation that $t\mapsto h_i(t)$ has the following properties:

(a) $h_i \geq 0$.
(b) $0 \leq \int_0^\infty \mathrm{d}t\, h_i(t) \leq 1+K$.
(c) h_i is differentiable with h_i' bounded.

Property (a) is evident. Property (b) follows from integration of (3.135):

$$\int_0^t \mathrm{d}s\, h_i(s) = \hat{\mathbb{E}}\big[|\Delta_i(0)| + K|\delta_i(0)|\big] - \hat{\mathbb{E}}\big[|\Delta_i(t)| + K|\delta_i(t)|\big]. \tag{3.145}$$

The proof of Property (c) is given in Appendix A.4. It follows from (a)–(c) that $\lim_{t\to\infty} h(t) = 0$. Hence, for every $\epsilon > 0$,

$$\begin{aligned}
&\forall\, i,j\in\mathbb{G} \text{ with } a(i,j)>0:\\
&\qquad \lim_{t\to\infty}\hat{\mathbb{P}}\Big(\{\Delta_i(t) < -\epsilon,\, \Delta_j(t) > \epsilon\} \cup \{\Delta_i(t) > \epsilon,\, \Delta_j(t) < -\epsilon\}\Big) = 0,\\
&\forall\, i\in\mathbb{G}:\\
&\qquad \lim_{t\to\infty}\hat{\mathbb{P}}\Big(\{\Delta_i(t) < -\epsilon,\, \delta_i(t) > \epsilon\} \cup \{\Delta_i(t) > \epsilon,\, \delta_i(t) < -\epsilon\}\Big) = 0.
\end{aligned} \tag{3.146}$$

In Appendix A.3 we will prove the following lemma:

Lemma 3.2.11 (Successful coupling). *For all $i,j\in\mathbb{G}$ and $\epsilon > 0$,*

$$\lim_{t\to\infty}\hat{\mathbb{P}}\Big(\{\Delta_i(t) < -\epsilon,\, \Delta_j(t) > \epsilon\} \cup \{\Delta_i(t) > \epsilon,\, \Delta_j(t) < -\epsilon\}\Big) = 0. \tag{3.147}$$

The proof of this lemma relies on the fact that $\hat{a}(\cdot,\cdot)$ is irreducible. Let

$$\begin{aligned} E_0 \times E_0 = &\Big\{\bar{z} \in E \times E\colon\ z^1_{(i,R_i)}(t) \geq z^2_{(i,R_i)}(t)\ \forall (i,R_i) \in \mathbb{G} \times \{A,D\}\Big\} \\ &\cup \Big\{\bar{z} \in E \times E\colon\ z^2_{(i,R_i)}(t) \geq z^1_{(i,R_i)}(t)\ \forall (i,R_i) \in \mathbb{G} \times \{A,D\}\Big\}. \end{aligned} \tag{3.148}$$

Then Lemma 3.2.11 together with (3.146) imply that $\lim_{t\to\infty} \hat{\mathbb{P}}\,(E_0 \times E_0) = 1$, which we express by saying that "one diffusion lies on top of the other".

Using Lemma 3.2.11 we can complete the proof of the successful coupling. Let $t_n \to \infty$ as $n \to \infty$ and suppose, by possibly going to further subsequences, that $\lim_{n\to\infty} \mu^1(t_n) = \nu^1_\theta$ and $\lim_{n\to\infty} \mu^2(t_n) = \nu^2_\theta$. Let $\bar{\nu}_\theta$ be the measure on $E \times E$ given by $\bar{\nu}_\theta = \nu^1_\theta \times \nu^2_\theta$. Using dominated convergence, invoking the preservation of translation invariance, and using the limiting distribution of $b^{(1)}_t(\cdot,\cdot)$ on $\{A,D\}$, we find

$$\begin{aligned} &\int_{E\times E} \mathrm{d}\bar{\nu}_\theta\, |\Delta_i| + K|\delta_i| \\ &= (1+K) \int_{E_0\times E_0} \mathrm{d}\bar{\nu}_\theta \lim_{n\to\infty} \sum_{j\in\mathbb{G}} \Big[b^{(1)}_{t_n}\big((i,R_i),(j,A)\big)\, |x^1_i - x^2_i| \\ &\qquad\qquad + b^{(1)}_{t_n}\big((i,R_i),(j,D)\big)\, |y^1_i - y^2_i| \Big] \\ &= \lim_{n\to\infty} (1+K) \int_{E_0\times E_0} \mathrm{d}\bar{\nu}_\theta \left| \sum_{j\in\mathbb{G}\times\{A,D\}} b^{(1)}_{t_n}\big((i,R_i),(j,R_j)\big)\, (z^1_{(j,R_j)} - z^2_{(j,R_j)}) \right| \\ &\leq \lim_{n\to\infty} (1+K) \int_E \mathrm{d}\nu^1_\theta \left| \sum_{j\in\mathbb{G}\times\{A,D\}} b^{(1)}_{t_n}\big((i,R_i),(j,R_j)\big)\, z^1_{(j,R_j)} - \theta \right| \\ &\quad + \lim_{n\to\infty} (1+K) \int_E \mathrm{d}\nu^2_\theta \left| \sum_{i\in\mathbb{G}\times\{A,D\}} b^{(1)}_{t_n}\big((i,R_i),(j,R_j)\big)\, z^2_{(j,R_j)} - \theta \right| = 0. \end{aligned} \tag{3.149}$$

Here, the last equality follows because both ν^1_θ and ν^2_θ are in $\mathcal{R}^{(1)}_\theta$ by Lemma 3.2.8. Thus, we see that $\bar{\nu}_\theta$ concentrates on the diagonal. Suppose now that there exists a sequence $(t_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \mathbb{E}[|\Delta_i(t_n)| + K|\delta_i(t_n)|] = \delta > 0$. Since $\{\mathcal{L}(\bar{Z}(t_n))\}_{n\in\mathbb{N}}$ is tight (recall (3.134)), by Prokhorov's theorem there exists a converging subsequence $\{\mathcal{L}(\bar{Z}(t_{n_k}))\}_{k\in\mathbb{N}}$. Let $\bar{\nu}_\theta$ denote the limiting measure. Then, by Lemma 3.2.8 and (3.149),

$$\delta = \lim_{k\to\infty} \mathbb{E}[|\Delta_i(t_{n_k})| + K|\delta_i(t_{n_k})|] = \int_{E\times E} \mathrm{d}\bar{\nu}_\theta\, [|\Delta_i| + K|\delta_i|] = 0. \tag{3.150}$$

Thus, $\lim_{t\to\infty} \mathbb{E}[|\Delta_i(t)|+K|\delta_i(t)|] = 0$, and we conclude that the coupling is successful. Hence, given the initial average density θ in (2.62), the equilibrium measure is unique if it exists. □

3. Stationarity of ν_θ and convergence to ν_θ.

Lemma 3.2.12 (Existence of equilibrium). *Let* $\mu(0) \in \mathcal{T}_\theta^{\mathrm{erg}}$*. Then* $\lim_{t\to\infty} \mu(t) = \nu_\theta$ *for some invariant measure* ν_θ*.*

Proof. To prove that the limit is an invariant measure, suppose that $\mu(0) = \mu = \delta_\theta$. Since the state space of $(Z(t))_{t\geq 0}$ is compact, each sequence $\{\mathcal{L}(Z(t_n))\}_{n\in\mathbb{N}}$ is tight. Hence, by Prokhorov's theorem, there exists a converging subsequence such that $\lim_{n\to\infty} \delta_\theta S_{t_n} = \nu_\theta$. Since $\delta_\theta \in \mathcal{R}_\theta^{(1)}$, Lemma 3.2.8 tells us that $\lim_{n\to\infty} \delta_\theta S_{t_n} \in \mathcal{R}_\theta^{(1)}$. To prove that ν_θ is invariant, fix any $s_0 \geq 0$. Let $\mu = \delta_\theta S_{s_0}$. Then, by Lemma 3.2.8, $\mu \in \mathcal{R}_\theta^{(1)}$ and, by Lemma 3.2.11, we can find a further subsequence such that $\lim_{k\to\infty} \mu(t_{n_k}) = \nu_\theta$. By the Feller property of the SSDE in (2.4)–(2.5), we obtain

$$\nu_\theta S_{s_0} = \lim_{n\to\infty} \delta_\theta S_{t_n} S_{s_0} = \lim_{k\to\infty} \delta_\theta S_{s_0} S_{t_{n_k}} = \lim_{k\to\infty} \mu S_{t_{n_k}} = \nu_\theta. \tag{3.151}$$

Hence, ν_θ is indeed an invariant measure.

To prove the convergence of $\mu(t)$ to ν_θ, note that $\nu_\theta \in \mathcal{R}_\theta^{(1)}$ by Lemma 3.2.8. Let $\nu = \nu_\theta$. Then, by the invariance of ν_θ, we have $\lim_{t\to\infty} \nu S_t = \nu_\theta$. By Lemma 3.2.10, we have $\lim_{t\to\infty} \mu S_t = \lim_{t\to\infty} \nu S_t = \nu_\theta$ for all $\mu \in \mathcal{R}_\theta^{(1)}$. □

4. Ergodicity, mixing and associatedness.

Lemma 3.2.13 (Properties of equilibrium). *Let* $\mu(0) \in \mathcal{R}_\theta^{(1)}$ *be ergodic under translations. Then* $\nu_\theta = \lim_{t\to\infty} \mu(t)$ *is ergodic and mixing under translations, and is associated.*

Proof. After a standard approximation argument, [46, Corollary 1.5 and subsequent discussion] implies that associatedness is preserved over time. Note that δ_θ is an associated measure and lies in $\mathcal{R}_\theta^{(1)}$. Hence, by Lemma 3.2.12, $\nu_\theta = \lim_{t\to\infty} \delta_\theta S_t$ and therefore ν_θ is associated.

We prove the ergodicity of ν_θ by showing that the random field of components is mixing. To prove that ν_θ is mixing, we use associatedness and decay of correlations. Let $B, B' \subset \mathbb{G}$ be finite, and let c_j, d_i be positive constants for $j \in B$, $i \in B'$. For $k \in \mathbb{G}$, define the random variables

$$Y_0 = \sum_{j\in B} c_j z_{(j,R_j)}, \qquad Y_k = \sum_{i\in B'} d_i z_{(i+k,R_{i+k})}. \tag{3.152}$$

Note that Y_0 and Y_k are associated under ν_θ because $(z_{(i,R_i)})_{(i,R_i)\in\mathbb{G}\times\{A,D\}}$ are associated. Therefore, by [61, Eq.(2.2)], it follows that for $s, t \in \mathbb{R}$,

$$\left|\mathbb{E}_{\nu_\theta}[e^{\mathrm{i}(sY_0+tY_n)}] - \mathbb{E}_{\nu_\theta}[e^{\mathrm{i}sY_0}]\,\mathbb{E}_{\nu_\theta}[e^{\mathrm{i}tY_n}]\right| \leq |st| \operatorname{Cov}_{\nu_\theta}(Y_0, Y_n). \tag{3.153}$$

Since $\mu \in \mathcal{R}_\theta^{(1)}$ by Lemma 3.2.1,

$$\begin{aligned}\mathrm{Cov}_{\nu_\theta}(Y_0,Y_k) &= \sum_{j\in B}\sum_{i\in B'} c_j d_i \lim_{t\to\infty}\mathrm{Cov}_\mu(z_{(j,R_j)}(t), z_{(i+k,R_{i+k})}(t))\\ &\leq \|g\| \sum_{j\in B}\sum_{i\in B'} c_j d_i\\ &\quad\times \int_0^\infty \mathrm{d}r \sum_{(l,R_l)\in\mathbb{G}\times\{A\}} b_r^{(1)}\big((j,R_j),(l,A)\big)\, b_r^{(1)}\big((i+k,R_{i+k}),(l,A)\big).\end{aligned} \tag{3.154}$$

The last integral gives the expected total time for two partition elements in the dual, starting in (j,R_j) and $(i+k,R_{i+k})$, to be active at the same site. To show that this integral converges to 0 as $\|k\|\to\infty$, we rewrite the sum as (recall (3.64)–(3.65))

$$\begin{aligned}&\mathbb{E}_{(i+k,R_{i+k}),(j,R_j)}\left[\left(\sum_{l\in\mathbb{G}} a_{T(r)}(j,l)\, a_{T'(r)}(i+k,l)\right) 1_{\mathcal{E}(r)}\, 1_{\mathcal{E}'(r)}\right]\\ &= \mathbb{E}_{(i+k,R_{i+k}),(j,R_j)}\left[\left(\sum_{l'\in\mathbb{G}} \hat{a}_{2M(r)}(i+k-j,l')\, a_{\Delta(r)}(l',0)\right) 1_{\mathcal{E}(r)}\, 1_{\mathcal{E}'(r)}\right]\\ &\leq \mathbb{E}_{(i+k,R_{i+k}),(j,R_j)}\left[\left(\sum_{l'\in\mathbb{G}} \hat{a}_{2M(r)}(i+k-j,l')\left[a_{\Delta(r)}(l',0)+a_{\Delta(r)}(0,l')\right]\right) 1_{\mathcal{E}(r)}\, 1_{\mathcal{E}'(r)}\right]\\ &= \mathbb{E}_{(i+k,R_{i+k}),(j,R_j)}\left[\hat{a}_{2M(r)+2\Delta(r)}(i+k-j,0) 1_{\mathcal{E}(r)}\, 1_{\mathcal{E}'(r)}\right].\end{aligned} \tag{3.155}$$

Because $\hat{a}(\cdot,\cdot)$ is symmetric, we have $\hat{a}_{2M(r)+2\Delta(r)}(i+k-j,0) \leq \hat{a}_{2M(r)+2\Delta(r)}(0,0)$. Since

$$T(t)+T'(t) \leq 2M(r)+2\Delta(r) \leq 2\left(T(t)+T'(t)\right), \tag{3.156}$$

and the Fourier transform in (3.74)–(3.75) implies that

$$\int_0^\infty \mathrm{d}r\, \mathbb{E}_{(i+k,R_{i+k}),(j,R_j)}\left[\hat{a}_{2M(r)+2\Delta(r)}(0,0) 1_{\mathcal{E}(r)}\, 1_{\mathcal{E}'(r)}\right] < \infty. \tag{3.157}$$

if and only if $I_{\hat{a}}<\infty$. Since we are in the transient regime, i.e., $I_{\hat{a}}<\infty$, we can use dominated convergence, in combination with the fact that $\lim_{\|k\|\to\infty}\hat{a}_t(i+k-j,0)=0$ for all i,j,t, to conclude that $\lim_{\|k\|\to\infty}\mathrm{Cov}_{\nu_\theta}(Y_0,Y_k)=0$. □

§3.2.4 Proof of the dichotomy

Theorem 2.3.1(a) follows from Lemmas 3.2.7, 3.2.12 and 3.2.13. The equality $\mathbb{E}_{\nu_\theta}[x_0] = \mathbb{E}_{\nu_\theta}[y_0]=\theta$ follows from the evolution equations in (2.4)–(2.5), the fact that ν_θ is an equilibrium measure, and the preservation of θ (see (2.63)). Theorem 2.3.1(b) follows from Lemma 3.2.5.

§3.3 Proofs: Long-time behaviour for Model 2

In Sections 3.3.1–3.3.4 we show that the results proved in Sections 3.2.1–3.2.4 carry over from model 1 to model 2. In Section 3.3.5 we show that symmetry of $a(\cdot,\cdot)$

is needed. In Section 3.3.6 we show what happens when for infinite seed-bank the fat-tailed wake-up time is modulated by a slowly varying function.

§3.3.1 Moment relations

Like in model 1, we start by relating the first and second moments of the system in (2.12)–(2.13) to the random walk that evolves according to the transition kernel $b^{(2)}(\cdot,\cdot)$ on $\mathbb{G}\times\{A,(D_m)_{m\in\mathbb{N}_0}\}$ given by (2.41). Also here these moment relations hold for all $g\in\mathcal{G}$. Moreover these moment relations holds for $\rho<\infty$ as well as for $\rho=\infty$. Below we write $\mathbb{E}_z$ for $\mathbb{E}_{\delta_z}$, the expectation when the process starts from the initial measure δ_z, $z\in E$.

Lemma 3.3.1 (First and second moment). *For $z\in E$, $t\geq 0$ and $(i,R_i),(j,R_j)\in\mathbb{G}\times\{A,(D_m)_{m\in\mathbb{N}_0}\}$,*

$$\mathbb{E}_z[z_{(i,R_i)}(t)]=\sum_{\substack{(k,R_k)\\ \in\mathbb{G}\times\{A,(D_m)_{m\in\mathbb{N}_0}\}}} b^{(2)}_t\big((i,R_i),(k,R_k)\big)\,z_{(k,R_k)} \tag{3.158}$$

and

$$\begin{aligned}&\mathbb{E}_z[z_{(i,R_i)}(t)z_{(j,R_j)}(t)]\\ &=\sum_{\substack{(k,R_k),(l,R_l)\\ \in\mathbb{G}\times\{A,(D_m)_{m\in\mathbb{N}_0}\}}} b^{(2)}_t\big((i,R_i),(k,R_k)\big)\,b^{(2)}_t\big((j,R_j),(l,R_l)\big)\,z_{(k,R_k)}z_{(l,R_l)}\\ &\quad+\int_0^t \mathrm{d}s\sum_{k\in\mathbb{G}} b^{(2)}_{(t-s)}((i,R_i),(k,A))\,b^{(2)}_{(t-s)}((j,R_j),(k,A))\,\mathbb{E}_z[g(x_k(s))].\end{aligned} \tag{3.159}$$

Proof. The proof follows from that of Lemma 3.2.1 after we replace $b^{(1)}(\cdot,\cdot)$ by $b^{(2)}(\cdot,\cdot)$ and use (2.12)–(2.13) instead of (2.4)–(2.5). □

Remark 3.3.2 (Density). From Lemma 3.3.1 we obtain that if μ is invariant under translations with $\mathbb{E}_\mu[x_0(0)]=\theta_x$ and $\mathbb{E}_\mu[y_{0,m}(0)]=\theta_{y,m}$ for all $m\in\mathbb{N}_0$, then

$$\begin{aligned}\mathbb{E}_\mu[z_{(i,R_i)}(t)]=&\,\theta_x\sum_{(k,R_k)\in\mathbb{G}\times\{A\}} b^{(2)}_t\big((i,R_i),(k,R_k)\big)\\ &+\sum_{m\in\mathbb{N}_0}\theta_{y,m}\sum_{(k,R_k)\in\mathbb{G}\times\{D_m\}} b^{(2)}_t\big((i,R_i),(k,R_k)\big)\end{aligned} \tag{3.160}$$

and

$$\begin{aligned}&\mathbb{E}_\mu[z_{(i,R_i)}(t)z_{(j,R_j)}(t)]\\ &=\sum_{\substack{(k,R_k),(l,R_l)\\ \in\mathbb{G}\times\{A,(D_m)_{m\in\mathbb{N}_0}\}}} b^{(2)}_t\big((i,R_i),(k,R_k)\big)\,b^{(2)}_t\big((j,R_j),(l,R_l)\big)\,\mathbb{E}_\mu[z_{(k,R_k)}z_{(l,R_l)}]\\ &\quad+\int_0^t \mathrm{d}s\sum_{k\in\mathbb{G}} b^{(2)}_{t-s}\big((i,R_i),(k,A)\big)\,b^{(2)}_{t-s}\big((j,R_j),(k,A)\big)\,\mathbb{E}_\mu[g(x_i(s))].\end{aligned} \tag{3.161}$$

- For $\rho < \infty$, the kernel $b^{(2)}(\cdot,\cdot)$ projected on the second component (= the seed-bank) corresponds to recurrent Markov chain. Therefore, by translation invariance in the first component, we have

$$\lim_{t\to\infty} \mathbb{E}_\mu[z_{(i,R_i)}(t)] = \frac{\theta_x + \sum_{m\in\mathbb{N}_0} K_m\theta_{y,m}}{1+\sum_{m\in\mathbb{N}_0} K_m} = \theta. \tag{3.162}$$

- For $\rho = \infty$, the kernel $b^{(2)}(\cdot,\cdot)$ viewed as a kernel on $\{A,(D_m)_{m\in\mathbb{N}_0}\}$ corresponds to a null-recurrent Markov chain. Hence, for all (i,R_i) and all D_m, $m\in\mathbb{N}_0$,

$$\lim_{t\to\infty}\sum_{k\in\mathbb{G}} b_t^{(2)}((i,R_i),(k,D_m)) = 0. \tag{3.163}$$

Since for $\rho=\infty$ we assume not only that $\mu\in\mathcal{T}_\theta^{\mathrm{erg}}$ but also that μ is colour regular, it follows that, for all $M\in\mathbb{N}_0$,

$$\begin{aligned}\lim_{t\to\infty}\mathbb{E}_\mu[z_{(i,R_i)}(t)] &= \lim_{t\to\infty}\theta_x\sum_{k\in\mathbb{G}} b_t^{(2)}\big((i,R_i),(k,R_k)\big)\\ &\quad+\sum_{m\in\mathbb{N}_0}\theta_{y,m}\sum_{(k,R_k)\in\mathbb{G}\times\{D_m\}} b_t^{(2)}\big((i,R_i),(k,R_k)\big)\\ &=\lim_{t\to\infty}\sum_{m=M}^{\infty}\theta_{y,m}\sum_{(k,R_k)\in\mathbb{G}\times\{D_m\}} b_t^{(2)}\big((i,R_i),(k,R_k)\big).\end{aligned} \tag{3.164}$$

Therefore

$$\lim_{t\to\infty}\mathbb{E}_\mu[z_{(i,R_i)}(t)] = \theta. \tag{3.165}$$

□

§3.3.2 The clustering case

In this section we prove convergence to a trivial equilibrium when $\rho<\infty$ and $I_{\hat a}=\infty$ and when $\rho=\infty$ and $I_{\hat a,\gamma}=\infty$. The proof follows along the same lines as in Section 3.2.2. Therefore we again first consider $g = dg_{\mathrm{FW}}$, and subsequently use a duality comparison argument to show that the results hold for $g\neq dg_{\mathrm{FW}}$ as well.

Case $g = dg_{\mathrm{FW}}$. We start by proving the equivalent of Lemma 3.2.4, which is Lemma 3.3.3 below.

Lemma 3.3.3 (Clustering). *Suppose that $\mu(0)\in\mathcal{T}_\theta^{\mathrm{erg}}$ and $g = dg_{\mathrm{FW}}$. Let $\mu(t)$ be the law at time t of the system defined in* (2.12)–(2.13)*. Then the following two statements hold:*

- *If $\rho<\infty$ and $I_{\hat a}=\infty$, i.e., $\hat a(\cdot,\cdot)$ is recurrent, then*

$$\lim_{t\to\infty}\mu(t) = \theta\,[\delta_{(1,1^{\mathbb{N}_0})}]^{\otimes\mathbb{G}} + (1-\theta)\,[\delta_{(0,0^{\mathbb{N}_0})}]^{\otimes\mathbb{G}}. \tag{3.166}$$

- *If $\rho = \infty$ and $I_{\hat{a},\gamma} = \infty$ then*

$$\lim_{t\to\infty} \mu(t) = \theta\, [\delta_{(1,1^{\mathbb{N}_0})}]^{\otimes\mathbb{G}} + (1-\theta)\, [\delta_{(0,0^{\mathbb{N}_0})}]^{\otimes\mathbb{G}}. \tag{3.167}$$

Proof. We distinguish between $\rho < \infty$ and $\rho = \infty$, which exhibit different behaviour.

Case $\rho < \infty$. The same dichotomy as for model 1 holds when the average wake-up time is finite (recall (2.20)–(2.21), (2.50)). Indeed, the argument in (3.69)–(3.79) can be copied with Ke, e replaced by $\chi, \chi/\rho$ and A, B by $\chi/(1+\rho), 1/(1+\rho)$. Under the *symmetry assumption* in (2.73) we have $\tilde{a}(\phi) = 0$. Hence only the law of large numbers in (3.70) is needed, not the central limit theorem in (3.72), which may fail (see Section 3.3.5).

Case $\rho = \infty$. When the average wake-up time is infinite, we need the assumptions in (2.60) and (2.76). By the standard law of large numbers for stable random variables (see e.g. [34, Section XIII.6]), we have

$$\lim_{k\to\infty} \frac{1}{k}\sum_{\ell=1}^{k} \sigma_\ell = \frac{1}{\chi} \quad \mathbb{P}\text{-a.s.}, \qquad \lim_{k\to\infty} \frac{1}{k^{1/\gamma}}\sum_{\ell=1}^{k} \tau_\ell = W \quad \text{in } \mathbb{P}\text{-probability}, \tag{3.168}$$

with W a stable law random variable on $(0,\infty)$ with exponent γ. Therefore

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{t^\gamma} N(t) = \lim_{t\to\infty} \frac{1}{t^\gamma} N'(t) = W^{-\gamma} \quad \text{in } \mathbb{P}\text{-probability}, \\ &\lim_{t\to\infty} \frac{1}{t^\gamma} T(t) = \lim_{t\to\infty} \frac{1}{t^\gamma} T'(t) = \frac{1}{\chi} W^{-\gamma} \quad \text{in } \mathbb{P}\text{-probability}, \\ &\lim_{t\to\infty} t^{1-\gamma}\, \mathbb{P}\big(\mathcal{E}(t)\big) = \lim_{t\to\infty} t^{1-\gamma}\, \mathbb{P}\big(\mathcal{E}'(t)\big) = \frac{1}{\chi}\, \mathbb{E}[W^{-\gamma}], \qquad t\to\infty. \end{aligned} \tag{3.169}$$

For the last statement to make sense, we must check the following.

Lemma 3.3.4 (Finite limits). $\mathbb{E}[W^{-\gamma}] < \infty$.

Proof. Let $W_k = k^{-1/\gamma}\sum_{l=1}^{k} \tau_l$. Then $W_k^{-\gamma} \leq k(\max_{1\leq i\leq k} \tau_i^\gamma)^{-1}$ and, since τ_i are i.i.d. random variables,

$$\mathbb{E}[W_k^{-\gamma}] \leq \int_0^\infty \mathrm{d}x\, \mathbb{P}\left(k\left(\max_{1\leq i\leq k} \tau_i^\gamma\right)^{-1} > x\right) = \int_0^\infty \mathrm{d}x\, \mathbb{P}\left(\tau_1^\gamma < \tfrac{k}{x}\right)^k. \tag{3.170}$$

To estimate the integral in the right-hand side of (3.170), we introduce three constants, T, C_1, C_2. Let $\epsilon \in (0,1)$ and choose $T \in \mathbb{R}_+$ such that, for all $t > T$,

$$|[\mathbb{P}(\tau > t)/(Ct^{-\gamma})] - 1| < \epsilon \tag{3.171}$$

Since $\mathbb{P}(\tau \leq t) = 1 - \chi^{-1}\sum_{m\in\mathbb{N}_0} K_m e_m\, \mathrm{e}^{-e_m t}$, we note that, under assumption (2.76), τ admits a continuous bounded density. Hence there exists a $C_1 \in \mathbb{R}_+$ such that $\mathbb{P}(\tau \le t) < C_1 t$. Finally, choose $C_2 \in \mathbb{R}_+$ such that $C_2 > \max(1, C_1^\gamma)$. Split

$$\begin{aligned}\int_0^\infty \mathrm{d}x\,\mathbb{P}\left(\tau_1^\gamma < \tfrac{k}{x}\right)^k &= \int_0^{k/T} \mathrm{d}x\,\mathbb{P}\left(\tau_1^\gamma < \tfrac{k}{x}\right)^k \\ &\quad + \int_{k/T}^{kC_2} \mathrm{d}x\,\mathbb{P}\left(\tau_1^\gamma < \tfrac{k}{x}\right)^k + \int_{kC_2}^\infty \mathrm{d}x\,\mathbb{P}\left(\tau_1^\gamma < \tfrac{k}{x}\right)^k.\end{aligned} \tag{3.172}$$

We estimate each of the three integrals separately. For the first integral, we use the estimate $(1-\mathbb{P}(\tau_1^\gamma \ge \frac{k}{x}))^k \le \exp[-k\mathbb{P}(\tau_1^\gamma \ge \frac{k}{x})]$ to obtain

$$\begin{aligned}\int_0^{k/T} \mathrm{d}x\,\mathbb{P}\left(\tau_1^\gamma < \tfrac{k}{x}\right)^k &= \int_0^{k/T} \mathrm{d}x\,\exp\left[-k\,\mathbb{P}\left(\tau_1 \ge \left(\tfrac{k}{x}\right)^{1/\gamma}\right)\right] \\ &\le \int_0^{k/T} \mathrm{d}x\,\mathrm{e}^{-(1-\epsilon)Cx} \\ &\le \frac{1}{(1-\epsilon)C}.\end{aligned} \tag{3.173}$$

For the second integral, we note that $t \mapsto t\,\mathbb{P}(\tau_1^\gamma > t)$ is a continuous function on $[\frac{1}{C_2}, T]$, and hence attains a minimum value $C_3 \in \mathbb{R}_+$ on $[\frac{1}{C_2}, T]$. Therefore

$$\begin{aligned}\int_{k/T}^{kC_2} \mathrm{d}x\,\mathbb{P}\left(\tau_1^\gamma < \tfrac{k}{x}\right)^k &= \int_{k/T}^{kC_2} \mathrm{d}x\,\left[1-\mathbb{P}\left(\tau_1^\gamma \ge \tfrac{k}{x}\right)\right]^k \\ &\le \int_{k/T}^{kC_2} \mathrm{d}x\,\exp\left[-x\left(\frac{k}{x}\mathbb{P}(\tau_1^\gamma \ge \tfrac{k}{x})\right)\right] \\ &\le \frac{1}{C_3}.\end{aligned} \tag{3.174}$$

For the third integral, we compute

$$\int_{kC_2}^\infty \mathrm{d}x\,\mathbb{P}\left(\tau_1^\gamma < \tfrac{k}{x}\right)^k \le \int_{kC_2}^\infty \mathrm{d}x\,\left(C_1^\gamma \tfrac{k}{x}\right)^{\frac{k}{\gamma}} = \int_0^{1/C_2} \mathrm{d}v\,\tfrac{k}{v^2}(C_1^\gamma v)^{\frac{k}{\gamma}} = \frac{C_1^\gamma k}{\frac{k}{\gamma}-1}\left(\frac{C_1^\gamma}{C_2}\right)^{\frac{k}{\gamma}-1}, \tag{3.175}$$

where in the first equality we substitute $v = \frac{k}{x}$. Since $C_2 > C_1^\gamma$, we see that the right-hand side tends to zero as $k \to \infty$. Hence

$$\mathbb{E}[W_k^{-\gamma}] \le \frac{1}{(1-\epsilon)(C/\gamma)} + \frac{1}{C_3} + \frac{C_1^\gamma k}{\frac{k}{\gamma}-1}\left(\frac{C_1^\gamma}{C_2}\right)^{k-1}, \tag{3.176}$$

and by dominated convergence it follows that $\mathbb{E}[W^{-\gamma}] = \lim_{k\to\infty}\mathbb{E}[W_k^{-\gamma}] < \infty$. □

By (2.73), we have $\hat{a}(\phi) = a(\phi)$ and $\tilde{a}(\phi) = 0$ in (3.66), and so (3.74) becomes, with the help of (3.169),

$$\mathbb{E}_{(0,A),(0,A)}\left[\left(\sum_{i\in\mathbb{G}} a_{T(t)}(0,i)\,a_{T'(t)}(0,i)\right)1_{\mathcal{E}(t)}\,1_{\mathcal{E}'(t)}\right] \asymp t^{-2(1-\gamma)}f(t), \qquad t\to\infty, \tag{3.177}$$

with (recall (3.68))

$$f(t) = \hat{a}_{ct^\gamma}(0,0) \tag{3.178}$$

for some $c \in (0,\infty)$. Here we use that deviations of $T(t)/t^\gamma$ and $T'(t)/t^\gamma$ away from order 1 are stretched exponentially costly in t [31], and therefore are negligible. Since $t \mapsto \hat{a}_t(0,0)$ is regularly varying at infinity (recall (2.60)), it follows that

$$\hat{a}_{ct^\gamma}(0,0) \asymp \hat{a}_{t^\gamma}(0,0), \qquad t \to \infty. \tag{3.179}$$

Combining (3.63) and (3.177)–(3.179), we get

$$I = \infty \quad \Longleftrightarrow \quad I_{\hat{a},\gamma} = \infty \tag{3.180}$$

with $I_{\hat{a},\gamma} = \int_1^\infty \mathrm{d}t\, t^{-2(1-\gamma)}\, \hat{a}_{t^\gamma}(0,0)$. Putting $s = t^\gamma$, we have

$$I_{\hat{a},\gamma} = \int_1^\infty \mathrm{d}s\, s^{-(1-\gamma)/\gamma} \hat{a}_s(0,0), \tag{3.181}$$

which is precisely the integral defined in (2.80). □

Case $g \neq dg_{\mathrm{FW}}$. To prove that the dichotomy criterion of Lemma 3.3.3 holds for general $g \in \mathcal{G}$ we need the equivalent of Lemma 3.2.5. Replacing (2.4)–(2.5) by (2.12)–(2.13), replacing $b^{(1)}$ by $b^{(2)}$ in the proof of Lemma 3.2.5, and using the moment relations in Lemma 3.3.1 instead of the moment relations in Lemma 3.2.1, we see that Lemma 3.3.3 also holds for $g \in \mathcal{G}$.

§3.3.3 The coexistence case

In this section we prove the coexistence results stated in Theorem 2.3.3. Like for model 1 the proofs hold for general $g \in \mathcal{G}$ and we need not distinguish between $g = dg_{\mathrm{FW}}$ and $g \neq dg_{\mathrm{FW}}$. For $\rho < \infty$, the argument is given in Section 3.3.3 and proceeds as in Section 3.2.3. It is organised along the same 4 Steps as the argument for model 1, plus an extra Step 5 that settles the statement in (2.87). For $\rho = \infty$, the argument is given in Section 3.3.3 and is also organised along 5 Steps, but structured differently. In Step 1 we define a set of measures that is preserved under the evolution. In Step 2 we use a coupling argument to show the existence of invariant measures. In Step 3 we show that these invariant measures have vanishing covariances in the seed-bank direction. In Step 4 we use the vanishing covariances to show uniqueness of the invariant measure by coupling. Finally, in Step 5 we show that the unique equilibrium measure is invariant, ergodic and mixing under translations, and is associated.

- **Proof of coexistence for finite seed-bank**

1. Properties of measures preserved under the evolution. For model 2 with $\rho < \infty$, the class of preserved measures is equivalent to $\mathcal{R}_\theta^{(1)}$ for model 1 and is now defined as follows.

Definition 3.3.5 (Preserved class of measures). Let $\mathcal{R}_\theta^{(2)}$ denote the set of measures $\mu \in \mathcal{T}$ satisfying, for all $(i,R_i),(j,R_j) \in \mathbb{G} \times \{A,(D_m)_{m\in\mathbb{N}_0}\}$,

(1)

$$\lim_{t\to\infty} \mathbb{E}_\mu[z_{(i,R_i)}(t)] = \theta, \tag{3.182}$$

(2)

$$\begin{aligned}\lim_{t\to\infty} \sum_{\substack{(k,R_k),(l,R_l)\\ \in\mathbb{G}\times\{A,(D_m)_{m\in\mathbb{N}_0}\}}} & b_t^{(2)}\big((i,R_i),(k,R_k)\big)\, b_t^{(2)}\big((j,R_j),(l,R_l)\big)\\ & \times \mathbb{E}_\mu[z_{(k,R_k)}z_{(l,R_l)}] = \theta^2.\end{aligned} \tag{3.183}$$

□

Like for model 1, properties (1) and (2) of Definition 3.3.5 hold if and only if

$$\begin{aligned}&\lim_{t\to\infty} \mathbb{E}_\mu\left[\left(\sum_{(k,R_k),(l,R_l)\in\mathbb{G}\times\{A,(D_m)_{m\in\mathbb{N}_0}\}} b_t^{(2)}((0,A),(k,R_k))\, z_{(k,R_k)} - \theta\right)^2\right] = 0\\ &\text{for some } (i,R_i)\in\mathbb{G}\times\{A,(D_m)_{m\in\mathbb{N}_0}\}.\end{aligned} \tag{3.184}$$

Also for model 2 with $\rho<\infty$ we have $\mathcal{T}_\theta^{\text{erg}} \subset \mathcal{R}_\theta^{(1)}$. To see why, note for all $t>0$ and $m\in\mathbb{N}_0$, $(x_i(t))_{i\in\mathbb{G}}$ and $(y_{i,m}(t))_{i\in\mathbb{G}}$ still are stationary time series. Hence with the help of the Herglotz theorem we can define spectral measures λ_A, λ_{D_m} for $m\in\mathbb{N}_0$ as in (3.112). Let $(RW_t)_{t\geq 0}$ be the random walk evolving according to $b^{(2)}(\cdot,\cdot)$. Introduce the sets

$$\begin{aligned}\mathcal{E}(t) &= \big\{\text{at time } t \text{ the random walk is active}\big\},\\ \mathcal{E}_m(t) &= \big\{\text{at time } t \text{ the random walk is dormant with colour } m\big\}.\end{aligned} \tag{3.185}$$

Note that

$$\begin{aligned}&\lim_{t\to\infty} \mathbb{E}_\mu\left[\left(\sum_{(k,R_k),(l,R_l)\in\mathbb{G}\times\{A,(D_m)_{m\in\mathbb{N}_0}\}} b_t^{(2)}((0,A),(k,R_k))\, z_{(k,R_k)} - \theta\right)^2\right]\\ &\leq \lim_{t\to\infty} \mathbb{P}_{(0,A)}(\mathcal{E}(t))\, \mathbb{E}_\mu\left[\left(\sum_{k\in Z^d} \frac{b_t^{(2)}((0,A),(k,A))}{\mathbb{P}_{(0,A)}(\mathcal{E}(t))}\, x_k - \frac{1}{\mathbb{P}_{(0,A)}(\mathcal{E}(t))}\frac{\theta_x}{1+\rho}\right)^2\right]\\ &+ \sum_{m\in\mathbb{N}_0} \mathbb{P}_{(0,A)}(\mathcal{E}_m(t))\, \mathbb{E}_\mu\left[\left(\sum_{k\in Z^d} \frac{b_t^{(2)}((0,A),(k,(D_m))}{\mathbb{P}_{(0,A)}(\mathcal{E}_m(t))}\, y_{k,m} - \frac{1}{\mathbb{P}_{(0,A)}(\mathcal{E}_m(t))}\frac{K_m\theta_{y,m}}{1+\rho}\right)^2\right].\end{aligned} \tag{3.186}$$

Hence we can use the same argument as in the proof of Lemma 3.2.7 to show that $\mathcal{T}_\theta^{\text{erg}} \subset \mathcal{R}_\theta^{(2)}$.

Also Lemma 3.2.8 carries over after we replace $b^{(1)}(\cdot,\cdot)$ by $b^{(2)}(\cdot,\cdot)$ and $\mathcal{R}_\theta^{(1)}$ by $\mathcal{R}_\theta^{(2)}$, as defined in (3.3.5).

2. Uniqueness of the equilibrium. To prove uniqueness of the equilibrium for given θ, we use a similar coupling as for model 1 in Section 3.2.3 in Step 3. Consider two copies of the system in (2.12)–(2.13) *coupled via their Brownian motions*:

$$\mathrm{d}x_i^k(t) = \sum_{j\in\mathbb{G}} a(i,j)\left[x_j^k(t)-x_i^k(t)\right]\mathrm{d}t + \sqrt{g(x_i^k(t))}\,\mathrm{d}w_i(t) \tag{3.187}$$

$$+\sum_{m\in\mathbb{N}_0} K_m e_m \left[y_{i,m}^k(t)-x_i^k(t)\right]\mathrm{d}t, \tag{3.188}$$

$$\mathrm{d}y_{i,m}^k(t) = e_m\left[x_i^k(t)-y_{i,m}^k(t)\right]\mathrm{d}t, \qquad m\in\mathbb{N}_0, \qquad k\in\{1,2\}. \tag{3.189}$$

Here, k labels the copy, and the two copies are driven by the same Brownian motions $(w_i(t))_{t\geq 0}$, $i\in\mathbb{G}$. As initial measures we choose $\mu^1(0),\mu^2(0)\in\mathcal{T}_\theta^{\mathrm{erg}}$.

Let

$$\bar{z}_i(t) = \left(z_i^1(t), z_i^2(t)\right), \qquad z_i^k(t) = \left(x_i^k(t), (y_{i,m}^k(t))_{m\in\mathbb{N}_0}\right), \quad k\in\{1,2\}. \tag{3.190}$$

By [67, Theorem 3.2], the coupled system $(\bar{z}_i(t))_{i\in\mathbb{G}}$ has a unique strong solution whose marginals are the single-component systems. Write $\hat{\mathbb{P}}$ to denote the law of the coupled system, and let $\Delta_i(t) = x_i^1(t)-x_i^2(t)$ and $\delta_{i,m}(t) = y_{i,m}^1(t)-y_{i,m}^2(t)$, $m\in\mathbb{N}_0$. The analogue of Lemma 3.2.9 reads:

Lemma 3.3.6 (Coupling dynamics $\rho<\infty$). *For every $t\geq 0$,*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,\hat{\mathbb{E}}&\left[|\Delta_i(t)| + \sum_{m\in\mathbb{N}_0} K_m|\delta_i(t)|\right]\\ &= -2\sum_{j\in\mathbb{G}} a(i,j)\,\hat{\mathbb{E}}\left[|\Delta_j(t)|\,1_{\{\operatorname{sgn}\Delta_i(t)\neq\operatorname{sgn}\Delta_j(t)\}}\right]\\ &\qquad -2\sum_{m\in\mathbb{N}_0} K_m e_m\,\hat{\mathbb{E}}\left[\left(|\Delta_i(t)|+|\delta_{i,m}(t)|\right)1_{\{\operatorname{sgn}\Delta_i(t)\neq\operatorname{sgn}\delta_{i,m}(t)\}}\right].\end{aligned} \tag{3.191}$$

Proof. Note that the left-hand side of (3.191) is well defined because $\rho<\infty$. The proof of Lemma 3.3.6 carries over from that of Lemma 3.2.9 after replacing (2.4)–(2.5) by (2.12)–(2.13). □

The analogue of Lemma 3.2.10 reads as follows.

Lemma 3.3.7 (Successful coupling $\rho<\infty$). *If $a(\cdot,\cdot)$ is transient, then the coupling is successful, i.e.,*

$$\lim_{t\to\infty}\hat{\mathbb{E}}\big[|\Delta_i(t)| + \sum_{m\in\mathbb{N}_0} K_m|\delta_{i,m}(t)|\big] = 0, \qquad \forall i\in\mathbb{G}. \tag{3.192}$$

Proof. This follows in the same way as in the proof of Lemma 3.2.10, by defining $-h_i(t)$ as in the right-hand side of (3.191). Using that the second line of (3.146) now holds for $\delta_{i,m}(t)$ and all $m\in\mathbb{N}_0$, we can finish the proof after replacing $b_t^{(1)}(\cdot,\cdot)$ in (3.149) by $b_t^{(2)}(\cdot,\cdot)$ and summing over the seed-banks D_m, $m\in\mathbb{N}_0$. □

3. Stationarity of the equilibrium ν_θ and convergence to ν_θ. Lemma 3.2.12 holds also for $\mu \in \mathcal{R}_\theta^{(2)}$. This follows after replacing $\mu \in \mathcal{R}_\theta^{(1)}$ by $\mu \in \mathcal{R}_\theta^{(2)}$ in the proof of Lemma 3.2.12, using the equivalent of Lemma 3.2.8 and invoking Lemma 3.3.7 instead of Lemma 3.2.10.

4. Ergodicity, mixing and associatedness. Also Lemma 3.2.13 holds, after replacing $b^{(1)}(\cdot,\cdot)$ by $b^{(2)}(\cdot,\cdot)$. The proof even simplifies, since we can invoke the symmetry of $a(\cdot,\cdot)$ in (3.155).

5. Variances under the equilibrium measure ν_θ. If $\limsup_{m\to\infty} e_m = 0$, then the claim in (2.87) is a direct consequence of the proof of Lemma 3.3.10 for $\rho = \infty$. If $\liminf_{m\to\infty} e_m > 0$, then the claim follows from the fact that $\mu \in \mathbb{R}_\theta^{(2)}$ and

$$\mathrm{Var}_{\nu_\theta}(y_{0,m}) = \int_0^t \mathrm{d}s \sum_{k\in\mathbb{G}} b_{t-s}^{(2)}\big((0,D_m),(k,A)\big)\, b_{t-s}^{(2)}\big((0,D_m),(k,A)\big)\, \mathbb{E}_\mu[g(x_i(s))]. \tag{3.193}$$

Since $e_m > 0$ for all $m \in \mathbb{N}_0$ and $\liminf_{m\to\infty} e_m > 0$, there is a positive probability that after the first steps the two random walks are both active at 0, i.e., are both in state $(0,A)$. Hence, for all $m \in \mathbb{N}_0$ there exists a constant $c > 0$ such that

$$\mathrm{Var}_{\nu_\theta}(y_{0,m}) \geq c\mathrm{Var}_{\nu_\theta}(x_0). \tag{3.194}$$

Since ν_θ is a non-trivial equilibrium, we have $\mathrm{Var}_{\nu_\theta}(x_0) > 0$.

• **Proof of coexistence for infinite seed-bank**

1. Properties of measures preserved under the evolution. For $\rho = \infty$, the class of preserved measures is also given by $\mathcal{R}_\theta^{(2)}$ (recall Definition 3.3.5). We show that if $\mu \in \mathcal{T}_\theta^{\mathrm{erg}}$ is colour regular, then $\mu \in \mathcal{R}_\theta^{(2)}$. Let the sets $\mathcal{E}_m(t)$, $t > 0$, $m \in \mathbb{N}_0$, be defined as in (3.185), and define λ_A and λ_{D_m} analogously to (3.112), like for $\rho < \infty$. The equivalent of (3.115) is

$$\begin{aligned}
&\mathbb{E}_\mu\left[\left(\frac{1}{\mathbb{P}_{(0,A)}(\mathcal{E}(t))}\sum_{k\in\mathbb{G}} b_t^{(2)}((0,A),(k,A))x_k - \theta_x\right)^2\right]\\
&\quad= \frac{1}{\mathbb{P}_{(0,A)}(\mathcal{E}(t))^2}\int_{[-\pi,\pi]^d} \mathbb{E}_{(0,A),(0,A)}\left[\mathrm{e}^{-T(t)(1-a(\phi))}1_{\mathcal{E}(t)}\mathrm{e}^{-T'(t)(1-\bar{a}(\phi))}1_{\mathcal{E}'(t)}\right]\mathrm{d}\lambda_A.
\end{aligned} \tag{3.195}$$

Using that $T(t), T'(t) \to \infty$ as $t \to \infty$ (see (3.169)), that $T(t)$, $T'(t)$, $\mathcal{E}(t)$, $\mathcal{E}'(t)$ are asymptotically independent and that $a(\cdot,\cdot)$ is irreducible, we still find

$$\lim_{t\to\infty} \mathbb{E}_\mu\left[\left(\frac{1}{\mathbb{P}_{(0,A)}(\mathcal{E}(t))}\sum_{k\in\mathbb{G}} b_t^{(2)}((0,A),(k,A))x_k - \theta_x\right)^2\right] = \lambda_A(\{0\}) \tag{3.196}$$

and, similarly,

$$\lim_{t\to\infty} \mathbb{E}_\mu \left[\left(\frac{1}{\mathbb{P}_{(0,A)}(\mathcal{E}_m(t))} \sum_{k\in\mathbb{G}} b_t^{(2)}((0,A),(k,A)) y_{k,m} - \theta_{y,m} \right)^2 \right] = \lambda_{D_m}(\{0\}). \tag{3.197}$$

Since μ is ergodic, we have $\lambda_A(\{0\}) = 0$ and $\lambda_{D_m}(\{0\}) = 0$ for all $m \in \mathbb{N}_0$ (recall (3.121)). By the colour regularity,

$$\lim_{t\to\infty} \theta_x \mathbb{P}_{(0,A)}(\mathcal{E}(t)) + \sum_{m\in\mathbb{N}_0} \theta_{y,m} \mathbb{P}_{(0,A)}(\mathcal{E}_m(t)) = \theta. \tag{3.198}$$

Therefore we can rewrite (3.186) as

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}_\mu \left[\left(\sum_{(k,R_k),(l,R_l)\in\mathbb{G}\times\{A,(D_m)_{m\in\mathbb{N}_0}\}} b_t^{(2)}((0,A),(k,R_k))\, z_{(k,R_k)} - \theta \right)^2 \right] \\
&\quad \le \lim_{t\to\infty} \mathbb{P}_{(0,A)}(\mathcal{E}(t))\, \mathbb{E}_\mu \left[\left(\sum_{k\in Z^d} \frac{b_t^{(2)}((0,A),(k,A))}{\mathbb{P}_{(0,A)}(\mathcal{E}(t))} (x_k - \theta_x) \right)^2 \right] \\
&\qquad + \sum_{m\in\mathbb{N}_0} \mathbb{P}_{(0,A)}(\mathcal{E}_m(t))\, \mathbb{E}_\mu \left[\left(\sum_{k\in Z^d} \frac{b_t^{(2)}((0,A),(k,(D_m))}{\mathbb{P}_{(0,A)}(\mathcal{E}_m(t))} (y_{k,m} - \theta_{y,m}) \right)^2 \right] \\
&\quad = \lim_{t\to\infty} \mathbb{P}_{(0,A)}(\mathcal{E}(t))\, \lambda_A(\{0\}) \; + \sum_{m\in\mathbb{N}_0} \mathbb{P}_{(0,A)}(\mathcal{E}_m(t))\, \lambda_{D_m}(\{0\}) = 0.
\end{aligned} \tag{3.199}$$

We conclude that indeed $\mu \in \mathcal{R}_\theta^{(2)}$.

Like for $\rho < \infty$, Lemma 3.2.8 carries over after we replace $b^{(1)}(\cdot,\cdot)$ by $b^{(2)}(\cdot,\cdot)$ and $\mathcal{R}_\theta^{(1)}$ by $\mathcal{R}_\theta^{(2)}$.

2. Existence of invariant measures ν_θ for $\rho = \infty$. Since the dynamics for $\rho = \infty$ and $\rho < \infty$ are the same, we can still use the coupling in (3.187)–(3.189). Also Lemma 3.3.6 holds for $\rho = \infty$, but if $\rho = \infty$, then the left-hand side of (3.191) can become infinite. Therefore we cannot use the line of argument used for model 1 to show that the coupling is successful for arbitrary colour regular initial measures $\mu_1,\ \mu_2 \in \mathcal{T}_\theta^{\mathrm{erg}}$. However, we can prove the following lemma.

Lemma 3.3.8 (Successful coupling). *If $\mu_1,\ \mu_2 \in \mathcal{T}_\theta^{\mathrm{erg}}$ are both colour regular and satisfy*

$$\hat{\mathbb{E}} \left[|\Delta_i(0)| + \sum_{m\in\mathbb{N}_0} K_m |\delta_i(0)| \right] < \infty, \tag{3.200}$$

then the coupling in (3.187)–(3.189) *is successful.*

Proof. We proceed similarly as in Step 3 for $\rho < \infty$. Note, in particular, that $h_i(t)$ (recall (3.191)) is bounded from above by $\hat{\mathbb{E}}\left[|\Delta_i(0)| + \sum_{m\in\mathbb{N}_0} K_m|\delta_i(0)|\right]$ (compare with (3.145)). Also for $\rho = \infty$ we obtain Lemma 3.2.11. Like for model 1, if we define

$$
\begin{aligned}
E_0 \times E_0 = & \Big\{ \bar{z} \in E \times E\colon\ z^1_{(i,R_i)}(t) \geq z^2_{(i,R_i)}(t)\ \forall (i,R_i) \in \mathbb{G} \times \{A, (D_m)_{m\in\mathbb{N}_0}\} \Big\} \\
& \cup \Big\{ \bar{z} \in E \times E\colon\ z^2_{(i,R_i)}(t) \geq z^1_{(i,R_i)}(t)\ \forall (i,R_i) \in \mathbb{G} \times \{A, (D_m)_{m\in\mathbb{N}_0}\} \Big\},
\end{aligned}
\tag{3.201}
$$

then we find $\lim_{t\to\infty} \mathbb{P}(E_0 \times E_0) = 1$ and hence the coupled diffusions $(Z^1(t))_{t\geq 0}$ and $(Z^2(t))_{t\geq 0}$ lay on top of each other as $t \to \infty$. However, in (3.149) the limiting distribution of $b^{(1)}_{t_n}(\cdot,\cdot)$ was used "to compensate" the factors K_m in $|\Delta_i| + \sum_{m\in\mathbb{N}_0} K_m |\delta_{i,m}|$. Since, for $\rho = \infty$, $b^{(1)}_{t_n}(\cdot,\cdot)$ does not have a well-defined limiting distribution for the projection on the colour components, we need a different strategy.

To obtain a successful coupling, as before, let $(t_n)_{n\in\mathbb{N}}$ be a subsequence such that $\nu^1_\theta = \lim_{n\to\infty} \mathcal{L}(Z^1(t_n))$ with $\mathcal{L}(Z^1(0)) = \mu^1$ and $\nu^2_\theta = \lim_{n\to\infty} \mathcal{L}(Z^2(t_n))$ with $\mathcal{L}(Z^2(0)) = \mu^2$. For $\mathbb{G} = \mathbb{Z}^d$, let $\Lambda_N = [0,N)^d \cap \mathbb{Z}^d$, $N \in \mathbb{N}$. (As noted before, for amenable groups $\mathbb{G}$, $(\Lambda_N)_{N\in\mathbb{N}}$ must be replaced by a so-called Fϕlner sequence.) Note that

$$
\mathbb{E}_{\nu^1_\theta}\left[\left(\frac{1}{|\Lambda_N|} \sum_{j\in\Lambda_N} x_j - \theta \right)^2 \right] = \frac{1}{|\Lambda_N|^2} \sum_{i,j\in\Lambda_N} \mathrm{Cov}_{\nu^1_\theta}(x_i,x_j). \tag{3.202}
$$

Since μ^1 is colour regular and $\mu^1 \in \mathcal{T}^{\mathrm{erg}}_\theta$, we have $\mu^1 \in \mathcal{R}^{(2)}_\theta$. Hence, by Lemma 3.3.1,

$$
\begin{aligned}
\mathrm{Cov}_{\nu^1_\theta}(x_i,x_j) &= \lim_{n\to\infty} \mathrm{Cov}_{\mu^1}(x_i(t_n), x_j(t_n)) \\
&\leq \lim_{n\to\infty} \|g\| \int_0^{t_n} \mathrm{d}s \sum_{k\in\mathbb{G}} b^{(2)}_{(t_n-s)}((i,A),(k,A))\, b^{(2)}_{(t-s)}((j,A),(k,A)) \\
&\leq \|g\| \int_0^\infty \mathrm{d}s \sum_{k\in\mathbb{G}} \mathbb{E}_{(i,A),(j,A)} \left[a_{T(s)}(i,k)\, 1_{\mathcal{E}(s)}\, a_{T'(s)}(j,k)\, 1_{\mathcal{E}'(s)} \right] \\
&\leq \|g\| \int_0^\infty \mathrm{d}s\ \mathbb{E}_{(i,A),(j,A)} \left[\hat{a}_{T(s)+T'(s)}(i-j,0)\, 1_{\mathcal{E}(s)}\, 1_{\mathcal{E}'(s)} \right].
\end{aligned}
\tag{3.203}
$$

Since $I_{\alpha,\gamma} < \infty$, we see that the last integral is finite. Since $\lim_{||i-j||\to\infty} \hat{a}_t(i-j,0) = 0$ for all $t > 0$, it follows by transience and dominated convergence that $\lim_{||i-j||\to\infty} \mathrm{Cov}_{\nu^1_\theta}(x_i,x_j) = 0$. Since $\mathrm{Cov}_{\nu^1_\theta}(x_i,x_j) \leq 1$ for all $i,j \in \mathbb{G}$, for all $\epsilon > 0$

there exists an $L \in \mathbb{N}$ such that

$$\lim_{N\to\infty} \mathbb{E}_{\nu_\theta^1}\left[\left(\frac{1}{|\Lambda_N|}\sum_{j\in\Lambda_N} x_j - \theta\right)^2\right] = \lim_{N\to\infty} \frac{1}{|\Lambda_N|^2}\sum_{i,j\in\Lambda_N} \mathrm{Cov}_{\nu_\theta^1}(x_i,x_j)$$
$$= \lim_{N\to\infty} \frac{1}{|\Lambda_N|^2}\sum_{\substack{i,j\in\Lambda_N\\ \|i-j\|\le L}} \mathrm{Cov}_{\nu_\theta^1}(x_i,x_j) + \frac{1}{|\Lambda_N|^2}\sum_{\substack{i,j\in\Lambda_N\\ \|i-j\|> L}} \mathrm{Cov}_{\nu_\theta^1}(x_i,x_j)$$
$$\le \lim_{N\to\infty} \frac{|\{i,j\in\Lambda_N\colon \|i-j\|\le L\}|}{|\Lambda_N|^2} + \epsilon \lim_{N\to\infty} \frac{|\{i,j\in\Lambda_N\colon \|i-j\|> L\}|}{|\Lambda_N|^2} < \epsilon. \tag{3.204}$$

We conclude that

$$\lim_{N\to\infty} \mathbb{E}_{\nu_\theta^1}\left[\left(\frac{1}{\Lambda_N}\sum_{j\in\Lambda_N} x_j - \theta\right)^2\right] = 0, \tag{3.205}$$

and the same holds for ν_2^θ. Let $\lim_{n\to\infty}\mathcal{L}(\bar{Z}(t_n)) = \bar{\nu}_\theta$ such that $\lim_{n\to\infty}\mathcal{L}(Z^1(t_n)) = \nu_\theta^1$ and $\lim_{n\to\infty}\mathcal{L}(Z^2(t_n)) = \nu_\theta^2$. Then by translation invariance of $\bar{\nu}_\theta$ and the fact that $\bar{\nu}_\theta(E_0\times E_0) = 1$, we find

$$\int_{E\times E} \mathrm{d}\bar{\nu}_\theta |\Delta_i| = \int_{E_0\times E_0} \mathrm{d}\bar{\nu}_\theta \frac{1}{|\Lambda_N|}\sum_{j\in\Lambda_N} |x_j^1 - x_j^2|$$
$$\le \int_{E_0} \mathrm{d}\nu_\theta^1 \left|\frac{1}{|\Lambda_N|}\sum_{j\in\Lambda_N} x_j^1 - \theta\right| + \int_{E_0} \mathrm{d}\nu_\theta^2 \left|\frac{1}{|\Lambda_N|}\sum_{j\in\Lambda_N} x_j^2 - \theta\right|. \tag{3.206}$$

Letting $N\to\infty$, we see by translation invariance of $\bar{\nu}_\theta$ that $\mathbb{E}_{\bar{\nu}_\theta}[|\Delta_i|] = 0$ for all $i\in\mathbb{G}$.

The result in (3.205) holds also for x_i replaced by $y_{i,m}$, $m\in\mathbb{N}_0$, since the integral in (3.203) can only become smaller when we start from a dormant site. Replacing $|\Delta_i|$ in (3.206) by $|\delta_{i,m}|$, we obtain, for all $m\in\mathbb{N}_0$,

$$\mathbb{E}_{\bar{\nu}_\theta}[|\delta_{i,m}|] = 0, \qquad \forall\, m\in\mathbb{N}_0. \tag{3.207}$$

We conclude that the coupling is successful. □

Let $(S_t)_{t\ge 0}$ denote the semigroup associated with (2.12)–(2.13). To prove the existence of an invariant measure, note that $E\times E$ is a compact space. Hence, if $t_n\to\infty$, then the sequence μS_{t_n} has a convergent subsequence. In Lemma 3.3.9 below we show that each weak limit point of the sequence μS_{t_n} is invariant under the evolution of (2.12)–(2.13).

Lemma 3.3.9 (Invariant measure). *Suppose that $\mu\in\mathcal{R}_\theta^{(2)}$ and that μ is colour regular. If $t_n\to\infty$ and $\mu S_{t_n}\to\nu_\theta$, then ν_θ is an invariant measure under the evolution in* (2.12)–(2.13)*.*

Proof. Fix $s > 0$. Let $\mu_1 = \mu$ and $\mu_2 = \mu S_s$. We couple μ_1 and μ_2 via their Brownian motions (see (3.187)–(3.189)). Note that, by the SSDE in (2.12)–(2.13),

$$
\begin{aligned}
&\hat{\mathbb{E}}\left[|\Delta_i(0)| + \sum_{m\in\mathbb{N}_0} K_m|\delta_{i,m}(0)|\right] \\
&= \mathbb{E}\left[|x_i(0) - x_i(s)| + \sum_{m\in\mathbb{N}_0} K_m|y_{i,m}(0) - y_{i,m}(s)|\right] \\
&= \mathbb{E}\Bigg[\Bigg|\int_0^s \sum_{j\in\mathbb{G}} a(i,j)\,[x_j(r) - x_i(r)]\,\mathrm{d}r + \int_0^s \sqrt{g(x_i(r))}\,\mathrm{d}w_i(r) \\
&\qquad + \int_0^s \sum_{m\in\mathbb{N}_0} K_m e_m\,[y_{i,m}(r) - x_i(r)]\,\mathrm{d}r\Bigg| \\
&\qquad + \sum_{m\in\mathbb{N}_0} K_m \int_0^s |e_m[y_{i,m}(r) - x_i(r)]|\,\mathrm{d}r\Bigg].
\end{aligned}
\tag{3.208}
$$

Using that all rates are finite and that, by Knight's theorem (see [62, Theorem V.1.9 p.183]), we can write the Brownian integral as a time-transformed Brownian motion, we see that $\hat{\mathbb{E}}[|\Delta_i(0)| + \sum_{m\in\mathbb{N}_0} K_m|\delta_{i,m}(0)|] < \infty$. Hence, by Lemma 3.200, we can successfully couple μ^1 and μ^2, and $\lim_{n\to\infty} \mu^2 S_{t_n} = \lim_{n\to\infty} \mu S_s S_{t_n} = \nu_\theta$. By the Feller property of the SSDE in (2.12)–(2.13), it follows that

$$
\nu_\theta S_s = \lim_{n\to\infty} \mu(t_n) S_s = \lim_{n\to\infty} \mu S_{t_n} S_s = \lim_{n\to\infty} \mu S_s S_{t_n} = \nu_\theta. \tag{3.209}
$$

We conclude that ν_θ is indeed an invariant measure for the SSDE in (2.12)–(2.13). □

3. Invariant measures have vanishing covariances in the seed-bank direction for $\rho = \infty$. In this step we prove that an invariant measure ν_θ has vanishing variances in the seed-bank direction. In Step 5 we use this property to successfully couple any two invariant measures.

Lemma 3.3.10 (Deterministic deep seed-banks). *If $\nu_\theta = \lim_{n\to\infty} \mu S_{t_n}$ for some colour regular $\mu \in \mathcal{R}_\theta^{(2)}$ and $t_n \to \infty$, then*

$$
\lim_{m\to\infty} \mathrm{Var}_{\nu_\theta}[y_{i,m}] = 0 \qquad \forall i \in \mathbb{G}. \tag{3.210}
$$

Proof. Since ν_θ is translation invariant, it is enough to show that $\lim_{m\to\infty} \mathrm{Var}_{\nu_\theta}[y_{0,m}] = 0$. Since $\mu(0) \in \mathcal{R}_\theta^{(2)}$, it follows from Lemma 3.3.1 that

$$
\begin{aligned}
&\lim_{m\to\infty} \mathrm{Var}_{\nu_\theta}[y_{0,m}] \\
&= \lim_{m\to\infty}\lim_{n\to\infty} \mathbb{E}_\mu\left[(y_{0,m}(t_n) - \mathbb{E}_\mu[y_{0,m}(t_n)])^2\right] \\
&= \lim_{m\to\infty}\lim_{n\to\infty} \int_0^{t_n} \mathrm{d}s \sum_{k\in\mathbb{G}} b^{(2)}_{t_n-s}((0,D_m),(k,A))\, b^{(2)}_{t_{n_k}-s}((0,D_m),(k,A))\mathbb{E}_z[g(x_k(s))].
\end{aligned}
\tag{3.211}
$$

Since g is positive and bounded, it is therefore enough to prove that

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_0^{t_n}\mathrm{d}u\sum_{k\in\mathbb{G}} b_u^{(2)}((0,D_m),(k,A))\, b_u^{(2)}((0,D_m),(k,A)) = 0. \tag{3.212}$$

Recall (see e.g. (3.203)) that $b_u^{(2)}((0,D_m),(k,A))\, b_u^{(2)}((0,D_m),(k,A))$ is the probability that two random walks, denoted by RW and RW' and moving according to $b^{(2)}(\cdot,\cdot)$, are at time u at the same site k and both active. Define

$$\tau = \big\{t\geq 0 : RW(t) = RW'(t) = (i,A) \text{ for some } i\in\mathbb{G}\big\}. \tag{3.213}$$

Then we can rewrite the left-hand side of (3.212) as

$$\begin{aligned}
&\lim_{m\to\infty}\lim_{n\to\infty}\int_0^{t_n}\mathrm{d}u\sum_{k\in\mathbb{G}} b_u^{(2)}((0,D_m),(k,A))\, b_u^{(2)}((0,D_m),(k,A))\\
&=\lim_{m\to\infty}\lim_{n\to\infty}\int_0^{t_n}\mathrm{d}u\,\mathbb{E}_{(0,D_m),(0,D_m)}\left[\sum_{k\in\mathbb{G}} 1_{\{RW(u)=k\}}\,1_{\{RW'(u)=k\}}\,1_{\mathcal{E}(u)}\,1_{\mathcal{E}'(u)}\right]\\
&=\lim_{m\to\infty}\lim_{n\to\infty}\int_0^{t_n}\mathrm{d}u\,\mathbb{E}_{(0,D_m),(0,D_m)}\Bigg[\sum_{k\in\mathbb{G}} 1_{\{RW(u)=k\}}\,1_{\{RW'(u)=k\}}\,1_{\mathcal{E}(t)}\,1_{\mathcal{E}'(t)}\\
&\qquad\qquad\qquad\times\big(1_{\{\tau<\infty\}}+1_{\{\tau=\infty\}}\big)\Bigg]\\
&=\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{E}_{(0,D_m),(0,D_m)}\Bigg[1_{\{\tau<\infty\}}\\
&\qquad\times\mathbb{E}_{(0,D_m),(0,D_m)}\left[\int_0^{t_n}\mathrm{d}u\sum_{k\in\mathbb{G}}1_{\{RW(u)=k\}}1_{\{RW'(u)=k\}}1_{\mathcal{E}(u)}1_{\mathcal{E}'(u)}\mid\mathcal{F}_\tau\right]\Bigg]\\
&=\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{E}_{(0,D_m),(0,D_m)}\Bigg[1_{\{\tau<\infty\}}\\
&\qquad\times\mathbb{E}_{(0,A),(0,A)}\left[\int_0^{t_n-\tau}\mathrm{d}u\sum_{k\in\mathbb{G}}1_{\{RW(u)=k\}}1_{\{RW'(u)=k\}}1_{\mathcal{E}(u)}1_{\mathcal{E}'(u)}\right]\Bigg]\\
&\leq\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{E}_{(0,D_m),(0,D_m)}\Bigg[1_{\{\tau<\infty\}}\\
&\qquad\times\mathbb{E}_{(0,A),(0,A)}\left[\int_0^{\infty}\mathrm{d}u\sum_{k\in\mathbb{G}}1_{\{RW(u)=k\}}1_{\{RW'(u)=k\}}1_{\mathcal{E}(u)}1_{\mathcal{E}'(u)}\right]\Bigg]\\
&=\lim_{m\to\infty}\mathbb{P}_{(0,D_m),(0,D_m)}(\tau<\infty)\, I_{\hat{a},\gamma},
\end{aligned}\tag{3.214}$$

where we use that $I_{\hat{a},\gamma}<\infty$, the strong Markov property, and the fact that for $\tau=\infty$ the product of the indicators equals 0 for all $u\in\mathbb{R}_{\geq 0}$. Therefore (3.210) holds if

$$\lim_{m\to\infty}\mathbb{P}_{(0,D_m),(0,D_m)}(\tau<\infty) = 0. \tag{3.215}$$

Define

$$\tau^* = \inf\big\{t\geq 0:\ \text{both } RW \text{ and } RW' \text{ are active at time } t\big\}. \tag{3.216}$$

Note that $\tau^* \leq \tau$. Theorefore we can write (recall that in model 2 the random walk kernel $a(\cdot,\cdot)$ is assumed to be symmetric),

$$
\begin{aligned}
&\lim_{m\to\infty} \mathbb{P}_{(0,D_m),(0,D_m)}(\tau<\infty)\\
&= \lim_{m\to\infty} \mathbb{E}_{(0,D_m),(0,D_m)}[1_{\{\tau<\infty\}}]\\
&= \lim_{m\to\infty} \mathbb{E}_{(0,D_m),(0,D_m)}\left[1_{\{\tau^*<\infty\}}\,\mathbb{E}_{(0,D_m)^2}\left[1_{\{\tau<\infty\}} \mid \mathcal{F}_{\tau^*}\right]\right]\\
&= \lim_{m\to\infty} \mathbb{E}_{(0,D_m),(0,D_m)}\left[\mathbb{E}^{RW(\tau^*),RW'(\tau^*)}\left[1_{\{\tau<\infty\}}\right]\right]\\
&= \lim_{m\to\infty} \sum_{k,l\in\mathbb{G}} \mathbb{P}_{(0,D_m),(0,D_m)}\left(RW(\tau^*)=(k,A),RW'(\tau^*)=(l,A)\right)\\
&\qquad\qquad \times \mathbb{E}_{(k,A),(l,A)}\left[1_{\{\tau<\infty\}}\right]\\
&= \lim_{m\to\infty} \sum_{k,l\in\mathbb{G}} \mathbb{E}_{(0,D_m),(0,D_m)}[\hat{a}_{T(\tau^*)}(0,k)\,\hat{a}_{T'(\tau^*)}(0,l)]\,\mathbb{E}_{(0,A),(l-k,A)}\left[1_{\{\tau<\infty\}}\right]\\
&= \lim_{m\to\infty} \sum_{k,l\in\mathbb{G}} \mathbb{E}_{(0,D_m),(0,D_m)}[\hat{a}_{T(\tau^*)}(0,k)\,\hat{a}_{T'(\tau^*)}(-k,l-k)]\,\mathbb{E}_{(0,A),(l-k,A)}\left[1_{\{\tau<\infty\}}\right]\\
&= \lim_{m\to\infty} \sum_{k,j\in\mathbb{G}} \mathbb{E}_{(0,D_m),(0,D_m)}[\hat{a}_{T(\tau^*)}(0,-k)\,\hat{a}_{T'(\tau^*)}(-k,j)]\,\mathbb{E}_{(0,A),(j,A)}\left[1_{\{\tau<\infty\}}\right]\\
&= \lim_{m\to\infty} \sum_{j\in\mathbb{G}} \mathbb{E}_{(0,D_m),(0,D_m)}[\hat{a}_{T(\tau^*)+T'(\tau^*)}(0,j)]\,\mathbb{E}_{(0,A),(j,A)}\left[1_{\{\tau<\infty\}}\right]\\
&= \lim_{m\to\infty} \sum_{\substack{j\in\mathbb{G}\\ \|j\|\leq L}} \mathbb{E}_{(0,D_m),(0,D_m)}[\hat{a}_{T(\tau^*)+T'(\tau^*)}(0,j)]\,\mathbb{E}_{(0,A),(j,A)}\left[1_{\{\tau<\infty\}}\right]\\
&\qquad + \lim_{m\to\infty} \sum_{\substack{j\in\mathbb{G}\\ \|j\|> L}} \mathbb{E}_{(0,D_m),(0,D_m)}[\hat{a}_{T(\tau^*)+T'(\tau^*)}(0,j)]\,\mathbb{E}_{(0,A),(j,A)}\left[1_{\{\tau<\infty\}}\right].
\end{aligned}
\tag{3.217}
$$

To prove that the expression in the right-hand side tends to zero, we fix $\epsilon>0$ and prove that there exists an $L\in\mathbb{N}$ such that both sums are smaller that $\frac{\epsilon}{2}$.

Claim 1: There exists an L such that

$$
\lim_{m\to\infty} \sum_{j\in\mathbb{G},\|j\|>L} \mathbb{E}_{(0,D_m)^2}[\hat{a}_{T(\tau^*)+T'(\tau^*)}(0,j)]\,\mathbb{E}_{(0,A),(j,A)}[1_{\{\tau<\infty\}}] < \frac{\epsilon}{2}. \tag{3.218}
$$

Using the symmetry of the kernel $a(\cdot,\cdot)$ in model 2, we find

$$\begin{aligned}\mathbb{E}_{(0,A),(j,A)}\left[1_{\{\tau<\infty\}}\right] &= \mathbb{E}_{(0,A),(j,A)}\left[\int_0^\infty \mathrm{d}s\, 1_{\{\tau\in\mathrm{d}s\}}\right]\\ &\le \mathbb{E}_{(0,A),(j,A)}\left[\int_0^\infty \mathrm{d}s \sum_{k\in\mathbb{G}} 1_{\mathcal{E}(s)}1_{\mathcal{E}'(s)}1_{\{RW=k\}}1_{\{RW'=k\}}\right]\\ &\le \mathbb{E}_{(0,A),(j,A)}\left[\int_0^\infty \mathrm{d}s \sum_{k\in\mathbb{G}} \hat{a}_{T(s)}(0,k)\,\hat{a}_{T'(s)}(j,k)\,1_{\mathcal{E}(s)}\,1_{\mathcal{E}'(s)}\right]\\ &\le \mathbb{E}_{(0,A),(j,A)}\left[\int_0^\infty \mathrm{d}s\, \hat{a}_{T(s)+T'(s)}(j,0)1_{\mathcal{E}(s)}1_{\mathcal{E}'(s)}\right].\end{aligned} \tag{3.219}$$

The last integral in the right-hand side is dominated by $I_{\hat{a},\gamma}$ (recall (2.100)). Since, for all $t\in\mathbb{R}_{\ge 0}$,

$$\lim_{\|j\|\to\infty} \hat{a}_t(0,j) = 0, \tag{3.220}$$

it follows by dominated convergence that for each $\epsilon>0$ we can find an L such that, for all $\|j\|>L$,

$$\mathbb{E}_{(0,A),(j,A)}\left[1_{\{\tau<\infty\}}\right] < \tfrac{\epsilon}{2}. \tag{3.221}$$

Hence, for L sufficiently large, we find

$$\begin{aligned}&\lim_{m\to\infty} \sum_{j\in\mathbb{G},|\|j\||>L} \mathbb{E}_{(0,D_m),(0,D_m)}[\hat{a}_{T(\tau^*)+T'(\tau^*)}(0,j)]\left[\mathbb{E}_{(0,A),(j,A)}\left[1_{\{\tau<\infty\}}\right]\right]\\ &\le \lim_{m\to\infty} \tfrac{\epsilon}{2} \sum_{j\in\mathbb{G},\|j\|>L} \mathbb{E}_{(0,D_m),(0,D_m)}[\hat{a}_{T(\tau^*)+T'(\tau^*)}(0,j)] \le \tfrac{\epsilon}{2}.\end{aligned} \tag{3.222}$$

Claim 2: For L given as in Claim 1,

$$\lim_{m\to\infty} \sum_{j\in\mathbb{G},\|j\|\le L} \mathbb{E}_{(0,D_m)^2}[\hat{a}_{T(\tau^*)+T'(\tau^*)}(0,j)]\,\mathbb{E}^{(0,A),(j,A)}[1_{\{\tau<\infty\}}] < \frac{\epsilon}{2}. \tag{3.223}$$

For the first sum, note that

$$\begin{aligned}&\lim_{m\to\infty} \sum_{\substack{j\in\mathbb{G}\\ \|j\|\le L}} \mathbb{E}_{(0,D_m),(0,D_m)}[\hat{a}_{(T(\tau^*)+T'(\tau^*))}(0,j)]\,\mathbb{E}_{(0,A),(j,A)}\left[1_{\{\tau<\infty\}}\right]\\ &\le \lim_{m\to\infty} \sum_{\substack{j\in\mathbb{G}\\ \|j\|\le L}} \mathbb{E}_{(0,D_m),(0,D_m)}[\hat{a}_{T(\tau^*)+T'(\tau^*)}(0,j)]\\ &= \lim_{m\to\infty} \sum_{\substack{j\in\mathbb{G}\\ \|j\|\le L}} \mathbb{E}_{(0,A),(0,D_m)}[\hat{a}_{T(\tau^*)+T'(\tau^*)}(0,j)],\end{aligned} \tag{3.224}$$

where in the last equality we condition on the first time one of the two random walks wakes up, and use the strong Markov property. We will show that the right-hand side tends to zero as $m\to\infty$. Recall that we assumed (2.76): $e_m \sim Bm^{-\beta}$ for $\beta>0$. Note that, in order for the random walks to be both active at the same time, the random walk starting in $(0, D_m)$ has to become active at least once. Hence, for all $t \geq 0$, we have

$$\lim_{m\to\infty} \mathbb{P}_{(0,D_m),(0,A)}(\tau^* \leq t) \leq \lim_{m\to\infty} 1 - e^{-e_m t} = 0. \tag{3.225}$$

By (3.169) and [31], we also have for the random walk starting in $(0, A)$ that

$$\lim_{t\to\infty} T(t) \sim ct^\gamma. \tag{3.226}$$

Fix $\epsilon > 0$. Since $\lim_{t\to\infty} \hat{a}_t(0, j) = 0$ for all $j \in \mathbb{G}$, we can find a $T^\star$ such that, for all $t > T^\star$,

$$\sum_{\substack{j\in\mathbb{G}\\ \|j\|\leq L}} \hat{a}_t(0, j) < \tfrac{\epsilon}{6}. \tag{3.227}$$

By (3.226), we can find a $\tilde{t} \in \mathbb{R}_{\geq 0}$ such that $\mathbb{P}_{(0,A)}(T(\tilde{t}) > T^\star) \geq 1 - \frac{\epsilon}{6}$. By (3.225), we can find an $M \in \mathbb{N}_0$ such that for all $m > M$,

$$\lim_{m\to\infty} \mathbb{P}_{(0,D_m),(0,A)}(\tau^* \leq \tilde{t}) < \tfrac{\epsilon}{6}, \tag{3.228}$$

and hence

$$\lim_{m\to\infty} \sum_{\substack{j\in\mathbb{G}\\ \|j\|\leq L}} \mathbb{E}_{(0,A),(0,D_m)}[\hat{a}_{T(\tau^*)+T'(\tau^*)}(0, j)] < \tfrac{\epsilon}{6} + \tfrac{\epsilon}{6} + \tfrac{\epsilon}{6} = \tfrac{\epsilon}{2}. \tag{3.229}$$

□

4. Uniqueness of the invariant measure ν_θ when $\rho = \infty$.

Lemma 3.3.11 (Uniqueness of and convergence to ν_θ.). *For all $\theta \in (0, 1)$ there exists a unique invariant measure ν_θ such that $\lim_{t\to\infty} \mu(t) = \nu_\theta$ for all colour regular $\mu(0) \in \mathcal{T}_\theta^{\text{erg}}$.*

Proof. Suppose that ν_θ^1 and ν_θ^2 and are two different weak limit points of $\mu(t_n)$ as $t_n \to \infty$, and that $\mu \in \mathcal{R}_\theta^{(2)}$ is colour regular. Let $(\bar{Z}(t))_{t\geq 0} = (Z^1(t), Z^2(t))_{t\geq 0}$ be the coupled process from (3.133) with $\mathcal{L}(\bar{Z}(0)) = \bar{\nu}_\theta$, $\mathcal{L}(Z^1(0)) = \nu_\theta^1$ and $\mathcal{L}(Z^2(0)) = \nu_\theta^2$. Define the process Y^1 by

$$\begin{aligned} &Y^1 = \big(Y^1(m)\big)_{m\in\{-1\}\cup\mathbb{N}_0},\\ &Y^1(-1) = (x_i^1(0))_{i\in\mathbb{G}}, \qquad Y^1(m) = (y_{i,m}^1(0))_{i\in\mathbb{G}} \text{ for } m \in \mathbb{N}_0. \end{aligned} \tag{3.230}$$

Thus, Y^1 has state space $[0, 1]^{\mathbb{G}}$ and $\mathcal{L}(Y^1) = \mathcal{L}(Z^1(0)) = \nu_\theta^1$. We can interpret Y^1 as a process that describes the states of the population *in the seed-bank direction.* Similarly, define the process Y^2 by

$$\begin{aligned} &Y^2 = \big(Y^2(m)\big)_{m\in\{-1\}\cup\mathbb{N}_0},\\ &Y^2(-1) = (x_i^2(0))_{i\in\mathbb{G}}, \qquad Y^2(m) = (y_{i,m}^2(0))_{i\in\mathbb{G}} \text{ for } m \in \mathbb{N}_0. \end{aligned} \tag{3.231}$$

Thus, Y^2 has state space $[0, 1]^{\mathbb{G}}$ and $\mathcal{L}(Y^2) = \mathcal{L}(Z^2(0)) = \nu_\theta^2$.

Define the σ-algebra's $\mathcal{B}^1_M$ and $\mathcal{B}^1$, respectively, $\mathcal{B}^2_M$ and $\mathcal{B}^2$ by

$$\mathcal{B}^k = \cap_{M\in\mathbb{N}_0}\mathcal{B}^k_M, \qquad \mathcal{B}^k_M = \sigma\big(\, y^k_{i,m} \colon\, i \in \mathbb{G},\, m \geq M\big), \quad k \in \{1,2\}. \tag{3.232}$$

Here, $\mathcal{B}^1$ and $\mathcal{B}^2$ are *the tail-σ-algebras in the seed-bank direction.* By Lemma 3.3.10, we have

$$\lim_{m\to\infty} \mathcal{L}_{\nu^1_\theta}(y_{i,m}) = \lim_{m\to\infty} \mathcal{L}_{\nu^2_\theta}(y_{i,m}) = \delta_\theta. \tag{3.233}$$

Hence, $\mathcal{B}^1 = \mathcal{B}^2$, both are trivial, and ν^1_θ and ν^2_θ agree on $\mathcal{B}$. Therefore Goldstein's Theorem [39] implies that there exists a successful coupling of Y^1 and Y^2. Consequently, there exists a random variable $T^{\mathrm{coup}} \in \{-1\} \cup \mathbb{N}_0$ such that, for all $m \geq T^{\mathrm{coup}}$, $Y^1(m) = Y^2(m)$, i.e., $|\delta_{i,m}(0)| = 0$ for all $i \in \mathbb{G}$ and $\mathbb{P}(T^{\mathrm{coup}} < \infty) = 1$. Hence

$$\hat{\mathbb{E}}\left[|\Delta_i(0)| + \sum_{m\in\mathbb{N}_0} K_m|\delta_i(0)|\right] = \hat{\mathbb{E}}\left[|\Delta_i(0)| + \sum_{m=0}^{T^{\mathrm{coup}}} K_m|\delta_i(0)|\right]. \tag{3.234}$$

However, we cannot conclude that the left-hand side of (3.234) is finite. Therefore, let $\bar{\nu}_\theta|_{\{T^{\mathrm{coup}}<T\}}$ denote the restriction of the measure $\bar{\nu}_\theta$ to the set $\{T^{\mathrm{coup}} < T\}$. Since $\{T^{\mathrm{coup}} < T\}$ is a translation-invariant event in the spatial direction, the measure $\bar{\nu}_\theta|_{\{T^{\mathrm{coup}}<T\}}$ is translation invariant. Moreover,

$$\begin{aligned} &\hat{\mathbb{E}}_{\bar{\nu}_\theta|_{\{T^{\mathrm{coup}}<T\}}}\left[|\Delta_i(0)| + \sum_{m\in\mathbb{N}_0} K_m|\delta_i(0)|\right] \\ &= \hat{\mathbb{E}}_{\bar{\nu}_\theta|_{\{T^{\mathrm{coup}}<T\}}}\left[|\Delta_i(0)| + \sum_{m=0}^{T} K_m|\delta_i(0)|\right] < \infty. \end{aligned} \tag{3.235}$$

Therefore we can use the dynamics in (3.191) and conclude that, for all $T \in \mathbb{N}$, $\hat{\mathbb{P}}_{\bar{\nu}_\theta|_{\{T^{\mathrm{coup}}<T\}}}(E_0 \times E_0) = 1$ (recall (3.201)). Since $\lim_{T\to\infty} \bar{\nu}_\theta|_{\{T^{\mathrm{coup}}<T\}} = \bar{\nu}_\theta$, it follows that

$$\hat{\mathbb{P}}_{\bar{\nu}_\theta}(E_0 \times E_0) = 1. \tag{3.236}$$

By (3.206) and (3.207), we conclude that $\nu^1_\theta = \nu^2_\theta$ and hence that all weak limit points of $(\mu(t))_{t\geq 0}$ are the same. Suppose now that $\mu^1(0) \in \mathcal{T}^{\mathrm{erg}}_\theta$ and $\mu^2(0) \in \mathcal{T}^{\mathrm{erg}}_\theta$ are two different colour regular initial measures. By the above argument, we know that $\lim_{t\to\infty} \mu^1(t) = \nu^1_\theta$ and $\lim_{t\to\infty} \mu^2(t) = \nu^2_\theta$. By Lemma 3.3.10, we know that ν^1_θ and ν^2_θ have the same trivial tail-σ-algebras in the seed-bank direction. Hence, repeating the above argument, we find that $\nu^1_\theta = \nu^2_\theta$. We conclude that for each colour regular initial measure $\mu \in \mathcal{T}^{\mathrm{erg}}_\theta$ the SSDE in (2.12)–(2.13) converges to a unique non-trivial equilibrium measure ν_θ. □

5. Ergodicity, mixing and associatedness. The equivalent of Lemma 3.2.13 for $\rho = \infty$ follows in the same way as for $\rho < \infty$.

§3.3.4 Proof of the dichotomy

Theorem 2.3.3(I)(a) follows from Lemma (3.3.7) and Steps 3-5 in Section 3.3.3. The equality $\mathbb{E}_{\nu_\theta}[x_0] = \mathbb{E}_{\nu_\theta}[y_{0,m}] = \theta$, $m \in \mathbb{N}_0$, follows from (2.12)–(2.13), the fact that ν_θ is an equilibrium measure, and the preservation of θ (see Section 2.3.2). Theorem 2.3.3(I)(b) follows by combining Lemma 3.3.3 with the analogue of Lemma 3.2.5. Theorem 2.3.3(II) follows from Lemmas 3.3.3, 3.3.10, 3.3.11, the analogue of Lemma 3.2.5, and Step 6 in Section 3.3.3. The equality $\mathbb{E}_{\nu_\theta}[x_0] = \mathbb{E}_{\nu_\theta}[y_{0,m}] = \theta$, $m \in \mathbb{N}_0$, follows from (3.165) in Step 1 of Section 3.3.3.

Corollary 2.3.4(1) corresponds to $\gamma \in (1,\infty)$ and $\rho < \infty$, and migration dominates. Corollary 2.3.4(2) corresponds to $\gamma \in [\frac{1}{2}, 1]$ and $\rho = \infty$, and $I_{\hat{a},\gamma}$ shows in interplay between migration and seed-bank. Corollary 2.3.4(3) corresponds to $\gamma \in (0, \frac{1}{2}, 1)$ and $\rho = \infty$, and the seed-bank dominates: $I_{\hat{a},\gamma} < \infty$ because $\hat{a}_t(0,0) \leq 1$.

§3.3.5 Different dichotomy for asymmetric migration

It remains to explain how the counterexample below Theorem 2.3.3 arises. We focus on the case when $\rho < \infty$, which implies $\mathbb{E}(\tau) < \infty$, but we assume $\mathbb{E}(\tau^2) = \infty$. Therefore the central limit theorem does not hold for $T(t)$, $T'(t)$, and $\Delta(t) \gg \sqrt{M(t)}$. Hence (3.75) must be replaced by

$$f(t) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \mathrm{d}\phi \, \mathrm{e}^{-[1+o(1)]\,2Bt\,[1-\hat{a}(\phi)]}\, \mathbb{E}\left[\cos\Big(\Delta(t)\tilde{a}(\phi)\Big)\right]. \tag{3.237}$$

The key observation is that if $\tilde{a}(\phi) \neq 0$ (due to the asymmetry of $a(\cdot,\cdot)$; recall (3.66)), then the expectation in (3.237) can change the integrability properties of $f(t)$.

Under the assumption that τ has a *one-sided stable distribution* with parameter $\gamma \in (1,2)$, we have (3.70) with $A = \chi/(1+\rho)$ and $B = 1/(1+\rho)$, while there exists a constant $C \in (0,\infty)$ such that (see [34, Chapter XVII])

$$\mathbb{E}[\cos(\Delta(t)\tilde{a}(\phi))] = \mathrm{e}^{-[1+o(t)]\,At|C\tilde{a}(\phi)|^\gamma}. \tag{3.238}$$

Substituting (3.238) into (3.237), we see that for large t the contribution to $f(t)$ comes from ϕ such that $\hat{a}(\phi) \to 1$ and $\tilde{a}(\phi) \to 0$. By our choice of the migration kernel in (2.90), this holds as $\phi = (\phi_1,\phi_2) \to (0,0)$. Using that $1 - \hat{a}(\phi) \sim \frac{1}{2}(\phi_1^2 + \phi_2^2)$ and $\tilde{a}(\phi) \sim \frac{1}{2}\eta(\phi_1+\phi_2)$ for $(\phi_1,\phi_2) \to (0,0)$, we find that (3.237) equals

$$f(t) = \frac{1}{(2\pi)^2} \int_{[-\pi,\pi]^2} \mathrm{d}\phi \, \mathrm{e}^{-[1+o(1)]\,\{Bt(\phi_1^2+\phi_2^2)+At[|\frac{1}{2}C\eta(\phi_1+\phi_2)|]^\gamma\}}, \qquad t \to \infty. \tag{3.239}$$

Hence the integral in (3.239) is determined by ϕ such that

$$B(\phi_1^2+\phi_2^2) + A\big[|\tfrac{1}{2}C\eta(\phi_1+\phi_2)|\big]^\gamma \leq \frac{c}{t}. \tag{3.240}$$

for c a positive constant, and we find that $f(t) \asymp t^{-(\frac{1}{\gamma}+\frac{1}{2})}$. Since $\gamma \in (1,2)$, $f(t)$ is much smaller than $\hat{a}_t(0,0) \asymp 1/t$, valid for two-dimensional simple random walk. Thus we see that $t \mapsto f(t)$ is integrable, while $t \mapsto \hat{a}_t(0,0)$ is not.

§3.3.6 Modulation of the law of the wake-up times by a slowly varying function

The integral in (2.96) is the *total hazard of coalescence* of two dual lineages:

- If $\gamma \in (0,1)$, then the probability for each of the lineages to be active at time s decays like $\asymp \varphi(s)^{-1}s^{-(1-\gamma)}$ [1]. Hence the expected total time they are active up to time s is $\asymp \varphi(s)^{-1}s^{\gamma}$. Because the lineages only move when they are active, the probability that the two lineages meet at time s is $\asymp a^{(N)}_{\varphi(s)^{-1}s^{\gamma}}(0,0)$. Hence the total hazard is $\asymp \int_1^\infty \mathrm{d}s\, \varphi(s)^{-2}s^{-2(1-\gamma)}\, a^{(N)}_{\varphi(s)^{-1}s^{\gamma}}(0,0)$. After the transformation $t = t(s) = \varphi(s)^{-1}s^{\gamma}$, we get the integral in (2.96), modulo a constant. (When carrying out this transformation, we need that $\lim_{s\to\infty} s\varphi'(s)/\varphi(s) = 0$, which is immediate from (2.95), and $\varphi(t(s))/\varphi(s) \asymp 1$ as $s \to \infty$, which is immediate from the bound we imposed on ψ together with the fact that $\lim_{s\to\infty} \log\varphi(s)/\log s = 0$.)

- If $\gamma = 1$, then the probability for each of the lineages to be active at time s decays like $\hat{\varphi}(s)^{-1}$ [1]. Hence the expected total time they are active up to time s is $\asymp s\hat{\varphi}(s)^{-1}$. Hence the total hazard is $\asymp \int_1^\infty \mathrm{d}s\, \hat{\varphi}(s)^{-2}\, a^{(N)}_{\hat{\varphi}(s)^{-1}s}(0,0)$. After the transformation $t = t(s) = \hat{\varphi}(s)^{-1}s$, we get the integral in (2.96), modulo a constant.

§3.4 Proofs: Long-time behaviour for Model 3

The arguments for model 2 in Section 3.3 all carry over with minor adaptations. The only difference is that for $\rho = \infty$ the clustering criterion changes. In this section we prove the new clustering criterion and comment on the modifications needed in the corresponding proofs for model 2 in Section 3.3.

§3.4.1 Moment relations

Like in model 1 and 2, we can relate the first and second moments of the system in (2.18)–(2.19) to the random walk that evolves according to the transition kernel $b^{(3)}(\cdot,\cdot)$ on $\mathbb{G} \times \{A, (D_m)_{m\in\mathbb{N}_0}\}$ given by (2.53). Replacing in Lemma 3.3.1 the kernel $b^{(2)}(\cdot,\cdot)$ by $b^{(3)}(\cdot,\cdot)$, we find the moment relation for model 3. Also here these moment relations hold for all $g \in \mathcal{G}$. Moreover these moment relations holds for $\rho < \infty$ as well as for $\rho = \infty$.

§3.4.2 The clustering case

To obtain the equivalent of Lemma 3.3.3, we need to replace the kernel $\hat{a}(\cdot,\cdot)$ by the convoluted kernel $(\hat{a} * \hat{a}^\dagger)(\cdot,\cdot)$. Each time one of the two copies of the random walk with migration kernel $a(\cdot,\cdot)$ moves from the active state to the dormant state, it makes a transition according to the displacement kernel $a^\dagger(\cdot,\cdot)$ (recall (2.97)). Therefore the expression in (3.60) needs to be replaced by

$$I = \int_0^\infty \mathrm{d}t \sum_{k,k'\in\mathbb{N}} \sum_{i,i'\in\mathbb{G}} \sum_{j\in\mathbb{G}} \mathbb{E}_{(0,A)} \Big[\hat{a}_{T(k,t)}(0,i)\, \hat{a}_{T'(k',t)}(0,i')\, \hat{a}^\dagger_k(i,j)\, \hat{a}^\dagger_{k'}(i',j)\, 1_{\mathcal{E}(k,t)}\, 1_{\mathcal{E}'(k',t)} \Big], \tag{3.241}$$

where $\hat{a}^\dagger_k(\cdot,\cdot)$ is the step-k transition kernel of the random walk with displacement kernel $\hat{a}^\dagger(\cdot,\cdot)$. Using the *symmetry* of both kernels, we can carry out the sum over j, i' and write

$$\begin{aligned} I &= \int_0^\infty \mathrm{d}t \sum_{k,k'\in\mathbb{N}} \sum_{j\in\mathbb{G}} \mathbb{E}_{(0,A)} \Big[\hat{a}_{T(k,t)+T'(k',t)}(0,j)\ \hat{a}^\dagger_{k+k'}(0,j)\, 1_{\mathcal{E}(k,t)}\, 1_{\mathcal{E}'(k',t)} \Big] \\ &= \int_0^\infty \mathrm{d}t \sum_{j\in\mathbb{G}} \mathbb{E}_{(0,A)} \Big[\hat{a}_{T(t)+T'(t)}(0,j)\ \hat{a}^\dagger_{N(t)+N'(t)}(0,j)\, 1_{\mathcal{E}(t)}\, 1_{\mathcal{E}'(t)} \Big] \\ &= \int_0^\infty \mathrm{d}t\, \mathbb{E}_{(0,A)} \Big[\big(\hat{a}_{T(t)+T'(t)} * \hat{a}^\dagger_{N(t)+N'(t)}\big)\,(0,0)\, 1_{\mathcal{E}(t)}\, 1_{\mathcal{E}'(t)} \Big]. \end{aligned} \tag{3.242}$$

The last expression is the analogue of (3.63).

For $\rho < \infty$, following the same line of argument as for model 2, we find with the help of (2.98) that

$$I \asymp \int_1^\infty \mathrm{d}t\, (\hat{a}_t * \hat{a}^\dagger_t)(0,0). \tag{3.243}$$

For $\rho = \infty$, with the help of the Fourier transform we compute

$$\begin{aligned} &\mathbb{E}_{(0,A)} \big[\big(a_{T(t)+T'(t)} * a^\dagger_{N(t)+N'(t)}\big)\,(0,0) \,\big] \\ &= \mathbb{E}_{(0,A)} \left[\frac{1}{(2\pi)^d} \int_{(-\pi,\pi]^d} \mathrm{d}\phi\, \mathrm{e}^{-(T(t)+T'(t))[1-\hat{a}(\phi)]}\ \hat{a}^\dagger(\phi)^{N(t)+N'(t)} \right] \\ &= \frac{1}{(2\pi)^d} \int_{(-\pi,\pi]^d} \mathrm{d}\phi\, \mathrm{e}^{-[1+o(1)]\,2ct^{-\gamma}\,[1-\hat{a}(\phi)]}\ \mathrm{e}^{-[1+o(1)]\,2t^{-\gamma}[1-\hat{a}^\dagger(\phi)]} \\ &\asymp (\hat{a}_{ct^{-\gamma}} * \hat{a}^\dagger_{t^{-\gamma}})(0,0) \asymp (\hat{a}_{t^{-\gamma}} * \hat{a}^\dagger_{t^{-\gamma}})(0,0), \end{aligned} \tag{3.244}$$

where we use (2.98), (3.169) and the fact that deviations of $T(t)/t^\gamma$ and $T'(t)/t^\gamma$ away from order 1 are stretched exponentially costly in t [31]. Hence

$$I \asymp \int_1^\infty \mathrm{d}t\, t^{-2(1-\gamma)} (\hat{a}_{t^\gamma} * \hat{a}^\dagger_{t^\gamma})(0,0). \tag{3.245}$$

Putting $s = t^\gamma$ we obtain, instead of (3.180),

$$I = \infty \quad \Longleftrightarrow \quad I_{\hat{a}*\hat{a}^\dagger,\gamma} = \infty \tag{3.246}$$

with

$$I_{\hat{a}*\hat{a}^\dagger,\gamma} = \int_1^\infty \mathrm{d}s\, s^{-(1-\gamma)/\gamma}\, (\hat{a}_s * \hat{a}^\dagger_s)(0,0), \tag{3.247}$$

which is precisely the integral in (2.100).

§3.4.3 The coexistence case

The coexistence results in Theorem 2.3.6 follow for both $\rho < \infty$ and $\rho = \infty$ by the same type of argument as the one we used for model 2 in Section 3.3.3. We replace (2.12)–(2.13) by (2.18)–(2.19), replace $b^{(2)}(\cdot,\cdot)$ (see 2.41) by $b^{(3)}(\cdot,\cdot)$ (see 2.53), and use the Fourier transform of $\hat{a} * \hat{a}^{\dagger}(\cdot,\cdot)$ instead of $\hat{a}(\cdot,\cdot)$. The key of the argument is that, in the coexistence case, for $\rho < \infty$ we have $I_{\hat{a}*\hat{a}^{\dagger}} < \infty$, while for $\rho = \infty$ we have $I_{\hat{a}*\hat{a}^{\dagger},\gamma} < \infty$.

§3.4.4 Proof of the dichotomy

This follows in exactly the same way as for model 2.