

## EXPLICIT ARITHMETIC INTERSECTION THEORY AND COMPUTATION OF NÉRON-TATE HEIGHTS

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ABSTRACT. We describe a general algorithm for computing intersection pairings on arithmetic surfaces. We have implemented our algorithm for curves over  $\mathbb{Q}$ , and we show how to use it to compute regulators for a number of Jacobians of smooth plane quartics, and to numerically verify the conjecture of Birch and Swinnerton-Dyer for the Jacobian of the split Cartan curve of level 13, up to squares.

### 1. INTRODUCTION

If  $A/K$  is an abelian variety over a global field  $K$ , then an ample symmetric divisor class  $c$  on  $A$  induces a non-degenerate quadratic form  $\hat{h}_c$  on  $A(K)$ , the *Néron-Tate height* or *canonical height* with respect to  $c$ . Given  $P \in A(K)$ , the height of  $P$  can be defined as

$$\hat{h}_c(P) = \lim_{n \rightarrow \infty} \frac{1}{n^2} h_c(nP),$$

where  $h_c$  is a Weil height on  $A$  induced by  $c$  (see [Nér65] and [HS00, Section B.5]). The Néron-Tate height also induces a symmetric bilinear pairing on  $A(K)$  given by

$$\hat{h}_c(P, Q) = \frac{1}{2} \left( \hat{h}_c(P + Q) - \hat{h}_c(P) - \hat{h}_c(Q) \right).$$

An algorithm to compute the Néron-Tate height is required, for instance, to compute generators of  $A(K)$ . More precisely, the canonical height endows  $A(K) \otimes_K \mathbb{R}$  with the structure of a Euclidean vector space and  $A(K)/A(K)_{\text{tors}}$  embeds into this vector space as a lattice  $\Lambda$ . Given generators of a subgroup of  $A(K)/A(K)_{\text{tors}}$  of finite index, we can find generators of the full group by saturating the corresponding sublattice of  $\Lambda$ . All known methods for this saturation step require an algorithm to compute the canonical height (see [Sik95, FS97, Sto02]). Another important application is the computation of the regulator of  $A/K$ , a quantity which appears in the conjecture of Birch and Swinnerton-Dyer. The regulator of  $A/K$  is the Gram determinant of a set of generators of  $\Lambda$  (for a certain choice of  $c$ ). If we only have generators of a finite index subgroup available, then we can still compute the regulator up to an integral square factor.

We can construct  $\hat{h}_c$  explicitly if we have explicit formulas for a map to projective space corresponding to the linear system of  $c$ . For instance, an explicit embedding of the Kummer variety of  $A$  has been used to give algorithms for the computation of Néron-Tate heights for elliptic curves [Sil88, MS16a] and Jacobians of hyperelliptic

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curves of genus 2 [FS97, Sto02, MS16b] and genus 3 [Sto17]. However, this approach becomes quickly infeasible if we increase the dimension of  $A$ .

But if  $J$  is the Jacobian variety of a smooth projective geometrically connected curve  $C/K$ , then there is an alternative way due to Faltings and Hriljac to describe the Néron-Tate height on  $J/K$  with respect to twice the class of a symmetric theta divisor as follows (see section 4.1 for details):

$$(1) \quad \hat{h}_{2\vartheta}([D], [E]) = - \sum_{v \in M_K} \langle D, E \rangle_v.$$

Here  $D$  and  $E$  are two divisors of degree 0 on  $C$  without common component,  $M_K$  denotes the set of places of  $K$ , and  $\langle D, E \rangle_v$  denotes the local Néron pairing of  $D$  and  $E$  at  $v$ , which is defined below in sections 2 (for the non-archimedean places) and 3 (for the archimedean places).

In this note, we show how to turn (1) into an algorithm for computing  $\hat{h}_{2\vartheta}$  when  $K = \mathbb{Q}$  (our algorithm can be generalised easily to work over general global fields). This was already done independently by the second-named and the third-named authors in [Hol12] and [Mül14] in the special case of hyperelliptic curves. But for Jacobians of non-hyperelliptic curves, no practical algorithms for computing Néron-Tate heights are known, and therefore no numerical evidence for the Birch and Swinnerton-Dyer conjecture has been collected.

In the present paper we develop such an algorithm and we give numerical evidence for the conjecture of Birch and Swinnerton-Dyer for the Jacobian of the Cartan modular curve of level 13. Our main contribution is a new way to compute the non-archimedean local Néron pairings. In fact, we give a new algorithm for computing the intersection pairing of two divisors without common component on a regular arithmetic surface, which might be of independent interest. In short, we lift divisors from the generic fibre to the arithmetic surface by saturating the defining ideals, and we use an inclusion-exclusion principle to deal with divisors intersecting on several affine patches. The archimedean local Néron pairings  $\langle D, E \rangle_\infty$  are computed in essentially the same way as in [Hol12] and [Mül14], by pulling back a translate of the Riemann theta function to  $C(\mathbb{C})$ . This requires explicitly computing period matrices and Abel-Jacobi maps on Riemann surfaces; we use the recent algorithms of Neurohr [Neu18, Chapter 4] and Molin-Neurohr [MN19].

The paper is organised as follows. In section 2 we introduce our algorithm to compute non-archimedean local Néron pairings. The computation of archimedean local Néron pairings is discussed in section 3. The topic of section 4 is how to apply these to compute canonical heights using (1). Finally, in section 5 we demonstrate the practicality of our algorithm by computing the Néron-Tate regulator, up to an integral square, for several Jacobians of smooth plane quartics including the split (or, equivalently, non-split) Cartan modular curve of level 13, and we numerically verify BSD for the latter curve up to an integral square.

## 2. THE NON-ARCHIMEDEAN NÉRON PAIRING

For simplicity of exposition, we restrict ourselves to curves over the rational numbers; everything we do generalises without substantial difficulty to global fields. For background on arithmetic surfaces and their intersection pairing, we refer to Liu's book [Liu02]. In this section we work over a fixed prime  $p$  of  $\mathbb{Z}$ . Let  $C/\mathbb{Q}_p$  be a smooth proper geometrically connected curve, and let  $\mathcal{C}/\mathbb{Z}_p$  be a proper regular

model of  $C$ . Because  $C$  is a regular surface, we have an intersection pairing between divisors on  $C$  having no components in common; if  $\mathcal{P}$  and  $\mathcal{Q}$  are distinct prime divisors the pairing is given by

$$\iota(\mathcal{P}, \mathcal{Q}) = \sum_{P \in \mathcal{C}^0} \text{length}_{\mathcal{O}_{C,P}} \left( \frac{\mathcal{O}_{C,P}}{\mathcal{O}_{C,P}(-\mathcal{P}) + \mathcal{O}_{C,P}(-\mathcal{Q})} \right) \log \#k(P);$$

here  $\mathcal{C}^0$  denotes the set of closed points of  $C$ , and  $k(P)$  denotes the residue field of the point  $P$ . We extend to arbitrary divisors with no common components by additivity.

In general, this intersection pairing fails to respect linear equivalence. However, if  $\mathcal{D}$  is a divisor on  $C$  whose restriction to the generic fibre  $C$  has degree 0, and  $Y$  is a divisor on  $C$  pulled back from a divisor on  $\text{Spec } \mathbb{Z}_p$ , then  $\mathcal{D} \cdot Y = 0$ . By the usual formalism with a moving lemma, this allows us to define the intersection pairing between any two divisors  $\mathcal{D}$  and  $\mathcal{E}$  on  $C$  as long as the restrictions of  $\mathcal{D}$  and  $\mathcal{E}$  to the generic fibre  $C$  have degree 0 and disjoint support.

If  $D$  is a divisor on  $C$ , we write  $\mathcal{D}$  for the unique horizontal divisor on  $C$  whose generic fibre is  $D$ . For a divisor  $D$  of degree 0 on  $C$ , we write  $\Phi(D)$  for a vertical  $\mathbb{Q}$ -divisor on  $C$  such that for every vertical divisor  $Y$  on  $C$ , we have  $\iota(Y, \mathcal{D} + \Phi(D)) = 0$ ; this  $\Phi(D)$  always exists, and is unique up to the addition of divisors pulled back from  $\text{Spec } \mathbb{Z}_p$  (see [Lan88, Theorem III.3.6]).

Let  $D$  and  $E$  be two divisors on  $C$ , of degree 0 and with disjoint support. Then the *local Néron pairing between  $D$  and  $E$*  is given by

$$\langle D, E \rangle_p := \iota(\mathcal{D} + \Phi(D), \mathcal{E} + \Phi(E)).$$

This pairing is bilinear and symmetric, but it does not respect linear equivalence; see [Lan88, Theorem III.5.2].

Our goal in this section is to compute the pairing  $\langle D, E \rangle_p$ , assuming that  $D$  and  $E$  are given to us (arranging suitable  $D$  and  $E$ , and identifying those primes  $p$  which may yield a non-zero pairing, will be discussed in section 4). A first step in applying the above definitions is to compute a regular model of  $C$  over  $\mathbb{Z}_p$ . Algorithms are available for this in **Magma**, one due to Steve Donnelly, and another to Tim Dokchitser [Dok18]. For our examples below we used Donnelly's implementation as slightly more functionality was available, but our emphasis in this section is on providing a general-purpose algorithm which should be easily adapted to take advantage of future developments in the computation of regular models.

**2.1. The naive intersection pairing.** To facilitate the computation of the local Néron pairing at non-archimedean places, we will introduce a *naive* intersection pairing, which coincides with the standard intersection pairing on regular schemes, and then give an algorithm to compute the naive intersection pairing in a fairly general setting.

**Situation 2.1.** We fix the following data:

- an integral domain  $R$  of dimension 2, flat and finitely presented over  $\mathbb{Z}$ ;
- effective Weil divisors  $\mathcal{D}$  and  $\mathcal{E}$  on  $C := \text{Spec } R$  with no common irreducible component in their support, defined by the vanishing of ideals  $I_{\mathcal{D}}$  and  $I_{\mathcal{E}}$  in  $R$  (i.e.,  $I_{\mathcal{D}} = \mathcal{O}_C(-\mathcal{D}) \subseteq \mathcal{O}_C$ , and analogously for  $\mathcal{E}$ );
- a constructible subset  $V$  of  $C$ .

For computational purposes, we suppose that a finite presentation of  $R$  is given, along with generators of  $I_{\mathcal{D}}$  and  $I_{\mathcal{E}}$ . Moreover, we suppose that  $V$  is given as a disjoint union of intersections of open and closed subsets.

**Definition 2.2.** Let  $P$  be a closed point of  $\mathcal{C}$  lying over  $p$ . The *naive intersection number of  $\mathcal{D}$  and  $\mathcal{E}$  at  $P$*  is given by

$$\iota_P^{naive}(\mathcal{D}, \mathcal{E}) := \text{length}_{\mathcal{O}_{\mathcal{C},P}} \left( \frac{\mathcal{O}_{\mathcal{C},P}}{I_{\mathcal{D},P} + I_{\mathcal{E},P}} \right) \log \#k(P),$$

where  $I_{\mathcal{D},p} = I_{\mathcal{D}} \otimes \mathcal{O}_{\mathcal{C},p}$  and likewise for  $\mathcal{E}$ . If  $W$  is any subset of  $\mathcal{C}$ , we define

$$\iota_W^{naive}(\mathcal{D}, \mathcal{E}) := \sum_{P \in W^0} \iota_P^{naive}(\mathcal{D}, \mathcal{E}),$$

where  $W^0$  denotes the set of closed points in  $W$  lying over  $p$ .

Note that if  $\mathcal{C}$  is regular at  $P$ , then  $\iota_P^{naive}(\mathcal{D}, \mathcal{E})$  is the usual intersection pairing  $\iota_P(\mathcal{D}, \mathcal{E})$  at  $P$ . If  $W$  and  $W'$  are disjoint subsets of  $\mathcal{C}$ , then

$$(2) \quad \iota_W^{naive}(\mathcal{D}, \mathcal{E}) + \iota_{W'}^{naive}(\mathcal{D}, \mathcal{E}) = \iota_{W \cup W'}^{naive}(\mathcal{D}, \mathcal{E}).$$

We present here an algorithm for computing the naive intersection pairing  $\iota_V^{naive}(\mathcal{D}, \mathcal{E})$  for  $V$  any constructible subset of  $\mathcal{C}$ . This seems to us a reasonable level of generality to work in; constructible subsets are the most general subsets easily described by a finite amount of data, and should be flexible enough for computing local Néron pairings for any reasonable way a regular model is given to us. Note that only being able to compute the intersection pairing at points would not be sufficient, as we would then need to sum over infinitely many points, and only being able to compute it for  $V$  affine gives complications where patches of the model overlap.

**Algorithm 2.3.** Suppose we are in Situation 2.1. The following is an algorithm to compute  $\iota_V^{naive}(\mathcal{D}, \mathcal{E})$ .

**First reduction step:** By (2) we may assume  $V$  is locally closed.

**Second reduction step:** Write  $V = Z_1 \setminus Z_2$ , a difference of closed subschemes with  $Z_2 \subseteq Z_1$  closed. Then by (2) we have

$$\iota_V^{naive}(\mathcal{D}, \mathcal{E}) = \iota_{Z_1}^{naive}(\mathcal{D}, \mathcal{E}) - \iota_{Z_2}^{naive}(\mathcal{D}, \mathcal{E}),$$

so we may assume  $V$  is closed.

**Third reduction step:** Write  $V = Z(f_1, \dots, f_r)$ , with  $f_i \in R$ . For a subset  $T \subseteq \{1, \dots, r\}$  define  $S_T = \text{Spec}((\prod_{i \in T} f_i)^{-1}R)$ . Then by inclusion-exclusion we have

$$\iota_V^{naive}(\mathcal{D}, \mathcal{E}) = \sum_{T \subseteq \{1, \dots, r\}} (-1)^{\#T} \iota_{S_T}^{naive}(\mathcal{D}, \mathcal{E}).$$

Since  $S_T$  is affine, we are reduced to the case where  $V$  is the whole of  $\mathcal{C} = \text{Spec } R$ .

**Concluding the algorithm:** Since forming quotients commutes with flat base-change, we obtain

$$\iota_{\mathcal{C}}^{naive}(\mathcal{D}, \mathcal{E}) = \text{length}_R \left( \frac{R \otimes_{\mathbb{Z}} \mathbb{Z}_p}{I_{\mathcal{D}} \otimes_{\mathbb{Z}} \mathbb{Z}_p + I_{\mathcal{E}} \otimes_{\mathbb{Z}} \mathbb{Z}_p} \right) \log \#k(p).$$

This can be computed using [Mül14, Algorithm 1]. For efficiency we compute this length working modulo a sufficiently large power of  $p$ , which will be determined in Remark 4.3.

*Remark 2.4.* Note that the third reduction step is exponential in  $r$ . In the examples we have computed, the largest value of  $r$  was 4.

**2.2. Computing the intersection pairing.** Let  $C/\mathbb{Q}_p$  be a smooth projective curve,  $\mathcal{C}/\mathbb{Z}_p$  a regular model, and  $\mathcal{D}, \mathcal{E}$  two divisors on  $\mathcal{C}$  without common component. In this section, we describe several approaches to computing the intersection pairing  $\iota(\mathcal{D}, \mathcal{E})$ , depending on how  $\mathcal{C}$  is given to us.

*Regular model given by affine charts and glueing data.* Suppose that the regular model  $\mathcal{C}$  is given as a list of affine charts  $C_1, \dots, C_n$  and glueing data. We partition  $\mathcal{C}$  into constructible subsets  $V_i$  by setting  $V_i = C_i \setminus (\bigcup_{j < i} C_j)$  for each  $i \in \{1, \dots, n\}$ . Then the intersection pairing is given by

$$\iota(\mathcal{D}, \mathcal{E}) = \sum_{i \in \{1, \dots, n\}} \iota_{V_i}^{naive}(\mathcal{D}, \mathcal{E}).$$

*Regular model as described by Magma.* Magma’s regular model implementation (due to Steve Donnelly) describes the model  $\mathcal{C}$  in a slightly different way. It constructs a regular model by repeatedly blowing up non-regular points and/or components in a proper model. In this way, it creates a list of affine patches  $U_i$  together with open immersions from the generic fibre of the  $U_i$  to  $C$ . For each  $i$ , it stores a constructible subset  $V_i \subseteq U_i$ , consisting of all regular points in the special fibre which did not appear in any of the previous affine patches. These  $V_i$  form a constructible partition of the special fibre of a regular model. In this case, we simply compute

$$\iota(\mathcal{D}, \mathcal{E}) = \sum_i \iota_{V_i}^{naive}(\mathcal{D}, \mathcal{E}).$$

**2.3. Computing the non-archimedean local Néron pairing.** Let  $C/\mathbb{Q}_p$  be a smooth projective curve,  $\mathcal{C}/\mathbb{Z}_p$  a regular model, and  $D$  and  $E$  degree 0 divisors on  $C$  with disjoint support. In this section we will describe how to compute the local Néron pairing  $\langle D, E \rangle_p$ .

First we compute the extensions of  $D$  and  $E$  to horizontal divisors  $\mathcal{D}$  and  $\mathcal{E}$  on  $\mathcal{C}$ . We break  $D$  and  $E$  into their effective and anti-effective parts, then choose some extensions of these ideals to  $\mathcal{C}$  (the associated subschemes may contain many vertical components). We then saturate these ideals with respect to the prime  $p$  to obtain (ideals for) horizontal divisors. This works by the following well-known lemma.

**Lemma 2.5.** *Let  $R$  be a  $\mathbb{Z}$ -algebra and  $I$  an ideal of  $R$ . The ideal sheaf of the schematic image of  $\text{Spec } R[1/p]/(I \otimes_R R[1/p])$  in  $\text{Spec } R$  is given by the saturation*

$$(I : p^\infty) = \{r \in R : \exists n : p^n r \in I\}.$$

*Proof.* It is immediate that  $(I : p^\infty) \otimes_R R[1/p] = I \otimes_R R[1/p]$ . We need to check that, for any ideal  $J \triangleleft R$  with  $J \otimes_R R[1/p] = I \otimes_R R[1/p]$ , we have  $J \subseteq (I : p^\infty)$ . Indeed, if  $j \in J$ , then we can write  $j$  as a finite sum of elements  $\frac{i}{p^{n_i}}$  with  $i \in I$ ,  $n_i \in \mathbb{N}$ , so  $p^{\max_i n_i} j \in I$ , as required.  $\square$

To compute the vertical correction term  $\Phi(D)$ , we use the algorithm from section 2.2 to compute the intersection of  $\mathcal{D}$  with every component of the fibre of  $\mathcal{C}$  over  $p$ , and then apply simple linear algebra as in [Mül14, §4.5] to find the coefficients of  $\Phi(D)$ .

Finally, we use again the algorithm in section 2.2 to compute

$$\langle D, E \rangle_p = \iota(\mathcal{D} + \Phi(D), \mathcal{E} + \Phi(E)) = \iota(\mathcal{D}, \mathcal{E}) + \iota(\Phi(D), \mathcal{E}).$$

### 3. THE ARCHIMEDEAN NÉRON PAIRING

**3.1. Green’s functions; definition of the pairing.** Let  $C/\mathbb{C}$  be a smooth projective connected curve of genus  $g$ , and let  $\varphi$  be a volume form on  $C$ . If  $E$  is a divisor on  $C$ , we write

$$g_{E,\varphi} : C(\mathbb{C}) \setminus \text{supp}(E) \rightarrow \mathbb{R}$$

for a Green’s function on  $C(\mathbb{C})$  with respect to  $E$  (see [Lan88, II, §1]). If  $E$  has degree 0, and  $\varphi'$  is another volume form, then  $g_{E,\varphi} - g_{E,\varphi'}$  is constant. If  $D = \sum_P n_P P$  is another divisor of degree 0 with support disjoint from  $E$ , then the *local Néron pairing* is given by

$$\langle D, E \rangle_\infty := \sum_P n_P g_{E,\varphi}(P).$$

This pairing is bilinear and symmetric, and is independent of the choice of  $\varphi$ ; see [Lan88, Theorem III.5.3]. As we evaluate  $g_{E,\varphi}$  in a divisor of degree 0, we can replace  $g_{E,\varphi}$  by  $g_{E,\varphi} + c$  for a constant  $c \in \mathbb{R}$  without changing  $\langle D, E \rangle_\infty$ .

**3.2. Theta functions; a formula for the pairing.** Let  $\{\omega_1, \dots, \omega_g\}$  be an orthonormal basis of  $H^0(C, \Omega^1)$  with respect to the scalar product  $(\omega, \eta) \mapsto \frac{i}{2} \int_{C(\mathbb{C})} \omega \wedge \bar{\eta}$  and let  $\varphi := \frac{i}{2g}(\omega_1 \wedge \bar{\omega}_1 + \dots + \omega_g \wedge \bar{\omega}_g)$  be the canonical volume form. We fix a base point  $P_0 \in C(\mathbb{C})$  and denote by  $\alpha : C(\mathbb{C}) \rightarrow J(\mathbb{C})$  the Abel-Jacobi map with respect to  $P_0$ . By abuse of notation, we also denote the additive extension of  $\alpha$  to divisors on  $C$  by  $\alpha$ . Following Hriljac, we construct a Green’s function by pulling back the logarithm of a translate of the Riemann theta function  $\theta$  along  $\alpha$ . Let  $\tau \in \mathbb{C}^{g \times g}$  be the small period matrix of  $J(\mathbb{C})$ ; it has symmetric positive definite imaginary part and satisfies  $J(\mathbb{C}) \cong \mathbb{C}^g / (\mathbb{Z}^g + \tau \mathbb{Z}^g)$ . We define

$$j : \mathbb{C}^g \twoheadrightarrow \mathbb{C}^g / (\mathbb{Z}^g + \tau \mathbb{Z}^g) \xrightarrow{\cong} J(\mathbb{C}).$$

Let  $\Theta$  denote the theta divisor on  $J$  corresponding to  $\alpha$ . By a theorem of Riemann (see [Lan83, Theorem 13.4.1]), there exists a divisor  $W$  on  $C$  such that  $2W$  is canonical and such that the translate  $\Theta_{-\alpha(W)}$  of  $\Theta$  by  $-\alpha(W)$  is the divisor of the normalised (in the notation of [Lan83, §13.1]) version of the Riemann theta function

$$(3) \quad F_{\Theta_{-\alpha(W)}}(z) := \theta(z, \tau) \exp\left(\frac{\pi}{2} z^T (\text{Im } \tau)^{-1} z\right).$$

This  $W$  is in fact unique up to linear equivalence, by [Mum83, Chapter II, Theorem 3.10].

For the remainder of this section, we suppose that  $E = E_1 - E_2$ , where  $E_1$  and  $E_2$  are *non-special*. This means that they are effective of degree  $g$  with  $h^0(C, \mathcal{O}(E_i)) = 1$ . Because of the bilinearity of the Néron pairing, the following gives a formula to compute  $\langle D, E \rangle_\infty$  for all  $D \in \text{Div}^0(C)$  with support disjoint from  $E$ .

**Proposition 3.1.** *Suppose that  $D = P_1 - P_2$  with  $P_1, P_2 \in C(\mathbb{C})$  not in the support of  $E$ . Then*

$$\langle D, E \rangle_\infty = -\log \left| \frac{\theta(z_{11}, \tau) \cdot \theta(z_{22}, \tau)}{\theta(z_{12}, \tau) \cdot \theta(z_{21}, \tau)} \right| - 2\pi \text{Im}(z_E)^T \text{Im}(\tau)^{-1} \text{Im}(z_D),$$

where  $z_D, z_E, z_{ij} \in \mathbb{C}^g$  satisfy  $j(z_D) = \alpha(D)$ ,  $j(z_E) = \alpha(E)$ , and  $j(z_{ij}) = \alpha(P_i - E_j + W)$ .

For the proof of Proposition 3.1 we need the notion of a Néron function on  $J(\mathbb{C})$ ; see [Lan83, §13.1]. For each divisor  $A \in \text{Div}(J)$ , there is a Néron function with respect to  $A$ , which is uniquely determined up to adding a constant. This is a continuous function  $\lambda_A : J(\mathbb{C}) \setminus \text{supp}(A) \rightarrow \mathbb{R}$ , and together they have the following properties:

- (NF1) if  $A, B \in \text{Div}(J)$ , then  $\lambda_{A+B} - \lambda_A - \lambda_B$  is constant;
- (NF2) if  $f \in \mathbb{C}(J)$ , then  $\lambda_{\text{div}(f)} + \log |f|$  is constant;
- (NF3) if  $A \in \text{Div}(J)$  and  $Q \in J(\mathbb{C})$ , then  $P \mapsto \lambda_{A_Q}(P) - \lambda_A(P - Q)$  is constant.

A result of Néron lets us express the Néron function of a divisor in terms of the normalised theta function associated to that divisor. In particular, we find the following.

**Lemma 3.2.** *We get a Néron function with respect to  $\Theta_{-\alpha(W)}$  by mapping  $P \in J(\mathbb{C})$  to*

$$\lambda_{\Theta_{-\alpha(W)}}(P) := -\log |\theta(z, \tau)| + \pi \text{Im}(z)^T \text{Im}(\tau)^{-1} \text{Im}(z),$$

where  $z \in \mathbb{C}^g$  is such that  $j(z) = P$ .

*Proof.* Let  $H$  denote the Hermitian form with matrix  $\text{Im}(\tau)^{-1}$ ; by [Lan83, Proposition 13.3.1] this is the Hermitian form (in the language of [Lan83, §13.1]) of the divisor  $\Theta_{-\alpha(W)}$ . Because of Néron’s theorem (see [Lan83, Theorem 13.1.1]) and because of (3), we get a Néron function by mapping  $P \in J(\mathbb{C})$  to

$$\begin{aligned} \lambda_{\Theta_{-\alpha(W)}}(P) &:= -\log |F_{\Theta_{-\alpha(W)}}(z)| + \frac{\pi}{2} H(z, z) \\ &= -\log |\theta(z, \tau)| - \log \left| \exp \left( \frac{\pi}{2} z^T (\text{Im } \tau)^{-1} z \right) \right| + \frac{\pi}{2} z^T \text{Im}(\tau)^{-1} \bar{z} \\ &= -\log |\theta(z, \tau)| - \frac{\pi}{2} (\text{Re}(z))^T (\text{Im } \tau)^{-1} \text{Re}(z) - \text{Im}(z)^T (\text{Im } \tau)^{-1} \text{Im}(z) \\ &\quad + \frac{\pi}{2} z^T \text{Im}(\tau)^{-1} \bar{z} \\ &= -\log |\theta(z, \tau)| + \pi \text{Im}(z)^T \text{Im}(\tau)^{-1} \text{Im}(z), \end{aligned}$$

where  $z \in \mathbb{C}^g$  is such that  $j(z) = P$ . □

*Proof of Proposition 3.1.* Let  $\Theta^- = [-1]^* \Theta$ . We first find a Néron function for  $\Theta_{\alpha(E_j)}^-$ , where  $j \in \{1, 2\}$ . Since we have

$$\Theta^- = \Theta_{-\alpha(2W)}$$

by [Lan83, Theorem 5.5.8], property (NF3) implies that

$$(4) \quad \lambda_j(P) := \lambda_{\Theta_{-\alpha(W)}}(P - \alpha(E_j) + \alpha(W))$$

is a Néron function with respect to  $\Theta_{\alpha(E_j)}^-$ , where  $\lambda_{\Theta_{-\alpha(W)}}$  is as in Lemma 3.2.

Since  $E_j$  is non-special, a result of Hriljac (see [Lan83, Theorem 13.5.2]) implies that

$$(5) \quad g_{E_j, \varphi} = \lambda_j \circ \alpha + c_j$$

for some constant  $c_j \in \mathbb{R}$ .

Using (5), (4), and Lemma 3.2, we conclude that

$$g_{E_j, \varphi}(P_i) = \lambda_{\Theta_{-\alpha(W)}}(\alpha(P_i) - \alpha(E_j) + \alpha(W)) + c_j \\ = -\log |\theta(z_{ij}, \tau)| + \pi \operatorname{Im}(z_{ij})^T \operatorname{Im}(\tau)^{-1} \operatorname{Im}(z_{ij}) + c_j.$$

The result now follows from

$$g_{E, \varphi}(D) = g_{E_1, \varphi}(P_1) - g_{E_2, \varphi}(P_1) - g_{E_1, \varphi}(P_2) + g_{E_2, \varphi}(P_2)$$

and the definition of the local Néron pairing. □

*Remark 3.3.* In [Mül14, Corollary 4.16] and [Hol12, §7.3] equivalent formulas for  $\langle D, E \rangle_\infty$  were given for the special case of hyperelliptic curves. Our Proposition 3.1 implies those results, if we use a Weierstrass point as the base point for the Abel-Jacobi map; in this case  $\alpha(W) = 0$ . Note that [Mül14, Corollary 4.16] is stated without the assumption that the curve is hyperelliptic, but is false in general. We have adapted and corrected the proof given there. Alternatively, one could also generalise the proof in [Hol12, §7].

*Remark 3.4.* In the proof of Proposition 3.1 the condition that  $E_1$  and  $E_2$  are non-special is only used to apply Hriljac’s theorem which constructs the Green’s function on  $C$  by pulling back a Néron function on  $J$  along the Abel-Jacobi map. If the divisor  $E_j$  is non-special, then the intersection of the translate of  $\Theta^-$  by  $\alpha(E_j)$  with the curve  $C$  recovers the divisor  $E_j$  (see [Lan83, Theorem 5.5.8]), hence we can pull back a Néron function with respect to  $\Theta_{\alpha(E_j)}^-$  to obtain a Green’s function for the divisor  $E_j$ . In contrast, if the divisor  $E_j$  is special, then this intersection can (set-theoretically) be much larger, so pulling back a Néron function does not give anything meaningful. Indeed, we have found examples where Proposition 3.1 is false for special  $E_1$  and  $E_2$ .

**3.3. Computing the archimedean local Néron pairing.** To compute  $\langle D, E \rangle_\infty$ , we use the `Magma` code written by Christian Neurohr for the computation of the small period matrix  $\tau$  associated to  $C(\mathbb{C})$  and the Abel-Jacobi map  $\alpha$ . See Neurohr’s thesis [Neu18] for a description of the algorithm. This code makes it possible to numerically approximate these objects efficiently to any desired precision. If  $C$  is superelliptic, then we instead use Neurohr’s implementation of the specialised algorithms of Molin-Neurohr [MN19] (<https://github.com/pascalmolin/hcperiods>). In both cases, the code requires as input a (possibly singular) plane model of  $C$ . This is easy to produce in practice, for instance via projection or by computing a primitive element of the function field of  $C$ .

The Riemann theta function can be computed using code already contained in `Magma`. It is also necessary to find the divisor  $W$  in Proposition 3.1. We first compute a canonical divisor and its image under  $\alpha$ . Then we run through all preimages under multiplication by 2 in  $\mathbb{C}^g / (\mathbb{Z}^g \oplus \tau \mathbb{Z}^g)$  until we find the correct  $W$  so that  $\Theta_{-\alpha(W)}$  is the divisor of the normalised Riemann theta function; see section 5.1. Once we have the correct  $\alpha(W)$ , we can compute  $\langle D, E \rangle_\infty$  easily via Proposition 3.1.

*Remark 3.5.* The implementation of Molin-Neurohr and the computation of theta functions in `Magma` are rigorous, which means that for superelliptic curves our algorithm returns a provably correct result to any desired precision, if we disregard possible precision loss. To handle the latter, one would have to use interval or ball arithmetic, as implemented, for instance, in `Arb` [Joh17]. Indeed, Molin and

Neurohr have implemented their algorithms in **Arb**, but we have not attempted to use this. In contrast, Neurohr's **Magma**-implementation of his algorithms for more general curves does not currently yield provably correct output; see the discussion in [Neu18, Section 4.10].

#### 4. THE GLOBAL HEIGHT PAIRING

**4.1. Faltings-Hriljac.** Let  $K$  be a global field, let  $C/K$  be a smooth, projective, geometrically connected curve of genus  $g > 0$  with Jacobian  $J = \text{Pic}_{C/K}^0$ , and let  $D$  and  $E$  be degree 0 divisors on  $C$  with disjoint support. If  $v \in M_K$  is a place of  $K$ , then according to [Lan88, III, §5], the local Néron pairing at  $v$  satisfies

$$\langle D, \text{div}(f) \rangle_v = -\log |f(D)|_v$$

for all rational functions  $f \in K(C)^\times$  and divisors  $D \in \text{Div}(C)$  of degree 0, with support disjoint from  $\text{div}(f)$ . Here the absolute values are normalised to satisfy the product formula and we define  $f(D) = \prod_j f(Q_j)^{m_j}$  if  $D = \sum m_j Q_j$ . Hence the global Néron pairing  $\sum_{v \in M_K} \langle D, E \rangle_v$  does respect linear equivalence and extends to a symmetric bilinear pairing on the rational points of  $J$ .

We now relate the global Néron pairing to Néron-Tate heights. Write  $T$  for the image of  $C^{g-1}$  in  $\text{Pic}_{C/K}^{g-1}$ . Choose a class  $w \in \text{Pic}_{C/K}^{g-1}(\bar{K})$  with  $2w$  equal to the canonical class of  $C$  in  $\text{Pic}_{C/K}^{2g-2}(K)$ . Then the class  $\vartheta$  of  $T_{-w}$  is a symmetric ample divisor class on  $J_{\bar{K}}$ , and  $2\vartheta$  is independent of the choice of  $w$  and is defined over  $K$ . The following theorem is due to Faltings and Hriljac [Fal84, Hri85, Gro86].

**Theorem 4.1.** *Let  $D$  and  $E$  be degree 0 divisors on  $C$  with disjoint support. Then*

$$\hat{h}_{2\vartheta}([D], [E]) = - \sum_{v \in M_K} \langle D, E \rangle_v.$$

In the following, we assume  $K = \mathbb{Q}$  for simplicity. We also assume that every element of  $J(\mathbb{Q})$  can be represented using a  $\mathbb{Q}$ -rational divisor; this always holds if  $C$  has a  $\mathbb{Q}_v$  rational divisor of degree 1 for all places  $v$  of  $\mathbb{Q}$ ; see [PS97, Proposition 3.3]. This assumption is convenient, as it allows us to compute the non-archimedean Néron pairings over  $\mathbb{Z}_p$ . If such representatives do not exist, we could work over finite extensions.

*Remark 4.2.* There is a similar decomposition of the  $p$ -adic height on  $J$  due to Coleman-Gross [CG89], where the local summand at a non-archimedean prime  $v \neq p$  is the Néron pairing at  $v$ , up to a constant factor, and there is no archimedean summand. Therefore we only need to combine Algorithm 2.3 with an algorithm to compute the summand at  $p$ , which is defined in terms of Coleman integrals, to get a method for the computation of the  $p$ -adic height on  $J$ . This would be interesting, for instance, in the context of quadratic Chabauty; see the discussion in [BDM<sup>+</sup>19, §1.7]. For hyperelliptic curves, such an algorithm is due to Balakrishnan-Besser [BB12].

**4.2. Finding suitable representatives.** Suppose we are given two points  $P, Q \in J(\mathbb{Q})$ , given by  $\mathbb{Q}$ -rational degree 0 divisors  $D$  (resp.,  $E$ ) representing  $P$  (resp.,  $Q$ ), and wish to compute the height pairing  $\hat{h}_{2\vartheta}(P, Q)$ . The local Néron pairings are only defined for divisors with disjoint support. If  $D$  and  $E$  have common support, we can move  $E$  away from  $D$  using strong approximation; see [Neu18, §4.9.4]. This algorithm computes a rational function  $f_P$  for  $P$  in the common support of both  $D$

and  $E$  such that  $v_P(\operatorname{div}(f_P)) = -1$  and such that  $\operatorname{supp}(\operatorname{div}(f_P)) \cap \operatorname{supp}(D) = \{P\}$ . We replace  $E$  by  $E + \sum_P v_P(E) \operatorname{div}(f_P)$ .

In practice, the following approach is often simpler: reduce multiples of  $E$  along a suitable divisor until this yields a divisor  $E'$  with support disjoint from  $D$ . Due to the bilinearity of the Néron pairings, we can replace  $E$  by  $E'$ ; see also [Mül14, §4.1]. In both approaches, the bottleneck is the computation of Riemann-Roch spaces (as described in [Hes02]). We can also use these methods to ensure that  $E$  can be written as the difference of non-special divisors.

**4.3. Identifying relevant primes.** Fix degree 0 divisors  $D$  and  $E$  with disjoint support. A priori the expression in Theorem 4.1 is an infinite sum; we must identify a finite set  $R$  of “relevant” places outside which we can guarantee that the local Néron pairing of  $D$  and  $E$  vanishes. This set  $R$  will be the union of three sets: the infinite place, the primes where  $C$  has bad reduction, and another finite set containing the other primes at which  $D$  and  $E$  meet.

**4.3.1. Bad primes.** We assume that  $C$  is given with an embedding  $i: C \rightarrow \mathbb{P}_{\mathbb{Q}}^n$  in some projective space, and we write  $\bar{C}$  for some proper model of  $C$  inside  $\mathbb{P}_{\mathbb{Z}}^n$ . For instance, we could always take  $n = 3$  in practice. The standard affine charts of  $\mathbb{P}_{\mathbb{Z}}^n$  induce an affine cover of  $\bar{C}$ , and we check non-smoothness of  $\bar{C}$  on each chart of the cover separately. Suppose that a chart of  $\bar{C}$  is given by an ideal  $I \triangleleft \mathbb{Z}[x_1, \dots, x_n]$ , and  $I$  is generated by  $f_1, \dots, f_r$ . Then a Gröbner basis for the Jacobian ideal of  $I$  will contain exactly one integer, and its prime factors are exactly those primes over which this affine patch fails to be smooth over  $\mathbb{Z}$ .

**4.3.2. Primes where  $D$  and  $E$  may meet.** We reduce to the case where  $D$  and  $E$  are effective. Then we proceed as above, embedding  $C$  in some projective space, and taking some model  $\bar{C}$ . On each affine chart, we take some proper models  $\bar{D}$  and  $\bar{E}$  of  $D$  and  $E$ . If  $\bar{C}$  is cut out by  $I$ , and  $\bar{D}$  and  $\bar{E}$  by ideals  $I_D$  and  $I_E$ , then a Gröbner basis for  $I + I_D + I_E$  has exactly one entry that is an integer (we denote it  $n_{D,E}$ ), and again the prime factors of  $n_{D,E}$  contain all the primes above which  $\bar{D}$  and  $\bar{E}$  meet.

*Remark 4.3.* The final step in Algorithm 2.3 computes lengths of modules over  $\mathbb{Z}$ . In fact, it is much more efficient to work modulo a large power of the prime  $p$ . The techniques just described to identify a finite set of relevant primes can also be used to bound the required precision. If either of the divisors concerned is supported on the special fibre, then it suffices to work modulo  $p^n$ , where  $n$  is the maximum of the multiplicities of the components. If both divisors  $D$  and  $E$  are horizontal, then the maximal power of the prime  $p$  dividing the integer  $n_{D,E}$  (defined just above) is an upper bound on the intersection number, and so provides a sufficient amount of  $p$ -adic precision. Note that resolving singularities by blowing up can only decrease the naive intersection multiplicity, and so this bound is also valid at bad places, as long as the regular model we use is obtained by blowing up  $\bar{C}$ .

*Remark 4.4.* The integer  $n_{D,E}$  can become very large, even if the equations for  $C$ ,  $D$ , and  $E$  have small coefficients (moving  $E$  by linear equivalence often makes the coefficients very much larger). As such, factoring it can become a bottleneck. In principle this factorisation should be avoidable; for example, one can treat the bad primes separately, then one has a global regular model over the remaining primes and the multiplicity can be computed there directly. Algorithms for computing

heights on genus 1 and 2 curves without factorisation can be found in [MS16a, MS16b].

## 5. EXAMPLES

We have implemented our algorithm in `Magma`. Besides testing it against the code in `Magma` (based on [FS97, Sto02, Mü14]) for some hyperelliptic Jacobians, we also tested it on a few Jacobians of smooth plane quartics, though the algorithm is by no means limited to genus 3. At present we can only compute the regulator up to an integral square, because our algorithm only lets us *compute* the Néron-Tate height—we cannot use it to *enumerate* points of bounded Néron-Tate height, which would be required for provably determining generators of  $J(\mathbb{Q})$  with the usual saturation techniques; see the introduction and [Sik95, Sto02]. If  $C$  is hyperelliptic of genus at most 3, then this is possible using the algorithms discussed in the introduction. For an Arakelov-theoretic approach to this problem see [Hol14].

**5.1. A torsion example.** Let  $C: X^3Y - X^2Y^2 - X^2Z^2 - XY^2Z + XZ^3 + Y^3Z = 0$  in  $\mathbb{P}_{\mathbb{Q}}^2$  from [BPS16, Example 12.9.1]. Its Jacobian is of rank 0 and has 51 rational torsion points. Its bad primes are 29 and 163, but the model over  $\mathbb{Z}_{29}$  and  $\mathbb{Z}_{163}$  given by the same equation is already regular.

Let  $D = D_1 - D_2$  and  $E = 3 \cdot E_1 - 3 \cdot E_2$ , where  $D_1 = (1 : 0 : 1)$ ,  $D_2 = (1 : 1 : 0)$ ,  $E_1 = (1 : 0 : 0)$ , and  $E_2 = (1 : 1 : 1)$ . We choose this  $E$  rather than  $E_1 - E_2$  because of the conditions imposed on  $E$  in 3.2. Then the computations for the intersections can be done on the affine patch where  $X \neq 0$  of  $C$ . Consider the ring

$$R = \mathbb{Z}[y, z]/(y - y^2 - z^2 - y^2z + z^3 + y^3z),$$

which is regular. The ideals  $I_{D_1} = (y, z - 1)$  and  $I_{3 \cdot E_1} = (y^3, z^3)$  are coprime in  $R$ , and hence there will be no intersection between  $D_1$  and  $E_1$  at any of the non-archimedean places. In the same way, there is no non-archimedean intersection between  $D_1$  and  $E_2$ , between  $D_2$  and  $E_1$ , and between  $D_2$  and  $E_2$ . Note that also  $\Phi(D)$  and  $\Phi(E)$  can be taken to be 0, as the special fibres of the regular models we computed are irreducible.

For the computation of the archimedean contribution, we first need a canonical divisor which, for practical reasons, has to be supported outside infinity (i.e.,  $X = 0$ ). For this purpose, we pick  $K = \text{div}((z - 1)^2/(y^2z^2) dz)$ .

Then we use Neurohr's algorithm [Neu18] to compute the small period matrix  $\tau$ , and  $\alpha(D_1), \alpha(D_2), \alpha(E_1), \alpha(E_2)$ , and  $\alpha(K)$ , where  $\alpha: C(\mathbb{C}) \rightarrow J(\mathbb{C})$  is the embedding whose base point is chosen by Neurohr's algorithm, which turned out to be the point  $(1 : -2 : -2.6615\dots)$  in this case. To find the appropriate divisor  $W$  with  $2W = K$  out of the  $2^6 = 64$  candidates, we try the 64 candidates for  $\alpha(W)$  and compute for which one the function  $\theta(z, \tau)$  has a pole at a point  $z \in \mathbb{C}^g$  satisfying  $j(z) = \alpha(D_1) + \alpha(D_2) - \alpha(W)$  (which is in  $\Theta$ ). Then we finally compute the expression in Proposition 3.1, and find that the archimedean contribution is approximately 0, or to be more precise, the result was approximately  $2 \cdot 10^{-29}$  when computing with 30 decimal digits of precision.

**5.2. An example in rank 1.** Let  $C$  be the smooth plane quartic curve over  $\mathbb{Q}$  given by

$$X^2Y^2 - XY^3 - X^3Z - 2X^2Z^2 + Y^2Z^2 - XZ^3 + YZ^3 = 0.$$

This is the curve from [BPS16, Example 12.9.2]. It has rank 1 and trivial rational torsion subgroup. Its bad primes are 41 and 347, but the model over  $\mathbb{Z}_{41}$  and  $\mathbb{Z}_{347}$  given by the same equation is already regular.

Let  $D = D_1 - D_2$  and  $E = 3 \cdot E_1 - 3 \cdot E_2$ , where  $D_1 = (1 : 0 : -1)$ ,  $D_2 = (1 : 1 : -1)$ ,  $E_1 = (1 : 1 : 0)$ , and  $E_2 = (1 : 4 : -3)$ . The computations for the intersections can be done on the affine patch of  $C$  where  $X \neq 0$ . Consider the ring

$$R = \mathbb{Z}[y, z]/(y^2 - y^3 - z - 2z^2 - y^2z^2 - z^3 - yz^3).$$

The sum of the two ideals  $I_{D_1} = (y, z + 1)$  and  $I_{E_2} = (y - 4, z + 3)$  inside  $R$  is  $(2, y, z + 1)$ . Hence, the only place where  $D_1$  and  $E_2$  could possibly intersect is the prime 2. At 2, the length of  $\mathbb{Z}_{(2)}[y, z]/(2, y, z + 1) \cong \mathbb{F}_2$  as an  $R_{(2)}$ -module is 1, so  $\iota(D_1, E_2) = \log(2)$ . There is no intersection between  $D_1$  and  $E_1$ , between  $D_2$  and  $E_1$ , and between  $D_2$  and  $E_2$ . Moreover,  $\Phi(D)$  and  $\Phi(E)$  can be taken to be 0 again. Hence, the intersection pairing  $\langle D, E \rangle_{\mathfrak{p}}$  equals  $-3 \log(2)$  if  $\mathfrak{p} = (2)$ , and 0 otherwise.

We computed the archimedean contribution in the same way as in the previous example, and we found it to be  $-0.013563$ . Hence, the Néron-Tate height pairing is  $\hat{h}_{2\vartheta}([D], [E]) = 2.0930$ .

We performed an analogous computation for the points  $F = (0 : 1 : 0) - D_2$  and  $G = 3 \cdot E_2 - 3 \cdot (0 : 1 : -1)$ , and found that  $\hat{h}_{2\vartheta}([F], [G]) = -0.59966$ . We computed this with 30 decimal digits of precision, and found numerically that  $-414 \cdot \hat{h}_{2\vartheta}([D], [E]) = 1445 \cdot \hat{h}_{2\vartheta}([F], [G])$ . We deduced that  $g = [E] - [F]$  is a possible generator for the Mordell-Weil group, and the relation between the heights suggested the relations  $[D] = 17 \cdot g$ ,  $[E] = 255 \cdot g$ ,  $[F] = -69 \cdot g$ , and  $[G] = 18 \cdot g$ , which we confirmed in the Mordell-Weil group. If  $g$  is indeed the generator of the Mordell-Weil group, then the regulator is 0.00048282.

**5.3. The split Cartan modular curve of level 13.** Let  $C$  denote the smooth plane quartic curve given by

$$(6) \quad (-Y - Z)X^3 + (2Y^2 + YZ)X^2 + (-Y^3 + Y^2Z - 2YZ^2 + Z^3)X + (2Y^2Z^2 - 3YZ^3) = 0.$$

By work of Baran [Bar14a, Bar14b] this curve is isomorphic to the modular curve  $X_s(13)$  which classifies elliptic curves whose Galois representation is contained in a normaliser of a split Cartan subgroup of  $\text{GL}_2(\mathbb{F}_{13})$ , as well as its non-split counterpart  $X_{ns}(13)$ . Assuming the Generalised Riemann Hypothesis, Bruin-Poonen-Stoll [BPS16, Example 12.9.3] prove that  $J(\mathbb{Q})$  has rank 3; an unconditional proof is given in [BDM<sup>+</sup>19]. By a result of Balakrishnan, Dogra, Tuitman, Vonk, and the third-named author [BDM<sup>+</sup>19], there are precisely seven rational points on  $C$ . Using reduction modulo small primes, Bruin-Poonen-Stoll show that the points

$$P_0 := (1 : 0 : 0), P_1 := (0 : 1 : 0), P_2 := (0 : 0 : 1), P_3 := (-1 : 0 : 1) \in C(\mathbb{Q})$$

have the property that

$$[P_1 - P_0], [P_2 - P_0], [P_3 - P_0]$$

on the Jacobian  $J$  of  $C$  generate a subgroup  $G$  of  $J(\mathbb{Q})$  of rank 3, which contains all differences of rational points. Therefore the regulator of  $J/\mathbb{Q}$  differs from the regulator of  $G$  multiplicatively by an integral square.

The height pairings that we obtain by using our code are:

	$[P_1 - P_0]$	$[P_2 - P_0]$	$[P_3 - P_0]$
$[P_1 - P_0]$	0.78401	0.59540	0.32516
$[P_2 - P_0]$	0.59540	0.98372	0.37437
$[P_3 - P_0]$	0.32516	0.37437	0.18861

Hence, the regulator is  $9.6703 \cdot 10^{-3}$  up to an integral square factor.

The work of Gross-Zagier [GZ86] and Kolyvagin-Logachev [KL89] implies that the rank part of BSD holds in this example, that the Shafarevich-Tate group is finite, and that the full conjecture of Birch and Swinnerton-Dyer holds up to an integer. We give numerical evidence that it holds up to an integral square. This is the first non-hyperelliptic example where the BSD invariants (except the order of the Shafarevich-Tate group) have been computed; for hyperelliptic examples see [FLS<sup>+</sup>01, vB].

In [BPS16, Example 12.9.3], it is already shown that  $J$  has no non-trivial rational torsion. It is verified easily that the model in  $\mathbb{Z}$  given by the same equation as in (6) is regular at all primes. Hence, all Tamagawa numbers equal 1. For the value of the  $L$ -function, we use that  $J$  is isogenous to the abelian variety  $A_f$  associated to a newform  $f \in S_2(\Gamma_0(169))$  with Fourier coefficients in  $\mathbb{Q}(\zeta_7)^+$ . Hence we have

$$L(J, s) = \prod_{\sigma} L(f^{\sigma}, s),$$

where  $\sigma$  runs through  $\text{Gal}(\mathbb{Q}(\zeta_7)^+/\mathbb{Q})$ . Computing the factors on the right-hand side using **Magma**, we obtained  $\lim_{s \rightarrow 1} L(J, s) \cdot (s-1)^{-3} \approx 0.76825$ .

For the real period, we used the code of Neurohr to compute a big period matrix  $\Lambda$  for  $J$ . One can then apply the methods of the first-named author [vB, Algorithm 13] to check that the differentials used for the computation of the big period matrix are three times a set of generators for the canonical sheaf. Hence, the real period is  $\frac{1}{27}$  times the covolume of the lattice generated by the six columns of  $\Lambda + \bar{\Lambda}$  inside  $\mathbb{R}^3$ . We computed the real period to be 79.444 and checked that this value agrees with the real volume of  $A_f$ .

Assuming our value for the regulator is correct, the BSD formula predicts that the size of the Shafarevich-Tate group is  $\frac{0.76825}{9.6703 \cdot 10^{-3} \cdot 79.444} \approx 1.0000$ , which is consistent with the result of [PS99] proving that the size of the group is a square in this case, if it is finite.

**5.4. An example with very bad reduction.** In all the examples we tried so far, the naive model over  $\mathbb{Z}$  happened to be regular. We wanted to try a curve where this was far from the case, but still with Jacobian of positive rank. We searched for a curve with some rational points and very bad reduction at a small prime, finding the genus 3 curve  $C$  over  $\mathbb{Q}$  given by

$$3x^3y + 5xy^2z + 5y^4 - 1953125z^4 = 0,$$

with rational points  $P_1 = (1 : 0 : 0)$  and  $P_2 = (0 : 25 : 1)$ . The bad primes are 3, 5, 17, 358166959, 523687087967. For the three largest prime factors, the naive models are already regular. The special fibre of the regular model produced by **Magma** over the prime 3 has four irreducible components, with multiplicities  $[1, 1, 2, 2]$  and

intersection matrix

$$\begin{bmatrix} -6 & 0 & 2 & 1 \\ 0 & -2 & 0 & 1 \\ 2 & 0 & -2 & 1 \\ 1 & 1 & 1 & -2 \end{bmatrix}.$$

The special fibre over the prime 5 has nine components, with multiplicities  $[1, 1, 1, 1, 1, 1, 2, 3, 3]$  and intersection matrix

$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & -2 \end{bmatrix}.$$

We define a degree 0 divisor  $D = P_1 - P_2$ , and compute the height pairing of  $D$  with itself, obtaining

$$\hat{h}_{2\vartheta}(D, D) \approx 3.2107.$$

In particular, this shows that  $D$  is not torsion on the Jacobian, hence the rank is at least 1 (probably, it equals 1) and the regulator is probably 3.2107, though of course there might exist a generator of smaller height.

The computation took around five minutes, with 90% of this time spent on the saturation step (Lemma 2.5). Each saturation carried out took around 1.5 seconds, but the complexity of the reduction types meant that many such steps were necessary.

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