On the coexistence of Landau levels and superconductivity

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1 Introduction

1.1 Preface

Quantum states of electrons in a strong magnetic field are massively degenerate Landau levels – the quantized cyclotron orbits. Their presence results in many important phenomena, the most striking of which is the quantum Hall effect: accurate to an extraordinary precision quantization of the Hall conductance in the units of e^2/h.

In the presence of superconductivity, the formation of Landau levels is hindered, as the magnetic field is repelled from a superconductor as a result of the Meissner effect. It is possible for the magnetic field to penetrate the superconductor in the case of type-II superconductors. Then, however, the magnetic field is accompanied by supercurrents circulating around the Abrikosov vortices. In the case of the *d*-wave superconductors – an unconventional type of gapless superconductors which is postulated to encompass the high temperature superconductors – it was shown that these supercurrents spoil the degeneracy of the Landau levels, transforming them into dispersive magnetic Bloch bands [1–11]. In this thesis we consider a different kind of gapless superconductors: Weyl superconductors, which are formed when conventional *s*-wave superconductivity occurs in Weyl semimetals [12, 13]. We show that the chiral symmetry present in such systems gives rise to a zeroth Landau level, which persists even in the presence of the Abrikosov vortices. We study the properties of this superconducting Landau level. One of them is the universal chiral magnetic effect in equilibrium. It is a signature of the topological properties of the Weyl fermion that can only be accessed in the Weyl superconductors, as it requires the presence of a non-vanishing equilibrium current, which is forbidden in the absence of superconductivity.

We also study a different type of superconductors, which arises when superconductivity is induced in the surface states of topological insulators – the Fu-Kane model [14]. Under certain circumstances, the excitation of such a system can also become gapless, which was recently demonstrated experimentally [15]. This system also possesses chiral symmetry, which, like in the Weyl superconductor, gives rise to a topologically protected zeroth Landau level, which is not spoiled by the vortices.

The cases above, discussed in detail in this thesis, are the first instances of the coexistence of Landau levels and superconductivity. They open up a path to a new range of possible effects that are only possible due to a combination of these two phenomena. In the remainder of this chapter we will introduce the basic concepts, which lie at the foundations of the results presented afterwards.

1.2 Landau levels

1.2.1 Particles in a magnetic field

When a charged particle is moving through a uniform magnetic field B, and its velocity v is perpendicular to the direction of the field, it follows the circular trajectory of a cyclotron orbit of radius

$$l_c = \frac{mv}{qB}\,, \tag{1.1}$$

where $q > 0$ is the particle's charge and m is its mass. If the particle is simultaneously moving parallel to the magnetic field, its motion in that direction is unaffected. As a result, the particle's trajectory will be a helix oriented along the magnetic field – Fig. 1.1. This phenomenon has striking consequences: for instance, when a charged particle coming from the Sun as a part of the solar wind reaches the Earth's magnetosphere, it is forced to follow the helical trajectory along the lines of the magnetic field, which guide it towards the Earth's poles. This provides the inhabitants of the Earth protection from the harmful effects of space radiation, as well as a marvelous spectacle of light produced once particles' motion is interrupted by the Earth's atmosphere.

This periodic cyclotron motion carries on to the lowest-level description of reality: quantum physics. There, however, due to the wave-like nature of particles, not all sizes of their orbits are permitted. The phase of a particle's wave must change by an integer multiple of 2π upon completion of one cycle, which puts a constraint that only allows orbits with energies equal to

$$E_n = \hbar\omega_c\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots \tag{1.2}$$

where $\omega_c = v/l_c = qB/m$ is the cyclotron frequency. The quantum states corresponding to these orbits are called the Landau levels. Noticeably,

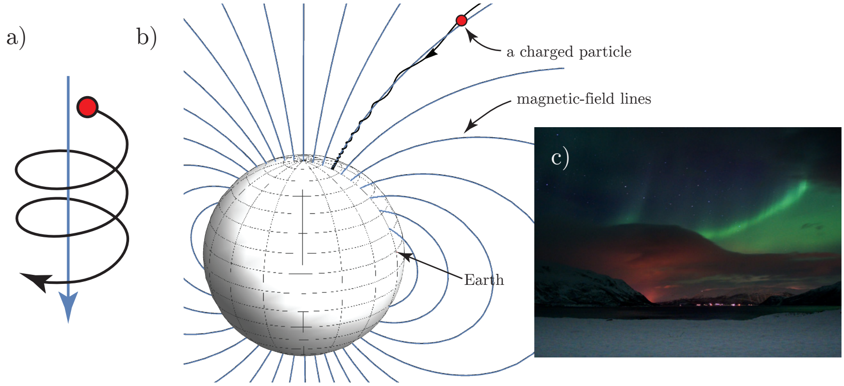

Figure 1.1: a) Trajectory of a charged particle in a uniform magnetic field, the direction of which is indicated by the blue arrow. The particle, represented as a red dot, can propagate freely along the magnetic field, however its motion in the perpendicular direction is confined. b) Trajectory of a charged particle in the Earth's magnetic field. The particle follows a helical path along the magnetic-field lines, until it reaches the atmosphere in the polar regions, where its kinetic energy is converted into radiation, producing the aurora. c) Picture of an aurora in Norway, taken by the author.

the lowest energy allowed for a particle $E_0 = \hbar\omega_c/2$ is greater than zero. This can be understood as the particle's zero-point motion.

The classical orbits can be centered at any point in the region of applied magnetic field. In a quantum case, this means that there are multiple states at each energy E_n. Their number, known as the degeneracy of a given energy level, is limited by the requirement that different quantum states must be orthogonal, and equals

$$\mathcal{N} = \frac{\Phi}{\Phi_0}\,, \tag{1.3}$$

where Φ is the total flux of applied magnetic field, and $\Phi_0 = h/q$ is the magnetic flux quantum.

As in the classical case, the motion of a particle parallel to the field remains unaffected. This manifests itself in a relation between the particle's momentum in that direction $p_\parallel$ and its energy, given by

$$E_n(p_\parallel) = \hbar\omega_c\left(n + \frac{1}{2}\right) + \frac{p_\parallel^2}{2m}\,, \tag{1.4}$$

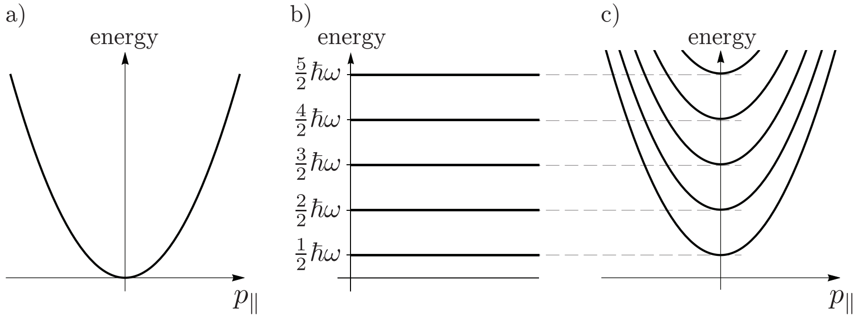

Figure 1.2: a) Dispersion relation of a free non-relativistic massive particle. b) Energies of the Landau levels at $p_{\|} = 0$. c) Dispersion relation of the Landau levels in the direction parallel to the magnetic field.

which is simply a sum of the energy of the Landau level, and the kinetic energy of motion in the direction parallel to the magnetic field – Fig. 1.2.

The group velocity of a particle equals to the derivative of its energy with respect to its momentum in the direction of motion. For the nth Landau level it then equals

$$v_n(p_{\|}) = \frac{\mathrm{d}E_n(p_{\|})}{\mathrm{d}p_{\|}} = \frac{p_{\|}}{m}\,, \tag{1.5}$$

which is a result familiar from classical physics. Unsurprisingly, as the velocity can be positive or negative, depending on the sign of momentum $p_{\|}$, the particle is free to propagate in either direction along the magnetic field, however, as it will be shown in Sec. 1.2.4, this property should not be taken for granted.

1.2.2 Weyl semimetal

In this thesis, we will consider a more exotic type of particles: Weyl fermions. They are massless relativistic particles. Unlike massive non-relativistic particles, whose energy is given by $E = |\boldsymbol{p}|^2/2$, their dispersion relation reads

$$E(\boldsymbol{p}) = \pm v_{\mathrm{F}}|\boldsymbol{p}|\,, \tag{1.6}$$

similar to that of a photon. v_{F} is the Fermi velocity, and $\boldsymbol{p}$ is the three-component momentum vector. This dispersion relation forms a three-dimensional double cone in the energy-momentum space, called a Weyl cone. Such particles are the basic building blocks of the Standard Model (in which case v_{F} is the speed of light), however, they are not found among fundamental particles, as the spontaneous symmetry breaking introduces coupling between different flavors of Weyl fermions, which gives rise to emergent particles of a different type – massive Dirac fermions[1].

Luckily, Weyl fermions can also be found as low-energy electronic excitations in certain types of materials: Weyl semimetals, which allows for an experimental study of their properties. In the following paragraphs, I will explain how this can happen. An electron propagating in a material undergoes scattering off the electrostatic potential of the crystalline lattice, thus its momentum is not conserved. However, due to the periodic structure of the lattice, a similar quantum number $\boldsymbol{k}$, called the quasi-momentum, is conserved. For each value of the quasi-momentum, the energy of an electron can take one of the enumerably many values $E_n(\boldsymbol{k})$, where n is called the band index. As a result, the set of all allowed energy values is split into energy bands labeled by n, which lies at the foundations of the electronic band theory – the basic tool for describing the vast majority of solid-state devices.

Typically, the energy bands are separated – if two of them are accidentally crossing at some values of $\boldsymbol{k}$, generally a perturbation to the system will make these crossings avoided. This, however, may not be the case if two bands are touching in an isolated point $\boldsymbol{k}_0$ in the quasi-momentum space. It can turn out that for quasi-momenta near $\boldsymbol{k}_0$ and up to a constant energy shift, the energies of the two bands touching are given by the relativistic dispersion relation of Eq. (1.6) with $\boldsymbol{p} = \boldsymbol{k} - \boldsymbol{k}_0$. Then the electronic excitations with energy and momentum close to the touching point, in this case called the Weyl point, are Weyl fermions.

The effective low-energy Hamiltonian that leads to the Weyl dispersion relation is

$$H = \chi v_{\mathrm{F}} \boldsymbol{p} \cdot \boldsymbol{\sigma} \,, \tag{1.7}$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the pseudo-spin operator, and $\chi \in \{+1, -1\}$ is called chirality. Apart from their unusual dispersion relation, the Weyl fermions have the property that they are topologically protected: there is no perturbation that can be added to the Hamiltonian (1.7) that will open an energy gap between the two bands that are touching. The only effect that a perturbation can have is to change the position of the Weyl point, or add anisotropy or tilting to the Weyl cone.

It is only possible to gap out a Weyl cone, if a pair of Weyl cones of opposite chirality occurs at a single point in the quasi-momentum space.

[1] An exception to this are the neutrinos, for which the origin of mass is currently unknown.

A situation like this is captured by a Hamiltonian

$$H = v_{\mathrm{F}} \tau_z \boldsymbol{p} \cdot \boldsymbol{\sigma} \,, \tag{1.8}$$

where τ_z is a Pauli matrix that distinguishes between the two Weyl fermions. Then, a perturbation $m\tau_x\sigma_0$ opens a gap of width $2m$, and the dispersion becomes

$$E(\boldsymbol{p}) = \pm v_{\mathrm{F}} \sqrt{m^2 + \boldsymbol{p}^2} \,, \tag{1.9}$$

which is that of a massive relativistic particle.

Another important property of a Weyl fermion is that its spin must always be oriented along its momentum – a phenomenon known as spin-momentum locking. In the context of a relativistic field theory it originates from the Lorentz invariance: the co-linear arrangement of spin and momentum is the only one that does not depend on the frame of reference. (Pseudo-)spin momentum-locking also applies to Weyl fermions in condensed-matter systems, even though the Lorentz invariance there is emergent and can be broken by perturbations. The universality of this phenomenon arises from the form of the Weyl Hamiltonian (1.7), which also determines the character of the spin-momentum locking, depending on the chirality χ: for $\chi = +1$ the spin and the momentum of a positive-energy Weyl fermion point in the same, while for $\chi = -1$ in the opposite direction.

1.2.3 Zeroth Landau level

Now we will turn our attention to the behavior of a Weyl fermion in the presence of the magnetic field. The classical picture is similar to that for a non-relativistic particle – the Lorentz force exerted on a moving Weyl fermion guides it along a circular, or helical, trajectory. The only difference is that in the expression for the radius of such trajectory, given in Eq. (1.1), the product mv must be replaced with the momentum component perpendicular to the magnetic field $p_\perp$, as the Weyl fermion is a massless particle.

When quantum mechanics is taken into account, however, the differences become more significant: as a result of the spin-momentum locking, when a Weyl fermion follows a closed cyclotron orbit and its velocity – and with it the quasi-momentum – makes a 360° rotation, so does its pseudo-spin. A full rotation of a spin-1/2 particle changes the phase of its wave by π. The same holds for the pseudo-spin. This results in a change in the wave-matching condition for the Landau level, with which

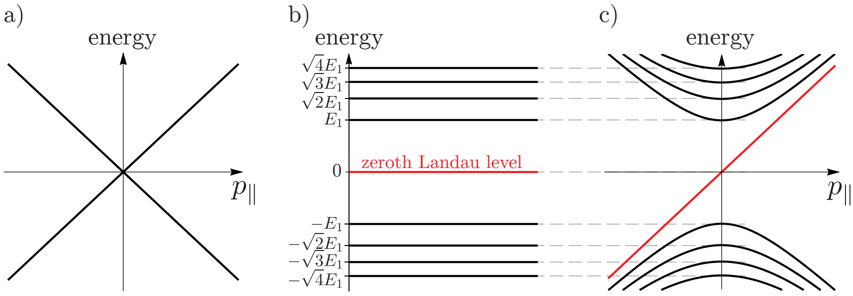

Figure 1.3: a) Dispersion relation of a Weyl fermion. b) Energies of the Landau levels at $p_{\|} = 0$. c) Dispersion relation of the Landau levels in the direction parallel to the magnetic field.

the quantized energy levels become

$$E_n = \text{sign}(n) v_{\text{F}} \sqrt{2e\hbar B|n|}\,, \qquad n \in \mathbb{Z}\,, \tag{1.10}$$

where we took $q = e$, the charge of an electron. The allowed energies are no longer just positive. This is a result of the fact that the Weyl cone extends to both positive and negative energies. Central to the work presented in this thesis is the $n = 0$ state, known as the zeroth Landau level, whose energy is exactly equal to zero (with respect to the Weyl point). Its apparent lack of zero-point motion is the direct effect of the spin-momentum locking discussed earlier. The zeroth Landau level is special – it is robust against a wide class of perturbations [16]. For instance, while for other Landau levels an inhomogeneity of the magnetic field results in a broadening of their energy band, the states in the zeroth Landau level remain all at zero energy. The cause of this is rooted in the topological nature of the Weyl fermion and is a consequence of the Atiyah-Singer index theorem [17].

1.2.4 Chiral anomaly

When the motion along the magnetic field is considered, the peculiarities of the zeroth Landau level become even more pronounced. The energies of the particle are then given by

$$E_n(p_{\|}) = \text{sign}(n) v_{\text{F}} \sqrt{2e\hbar B|n| + p_{\|}^2}\,, \qquad n \in \mathbb{Z} \setminus \{0\}\,. \tag{1.11}$$

and

$$E_0(p_{\|}) = \chi v_{\text{F}} p_{\|}\,, \tag{1.12}$$

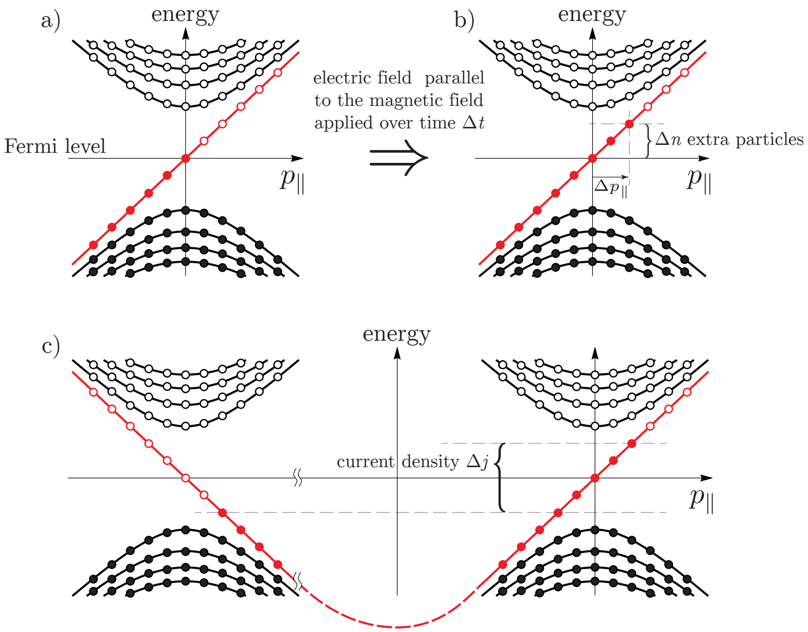

Figure 1.4: a) The distribution of electrons in the Landau levels of a Weyl fermion in the ground state. The filled circles represent states occupied by electrons, whereas the empty circles – unoccupied states. In the ground state all the states up the Fermi level are occupied and all those above it – empty. The velocity of each electron is given by the slope of the dispersion relation of the Landau level it lies in. For all but the zeroth Landau level, the current carried by electrons with positive velocity is cancelled by the current carried by electrons with negative velocity, resulting in zero net current. However, all the electrons in the zeroth Landau level are moving in the same direction, resulting in an infinite ground-state current. b) The distribution of electrons after applying a pulse of electric field parallel to the magnetic field over time Δt. Each electron acquires an additional momentum $\Delta p_{\|}$, which makes some of the previously unoccupied states, occupied. This produces a certain number of extra particles, violating the particle conservation – a phenomenon known as the chiral anomaly. c) A resolution to the lack of particle conservation 'paradox' in a Weyl semimetal. Each Weyl cone with chirality $+1$ is accompanied by a cone with chirality -1. When a certain number of electrons is produced in one of the Weyl cones, due to chiral anomaly, the same number vanishes from the other Weyl cone, ensuring total particle conservation. This results in a unequal number of left- and right-moving electrons, which produces a non-vanishing current density Δj. The appearance of this current is known as the chiral magnetic effect.

which means that the zeroth Landau level can only propagate in one direction along the magnetic field, depending on its chirality χ – Fig. 1.3. This has two puzzling consequences. Firstly, such a dispersion relation leads to an infinite current in the ground state: at zero temperature, electrons occupy all the lowest-energy states up to a certain energy, called the Fermi level. Then for all Landau levels with $n \neq 0$, at each energy there is the same number of electrons propagating in either direction along the magnetic field, thus the net current they carry vanishes. All the electrons in the zeroth Landau level, however, propagate in the same direction, so their total contribution to the current diverges. In quantum field theories, diverging ground-state (also called the vacuum in this context) properties, such as energy or charge density, are not unusual. A common practice is to define all the observables with respect to the vacuum, by subtracting their – possibly infinite – vacuum expectation values. In the condensed-matter context, however, there is no such freedom, as the ground state properties are directly accessible experimentally. The non-vanishing ground-state current thus constitutes a contradiction as it is not allowed by the Bloch theorem[2] [18–21].

A second problem arises when, in addition to the magnetic field, an electric field $\boldsymbol{\mathcal{E}}$ is applied in the same direction for some finite time Δt. It will exert a force $\boldsymbol{F} = e\boldsymbol{\mathcal{E}}$ on each electron, which will change its momentum by $\Delta p_{\|} = e\mathcal{E}\Delta t$. If the Fermi level was originally between the $n = -1$ and $n = 1$ Landau levels, it will effectively shift by $\Delta\mu = \chi v_{\mathrm{F}} e\mathcal{E}\Delta t$, which implies that the number of particles in the system will change. This change is equal to

$$\Delta n = \chi \frac{e\mathcal{E}\Delta t}{h/L}\frac{\Phi}{\Phi_0} = \chi \frac{e^2}{h^2}\boldsymbol{\mathcal{E}} \cdot \boldsymbol{B} V \Delta t \,, \tag{1.13}$$

where V is the total volume of the system [55]. This violation of particle-number conservation is known as the chiral anomaly. In particle physics, the chiral anomaly affects the decay rate of the neutral pion π^0, which was first explained by Adler [22], and Bell and Jackiw [23]. In the condensed-matter context, however, the Weyl fermions arise from ordinary electrons, the number of which must be conserved, which, once again, leads to a contradiction.

The resolution to both of the issues discussed above is the same: in the electronic band structure of a material, there must always be an even number of Weyl points, and the number of those with $+1$ chirality must be equal to those with -1 chirality. In that case the ground-state current

[2] The theorem allows for persistent currents, however their magnitude vanishes in the thermodynamic limit.

carried by the electrons in the zeroth Landau level of the Weyl fermions with $+1$ chirality is canceled by that of the Weyl fermions with -1 chirality, in agreement with the Bloch theorem – Fig. 1.4. In the same way, the total change of the number of electron due to the chiral anomaly for all the Weyl fermions is zero, as expected: the electric field simply pumps the electrons from one Weyl fermion to the other.

1.2.5 Chiral magnetic effect

Even though the total number of electrons is conserved, the chiral anomaly can still lead to observable consequences, as it creates an imbalance between the Weyl fermions of opposite chiralities. Since the current carried by electrons in the zeroth Landau level depends on its chirality, such imbalance produces an electric current. This phenomenon is known as the chiral magnetic effect (CME). For each Weyl fermion, the resulting current density is equal to

$$\Delta j = \chi \frac{e^2}{h} \Delta\mu \,, \tag{1.14}$$

where the proportionality coefficient only depends on the fundamental constants: e – the elementary charge, and h – the Planck's constant. This independence of any details of the system is a case of universality. Universal responses are often found in systems with topological properties. The most prominent example of such a response is the quantized Hall conductance in the quantum Hall effect, which, similarly, assumes values of integer multiples of e^2/h – independent of the sample. The accuracy of this result lead to establishing a new definition of the kilogram in 2019, based on the value of Planck's constant, rather than a physical artifact.

Unfortunately, the chiral magnetic effect – a universal response of the Weyl semimetal – is difficult to access: the electric field applied brings the system out of equilibrium – a state, which must decay once the electric field is removed, due to random scattering processes. In order to maintain the imbalance of chiralities, the electric field must be applied continuously, which keeps the system in a steady state in which the rate of scattering between the Weyl cones balances out the pumping of electrons. Then, if the characteristic relaxation time associated with this scattering is τ, the current density contribution of a single Weyl cone can be shown to be

$$\Delta j = \chi v_F \frac{e^3}{h^2} \boldsymbol{\mathcal{E}} \cdot \boldsymbol{B} \tau \,. \tag{1.15}$$

This results in an appearance of negative magnetoresistance in Weyl semimetals – an increase in conductance proportional to the applied magnetic field, regarded as the basic transport signature of Weyl fermions in condensed matter systems [24, 25]. It has been demonstrated experimentally in agreement with the presented theory [26, 27]. In this phenomenon, however, the universality of CME becomes obscured, as the observed effect depends on the relaxation time τ, which is an effective parameter depending on the microscopic details of the system.

1.3 Superconductivity

1.3.1 Bogoliubov-de Gennes formalism

In normal materials the electric current must vanish in equilibrium, which prevents the access to the chiral magnetic effect in such a situation. Superconductors, however, can support a non-vanishing current in equilibrium. This opens a tantalizing possibility that in such systems the equilibrium CME can be accessed. It would then allow for a direct observation of the universal coefficient e^2/h of Eq. (1.14). This possibility was the main motivation for the work presented in this thesis.

To demonstrate how it can happen, I will first explain the effect of superconductivity on Weyl semimetals. In the ground state of a superconductor, the electrons form a condensate of Cooper pairs, which allows for a frictionless flow of the current. The presence of the condensate modifies the dynamics of excitations in a superconductor: if energy is supplied to the system, a Cooper pair can split into two independent electrons. A converse process is also possible, in which two electrons combine and form a Cooper pair, which becomes a part of the condensate. In the simplest case, a Cooper pair consists of two electrons of opposite quasi-momenta and spin. This is the case in the s-wave superconductors. In the mean-field approximation, these processes are captured by the Bogoliubov-de Gennes (BdG) formalism, in which the number of degrees of freedom is doubled: in addition to electrons, one introduces new particles – holes. Then, for an s-wave superconductor, a given state of the system can be represented in two ways: the electron at energy E (with respect to the Fermi level), quasi-momentum $\boldsymbol{k}$ and spin $\boldsymbol{s}$, is equivalent to the absence of a hole at energy $-E$, quasi-momentum $-\boldsymbol{k}$ and spin $-\boldsymbol{s}$. With this redundancy, a multi-particle process of a decay of a Cooper pair, which results in the creation of two electrons of opposite quasi-momenta and spins, is equivalently described as a scattering process of a hole at quasi-momentum $\boldsymbol{k}$ and spin $\boldsymbol{s}$ into an electron with the same quasi-momentum and spin, which is a single-particle process – Fig. 1.5. In this formal-

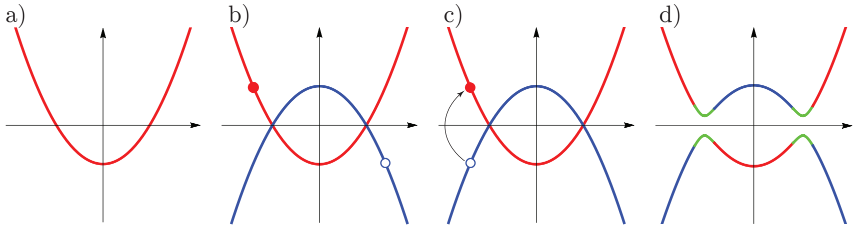

Figure 1.5: a) Dispersion relation of an electron in a normal material. b) Dispersion relation for a normal material in the BdG formalism: dispersion relation of an electron is in red, and that of a hole is in blue. Electron at energy E, quasi-momentum $\boldsymbol{k}$ and spin $\boldsymbol{s}$ (marked by a red dot), can be equivalently described as absence of a hole at energy $-E$, quasi-momentum $-\boldsymbol{k}$ and spin $-\boldsymbol{s}$ (marked by a blue circle). c) Decay of a Cooper pair: a hole at quasi-momentum $\boldsymbol{k}$ and spin $\boldsymbol{s}$ is scattered into an electron with the same quasi-momentum and spin. d) Resulting dispersion relation of a superconductor, given by Eq. (1.18)

ism, the dynamics of the system is captured by a single-body Bogoliubov de-Gennes Hamiltonian

$$\mathcal{H}(\boldsymbol{k}) = \begin{pmatrix} H_0(\boldsymbol{k}) & \Delta \\ \Delta^\dagger & -s_y H_0^*(-\boldsymbol{k}) s_y \end{pmatrix}, \tag{1.16}$$

where the top block describes the behavior of electrons, the bottom block that of the holes and Δ – the superconducting pairing – introduces the scattering between them induced by the Cooper pair condensate. Operator s_y is the Pauli matrix acting on the spin.

1.3.2 Weyl superconductor

Typically, the superconducting pairing induces an energy gap in the system. For instance, if an electron is described by a simple quadratic Hamiltonian

$$H_0 = \frac{k^2}{2m} - \mu\,, \tag{1.17}$$

the BdG Hamiltonian results in the dispersion

$$E = \pm\sqrt{\left(\frac{k^2}{2m^*} - \mu\right)^2 + \Delta^2}\,, \tag{1.18}$$

which is gapped. This gap is in fact a crucial feature of the superconductors, as the lack of low-energy excitations means that they cannot participate in the dissipation of energy, which suppresses the production of heat when the current is passing through the system.

However, if the superconducting pairing is present in a Weyl semimetal, in which case the system is dubbed a Weyl superconductor [12, 13], the appearance of such a gap can be generally avoided. This is due to the topological protection of the Weyl cones. As explained in Sec. 1.2.2, it is only possible for a small perturbation to open a gap of a Weyl cone if there are two Weyl cones of opposite chirality at the same quasi-momentum $\boldsymbol{k}_0$. In the BdG formalism, the band structure is doubled, thus additional Weyl cones appear originating from the dispersion relation of the holes. A hole Weyl cone at quasi-momentum $\boldsymbol{k}_0$ with chirality χ appears as a redundant representation of an electronic Weyl cone at quasi-momentum $-\boldsymbol{k}_0$ and chirality $-\chi$. Thus, in the BdG formalism, two Weyl cones of opposite chirality – one for electron and one for hole – occur at the same quasi-momentum, in which case a small superconducting pairing can open a gap, only if two Weyl cones of the same chirality occur at opposite quasi-momenta, $\boldsymbol{k}_0$ and $-\boldsymbol{k}_0$, in the original electronic band structure of a Weyl semimetal. Such a configuration requires either fine-tuning, or the presence of certain symmetry – in this case time-reversal symmetry, combined with a broken inversion symmetry. Thus, generically the band structure of a Weyl superconductor remains gapless.

In normal materials, the current from CME vanishes in equilibrium, because there is the same number of Weyl cones of each chirality. Then, if the chemical potential is changed by $\Delta\mu$, it changes by the same amount for all the Weyl cones, and the contributions – given by Eq. (1.14) – from the Weyl cones of opposite chiralities cancel out. If the superconducting pairing fails to open a gap of any of the Weyl cones in the Weyl superconductor, the situation is the same there. However, as explained before, a gap can open for a given electronic Weyl cone at momentum $\boldsymbol{k}_0$, if there also exists another one with the same chirality at the opposite momentum. The superconducting pairing then opens a gap in both of these two Weyl cones simultaneously. Alternatively, a gap can also open for a single Weyl cone, if it occurs at $\boldsymbol{k}_0 = 0$. Such a gap is analogous to the Majorana mass in the high-energy physics context. In either of these situations, the number of Weyl fermions of a single chirality is reduced – by either 2 or 1, depending on the number of Weyl cones involved. While for an infinitesimal Δ such a case is only possible with fine-tuning, it is not so when Δ is finite – the regime with an unequal number of Weyl fermions with different chiralities occurs in finite regions in the parameter space. In such a regime then, if the zeroth Landau level forms when the magnetic field is applied, the chiral magnetic effect should manifest itself in equilibrium.

1.3.3 Vortex lattice

As discovered by Meissner in 1933, a magnetic field is repelled from a superconductor. This poses both practical and conceptual difficulties in the observation of the equilibrium CME: if the magnetic field cannot penetrate the superconductor, the Landau levels cannot develop and CME will not occur. A way around this is to consider a thin sample, as the magnetic field can penetrate the superconductor up to a certain distance called the penetration depth λ. For a superconducting slab much thinner than λ, the magnetic field applied parallel to its surface will penetrate it almost uniformly. Such a situation was considered by O'Brien et al. [47] who found that when a Weyl superconductor is driven into a regime with just a single Weyl cone present, the equilibrium CME occurs, with the current given by

$$\Delta j = \kappa \frac{e^2}{h} \Delta\mu \,, \tag{1.19}$$

where κ is a non-universal coefficient describing the effective charge κe of the low-energy excitations. This factor appears due to change in the degeneracy of the Landau level, in which the electron charge is replaced by the effective charge.

If a sample is thicker, and the magnetic field is strong enough, the superconductor may enter a mixed phase, in which the magnetic field penetrates the superconductor in the form of vortices of circulating supercurrent – Abrikosov vortices. In each vortex core, the size of which is approximately equal to the superconducting coherence length ξ, the superconductivity is destroyed, whereas the magnetic field decays exponentially with the distance from the vortex core, with the decay length λ. The number of vortices is equal to the number of superconducting flux quanta $h/(2e) = \frac{1}{2}\Phi_0$ of the applied magnetic field. If $\lambda > \xi/\sqrt{2}$, the mixed phase is energetically favorable to the Meissner effect, which is the defining property of a type-II superconductor. Moreover, if $\lambda \gg \sqrt{h/(eB)} \gg \xi$, the vortex cores occupy a negligible volume of the system, and the distances between them are small compared to λ, which yields an almost homogeneous magnetic field.

One can ask whether the Landau levels can develop in this mixed phase. A similar question was considered in the context of unconventional d-wave superconductors, whose superconducting pairing depends on the quasi-momentum, and vanishes on certain lines in the quasi-momentum space.

On these lines the superconducting gap closes, which gives rise to a linear dispersion relation around certain points in the quasi-momentum space, similar to that of a Weyl fermion. The robustness of the zeroth Landau level of massless relativistic fermions lead Gor'kov and Schrieffer [29] to a conclusion that the zeroth Landau level would also occur in the d-wave superconductor. The same prediction was also made by Anderson [30]. This, however, turned out not to be the case. Franz and Tešanović [1] showed that scattering of the quasiparticles from circulating vortex currents broadens the zeroth Landau level, leading to a dispersive magnetic Bloch band.

One of the questions considered in this thesis was whether the same fate awaits the zeroth Landau level in Weyl superconductors. If that was the case, it would close the way to access the equilibrium CME in the mixed phase. It turned out, however, otherwise: in the Weyl superconductors, the topological protection of the zeroth Landau level holds up against the effects of the vortices. Moreover, this leads to a CME current given by Eq. (1.14) with the universal coefficient e^2/h, independent of the effective charge κe, as it was in a thin slab.

1.4 Outline of this thesis

Chapter 2

In this chapter, we address the question whether Landau levels can emerge in a superconductor in the presence of Abrikosov vortices, induced by the applied magnetic field. We show that in a Weyl superconductor – a type of gapless superconductor, in which the low-energy excitations are Weyl fermions – it is possible. This is a surprising result, as the quasiparticle scattering off the superconducting vortices is known to spoil the Landau levels in a different type of gapless superconductors, the d-wave superconductor. We show that in the Weyl superconductor the situation is different: as a result of topological properties of the Weyl fermions, the zeroth Landau level is robust against such scattering. The particles in the zeroth Landau level can propagate along the magnetic field lines, which allows them to carry energy in that direction. This manifests itself in a contribution to the thermal conductance, which takes a universal value $G = \frac{1}{2}g_0\Phi/\Phi_0$, where g_0 is the thermal conductance quantum, Φ_0 is the superconducting flux quantum, and Φ is the total flux of the applied magnetic field.

Chapter 3

In this chapter we further explore the consequences of the formation of the zeroth Landau level in a Weyl superconductor. We study how it affects electric and thermoelectric transport properties along the applied magnetic field. We show that the vortex lattice carries an electric current $I = \frac{1}{2}(Q_{\text{eff}}^2/h)(\Phi/\Phi_0)V$ between two normal metal contacts at voltage difference V, with Φ the magnetic flux through the system, Φ_0 the superconducting flux quantum, and $Q_{\text{eff}} < e$ the renormalized charge of the Weyl fermions in the superconducting Landau level. Because the charge renormalization is energy-dependent, a nonzero thermo-electric coefficient appears even in the absence of energy-dependent scattering processes.

Chapter 4

In this chapter we study the chiral magnetic effect in the zeroth Landau level in the Weyl superconductor: the appearance of current I along the lines of magnetic flux Φ, due to an imbalance between Weyl fermions of opposite chirality. In Weyl semimetals, the presence of such a current is only possible out of equilibrium, which makes it only accessible through indirect observations, e.g. through the negative magnetoresistance measurements. We show that in a Weyl superconductor, the chiral magnetic effect is accessible in equilibrium, manifesting itself as a universal current contribution $dI/d\Phi = (e/h)^2\mu$ (at equilibrium chemical potential μ relative to the Weyl point), when quasiparticles of one of the two chiralities are confined in vortex cores. The confined states are charge-neutral Majorana fermions.

Chapter 5

Shared contribution with Gal Lemut, who was responsible for the numerical simulations.

In this chapter we consider a different type of a superconductor: the Fu-Kane heterostructure, which consists of a superconductor on top of the surface of a topological insulator. The gapless surface excitations of a topological insulator acquire an energy gap in the presence of superconductivity. It is known that when a vortex-forming magnetic field is applied perpendicular to the surface, each vortex binds a zero-energy Majorana mode exponentially localized at its core. We examine the consequences of applying a supercurrent parallel to the surface. When the magnitude of the supercurrent exceeds a critical value, the surface quasiparticle gap closes, which drives a deconfinement transition of the Majorana bound states. In the deconfined phase at zero chemical potential, the Majorana fermions form a dispersionless Landau level, protected by chiral symmetry against broadening due to vortex scattering. The coherent superposition of electrons and holes in the Majorana Landau level is detectable as a local density of states oscillation with a known wave vector. The striped pattern also provides a means to measure the chirality of the Majorana fermions.

Chapter 6

Shared contribution with Gal Lemut, who was responsible for the numerical simulations.

In this chapter we study a mathematical problem of discretizing the single-cone Dirac equation. Such a problem arises when performing a computer simulation of gapless excitations on the surface of a topological insulator or superconductor. The simplest approach results in so-called fermion doubling: appearance of an additional gapless low energy excitations in the simulated system. It is known that this cannot be avoided without breaking locality or chiral symmetry of the model. In this chapter we examine a special staggered discretization by Stacey [110], which avoids the fermion doubling. In this approach the Dirac equation is discretized as a generalized eigenvalue problem $\mathcal{H}\psi = E\mathcal{P}\psi$. While maintaining the chiral symmetry, this formulation breaks locality, as it can be cast in the form of an ordinary eigenvalue problem with a non-local Hamiltonian. We show that despite this shortcoming, the resulting theory possesses a locally-conserved particle current. As the discretization maintains the symmetries of the original Dirac equation, this permits the study of the spectral statistics of Dirac fermions in each of the four symmetry classes A, AII, AIII and D of random-matrix theory.