Measures and matching for number systems

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CHAPTER 4

Matching for flipped α-CF

This chapter is based on: [KLMM20].

Abstract

As a natural counterpart to Nakada's α-continued fraction maps, we study a one-parameter family of continued fraction transformations with an indifferent fixed point. We prove that matching holds for Lebesgue almost every parameter in this family and that the exceptional set has Hausdorff dimension 1. Due to this matching property, we can construct a planar version of the natural extension. We use this construction to obtain an explicit expression for the density of the unique infinite σ-finite absolutely continuous invariant measure, and we also compute the Krengel entropy, the return sequence and the wandering rate of the maps for a large part of the parameter space.

§4.1 Motivation and context

Over the past decades the dynamical phenomenon of matching, or synchronisation as described in Definition 1.2.7, has surfaced increasingly often in the study of the dynamics of interval maps. Recall that a map T is said to have matching if for any discontinuity point c of the map T or its derivative T' the orbits of the left and right limits of c eventually meet. That is, there exist non-negative integers M and N, called *matching exponents*, such that

$$T^M(c^-) = T^N(c^+), \tag{4.1}$$

where

$$c^- = \lim_{x \uparrow c} T(x) \quad \text{and} \quad c^+ = \lim_{x \downarrow c} T(x).$$

General results on the implications of matching are scarce. There are many results however on the consequences of matching for specific families of interval maps. In [KS12, BSORG13, BCK17, BCMP18, CM18, DK17] matching was considered for various families of piecewise linear maps in relation to expressions for the invariant densities, entropy and multiple tilings. Another type of transformation for which matching has proven to be convenient is for continued fraction maps, most notably for Nakada's α-continued fraction maps. This family was introduced in [N81] by defining for each $\alpha \in \left[\frac{1}{2}, 1\right]$ the map $S_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha]$ by $S_\alpha(0) = 0$ and for $x \neq 0$,

$$S_\alpha(x) = \frac{1}{|x|} - \left\lfloor \frac{1}{|x|} + 1 - \alpha \right\rfloor. \tag{4.2}$$

In [N81] Nakada constructed a planar natural extension of S_α and proved the existence of a unique absolutely continuous invariant probability measure. In [LM08] the family was extended to include the parameters $\alpha \in \left[0, \frac{1}{2}\right)$. On this part of the parameter space the planar natural extension strongly depends on the matching property, and it is much more complicated. This also affects the behaviour of the metric entropy as a function of α, which is described in detail in [LM08, NN08, CMPT10, KSS12, CT13, T14]. In [DKS09, KU10, CIT18] matching was successfully considered for other families of continued fraction transformations.

The matching behaviour of these different families has some striking similarities. The parameter space usually breaks down into maximal intervals on which the exponents M and N from (4.1) are constant, called *matching intervals*. These matching intervals usually cover most of the space, leaving a Lebesgue null set. The set where matching fails, called the *exceptional set*, is often of positive Haussdorff dimension, see [CT12, KSS12, BCIT13, BCK17, DK17] for example.

So far, matching has been considered only for dynamical systems with a finite absolutely continuous invariant measure. In this article, we introduce and study the matching behaviour and its consequences for a one-parameter family of continued fraction transformations on the interval that have a unique absolutely continuous, σ-finite invariant measure that is infinite. This family of *flipped α-continued fraction transformations* we introduce arises naturally as a counterpart to Nakada's α-continued fraction maps. Due to matching we obtain a nice planar natural extension on a large part of the parameter space, which allows us to explicitly compute dynamical features of the maps, such as the invariant density, Krengel entropy and wandering rate.

The family of maps $\{T_\alpha\}_{\alpha\in(0,1)}$ we consider is defined as follows. For each $\alpha \in (0,1)$ let

$$D_\alpha = \bigcup_{n\geq 1} \left[\frac{1}{n+\alpha}, \frac{1}{n}\right] \subseteq [0,1], \tag{4.3}$$

and $I_\alpha := [\min\{\alpha, 1-\alpha\}, 1]$, and define the map $T_\alpha : I_\alpha \to I_\alpha$ by

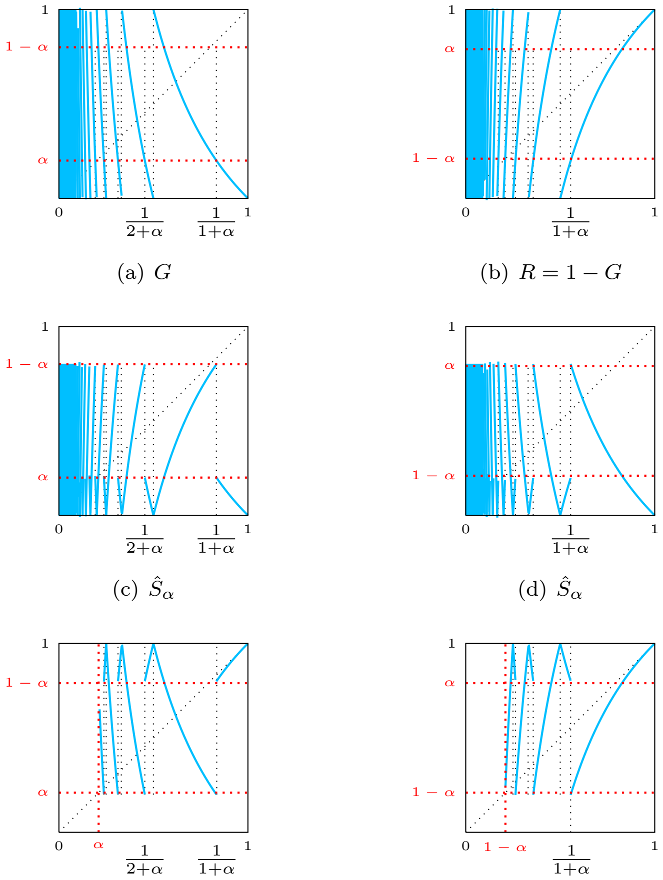
$$T_\alpha(x) = \begin{cases} G(x) = \dfrac{1}{x} - \left\lfloor \dfrac{1}{x} \right\rfloor, & \text{if } x \in D_\alpha^{\mathsf{c}}, \\[2ex] 1 - G(x) = -\dfrac{1}{x} + \left(1 + \left\lfloor \dfrac{1}{x} \right\rfloor\right), & \text{if } x \in D_\alpha, \end{cases}$$

where $G(x) = \frac{1}{x} \pmod 1$ is the *Gauss map* and D_α^{c} denotes the complement of D_α in $[0,1]$. Note that for $\alpha = 0$ one recovers the Gauss map G, and $\alpha = 1$ gives $1 - G$, which is a shifted version of the *Rényi map* or *backwards continued fraction map*. Since these transformations have already been studied extensively, we omit them from our analysis. Figures 4.1(c) and 4.1(f) show the graphs of the maps T_α for a parameter $\alpha < \frac{1}{2}$ and a parameter $\alpha > \frac{1}{2}$, respectively. We could define T_α on the whole interval $[0,1]$, but since the dynamics of T_α is attracted to the interval I_α we just take that as the domain. Since I_α is bounded away from 0, any map T_α has only a finite number of branches. Note also that each map T_α has an indifferent fixed point at 1.

We call these transformations flipped α-continued fraction maps, due to their relation to the family of maps described in [MMY97]. The authors defined for each $\alpha \in [0,1]$ the *folded α-continued fraction map* $\hat{S}_\alpha : [0, \max\{\alpha, 1-\alpha\}] \to [0, \max\{\alpha, 1-\alpha\}]$ by $\hat{S}_\alpha(0) = 0$ and for $x \neq 0$,

$$\hat{S}_\alpha(x) = \left|\frac{1}{x} - \left\lfloor \frac{1}{x} + 1 - \alpha \right\rfloor\right| = \begin{cases} G(x) = \dfrac{1}{x} - \left\lfloor \dfrac{1}{x} \right\rfloor, & \text{if } x \in D_\alpha, \\[2ex] 1 - G(x) = -\dfrac{1}{x} + \left(1 + \left\lfloor \dfrac{1}{x} \right\rfloor\right), & \text{if } x \in D_\alpha^{\mathsf{c}}. \end{cases}$$

The dynamical properties of the folded α-continued fraction maps are essentially equal to those of Nakada's α-continued fraction maps. The name represents the idea that these maps 'fold' the interval $[\alpha-1, \alpha]$ onto $[0, \max\{\alpha, 1-\alpha\}]$. As shown in Figure 4.1 the families $\{T_\alpha\}$ and $\{\hat{S}_\alpha\}$ are obtained by flipping the Gauss map on complementary parts of the unit interval and, as such, both families are particular instances of what are called *D-continued fraction maps* in [DHKM12]. Furthermore, for $\alpha = \frac{1}{2}$ the transformation T_α seems to be closely related, but not isomorphic, to the object of study of [DK00].



(a) G (b) $R = 1 - G$

(c) $\hat{S}_\alpha$ (d) $\hat{S}_\alpha$

(e) T_α (f) T_α

Figure 4.1: The Gauss map G and the flipped map $R = 1 - G$ in (a) and (b). The folded α-continued fraction map $\hat{S}_\alpha$ and the flipped α-continued fraction map T_α for $\alpha < \frac{1}{2}$ in (c) and (e) and for $\alpha > \frac{1}{2}$ in (d) and (f).

The first main result of this article is on the matching behaviour of the family T_α.

4.1.1 Theorem. *The set of parameters $\alpha \in (0, 1)$ for which the transformation T_α does not have matching is a Lebesgue null set of full Hausdorff dimension.*

We also give an explicit description of the matching intervals by relating them to the matching intervals of Nakada's α-continued fraction transformations. The matching behaviour allows us to construct a planar version of the natural extension for $\alpha \in [0, \frac{1}{2}\sqrt{2}]$ leading to the following result.

4.1.2 Theorem. *Let $0 \leq \alpha \leq \frac{1}{2}\sqrt{2}$, let $\mathcal{B}_\alpha$ be the Borel σ-algebra on I_α and let*

$g = \frac{\sqrt{5}-1}{2}$. *The absolutely continuous measure* μ_α *on* $(I_\alpha, \mathcal{B}_\alpha)$ *with density*

$$f_\alpha(x) = \begin{cases} \frac{1}{x}\mathbf{1}_{[\alpha,\frac{\alpha}{1-\alpha}]}(x) + \frac{1}{1+x}\mathbf{1}_{[\frac{\alpha}{1-\alpha},1-\alpha]}(x) + \frac{2}{1-x^2}\mathbf{1}_{[1-\alpha,1]}(x), & \text{for } \alpha \in [0,\frac{1}{2}), \\ \frac{1}{1-x}\mathbf{1}_{[1-\alpha,\alpha]}(x) + \frac{1}{x(1-x)}\mathbf{1}_{[\alpha,\frac{1-\alpha}{\alpha}]}(x) + \frac{x^2+1}{x(1-x^2)}\mathbf{1}_{[\frac{1-\alpha}{\alpha},1]}(x), & \text{for } \alpha \in [\frac{1}{2},g), \\ (\frac{1}{1-x} + \frac{1}{x+\frac{1}{g-1}})\mathbf{1}_{[1-\alpha,\frac{2\alpha-1}{\alpha}]}(x) + \frac{1}{1-x}\mathbf{1}_{[\frac{2\alpha-1}{\alpha},\alpha]}(x) + & \\ +(\frac{1}{1-x} + \frac{1}{x} - \frac{1}{x+\frac{1}{g}})\mathbf{1}_{[\alpha,\frac{2\alpha-1}{1-\alpha}]}(x) + \frac{x^2+1}{x(1-x^2)}\mathbf{1}_{[\frac{2\alpha-1}{1-\alpha},1]}(x), & \text{for } \alpha \in [g,\frac{2}{3}), \\ (\frac{1}{1-x} + \frac{1}{x+\frac{1}{g-1}})\mathbf{1}_{[1-\alpha,\frac{2\alpha-1}{\alpha}]}(x) + \frac{1}{1-x}\mathbf{1}_{[\frac{2\alpha-1}{\alpha},\alpha]}(x) + & \\ +(\frac{1}{1-x} + \frac{1}{x} - \frac{1}{x+\frac{1}{g}})\mathbf{1}_{[\alpha,\frac{1-\alpha}{2\alpha-1}]}(x) + & \\ +(\frac{1}{1-x} + \frac{1}{x+1} - \frac{1}{x+\frac{1}{g}} + \frac{1}{x} - \frac{1}{x+\frac{1}{g+1}})\mathbf{1}_{[\frac{1-\alpha}{2\alpha-1},1]}(x), & \text{for } \alpha \in [\frac{2}{3},\frac{1}{2}\sqrt{2}], \end{cases}$$

is the unique (up to scalar multiplication) σ*-finite, infinite absolutely continuous invariant measure for* T_α*. Furthermore, for* $\alpha \in (0,g]$ *the Krengel entropy equals* $\frac{\pi^2}{6}$*. For* $\alpha \in (0,\frac{1}{2}\sqrt{2})$ *the wandering rate is given by* $w_n(T_\alpha) \sim \log n$ *and the return sequence by* $a_n(T_\alpha) \sim \frac{n}{\log n}$*.*

The value $\frac{1}{2}\sqrt{2}$ is the endpoint of the fourth matching interval. As α grows beyond $\frac{1}{2}\sqrt{2}$ the matching intervals get smaller and smaller with higher matching exponents and the natural extension domain develops a more fractal structure, see Figure 4.4. As a consequence, it is in principle still possible to obtain results similar to those from Theorem 4.1.2 for bigger values of α, but the natural extension and the computations involved become increasingly complicated.

The chapter is outlined as follows. In the next section we give some preliminaries on continued fractions and explain how the maps T_α can be used to generate them for numbers in the interval I_α. We also prove that the maps T_α fall into the family of what are called AFN-maps in [Z98]. In the third section we study the phenomenon of matching, leading to Theorem 4.1.1, and we give an explicit description of the matching intervals. The fourth section is devoted to defining a planar natural extension for the maps T_α for $\alpha \leq \frac{1}{2}\sqrt{2}$. This is then used to obtain the invariant densities appearing in Theorem 4.1.2. In the last section we compute the Krengel entropy, the wandering rate and the return sequence for T_α, giving the last part of Theorem 4.1.2.

§4.2 More CF-maps

§4.2.1 Semi-regular CF-expansions

In 1913, Perron introduced the notion of *semi-regular* continued fraction expansions, which are finite or infinite expressions for real numbers of the following form:

$$x = d_0 + \cfrac{\epsilon_0}{d_1 + \cfrac{\epsilon_1}{d_2 + \ddots + \cfrac{\epsilon_{n-1}}{d_n + \ddots}}},$$

where $d_0 \in \mathbb{Z}$ and for each $n \geq 1$, $\epsilon_{n-1} \in \{-1, 1\}$, $d_n \in \mathbb{N}$ and $d_n + \epsilon_n \geq 1$; see for example [P57]. We denote the semi-regular continued fraction expansion of a number x by

$$x = [d_0; \epsilon_0/d_1, \epsilon_1/d_2, \epsilon_2/d_3, \ldots].$$

The maps T_α generate semi-regular continued fraction expansions of real numbers by iteration. Define for any $\alpha \in (0,1)$ and any $x \in I_\alpha$ the partial quotients $d_k = d_k(x) = d_1(T_\alpha^{k-1}(x))$ and the signs $\epsilon_k = \epsilon_k(x) = \epsilon_1(T_\alpha^{k-1}(x))$ by setting

$$d_1(x) := \begin{cases} \lfloor \frac{1}{x} \rfloor, & \text{if } x \in D_\alpha^{\mathsf{c}}, \\ \lfloor \frac{1}{x} \rfloor + 1, & \text{otherwise;} \end{cases} \quad \text{and} \quad \epsilon_1(x) := \begin{cases} 1, & \text{if } x \in D_\alpha^{\mathsf{c}}, \\ -1, & \text{otherwise.} \end{cases}$$

With this notation the map T_α can be written as $T_\alpha(x) = \epsilon_1(x)\big(\frac{1}{x} - d_1(x)\big)$, implying

$$x = \frac{1}{d_1 + \epsilon_1 T_\alpha(x)} = \cfrac{1}{d_1 + \cfrac{\epsilon_1}{d_2 + \ddots + \cfrac{\epsilon_{n-1}}{d_n + \epsilon_n T_\alpha^n(x)}}}. \tag{4.4}$$

Denote by $(p_n/q_n)_{n\geq 1}$ the sequence of convergents of such an expansion, that is,

$$p_n/q_n = [0; 1/d_1, \epsilon_1/d_2, \ldots, \epsilon_{n-1}/d_n].$$

Since we obtained T_α from the Gauss map, by flipping on the domain D_α from (4.3), it follows from [DHKM12, Theorem 1] that for any $x \in I_\alpha$ we have: $\lim\limits_{n\to\infty} \dfrac{p_n}{q_n} = x$. Therefore, we can write

$$x = \cfrac{1}{d_1 + \cfrac{\epsilon_1}{d_2 + \ddots + \cfrac{\epsilon_{n-1}}{d_n + \ddots}}} =: [0; 1/d_1, \epsilon_1/d_2, \epsilon_2/d_3, \ldots]_\alpha,$$

which we call the *flipped α-continued fraction expansion* of x.

In case $\epsilon_n = 1$ for all $n \geq 1$ the continued fraction expansion is called *regular* and we use the common notation $[a_1, a_2, a_3, \ldots]$ for them. Regular continued fraction expansions are generated by the Gauss map $G : [0, 1] \to [0, 1]$ given by $G(0) = 0$ and $G(x) = \frac{1}{x} \pmod 1$ if $x \neq 0$. Therefore, G acts as a shift on the regular continued fraction expansions:

$$x = [a_1, a_2, a_3, \ldots] \quad \Rightarrow \quad G(x) = [a_2, a_3, a_4, \ldots].$$

It is well known that the regular continued fraction expansion of a number x is finite if and only if $x \in \mathbb{Q}$. For any $x \in [0, \frac{1}{2}]$ the following correspondence between the regular continued fraction expansions of x and $1 - x$ holds:

$$x = [a_1, a_2, a_3, \ldots] \quad \Leftrightarrow \quad 1 - x = [1, a_1 - 1, a_2, a_3, \ldots]. \tag{4.5}$$

We will need this property later.

On sequences of digits $(a_n)_{n\geq1} \in \mathbb{N}^{\mathbb{N}}$ the *alternating ordering* is defined by setting $(a_n)_{n\geq1} \prec (b_n)_{n\geq1}$ if and only if for the smallest index $m \geq 1$ such that $a_m \neq b_m$ it holds that $(-1)^m a_m < (-1)^m b_m$. The same definition holds for finite strings of digits of the same length. The alternating ordering on continued fraction expansions is consistent with standard ordering on the real line, i.e.,

$$(a_n)_{n\geq1} \prec (b_n)_{n\geq1} \quad \Leftrightarrow \quad [a_1, a_2, a_3, \ldots] < [b_1, b_2, b_3, \ldots].$$

The next proposition will be needed in the following section.

4.2.1 Proposition. Let $\alpha \in (0, 1)$ and $x \in I_\alpha$ be given. Then $x \in \mathbb{Q}$ if and only if there is an $N \geq 0$ such that $T_\alpha^N(x) = 1$.

Proof. If there is an $N \geq 0$ such that $T_\alpha^N(x) = 1$, then it follows immediately from (4.4) that $x \in \mathbb{Q}$. Suppose $x \in \mathbb{Q}$. Note that $T_\alpha^n(x) \in \mathbb{Q} \cap I_\alpha$ for all $n \geq 0$ and write $T_\alpha^n(x) = \frac{s_n}{t_n}$ with $s_n, t_n \in \mathbb{N}$ and t_n as small as possible. Assume for a contradiction that $T_\alpha^n(x) \neq 1$ for all $n \geq 1$. Then $s_n < t_n$ and since either $T_\alpha^{n+1}(x) = \frac{t_n - ks_n}{s_n}$ or $T_\alpha^{n+1}(x) = \frac{(k+1)s_n - t_n}{s_n}$, we get $0 < t_{n+1} < t_n$. This gives a contradiction. $\square$

§4.2.2 AFN-maps

We start our investigation into the dynamical properties of the maps T_α by showing that they fall into the category of AFN-maps considered in [Z98, Z00]. Let λ denote the one-dimensional Lebesgue measure and let X be a finite union of bounded intervals. A map $T : X \to X$ is called an *AFN-map* if there is a finite partition $\mathcal{P}$ of X consisting of non-empty, open intervals I_i, such that the restriction $T|_{I_i}$ is continuous, strictly monotone and twice differentiable. Moreover, T has to satisfy the following three properties:

(A) Adler's condition: $\frac{T''}{(T')^2}$ is bounded on $\cup_i I_i$;

(F) The finite image condition: $T(\mathcal{P}) := \{T(I_i) : I_i \in \mathcal{P}\}$ is finite;

(N) The repelling indifferent fixed point condition: there exists a finite set $\mathcal{Z} \subseteq \mathcal{P}$, such that each $Z_i \in \mathcal{Z}$ has an indifferent fixed point x_{Z_i}, that is,

$$\lim_{x \to x_{Z_i}, x \in Z_i} T_\alpha(x) = x_{Z_i} \quad \text{and} \quad \lim_{x \to x_{Z_i}, x \in Z_i} T'(x) = 1,$$

T' decreases on $(-\infty, x_{Z_i}) \cap Z_i$ and increases on $(x_{Z_i}, \infty) \cap Z_i$. Lastly, T is assumed to be uniformly expanding on sets bounded away from $\{x_{Z_i} : Z_i \in \mathcal{Z}\}$.

For the maps T_α we can take $\mathcal{P}$ to be the collection of intervals of monotonicity (or cylinder sets) of T_α, defined for each $\epsilon \in \{-1, 1\}$ and $d \geq 1$ by

$$\Delta(\epsilon, d) = int\{x \in I_\alpha \, : \, \epsilon_1(x) = \epsilon \text{ and } d_1(x) = d\}, \tag{4.6}$$

where we use *int* to denote the interior of the set.

4.2.2 Lemma. *For each $\alpha \in (0,1)$ the map T_α is an AFN-map.*

Proof. Let $\mathcal{P} = \{\Delta(\epsilon, d)\}$. Then T_α is continuous, strictly monotone and twice differentiable on each of the intervals in $\mathcal{P}$. We check the three other conditions. For **(A)** note that $T'_\alpha(x) = \pm\frac{1}{x^2}$, so that $\left|\frac{T''_\alpha(x)}{(T'_\alpha(x))^2}\right| = \left|\frac{2x^4}{x^3}\right| = \pm 2x \leq 2$ for any x for which T'_α is defined. Also, for any $J \in \mathcal{P}$ we have

$$T_\alpha(J) \in \big\{(\alpha, 1), (1-\alpha, 1), (\alpha, T_\alpha(\alpha)), (1-\alpha, T_\alpha(1-\alpha)), (T_\alpha(\alpha), 1), (T_\alpha(1-\alpha), 1)\big\},$$

giving **(F)**. Finally, T_α has only 1 as an indifferent fixed point. Since $T'_\alpha(x) = 1/x^2 > 1$ for any $x \in I_\alpha \setminus \{1\}$ where $T'_\alpha(x)$ is defined, we see that T'_α decreases near 1 and also **(N)** holds. □

Using [Z98, Theorem 1] we then obtain the following result.

4.2.3 Proposition. For each $\alpha \in \big(0,1\big)$ there exists a unique absolutely continuous, infinite, σ-finite T_α-invariant measure μ_α that is ergodic and conservative for T_α.

Proof. Since T_α is an AFN-map, [Z98, Theorem 1] immediately implies that there are finitely many disjoint open sets $X_1, \dots, X_N \subseteq I_\alpha$, such that $T_\alpha(X_i) = X_i \pmod{\lambda}$ and $T|_{X_i}$ is conservative and ergodic with respect to λ. Each X_i is a finite union of open intervals and supports a unique (up to a constant factor) absolutely continuous T_α-invariant measure. Moreover, this invariant measure is infinite if and only if X_i contains an interval $(1-\delta, 1)$ for some $\delta > 0$. Since each open interval contains a rational point in its interior, Proposition 4.2.1 together with the forward invariance of the sets X_i implies that there can only be one set X_i and that this set contains an interval of the form $(1-\delta, 1)$. Hence, there is a unique (up to a constant factor) absolutely continuous invariant measure μ_α that is infinite, σ-finite, ergodic and conservative for T_α. □

From Proposition 4.2.3 and [Z00, Theorem 1] it follows that each map T_α is *pointwise dual-ergodic*, i.e., there are positive constants $a_n(T_\alpha)$, $n \geq 1$, such that for each $f \in L^1(I_\alpha, \mathcal{B}_\alpha, \mu_\alpha)$, where $\mathcal{B}_\alpha$ is the Borel $\sigma-$algebra on I_α,

$$\lim_{n \to \infty} \frac{1}{a_n(T_\alpha)} \sum_{k=0}^{n-1} P^k_{T_\alpha} f = \int_{I_\alpha} f \, d\mu_\alpha \quad \mu_\alpha\text{-a.e.}, \tag{4.7}$$

where P_{T_α} denotes the Perron-Frobenius operator of the map T_α, given in Definition 1.2. The sequence $(a_n(T_\alpha))_{n\geq 1}$ is called the *return sequence* of T_α and will be given for $\alpha \in \left(0, \frac{1}{2}\sqrt{2}\right)$ in Section 4.5.

§4.3 Matching almost everywhere

In this section we prove that matching holds for almost every $\alpha \in (0,1)$. The discontinuity points of the map T_α are of the form $\frac{1}{k+\alpha}$ for some positive integer k. For any such point,

$$c^- = \lim_{x \uparrow \frac{1}{k+\alpha}} T_\alpha(x) = \alpha \quad \text{and} \quad c^+ = \lim_{x \downarrow \frac{1}{k+\alpha}} T_\alpha(x) = -(k+\alpha) + k + 1 = 1 - \alpha.$$

Recall the definition of matching from equation (4.1): matching for T_α holds if there exist non-negative integers M, N such that

$$T_\alpha^M(\alpha) = T_\alpha^N(1-\alpha). \tag{4.8}$$

Some authors also require the evaluation of the derivative of the iterates in the left and right limits of the critical points to coincide. In our case, we do not need this constraint, since we prove that matching is a local property.

In the next proposition we show that the first half of the parameter space consists of a single matching interval.

4.3.1 Proposition. For $\alpha \in \left(0, \frac{1}{2}\right)$ it holds that $T_\alpha(\alpha) = T_\alpha^2(1-\alpha)$.

Proof. Fix $\alpha = [a_1, a_2, \dots] \in \left(0, \frac{1}{2}\right)$. First note that $\frac{1}{2} < 1 - \alpha < \frac{1}{1+\alpha}$, so that by (4.5) we obtain that

$$T_\alpha(1-\alpha) = G(1-\alpha) = \frac{\alpha}{1-\alpha} = [a_1 - 1, a_2, a_3, \dots].$$

Hence

$$\frac{1}{a_1+1} < \alpha < \frac{1}{a_1+\alpha} \Leftrightarrow \frac{1}{a_1} < \frac{\alpha}{1-\alpha} < \frac{1}{a_1 - 1 + \alpha},$$

which gives that either α and $T_\alpha(1-\alpha)$ are both in D_α or in D_α^{c}. In both cases,

$$T_\alpha(\alpha) = T_\alpha^2(1-\alpha).$$

□

For $\alpha > \frac{1}{2}$ the situation is much more complicated. One explanation for this difference comes from two operations that convert one semi-regular continued fraction expansion of a number into another: singularisation and insertion. Both operations were introduced in [P57] and later appeared in many other places in the literature, see e.g. [K91, DK00, HK02, S04, DHKM12]. *Singularisation* deletes one of the convergents $\frac{p_n}{q_n}$ from the sequence while altering the ones before and after; *insertion* inserts the mediant $\frac{p_n+p_{n+1}}{q_n+q_{n+1}}$ of $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$ into the sequence. It follows from [DHKM12, Section

2.1] that for $\alpha < \frac{1}{2}$ the flipped α-continued fraction expansions of numbers in I_α can be obtained from their regular continued fraction expansions by insertions only, while for $\alpha > \frac{1}{2}$ one needs singularisations as well.

Define the map $R : [0,1] \to [0,1]$ by $R(x) = 1 - G(x)$, see Figure 4.1(b). Before we prove that matching holds Lebesgue almost everywhere, we describe the effect of R on the regular continued fraction expansions of numbers in $(0,1)$.

4.3.2 Lemma. *Let $x \in (0,1)$ have regular continued fraction expansion $x = [x_1, x_2, x_3, \ldots]$. Then for each $j \geq 1$,*

$$R^{x_2+x_4+\cdots+x_{2j}}(x) = [x_{2j+1}+1, x_{2j+2}, x_{2j+3}, \ldots]$$

and if $0 < \ell < x_{2j}$, then

$$R^{x_2+x_4+\cdots+x_{2j-2}+\ell}(x) = [1, x_{2j}-\ell, x_{2j+1}, x_{2j+2}, \ldots].$$

Proof. By (4.5) it holds that

$$R(x) = 1 - G(x) = 1 - [x_2, x_3, x_4, \ldots] = \begin{cases} [x_3+1, x_4, x_5, \ldots] = R^{x_2}(x), & \text{if } x_2 = 1, \\ [1, x_2-1, x_3, x_4, \ldots], & \text{if } x_2 > 1. \end{cases}$$

The statement then easily follows by induction. $\square$

4.3.3 Remark. The previous lemma implies that R preserves the parity of the regular continued fraction digits. More precisely, if $x \in (0,1)$, then (except for possibly the first two digits) the regular continued fraction expansion of $R(x)$ has regular continued fraction digits of x with even indices in even positions and regular continued fraction digits with odd indices in odd positions.

The map T_α equals the map R on D_α and G on D_α^{c}. The next lemma specifies the times n at which the orbit of α (or $1-\alpha$) can enter D_α^{c} for the first time.

4.3.4 Lemma. *Let $\alpha = [1, a_1, a_2, a_3, \ldots] \in \left(\frac{1}{2}, 1\right)$. If $m := \min\{i \geq 0 : T_\alpha^i(\alpha) \in D_\alpha^{\mathsf{c}}\}$ exists, then $m = a_1 + a_3 + \cdots + a_{2j+1} - 1$ where j is the unique integer such that*

$$a_1 + a_3 + \cdots + a_{2j-1} - 1 < m \leq a_1 + a_3 + \cdots + a_{2j+1} - 1.$$

Similarly, if $k := \min\{i \geq 0 : T_\alpha^i(1-\alpha) \in D_\alpha^{\mathsf{c}}\}$ exists, then $k = a_2 + a_4 + \cdots + a_{2j} - 1$ where j is the unique integer such that

$$a_2 + a_4 + \cdots + a_{2j-2} - 1 < k \leq a_2 + a_4 + \cdots + a_{2j} - 1.$$

Proof. For the first statement, by the definition of m we know that $T_\alpha^i(\alpha) = R^i(\alpha)$ for all $i \leq m$. From Lemma 4.3.2 it then follows that if $m = a_1 + a_3 + \cdots + a_{2j-1}$, then

$$T_\alpha^m(\alpha) = [a_{2j}+1, a_{2j+1}, a_{2j+2}, \ldots],$$

and if $m = a_1 + a_3 + \cdots + a_{2j-1} + \ell$ for some $0 < \ell \leq a_{2j+1} - 1$, then

$$T_\alpha^m(\alpha) = [1, a_{2j+1}-\ell, a_{2j+2}, a_{2j+3}, \ldots].$$

Recall that $D_\alpha^{\mathsf{c}} = \bigcup_d \Delta(1,d)$. The right boundary point of any cylinder $\Delta(1,d) = (\frac{1}{d+1}, \frac{1}{d+\alpha})$ has regular continued fraction expansion $[d,1,a_1,a_2,\ldots]$. Since the regular continued fraction expansion of the left boundary point also starts with the digits $d,1$, any $x \in D_\alpha^{\mathsf{c}}$ has a regular continued fraction expansion of the form $[x_1,1,x_3,\ldots]$. In particular this holds for $T_\alpha^m(\alpha)$, which implies that either $a_{2j+1} = 1$ or $\ell = a_{2j+1} - 1$. In both cases, $m = a_1 + a_3 + \cdots + a_{2j+1} - 1$. For the second part of the lemma, recall from (4.5) that $1 - \alpha = [a_1+1, a_2, a_3, \ldots]$. The proof of the second part then goes along the same lines as above. □

Recall from the introduction the definition of matching intervals as the maximal parameter intervals on which the matching exponents M, N from (4.1) are constant. We can obtain a complete description of the matching intervals by relating them to the matching intervals of Nakada's α-continued fraction maps from (4.2). First we recall some notation and results on matching for the maps from (4.2). Any rational number $a \in \mathbb{Q} \cap (0,1)$ has two regular continued fraction expansions:

$$a = [a_1, \ldots, a_n] = [a_1, \ldots, a_n - 1, 1], \quad a_n \geq 2.$$

The *quadratic interval* I_a associated to a is the interval with endpoints

$$[\overline{a_1, \ldots, a_n}] \quad \text{and} \quad [\overline{a_1, \ldots, a_n - 1, 1}].$$

The quadratic interval I_1 is defined separately by $I_1 = (g, 1)$, where $g = \frac{\sqrt{5}-1}{2}$. A quadratic interval I_a is called *maximal* if it is not properly contained in any other quadratic interval. By [CT12, Theorem 1.3] maximal intervals correspond to matching intervals for Nakada's α-continued fraction maps.

Let $\mathcal{R} = \{a \in \mathbb{Q} \cap (0,1] : I_a \text{ is maximal}\}$ and $a = [a_1, \ldots, a_n] \in \mathcal{R}$ with $a_n \geq 2$. The map $x \mapsto \frac{1}{1+x}$ is the inverse of the right most branch of the Gauss map. Therefore, $\frac{1}{1+a} = [1, a_1, a_2, \ldots, a_n - 1, 1] = [1, a_1, a_2, \ldots, a_n]$. Write

$$J_a^L = \Big([1, \overline{a_1, a_2, \ldots, a_n - 1, 1}], [1, a_1, a_2, \ldots, a_n - 1, 1]\Big),$$
$$J_a^R = \Big([1, a_1, a_2, \ldots, a_n], [1, \overline{a_1, a_2, \ldots, a_n}]\Big),$$

if n is odd and

$$J_a^L = \Big([1, \overline{a_1, a_2, \ldots, a_n}], [1, a_1, a_2, \ldots, a_n]\Big),$$
$$J_a^R = \Big([1, a_1, a_2, \ldots, a_n - 1, 1], [1, \overline{a_1, a_2, \ldots, a_n - 1, 1}]\Big),$$

if n is even, so that $\frac{1}{1+I_a} = J_a^L \cup J_a^R \cup \{\frac{1}{1+a}\}$. Finally, let

$$M = a_1 + a_3 + \cdots + a_n \quad \text{and} \quad N = a_2 + a_4 + \cdots + a_{n-1} + 2 \tag{4.9}$$

if n is odd and

$$M = a_1 + a_3 + \cdots + a_{n-1} + 1 \quad \text{and} \quad N = a_2 + a_4 + \cdots + a_n + 1 \tag{4.10}$$

if n is even. The next theorem states that the intervals J_a^L and J_a^R are matching intervals for the flipped α-continued fraction maps with matching exponents that depend on M and N.

4.3.5 Theorem. *Let $a \in \mathcal{R}$ and let M and N be as in (4.9) and (4.10). For each $\alpha \in J_a^L$ the map T_α satisfies $T_\alpha^M(\alpha) = T_\alpha^N(1-\alpha)$ and for each $\alpha \in J_a^R$ the map T_α satisfies $T_\alpha^{M+1}(\alpha) = T_\alpha^{N-1}(1-\alpha)$.*

Proof. First we consider the special maximal quadratic interval $I_1 = (g, 1)$ separately, for which n is odd and $J_1^L = \emptyset$ and $J_1^R = \left(\frac{1}{2}, g\right)$. Let $\alpha \in J_1^R$. Then $\alpha = [1, 1, a_2, a_3, \ldots]$ and $1 - \alpha = [2, a_2, a_3, \ldots]$. Note that $M = 1$ and $N = 2$. From $\alpha < g$ it follows that $\alpha^2 + \alpha - 1 < 0$. This implies that $1 - \alpha > \frac{1}{2+\alpha}$, so that $T_\alpha^{N-1}(1-\alpha) = T_\alpha(1-\alpha) = R(1-\alpha)$. It also implies that $\frac{1}{2} < \alpha < \frac{1}{1+\alpha}$ and that

$$T_\alpha(\alpha) = G(\alpha) = \frac{1}{\alpha} - 1 > \frac{1}{1+\alpha},$$

so that $T_\alpha^{M+1}(\alpha) = T_\alpha^2(\alpha) = R \circ G(\alpha)$, which by Lemma 4.3.2 equals $T_\alpha^{N-1}(1-\alpha)$.

Fix $a \in \mathcal{R}\backslash\{1\}$ and write $a = [a_1, a_2, \ldots, a_n] = [a_1, a_2, \ldots, a_n - 1, 1]$ for its regular continued fraction expansions. We only prove the statement for J_a^L, since the proof for J_a^R is similar. Assume without loss of generality that n is odd. The proof is analogous for n even and the parity is fixed only to determine the endpoints of J_a^L. We start by proving that matching cannot occur for indices smaller than M and N.

Write $\mathbf{a} = a_1, a_2, \ldots, a_{n-1}, a_n - 1, 1 \in \mathbb{N}^{n+1}$ and let $\alpha \in J_a^L = ([1, \bar{\mathbf{a}}], [1, \mathbf{a}])$. Then there is some finite or infinite string of positive integers $\mathbf{w} = a_{n+2}, a_{n+3}, \ldots$, such that $\alpha = [1, \mathbf{a}, \mathbf{w}]$. The assumption that n is odd together with the fact that the Gauss map preserves the alternating ordering imply that

$$\bar{\mathbf{a}} \succ \mathbf{w}. \tag{4.11}$$

Assume that $m = \min\{i \geq 0 : T_\alpha^i(\alpha) \in D_\alpha^c\}$ exists. By Lemma 4.3.4 there is a j, such that $m = a_1 + a_3 + \cdots + a_{2j-1} - 1$. Assume that $2j - 1 < n$. By the definition of m it holds that $T_\alpha^m(\alpha) \in D_\alpha^c$. So, using Lemma 4.3.2 we obtain that

$$[1, a_{2j}, a_{2j+1}, \ldots] = G(T_\alpha^m(\alpha)) = T_\alpha^{m+1}(\alpha) > \alpha = [1, \mathbf{a}, \mathbf{w}].$$

Since $a \in \mathcal{R}$, the result from [CT12, Proposition 4.5.2] implies that for any two non-empty strings $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{a} = \mathbf{uv}$, the inequality

$$\mathbf{v} \succ \mathbf{uv} \tag{4.12}$$

holds. Thus, if we take $\mathbf{v} = a_{2j}, a_{2j+1}, \ldots, a_n - 1, 1$ and $\mathbf{u} = a_1, a_2, \ldots, a_{2j-1}$, then we find $\mathbf{v} \preceq \mathbf{uv}$, which contradicts (4.12). Hence, if m exists, then $m \geq M - 1$. In a similar way we can deduce that if $k = \min\{i \geq 0 : T_\alpha^i(1-\alpha) \in D_\alpha^c\}$ exists, then $k \geq N - 2$.

Now assume that there exist $\ell < M - 1$ and $i < N - 2$, such that

$$T_\alpha^\ell(\alpha) = R^\ell(\alpha) = R^i(1-\alpha) = T_\alpha^i(1-\alpha). \tag{4.13}$$

Recall from Remark 4.3.3 that R preserves the parity of the regular continued fraction digits. Since $\alpha = [1, a_1, a_2, \ldots]$ and $1-\alpha = [a_1+1, a_2, a_3, \ldots]$, the assumption (4.13) then implies the existence of an even index $2 \le j \le n-1$ and an odd index $1 \le \ell < n$, such that

$$a_j, a_{j+1}, a_{j+2}, \ldots = a_\ell, a_{\ell+1}, a_{\ell+2}, \ldots.$$

This implies that a has an ultimately periodic regular continued fraction expansion, which contradicts the fact that $a \in \mathbb{Q}$. Hence, matching cannot occur with indices $\ell < M-1$ and $i < N-2$.

Next consider $T_\alpha^M(\alpha)$ and $T_\alpha^{N-2}(1-\alpha)$. From Lemma 4.3.2 applied to $\alpha = [1, a_1, a_2, \ldots]$, i.e., $x_i = a_{i-1}$, we get that

$$T_\alpha^{M-1}(\alpha) = R^{a_1+a_3+\cdots+a_n-1}(\alpha) = [2, \mathbf{w}].$$

From $\alpha = [1, \mathbf{a}, \mathbf{w}] > g$, it follows that $G(\alpha) = [\mathbf{a}, \mathbf{w}] < g$. Combining this with the fact that the property from (4.11) implies $\mathbf{w} \prec \mathbf{aw} \prec 1\mathbf{aw}$. Hence, $T_\alpha^{M-1}(\alpha) > [2, 1, \mathbf{a}, \mathbf{w}] = \frac{1}{2+\alpha}$ which gives $T_\alpha^{M-1}(\alpha) \in (\frac{1}{2+\alpha}, \frac{1}{2})$. This implies that $T_\alpha^M(\alpha) = R(T_\alpha^{M-1}(\alpha)) = R([2, \mathbf{w}])$. For $1-\alpha = [a_1+1, a_2, a_3, \ldots, a_{n-1}, a_n - 1, 1, \mathbf{w}]$ we get from Lemma 4.3.2 that

$$T_\alpha^{N-2}(1-\alpha) = R^{N-2}(1-\alpha) = [a_n, 1, \mathbf{w}].$$

Again using that $\mathbf{w} \prec \mathbf{aw}$ gives $T_\alpha^{N-2}(1-\alpha) \in \Delta(1, a_n) = ([a_n, 1], [a_n, 1, \mathbf{a}, \mathbf{w}])$. Since $\alpha > g$, it follows that $T_\alpha^N(1-\alpha) = R \circ G(T_\alpha^{N-2}(1-\alpha))$. Then, again by using Lemma 4.3.2, we obtain

$$T_\alpha^N(1-\alpha) = R([1, \mathbf{w}]) = R([2, \mathbf{w}]) = T_\alpha^M(\alpha).$$

For $\alpha \in J_a^R$, one can show similarly that $T_\alpha^{M-1}(\alpha) = [1, 1, \mathbf{w}] \in \Delta(1, 1)$. Since $\alpha > g$, this gives $T_\alpha^{M+1}(\alpha) = R \circ G(T_\alpha^{M-1}(\alpha)) = R([1, \mathbf{w}])$. On the other hand, $T_\alpha^{N-2}(1-\alpha) = R_\alpha^{N-2}(1-\alpha) = [a_n+1, \mathbf{w}] > \frac{1}{a_n+1+\alpha}$. So,

$$T_\alpha^{N-1}(1-\alpha) = R_\alpha^{N-1}(1-\alpha) = R([a_n+1, \mathbf{w}]) = R([1, \mathbf{w}]) = T_\alpha^{M+1}(\alpha).$$

□

From this theorem we obtain the result from Theorem 4.1.1 on the size of the set of non-matching parameters. We use $\dim_H(A)$ to denote the Hausdorff dimension of a set A and let $\mathcal{E}$ denote the non-matching set, that is,

$$\mathcal{E} = \{\alpha \in (0,1) \,:\, T_\alpha \text{ does not have the matching property}\}.$$

Proof of Theorem 4.1.1. We use known results on the exceptional set $\mathcal{N}$ of non-matching parameters for Nakada's α-continued fraction maps from (4.2). It is proven in [CT12] and [KSS12] that $\lambda(\mathcal{N}) = 0$ and in [CT12, Theorem 1.2] that $\dim_H(\mathcal{N}) = 1$. Since the bi-Lipschitz map $x \mapsto \frac{1}{1+x}$ on $(0,1)$ preserves Lebesgue null sets and Hausdorff dimension, the same properties hold for the set $E := \frac{1}{1+\mathcal{N}}$. Note that T_α has matching for all $\alpha \in E^{\mathsf{c}}$, since according to Theorem 4.3.5 either α is in a matching interval or it is of the form $\frac{1}{1+a}$ for some rational number a and then both α and $1-\alpha$ eventually get mapped to 1. Hence, $\mathcal{E} \subseteq E$ and it follows that $\lambda(\mathcal{E}) = 0$.

Now consider $E \setminus \mathcal{E}$. Let $a \in \mathcal{N}$. By [KSS12, Section 4], this is equivalent to $G^n(a) \geq a$ for all $n \geq 1$. Let $\alpha := \frac{1}{1+a}$ and write $[1, a_1, a_2, \ldots]$ for its regular continued fraction expansion. Suppose there exists a minimal $m \geq 0$ such that $T_\alpha^m(\alpha) \in D_\alpha^{\mathsf{c}}$. Then there exists a positive integer d such that

$$\frac{1}{d+1} < T_\alpha^m(\alpha) < \frac{1}{d+\alpha}.$$

By Lemma 4.3.2 the inequality implies in particular that for some $j > 2$

$$[a_j, a_{j+1}, \ldots] < [a_1, a_2, \ldots],$$

i.e., $G^{j-1}(a) < a$, which contradicts the assumption on a. Hence, $T_\alpha^k(\alpha) \notin D_\alpha^{\mathsf{c}}$ for all $k \geq 0$. Since the regular continued fraction expansion of $1-\alpha$ is given by $1-\alpha = [a_1+1, a_2, \ldots]$, the same conclusion holds for $1-\alpha$, that is, $T_\alpha^k(\alpha) \notin D_\alpha^{\mathsf{c}}$ for all $k \geq 0$. Hence, $T_\alpha^k(\alpha) = R^k(\alpha)$ and $T_\alpha^k(1-\alpha) = R^k(1-\alpha)$ for all k. Assume that $\alpha \notin \mathcal{E}$, so there are positive integers M, N such that $T_\alpha^M(\alpha) = T_\alpha^N(1-\alpha)$. By Remark 4.3.3 there is an odd index $\ell \geq 1$ and an even index $k \geq 2$ such that

$$a_\ell, a_{\ell+1}, \ldots = a_k, a_{k+1}, \ldots.$$

Therefore α is ultimately periodic and thus a preimage of a quadratic irrational. This implies that $\dim_H(E \setminus \mathcal{E}) = 0$ and hence $\dim_H(\mathcal{E}) = 1$. □

These matching results are the main reason for the existence of the nice geometric versions of the natural extensions that we investigate in the next section.

§4.4 Natural extensions

For non-invertible dynamical systems, especially for continued fraction transformations, the natural extension is a very useful tool to obtain dynamical properties of the system, see Definition 1.2.6. Canonical constructions of the natural extension were first studied by Rohlin in [R61]. Based on these results it was shown in [S88, ST91] that for infinite measure systems like T_α a natural extension always exists and that any two natural extensions of the same system are necessarily isomorphic. Moreover, many ergodic properties carry over from the natural extension to the original map. The amount of information on the original system that can be gained from the natural extension, depends to a large extent on the version of the natural extension one considers. For continued fraction maps, there is a canonical construction, described in Section 1.9, that has led to many useful observations; see for example [N81, K91, KSS12, AS13, H02]. It turns out that a similar construction also works for the family $\{T_\alpha\}_{\alpha\in(0,1)}$.

In this section we construct a natural extension for the system $(I_\alpha, \mathcal{B}_\alpha, \mu_\alpha, T_\alpha)$, where $\mathcal{B}_\alpha$ is the Borel σ−algebra on I_α and μ_α is the measure from Proposition 4.2.3.

This natural extension is given by the dynamical system $(\mathcal{D}_\alpha, \mathcal{B}(\mathcal{D}_\alpha), \nu_\alpha, \mathcal{T}_\alpha)$, where $\mathcal{D}_\alpha$ is some domain in $\mathbb{R}^2$ that needs to be determined, $\mathcal{B}(\mathcal{D}_\alpha)$ is the Borel σ-algebra on $\mathcal{D}_\alpha$, ν_α is the measure defined by

$$\nu_\alpha(A) = \iint_A \frac{1}{(1+xy)^2}\, d\lambda^2(x,y) \quad \text{for any } A \in \mathcal{B}(\mathcal{D}_\alpha), \tag{4.14}$$

where λ^2 is the two-dimensional Lebesgue measure, and $\mathcal{T}_\alpha : \mathcal{D}_\alpha \to \mathcal{D}_\alpha$ is given by

$$\mathcal{T}_\alpha(x,y) = \left(T_\alpha(x), \frac{\epsilon_1(x)}{d_1(x)+y} \right).$$

To prove that $(\mathcal{D}_\alpha, \mathcal{B}(\mathcal{D}_\alpha), \nu_\alpha, \mathcal{T}_\alpha)$ is the natural extension of $(I_\alpha, \mathcal{B}_\alpha, \mu_\alpha, T_\alpha)$ we need to show that ν_α is $\mathcal{T}_\alpha$-invariant and that all of the following properties hold ν_α-almost everywhere:

(ne1) $\mathcal{T}_\alpha$ is invertible;

(ne2) the projection map $\pi : \mathcal{D}_\alpha \to I_\alpha$ is measurable and surjective;

(ne3) $\pi \circ \mathcal{T}_\alpha = T_\alpha \circ \pi$, where π is the projection onto the first coordinate;

(ne4) $\bigvee_{n=0}^{\infty} \mathcal{T}_\alpha^n \pi^{-1}(\mathcal{B}_\alpha) = \mathcal{B}(\mathcal{D}_\alpha)$, where $\bigvee_{n=0}^{\infty} \mathcal{T}_\alpha^n \pi^{-1}(\mathcal{B}_\alpha)$ is the smallest σ-algebra containing the σ-algebras $\mathcal{T}_\alpha^n \pi^{-1}(\mathcal{B}_\alpha)$ for all $n \geq 0$.

The shape of $\mathcal{D}_\alpha$ will depend on the orbits of α and $1-\alpha$ up to the moment of matching. As might be imagined in light of Proposition 4.3.1 and Theorem 4.3.5, the situation for $0 < \alpha < \frac{1}{2}$ is simpler than for $\frac{1}{2} < \alpha < 1$. We will provide a detailed description and proof for $0 < \alpha < \frac{1}{2}$ and list some analytical and numerical results for $\frac{1}{2} < \alpha < 1$.

§4.4.1 For $\alpha < \frac{1}{2}$

We claim that for $\alpha < \frac{1}{2}$ the domain of the natural extension is given by

$$\mathcal{D}_\alpha := \left[\alpha, \frac{\alpha}{1-\alpha}\right] \times [0,\infty) \cup \left(\frac{\alpha}{1-\alpha}, 1-\alpha\right] \times [0,1] \cup \left(1-\alpha, 1\right] \times [-1,1],$$

see Figure 4.2. Before we check (ne1)–(ne4), we introduce some notation. Partition D_α according to the cylinder sets of T_α described in (4.6). Let

$$\tilde{\Delta}(-1,2) = \Delta(-1,2) \times (-1,1), \quad \tilde{\Delta}(1,1) = \left(\frac{1}{2}, 1-\alpha\right) \times (0,1) \cup \left(1-\alpha, \frac{1}{1+\alpha}\right) \times (-1,1),$$

and for $(\epsilon, d) \notin \{(1,1), (-1,2)\}$,

$$\tilde{\Delta}(\epsilon,d) = \begin{cases} \Delta(\epsilon,d) \times (0,1), & \text{if } \Delta(\epsilon,d) \subseteq \left[\frac{\alpha}{1-\alpha}, 1-\alpha\right], \\ \Delta(\epsilon,d) \times (0,\infty), & \text{if } \Delta(\epsilon,d) \subseteq \left[\alpha, \frac{\alpha}{1-\alpha}\right], \end{cases}$$

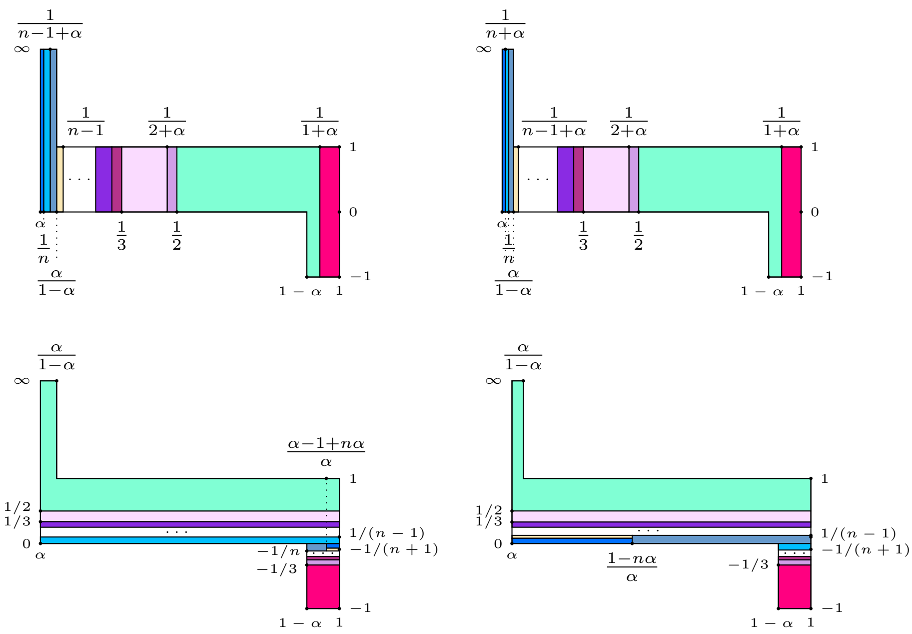
and for the ϵ and d such that $\frac{\alpha}{1-\alpha} \in \Delta(\epsilon, d)$,

$$\tilde{\Delta}_L(\epsilon, d) = \left(\Delta(\epsilon, d) \cap \left[\alpha, \frac{\alpha}{1-\alpha}\right]\right) \times (0, \infty), \quad \tilde{\Delta}_R(\epsilon, d) = \left(\Delta(\epsilon, d) \cap \left[\frac{\alpha}{1-\alpha}, 1\right]\right) \times (0, 1).$$

Due to the matching property described in Proposition 4.3.1 we have, up to a Lebesgue measure zero set,

- $\mathcal{T}_\alpha(\tilde{\Delta}(1,1)) = \left(\alpha, \frac{\alpha}{1-\alpha}\right) \times \left(\frac{1}{2}, \infty\right) \cup \left(\frac{\alpha}{1-\alpha}, 1\right) \times \left(\frac{1}{2}, 1\right)$,
- $\bigcup_{d \geq 2} \mathcal{T}_\alpha(\tilde{\Delta}(-1, d)) = (1-\alpha, 1) \times (-1, 0)$,
- $\bigcup_{d \geq 1} \mathcal{T}_\alpha(\tilde{\Delta}(1, d)) = (\alpha, 1) \times (0, 1)$,

where we have included the sets $\tilde{\Delta}_L(\epsilon, d)$ and $\tilde{\Delta}_R(\epsilon, d)$ in the appropriate union. Hence, $\mathcal{T}_\alpha$ is Lebesgue almost everywhere invertible, which gives (ne1).



(a) $\mathcal{D}_\alpha$ (above) and $\mathcal{T}_\alpha(\mathcal{D}_\alpha)$ (below) for $\frac{1}{n+\alpha} < \alpha < \frac{1}{n}$, $n \geq 2$

(b) $\mathcal{D}_\alpha$ (above) and $\mathcal{T}_\alpha(\mathcal{D}_\alpha)$ (below) for $\frac{1}{n+1} < \alpha < \frac{1}{n+\alpha}$, $n \geq 2$

Figure 4.2: The transformation $\mathcal{T}_\alpha$ maps areas on the top to areas on the bottom with the same colours, respectively.

The properties (ne2) and (ne3) follow immediately. Left to prove are (ne4) and the fact that ν_α is $\mathcal{T}_\alpha$-invariant. To prove that ν_α is invariant for $\mathcal{T}_\alpha$, it suffices to check that $\nu_\alpha(A) = \nu_\alpha(\mathcal{T}_\alpha^{-1}(A))$ for any rectangle $A = [a, b] \times [c, d] \subseteq \mathcal{T}_\alpha(D)$ for any $D = \tilde{\Delta}(\epsilon, d)$, $D = \tilde{\Delta}_L(\epsilon, d)$ or $D = \tilde{\Delta}_R(\epsilon, d)$. This computation is very similar to the corresponding ones for natural extensions of other continued fraction maps that can be found in literature, see e.g. [N81, Theorem 1]. We reproduce it here for the convenience of the reader. For any such rectangle A we have on the one hand,

$$\begin{aligned}\nu_\alpha(A) = &\iint_A \frac{1}{(1+xy)^2}\,d\lambda^2(x,y) = \int_{[c,d]} \frac{b}{1+by} - \frac{a}{1+ay}\,d\lambda(y)\\ &= \ln\left(\frac{1+bd}{1+bc}\right) - \ln\left(\frac{1+ad}{1+ac}\right) = \ln\left(\frac{1+ac+bd+abcd}{1+bc+ad+abcd}\right).\end{aligned}$$

If there is a $k \geq 1$ such that $\pi(D) \subseteq \Delta(1,k)$, then

$$\begin{aligned}\nu_\alpha\left(\mathcal{T}_\alpha^{-1}(A)\right) &= \iint_{\mathcal{T}_\alpha^{-1}(A)} \frac{1}{(1+xy)^2}\,d\lambda^2(x,y)\\ &= \int_{[\frac{1}{d}-k,\frac{1}{c}-k]} \frac{1}{k+a+y} - \frac{1}{k+b+y}d\lambda(y)\\ &= \ln\left(\frac{k+a+\frac{1}{c}-k}{k+a+\frac{1}{d}-k}\right) - \ln\left(\frac{k+b+\frac{1}{c}-k}{k+b+\frac{1}{d}-k}\right)\\ &= \ln\left(\frac{1+ac+bd+abcd}{1+bc+ad+abcd}\right).\end{aligned}$$

If there is a $k \geq 2$ such that $\pi(D) \subseteq \Delta(-1,k)$, then

$$\begin{aligned}\nu_\alpha\left(\mathcal{T}_\alpha^{-1}(A)\right) &= \iint_{\mathcal{T}_\alpha^{-1}(A)} \frac{1}{(1+xy)^2}d\lambda^2(x,y)\\ &= \int_{[-k-\frac{1}{c},-k-\frac{1}{d}]} \frac{1}{k-b+y} - \frac{1}{k-a+y}d\lambda(y)\\ &= \ln\left(\frac{k-b-k-\frac{1}{d}}{k-b-k-\frac{1}{c}}\right) - \ln\left(\frac{k-a-k-\frac{1}{d}}{k-a-k-\frac{1}{c}}\right)\\ &= \ln\left(\frac{1+ac+bd+abcd}{1+bc+ad+abcd}\right).\end{aligned}$$

In both cases $\nu_\alpha(A) = \nu_\alpha(\mathcal{T}_\alpha^{-1}(A))$ proving that ν_α is a $\mathcal{T}_\alpha$-invariant measure.

To prove that (ne4) holds, it is enough to show that $\bigvee_{n=0}^\infty \mathcal{T}_\alpha^n\pi^{-1}(\mathcal{B}_\alpha)$ separates points, i.e., that for λ^2-almost all $(x,y),(x',y') \in \mathcal{D}_\alpha$ with $(x,y) \neq (x',y')$ there are disjoint sets $A, B \in \bigvee_{n=0}^\infty \mathcal{T}_\alpha^n\pi^{-1}(\mathcal{B}_\alpha)$ with $(x,y) \in A$ and $(x',y') \in B$. Since $\mathcal{B}_\alpha$ is separating, the property is clear if $x \neq x'$. Furthermore, note that for λ-almost all values of y there is an ε and a d, such that on a neighbourhood of (x,y), the inverse of $\mathcal{T}_\alpha$ is given by

$$\mathcal{T}_\alpha^{-1}(x,y) = \left(\frac{1}{d+\varepsilon x}, \frac{\varepsilon}{y} - d\right).$$

The map $\frac{\varepsilon}{y} - d$ is expanding and $\mathcal{T}_\alpha^{-1}$ maps horizontal strips to vertical strips. Hence, we can also separate points that agree on the x-coordinate, giving (ne4). Therefore, we have obtained the following result.

4.4.1 Theorem. *Let $\alpha \in \left(0, \frac{1}{2}\right)$. The dynamical system $(\mathcal{D}_\alpha, \mathcal{B}(\mathcal{D}_\alpha), \nu_\alpha, \mathcal{T}_\alpha)$ is a version of the natural extension of the dynamical system $(I_\alpha, \mathcal{B}_\alpha, \mu_\alpha, T_\alpha)$ where $\mu_\alpha := \nu_\alpha \circ \pi^{-1}$.*

The measure $\mu_\alpha = \nu_\alpha \circ \pi^{-1}$ is the unique invariant measure for T_α that is absolutely continuous with respect to λ from Proposition 4.2.3. Projecting on the first coordinate gives the following explicit expression for the density f_α of μ_α:

$$\begin{aligned} f_\alpha(x) &= \frac{1}{x}\mathbf{1}_{[\alpha, \frac{\alpha}{1-\alpha}]}(x) + \frac{1}{1+x}\mathbf{1}_{[\frac{\alpha}{1-\alpha}, 1]}(x) + \frac{1}{1-x}\mathbf{1}_{[1-\alpha, 1]}(x) \\ &= \frac{1}{x}\mathbf{1}_{[\alpha, \frac{\alpha}{1-\alpha}]}(x) + \frac{1}{1+x}\mathbf{1}_{[\frac{\alpha}{1-\alpha}, 1-\alpha]}(x) + \frac{2}{1-x^2}\mathbf{1}_{[1-\alpha, 1]}(x). \end{aligned} \tag{4.15}$$

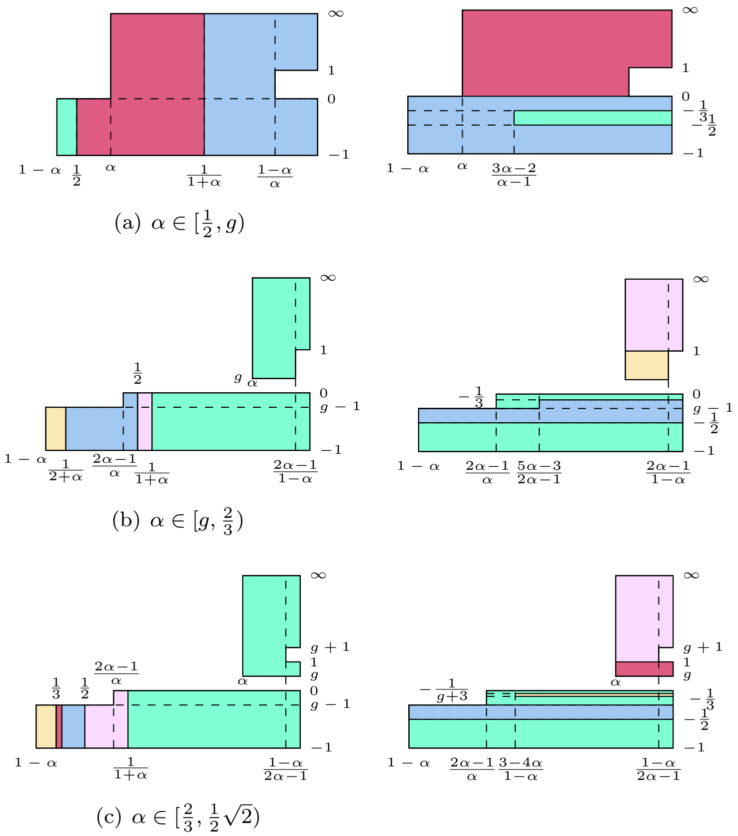
Here, by "unique", we of course mean unique up to scalar multiples. We choose to work with the above expression, because it comes from projecting the canonical measure (4.14) for the natural extension, and is thus a natural choice.

§4.4.2 For $\alpha \geq \frac{1}{2}$

As indicated by Theorem 4.3.5 the situation for $\alpha \geq \frac{1}{2}$ becomes increasingly complicated. Figure 4.3 shows the natural extension domain $\mathcal{D}_\alpha$ for $\alpha \in \left[\frac{1}{2}, \frac{1}{2}\sqrt{2}\right)$ with the action of $\mathcal{T}_\alpha$ and Table 4.1 provides the corresponding densities. We do not provide further details as the proofs are exactly like the one for $0 < \alpha < \frac{1}{2}$.

α	**Density** f_α
$[\frac{1}{2}, g)$	$\frac{1}{1-x}\mathbf{1}_{[1-\alpha,\alpha]}(x) + \frac{1}{x(1-x)}\mathbf{1}_{[\alpha, \frac{1-\alpha}{\alpha}]}(x) + \frac{x^2+1}{x(1-x^2)}\mathbf{1}_{[\frac{1-\alpha}{\alpha}, 1]}(x)$
$[g, \frac{2}{3})$	$\left(\frac{1}{1-x} + \frac{1}{x+\frac{1}{g-1}}\right)\mathbf{1}_{[1-\alpha, \frac{2\alpha-1}{\alpha}]}(x) + \frac{1}{1-x}\mathbf{1}_{[\frac{2\alpha-1}{\alpha}, \alpha]}(x) +$
	$+\left(\frac{1}{1-x} + \frac{1}{x} - \frac{1}{x+\frac{1}{g}}\right)\mathbf{1}_{[\alpha, \frac{2\alpha-1}{1-\alpha}]}(x) + \frac{x^2+1}{x(1-x^2)}\mathbf{1}_{[\frac{2\alpha-1}{1-\alpha}, 1]}(x)$
$[\frac{2}{3}, \frac{1}{2}\sqrt{2}]$	$\left(\frac{1}{1-x} + \frac{1}{x+\frac{1}{g-1}}\right)\mathbf{1}_{[1-\alpha, \frac{2\alpha-1}{\alpha}]}(x) + \frac{1}{1-x}\mathbf{1}_{[\frac{2\alpha-1}{\alpha}, \alpha]}(x) +$
	$+\left(\frac{1}{1-x} + \frac{1}{x} - \frac{1}{x+\frac{1}{g}}\right)\mathbf{1}_{[\alpha, \frac{1-\alpha}{2\alpha-1}]}(x) +$
	$+\left(\frac{1}{1-x} + \frac{1}{x+1} - \frac{1}{x+\frac{1}{g}} + \frac{1}{x} - \frac{1}{x+\frac{1}{g+1}}\right)\mathbf{1}_{[\frac{1-\alpha}{2\alpha-1}, 1]}(x)$

Table 4.1: Invariant densities for $\alpha \in \left[\frac{1}{2}, \frac{1}{2}\sqrt{2}\right]$.



(a) $\alpha \in [\frac{1}{2}, g)$

(b) $\alpha \in [g, \frac{2}{3})$

(c) $\alpha \in [\frac{2}{3}, \frac{1}{2}\sqrt{2})$

Figure 4.3: The maps $\mathcal{T}_\alpha$ for various values of α. Areas on the left are mapped to areas on the right with the same color.

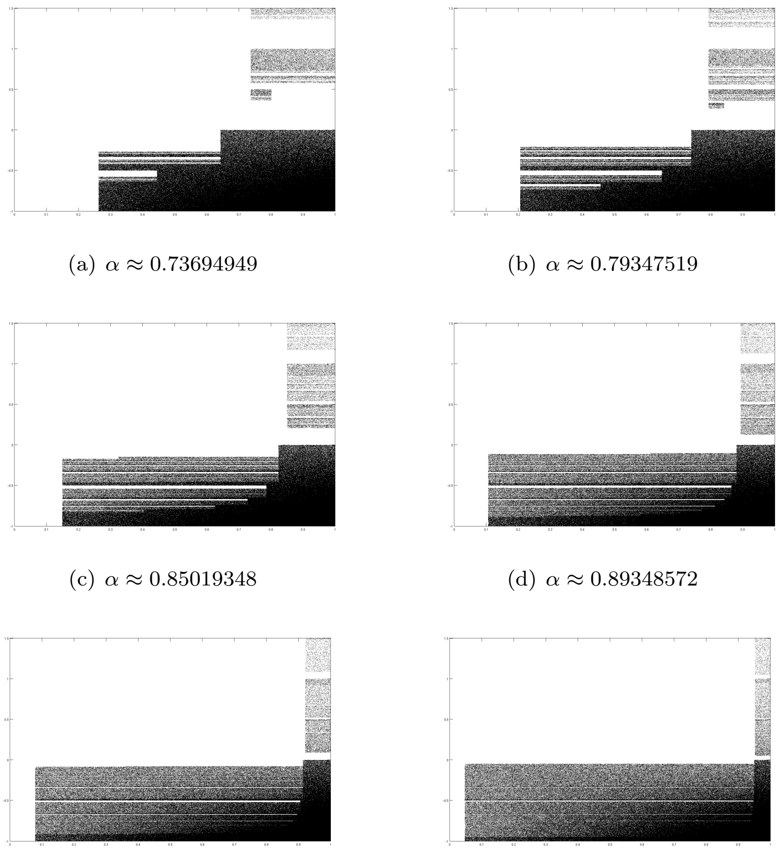
As α increases even further, the domain $\mathcal{D}_\alpha$ starts to exhibit a fractal structure. Figure 4.4 shows numerical simulations for various values of $\alpha > \frac{1}{2}\sqrt{2}$.

§4.5 Entropy, wandering rate and isomorphisms

With an explicit expression for the density of μ_α at hand, we can compute several dynamical quantities associated to the systems T_α. In this section we compute the Krengel entropy, return sequence and wandering rate of T_α for a large part of the parameter space $(0, 1)$.

In [K67] Krengel extended the notion of metric entropy to infinite, measure preserving and conservative systems $(X, \mathcal{B}, \mu, T)$ by considering the metric entropy on finite measure induced systems. More precisely, if A is a sweep-out set for T with $\mu(A) < \infty$, T_A the induced transformation of T on A and μ_A the restriction of μ to A, then the *Krengel entropy* of T is defined to be

$$h_{\mathrm{Kr},\mu}(T) = \mu(A) h_{\mu_A}(T_A),$$



(a) $\alpha \approx 0.73694949$ (b) $\alpha \approx 0.79347519$

(c) $\alpha \approx 0.85019348$ (d) $\alpha \approx 0.89348572$

(e) $\alpha \approx 0.92087668$ (f) $\alpha \approx 0.95234649$

Figure 4.4: Numerical simulations of $\mathcal{D}_\alpha$ for $\alpha > \frac{1}{2}\sqrt{2}$.

where $h_{\mu_A}(T_A)$ is the metric entropy of the system $(A, \mathcal{B} \cap A, T_A, \mu_A)$. Krengel proved in [K67] that this quantity is independent of the choice of A. In [Z00, Theorem 6] it is shown that if T is an AFN-map the Krengel entropy can be computed using Rohlin's formula:

$$h_{\mathrm{Kr},\mu}(T) = \int_X \log(|T'|)d\mu. \tag{4.16}$$

The following theorem follows from Lemma 4.2.2, (4.15) and Table 4.1.

4.5.1 Theorem. *For any $\alpha \in \big(0, g\big]$ the system $(I_\alpha, \mathcal{B}_\alpha, \mu_\alpha, T_\alpha)$ has $h_{Kr,\mu_\alpha}(T_\alpha) = \frac{\pi^2}{6}$.*

Proof. First fix $\alpha \in \big(0, \frac{1}{2}\big)$. By Lemma 4.2.2 we can use formula (4.16) to compute the Krengel entropy of T_α. For this computation we use some properties of the dilogarithm function, which is defined by

$$\mathrm{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2} \qquad \text{for } |x| \leq 1,$$

and satisfies (see [L81] for more information)

- $\mathrm{Li}_2(0) = 0$;
- $\mathrm{Li}_2(-1) = -\pi^2/12$;
- $\mathrm{Li}_2(x) + \mathrm{Li}_2(-\frac{x}{1-x}) = -\frac{1}{2}\log^2(1-x)$.

Using the density from (4.15) and these three properties of Li_2 we get

$$\begin{aligned}
\int_{I_\alpha} \log(|T'_\alpha|)d\mu_\alpha &= -2\Big(\int_\alpha^{\frac{\alpha}{1-\alpha}} \frac{\log x}{x}dx + \int_{\frac{\alpha}{1-\alpha}}^{1-\alpha} \frac{\log x}{1+x}dx + 2\int_{1-\alpha}^{1} \frac{\log x}{1-x^2}dx\Big) \\
&= [-\log^2 x]_\alpha^{\frac{\alpha}{1-\alpha}} - 2[\mathrm{Li}_2(-x) + \log x \log(x+1)]_{\frac{\alpha}{1-\alpha}}^{1-\alpha} \\
&\quad - 2[\mathrm{Li}_2(1-x) + \mathrm{Li}_2(-x) + \log x \log(x+1)]_{1-\alpha}^{1} \\
&= -\log^2\Big(\frac{\alpha}{1-\alpha}\Big) + \log^2(\alpha) + 2\,\mathrm{Li}_2\Big(\frac{-\alpha}{1-\alpha}\Big) + 2\log\Big(\frac{\alpha}{1-\alpha}\Big)\log\Big(\frac{1}{1-\alpha}\Big) \\
&\quad - 2\,\mathrm{Li}_2(-1) + 2\,\mathrm{Li}_2(\alpha) \\
&= \log^2(\alpha) - \log^2\big(\frac{\alpha}{1-\alpha}\big) - \log^2(1-\alpha) - 2\log\big(\frac{\alpha}{1-\alpha}\big)\log(1-\alpha) + \frac{\pi^2}{6} \\
&= \log^2(\alpha) - \log^2(\alpha) + 2\log(\alpha)\log(1-\alpha) - 2\log^2(1-\alpha) + \\
&\quad - 2\log(\alpha)\log(1-\alpha) + 2\log^2(1-\alpha) + \frac{\pi^2}{6} \\
&= \frac{\pi^2}{6}.
\end{aligned}$$

A similar computation yields $h_{\mathrm{Kr},\mu_\alpha}(T_\alpha) = \frac{\pi^2}{6}$ for $\alpha \in [\frac{1}{2}, g]$. □

4.5.2 Remark. Numerical evidence using the densities from Table 4.1 suggests that $h_{\mathrm{Kr},\mu_\alpha}(T_\alpha) = \frac{\pi^2}{6}$ for $\alpha \in (g, \frac{1}{2}\sqrt{2})$ as well. Even though we were not able to calculate the Krengel entropy for $\alpha \in (g, \frac{1}{2}\sqrt{2})$ explicitly, we conjecture that in fact $h_{\mathrm{Kr},\mu_\alpha}(T_\alpha) = \frac{\pi^2}{6}$ for all $\alpha \in (0,1)$. This claim is supported by the fact that the Krengel entropy for Nakada's α-continued fraction maps S_α from (4.2) is $\frac{\pi^2}{6}$ as well, see [KSS12, Theorem 2].

The *return sequence* of T_α is the sequence $(a_n(T_\alpha))_{n\geq 1}$ of positive real numbers satisfying (4.7). The pointwise dual ergodicity of each map T_α implies that such a sequence, which is unique up to asymptotic equivalence, exists. The *asymptotic type* of T_α corresponds to the family of all sequences asymptotically equivalent to some positive multiple of $(a_n(T_\alpha))_{n\geq 1}$. The return sequence of a system is related to its wandering rate, which quantifies how big the system is in relation to its subsets of finite measure. To be more precise, if $(X, \mathcal{B}, \mu, T)$ is a conservative, ergodic, measure preserving system and $A \in \mathcal{B}$ a set of finite positive measure, then the wandering rate of A with respect to T is the sequence $(w_n(A))_{n\geq 1}$ given by

$$w_n(A) := \mu\Big(\bigcup_{k=0}^{n-1} T^{-k}A\Big).$$

It follows from [Z00, Theorem 2] that for each of the maps T_α there exists a positive sequence $(w_n(T_\alpha))$ such that $w_n(T_\alpha) \uparrow \infty$ and $w_n(T_\alpha) \sim w_n(A)$ as $n \to \infty$ for all sets A that have positive, finite measure and are bounded away from 1. The asymptotic equivalence class of $(w_n(T_\alpha))$ defines the *wandering rate* of T_α. Using the machinery from [Z00] and the explicit formula of the density we compute both the return sequence and the wandering rate of the maps T_α.

4.5.3 Proposition. For all $\alpha \in (0,1)$ there is a constant $c_\alpha > 0$ such that

$$w_n(T_\alpha) \sim c_\alpha \log n \quad \textit{and} \quad a_n(T_\alpha) \sim \frac{n}{c_\alpha \log n}.$$

If $\alpha \in (0, \frac{1}{2}\sqrt{2})$, then $c_\alpha = 1$.

Proof. Using the Taylor expansion of the maps T_α, one sees that for $x \to 1$ we have $T_\alpha(x) = x - (x-1)^2 + o((x-1)^2)$. Hence, T_α admits what are called *nice expansions* in [Z00]. For $x \in \big(\frac{1}{1+\alpha}, 1\big]$ we can write $f_\alpha(x) = \frac{x-2}{x-1}H(x)$, where the function $x \mapsto \frac{x-2}{x-1}$ corresponds to the map called G in [Z00, Theorem A]. It then follows by [Z00, Theorems 3 and 4] that the wandering rate is

$$w_n(T_\alpha) \sim c_\alpha \log n \tag{4.17}$$

and the return sequence is

$$a_n(T_\alpha) \sim \frac{n}{c_\alpha \log n}, \tag{4.18}$$

for $c_\alpha = \lim_{x\uparrow 1} H(x)$. For $\alpha \in \big(0, \frac{1}{2}\sqrt{2}\big)$ the explicit formula for the densities from (4.15) and Table 4.1 gives $c_\alpha = 1$. $\square$

We have now established all parts of Theorem 4.1.2.

Proof of Theorem 4.1.2. The densities are given by (4.15) and listed in Table 4.1. The entropy is given by Theorem 4.5.1 and the wandering rate and return sequence in Proposition 4.5.3. $\square$

4.5.4 Remark. (i) As in Remark 4.5.2 we suspect that in fact $c_\alpha = 1$ for all $\alpha \in (0,1)$.

(ii) Since all the results from [Z00] apply to our family, we can use these to get an even more detailed description of the ergodic behaviour of the maps T_α. We briefly mention a few more results for $\alpha \in (0, 1/2\sqrt{2}]$. Since the return sequence $(a_n(T_\alpha))_{n\geq 1}$

is regularly varying with index 1, by [Z00, Theorem 5] and [A97, Corollary 3.7.3], we have

$$\frac{\log n}{n}\sum_{k=0}^{n-1} f\circ T_\alpha^k \xrightarrow{\mu_\alpha} \int_{I_\alpha} f\,d\mu_\alpha, \qquad \text{for } f\in L^1(I_\alpha,\mathcal{B}_\alpha,\mu_\alpha) \text{ and } \int_{I_\alpha} f\,d\mu_\alpha \neq 0. \quad (4.19)$$

where the convergence is in measure since the regularly varying index 1 of the return sequence $(a_n(T_\alpha))_{n\geq 1}$ turns the right hand side of the Darling-Kac Theorem into a constant. In other words, a weak law of large numbers holds for T_α.

In addition, we can obtain asymptotics for the excursion times to the interval $\left[\frac{1}{1+\alpha},1\right]$, corresponding to the rightmost branch of T_α. Let Y be a sweep-out set, T_Y the induced map on Y and $\varphi : x\mapsto \min\{n\geq 1 : T^n(x)\in Y\}$ the first return map. Write $\varphi_n^Y := \sum_{k=0}^{n-1}\varphi\circ T_Y^k$ and note that the asymptotic inverse of the sequence $(a_n(T_\alpha))_{n\geq 1}$ is $(n\log n)_{n\geq 1}$, so that the statement from (4.19) is equivalent to the following dual:

$$\frac{1}{n\log n}\varphi_n^Y \xrightarrow{\mu_\alpha} \frac{1}{\mu_\alpha(Y)}.$$

If we induce on $Y := \left[\min\{\alpha,1-\alpha\},\frac{1}{1+\alpha}\right]$, then φ_n^Y sums the lengths (increased by n) of the first n blocks of consecutive digits $(\epsilon,d)=(-1,2)$, and we obtain

$$\frac{1}{n\log n}\varphi_n^Y - \frac{1}{\log n} \xrightarrow{\mu_\alpha} \frac{1}{\mu_\alpha(Y)}.$$

From Theorem 4.1.2, it follows that for $\alpha<\frac{1}{2}$, $\mu_\alpha(Y)=\log(2+\alpha)$. Note that for α decreasing the right hand side is increasing, meaning we spend on average more time in $\Delta(-1,2)$. Intuitively, for a smaller α, every time we enter $\Delta(-1,2)$ we are closer to the indifferent fixed point, and it takes longer before we manage to escape from it.

Note that the Krengel entropy, return sequence and wandering rate we found do not display any dependence on α. These quantities give isomorphism invariants for dynamical systems with infinite invariant measures. Two measure preserving dynamical systems $(X,\mathcal{B},\mu,T)$ and $(Y,\mathcal{C},\nu,S)$ on σ-finite measure spaces are called *c-isomorphic* for $c\in(0,\infty]$ if there are sets $N\in\mathcal{B}$, $M\in\mathcal{C}$ with $\mu(N)=0=\nu(M)$ and $T(X\backslash N)\subseteq X\backslash N$ and $S(Y\backslash M)\subseteq Y\backslash M$ and if there is a map $\phi : X\backslash N\to Y\backslash M$ that is invertible, bi-measurable and satisfies $\phi\circ T = S\circ\phi$ and $\mu\circ\phi^{-1}=c\cdot\nu$. Invariants for c-isomorphisms are the asymptotic proportionality classes of the return sequence (see [A97, Propositions 3.7.1 and 3.3.2] and [Z00, Remark 8]) and the normalised wandering rates, which combine the Krengel entropy with the wandering rates (see e.g. [T83, Z00]). It follows from Theorem 4.1.2 that all these quantities are equal for all T_α, $\alpha\in(0,\frac{1}{2}\sqrt{2})$. Using the idea from [K14], however, we find many pairs α and α' such that T_α and $T_{\alpha'}$ are not c-isomorphic for any $c\in(0,\infty]$. Consider for example any $\alpha\in\left(\sqrt{2}-1,\frac{1}{2}\right)$, so that $\alpha\in\left(\frac{1}{2+\alpha},\frac{1}{2}\right)$, and any $\alpha'\in\left(\frac{1}{3},\frac{3-\sqrt{5}}{2}\right)$, so that $T_{\alpha'}(\alpha')>1-\alpha'$, see Figure 4.5. For a contradiction, suppose that there is a c-isomorphism $\phi : I_\alpha\to I_{\alpha'}$ for some $c\in(0,\infty]$. Let $J=[\alpha,1-\alpha]$ and note that any $x\in J$ has precisely one pre-image under T_α. Since $\phi\circ T_\alpha = T_{\alpha'}\circ\phi$ and ϕ is invertible, any element of the set $\phi(J)$ must also have precisely one pre-image. Since

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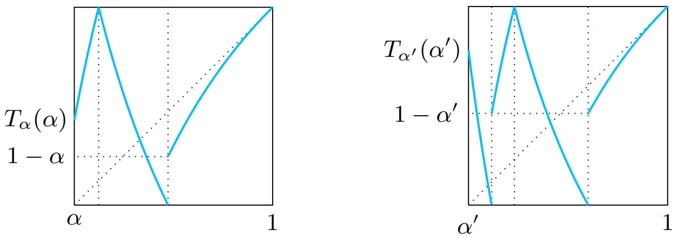
$T_{\alpha'}(\alpha') > 1-\alpha'$, there are no such points, so $\mu_{\alpha'}(\phi(J)) = 0$. On the other hand, since J is bounded away from 1, it follows that $0 < \mu_\alpha(J) < \infty$. Hence, there can be no c, such that $\mu_{\alpha'} \circ \phi^{-1} = c \cdot \mu_\alpha$. Obviously a similar argument holds for many other combinations of α and α', even for $\alpha > \frac{1}{2}$, and in case the argument does not work for T_α and $T_{\alpha'}$, one can also consider iterates T_α^n, $T_{\alpha'}^n$. Hence, even though the above discussed isomorphism invariants are equal for all $\alpha \in \left(0, \frac{1}{2}\right)$, it is not generally the case that any two maps T_α are c-isomorphic. We conjecture that for almost all pairs (α, α'), the maps T_α and $T_{\alpha'}$ are not c-isomorphic.



(a) $\alpha \in \left(\sqrt{2}-1, \frac{1}{2}\right)$ (b) $\alpha' \in \left(\frac{1}{3}, \frac{3-\sqrt{5}}{2}\right)$

Figure 4.5: Maps T_α and $T_{\alpha'}$ that are not c-isomorphic for any $c \in (0, \infty]$.

CHAPTER 4

§4.6 Remarks

1. The theory of piecewise affine interval maps is richer than the one available for smooth transformations. This is in part due to the fact that the former present a piecewise constant derivatives, leading for instance to a convenient representation of the Perron-Frobenius operator. However, for the specific case of flipped α-continued fraction maps, the contrary is true. Indeed, as remarked at the end of Chapter 2, the c-Lüroth map $T_{1,\alpha}$ can be seen as a linearisation of the flipped α-continued fraction map T_α for $\alpha \in (0, \frac{1}{2})$. While the density, of the absolutely continuous invariant measure, is explicitely given for T_α in Theorem 4.1.2 as a piecewise smooth function, for $T_{1,\alpha}$ we only know from [K90] that the density is a step function, and from Corollary 2.4.13 we can deduce that it is defined on a finite partition for rational parameters α and on a countable one for irrationals.

2. The consequences of matching on the structure of the density of an absolutely continuous invariant measure are given in [BCMP18, Theorem 1.2] for piecewise affine (or smooth) interval maps admitting a probability measure. This chapter provides an analogous result for a specific class of continued fraction maps with a σ-finite infinite invariant measure. In Chapter 5, we extend the notion of matching and explore its consequences for random interval maps.