Measures and matching for number systems
Maggioni, M.

Author: Maggioni, M.
Title: Measures and matching for number systems
Issue Date: 2021-03-04

CHAPTER 3

Measures for Random Systems

This chapter is based on: [KM18].

Abstract

For random systems T, that are expanding on average and given by piecewise affine interval maps, we explicitly construct the density functions of absolutely continuous T-invariant measures. In case the random transformation uses only expanding maps our procedure produces all invariant densities of the system. Examples include random tent maps, random W-shaped maps, random β-transformations and random c-Lüroth maps.

§3.1 Motivation and context

The Perron-Frobenius operator has been used since the seminal paper [LY73] of Lasota and Yorke to establish the existence of absolutely continuous invariant measures for deterministic dynamical systems. Later on, the same approach was successfully used in the setting of random systems. In this context, instead of a single map, a family of transformarions is considered from which at each iteration one is selected according to a probabilistic regime and applied. In [P84] Pelikan gave sufficient conditions under which a random system with a finite number of piecewise C^2-transformations on the interval has absolutely continuous invariant measures, and he discussed the possible number of ergodic components. Around the same time, a similar result was obtained by Morita in [M85], allowing for the possibility to choose from an infinite family of maps. In recent years these results have been generalised in various ways, see for example [B00, GB03, BG05, I12].

As shown in Chapter 2, finding an explicit formula for the density functions of these absolutely continuous invariant measures is not simple. Unless in the scenario of Markov maps, the Perron-Frobenius operator can only help if one can make an educated guess. An explicit expression for the invariant density is therefore available only for specific families of maps. In 1957 Rényi gave in [R57] an expression for the invariant density of the β-transformation $x \mapsto \beta x \pmod 1$ in case $\beta = \frac{1+\sqrt{5}}{2}$, the golden mean. Later Parry and Gel'fond gave a general formula for the invariant density of the β-transformation in [P60, G59]. In [DK10] generalisations of the β-transformation were considered. A more general set-up allowing different slopes was proposed in [K90] by Kopf. He introduced for any piecewise affine, expanding interval map, satisfying some minor restraints, a matrix M and associated each absolutely continuous invariant measure of the system to a vector from the null space of M. Twenty years later, Góra developed in [G09] a similar procedure for deterministic piecewise affine eventually expanding interval maps. Unless the map in question has many onto branches, the matrix involved in the procedure from [G09] is of higher dimension than the one used in [K90].

This chapter concerns finding explicit expressions for the invariant densities of random systems. We consider any finite or countable family $\{T_j : [0,1] \to [0,1]\}_{j\in\Omega}$ of piecewise affine maps that are expanding on average. The random system T is given by choosing at each step one of these maps according to a probability vector $\mathbf{p} = (p_j)_{j\in\Omega}$. We provide a procedure to construct explicit formulae for invariant probability densities of T. This is the content of Theorem 3.4.1. The results from Theorem 3.4.1 cover those from [K14] and [S19] regarding the expression for the invariant density for random β-transformations. In case we assume that all maps T_j are expanding, we obtain the stronger result that the procedure leading to Theorem 3.4.1 actually produces all absolutely continuous invariant measures of T. We prove this in Theorem 3.5.3.

The chapter is outlined as follows. In the second section we specify our set-up and introduce the necessary assumptions and notation. The third section is devoted to the definition of a matrix M and to the proof that the null space of M is non-trivial. In the fourth section we prove Theorem 3.4.1, relating each non-trivial vector γ from the null space of M to the density h_γ of an absolutely continuous invariant measure of the system T. In the fifth section we prove Theorem 3.5.3 on when we get all invariant densities. It is in this section that the extra difficulties that we had to overcome for dealing with random systems instead of deterministic ones, are most visible. In the sixth section we apply the results to some examples, that include random tent maps, random W-shaped maps and random β-transformations. In the last section we apply the results for the random c-Lüroth maps introduced in Chapter 2.

§3.2 Affine random interval systems

Let $R : \Omega^{\mathbb{N}} \times [0,1] \to \Omega^{\mathbb{N}} \times [0,1]$ be a pseudo-skew product as defined in Definition 1.2.8, with associated probability vector $\mathbf{p} = (p_j)_{j\in\Omega}$ and piecewise affine maps $\{T_j : [0,1] \to [0,1]\}_{j\in\Omega}$. Let $\pi_2 : \Omega^{\mathbb{N}} \times [0,1] \to [0,1]$ be the canonical projection on the second component, i.e., $\pi_2(\omega, x) = x$ and let T be the random system $T = \pi_2 \circ R$, such that

$$T(\omega, x) = T_{\omega_1}(x) \text{ with probability } p_{\omega_1}.$$

We put some assumptions on the systems T we consider.

(A1) Assume that the set of all the critical points of the maps T_j is *finite*.

Call these critical points $0 = z_0 < z_1 < \cdots < z_N = 1$. The points z_i together specify a common partition $\{I_i\}_{1\le i\le N}$ of subintervals of $[0,1]$, such that all maps T_j are monotone on each of the intervals I_i. Hence, there exist $k_{i,j}, d_{i,j} \in \mathbb{R}$ such that the maps $T_{i,j} := T_j|_{I_i}$ are given by

$$T_{i,j}(x) = k_{i,j}x + d_{i,j}.$$

(A2) Assume that T is *expanding on average* with respect to $\mathbf{p}$, i.e., assume that there is a constant $0 < \rho < 1$, such that for all $x \in [0,1]$, $\sum_{j\in\Omega} \frac{p_j}{|T_j'(x)|} \le \rho < 1$. This is equivalent to assuming that for each $1 \le i \le N$,

$$\sum_{j\in\Omega} \frac{p_j}{|k_{i,j}|} \le \rho < 1.$$

Recall that a measure $\mu_{\mathbf{p}}$ on $[0,1]$ is an absolutely continuous stationary measure for T and $\mathbf{p}$ if there is a density function h, such that for each Borel set $B \subseteq [0,1]$ we have

$$\mu_{\mathbf{p}}(B) = \int_B h\, d\lambda = \sum_{j\in\Omega} p_j \mu_{\mathbf{p}}(T_j^{-1}B), \tag{3.1}$$

where λ denotes the one-dimensional Lebesgue measure. Under these conditions the random system T satisfies the conditions (a) and (b) from [I12], which studies the existence of invariant densities h satisfying the random Perron-Frobenius equation

$$P_T h = \sum_{j\in\Omega} p_j P_{T_j} h. \tag{3.2}$$

for P_{T_j} the Perron-Frobenius operator defined in 1.2. The operator P_T is linear and positive. We call an $L^1(\lambda)$-function h *T-invariant* for the random system T if it is a fixed point of P_T, i.e., if it satisfies $P_T h = h$ Lebesgue almost everywhere. A density function h is the density of an absolutely continuous stationary measure $\mu_{\mathbf{p}}$ satisfying (3.1) if and only if it is a fixed point of P_T. From [I12, Theorem 5.2] it follows that a stationary measure $\mu_{\mathbf{p}}$ of the form (3.1), and hence a T-invariant function h, exists. Inoue obtained this result by showing that the operator P_T, applied to functions of bounded variation, satisfies a Lasota-Yorke type inequality. From the famous Ionescu-Tulcea and Marinescu Theorem one can then deduce much more than mere existence of an absolutely continuous invariant measure, it says that P_T as an operator on the space of functions of bounded variation is quasi-compact. The specific implications of the quasi-compactness of P_T that we use in this paper are the following. The eigenvalue 1 of P_T has a finite dimensional eigenspace. In other words, the subspace of $L^1(\lambda)$ of T-invariant functions is a finite-dimensional sublattice of the space of functions of bounded variation. As such, it has a finite base $H = \{v_1, \dots, v_r\}$ of T-invariant density functions of bounded variation, each corresponding to an ergodic measure, so that any other T-invariant $L^1(\lambda)$-function h can be written as a linear combination of the v_i: $h = \sum_{i=1}^r c_i v_i$ for some constants $c_i \in \mathbb{R}$. Furthermore, if we set $U_i := \{x : v_i(x) > 0\}$ for the *support* of the function v_i, then each U_i is forward invariant under T in the sense that

$$\lambda\Big(U_i \triangle \bigcup_{j\in\Omega} T_j(U_i)\Big) = 0, \tag{3.3}$$

where $\triangle$ denotes the symmetric difference. Also, the sets U_i are mutually disjoint and none of the sets U_i can properly contain another forward invariant set. We will use these properties in the proofs from Section 3.5. An account of these implications on the operator P_T can be found in [P84, M85, I12], for example. For more information, we also refer to standard textbooks like [BG97] and [LM94].

In this article we find T-invariant functions $h : [0,1] \to \mathbb{R}$ by linking them to the vectors from the null space of a matrix M. To guarantee that this null space is non-trivial, we need to assume that not all the lines $x \mapsto k_{i,j}x + d_{i,j}$, $1 \le i \le N$, with respective weights p_j, have a common intersection point with the diagonal. More precisely, consider for each interval I_i the weighted intersection point with the diagonal

$$x = \sum_{j\in\Omega} p_j \Big(\frac{x}{k_{i,j}} - \frac{d_{i,j}}{k_{i,j}}\Big).$$

Our third assumption states that for each i there is an n, such that these points do not coincide.

(A3) Assume that for each $1 \le i \le N$, there is an $1 \le n \le N$, such that

$$\frac{\sum_{j\in\Omega} \frac{p_j}{k_{i,j}} d_{i,j}}{1 - \sum_{j\in\Omega} \frac{p_j}{k_{i,j}}} \neq \frac{\sum_{j\in\Omega} \frac{p_j}{k_{n,j}} d_{n,j}}{1 - \sum_{j\in\Omega} \frac{p_j}{k_{n,j}}}.$$

Note that if $d_{i,j} < 0$, then $k_{i,j} > -d_{i,j}$ and if $d_{i,j} > 1$, then $k_{i,j} < 1 - d_{i,j}$. Hence, in all cases $|d_{i,j}| < |k_{i,j}| + 1$ and by (A2),

$$\sum_{j\in\Omega} \frac{p_j}{|k_{i,j}|} |d_{i,j}| \leq 1 + \rho. \tag{3.4}$$

So, the quantities in (A3) are all finite. Our last assumption is on the orbits of the points 0 and 1.

(A4) For each j, assume that

$$d_{1,j} = \begin{cases} 0, & \text{if } k_{1,j} > 0, \\ 1, & \text{if } k_{1,j} < 0, \end{cases} \qquad \text{and} \qquad d_{N,j} = \begin{cases} 1 - k_{N,j}, & \text{if } k_{N,j} > 0, \\ -k_{N,j}, & \text{if } k_{N,j} < 0. \end{cases}$$

In other words, the points 0 and 1 are mapped to 0 or 1 under all maps T_j, making the system continuous at the origin, when we consider it as acting on the circle $\mathbb{R}/\mathbb{Z}$ with the points 0 and 1 identified. Since we can deal with finitely many discontinuities, there is no actual need for these last assumptions, but they make computations easier. Any system not satisfying it can be extended to a system that does satisfy this condition and for which no absolutely continuous invariant measure puts weight on the added pieces. See Figure 3.1 for an illustration and see Section 3.6.4 for a concrete example, given by the random (α, β)-transformation.

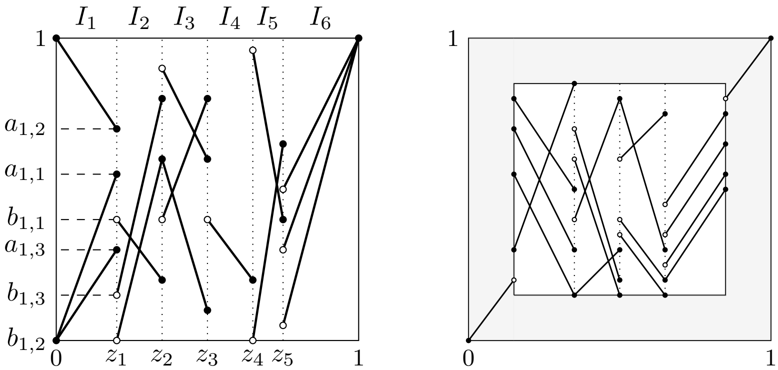


Figure 3.1: On the left is an arbitrary map T satisfying the above conditions. On the right we see a random map T in the white box that does not satisfy (A4). By adding the branches in the grey part and rescaling, we obtain a system that does satisfy these conditions. Note that any point in the grey part (except for 0 and 1) moves to the white part after a finite number of iterations and stays there. Hence, any invariant density will equal 0 on the grey part.

Finally, we include an assumption stating that the weighted inverse derivative cannot be 0 anywhere.

(A5) Assume that for any $x \in [0, 1]$, the weighted inverse derivative satisfies $\sum_{j\in\Omega} \frac{p_j}{T_j'(x)} \neq 0$. This is equivalent to assuming that for each $1 \leq i \leq N$,

$$\sum_{j\in\Omega} \frac{p_j}{k_{i,j}} \neq 0.$$

Conditions (A3) and (A5) are sufficient to get our main results, but probably not necessary. Note that (A5) is automatically fulfilled for any deterministic Lasota-Yorke map (and in particular for any deterministic piecewise linear map) and also for any random system for which on each interval I_i the derivatives of all maps T_j have the same sign. The last section contains an example that does not satisfy (A5) for a specific choice of $\mathbf{p}$. We will see that the procedure which leads to our main results still gives all invariant densities in that case. Moreover, if (A5) is not satisfied for some probability vector $\mathbf{p}$, then changing $\mathbf{p}$ slightly already lifts this restriction.

§3.3 The matrix equation

An invariant measure reflects the dynamics of a system. For the maps T_j, $j \in \Omega$, the dynamics is determined by the orbits of the critical points, which are the endpoints of the lines $x \mapsto k_{i,j}x + d_{i,j}$, $1 \leq i \leq N$. We start this section by defining some quantities that keep track of their orbits.

Let Ω^* be the set of all finite strings of elements from Ω together with the empty string ε. For $t \geq 0$, let $\Omega^t \subseteq \Omega^*$ denote the subset of those strings that have length t. So in particular, $\Omega^0 = \{\varepsilon\}$. Let $|\omega|$ denote the length of the string ω. For any string $\omega \in \Omega^*$ with $|\omega| \geq t$, we let ω_1^t denote the starting block of length t. For two strings ω, $\omega' \in \Omega^*$ we simply write $\omega\omega'$ for their concatenation. Each element $\omega \in \Omega^t$ defines a possible start of an orbit of a point in $[0,1]$ by composition of maps: for $x \in [0,1]$ and $\omega = \omega_1 \cdots \omega_t \in \Omega^t$, define

$$T_\omega(x) = T_{\omega_t} \circ T_{\omega_{t-1}} \circ \cdots \circ T_{\omega_1}(x)$$

and set $T_\varepsilon(x) = x$. For $\omega \in \Omega^*$, set $\tau_\omega(y,0) = 1$ and for $1 \leq t \leq |\omega|$, set

$$\tau_\omega(y,t) := \frac{p_{\omega_t}}{k_{i,\omega_t}}, \qquad \text{if } T_{\omega_1^{t-1}}(y) \in I_i.$$

Define

$$\delta_\omega(y,t) := \prod_{n=0}^{t} \tau_\omega(y,n). \tag{3.5}$$

Then $\delta_\omega(y,t)$ is the weighted slope of the map $T_{\omega_1^t}$ at the point y. Note that $\tau_\omega(y,t)$ and $\delta_\omega(y,t)$ only depend on the block ω_1^t and not on what comes after. Moreover, for a concatenation ωj, given by any block ω with $|\omega| = t-1$ and any $j \in \Omega$, it holds that $\tau_{\omega j}(y,t) = \tau_j(T_\omega(y),1)$ and $\delta_{\omega j}(y,t) = \tau_{\omega j}(y,t)\delta_\omega(y,t-1)$. By assumption (A2) we have that for any $y \in [0,1]$,

$$\begin{aligned} \Big|\sum_{t\geq 0}\sum_{\omega\in\Omega^t} \delta_\omega(y,t)\Big| &\leq 1 + \sum_{t\geq 1}\sum_{\omega\in\Omega^{t-1}}\sum_{j\in\Omega} |\delta_\omega(y,t-1)||\tau_{\omega j}(y,t)| \\ &\leq 1 + \sum_{t\geq 1}\sum_{\omega\in\Omega^{t-1}} |\delta_\omega(y,t-1)|\rho \leq \frac{1}{1-\rho}. \end{aligned} \tag{3.6}$$

Let $\mathbf{1}_A$ denote the characteristic function of the set A and set

$$\mathrm{KI}_n(y) := \sum_{t\geq 1}\sum_{\omega\in\Omega^t} \delta_\omega(y,t)\mathbf{1}_{I_n}(T_{\omega_1^{t-1}}(y)) \qquad \text{for } 1 \leq n \leq N.$$

$\mathrm{KI}_n(y)$ keeps track of all the number of visits of the random orbit of y to the interval I_n and adds the corresponding weighted slopes. For $1 \le i \le N-1$, set $A_i := I_1 \cup ... \cup I_i$ and $B_i := I_{i+1} \cup ... \cup I_N$. We define

$$\begin{aligned} \mathrm{KA}_i(y) &:= \sum_{t\ge 0}\sum_{\omega\in\Omega^t} \delta_\omega(y,t)\mathbf{1}_{A_i}(T_\omega(y)), \\ \mathrm{KB}_i(y) &:= \sum_{t\ge 0}\sum_{\omega\in\Omega^t} \delta_\omega(y,t)\mathbf{1}_{B_i}(T_\omega(y)). \end{aligned} \tag{3.7}$$

By (3.6) $|\mathrm{KI}_n|$, $|\mathrm{KA}_i|$ and $|\mathrm{KB}_i|$ are finite for all $y \in [0,1]$. For each $1 \le n \le N$, let S_n be the average inverse of the slope:

$$S_n := \sum_{j\in\Omega} \frac{p_j}{k_{n,j}},$$

which is non-zero by (A5), so that S_n^{-1} is well defined. The next two lemmata give some identities that we will use later.

3.3.1 Lemma. *For each $y \in [0,1]$ and $1 \le i \le N-1$ we have*

$$\mathrm{KA}_i(y) = \sum_{n=1}^{i} S_n^{-1}\,\mathrm{KI}_n(y) \qquad \textit{and} \qquad \mathrm{KB}_i(y) = \sum_{n=i+1}^{N} S_n^{-1}\,\mathrm{KI}_n(y).$$

Proof. For any $1 \le n \le N$ we have

$$\begin{aligned} \sum_{t\ge 0}\sum_{\omega\in\Omega^t} \delta_\omega(y,t)\mathbf{1}_{I_n}(T_\omega(y)) &= \sum_{t\ge 0}\sum_{\omega\in\Omega^t} \Big(\sum_{j\in\Omega}\frac{p_j}{k_{n,j}}\Big)^{-1}\Big(\sum_{j\in\Omega}\frac{p_j}{k_{n,j}}\Big)\delta_\omega(y,t)\mathbf{1}_{I_n}(T_\omega(y)) \\ &= \sum_{t\ge 0}\sum_{\omega\in\Omega^t} S_n^{-1}\sum_{j\in\Omega}\tau_{\omega j}(y,t+1)\delta_\omega(y,t)\mathbf{1}_{I_n}(T_\omega(y)) \\ &= S_n^{-1}\sum_{t\ge 0}\sum_{\omega\in\Omega^{t+1}} \delta_\omega(y,t+1)\mathbf{1}_{I_n}(T_{\omega_1^t}(y)) = S_n^{-1}\,\mathrm{KI}_n(y). \end{aligned} \tag{3.8}$$

Putting this in the definition of $\mathrm{KA}_i(y)$ from (3.7) gives the first part of the lemma. Using (3.8), we also get that

$$\mathrm{KA}_i(y) + \mathrm{KB}_i(y) = \sum_{t\ge 0}\sum_{\omega\in\Omega^t}\delta_\omega(y,t) = \sum_{n=1}^{N} S_n^{-1}\,\mathrm{KI}_n(y). \tag{3.9}$$

The result for KB_i follows. □

Define

$$K_n := S_n^{-1} - 1 \qquad \text{and} \qquad D_n := S_n^{-1}\Big(\sum_{j\in\Omega}\frac{p_j}{k_{n,j}}d_{n,j}\Big),$$

So that

$$\frac{D_n}{K_n} = \frac{\sum_{j\in\Omega}\frac{p_j}{k_{n,j}}d_{n,j}}{1-\sum_{j\in\Omega}\frac{p_j}{k_{n,j}}}.$$

Assumption (A3) now implies that for each $1 \le i \le N$, there is an $1 \le n \le N$, such that $\frac{D_i}{K_i} \neq \frac{D_n}{K_n}$. We have the following properties for K_n and D_n.

3.3.2 Lemma. *Let $y \in [0,1]$. Then*

$$\sum_{n=1}^{N} K_n \,\mathrm{KI}_n(y) = 1 \qquad \textit{and} \qquad -\sum_{n=1}^{N} D_n \,\mathrm{KI}_n(y) = y.$$

Proof. For the first part, note that by (3.9) we have

$$\sum_{n=1}^{N} S_n^{-1} \,\mathrm{KI}_n(y) = 1 + \sum_{t\ge 1} \sum_{\omega\in\Omega^t} \delta_\omega(y,t) = 1 + \sum_{n=1}^{N} \mathrm{KI}_n(y). \tag{3.10}$$

For the second part, let $1 \le i \le N$ be such that $y \in I_i$. Then for $j \in \Omega$ we get $T_{i,j}(y) = k_{i,j} y + d_{i,j}$, and thus

$$y = \sum_{j\in\Omega} \Big(\frac{p_j}{k_{i,j}} T_{i,j}(y) - \frac{p_j}{k_{i,j}} d_{i,j} \Big).$$

For $t \ge 1$ and $\omega \in \Omega^*$ with $|\omega| \ge t$, set

$$\theta_\omega(y,t) := -\frac{p_{\omega_t}}{k_{n,\omega_t}} d_{n,\omega_t} \qquad \text{if } \; T_{\omega_1^{t-1}}(y) \in I_n. \tag{3.11}$$

Then

$$y = \sum_{\omega\in\Omega} \tau_\omega(y,1) T_\omega(y) + \theta_\omega(y,1). \tag{3.12}$$

Since $\tau_j(T_\omega(y),1) = \tau_{\omega j}(y,2)$ and $\theta_j(T_\omega(y),1) = \theta_{\omega j}(y,2)$, we obtain for $\omega \in \Omega$ that

$$T_\omega(y) = \sum_{j\in\Omega} \tau_{\omega j}(y,2) T_{\omega j}(y) + \theta_{\omega j}(y,2). \tag{3.13}$$

Repeated application of (3.13) in (3.12), together with the definition of δ_ω from (3.5), yields after n steps,

$$y = \sum_{t=1}^{n+1} \sum_{\omega\in\Omega^t} \delta_\omega(y,t-1)\theta_\omega(y,t) + \sum_{\omega\in\Omega^{n+1}} \delta_\omega(y,n+1) T_\omega(y).$$

From (3.6) we obtain that $\lim_{n\to\infty} \sum_{\omega\in\Omega^{n+1}} \big|\delta_\omega(y,n+1)T_\omega(y)\big| = 0$. Hence, by (A2), (3.4) and (3.6),

$$
\begin{aligned}
y &= \sum_{t\geq 0}\sum_{\omega\in\Omega^{t+1}} \delta_\omega(y,t)\theta_\omega(y,t+1) &(3.14)\\
&= -\sum_{n=1}^{N}\sum_{t\geq 0}\sum_{\omega\in\Omega^{t}} \delta_\omega(y,t)\mathbf{1}_{I_n}(T_\omega(y))\Big(\sum_{j\in\Omega}\frac{p_j}{k_{n,j}}d_{n,j}\Big)\\
&= -\sum_{n=1}^{N} S_n^{-1}\Big(\sum_{j\in\Omega}\frac{p_j}{k_{n,j}}d_{n,j}\Big)\sum_{t\geq 0}\sum_{\omega\in\Omega^{t}} \delta_\omega(y,t)\Big(\sum_{j\in\Omega}\frac{p_j}{k_{n,j}}\Big)\mathbf{1}_{I_n}(T_\omega(y))\\
&= -\sum_{n=1}^{N} D_n\sum_{t\geq 0}\sum_{\omega\in\Omega^{t}} \delta_\omega(y,t)\Big(\sum_{j\in\Omega}\tau_{\omega j}(y,t+1)\Big)\mathbf{1}_{I_n}(T_\omega(y))\\
&= -\sum_{n=1}^{N} D_n\sum_{t\geq 0}\sum_{\omega\in\Omega^{t+1}} \delta_\omega(y,t+1)\mathbf{1}_{I_n}(T_{\omega_1^t}(y)) = -\sum_{n=1}^{N} D_n\,\mathrm{KI}_n(y). &(3.15)
\end{aligned}
$$

□

For the invariant densities, we need to keep track of the orbits of the limits from the left and from the right of each partition point. Set, for $1\leq i\leq N-1$ and $j\in\Omega$,

$$
a_{i,j} := k_{i,j}z_i + d_{i,j} = \lim_{x\uparrow z_i} T_j(x), \qquad \text{and} \qquad b_{i,j} := k_{i+1,j}z_i + d_{i+1,j} = \lim_{x\downarrow z_i} T_j(x).
$$

See also Figure 3.1.

3.3.3 Definition. The $N\times(N-1)$-matrix $M=(\mu_{n,i})$ given by

$$
\mu_{n,i} := \begin{cases}
\sum_{j\in\Omega}\Big[\frac{p_j}{k_{i,j}} + \frac{p_j}{k_{i,j}}\mathrm{KI}_n(a_{i,j}) - \frac{p_j}{k_{i+1,j}}\mathrm{KI}_n(b_{i,j})\Big], & \text{for } n=i,\\
\sum_{j\in\Omega}\Big[\frac{p_j}{k_{i,j}}\mathrm{KI}_n(a_{i,j}) - \frac{p_j}{k_{i+1,j}} - \frac{p_j}{k_{i+1,j}}\mathrm{KI}_n(b_{i,j})\Big], & \text{for } n=i+1,\\
\sum_{j\in\Omega}\Big[\frac{p_j}{k_{i,j}}\mathrm{KI}_n(a_{i,j}) - \frac{p_j}{k_{i+1,j}}\mathrm{KI}_n(b_{i,j})\Big], & \text{else},
\end{cases}
$$

is called the *fundamental matrix* of the random piecewise affine system T.

Note that assumption (A2) together with the fact that $|\mathrm{KI}_n(y)|<\infty$ for all $y\in[0,1]$ implies that all entries of M are finite. In the next section we associate invariant functions h_γ to vectors $\gamma\in\mathbb{R}^{N-1}$ in the null space of M. Here we prove that the null space of M is non-trivial.

3.3.4 Lemma. *The system $M\gamma=0$ admits at least one non-trivial solution.*

Proof. Since M has dimension $N\times(N-1)$, by the Rouché-Capelli Theorem the associated homogeneous system admits a non-trivial solution if and only if the rank of M is at most $N-2$. Below we will give non-trivial linear dependence relations between all combinations of $N-1$ out of N rows. It follows that any minor of order $N-1$ of M is zero and thus that the rank of M is at most $N-2$. We first show that for every $1 \le i \le N-1$,

$$\sum_{n=1}^{N} K_n \mu_{n,i} = 0 \qquad \text{and} \qquad \sum_{n=1}^{N} D_n \mu_{n,i} = 0.$$

Indeed by Lemma 3.3.2,

$$\begin{aligned}\sum_{n=1}^{N} & K_n \mu_{n,i} = \\ &= \sum_{j\in\Omega} \left[\frac{p_j}{k_{i,j}} K_i - \frac{p_j}{k_{i+1,j}} K_{i+1} + \frac{p_j}{k_{i,j}} \sum_{n=1}^{N} K_n \operatorname{KI}_n(a_{i,j}) - \frac{p_j}{k_{i+1,j}} \sum_{n=1}^{N} K_n \operatorname{KI}_n(b_{i,j}) \right] \\ &= S_i(S_i^{-1}-1) - S_{i+1}(S_{i+1}^{-1}-1) + S_i - S_{i+1} = 0.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{n=1}^{N} & D_n \mu_{n,i} = \\ &= \sum_{j\in\Omega} \left[\frac{p_j}{k_{i,j}} D_i - \frac{p_j}{k_{i+1,j}} D_{i+1} + \frac{p_j}{k_{i,j}} \sum_{n=1}^{N} D_n \operatorname{KI}_n(a_{i,j}) - \frac{p_j}{k_{i+1,j}} \sum_{n=1}^{N} D_n \operatorname{KI}_n(b_{i,j}) \right] \\ &= \sum_{j\in\Omega} \left(S_i S_i^{-1} \frac{p_j}{k_{i,j}} d_{i,j} - S_{i+1} S_{i+1}^{-1} \frac{p_j}{k_{i+1,j}} d_{i+1,j} - \frac{p_j}{k_{i,j}} a_{i,j} + \frac{p_j}{k_{i+1,j}} b_{i,j} \right) = 0.\end{aligned}$$

Consequently, for every $1 \le l \le N$ and every $1 \le i \le N-1$,

$$\sum_{n=1, n\neq l}^{N} (D_l K_n - D_n K_l)\mu_{n,i} = 0.$$

By assumption (A3) this gives non-trivial linear dependence relations between all combinations of $N-1$ out of N rows, giving the result. □

3.3.5 Remark. Note that if $S_n = 0$ for some $1 \le n \le N$, then the quantities K_n and D_n are not well defined. In this case $\mu_{n,i} = \sum_{j\in\Omega} \frac{p_j}{k_{i,j}} \operatorname{KI}_n(a_{i,j}) - \frac{p_j}{k_{i+1,j}} \operatorname{KI}_n(b_{i,j})$ for each $1 \le i \le N-1$ and by the definition of KI_n we can write for any $y \in [0,1]$ that

$$\begin{aligned}\operatorname{KI}_n(y) &= \sum_{t\ge 1} \sum_{\omega\in\Omega^{t-1}} \sum_{j\in\Omega} \delta_\omega(y, t-1) \frac{p_j}{k_{n,j}} 1_{I_n}(T_{\omega_1^{t-1}}(y)) \\ &= \sum_{t\ge 1} \sum_{\omega\in\Omega^{t-1}} \delta_\omega(y, t-1) 1_{I_n}(T_{\omega_1^{t-1}}(y)) S_n = 0.\end{aligned}$$

Hence, $\mu_{n,i} = 0$ for each i. From this, it is clear that if $S_n = 0$ for at least two indices n, then a non-trivial vector γ such that $M\gamma = 0$ still exists. If there is a unique ℓ with $S_\ell = 0$, then to obtain a non-trivial solution one still needs to find suitable constants c_n such that $\sum_{n=1,n\neq\ell}^{N} c_n \mu_{n,i} = 0$ for each i.

Any vector γ from the null space of M satisfies the following orthogonal relations, linking γ to the functions KA_i and KB_i.

3.3.6 Lemma. *For all $1 \le i \le N-1$ we have the following orthogonal relations:*

$$\gamma_i + \sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \left[\frac{p_j}{k_{m,j}} \mathrm{KA}_i(a_{m,j}) - \frac{p_j}{k_{m+1,j}} \mathrm{KA}_i(b_{m,j}) \right] = 0;$$

and

$$\gamma_i - \sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \left[\frac{p_j}{k_{m,j}} \mathrm{KB}_i(a_{m,j}) - \frac{p_j}{k_{m+1,j}} \mathrm{KB}_i(b_{m,j}) \right] = 0.$$

Proof. If γ is a solution of the system $M\gamma = 0$, then $\sum_{m=1}^{N-1} \gamma_m \mu_{n,m} = 0$ for all n. Lemma 3.3.1 gives for $n = 1$,

$$\begin{aligned}
0 &= S_1^{-1} \sum_{m=1}^{N-1} \gamma_m \mu_{1,m} \\
&= S_1^{-1} \gamma_1 \sum_{j\in\Omega} \frac{p_j}{k_{1,j}} + S_1^{-1} \sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \left(\frac{p_j}{k_{m,j}} \mathrm{KI}_1(a_{m,j}) - \frac{p_j}{k_{m+1,j}} \mathrm{KI}_1(b_{m,j}) \right) \\
&= \gamma_1 + \sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \left(\frac{p_j}{k_{m,j}} \mathrm{KA}_1(a_{m,j}) - \frac{p_j}{k_{m+1,j}} \mathrm{KA}_1(b_{m,j}) \right).
\end{aligned}$$

For $2 \le n \le N-1$ we obtain similarly

$$\begin{aligned}
0 = S_n^{-1} \sum_{m=1}^{N-1} \gamma_m \mu_{n,m} =&\; S_n^{-1} \sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \left(\frac{p_j}{k_{m,j}} \mathrm{KI}_n(a_{m,j}) - \frac{p_j}{k_{m+1,j}} \mathrm{KI}_n(b_{m,j}) \right) \\
&+ S_n^{-1} \left(\gamma_n \sum_{j\in\Omega} \frac{p_j}{k_{n,j}} - \gamma_{n-1} \sum_{j\in\Omega} \frac{p_j}{k_{n,j}} \right) \\
=&\; S_n^{-1} \sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \left(\frac{p_j}{k_{m,j}} \mathrm{KI}_n(a_{m,j}) - \frac{p_j}{k_{m+1,j}} \mathrm{KI}_n(b_{m,j}) \right) + \gamma_n - \gamma_{n-1}.
\end{aligned} \tag{3.16}$$

Then summing over all $1 \le n \le i$ and using (3.16) and Lemma 3.3.1 gives

$$\begin{aligned}
0 &= \sum_{n=1}^{i} S_n^{-1} \sum_{m=1}^{N-1} \gamma_m \mu_{n,m} \\
&= \gamma_i + \sum_{n=1}^{i} S_n^{-1} \sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \left(\frac{p_j}{k_{m,j}} \mathrm{KI}_n(a_{m,j}) - \frac{p_j}{k_{m+1,j}} \mathrm{KI}_n(b_{m,j}) \right) \\
&= \gamma_i + \sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \left(\frac{p_j}{k_{m,j}} \mathrm{KA}_i(a_{m,j}) - \frac{p_j}{k_{m+1,j}} \mathrm{KA}_i(b_{m,j}) \right).
\end{aligned}$$

This gives the relations for KA_i.

From $\sum_{m=1}^{N-1} \gamma_m \mu_{n,m} = 0$ for all n it also follows that $\sum_{m=1}^{N-1} \gamma_m \sum_{n=1}^{N} \mu_{n,m} = 0$. From this we obtain that

$$\sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \frac{p_j}{k_{m,j}} \Big(1 + \sum_{n=1}^{N} \mathrm{KI}_n(a_{m,j})\Big) = \sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \frac{p_j}{k_{m+1,j}} \Big(1 + \sum_{n=1}^{N} \mathrm{KI}_n(b_{m,j})\Big).$$

Then (3.10) from the proof of Lemma 3.3.2 gives that

$$\sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \frac{p_j}{k_{m,j}} \sum_{n=1}^{N} S_n^{-1}\,\mathrm{KI}_n(a_{m,j}) = \sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \frac{p_j}{k_{m+1,j}} \sum_{n=1}^{N} S_n^{-1}\,\mathrm{KI}_n(b_{m,j}).$$

Hence, by Lemma 3.3.1 we get for each i that

$$\sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \frac{p_j}{k_{m,j}} (\mathrm{KA}_i(a_{m,j}) + \mathrm{KB}_i(a_{m,j})) =$$
$$= \sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \frac{p_j}{k_{m+1,j}} (\mathrm{KA}_i(b_{m,j}) + \mathrm{KB}_i(b_{m,j})).$$

This gives the orthogonal relations for KB_i. □

In the proofs of our main results we only use the second part of Lemma 3.3.6, i.e., the orthogonal relations for KB_i, but since we obtain the orthogonal relations for KA_i and KB_i more or less simultaneously, we have listed them both.

§3.4 An explicit formula for invariant measures

We now state our main result. For $y \in [0,1]$, define the $L^1(\lambda)$-function $L_y : [0,1] \to \mathbb{R}$ by

$$L_y(x) = \sum_{t\geq 0} \sum_{\omega\in\Omega^t} \delta_\omega(y,t) \mathbf{1}_{[0,T_\omega(y))}(x). \tag{3.17}$$

3.4.1 Theorem. *Let T be a random piecewise affine system on the unit interval $[0,1]$ that satisfies the assumptions (A1) to (A5) from Section 3.2. Let M be the corresponding fundamental matrix and let $\gamma = (\gamma_1, \dots, \gamma_{N-1})^\intercal$ be a non-trivial solution of the system $M\gamma = 0$. For each $1 \leq m \leq N-1$, define the function $h_m : [0,1] \to \mathbb{R}$ by*

$$h_m(x) := \sum_{\ell\in\Omega} \left[\frac{p_\ell}{k_{m,\ell}} L_{a_{m,\ell}}(x) - \frac{p_\ell}{k_{m+1,\ell}} L_{b_{m,\ell}}(x) \right]. \tag{3.18}$$

Then a T-invariant function is given by

$$h_\gamma : [0,1] \to \mathbb{R}, x \mapsto \sum_{m=1}^{N-1} \gamma_m h_m(x), \tag{3.19}$$

and $h_\gamma \neq 0$.

To show that $P_T h_\gamma = h_\gamma$ λ-a.e. we have to determine for each $x \in [0,1]$ and each branch $T_{i,j}$, whether or not x has an inverse image in the branch $T_{i,j}$. Let

$$x_{i,j} := \frac{x - d_{i,j}}{k_{i,j}}$$

be the inverse of x under the map $T_{i,j} : \mathbb{R} \to \mathbb{R}$. By the definitions in (3.18) and (3.19), we have to show that

$$\begin{aligned} h_\gamma(x) &= \sum_{j\in\Omega}\sum_{i=1}^{N} \frac{p_j}{|k_{i,j}|} h_\gamma(x_{i,j}) \mathbf{1}_{I_i}(x_{i,j}) \\ &= \sum_{j\in\Omega}\sum_{i=1}^{N} \frac{p_j}{|k_{i,j}|} \mathbf{1}_{I_i}(x_{i,j}) \sum_{m=1}^{N-1} \gamma_m \sum_{\ell\in\Omega} \left(\frac{p_\ell}{k_{m,\ell}} L_{a_{m,\ell}}(x_{i,j}) - \frac{p_\ell}{k_{m+1,\ell}} L_{b_{m,\ell}}(x_{i,j}) \right). \end{aligned} \tag{3.20}$$

The parts for $L_{a_{m,\ell}}$ and $L_{b_{m,\ell}}$ behave similarly. That is why we first study

$$\sum_{j\in\Omega}\sum_{i=1}^{N} \frac{p_j}{|k_{i,j}|} \mathbf{1}_{I_i}(x_{i,j}) L_y(x_{i,j})$$

for general $y \in [0,1]$ through several lemmas. We introduce some notation to manage the long expressions. For $1 \le i \le N-1$, let

$$\eta_i := \sum_{j\in\Omega} \frac{p_j(\mathbf{1}_{(0,\infty)}(k_{i,j}) - a_{i,j})}{k_{i,j}} \quad \text{and} \quad \phi_i := \sum_{j\in\Omega} \frac{p_j(-\mathbf{1}_{(-\infty,0)}(k_{i+1,j}) + b_{i,j})}{k_{i+1,j}}.$$

For $y \in [0,1]$ let $1 \le n \le N$ be the index such that $y \in I_n$ and set

$$C(y) := \sum_{j\in\Omega} \left(\sum_{i=1}^{n-1} \frac{p_j}{|k_{i,j}|} + \frac{p_j}{|k_{n,j}|} \mathbf{1}_{(-\infty,0)}(k_{n,j}) \right). \tag{3.21}$$

3.4.2 Lemma. *Let $y \in [0,1]$. Then*

$$y = \sum_{t\ge0}\sum_{\omega\in\Omega^t} \delta_\omega(y,t) C(T_\omega(y)) - \sum_{i=1}^{N-1} (\eta_i + \phi_i)\, \mathrm{KB}_i(y).$$

Proof. Let $y \in [0,1]$ be given and recall the definition of $\theta_\omega(z,t)$ from (3.11). If $y \in I_n$, then

$$
\begin{aligned}
&C\ (y) - \sum_{i=1}^{N-1} (\eta_i + \phi_i) \mathbf{1}_{B_i}(y) \\
&= \sum_{j\in\Omega} \frac{p_j}{|k_{n,j}|} \mathbf{1}_{(-\infty,0)}(k_{n,j}) \\
&\quad + \sum_{j\in\Omega} \sum_{i=1}^{n-1} \left(\frac{p_j}{|k_{i,j}|} - \frac{p_j(\mathbf{1}_{(0,\infty)}(k_{i,j}) - a_{i,j})}{k_{i,j}} - \frac{p_j(-\mathbf{1}_{(-\infty,0)}(k_{i+1,j}) + b_{i,j})}{k_{i+1,j}} \right) \\
&= \sum_{j\in\Omega} \left(-\frac{p_j}{k_{n,j}} b_{n-1,j} + \frac{p_j}{|k_{1,j}|} - \frac{p_j}{k_{1,j}} \mathbf{1}_{(0,\infty)}(k_{1,j}) + \frac{p_j}{k_{1,j}} a_{1,j} + \sum_{i=2}^{n-1} \frac{p_j}{k_{i,j}} (a_{i,j} - b_{i-1,j}) \right) \\
&= - \sum_{j\in\Omega} \frac{p_j}{k_{n,j}} d_{n,j} = \sum_{j\in\Omega} \theta_j(y,1),
\end{aligned}
$$

where we have used the assumptions from (A4) in the second to last step. So, for any $t \geq 0$ and $\omega \in \Omega^t$, we get that

$$
C(T_\omega(y)) - \sum_{i=1}^{N-1} (\eta_i + \phi_i) \mathbf{1}_{B_i}(T_\omega(y)) = \sum_{j\in\Omega} \theta_{\omega j}(y, t+1), \tag{3.22}
$$

where ωj denotes the concatenation of ω with $j \in \Omega$. Recall from the first line of (3.14) that

$$
y = \sum_{t\geq 0} \sum_{\omega\in\Omega^t} \delta_\omega(y,t) \sum_{j\in\Omega} \theta_{\omega j}(y, t+1).
$$

Combining this with (3.22) and the definition of KB_i from (3.7) then gives the result. $\square$

For each $1 \leq i \leq N-1$, define the functions $E_i, F_i : [0,1] \to \mathbb{R}$ by

$$
\begin{aligned}
E_i(x) &:= \sum_{j\in\Omega} \frac{p_j}{k_{i,j}} \Big(-\mathbf{1}_{[a_{i,j},1]}(x) \mathbf{1}_{(0,\infty)}(k_{i,j}) + \mathbf{1}_{[0,a_{i,j})}(x) \mathbf{1}_{(-\infty,0)}(k_{i,j}) \Big), \\
F_i(x) &:= \sum_{j\in\Omega} \frac{p_j}{k_{i+1,j}} \Big(-\mathbf{1}_{[0,b_{i,j})}(x) \mathbf{1}_{(0,\infty)}(k_{i+1,j}) + \mathbf{1}_{[b_{i,j},1]}(x) \mathbf{1}_{(-\infty,0)}(k_{i+1,j}) \Big),
\end{aligned}
$$

and let $E_N, F_0 : [0,1] \to \mathbb{R}$ be the zero functions. Then for each $2 \leq i \leq N-1$, we have that for Lebesgue almost every $x \in [0,1]$,

$$
E_i(x) + F_{i-1}(x) = \sum_{j\in\Omega} \frac{p_j}{|k_{i,j}|} (\mathbf{1}_{I_i}(x_{i,j}) - 1),
$$

where we have used (A4) for $i = 1, N$. In fact, equality holds for all but countably many points.

3.4.3 Lemma. *For $y \in [0,1]$ we have that for Lebesgue almost every $x \in [0,1]$,*

$$\sum_{j\in\Omega}\sum_{i=1}^{N}\frac{p_j}{|k_{i,j}|}\mathbf{1}_{I_i}(x_{i,j})L_y(x_{i,j}) =$$
$$= \sum_{i=1}^{N-1}(E_i(x)+\eta_i+F_i(x)+\phi_i)\,\mathrm{KB}_i(y)+y+L_y(x)-\mathbf{1}_{[0,y)}(x).$$

Proof. For $y \in [0,1]$, let $1 \le n \le N$ be the index such that $y \in I_n$. By Fubini's Theorem, we get

$$\sum_{j\in\Omega}\sum_{i=1}^{N}\frac{p_j}{|k_{i,j}|}\mathbf{1}_{I_i}(x_{i,j})L_y(x_{i,j}) = \sum_{t\geq 0}\sum_{\omega\in\Omega^t}\delta_\omega(y,t)\sum_{i=1}^{N}\sum_{j\in\Omega}\frac{p_j}{|k_{i,j}|}\mathbf{1}_{I_i\cap[0,T_\omega(y))}(x_{i,j}). \tag{3.23}$$

For Lebesgue almost every $x \in [0,1]$ it holds that

$$\begin{aligned}\sum_{j\in\Omega}\frac{p_j}{|k_{n,j}|}&\mathbf{1}_{(-\infty,0)}(k_{n,j})+\sum_{j\in\Omega}\frac{p_j}{k_{n,j}}\mathbf{1}_{[0,T_j(y))}(x)+F_{n-1}(x)\\ &=\sum_{j\in\Omega}\Bigg(\frac{p_j}{|k_{n,j}|}\mathbf{1}_{(-\infty,0)}(k_{n,j})(1-\mathbf{1}_{[0,T_j(y))}(x)-\mathbf{1}_{[b_{n-1,j},1]}(x))\\ &\quad+\frac{p_j}{|k_{n,j}|}\mathbf{1}_{(0,\infty)}(k_{n,j})(\mathbf{1}_{[0,T_j(y))}(x)-\mathbf{1}_{[0,b_{n-1,j})}(x))\Bigg)\\ &=\sum_{j\in\Omega}\frac{p_j}{|k_{n,j}|}\mathbf{1}_{I_n\cap[0,y)}(x_{n,j}).\end{aligned} \tag{3.24}$$

Since $y \in I_n$ we have for Lebesgue almost every $x \in [0,1]$ that

$$\sum_{i=1}^{N-1}(E_i(x)+F_i(x))\mathbf{1}_{B_i}(y)=\sum_{i=1}^{n-1}\sum_{j\in\Omega}\frac{p_j}{|k_{i,j}|}(\mathbf{1}_{I_i}(x_{i,j})-1)+F_{n-1}(x).$$

Combining this with (3.24) and the definition of $C(y)$ from (3.21) we obtain that for each $y \in [0,1]$, there is a set of $x \in [0,1]$ of full Lebesgue measure, for which

$$\begin{aligned}\sum_{j\in\Omega}\sum_{i=1}^{N}&\frac{p_j}{|k_{i,j}|}\mathbf{1}_{I_i\cap[0,y)}(x_{i,j})\\ &=\sum_{j\in\Omega}\sum_{i=1}^{n-1}\frac{p_j}{|k_{i,j}|}\mathbf{1}_{I_i}(x_{i,j})+\sum_{j\in\Omega}\frac{p_j}{|k_{n,j}|}\mathbf{1}_{(-\infty,0)}(k_{n,j})+\sum_{j\in\Omega}\frac{p_j}{k_{n,j}}\mathbf{1}_{[0,T_j(y))}(x)+F_{n-1}(x)\\ &=\sum_{i=1}^{N-1}(E_i(x)+F_i(x))\mathbf{1}_{B_i}(y)+C(y)+\sum_{j\in\Omega}\tau_j(y,1)\mathbf{1}_{[0,T_j(y))}(x).\end{aligned}$$

CHAPTER 3

Hence, by (3.23) we also have that for Lebesgue almost every $x \in [0,1]$,

$$\sum_{j\in\Omega}\sum_{i=1}^{N}\frac{p_j}{|k_{i,j}|}\mathbf{1}_{I_i}(x_{i,j})L_y(x_{i,j}) = \sum_{i=1}^{N-1}(E_i(x)+F_i(x))\sum_{t\geq 0}\sum_{\omega\in\Omega^t}\delta_\omega(y,t)\mathbf{1}_{B_i}(T_\omega(y)) \\ +\sum_{t\geq 0}\sum_{\omega\in\Omega^t}\delta_\omega(y,t)C(T_\omega(y))+\sum_{t\geq 1}\sum_{\omega\in\Omega^t}\delta_\omega(y,t)\mathbf{1}_{[0,T_\omega(y))}(x).$$

The statement now follows from the definition of KB_i from (3.7) and Lemma 3.4.2. □

Proof of Theorem 3.4.1. First note that for all $1 \leq i \leq N-1$ and all $x \in [0,1]$,

$$E_i(x)+\eta_i = \sum_{j\in\Omega}\frac{p_j}{k_{i,j}}\Big(\mathbf{1}_{[0,a_{i,j})}(x)-a_{i,j}\Big)$$

and

$$F_i(x)+\phi_i = \sum_{j\in\Omega}\frac{p_j}{k_{i+1,j}}\Big(-\mathbf{1}_{[0,b_{i,j})}(x)+b_{i,j}\Big).$$

Together they give that

$$\sum_{\ell\in\Omega}\Big(\frac{p_\ell}{k_{m,\ell}}(-\mathbf{1}_{[0,a_{m,\ell})}(x)+a_{m,\ell})-\frac{p_\ell}{k_{m+1,\ell}}(-\mathbf{1}_{[0,b_{m,\ell})}(x)+b_{m,\ell})\Big) \\ = -(E_m(x)+\eta_m+F_m(x)+\phi_m).$$

Using this together with Lemma 3.4.3 and Fubini's Theorem, we get by (3.20) that for Lebesgue almost every $x \in [0,1]$,

$$P_T h_\gamma(x) = \\ = \sum_{m=1}^{N-1}\gamma_m\sum_{i=1}^{N-1}(E_i(x)+\eta_i+F_i(x)+\phi_i)\sum_{\ell\in\Omega}\Big(\frac{p_\ell}{k_{m,\ell}}\mathrm{KB}_i(a_{m,\ell})-\frac{p_\ell}{k_{m+1,\ell}}\mathrm{KB}_i(b_{m,\ell})\Big) \\ -\sum_{m=1}^{N-1}\gamma_m(E_m(x)+\eta_m+F_m(x)+\phi_m)+h_\gamma(x).$$

From the second part of Lemma 3.3.6 we can deduce by multiplying with $E_i(x)+\eta_i+F_i(x)+\phi_i$ and summing over all i that

$$\sum_{i=1}^{N-1}(E_i(x)+\eta_i+F_i(x)+\phi_i)\gamma_i \\ = \sum_{i=1}^{N-1}(E_i(x)+\eta_i+F_i(x)+\phi_i)\sum_{m=1}^{N-1}\gamma_m\sum_{j\in\Omega}\Big(\frac{p_j}{k_{m,j}}\mathrm{KB}_i(a_{m,j})-\frac{p_j}{k_{m+1,j}}\mathrm{KB}_i(b_{m,j})\Big).$$

Hence, we have obtained that h_γ is a T-invariant function in $L^1(\lambda)$.

It remains to show that $h_\gamma \neq 0$. Recall from Section 3.2 that any T-invariant $L^1(\lambda)$-function is of bounded variation. So, at any point $y \in [0,1]$ the limits $\lim_{x \uparrow y} h_\gamma(x)$ and $\lim_{x \downarrow y} h_\gamma(x)$ exist. Consider $1 \leq \ell \leq N-1$ and assume $z_\ell \in I_\ell$. Then for all $y \in [0,1]$, by (3.6) and (3.7), we obtain by the Dominated Convergence Theorem,

$$\lim_{x \downarrow z_\ell} L_y(x) = \sum_{t \geq 0} \sum_{\omega \in \Omega^t} \delta_\omega(y,t) \lim_{x \downarrow z_\ell} \mathbf{1}_{[0, T_\omega(y))}(x) = \sum_{t \geq 0} \sum_{\omega \in \Omega^t} \delta_\omega(y,t) \mathbf{1}_{B_\ell}(T_\omega(y)) = \mathrm{KB}_\ell(y).$$

From this, Lemma 3.3.6 and the Dominated Convergence Theorem again we then get

$$\begin{aligned} \lim_{x \downarrow z_\ell} h_\gamma(x) &= \sum_{m=1}^{N-1} \gamma_m \sum_{j \in \Omega} \lim_{x \downarrow z_\ell} \left[\frac{p_j}{k_{m,j}} L_{a_{m,j}}(x) - \frac{p_j}{k_{m+1,j}} L_{b_{m,j}}(x) \right] \\ &= \sum_{m=1}^{N-1} \gamma_m \sum_{j \in \Omega} \left[\frac{p_j}{k_{m,j}} \mathrm{KB}_\ell(a_{m,j}) - \frac{p_j}{k_{m+1,j}} \mathrm{KB}_\ell(b_{m,j}) \right] \\ &= \gamma_\ell. \end{aligned} \tag{3.25}$$

If, on the other hand, $z_\ell \in I_{\ell+1}$, then we obtain similarly that $\lim_{x \uparrow z_\ell} L_y(x) = \mathrm{KB}_\ell(y)$ and thus that $\lim_{x \uparrow z_\ell} h_\gamma(x) = \gamma_\ell$. Hence, $h_\gamma = 0$ implies $\gamma = 0$. This proves the theorem. □

3.4.4 Remark. Theorem 3.4.1 assigns to each solution γ of $M\gamma = 0$ a T-invariant $L^1(\lambda)$-function $h_\gamma \neq 0$. If $\gamma \neq 0$, then from h_γ we can get invariant densities for T as follows. If h_γ is positive or negative, then we can scale h_γ to an invariant density function. If not, then we can write $h_\gamma = h^+ - h^-$ for two positive functions $h^+ : [0,1] \to [0,\infty)$ and $h^- : [0,1] \to [0,\infty)$ and by the linearity and the positivity of P_T it follows that

$$h^+ - h^- = h_\gamma = P_T h_\gamma = P_T h^+ - P_T h^-.$$

Hence, h^+ and h^- can both be normalised to obtain invariant densities for T.

3.4.5 Remark. In order to compute h_γ, one needs to compute the fundamental matrix M and a vector γ first. Lemma 3.3.4 implies that when N is small, the computation of γ is straightforward. Indeed, for $N = 2$, M is the null-vector, and we can take $\gamma = 1$. This is illustrated by the example of the random tent maps from Section 3.6.1. For $N = 3$, it is enough to compute only one row of M and take $\gamma = \begin{pmatrix} -\mu_{i,2} & \mu_{i,1} \end{pmatrix}^\intercal$. We see an illustration of this fact in Sections 3.6.3 and 3.6.4 on random β-transformations. For larger N, the computation of M can still be simplified by using the relations from Lemma 3.3.2.

To end this section we give a small example to show that condition (A5) is not necessary for Theorem 3.4.1 to hold. Consider the random system with $\Omega = \{0,1\}$, $T_0(x) = 2x \pmod 1$ the doubling map, $T_1(x) = 1 - T_0(x)$ and $p_0 = p_1 = \frac{1}{2}$. Then $N = 2$ and for both $n = 1,2$ we have $S_n = \frac{1}{2} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2} = 0$. Hence $M = \begin{pmatrix} 0 & 0 \end{pmatrix}^\intercal$ and any $\gamma = \gamma_1 \in \mathbb{R} \setminus \{0\}$ is a non-trivial solution to $M\gamma = 0$. Since all critical points of T_0 and T_1 are mapped to 0 or 1, the function h_1 from (3.18) will be of the form $c \cdot \mathbf{1}_{[0,1)}$ for some $c \neq 0$ and the function $h_\gamma = \frac{\gamma}{c} \cdot \mathbf{1}_{[0,1)}$ is indeed invariant for T.

§3.5 All invariant measures

The aim of this section is twofold. Firstly, we prove that the way T is defined on the partition points z_ℓ does not influence the final result. In other words, the set of invariant functions we obtain from Theorem 3.4.1 if $z_\ell \in I_\ell$ is equal to the set of invariant functions we obtain if we choose $z_\ell \in I_{\ell+1}$. This is the content of Proposition 3.5.1. The amount of work it takes to compute the matrix M and the invariant functions h_γ depend on whether $z_\ell \in I_\ell$ or $z_\ell \in I_{\ell+1}$. Proposition 3.5.1 tells us that we are free to choose the most convenient option. We shall see several examples below. Next we will use Proposition 3.5.1 to prove that, under the additional assumption that all maps T_j are expanding, Theorem 3.4.1 actually produces all absolutely continuous invariant measures of T. We do this by proving in Theorem 3.5.3 that the map $\gamma \mapsto h_\gamma$ is a bijection between the null space of M and the subspace of $L^1(\lambda)$ of all T-invariant functions.

3.5.1 Proposition. Let T be a random system with partition $\{I_i\}_{1\le i\le N}$ and corresponding partition points $z_0, \dots, z_N$. Let $\{\hat{I}_i\}_{1\le i\le N}$ be another partition of $[0,1]$ given by $z_0, \dots, z_N$ and differing from $\{I_i\}_{1\le i\le N}$ only on one or more of the points $z_1, \dots, z_{N-1}$. Let $\hat{T}$ be the corresponding random system, i.e., $\hat{T}(x) = T(x)$ for all $x \neq z_i$, $1 \le i \le N-1$. Let $\hat{M}$ be the fundamental matrix of $\hat{T}$. There is a 1-to-1 correspondence between the solutions γ of $M\gamma = 0$ and the solutions $\hat{\gamma}$ of $\hat{M}\hat{\gamma} = 0$. Moreover, the functions h_γ and $\hat{h}_{\hat{\gamma}}$ coincide.

Proof. First assume that there is only one point z_ℓ on which $\{I_i\}_{1\le i\le N}$ and $\{\hat{I}_i\}_{1\le i\le N}$ differ. We show that any column of $\hat{M}$ is a linear combination of columns of M. More precisely, we show that the i-th column of $\hat{M}$ is a linear combination of the i-th and the ℓ-th column of M. Assume without loss of generality that $z_\ell \in I_\ell$ and therefore $z_\ell \in \hat{I}_{\ell+1}$. This implies that $T_j(z_\ell) = a_{\ell,j}$, whereas $\hat{T}_j(z_\ell) = b_{\ell,j}$. This difference is reflected in the values of the quantities $\mathrm{KI}_n(a_{i,s})$ and $\mathrm{KI}_n(b_{i,s})$ appearing in the matrix M in case $a_{i,s}$ or $b_{i,s}$ enters z_ℓ under some iteration of T. We will describe these changes, but first we define some quantities.

For any $y \in \{a_{i,j}, b_{i,j} \,:\, 1 \le i \le N-1,\, j \in \Omega\}$ let $\Omega_y \subseteq \Omega^*$ be the collection of paths that lead y to z_ℓ, i.e., $\omega \in \Omega_y$ if and only if there is a $0 \le t < |\omega|$, such that $T_{\omega_1^t}(y) = z_\ell$. Let

$$\Omega_y^t := \{\omega \in \Omega^* \mid \exists\, \eta \in \Omega_y \,:\, \omega = \eta_1^t,\, T_\omega(y) = z_\ell \text{ and } T_{\omega_1^s}(y) \neq z_\ell \text{ for } s < t\}.$$

Then Ω_y^t is the collection of words of length t that lead y to z_ℓ via a path that does not lead y to z_ℓ before time t. We are interested in the difference between the quantities $\mathrm{KI}_n(y)$ and $\hat{\mathrm{KI}}_n(y)$ and we let C_n^y denote the part that they have in common, i.e., set

$$C_n^y := \sum_{t\ge1} \sum_{\omega \in \Omega_y^t \cup \Omega^t \setminus \Omega_y} \delta_\omega(y,t) \mathbf{1}_{I_n}(T_{\omega_1^{t-1}}(y)).$$

Then for $n \neq \ell$, we get

$$\begin{aligned}\mathrm{KI}_n(y) &= C_n^y + \sum_{t\geq 0}\sum_{\omega\in\Omega_y^t}\sum_{u\geq 1}\sum_{\eta\in\Omega^u}\delta_\omega(y,t)\delta_\eta(z_\ell,u)\mathbf{1}_{I_n}(T_{\eta_1^{u-1}}(z_\ell))\\ &= C_n^y + \sum_{t\geq 0}\sum_{\omega\in\Omega_y^t}\sum_{u\geq 1}\sum_{\eta\in\Omega^u}\sum_{j\in\Omega}\delta_\omega(y,t)\frac{p_j}{k_{\ell,j}}\delta_\eta(a_{\ell,j},u)\mathbf{1}_{I_n}(T_{\eta_1^{u-1}}(a_{\ell,j}))\\ &= C_n^y + \sum_{t\geq 0}\sum_{\omega\in\Omega_y^t}\delta_\omega(y,t)\sum_{j\in\Omega}\frac{p_j}{k_{\ell,j}}\mathrm{KI}_n(a_{\ell,j}),\end{aligned}$$

and similarly, for $n = \ell$ we obtain

$$\mathrm{KI}_\ell(y) = C_\ell^y + \sum_{t\geq 0}\sum_{\omega\in\Omega_y^t}\delta_\omega(y,t)\sum_{j\in\Omega}\frac{p_j}{k_{\ell,j}}(1+\mathrm{KI}_\ell(a_{\ell,j})).$$

If we set $Q(y) = \sum_{t\geq 0}\sum_{\omega\in\Omega_y^t}\delta_\omega(y,t)$ as the constant that keeps track of all the paths that lead y to z_ℓ for the first time, then we can write

$$\begin{aligned}\mathrm{KI}_n(y) &= C_n^y + Q(y)\sum_{j\in\Omega}\frac{p_j}{k_{\ell,j}}\mathrm{KI}_n(a_{\ell,j}), \text{ for } n\neq\ell,\\ \mathrm{KI}_\ell(y) &= C_\ell^y + Q(y)\sum_{j\in\Omega}\frac{p_j}{k_{\ell,j}}(1+\mathrm{KI}_\ell(a_{\ell,j})).\end{aligned} \tag{3.26}$$

On the other hand, for $\mathrm{K\hat{I}}_n(y)$ we get

$$\begin{aligned}\mathrm{K\hat{I}}_n(y) &= C_n^y + Q(y)\sum_{j\in\Omega}\frac{p_j}{k_{\ell+1,j}}\mathrm{K\hat{I}}_n(b_{\ell,j}), \text{ for } n\neq\ell+1,\\ \mathrm{K\hat{I}}_{\ell+1}(y) &= C_{\ell+1}^y + Q(y)\sum_{j\in\Omega}\frac{p_j}{k_{\ell+1,j}}(1+\mathrm{K\hat{I}}_{\ell+1}(b_{\ell,j})).\end{aligned} \tag{3.27}$$

If $b_{\ell,j}$ does not return to z_ℓ, then $\mathrm{KI}_n(b_{\ell,j}) = \mathrm{K\hat{I}}_n(b_{\ell,j})$. Set

$$B := \{j\in\Omega\,:\,\Omega_{b_{\ell,j}}\neq\emptyset\}.$$

Then

$$\mathrm{K\hat{I}}_n(y) = C_n^y + Q(y)\sum_{j\notin B}\frac{p_j}{k_{\ell+1,j}}\mathrm{KI}_n(b_{\ell,j}) + Q(y)\sum_{j\in B}\frac{p_j}{k_{\ell+1,j}}\mathrm{K\hat{I}}_n(b_{\ell,j}), \text{ for } n\neq\ell+1,$$

$$\begin{aligned}&\mathrm{K\hat{I}}_{\ell+1}(y) =\\ &= C_{\ell+1}^y + Q(y)\sum_{j\notin B}\frac{p_j}{k_{\ell+1,j}}(1+\mathrm{KI}_{\ell+1}(b_{\ell,j})) + Q(y)\sum_{j\in B}\frac{p_j}{k_{\ell+1,j}}(1+\mathrm{K\hat{I}}_{\ell+1}(b_{\ell,j})).\end{aligned}$$

To determine the difference between $\mathrm{KI}_n(y)$ and $\mathrm{K\hat{I}}_n(y)$, we would like an expression of $\mathrm{K\hat{I}}_n(b_{\ell,j})$ in terms of $\mathrm{KI}_n(b_{\ell,j})$ for $j\in B$. Fix $n\neq\ell+1$ for a moment and set for each $j\in B$,

$$A_j = C_n^{b_{\ell,j}} + Q(b_{\ell,j})\sum_{i\notin B}\frac{p_i}{k_{\ell+1,i}}\mathrm{KI}_n(b_{\ell,i}).$$

CHAPTER 3

Then we can find expressions of $\mathrm{K\hat{I}}_n(b_{\ell,j})$ in terms of the values $\mathrm{KI}_n(b_{\ell,i})$ by solving the following system of linear equations:

$$\mathrm{K\hat{I}}_n(b_{\ell,j}) = A_j + Q(b_{\ell,j}) \sum_{i\in B} \frac{p_i}{k_{\ell+1,i}} \mathrm{K\hat{I}}_n(b_{\ell,i}), \quad j \in B.$$

A solution is easily computed through Cramer's method, which gives for $j \in B$

$$\mathrm{K\hat{I}}_n(b_{\ell,j}) = \frac{A_j\left(1 - \sum_{u\in B\setminus\{j\}} Q(b_{\ell,u})\frac{p_u}{k_{\ell+1,u}}\right) + Q(b_{\ell,j}) \sum_{u\in B\setminus\{j\}} \frac{p_u}{k_{\ell+1,u}} A_u}{1 - \sum_{i\in B} Q(b_{\ell,i})\frac{p_i}{k_{\ell+1,i}}}. \tag{3.28}$$

Set

$$B_\ell := 1 - \sum_{j\in\Omega} Q(b_{\ell,j})\frac{p_j}{k_{\ell+1,j}}.$$

Below we will use B_ℓ^{-1}. If $|Q(b_{\ell,j})| \le 1$, then

$$\Big|\sum_{j\in\Omega} Q(b_{\ell,j})\frac{p_j}{k_{\ell+1,j}}\Big| \le \sum_{j\in\Omega} |Q(b_{\ell,j})|\frac{p_j}{|k_{\ell+1,j}|} \le \sum_{j\in\Omega} \frac{p_j}{|k_{\ell+1,j}|} \le \rho < 1,$$

so in this case $B_\ell \neq 0$ and B_ℓ^{-1} is well defined. We now show that $|Q(b_{\ell,j})| \le 1$. If $b_{\ell,j} = z_\ell$, then $\Omega^t_{b_{\ell,j}} = \emptyset$ for any $t \ge 1$, and so $Q(b_{\ell,j}) = 1$. If $b_{\ell,j} \neq z_\ell$, then $Q(b_{\ell,j}) = \sum_{t\ge1}\sum_{\omega\in\Omega^t_{b_{\ell,j}}} \delta_\omega(b_{\ell,j}, t)$. By the expanding on average property (A2), for any $y \in I$, any $t \ge 0$ and any $\omega \in \Omega^{\mathbb{N}}$,

$$|\delta_\omega(y,t)| > \sum_{j\in\Omega} |\delta_\omega(y,t)\tau_j(T_{\omega_1^t}(y),1)| = \sum_{j\in\Omega} |\delta_{\omega j}(y,t+1)|. \tag{3.29}$$

Note that by the definition of $Q(b_{\ell,j})$ the union

$$\bigcup_{t\ge1} \bigcup_{\omega\in\Omega^t_{b_{\ell,j}}} [\omega] \subseteq \Omega^{\mathbb{N}} \tag{3.30}$$

is a disjoint union of cylinder sets. Hence, by repeated application of (3.29) we obtain for each $n \ge 1$ that

$$\begin{aligned}
1 = |\delta_\epsilon(b_{\ell,j},0)| &> \sum_{i_1\in\Omega} |\delta_{i_1}(b_{\ell,j},1)| = \sum_{i_1\in\Omega_{b_{\ell,j}}} |\delta_{i_1}(b_{\ell,j},1)| + \sum_{i_1\in\Omega^c_{b_{\ell,j}}} |\delta_{i_1}(b_{\ell,j},1)| \\
&> \sum_{i_1\in\Omega_{b_{\ell,j}}} |\delta_{i_1}(b_{\ell,j},1)| + \sum_{i_1\in\Omega^c_{b_{\ell,j}}} \sum_{i_2\in\Omega} |\delta_{i_1 i_2}(b_{\ell,j},2)| \\
&= \sum_{t=1}^{2} \sum_{\omega\in\Omega^t_{b_{\ell,j}}} |\delta_\omega(b_{\ell,j},t)| + \sum_{\omega\in(\Omega_{b_{\ell,j}}\cup\Omega^2_{b_{\ell,j}})^c} |\delta_\omega(b_{\ell,j},2)| \\
&> \cdots > \sum_{t=1}^{n} \sum_{\omega\in\Omega^t_{b_{\ell,j}}} |\delta_\omega(b_{\ell,j},t)| + \sum_{\omega\in(\cup_{t=1}^n\Omega^t_{b_{\ell,j}})^c} |\delta_\omega(b_{\ell,j},n)|.
\end{aligned}$$

Since this holds for each n, we get $|Q(b_{\ell,j})| \leq 1$ and $B_\ell \neq 0$.

For $i \notin B$ it holds that $\mathrm{KI}_n(b_{\ell,i}) = C_n^{b_{\ell,i}}$. Then by the definition of B_ℓ, we get

$$\begin{aligned}
\sum_{j\in B} \frac{p_j}{k_{\ell+1,j}} \hat{\mathrm{KI}}_n(b_{\ell,j}) &= B_\ell^{-1} \sum_{j\in B} \frac{p_j}{k_{\ell+1,j}} A_j \\
&= B_\ell^{-1} \sum_{j\in B} \frac{p_j}{k_{\ell+1,j}} \Big(C_n^{b_{\ell,j}} + Q(b_{\ell,j}) \sum_{i\notin B} \frac{p_i}{k_{\ell+1,i}} C_n^{b_{\ell,i}} \Big) \\
&= B_\ell^{-1} \sum_{j\in B} \frac{p_j}{k_{\ell+1,j}} C_n^{b_{\ell,j}} + B_\ell^{-1}(1-B_\ell) \sum_{i\notin B} \frac{p_i}{k_{\ell+1,i}} C_n^{b_{\ell,i}} \\
&= B_\ell^{-1} \sum_{j\in \Omega} \frac{p_j}{k_{\ell+1,j}} C_n^{b_{\ell,j}} - \sum_{i\notin B} \frac{p_i}{k_{\ell+1,i}} C_n^{b_{\ell,i}}.
\end{aligned} \tag{3.31}$$

We obtain similar expressions for $n = \ell + 1$. For each $1 \leq i \leq N-1$, let

$$Q_i := \sum_{j\in\Omega} \Big(\frac{p_j}{k_{i,j}} Q(a_{i,j}) - \frac{p_j}{k_{i+1,j}} Q(b_{i,j}) \Big).$$

We show that for each $1 \leq n \leq N$ and $1 \leq i \leq N-1$ we have

$$\hat{\mu}_{n,i} = \mu_{n,i} - Q_i B_\ell^{-1} \mu_{n,\ell},$$

i.e., the i-th column of $\hat{M}$ is a linear combination of the i-th and the ℓ-th column of M. We give the proof only for $n \notin \{\ell, \ell+1, i, i+1\}$, since the other cases are very similar. To prove this, we first rewrite $\mu_{n,i} - Q_i B_\ell^{-1} \mu_{n,\ell}$. Therefore, note that

$$\begin{aligned}
&\sum_{j\in\Omega} \frac{p_j}{k_{\ell,j}} \mathrm{KI}_n(a_{\ell,j}) - B_\ell^{-1} \Big(\sum_{j\in\Omega} \frac{p_j}{k_{\ell,j}} \mathrm{KI}_n(a_{\ell,j}) - \sum_{j\in B} \frac{p_j}{k_{\ell+1,j}} Q(b_{\ell,j}) \sum_{i\in\Omega} \frac{p_i}{k_{\ell,i}} \mathrm{KI}_n(a_{\ell,i}) \Big) \\
&\qquad = \sum_{j\in\Omega} \frac{p_j}{k_{\ell,j}} \mathrm{KI}_n(a_{\ell,j})(1 - B_\ell^{-1} B_\ell) = 0.
\end{aligned}$$

Then we obtain from the definition of M, (3.26) and the above equation that

$$\begin{aligned}
\mu_{n,i} - Q_i B_\ell^{-1} \mu_{n,\ell} =& \sum_{j\in\Omega} \Big(\frac{p_j}{k_{i,j}} C_n^{a_{i,j}} - \frac{p_j}{k_{i+1,j}} C_n^{b_{i,j}} \Big) + Q_i \sum_{j\in\Omega} \frac{p_j}{k_{\ell,j}} \mathrm{KI}_n(a_{\ell,j}) \\
&- Q_i B_\ell^{-1} \sum_{j\in\Omega} \frac{p_j}{k_{\ell,j}} \mathrm{KI}_n(a_{\ell,j}) + Q_i B_\ell^{-1} \sum_{j\notin B} \frac{p_j}{k_{\ell+1,j}} \mathrm{KI}_n(b_{\ell,j}) \\
&+ Q_i B_\ell^{-1} \sum_{j\in B} \frac{p_j}{k_{\ell+1,j}} \Big(C_n^{b_{\ell,j}} + Q(b_{\ell,j}) \sum_{u\in\Omega} \frac{p_u}{k_{\ell,u}} \mathrm{KI}_n(a_{\ell,u}) \Big) \\
=& \sum_{j\in\Omega} \Big(\frac{p_j}{k_{i,j}} C_n^{a_{i,j}} - \frac{p_j}{k_{i+1,j}} C_n^{b_{i,s}} \Big) + Q_i B_\ell^{-1} \sum_{j\in\Omega} \frac{p_j}{k_{\ell+1,j}} C_n^{b_{\ell,j}}.
\end{aligned}$$

For $\hat{\mu}_{n,i}$ we get by combining (3.27) and (3.31) that

$$\begin{aligned}
\hat{\mu}_{n,i} =& \sum_{j\in\Omega} \Big(\frac{p_j}{k_{i,j}} C_n^{a_{i,j}} + \frac{p_j}{k_{i+1,j}} C_n^{b_{i,j}} \Big) + Q_i \sum_{j\notin B} \frac{p_j}{k_{\ell+1,j}} \mathrm{KI}_n(b_{\ell,j}) \\
&+ Q_i B_\ell^{-1} \sum_{j\in\Omega} \frac{p_j}{k_{\ell+1,j}} C_n^{b_{\ell,j}} - Q_i \sum_{j\notin B} \frac{p_j}{k_{\ell+1,j}} C_n^{b_{\ell,j}} = \mu_{n,i} - Q_i B_\ell^{-1} \mu_{n,\ell}.
\end{aligned}$$

One now easily checks that if $\gamma = (\gamma_1, \ldots, \gamma_{N-1})^\intercal$ is a solution of $M\gamma = 0$, then the vector $\hat{\gamma} = (\hat{\gamma}_1, \ldots, \hat{\gamma}_{N-1})^\intercal$ given by

$$\hat{\gamma}_\ell = \gamma_\ell + \sum_{i=1}^{N-1} \frac{Q_i}{B_\ell - Q_\ell} \gamma_i \tag{3.32}$$

and $\hat{\gamma}_i = \gamma_i$ if $i \neq \ell$, satisfies $\hat{M}\hat{\gamma} = 0$. The fact that $B_\ell - Q_\ell \neq 0$ follows in the same way as that $B_\ell \neq 0$. Hence, there is a 1-to-1 relation between the solutions γ of $M\gamma = 0$ and $\hat{\gamma}$ of $\hat{M}\hat{\gamma} = 0$.

It remains to prove that the functions h_γ and $\hat{h}_{\hat{\gamma}}$ coincide. For that we need to consider the functions L_y. As we did for KI_n, let L^y denote the parts that L_y and $\hat{L}_y$ have in common, i.e., set

$$L^y = \sum_{t \geq 0} \sum_{\omega \in \Omega_y^t \cup \Omega^t \setminus \Omega_y} \delta_\omega(y, t) \mathbf{1}_{[0, T_\omega(y))}.$$

Set $A := \{j \in \Omega \, : \, \Omega_{a_{\ell,j}} \neq \emptyset\}$. Then

$$\begin{aligned} L_y &= L^y + Q(y) \sum_{t \geq 1} \sum_{\omega \in \Omega^t} \delta_\omega(z_\ell, t) \mathbf{1}_{[0, \hat{T}_\omega(z_\ell))} \\ &= L^y + Q(y) \Big(\sum_{j \in \Omega} \mathbf{1}_{[0, a_{\ell,j})} + \sum_{t \geq 1} \sum_{\omega \in \Omega^t} \frac{p_j}{k_{\ell,j}} \delta_\omega(b_{\ell,j}, u) \mathbf{1}_{[0, \hat{T}_\omega(a_{\ell,j}))} \Big) \\ &= L^y + Q(y) \sum_{j \notin A} \frac{p_j}{k_{\ell,j}} L_{a_{\ell,j}} + Q(y) \sum_{j \in A} \frac{p_j}{k_{\ell,j}} L_{a_{\ell,j}}. \end{aligned}$$

By Cramer's rule we obtain for each $j \in A$, that (compare (3.31))

$$\sum_{j \in A} \frac{p_j}{k_{\ell,j}} L_{a_{\ell,j}} = (B_\ell - Q_\ell)^{-1} \sum_{j \in \Omega} \frac{p_j}{k_{\ell,j}} L^{a_{\ell,j}} - \sum_{j \notin A} \frac{p_j}{k_{\ell,j}} L_{a_{\ell,j}}. \tag{3.33}$$

Similarly, we obtain that

$$\hat{L}_y = L^y + Q(y) \sum_{j \notin B} \frac{p_j}{k_{\ell+1,j}} L_{b_{\ell,j}} + Q(y) \sum_{j \in B} \frac{p_j}{k_{\ell+1,j}} \hat{L}_{b_{\ell,j}} \tag{3.34}$$

and

$$\sum_{j \in B} \frac{p_j}{k_{\ell+1,j}} \hat{L}_{b_{\ell,j}} = B_\ell^{-1} \sum_{j \in \Omega} \frac{p_j}{k_{\ell+1,j}} L^{b_{\ell,j}} - \sum_{j \notin B} \frac{p_j}{k_{\ell+1,j}} L^{b_{\ell,j}}. \tag{3.35}$$

To prove that $h_\gamma = \hat{h}_{\hat{\gamma}}$, note that on the one hand,

$$h_\gamma = \sum_{m=1}^{N-1} \gamma_m \sum_{j \in \Omega} \Big(\frac{p_j}{k_{m,j}} L^{a_{m,j}} - \frac{p_j}{k_{m+1,j}} L^{b_{m,j}} \Big) + \sum_{m=1}^{N-1} \gamma_m Q_m \sum_{j \in \Omega} \frac{p_j}{k_{\ell,j}} L_{a_{\ell,j}}.$$

On the other hand, using equations (3.32), (3.34) and (3.35) we obtain for $\hat{h}_{\hat{\gamma}}$ that

$$\begin{aligned}
\hat{h}_{\hat{\gamma}} &= \sum_{m=1}^{N-1} \gamma_m \sum_{s\in\Omega} \Big(\frac{p_s}{k_{m,s}} L^{a_{m,s}} - \frac{p_s}{k_{m+1,s}} L^{b_{m,s}} \Big) \\
&\quad + \sum_{m=1}^{N-1} \gamma_m Q_m \Big(1 + \frac{Q_\ell}{B_\ell - Q_\ell} \Big) \sum_{s\in\Omega} \frac{p_s}{k_{\ell+1,s}} \hat{L}_{b_{\ell,s}} \\
&\quad + \sum_{m=1}^{N-1} \gamma_m \frac{Q_m}{B_\ell - Q_\ell} \sum_{s\in\Omega} \Big(\frac{p_s}{k_{\ell,s}} L^{a_{\ell,s}} - \frac{p_s}{k_{\ell+1,s}} L^{b_{\ell,s}} \Big) \\
&= \sum_{m=1}^{N-1} \gamma_m \sum_{s\in\Omega} \Big(\frac{p_s}{k_{m,s}} L^{a_{m,s}} - \frac{p_s}{k_{m+1,s}} L^{b_{m,s}} \Big) \\
&\quad + \sum_{m=1}^{N-1} \gamma_m Q_m \frac{B_\ell}{B_\ell - Q_\ell} B_\ell^{-1} \sum_{s\in\Omega} \frac{p_s}{k_{\ell+1,s}} L^{b_{\ell,s}} \\
&\quad + \sum_{m=1}^{N-1} \gamma_m \frac{Q_m}{B_\ell - Q_\ell} \sum_{s\in\Omega} \Big(\frac{p_s}{k_{\ell,s}} L^{a_{\ell,s}} - \frac{p_s}{k_{\ell+1,s}} L^{b_{\ell,s}} \Big) \\
&= \sum_{m=1}^{N-1} \gamma_m \sum_{s\in\Omega} \Big(\frac{p_s}{k_{m,s}} L^{a_{m,s}} - \frac{p_s}{k_{m+1,s}} L^{b_{m,s}} \Big) + \sum_{m=1}^{N-1} \gamma_m \frac{Q_m}{B_\ell - Q_\ell} \sum_{s\in\Omega} \frac{p_s}{k_{\ell,s}} L^{a_{\ell,s}}.
\end{aligned}$$

By (3.33) this implies that $h_\gamma = \hat{h}_{\hat{\gamma}}$.

If the partitions $\{I_n\}_{1\le n\le N}$ and $\{\hat{I}_n\}_{1\le n\le N}$ differ in more than one partition point z_ℓ, we can obtain the results from the above by changing one partition point at a time. □

The next lemma states that adding extra points to the set $z_0, \dots, z_N$ does not influence the set of densities obtained from Theorem 3.4.1. This lemma is one of the ingredients of the proof of Theorem 3.5.3 below.

3.5.2 Lemma. *Let T be a random system with partition $\{I_i\}_{1\le i\le N}$ and corresponding partition points $z_0, \dots, z_N$. Consider a refinement of the partition, given by adding extra points $z_1^\dagger, \dots, z_s^\dagger$, for some $s \in \mathbb{N}$. Let $T^\dagger$ be the corresponding random system, i.e., $T^\dagger(x) = T(x)$ for all $x \in [0,1]$, and let $M^\dagger$ be the fundamental matrix of $T^\dagger$. There is a 1-to-1 correspondence between the solutions γ of $M\gamma = 0$ and the solutions $\gamma^\dagger$ of $M^\dagger\gamma^\dagger = 0$. Moreover, the functions h_γ and $h^\dagger_{\gamma^\dagger}$ coincide.*

Proof. Let $Z^\dagger := \{z_1^\dagger, \dots, z_s^\dagger\}$. By introducing these extra points the fundamental matrix $M^\dagger$ of $T^\dagger$ becomes an $(N+s)\times(N+s-1)$ matrix. It is possible to construct this matrix from M in s steps

$$M \to M_1^\dagger \to M_2^\dagger \to \cdots \to M_s^\dagger = M^\dagger,$$

by adding one of the points from $Z^\dagger$ to the partition of T at a time. All of these steps work in exactly the same way, so it is enough to prove the result for $s = 1$. Therefore,

Chapter 3

assume $Z^\dagger = \{z^\dagger\}$. There is an $1 \leq i \leq N$ such that $z^\dagger$ splits the interval I_i into two subintervals, say I_i^L and I_i^R. By Proposition 3.5.1, it is irrelevant whether $z^\dagger \in I_i^L$ or $z^\dagger \in I_i^R$. By construction, $z^\dagger$ is a continuity point of $T^\dagger = T$, so

$$a_{i,j}^\dagger = b_{i,j}^\dagger = k_{i,j} z^\dagger + d_{i,j},$$

and for each n we have

$$\sum_{j \in \Omega} \left[\frac{p_j}{k_{i,j}} \operatorname{KI}_n(a_{i,j}^\dagger) - \frac{p_j}{k_{i,j}} \operatorname{KI}_n(b_{i,j}^\dagger) \right] = 0.$$

Therefore $M^\dagger$ has, with respect to M, an extra column at the ith position, whose entries are all zeroes except for the diagonal and subdiagonal entries, which are given by $\sum_{j\in\Omega} \frac{p_j}{k_{i,j}}$ and $-\sum_{j\in\Omega} \frac{p_j}{k_{i,j}}$, respectively. Moreover, the ith and $(i+1)$th row of $M^\dagger$ are obtained by splitting the ith row of M into two, such that $\operatorname{KI}_i(a_{n,j}) = \operatorname{KI}_i^\dagger(a_{n,j}) + \operatorname{KI}_{i+1}^\dagger(a_{n,j})$ for all n, and analogously for $b_{n,j}$.

CHAPTER 3

The null space of $M^\dagger$ equals the null space of the $(N+1) \times N$ matrix A obtained from $M^\dagger$ by replacing the $(i+1)$th row by the sum of the ith and the $(i+1)$th row. Then all the entries of the ith column of A are 0 except for the diagonal entry, and the matrix M appears as a submatrix of A, by deleting the ith column and the ith row. Hence, any solution γ of $M\gamma = 0$ can be transformed in a solution $\gamma^\dagger$ of $M^\dagger\gamma^\dagger = 0$ by setting $\gamma_j^\dagger = \gamma_j$ for $j \neq i$ and by using the relation $\sum_{j=1}^N A_{i,j}\gamma_j^\dagger = 0$ for $\gamma_i^\dagger$. This gives the first part of the lemma.

Finally, for corresponding solutions γ and $\gamma^\dagger$ the associated densities h_γ and $h_{\gamma^\dagger}^\dagger$ coincide, since

$$\sum_{j \in \Omega} \left[\frac{p_j}{k_{i,j}} L_{a_{i,j}^\dagger}(x) - \frac{p_j}{k_{i,j}} L_{b_{i,j}^\dagger}(x) \right] = 0.$$

□

The next theorem says that in case all maps T_j are expanding, Theorem 3.4.1 in fact produces all absolutely continuous invariant measures for the system T.

3.5.3 Theorem. *Let $\Omega \subseteq \mathbb{N}$ and let T be a random piecewise affine system satisfying assumptions (A1), (A3) and (A4). Assume furthermore that $|k_{i,j}| > 1$ for each $j \in \Omega$ and $1 \leq i \leq N$. An $L^1(\lambda)$-function h is an invariant function for the random system T if and only if $h = h_\gamma$ for some solution γ of the system $M\gamma = 0$.*

An essential ingredient in the proof of this theorem is the extension of a result by Boyarksy, Góra and Islam from [GBI06] given in the next lemma. [GBI06, Theorem 3.6] states that in case we have a random system consisting of two maps that are both expanding, the supports of the invariant densities of T are a finite union of intervals. As the next lemma shows, this result in fact goes through for any finite or countable number of maps with only a small change in the proof. In case of piecewise affine maps, some small steps can be simplified a bit. We have included the proof for the convenience of the reader.

3.5.4 Lemma (cf. Lemma 3.4 and Theorem 3.6 from [GBI06]). *Let $\Omega \subseteq \mathbb{N}$ and let T be a random system of piecewise affine maps satisfying (A1) and such that for each $j \in \Omega$ the map T_j is expanding, i.e., it satisfies $|k_{i,j}| > 1$ for all $1 \le i \le N$. If h is a T-invariant density, then the support of h is a finite union of open intervals.*

Proof. Let $H = \{v_1, \dots, v_r\}$ be the base of the subspace of $L^1(\lambda)$ of T-invariant functions, consisting of density functions of bounded variation, mentioned in Section 3.2. Since any invariant function h for T can be written as $h = \sum_{n=1}^{r} c_n v_n$ for some constants $c_n \in \mathbb{R}$, it is enough to prove the result for elements in H. Therefore, let $h \in H$ and let $U := \operatorname{supp}(h)$ denote the support of h. Since h is a function of bounded variation, we can take h to be lower semicontinuous and U can be written as a countable union of open intervals, each separated by an interval of positive length: $U = \bigcup_{k \ge 1} U_k$. Assume without loss of generality that $\lambda(U_{k+1}) \le \lambda(U_k)$ for each $k \ge 1$. Let $Z := \{z_1, \dots, z_{N-1}\}$ and let $\mathcal{D}$ be the set of indices k, such that U_k contains one of the points $z \in Z$, i.e.,

$$\mathcal{D} = \{k \ge 1 \,|\, \exists\, z \in Z \,:\, z \in U_k\}.$$

We first show that $\mathcal{D} \neq \emptyset$ by proving that $Z \cap U_1 \neq \emptyset$. Suppose on the contrary that U_1 does not contain a point z, then for each $j \in \Omega$, $T_j(U_1)$ is an interval and since each T_j is expanding, we have $\lambda(T_j(U_1)) > \lambda(U_1)$. By the property from (3.3) that U is forward invariant, we know that $T_j(U_1) \subseteq U$ for each j, so it must be contained in one of the intervals U_k. This gives a contradiction.

Now, let J be the smallest interval in the set

$$\{U_k \cap I_n \,:\, k \in \mathcal{D},\, 1 \le n \le N\}.$$

Note that this is a finite set, since Z and $\mathcal{D}$ are both finite. Moreover, by the above this set is not empty, so J exists. Since each U_k is an open interval, we have $\lambda(J) > 0$. Let $\mathcal{F} = \{k \ge 1 \,:\, \lambda(U_k) \ge \lambda(J)\}$, where k is not necessarily in J, and let $S = \bigcup_{k \in \mathcal{F}} U_k$. Since any connected component U_k of S has Lebesgue measure bigger than $\lambda(J)$, S is a finite union of open intervals. We first prove that $T_j(S) \subseteq S$ for any $j \in \Omega$. Let $U_k \subseteq S$ and suppose first that $k \notin \mathcal{D}$. Then for each $j \in \Omega$, as above $T_j(U_k)$ is an interval with $\lambda(T_j(U_k)) > \lambda(U_k) \ge \lambda(J)$. So, $T_j(U_k)$ is contained in another interval U_i that satisfies $\lambda(U_i) > \lambda(J)$ and thus satisfies $U_i \subseteq S$. Hence, $T_j(U_k) \subseteq S$. If, on the other hand, $k \in \mathcal{D}$, then $T_j(U_k)$ consists of a finite union of intervals and since T_j is expanding, the Lebesgue measure of each of these intervals exceeds $\lambda(J)$. Hence, each of the connected components of $T_j(U_k)$ is contained in some interval U_i that satisfies $\lambda(U_i) > \lambda(J)$ and therefore $U_i \subseteq S$. Hence, also in this case $T_j(U_k) \subseteq S$, implying that $T_j(S) \subseteq S$ for all $j \in \Omega$.

Obviously, $S \subseteq U$. Using the fact that $T_j(S) \subseteq S$ for all $j \in \Omega$, we will now show that $U \subseteq S$. Suppose this is not the case and let U_s be the largest interval in $U \setminus S$. Since $U_k \subseteq S$ for any $k \in \mathcal{D}$, we have $s \notin \mathcal{D}$. So, again, for each $j \in \Omega$ the set $T_j(U_s)$ is an interval with $\lambda(T_j(U_s)) > \lambda(U_s)$ and hence, $T_j(U_s) \subseteq S$. Thus $U_s \subseteq T_j^{-1}(S)$ and since $U_s \not\subseteq S$, we have $U_s \subseteq T_j^{-1}(S) \setminus S$. Let $\mu_{\mathbf{p}}$ be the absolutely continuous

Chapter 3

T-invariant measure with density h. We show that $\mu_{\mathbf{p}}(T_j^{-1}(S) \setminus S) = 0$. Since for each $j \in \Omega$ we have

$$S \subseteq T_j^{-1}(T_j(S)) \subseteq T_j^{-1}(S),$$

we obtain from (3.1) that

$$\begin{aligned} 0 = \mu_{\mathbf{p}}(S) - \mu_{\mathbf{p}}(S) &= \sum_{j\in\Omega} p_j \mu_{\mathbf{p}}(T_j^{-1}(S)) - \sum_{j\in\Omega} p_j \mu_{\mathbf{p}}(S) \\ &= \sum_{j\in\Omega} p_j(\mu_{\mathbf{p}}(T_j^{-1}(S)) - \mu_{\mathbf{p}}(S)) = \sum_{j\in\Omega} p_j \mu_{\mathbf{p}}(T_j^{-1}(S) \setminus S). \end{aligned}$$

Since $p_j > 0$ for all j, we have that $\mu_{\mathbf{p}}(T_j^{-1}(S) \setminus S) = 0$ for each j. Hence, $\mu_{\mathbf{p}}(U_s) = 0$, which contradicts the fact that $U_s \subseteq U$. □

3.5.5 Remark. The article [GBI06] contains an example that shows that the previous lemma is not necessarily true if we drop the assumption that all maps T_j are expanding. In [GBI06, Example 3.7] the authors describe a random system T using an expanding and a non-expanding map, of which for a certain probability vector $\mathbf{p}$ the support of the invariant density is a countable union of intervals. The fact that the supports of the elements from H are finite unions of open intervals plays an essential role in the proof of Theorem 3.5.3 as we shall see now.

Proof of Theorem 3.5.3. We will show that the linear mapping from the null space of M to the subspace of $L^1(\lambda)$ of all T-invariant functions is a linear isomorphism. Let $H = \{v_1, \dots, v_r\}$ again be the basis of density functions of bounded variation, whose corresponding measures are ergodic, for the subspace of T-invariant $L^1(\lambda)$-functions mentioned in Section 3.2. Recall that any invariant function h for T can be written as $h = \sum_{n=1}^r c_n v_n$ for some constants $c_n \in \mathbb{R}$.

The injectivity follows from the proof of Theorem 3.4.1, where we showed that $h_\gamma = 0$ implies $\gamma = 0$. We prove surjectivity by providing for each $h \in H$ a vector γ such that $h_\gamma = h$. We will do this by altering T in several steps, so that we finally obtain a system T_U that has a vector γ_U associated to it for which the corresponding density $h^U_{\gamma_U}$ vanishes outside the support U of h. Then, using Proposition 3.5.1 and Lemma 3.5.2 we transform the solution γ_U to a solution γ for T that produces the original density h.

Fix $h \in H$, and let $U := \operatorname{supp}(h)$. Let $Z = \{z_1, \dots, z_{N-1}\}$ again be the set of critical points of the system. Following [K90, Theorem 2], we classify the points in Z as follows:

$$\begin{aligned} Z_1 =& \{z_i \in Z \,|\, z_i \text{ is in the interior of } U\}, \\ Z_2 =& \{z_i \in Z \,|\, z_i \text{ is a left/right endpoint of a subinterval of } U \text{ and } z_i \in I_{i+1}\, (z_i \in I_i)\}, \\ Z_3 =& \{z_i \in Z \,|\, z_i \text{ is a left/right endpoint of a subinterval of } U \text{ and } z_i \in I_i\, (z_i \in I_{i+1})\}, \\ Z_4 =& \{z_i \in Z \,|\, z_i \text{ is an exterior point for } U\}. \end{aligned}$$

We now modify the partition $\{I_i\}_{1\le i\le N}$ on the points in Z_3, so that it corresponds better to the set U. Let $\{\hat{I}_i\}_{1\le i\le N}$ be a partition of $[0,1]$ given by $z_0, \dots, z_N$ and

$$\boxed{\begin{array}{c}\{I_i\}_{1\le i\le N}\\ T\\ M\gamma=0\end{array}} \longrightarrow \boxed{\begin{array}{c}\{\hat{I}_i\}_{1\le i\le N}\\ \hat{T}\\ \hat{M}\hat{\gamma}=0\end{array}} \longrightarrow \boxed{\begin{array}{c}\{\hat{I}_i^\dagger\}_{1\le i\le N}\\ \hat{T}^\dagger\\ \hat{M}^\dagger\hat{\gamma}^\dagger=0\end{array}} \longrightarrow \boxed{\begin{array}{c}\{\hat{I}_i^\dagger\}_{1\le i\le N}\\ T_U\\ M_U\gamma_U=0\end{array}}$$

Figure 3.2: The steps we take in transforming T to T_U.

differing from $\{I_i\}_{1\le i\le N}$ only for $z_i \in Z_3$, i.e., $z_i \in \hat{I}_i$ if and only if $z_i \notin I_i$. Let $\hat{T}$ be the corresponding random system, i.e., $\hat{T}(x) = T(x)$ for all $x \notin Z_3$. By Proposition 3.5.1, the corresponding matrices M and $\hat{M}$ have vectors in their null spaces that differ only on the entries i for which $z_i \in Z_3$, but such that they define the same density.

There might be boundary points of U that are not in Z. Let $Z^\dagger$ be the set of such points. From Lemma 3.5.4 it follows that U is a finite union of open intervals, so the set $Z^\dagger$ is finite. Consider the partition $\{\hat{I}_i^\dagger\}$ given by the points in $Z \cup Z^\dagger$ and let $\hat{T}^\dagger$ be the system with this partition and given by $\hat{T}^\dagger(x) = \hat{T}(x)$ for all x. By Lemma 3.5.2, the corresponding matrices $\hat{M}$ and $\hat{M}^\dagger$ have vectors in their null spaces that differ only on the extra entries corresponding to points $z^\dagger \in Z^\dagger$, but such that they define the same density.

Define a new piecewise affine random system T_U by modifying $\hat{T}^\dagger$ outside of U. To be more precise, we let $T_U(x) = \hat{T}^\dagger(x)$ for all $x \in U$ and on each connected component of $[0,1]\setminus U$ we assume all maps $T_{U,j}$ to be equal and onto, i.e., mapping the interval onto $[0,1]$. Recall from (3.3) that the set U is forward invariant under T. Then any invariant function of T_U vanishes on $[0,1]\setminus U$ λ-almost everywhere, since the set of points $x \in [0,1]\setminus U$, such that $T^n(x) \in [0,1]\setminus U$ for all $n \ge 0$ is a self-similar set of Hausdorff dimension less than 1. From Theorem 3.4.1 we get a non-trivial solution γ_U of $M_U\gamma_U = 0$ with a corresponding function h_U that vanishes on $[0,1]\setminus U$. Since $\hat{T}$ and T_U coincide on U, the function h_U is also invariant for $\hat{T}$ and hence for T. From the fact that U is the support of one of the densities in the basis H and $\operatorname{supp}(h_U) \subseteq U$, we then conclude that $h_U = h$, up to possibly a set of Lebesgue measure 0.

It remains to show that γ_U can be transformed into a vector from the null space of M, leading to the same density h_U. We first show that $\hat{M}^\dagger\gamma_U = 0$. Note that for $z_i \in Z_4$, since h_U is of bounded variation,

$$\lim_{x\uparrow z_i} h_U(x) = 0 = \lim_{x\downarrow z_i} h_U(x).$$

Hence, by the calculations in (3.25) $\gamma_{U,i} = 0$. Similarly, for $z_i \in Z_2 \cup Z_3$ we have that either $\lim_{x\uparrow z_i} h_U(x) = 0$ or $\lim_{x\downarrow z_i} h_U(x) = 0$, which again by the calculations in (3.25) gives $\gamma_{U,i} = 0$. Hence, $\gamma_{U,i} = 0$ for each i such that $z_i \in Z_2 \cup Z_3 \cup Z_4$. Similarly, $\gamma_{U,i} = 0$ for each i such that $z_i \in Z^\dagger$. In the multiplication $\hat{M}^\dagger\gamma_U$ the orbits of the points $a_{i,j}$ and $b_{i,j}$ which are different under $\hat{T}^\dagger$ and T_U are multiplied by 0. Since U is forward invariant, all orbits of points $a_{i,j}$ and $b_{i,j}$ corresponding to i such that $z_i \in Z_1$ will stay in U and will thus be equal under $\hat{T}^\dagger$ and T_U. These facts imply that also $\hat{M}^\dagger\gamma_U = 0$ and that the corresponding invariant density for $\hat{T}^\dagger$ is again h_U.

From Lemma 3.5.2 it follows that there is a vector $\hat{\gamma}$ in the null space of $\hat{M}$ with $\hat{h}_{\hat{\gamma}} = h_U$. Finally, Proposition 3.5.1 then tells us how we can modify $\hat{\gamma}$ to get a vector γ in the null space of M with $h_\gamma = \hat{h}_{\hat{\gamma}} = h_U = h$. $\square$

§3.6 Examples

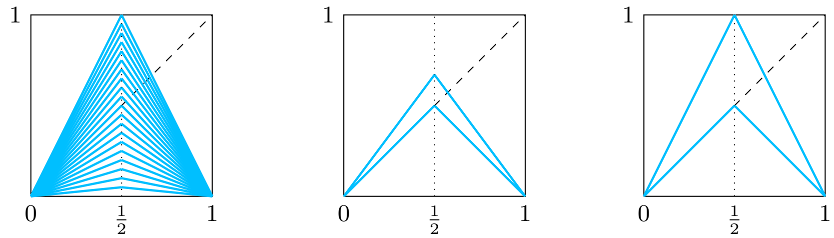
We apply Theorems 3.4.1 and 3.5.3 to various examples.

§3.6.1 Random tent maps

For any countable set of slopes $\{k_j\}_{j\in\Omega}$ with $k_j \in (0,2)$ for each j, consider the family $T := \{T_j\}_{j\in\Omega}$, where each T_j is a tent map of slope k_j, i.e., $T_j : [0,1] \to [0,1]$ is given by

$$T_j(x) = \begin{cases} k_j x, & \text{if } x \in [0,1/2], \\ k_j - k_j x, & \text{if } x \in (1/2,1], \end{cases}$$

see Figure 3.3(a). Let $\mathbf{p} = (p_j)_{j\geq 0}$ be a probability vector such that T is expanding



(a) Countably many tent maps. (b) Two tent maps. (c) Linear logistic maps.

Figure 3.3: Random families of tent maps.

on average, i.e. $\sum_{j\in\mathbb{N}} \frac{p_j}{k_j} < 1$, so (A2) holds. One easily verifies that then conditions (A3) and (A5) hold as well. For $N = 2$ set

$$z_0 = 0, \quad z_1 = \frac{1}{2}, \quad z_2 = 1,$$

and $I_1 = [z_0, z_1]$, $I_2 = (z_1, z_2]$. Since z_1 is the only discontinuity point, the fundamental matrix M is the null vector. As a consequence, we can choose $\gamma = 1$, to obtain the invariant density

$$h_\gamma = c \sum_{j\in\Omega} \frac{2p_j}{k_j} L_{k_j/2},$$

for some normalising constant c. If we set for each $j \in \mathbb{N}$ and $w \in \Omega^t$, $t \geq 0$,

$$\ell_{\omega,j} = \#\Big\{1 \leq n \leq t \,:\, T_{\omega_1^{n-1}}\Big(\frac{k_j}{2}\Big) \in \Big(\frac{1}{2}, 1\Big]\Big\},$$

then this becomes

$$h_\gamma = c \sum_{j\in\Omega} \frac{2p_j}{k_j} \sum_{t\geq 0} \sum_{\omega\in\Omega^t} (-1)^{\ell_{\omega,j}} \prod_{n=0}^{t} \frac{p_{\omega_n}}{k_{\omega_n}} \mathbf{1}_{[0,T_\omega(\frac{k_j}{2}))}. \tag{3.36}$$

If we assume that $k_j > 1$ for all j, then it follows from Theorem 3.5.3 that the density from (3.36) is the unique invariant density for $(T, \mathbf{p})$. If we do not assume this, then we can still draw the same conclusion in case there are only finitely many maps. Namely, to satisfy the condition (A2) there has to be at least one j such that $k_j > 1$. The existence and uniqueness of an absolutely continuous invariant measure for the map T_j is then guaranteed by the results from [LY73, LY78]. In case the set $\{k_j\}_{j\in\mathbb{N}}$ is finite, it then follows from [P84, Corollary 7] that there is only one invariant density for $(T, \mathbf{p})$.

In [AGH18] the authors considered random combinations of logistic maps. In [AGH18, Theorem 4.2] they proved that the random system $\{f_0, f_1\}$ with $f_0(x) = 2x(1-x)$ and $f_1(x) = 4x(1-x)$ has a σ-finite absolutely continuous invariant measure that is infinite in case the map f_0 is chosen with probability $p_0 > \frac{1}{2}$. The linear analogue of this system shows a different picture. Fix $a \in (1, 2]$ and consider the random system with two maps $T_0(x) = \min\{x, 1-x\}$ and $T_{a,1}(x) = \min\{ax, a-ax\}$. See Figure 3.3(b) for an example with $a = \frac{4}{3}$. For any $p \in (0, 1)$, set $p_0 = p$ and $p_1 = 1-p$ and note that $p_0 + \frac{p_1}{a} < 1$. The assumptions (A1)-(A5) are then met and the random system $T = \{T_0, T_{a,1}\}$ has a finite absolutely continuous invariant measure for any such p. A straightforward computation yields $L_{\frac{1}{2}} = \frac{1}{1-p}\mathbf{1}_{[0,\frac{1}{2})} + \frac{1}{a}L_{\frac{a}{2}}$, so that up to a normalising constant, the unique absolutely continuous invariant density is then

$$h_{\gamma,a} = \frac{2p}{1-p}\mathbf{1}_{[0,\frac{1}{2})} + \frac{2}{a}L_{\frac{a}{2}}. \tag{3.37}$$

In particular, for $a = 2$ as shown in Figure 3.3(c) we get

$$h_{\gamma,2} = (1+p)\mathbf{1}_{[0,\frac{1}{2}]} + (1-p)\mathbf{1}_{(\frac{1}{2},1]}.$$

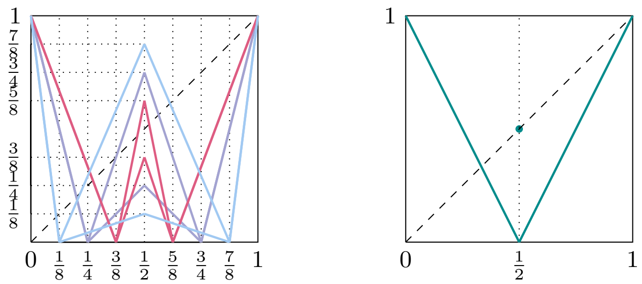
Note that for $p = 1$ we have a deterministic, non-expanding interval map that does not satisfy the requirements from [K90]. However, the limit $\lim_{p\to 1} h_{\gamma,2} = 2\cdot\mathbf{1}_{[0,\frac{1}{2}]}$ is an invariant density for the system. On the other hand, for a fixed $p \in (0, 1)$ the limit $\lim_{a\to 1} h_{\gamma,a}$ is not an absolutely continuous measure. To see this, note that $h_{\gamma,a}$ is determined by the random orbits of $\frac{a}{2}$ and that $1 - \frac{a}{2} \leq T_\omega(\frac{a}{2}) \leq \frac{a}{2}$ for any ω. Hence, by (3.37) and the definition of the L-functions in (3.17) it follows that $h_{\gamma,a} = 0$ on $(\frac{a}{2}, 1]$, while on $[0, 1-\frac{a}{2})$ we have $h_{\gamma,a} = v$ on $[0, 1-\frac{a}{2})$ for some constant $v \in \mathbb{R}$. For any point in $x \in [0, 1-\frac{a}{2})$, the random Perron-Frobenius operator from (3.2) now yields

$$v = h_{\gamma,a}(x) = P_T h_{\gamma,a}(x) = pv + (1-p)\frac{v}{a},$$

which holds if and only if $v = 0$. It follows that for any $a \in (1, 2]$ and any $p \in (0, 1)$, $\mathrm{supp}(h_{\gamma,a}) \subseteq [1-\frac{a}{2}, \frac{a}{2}]$. As a consequence $\lim_{a\to 1} h_\gamma = \delta_{\frac{1}{2}}$, where $\delta_{\frac{1}{2}}$ is the Dirac delta function at $\frac{1}{2}$.

§3.6.2 A random family of W-shaped maps

Keller introduced in [K82] a family of piecewise expanding W-maps to study the phenomenon of instability of absolutely continuous invariant measures. Later the stability of W-shaped maps was studied in other papers as well, see for example [LGB+13, EM12]. Here we construct a random family of W-shaped maps, where each element of the collection is an expanding on average random map $W_a := \{W_{a,0}, W_{a,1}\}$ defined on the unit interval. We give an absolutely continuous invariant probability measure.



(a) W_4, W_8 and $W_{\frac{8}{3}}$ (b) W_2

Figure 3.4: Examples of random systems W_a for various values of a.

For $a > 2$, let $\Omega = \{0, 1\}$ and $N = 4$. Set

$$z_0 = 0, \quad z_1 = 1/a, \quad z_2 = 1/2, \quad z_3 = (a-1)/a, \quad z_4 = 1$$

and

$$I_1 = [z_0, z_1], \quad I_2 = (z_1, z_2], \quad I_3 = (z_2, z_3), \quad I_4 = [z_3, z_4].$$

Let

$$W_{a,0}(x) = \begin{cases} 1 - ax, & \text{if } x \in I_1, \\ \frac{2}{a-2}x - \frac{2}{(a-2)a}, & \text{if } x \in I_2, \\ W_{a,0}(1-x), & \text{otherwise,} \end{cases}$$

and

$$W_{a,1}(x) = \begin{cases} 1 - ax, & \text{if } x \in I_1, \\ \frac{2(a-1)}{a-2}x - \frac{2(a-1)}{(a-2)a}, & \text{if } x \in I_2, \\ W_{a,1}(1-x), & \text{otherwise.} \end{cases}$$

For $a > 4$ the map $W_{a,0}$ presents two contractive branches. Let $1 > p > \frac{(a-4)(a-1)}{(a-2)^2}$ be arbitrary, and let $p_{a,0} = 1 - p$ and $p_{a,1} = p$. With this choice of probability vector the random map W_a satisfies (A1)-(A5). The fundamental matrix M is given by

$$M = \begin{pmatrix} \frac{1-a}{a^2} - \frac{C}{a} & \frac{p_{a0}(2-a)(a-1)}{a^2} + \frac{p_{a1}(2-a)}{a^2(a-1)} & -\frac{C}{a} + \frac{1}{a^2} \\ C & -C & 0 \\ 0 & -C & C \\ \frac{1}{a^2(a-1)} - \frac{C}{a(a-1)} & \frac{p_{a0}(2-a)}{a^2(a-1)} - \frac{p_{a1}(2-a)(a^2-a-1)}{a^2(a-1)^2} & -\frac{C}{a(a-1)} + \frac{1+a-a^2}{a^2(a-1)} \end{pmatrix}$$

for some constant C. Its null space consists of all vectors of the form

$$s\begin{pmatrix}1 & 1 & 1\end{pmatrix}^{\mathsf{T}}, \quad s \in \mathbb{R}.$$

From

$$L_0 = \frac{1}{1-a}, \qquad L_{\frac{1}{a}} = \frac{1}{a(a-1)} + \mathbf{1}_{[0,\frac{1}{a}]} \quad \text{and} \quad L_{\frac{a-1}{a}} = -\frac{1}{a(a-1)} + \mathbf{1}_{[0,\frac{a-1}{a}]},$$

we get the invariant density

$$h_{a,p} = c\left[((a-1) - p(a-2)) \cdot \mathbf{1}_{[0,\frac{1}{a})} + \mathbf{1}_{[\frac{1}{a},\frac{a-1}{a}]} + \left(1 - p\frac{a-2}{a-1}\right) \cdot \mathbf{1}_{(\frac{a-1}{a},1]}\right],$$

for the normalising constant

$$c = \frac{a(a-1)}{2(a-1)^2 - pa(a-2)}.$$

Theorem 3.5.3 implies that if $a < 4$, then this is the unique absolutely continuous invariant density for W_a. Note that

$$\lim_{a\to 2} h_{a,p}(x) = \frac{1}{2}\mathbf{1}_{[0,1]}(x) + \frac{1}{2}\delta_{\frac{1}{2}}(x).$$

On the other hand, for the limit map W_2 shown in Figure 3.4(b) Lebesgue measure is the only absolutely continuous invariant measure.

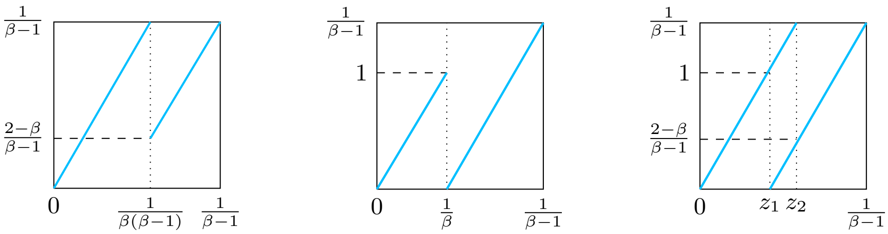
§3.6.3 Random β-transformations

Recall from 1.3.1 the definition of β-expansions. One of the more striking results is that Lebesgue almost all $x \in \left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right]$ have uncountably many different β-expansions (see [EJK90, S03, DdV07]). In [DK03] Dajani and Kraaikamp introduced a random system that produces for each $x \in \left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right]$ all its possible β-expansions. We will define this system for $1 < \beta < 2$ for simplicity, but everything easily extends to $\beta > 2$. Set

$$z_0 = 0, \quad z_1 = \frac{1}{\beta}, \quad z_2 = \frac{1}{\beta(\beta-1)}, \quad z_3 = \frac{1}{\beta-1},$$

and let

$$T_0(x) = \begin{cases} \beta x, & \text{if } x \in [z_0, z_2], \\ \beta x - 1, & \text{if } x \in (z_2, z_3], \end{cases} \quad \text{and} \quad T_1(x) = \begin{cases} \beta x, & \text{if } x \in [0, z_1), \\ \beta x - 1, & \text{if } x \in [z_1, z_3], \end{cases}$$



(a) T_0 (b) T_1 (c) T

Figure 3.5: In (a) we see the lazy β-transformation T_0, in (b) the greedy β-transformation T_1 and in (c) we see them combined. Whether or not $1 > \frac{2-\beta}{\beta-1}$ depends on the chosen value of β.

see Figure 5.1. The map T_0 is called the *lazy β-transformation* and the map T_1 is the *greedy β-transformation*. We do not bother to rescale the system to the unit interval $[0, 1]$, since this has no effect on the computations.

One of the reasons why people are interested in the random β-transformation is for its relation to the infinite Bernoulli convolution, see [DdV05, DK13, K14]. The density of the absolutely continuous invariant measures has been the subject of several papers. For a special class of values β an explicit expression of the density of $\mu_{\mathbf{p}}$ was found in [DdV07] using a Markov chain. In [K14] Kempton produced an explicit formula for the invariant density for all $1 < \beta < 2$ in case $p_0 = p_1 = \frac{1}{2}$ by constructing a natural extension of the system. He states that there is a straightforward extension of this method to $\beta > 2$. Recently Suzuki obtained a formula for the density of $\mu_{\mathbf{p}}$ for all $\beta > 1$ and any $\mathbf{p}$ in [S19]. Since the random β-transformation satisfies the assumptions (A1)-(A5) for any probability vector $\mathbf{p} = (p_0, p_1)$, we can also obtain the invariant density from Theorem 3.4.1. To illustrate our method we calculate the density for $\beta \in (1, 2)$ and $p_0 = p_1 = \frac{1}{2}$.

Let $\Omega = \{0, 1\}$, $N = 3$ and set

$$I_1 = [z_0, z_1), \quad I_2 = [z_1, z_2], \quad I_3 = (z_2, z_3].$$

Define the left and right limits at each point of discontinuity:

$$\begin{aligned} a_{1,0} &= 1, & b_{1,0} &= 1, & a_{2,0} &= \tfrac{1}{\beta-1}, & b_{2,0} &= \tfrac{2-\beta}{\beta-1}, \\ a_{1,1} &= 1, & b_{1,1} &= 0, & a_{2,1} &= \tfrac{2-\beta}{\beta-1}, & b_{2,1} &= \tfrac{2-\beta}{\beta-1}. \end{aligned}$$

As pointed out in Remark 3.4.5, to determine γ it would suffice to compute only one row of M, but for the sake of completeness we give M below. Let $\mathrm{KI}_n(1) = c_n$. By the symmetry of the system, for each $x \in [z_0, z_3]$ and all $(i, j) \in \{1, 2, 3\} \times \{0, 1\}$,

$$T_{i,j}(z_3 - x) = z_3 - T_{4-i,1-j}(x). \tag{3.38}$$

If for any $\omega = \omega_1 \dots \omega_t \in \{0, 1\}^*$, we let $\bar{\omega} \in \{0, 1\}^*$ denote the string $\bar{\omega} = (1 - \omega_1) \dots (1 - \omega_t)$, then (3.38) implies that $T_\omega(1) \in I_n$ if and only if $T_{\bar{\omega}}\left(\frac{2-\beta}{\beta-1}\right) \in I_{4-n}$ and so $\mathrm{KI}_n\left(\frac{2-\beta}{\beta-1}\right) = c_{4-n}$.

We obtain

$$M = \begin{pmatrix} \frac{1}{\beta} + \frac{1}{2\beta}(c_1 - \frac{1}{\beta-1}) & -\frac{1}{2\beta}c_3 \\ -\frac{1}{\beta} + \frac{1}{2\beta}c_2 & \frac{1}{\beta} - \frac{1}{2\beta}c_2 \\ \frac{1}{2\beta}c_3 & -\frac{1}{\beta} - \frac{1}{2\beta}(c_1 - \frac{1}{\beta-1}) \end{pmatrix}.$$

The null space consists of all vectors of the form

$$s\begin{pmatrix}1 & 1\end{pmatrix}^\intercal, \quad s \in \mathbb{R}.$$

From Theorem 3.5.3 we then know that the system T has a unique invariant density. We obtain

$$h_\gamma = \frac{c}{2\beta}\sum_{t\geq 0}\sum_{\omega\in\{0,1\}^t}\left(\frac{1}{2\beta}\right)^t\left(\mathbf{1}_{[0,T_\omega(1))} + \mathbf{1}_{[T_\omega(\frac{2-\beta}{\beta-1}),\frac{1}{\beta-1}]}\right),$$

for some normalising constant c. This matches the density found in [K14, Theorem 2.1] except for possibly countably many points.

If we set $p_0 \neq \frac{1}{2}$, the computations are less straightforward. Nevertheless, we can obtain a nice closed formula for the density in specific instances. Let $p_0 = p \in [0,1]$ be arbitrary and consider $\beta = \frac{1+\sqrt{5}}{2}$, the golden mean. Then β satisfies $\beta^2 - \beta - 1 = 0$ and the system has the nice property that $T_{2,0}(z_1) = z_2$ and $T_{2,1}(z_2) = z_1$ for $z_1 = \frac{1}{\beta}$ and $z_2 = 1$. Also note that $\frac{1}{\beta-1} = \beta$. This specific case has also been studied in [DdV07, Example 1]. The resulting matrix M is given by

$$M = \frac{\beta}{\beta^2 - p(1-p)}\begin{pmatrix} p^2 & -p(1-p) \\ -p & (1-p) \\ (1-p)p & -(1-p)^2 \end{pmatrix},$$

and its null space consists of all vectors of the form

$$s\begin{pmatrix}1-p & p\end{pmatrix}^\intercal, \quad s \in \mathbb{R}.$$

For the functions L_y we obtain $L_0 = 0$, $L_\beta = \beta^2$ and

$$\begin{aligned} L_{\frac{1}{\beta}} &= \frac{p^2\beta^2}{\beta^2 - p(1-p)} + \frac{\beta^2}{\beta^2 - p(1-p)}\mathbf{1}_{[0,\frac{1}{\beta})} + \frac{p\beta}{\beta^2 - p(1-p)}\mathbf{1}_{[0,1)}, \\ L_1 &= \frac{p\beta^3}{\beta^2 - p(1-p)} + \frac{(1-p)\beta}{\beta^2 - p(1-p)}\mathbf{1}_{[0,\frac{1}{\beta})} + \frac{\beta^2}{\beta^2 - p(1-p)}\mathbf{1}_{[0,1)}. \end{aligned}$$

The unique invariant density turns out to be

$$h_\gamma = \frac{\beta^2}{1+\beta^2}\big((1-p)\beta\cdot\mathbf{1}_{[0,\beta-1]} + \mathbf{1}_{(\beta-1,1)} + p\beta\cdot\mathbf{1}_{[1,\beta]}\big),$$

which for $p = \frac{1}{2}$ corresponds to

$$h_\gamma = \frac{\beta^2}{2(1+\beta^2)}(\beta\cdot\mathbf{1}_{[0,\beta-1]} + 2\cdot\mathbf{1}_{(\beta-1,1)} + \beta\cdot\mathbf{1}_{[1,\beta]}).$$

§3.6.4 The random (α,β)-transformation

As an example of a system that is not everywhere expanding, but is expanding on average, we consider a random combination of the greedy β-transformation and the non-expanding (α,β)-transformation introduced in [DHK09]. More specifically, let $0<\alpha<1$ and $1<\beta<2$ be given and

$$z_0=0, \quad z_1=1/\beta, \quad z_2=1.$$

Define the (α,β)-transformation T_0 on the interval $[0,1]$ by

$$T_0(x)=\begin{cases}\beta x, & \text{if } x\in[0,z_1),\\ \frac{\alpha}{\beta}(\beta x-1), & \text{if } x\in[z_1,z_2].\end{cases}$$

Let $T_1:[0,1]\to[0,1]$ be the greedy β-transformation again, given by $T_1(x)=\beta x \pmod 1$. For any $0<p<\frac{\alpha(\beta-1)}{\beta-\alpha}$ the random system T with probability vector $\mathbf{p}=(p,1-p)$ satisfies the conditions (A1), (A2), (A3) and (A5). The assumptions on the boundary points from (A4) do not hold, but this is easily solved by adding an extra interval $(z_2,z_3]$ for $z_3=\frac{1}{\beta-1}$ and extending T_0 and T_1 to it by setting $T_0(x)=T_1(x)=\beta x-1$.

This random system T does not satisfy the conditions of Theorem 3.5.3 and we can therefore not conclude directly that Theorem 3.4.1 produces all invariant densities for T. However, the set $\Omega=\{0,1\}$ is finite and the map T_1 is expanding with $T_1'(x)=\beta>1$ for all x and therefore T satisfies the conditions from [P84, Corollary 7] on the number of ergodic components of the pseudo skew-product R. Since the greedy β-transformation T_1 has a unique absolutely continuous invariant measure, this corollary implies that also the random system T has a unique invariant density. We use Theorem 3.4.1 to get this density.

Let $0<p<\frac{\alpha(\beta-1)}{\beta-\alpha}$ be arbitrary and set

$$I_1=[z_0,z_1), \quad I_2=[z_1,z_2], \quad I_3=(z_2,z_3].$$

The left and right limits at each point of discontinuity are given by:

$$\begin{array}{llll} a_{1,0}=1, & b_{1,0}=0, & a_{2,0}=\alpha-\frac{\alpha}{\beta}, & b_{2,0}=\beta-1,\\ a_{1,1}=1, & b_{1,1}=0, & a_{2,1}=\beta-1, & b_{2,1}=\beta-1.\end{array}$$

By construction, none of the points in $[0,1]$ will ever enter the interval I_3, therefore $\mathrm{KI}_3(y)=0$ for all $y\in[0,1]$. As a consequence, the last row of the 3×2 fundamental matrix M is given by $\mu_{3,1}=0$ and $\mu_{3,2}=-\frac{1}{\beta}$. This fact, together with the fact that we know from Lemma 3.3.4 that the null space of M is non-trivial, forces the first column of M to be zero, i.e., $\mu_{1,1}=\mu_{2,1}=\mu_{3,1}=0$. Hence, the null space of M consists of all vectors of the form

$$s\begin{pmatrix}1 & 0\end{pmatrix}^{\mathsf{T}}, \quad s\in\mathbb{R},$$

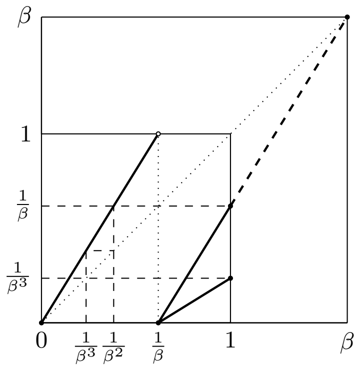


Figure 3.6: The random (α, β)-transformation for $\beta = \frac{1+\sqrt{5}}{2}$ and $\alpha = \frac{1}{\beta}$.

and the unique invariant density of the system T is

$$h_\gamma = \frac{c}{\beta} L_1 = \frac{c}{\beta} \sum_{t \geq 0} \sum_{\omega \in \Omega^t} \delta_\omega(1,t) \mathbf{1}_{[0, T_\omega(1))},$$

for some normalising constant c. In case we choose $\beta = \frac{1+\sqrt{5}}{2}$ and $\alpha = \frac{1}{\beta}$ as in Figure 3.6, we can compute further to get

$$h_\gamma = \frac{\beta^2}{\beta^2 + 1 + 2p} \left(p\beta \mathbf{1}_{[0,1/\beta^3]} + p\mathbf{1}_{[0,1/\beta^2]} + \frac{1}{\beta} \mathbf{1}_{[0,1/\beta]} + \mathbf{1}_{[0,1]} \right).$$

§3.7 c-Lüroth expansions

Recall the definition of Lüroth maps given 1.3.2 in Chapter 1 and then used in 2.2 in Chapter 2. Over the years, many people have considered digit properties of Lüroth expansions, such as digit frequencies and the sizes of sets of numbers for which the digit sequence $(d_n)_{n \geq 1}$ is bounded. See for example [BI09, FLMW10, SF11, MT13, GL16]. The set of points that have all Lüroth digits bounded by some integer D corresponds to the set of points that avoid the set $[0, \frac{1}{D}]$ under all iterations of the map T_L. For the deterministic system T_L, such a set is usually a fractal no matter how large we take the upper bound D. This situation can be modified by dealing with a random setting. More specifically, recall the c-Lüroth maps and expansions defined in Section **??**. The pseudo-skew product L_c, for $c > 0$, is constructed in such a way that the combination of T_L and T_A prevent any point of the interval to visit the subinterval $[0, c)$, giving c-Lüroth expansions with bounded digits.

We give an example for $c = \frac{1}{3}$, in which all $x \in \left[\frac{1}{3}, 1\right]$ have a random $\frac{1}{3}$-Lüroth expansion that uses only digits 2 and 3. Using the density given by Theorem 3.4.1 we can compute the frequency of each of these digits for any typical point $x \in \left[\frac{1}{3}, 1\right]$.

Let $T_0 = T_{0,\frac{1}{3}}$ and $T_1 = T_{1,\frac{1}{3}}$. Consider the partition of $I_{\frac{1}{3}}$ by setting

$$I_1 = \left[\frac{1}{3}, \frac{7}{18}\right] \quad I_2 = \left(\frac{7}{18}, \frac{4}{9}\right] \quad I_3 = \left(\frac{4}{9}, \frac{1}{2}\right]$$
$$I_4 = \left(\frac{1}{2}, \frac{2}{3}\right] \quad I_5 = \left(\frac{2}{3}, \frac{5}{6}\right] \quad I_6 = \left(\frac{5}{6}, 1\right].$$

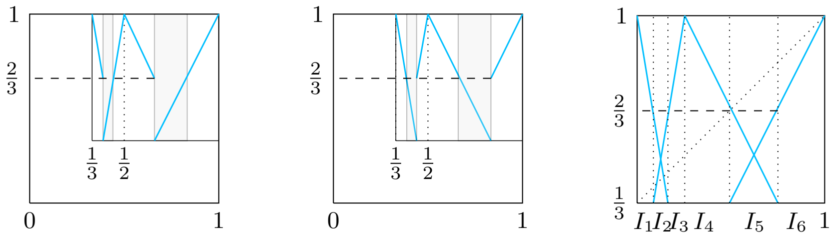
Note that

$$T_0(x) = \begin{cases} T_L(x) & \text{if } x \in I_2 \cup I_3 \cup I_5 \cup I_6, \\ \\ T_A(x) & \text{if } x \in I_1 \cup I_4, \end{cases}$$

and

$$T_1(x) = \begin{cases} T_A(x) & \text{if } x \in I_1 \cup I_2 \cup I_4 \cup I_5, \\ \\ T_L(x) & \text{if } x \in I_3 \cup I_6. \end{cases}$$

See Figure 3.7. Let $\mathbf{p} = (p, 1-p)$, for some $0 < p < 1$.



(a) T_0 (b) T_1 (c) T

Figure 3.7: The systems T_0, T_1 and $L_{\frac{1}{3}}$ on the interval $I_{\frac{1}{3}} = [\frac{1}{3}, 1]$.

To use Theorem 3.4.1, we need to determine the orbits of all the points $a_{n,j}$ and $b_{n,j}$, which in this case are $\frac{1}{3}$, $\frac{2}{3}$ and 1. One easily checks that all $\mathrm{KI}_n(a_{i,j})$ and $\mathrm{KI}_n(b_{i,j})$ are zero, except for

$$\mathrm{KI}_1\left(\frac{1}{3}\right) = -\frac{1}{6}, \qquad \mathrm{KI}_6\left(\frac{1}{3}\right) = -\frac{1}{6}, \qquad \mathrm{KI}_6(1) = 1 \quad \text{and} \quad \mathrm{KI}_4\left(\frac{2}{3}\right) = -\frac{1}{3}.$$

The fundamental matrix M of the system is therefore given by

$$M = \begin{pmatrix} \frac{p-6}{36} & \frac{1-p}{36} & 0 & \frac{p}{12} & \frac{1-p}{12} \\ \frac{1-2p}{6} & \frac{2p-1}{6} & 0 & 0 & 0 \\ 0 & -\frac{1}{6} & \frac{1}{6} & 0 & 0 \\ \frac{p}{18} & \frac{1-p}{18} & \frac{1}{2} & \frac{p-3}{6} & \frac{1-p}{6} \\ 0 & 0 & 0 & \frac{1-2p}{2} & \frac{2p-1}{2} \\ \frac{p}{36} & \frac{1-p}{36} & \frac{2}{3} & \frac{p}{12} & -\frac{p+5}{12} \end{pmatrix},$$

and its null space consists of all vectors of the form

$$s\begin{pmatrix}3 & 3 & 3 & 5 & 5\end{pmatrix}^{\mathsf{T}}, \quad s \in \mathbb{R}.$$

Again this is a one-dimensional space, so by Theorem 3.5.3 T has a unique invariant density. The corresponding measure $m_{\mathbf{p}} \times \mu_{\mathbf{p}}$ is necessarily ergodic for $L_{\frac{1}{3}}$. In the following we denote by $\boldsymbol{L}$ the L functions from (3.17) to distinguish them from the pseudo-skew product map $L_{\frac{1}{3}}$. From

$$\boldsymbol{L}_{\frac{1}{3}} = -\frac{1}{3}, \qquad \boldsymbol{L}_{\frac{2}{3}} = \frac{2}{3} \cdot \mathbf{1}_{[\frac{1}{3},\frac{2}{3}]} \quad \text{and} \quad \boldsymbol{L}_1 = 2$$

we get the invariant density

$$h_\gamma = \frac{3}{8}(3 \cdot \mathbf{1}_{[\frac{1}{3},\frac{2}{3}]} + 5 \cdot \mathbf{1}_{(\frac{2}{3},1]}).$$

For any point $(\omega, x) \in \{0,1\}^{\mathbb{N}} \times \left[\frac{1}{3}, 1\right]$ the frequency of the digit 2 in its random Lüroth expansion is given by

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_{\{0,1\}^{\mathbb{N}} \times (\frac{1}{2},1]}(L^k_{\frac{1}{3}}(\omega, x)).$$

Since $m_{\mathbf{p}} \times \mu_{\mathbf{p}}$ is ergodic, by Birkhoff's Ergodic Theorem we have that for $m_{\mathbf{p}} \times \mu_{\mathbf{p}}$-a.e. $(\omega, x) \in \{0,1\}^{\mathbb{N}} \times \left[\frac{1}{3}, 1\right]$ the frequency of 2 in the associated random Lüroth expansion is

$$\int_{(\frac{1}{2},1]} h_\gamma d\lambda = \frac{13}{16},$$

giving also that the frequency of the digit 3 is $\frac{3}{16}$.

Even though condition (A5) is not satisfied for $p = \frac{1}{2}$, the fundamental matrix M can still be computed and its null space is still given by $s\begin{pmatrix}3 & 3 & 3 & 5 & 5\end{pmatrix}^{\mathsf{T}}$, $s \in \mathbb{R}$. Moreover, the function $h_\gamma = \frac{3}{8}(3 \cdot \mathbf{1}_{[\frac{1}{3},\frac{2}{3}]} + 5 \cdot \mathbf{1}_{(\frac{2}{3},1]})$ is still the unique invariant density. We believe that Theorem 3.4.1 and Theorem 3.5.3 should still hold without the assumption (A5)

3.7.1 Remark. Note that for any rational c, the density of $\mu_{\mathbf{p}}$ can also be recovered using the matrix form P of the Perron-Frobenius operator, which is the approach used in 2.4. However, it is often the case that the matrix P is much larger than our fundamental matrix M. For instance, consider again Example 2.4.12 from Chapter 2. The Perron-Frobenius matrix P for the c-Lüroth transformation L_c for $c = \frac{12}{25}$ is a 13×13 square matrix. The corresponding fundamental matrix M is the 4×3 matrix

$$M = \begin{pmatrix} \frac{1}{6} & -\frac{p}{12} & -\frac{1-p}{12} \\ \frac{1}{2} & -\frac{1}{2} + \frac{p}{4} & -\frac{1-p}{4} \\ 0 & \frac{1-2p}{2} & -\frac{1-2p}{2} \\ \frac{2}{3} & \frac{p}{6} & -\frac{1}{2} + \frac{1-p}{6}. \end{pmatrix}.$$

For

$$\begin{array}{lll} \mathrm{KI}_5(1) = 1 & \mathrm{KI}_2(c) = \frac{1}{6} & \mathrm{KI}_3(c) = -\frac{1}{50} \\ \mathrm{KI}_5(c) = \frac{8}{75} & \mathrm{KI}_3(1-c) = -\frac{492}{1025} & \mathrm{KI}_5(1-c) = -\frac{451}{1025}, \end{array}$$

and $\mathrm{KI}_n(y) = 0$ for any other combination of $n \in \{1, 2, 3, 4, 5\}$ and $y \in \{c, 1-c\}$ not listed. Note that, due to the periodicity of the random orbit of c, the computation of the quantities KI uses the equality

$$\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2^{10}}\right)^n = \frac{1024}{1025}.$$

Furthermore, from Proposition 2.2.6 one finds that for irrational cutting points L_c does not admit a Markov partition so, while the aforementioned method, using the Perron-Frobenius operator in matrix form, does not apply anymore, Theorem 3.4.1 is also capable of handling these situations.

§3.8 Remarks

The procedure proposed in Section 3.4, and in particular the computation of the quantities KI_n and the functions L_y, seems at first glance quite complicated. However, this is not the case for an extensive class of transformations. This includes the random β-transformations studied in Sections 3.6.3, 3.6.4, the c-Lüroth maps introduced in Chapter 2 and the other families of examples proposed in Section 3.6. Moreover, for Markov maps the computation becomes pretty straightforward. We will see in Chapter 5 that this is also true for random interval maps having random matching. Furthermore, we will show how the entire procedure can be actually even implemented in Python, see Chapter 5 Section 5.6.