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Measures and matching for number systems

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CHAPTER 2

Random c-Lüroth Expansions

This chapter is based on: [KM21].

Abstract

We introduce a family of random c-Lüroth transformations $\{L_c\}_{c\in[0,\frac{1}{2}]}$, obtained by randomly combining the standard and alternating Lüroth maps with probabilities p and $1-p$, $0<p<1$. We prove that the pseudo-skew product map L_c produces for each $c \leq \frac{2}{5}$ and for Lebesgue almost all $x \in [c,1]$ uncountably many different generalised Lüroth expansions that can be investigated simultaneously. Moreover, for $c=\frac{1}{\ell}$, for some $\ell \in \mathbb{N}_{\geq 3} \cup \{\infty\}$, Lebesgue almost all x have uncountably many universal generalised Lüroth expansions with digits less than or equal to ℓ. For $c=0$ we show that typically the speed of convergence to an irrational number x, of the sequence of Lüroth approximants generated by L_0, is equal to that of the standard Lüroth approximants; and that the quality of the approximation coefficients depends on p and varies continuously between the values for the alternating and the standard Lüroth map. Furthermore, we show that for each $c \in \mathbb{Q}$ the map L_c admits a Markov partition. For specific values of $c>0$, we compute the density of the stationary measure and we use it to study the typical speed of convergence of the approximants and the digit frequencies.

§2.1 Motivation and context

In 1883 Lüroth showed in [L83] that each $x \in [0,1]$ can be expressed in the form

$$x = \frac{1}{d_1} + \frac{1}{d_1(d_1-1)d_2} + \ldots = \sum_{m\geq 1}(d_m - 1)\prod_{j=1}^{m}\frac{1}{d_j(d_j-1)}, \tag{2.1}$$

where $d_m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ for each m (and with $\frac{1}{\infty} = 0$). Such expressions are now called *Lüroth expansions*. By considering the numbers

$$\frac{p_n}{q_n} := \sum_{m=1}^{n}(d_m - 1)\prod_{j=1}^{m}\frac{1}{d_j(d_j-1)}, \quad n \geq 1, \tag{2.2}$$

one obtains a sequence of rationals converging to the number x, making Lüroth expansions suitable for finding rational approximations of irrational numbers. Since their introduction in 1883 much research has been done on the approximation properties of Lüroth expansions from various perspectives. In this article we address these questions by adopting a random dynamical systems approach. It turns out that this yields for each x many different number expansions similar to the Lüroth expansion from (2.1), without compromising the quality of approximation. Before we state our results, we first give a brief summary of a selection of the known results.

A Lüroth expansion is called *ultimately periodic* if there exist $n \geq 0$ and $r \geq 1$ such that $d_{n+j} = d_{n+r+j}$ for all $j \geq 1$ (and *periodic* if $n = 0$). One of the most basic results on Lüroth expansions, obtained in [L83], is on periodicity.

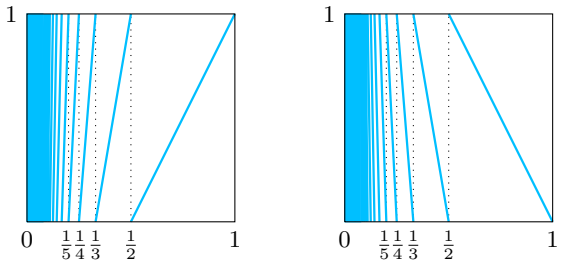
2.1.1 Theorem (page 416, [L83]). *A real number $x \in (0,1)$ has an ultimately periodic Lüroth expansion if and only if $x \in \mathbb{Q}$.*

Many other properties of Lüroth expansions were obtained using a dynamical system. Lüroth expansions can be obtained dynamically by iterating the *Lüroth transformation* $T_L : [0,1] \to [0,1]$ given by $T_L(0) = 0$, $T_L(1) = 1$ and

$$T_L(x) = \left\lceil \frac{1}{x} \right\rceil \left(\left\lceil \frac{1}{x} \right\rceil - 1 \right) x - \left(\left\lceil \frac{1}{x} \right\rceil - 1 \right)$$

for $x \neq 0, 1$, where $\lceil x \rceil$ denotes the smallest integer not less than x. See Figure 2.1(a) for the graph.

The digits d_n, $n \geq 1$, are obtained by setting $d_n(x) = k$ if $T_L^{n-1}(x) \in \left[\frac{1}{k}, \frac{1}{k-1}\right)$, $k \geq 2$, $d_n(x) = 2$ if $T_L^{n-1}(x) = 1$ and $d_n(x) = \infty$ if $T_L^{n-1}(x) = 0$. Hence, the map T_L produces for each $x \in [0,1]$ a Lüroth expansion as in (2.1). From the graph of T_L one sees immediately that Lebesgue almost all numbers $x \in [0,1]$ have a unique Lüroth expansion and if x does not have a unique expansion, then it has exactly two different ones, one with $d_n = d$ and $d_{n+j} = \infty$ and one with $d_n = d+1$ and $d_{n+j} = 2$ for some n, d and all $j \geq 1$. This holds for any number $x \in [0,1]$ for which there is an



(a) T_L (b) T_A

Figure 2.1: The standard and the alternating Lüroth maps in (a) and (b), respectively.

$n \geq 1$ such that $T_L^n(x) = 0$. By identifying these two expansions we can speak of the unique Lüroth expansion of any number $x \in [0, 1]$.

From the dynamics of T_L we get information on the digit frequencies in Lüroth expansions. The map T_L is measure preserving and ergodic with respect to the Lebesgue measure λ on $[0, 1]$. It is then a straightforward application of Birkhoff's Ergodic Theorem that, in the Lüroth expansion of Lebesgue almost every x, the frequency of the digit d equals $\frac{1}{d(d-1)}$, which corresponds to the length of the interval $\left[\frac{1}{d}, \frac{1}{d-1}\right)$. It was proven by Šalát in [S68] that for any $D \geq 2$ the set of points $x \in (0, 1)$ for which all Lüroth expansion digits are bounded by D has Hausdorff dimension < 1 with the dimension approaching 1 as $D \to \infty$. The articles [S68, BBDK94, BI09, SF11, CWZ13, MT13, SFM17] all consider Lüroth expansions with certain restrictions on the digits d_n.

The quality of approximation by Lüroth expansions depends on the *approximants* or *convergents* $\frac{p_n}{q_n}$ given in (2.2). In [BI09] the authors give a multifractal analysis of the speed with which the sequence $\left(\frac{p_n}{q_n}\right)_n$ converges to the corresponding x using the Lyapunov exponent. The Lyapunov exponent of T_L at $x \in (0, 1)$ is defined by

$$\Lambda_L(x) = \lim_{n\to\infty} \frac{1}{n} \log \prod_{k=0}^{n-1} |T_L'(T_L^k(x))|,$$

whenever this limit exists. It follows from another application of Birkhoff's Ergodic Theorem that for λ-a.e. $x \in (0, 1)$,

$$\Lambda(x) = \sum_{d=2}^{\infty} \frac{\log(d(d-1))}{d(d-1)}.$$

One of the results from [BI09], which we mention here for further reference, is the following.

2.1.2 Theorem ([BI09]). *For λ-a.e. $x \in (0,1)$,*

$$\lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{p_n}{q_n} \right| = -\sum_{d=2}^{\infty} \frac{\log(d(d-1))}{d(d-1)}.$$

Moreover, the range of possible values of this rate is $(-\infty, -\log 2]$.

Another way to express the quality of the approximations is via the limiting behaviour of the *approximation coefficients*

$$\theta_n^L(x) := q_n \left| x - \frac{p_n}{q_n} \right|, \tag{2.3}$$

where $q_n = d_n \prod_{i=1}^{n-1} d_i(d_i - 1)$. In [DK96] the authors proved the following result.

2.1.3 Theorem (Theorem 2, [DK96]). *For λ-a.e. $x \in [0,1]$ and for every $z \in (0,1]$ the limit*

$$\lim_{N\to\infty} \frac{\#\{1 \le j \le N \,:\, \theta_j^L(x) < z\}}{N}$$

exists and equals

$$F_L(z) := \sum_{k=2}^{\lfloor \frac{1}{z} \rfloor + 1} \frac{z}{k} + \frac{1}{\lfloor \frac{1}{z} \rfloor + 1}. \tag{2.4}$$

We refer to e.g. [SYZ14, V14, G16, GL16, ZC16, S17, LCTW18, SX18, TW18] for results on other properties of Lüroth expansions.

In [BBDK94] the authors placed the map T_L in the larger framework of *generalised Lüroth series transformations* (GLS). A GLS transformation is a piecewise affine onto map $T_{\mathcal{P},\varepsilon} : [0,1] \to [0,1]$ given by an at most countable interval partition $\mathcal{P}$ on $[0,1]$ and a vector $\varepsilon = (\varepsilon_n)_n \in \{0,1\}^{\#\mathcal{P}}$ specifying for each partition element the orientation of $T_{\mathcal{P},\varepsilon}$ on that interval. The Lüroth transformation can be obtained by taking the partition $\mathcal{P}_L = \left\{ \left[\frac{1}{n}, \frac{1}{n-1} \right) \right\}_{n\ge 2}$ and orientation vector $\varepsilon = (0)_{n\ge 1}$, i.e., all branches are orientation preserving. In [BBDK94] the authors considered all GLS transformations $T_{\mathcal{P}_L,\varepsilon}$ with partition $\mathcal{P}_L$. Besides the Lüroth transformation, another specific instance of this family is the *alternating Lüroth map* $T_A : [0,1] \to [0,1]$ given by $T_A(x) = 1 - T_L(x)$, see Figure 2.1(b), which has $\varepsilon = (1)_{n\ge 1}$, so that all branches orientation reversing. Similar to the Lüroth expansion from (2.1), iterations of any GLS transformation $T_{\mathcal{P}_L,\varepsilon}$ yield number expansions for $x \in [0,1]$ of the form

$$x = \sum_{n=1}^{\infty} (-1)^{\sum_{i=1}^{n-1} s_i} \frac{d_n - 1 + s_n}{\prod_{i=1}^{n} d_i(d_i - 1)}, \tag{2.5}$$

where $s_n \in \{0,1\}$ and $d_n \ge 2$, called *generalised Lüroth expansions*. Here we let $\sum_{i=1}^{0} s_i = 0$. For each map $T_{\mathcal{P}_L,\varepsilon}$ the authors of [BBDK94] considered the approximation coefficients $\theta_n^{\mathcal{P}_L,\varepsilon}$ and the corresponding distribution function $F_{\mathcal{P}_L,\varepsilon}$ and they found the following.

2.1.4 Theorem (Theorem 4, [BBDK94]). *The distribution function of θ_n^A for the map T_A is given for $0 < z \leq 1$ by*

$$F_A(z) = \sum_{k=2}^{\lfloor \frac{1}{z} \rfloor} \frac{z}{k-1} + \frac{1}{\lfloor \frac{1}{z} \rfloor}.$$

For any GLS transformation $T_{\mathcal{P}_L,\varepsilon}$ it holds that

$$F_A \leq F_{\mathcal{P}_L,\varepsilon} \leq F_L,$$

Furthermore, the first moments of F_L and F_A are given respectively by

$$M_L := \int_{[0,1]} 1 - F_L \, d\lambda = \frac{\zeta(2)}{2} - \frac{1}{2} \quad \text{and} \quad M_A := \int_{[0,1]} 1 - F_A \, d\lambda = 1 - \frac{\zeta(2)}{2},$$

where $\zeta(2)$ is the zeta function evaluated at 2.

The authors of [BBDK94] remark that they suspect that the set of values that the limit $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \theta_i^{\mathcal{P}_L,\varepsilon}$ can take is a fractal subset of the interval $[M_A, M_L]$. Other results on the map T_A can be found e.g. in [KKK90, KKK91]. For results on different families of GLS transformations, see e.g. [KMS11, M11, CWY14, CW14].

In this chapter we adopt a random approach to Lüroth expansions. We introduce a family of random Lüroth systems $\{L_{c,p}\}_{c\in[0,\frac{1}{2}],0\leq p\leq 1}$ that are obtained from randomly combining the maps T_L and T_A. The parameter c is the cutting point, that defines the interval $[c, 1]$ on which each $L_{c,p}$ is defined. More precisely, we overlap T_L and T_A on the interval $[c, 1]$ and remove from both maps the pieces that map points into $[0, c)$. The parameter p reflects the probability with which we apply the map T_L. To be precise, let $T_0 := T_L$ and $T_1 := T_A$ and let σ denote the left shift on sequences. Then the *random c-Lüroth transformation* $L_{c,p} : \{0,1\}^{\mathbb{N}} \times [c,1] \to \{0,1\}^{\mathbb{N}} \times [c,1]$ is defined by

$$L_{c,p}(\omega, x) = \big(\sigma(\omega), T_{\omega_1}(x)1_{[c,1]}(T_{\omega_1}(x)) + T_{1-\omega_1}(x)1_{[0,c)}(T_{\omega_1}(x))\big).$$

By iteration, each map $L_{c,p}$ produces for each pair (ω, x) a generalised Lüroth expansion for x as in (2.5). So for typical $x \in [c, 1]$ multiple generalised Lüroth expansions of the form (2.5) are obtained. If $c = 0$ the corresponding random Lüroth expansions have digits in the set $\mathbb{N}_{\geq 2} \cup \{\infty\}$, but for $c > 0$ the available set of digits is bounded from above. This makes the two cases inherently different. We summarise our main results in the following three theorems.

1 Theorem. *Let $c \in \left[0, \frac{1}{2}\right]$.*

(i) If $x \in [c, 1] \setminus \mathbb{Q}$, then no generalised Lüroth expansion of x produced by $L_{c,p}$ is ultimately periodic.

(ii) If $x \in [c, 1] \cap \mathbb{Q}$ then, depending on the values of x and c, the map $L_{c,p}$ can produce any of the following number of different generalised Lüroth expansions:

- *a unique expansion,*
- *a finite or countable number of ultimately periodic expansions,*
- *countably many ultimately periodic expansions and uncountably many expansions that are not ultimately periodic.*

We also give a characterisation on when each of these cases occurs. This result does not depend on the value of p. The following results further explore the properties of the produced generalised Lüroth expansions in case $c = 0$ and $c > 0$. Note that generalised Lüroth expansions are given by a sequence of pairs $((s_n, d_n))_n$ with $s_n \in \{0, 1\}$ and $d_n \geq 2$, rather than just the sequence $(d_n)_n$. We call a generalised Lüroth expansion generated by a map $L_{c,p}$ *universal* if any possible block of digits $(s_1, d_1), \ldots, (s_n, d_n)$ of any length n from the alphabet associated to $L_{c,p}$ occurs in the expansion.

2 Theorem. *Let $c = 0$ and $0 < p < 1$.*

(i) The map $L_{0,p}$ generates for Lebesgue almost every $x \in [0, 1]$ uncountably many universal generalised Lüroth expansions.

(ii) The speed of convergence of the sequence $\left(\frac{p_n}{q_n}\right)_n$ to x for any generalised Lüroth expansion produced by $L_{0,p}$ typically satisfies

$$\lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{p_n}{q_n} \right| = -\sum_{d=2}^{\infty} \frac{\log(d(d-1))}{d(d-1)}$$

and the range of possible values of this rate is $(-\infty, -\log 2]$. In particular, this rate does not depend on p.

(iii) Typically the approximation coefficients generated by $L_{0,p}$ satisfy

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \theta_i^{0,p} = p\frac{2\zeta(2) - 3}{2} + \frac{2 - \zeta(2)}{2},$$

where $\zeta(2)$ is the zeta function evaluated at 2. In particular, this limit can attain any value in the interval $[M_A, M_L]$.

The result in (ii) is given by considering the Lyapunov exponent of the random system $L_{0,p}$ as was done for Theorem 2.1.2. We see that the speed of convergence is not compromised by adding randomness to the system. For (iii) we note that instead of a fractal set inside $[M_A, M_L]$ we can obtain the full interval by adding randomness.

3 Theorem. *Let $c > 0$ and $0 < p < 1$.*

(i) If $0 < c \leq \frac{2}{5}$, then the map $L_{c,p}$ generates for every irrational $x \in [c, 1]$ uncountably many different generalised Lüroth expansions.

(ii) If $c = \frac{1}{\ell}$ for some $\ell \in \mathbb{N}_{\geq 3}$, then $L_{c,p}$ generates for every irrational $x \in [c, 1]$ uncountably many universal generalised Lüroth expansions.

Note that results corresponding to (ii) and (iii) from Theorem 2 in case $c > 0$ are missing. For $c > 0$ the speed of convergence of the sequence $\left(\frac{p_n}{q_n}\right)_n$ is still governed by the Lyapunov exponent of the map $L_{c,p}$, but this is not so easily computed. For $c = 0$ we are in the lucky circumstance that $m_p \times \lambda$ is an invariant measure for $L_{0,p}$, where m_p is the $(p, 1-p)$-Bernoulli measure on $\{0,1\}^{\mathbb{N}}$ and λ is the one-dimensional Lebesgue measure. General results give the existence of an invariant measure of the form $m_p \times \mu_{p,c}$ where $\mu_{p,c} \ll \lambda$. In most cases the random systems $L_{c,p}$ satisfy the conditions from Theorem 3.4.1, which gives an expression for the density $\frac{\mathrm{d}\,\mu_{p,c}}{\mathrm{d}\,\lambda}$ in individual cases. In the last section we discuss some values of c for which we can determine a nice formula for this density. We then get a result similar to Theorem 2(ii) and compute the frequency of the digits d in the generalised Lüroth expansions.

The chapter is organised as follows. In Section 2.2 we describe how to obtain generalised Lüroth expansions from the random maps $L_{c,p}$ and we characterise numbers with ultimately periodic expansions. Here we prove Theorem 1. Section 2.3 is dedicated to the case $c = 0$. Theorem 2(i) is proved in Proposition 2.3.1, part (ii) is covered by Proposition 2.3.3 and part (iii) is done in Proposition 2.3.4. In Section 2.4 we focus on $c > 0$. Theorem 3 corresponds to the content of Theorem 2.4.3 and Theorem 2.4.7. Proposition 2.4.10 contains the result that $L_{c,p}$ admits a Markov partition for any $c \in \left(0, \frac{1}{2}\right] \cap \mathbb{Q}$ and is followed by various examples in which we explicitly compute the density of the measure $\mu_{c,p}$ and in some cases also the typical speed of convergence of the convergents $\frac{p_n}{q_n}$ as well as the frequency of the digits. The last section stresses the need for computable procedures for obtaining the densities for absolutely continuous invariant measures of random interval maps and reveals links with all the remaining chapters of the dissertation.

§2.2 Random c-Lüroth transformations

In this section we introduce the family $\{L_{c,p}\}_{c\in[0,\frac{1}{2}],0\le p\le 1}$ of random (c,p)-Lüroth transformations and show how these maps produce generalised Lüroth expansions for all $x \in [c, 1]$. Since the probability p does not play a role in this section, we drop the subscript for now and refer to the map L_c as the random c-Lüroth transformation instead. First recall some notation for sequences. Let $\mathcal{A}$ be an at most countable set of symbols, called *alphabet*. Let $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ denote the left shift on sequences, so for a sequence $a = (a_i)_{i\ge1} \in \mathcal{A}^{\mathbb{N}}$ we have $\sigma(a) = a'$, where $a'_i = a_{i+1}$ for all i. (With slight abuse of notation we will always use σ to denote the left shift on sequences, regardless of the underlying alphabet.) For a finite string $a = a_1 \dots a_k \in \mathcal{A}^k$ and $1 \le n \le k$ or an infinite sequence $\mathbf{a} = (a_n)_{n\ge1} \in \mathcal{A}^{\mathbb{N}}$ and $n \ge 1$ we denote by a_1^n the initial part $a_1^n = a_1 \cdots a_n$. We call a sequence $\mathbf{a} = (a_i)_{i\ge1}$ *ultimately periodic* if there exist an $n \ge 0$ and an $r \ge 1$ such that $a_{n+j} = a_{n+r+j}$ for all $j \ge 1$ and *periodic* if $n = 0$. Finally, we denote cylinder sets in $\mathcal{A}^{\mathbb{N}}$ using square brackets, i.e.,

$$[b_1, \dots, b_k] = \{\mathbf{a} \in \mathcal{A}^{\mathbb{N}} : a_j = b_j, \text{ for all } 1 \le j \le k\}.$$

For $c \in [0, \frac{1}{2}]$ and $n \geq 1$ let $z_n = \frac{1}{n}$,

$$z_n^+ := z_n + cz_nz_{n-1} \quad \text{and} \quad z_n^- := z_n - cz_nz_{n+1}. \tag{2.6}$$

Then $z_n \leq z_n^+ \leq z_{n-1}^- \leq z_{n-1}$. Define

$$T_{0,c}(x) = \begin{cases} T_A(x), & \text{if } x \in \{0\} \cup \bigcup_{n=2}^{\infty} [z_n, z_n^+), \\ T_L(x), & \text{if } x \in \{1\} \cup \bigcup_{n=2}^{\infty} [z_n^+, z_{n-1}), \end{cases}$$

and

$$T_{1,c}(x) = \begin{cases} T_A(x), & \text{if } x \in \{0\} \cup \bigcup_{n=2}^{\infty} [z_n, z_{n-1}^-], \\ T_L(x), & \text{if } x \in \{1\} \cup \bigcup_{n=2}^{\infty} (z_{n-1}^-, z_{n-1}). \end{cases}$$

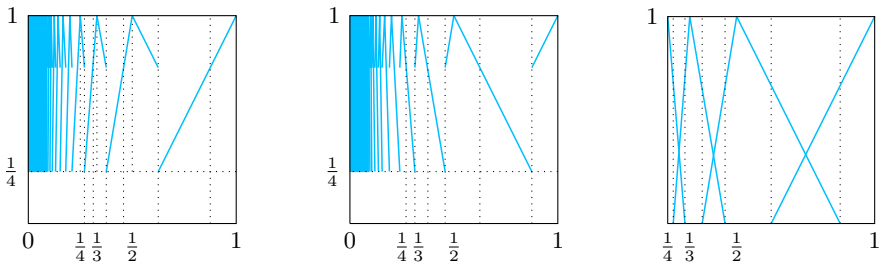
As can be seen from Figure 2.2, the interval $[c, 1]$ is an attractor for the dynamics of both $T_{0,c}$ and $T_{1,c}$, so we choose $[c, 1]$ as their domains. Note that $T_{0,c}$ and $T_{1,c}$ assume the same values on the intervals $[z_n, z_n^+)$ and $(z_n^-, z_n]$ and differ on $[z_n^+, z_{n-1}^-]$ for $n > 1$. We denote the union of the subintervals on which $T_{0,c}$ and $T_{1,c}$ assume different values by S, i.e.,

$$S = [c, 1] \cap \bigcup_{n>1} [z_n^+, z_{n-1}^-], \tag{2.7}$$

and call this the *switch region*. For $c = 0$ we see that $S = (0, 1] \setminus \{z_n : n \geq 1\}$ and $T_{0,0} = T_L$ and $T_{1,0} = T_A$ except for the points z_n where $T_{0,0}(z_n) = T_{1,0}(z_n) = 1$. We combine these two maps to make a random dynamical system by defining the *random c-Lüroth transformation* $L_c : \{0, 1\}^{\mathbb{N}} \times [c, 1] \to \{0, 1\}^{\mathbb{N}} \times [c, 1]$ by

$$L_c(\omega, x) = (\sigma(\omega), T_{\omega_1,c}(x)).$$

See Figure 2.2(c), for an example with $c = \frac{1}{4}$.



(a) $T_{0,\frac{1}{4}}$ (b) $T_{1,\frac{1}{4}}$ (c) $L_{\frac{1}{4}}$

Figure 2.2: The maps $T_{0,\frac{1}{4}}$, $T_{1,\frac{1}{4}}$, and the random c-Lüroth map $L_{\frac{1}{4}}$ on $\left[\frac{1}{4}, 1\right]$.

To denote the orbit along a specific path $\omega \in \{0,1\}^{\mathbb{N}}$ we use the following notation. For a finite string $\omega \in \{0,1\}^k$ and $0 \le n \le k$, let

$$T_{\omega,c} = T_{\omega_k,c} \circ T_{\omega_{k-1},c} \circ \cdots \circ T_{\omega_1,c} \quad \text{and} \quad T^n_{\omega,c} = T_{\omega^n_1,c} = T_{\omega_n,c} \circ T_{\omega_{n-1},c} \circ \cdots \circ T_{\omega_1,c}. \tag{2.8}$$

Similarly, for an infinite path $\omega \in \{0,1\}^{\mathbb{N}}$ and $n \ge 1$ we let

$$T^n_{\omega,c} = T_{\omega^n_1,c} = T_{\omega_n,c} \circ T_{\omega_{n-1},c} \circ \cdots \circ T_{\omega_1,c}. \tag{2.9}$$

For any ω we also set $T^0_{\omega,c} = T_{\omega^0_1,c} = id$. Note that we have defined $T_{0,c}$ and $T_{1,c}$ in such a way that $T_{j,c}(x) \neq 0$ for any j, c, x. With this notation we can obtain *c-Lüroth expansions* of points in $[c,1]$ by defining for each $(\omega, x) \in \{0,1\}^{\mathbb{N}} \times [c,1]$ two sequences $(s_i)_{i\ge1}$ (the signs) and $(d_i)_{i\ge1}$ (the digits) as follows. For $(\omega, x) \in \{0,1\}^{\mathbb{N}} \times [c,1]$ set for $c > 0$ and for $i \ge 1$,

$$s_i = s_i(\omega, x) = \begin{cases} 0, & \text{if } L^{i-1}_c(\omega, x) \in [0] \times S \cup \{0,1\}^{\mathbb{N}} \times \big(\cup_n \, (z^-_n, z_n) \cup \{1\} \big), \\ 1, & \text{if } L^{i-1}_c(\omega, x) \in [1] \times S \cup \{0,1\}^{\mathbb{N}} \times \big(\cup_n \, [z_n, z^+_n) \cup \{0\} \big). \end{cases}$$

For $c = 0$, set $s_i = \omega_i$ for each i. For $c \ge 0$, $x \neq 0$ and for $i \ge 1$ set

$$d_i = d_i(\omega, x) = \begin{cases} 2, & \text{if } T^{i-1}_{\omega,c}(x) = 1, \\ n, & \text{if } T^{i-1}_{\omega,c}(x) \in [z_n, z_{n-1}),\ n \ge 2. \end{cases}$$

Then one can write for $x \neq 0$ and each $i \ge 1$ that

$$T^i_{\omega,c}(x) = (-1)^{s_i} d_i(d_i - 1) T^{i-1}_{\omega,c}(x) + (-1)^{s_i+1}(d_i - 1 + s_i).$$

By inversion and iteration we obtain what we call the *c-Lüroth expansion* of (ω, x):

$$x = \sum_{n=1}^{\infty} (-1)^{\sum_{i=1}^{n-1} s_i(\omega,x)} \frac{d_n(\omega,x) - 1 + s_n(\omega,x)}{\prod_{i=1}^{n} d_i(\omega,x)(d_i(\omega,x) - 1)}, \tag{2.10}$$

where $\sum_{i=1}^{0} s_i(\omega, x) = 0$ and the sum converges since $d_i \ge 2$ and $T^i_{\omega,c}(x) \in (0,1]$ for all ω, x, i. Note that this expansion of x is of the form (2.5), i.e., it is a generalised Lüroth expansion.

2.2.1 Remark. (i) Due to the fact that $T_{j,c}(x) \neq 0$ for all j, c and $x \neq 0$, the maps L_c do not produce finite generalised Lüroth expansions. That is, L_c assigns to each (ω, x) an infinite sequence $((s_n(\omega,x), d_n(\omega,x)))_n$ with $s_n \in \{0,1\}$ and $d_n \ge 2$.

(ii) As an immediate consequence of the above, we see that for each $D \ge 2$ *every* $x \in \left[\frac{1}{D}, 1\right]$ has a generalised Lüroth expansion that only uses digits $d_n \le D$ and that is generated by any random c-Lüroth system with $c \ge \frac{1}{D}$. This is in contrast to the deterministic case, where by the result of Šalát in [S68] for any D the set of points x that has D as an upper bound for the Lüroth digits has Hausdorff dimension strictly less than 1.

(iii) If $x \in S$ and $x \notin \left\{ \frac{2n-1}{2n(n-1)} \,:\, n \ge 1 \right\}$ (so $T_L(x) \neq T_A(x)$), then $d_2(\omega, x) = 2$ for all ω with $T_{\omega_1}(x) > T_{1-\omega_1}(x)$ and $d_2(\omega, x) > 2$ otherwise. For $c = 0$ this implies,

since $(s_n)_{n\geq 1} = (\omega_n)_{n\geq 1}$ and $S = (0,1] \setminus \{\frac{1}{n} : n \geq 1\}$, that Lebesgue almost all $x \in [0,1]$ have uncountably many different random Lüroth expansions. In Section 2.4 we see that a similar statement holds for $c > 0$ and we get that most points even have uncountably many different generalised Lüroth expansions with all $d_n \leq D$.

Similar to the deterministic case, we call a c-Lüroth expansion of (ω, x) *(ultimately) periodic* if the corresponding sequence $((s_i, d_i))_{i\geq 1}$ is (ultimately) periodic. From the expression (2.10) it is clear that the c-Lüroth expansion of (ω, x) is ultimately periodic if and only if there are $n \geq 0$ and $r \geq 1$ such that

$$T_{\omega,c}^{n+j}(x) = T_{\omega,c}^{n+r+j}(x),$$

for all $j \geq 0$, implying that $x \in \mathbb{Q}$. We define the following weaker notion.

2.2.2 Definition (Returning points). Let $c \in \left[0, \frac{1}{2}\right]$. We call a number $x \in [c,1]$ *returning* for L_c if for every $\omega \in \{0,1\}^{\mathbb{N}}$ there exist an $n = n(\omega) \geq 0$ and an $r = r(\omega) \geq 1$ such that $T_{\omega,c}^{n}(x) = T_{\omega,c}^{n+r}(x)$.

2.2.3 Lemma. *Let $c \in \left[0, \frac{1}{2}\right]$. If $x \in \mathbb{Q} \cap [c,1]$, then x is a returning point for L_c. Hence, the set of returning points is dense in $[c,1]$.*

Proof. Let $\frac{N}{Q}$ be the reduced rational representing x, for $N, Q \in \mathbb{N}$. Then for any $\omega \in \{0,1\}^{\mathbb{N}}$ and any $t \in \mathbb{N}$ there exist $a, b \in \mathbb{Z}$ such that $T_{\omega,c}^{t}(x) = a\frac{N}{Q} + b \in [c,1]$. This implies that $aN + bQ \in \{0, 1, \ldots, Q\}$. It then follows by Dirichlet's Box Principle that for some $n \in \mathbb{N}$ there exists an $r \geq 1$ such that $T_{\omega,c}^{n}(x) = T_{\omega,c}^{n+r}(x)$. □

Contrarily to the deterministic case, the fact that there are n, r with $T_{\omega,c}^{n}(x) = T_{\omega,c}^{n+r}(x)$ does not necessarily imply that $T_{\omega,c}^{n+j}(x) = T_{\omega,c}^{n+r+j}(x)$ for all $j \geq 0$, since one can make a different choice when arriving at a point in S for the second time. To characterise the ultimately periodic c-Lüroth expansions we define *loops*.

2.2.4 Definition (Loop). A string $\mathbf{u}$ of symbols in $\{0,1\}$ is called a *loop* for $x \in [c,1]$ at the point $y \in [c,1]$ if there exist $\omega \in \{0,1\}^{\mathbb{N}}$ and $n = n(\omega) \geq 0$, $r = r(\omega) \geq 1$ such that

$$\omega_{n+1}^{n+r} = \mathbf{u}, \quad T_{\omega,c}^{n}(x) = y = T_{\omega,c}^{n+r}(x) \quad \text{and} \quad T_{\omega,c}^{n+j}(x) \neq y \quad \text{for } 1 \leq j < r. \tag{2.11}$$

We say that x admits the loop $\mathbf{u}$ at y.

For each $x, y \in [c,1]$ we define an equivalence relation on the collection of loops $\{\mathbf{u}\}$ of x at y by setting $\mathbf{u}_1 \sim \mathbf{u}_2$ if the corresponding paths $\omega_1, \omega_2 \in \{0,1\}^{\mathbb{N}}$ satisfying (2.11) both assign the same strings of signs and digits, i.e., if

$$(s(\omega_1, x), d(\omega_1, x))_{n(\omega_1)+1}^{n(\omega_1)+r} = (s(\omega_2, x), d(\omega_2, x))_{n(\omega_2)+1}^{n(\omega_2)+r}, \tag{2.12}$$

where $r = r(\omega_1) = r(\omega_2)$. We need this definition since for $x \in [c,1] \setminus S$, $T_{\omega,c}(x)$ is independent of the choice of $\omega \in \{0,1\}$, i.e., $T_{0,c}(x) = T_{1,c}(x)$ and the corresponding sign and digit only depend on the position of x, and not on ω. As a consequence, it is necessary that $T_{\omega,c}^{n}(x) \in S$ for some $\omega \in \{0,1\}^{\mathbb{N}}$ and $n \geq 0$, to have more than one loop (that is, more than one equivalence class).

2.2.5 Proposition. *Let $c \in \left[0, \frac{1}{2}\right]$ and $x \in \mathbb{Q} \cap [c, 1]$.*

(i) Suppose that there exists an $\omega \in \{0,1\}^{\mathbb{N}}$ such that $T^n_{\omega,c}(x) \notin S$ for all $n \geq 0$. Then x has a unique and ultimately periodic c-Lüroth expansion.

(ii) Suppose that for each $y \in [c, 1]$ the point x admits at most one loop at y. Then all c-Lüroth expansions of x are ultimately periodic, so there are at most countably many of them.

(iii) Suppose there is a $y \in [c, 1]$, such that x admits at least two loops $\mathbf{u}_1 \nsim \mathbf{u}_2$ at y. Then x has uncountably many c-Lüroth expansions that are not ultimately periodic, and countably many ultimately periodic c-Lüroth expansions.

Proof. For (i) note that if there exists an $\omega \in \{0,1\}^{\mathbb{N}}$ such that $T^n_{\omega,c}(x) \notin S$ for all $n \geq 0$, then $T^n_{\omega,c}(x)$ is independent of the choice of ω for any $n \geq 0$ and any path $\omega \in \{0,1\}^{\mathbb{N}}$ yields the same c-Lüroth expansion. The result then follows from Lemma 2.2.3.

For (ii) suppose by contradiction that there exists an $\omega \in \{0,1\}^{\mathbb{N}}$ such that (ω, x) presents a c-Lüroth expansion that is not ultimately periodic. Since $x \in \mathbb{Q}$, the set $\{T^n_{\omega,c}(x)\}_{n\geq 0}$ consists of finitely many points, and so, in particular, there exists a point $y = T^j_{\omega,c}(x)$ for some $j \geq 0$, that is visited infinitely often. Let $\{j_i\}_{i\in\mathbb{N}}$ be the sequence such that $T^{j_i}_{\omega,c} = y$ for every j_i. Since (ω, x) does not have an ultimately periodic expansion, there exists a k such that

$$(s(\omega,x), d(\omega,x))_{j_{k-1}+1}^{j_k} \neq (s(\omega,x), d(\omega,x))_{j_k+1}^{j_{k+1}},$$

which means in particular that the loops $\omega_{j_{k-1}+1}^{j_k}$ and $\omega_{j_k+1}^{j_{k+1}}$ are not in the same equivalence class, contradicting the assumption on the number of admissible loops at y. The second part follows since the sequence $((s_n(\omega,x), d_n(\omega,x)))_n$ takes its digits in an at most countable alphabet.

For (iii) let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two loops of x at y with $\mathbf{u}_1 \nsim \mathbf{u}_2$. Consider $\omega \in \{0,1\}^{\mathbb{N}}$ satisfying (2.11), i.e., such that

$$\omega_{n+1}^{n+r} = \mathbf{u}_1, \quad \text{and} \quad T^n_{\omega,c}(x) = T^{n+r}_{\omega,c}(x) = y,$$

for some $n, r \in \mathbb{N}$. Now take any sequence $(\ell_j)_{j\geq 1} \subseteq \mathbb{N}^{\mathbb{N}}$ and consider the path $\tilde{\omega} \in \{0,1\}^{\mathbb{N}}$ defined by the concatenation

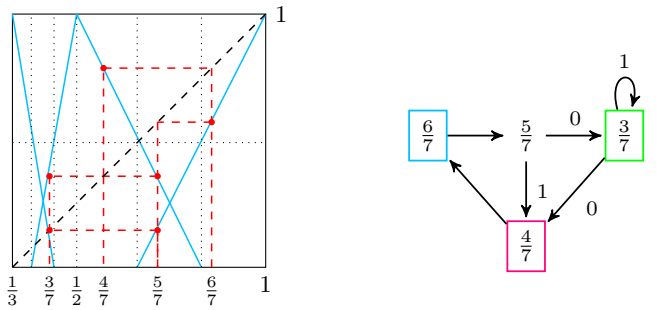
$$\tilde{\omega} = \omega_1^n \mathbf{u}_1^{\ell_1} \mathbf{u}_2^{\ell_2} \mathbf{u}_1^{\ell_3} \mathbf{u}_2^{\ell_4} \mathbf{u}_1^{\ell_5} \mathbf{u}_2^{\ell_6} \dots \ .$$

If (ℓ_j) is not ultimately periodic, then it is guaranteed by (2.12) that the sequence $(s_i(\tilde{\omega}, x), d_i(\tilde{\omega}, x))_{i\in\mathbb{N}}$ is not ultimately periodic and as a result $(\tilde{\omega}, x)$ presents a c-Lüroth expansion that is not ultimately periodic. Since there are uncountably many such sequences (ℓ_j) each yielding a different corresponding sequence of signs and digits, the first part of the statement follows. Taking any ultimately periodic sequence (ℓ_j) instead will yield an ultimately periodic c-Lüroth expansion. □

2.2.6 Proposition. *Let $c \in \left[0, \frac{1}{2}\right]$. If $x \in [c,1] \setminus \mathbb{Q}$, then any c-Lüroth expansion of x is infinite and not ultimately periodic.*

Proof. Let $x \in [c,1]\setminus\mathbb{Q}$ be given and assume that there exists an $\omega \in \{0,1\}^{\mathbb{N}}$ such that the corresponding c-Lüroth expansion of (ω, x) is ultimately periodic. Then there is an $n \geq 0$ and an $r \geq 1$ such that $T^n_{\omega,c}(x) = T^{n+r}_{\omega,c}(x)$. Hence, there are $a, b, c, d \in \mathbb{Z}$ such that $ax + b = cx + d$, implying that $x \in \mathbb{Q}$, contradicting the choice of x. □

2.2.7 Example. To illustrate Proposition 2.2.5 we give an example of the various possibilities for periodicity of expansions of rational numbers. Let $c = \frac{1}{3}$ and first consider the rational number $\frac{6}{7}$. See Figure 2.3(a) for the random map $L_{\frac{1}{3}}$ with the possible orbits of $\frac{6}{7}$. Figure 2.3(b) is a visualisation of the random orbits of $\frac{6}{7}$. We explicitly identify paths $\omega \in \{0,1\}^{\mathbb{N}}$ that produce c-Lüroth expansions of $\frac{6}{7}$ that are periodic, ultimately periodic and not ultimately periodic and list them in Table 2.1.



(a) (b)

Figure 2.3: The random Lüroth map L_c for $c = \frac{1}{3}$ with the random orbits of $\frac{6}{7}$ in red in (a) and another visualisation of the orbits of $\frac{6}{7}$ in (b). The digits with the arrows indicate which one of the maps $T_{0,c}$ or $T_{1,c}$ is applied. If there is no digit, then both maps yield the same orbit point.

ω	**Expansion of** $\frac{6}{7}$
$(011)^\infty$	$((0,2),(1,2)^2)^\infty$ is periodic
001^∞	$((0,2)^2,(1,3)^\infty)$ is ultimately periodic
$0^2 10^4 1^2 0^4 1^3 0^4 1^4 \ldots$	$((0,2)^2,(1,3),(0,3),(1,2),(0,2)^2,(1,3)^2,(0,3),(1,2),\ldots)$ is not ultimately periodic

Table 2.1: Examples of ω's and the corresponding type of the c-Lüroth expansions.

For the point $\frac{3}{4}$ it holds that for any $\omega \in \{0,1\}^{\mathbb{N}}$, $T^1_{\omega,c}\left(\frac{3}{4}\right) = \frac{1}{2}$ and $T^n_{\omega,c}\left(\frac{3}{4}\right) = 1$ for any $n \geq 2$. Hence, $\frac{3}{4}$ has precisely two c-Lüroth expansions (for any c) that are given

by the sequences

$$((0,2),(1,2),(0,2)^\infty) \quad \text{and} \quad ((1,2)^2,(0,2)^\infty).$$

Hence in case (ii) of Proposition 2.2.5 there are rational x that have only a finite number of ultimately periodic expansions and we cannot improve on the statement without giving a further description of the specific positions of the random orbit points.

2.2.8 Remark. Lemma 2.2.3 gives a further bound on the number of admissible digits $(d_n)_{n\in\mathbb{N}}$ in the c-Lüroth expansions of a rational number x. More precisely, if $c = 0$, any c-Lüroth expansion of $x = \frac{N}{Q} \in \mathbb{Q}\cap[0,1]$ has at most $Q+1$ different digits. Differently, by Proposition 2.2.6, for irrational numbers the set of admissible digits is $\mathbb{N}_{\geq 2}$. Note that for $\frac{1}{\ell+1} \leq c < \frac{1}{\ell}$, the bound is given by the minimum between the previous quantities and ℓ.

The proof of Theorem 1 is now given by Proposition 2.2.5 and Proposition 2.2.6.

§2.3 Approximations of irrationals

The approximation properties of generalised c-Lüroth expansions can be studied via the dynamical properties of the associated random system. For this one needs to have an accurate description of an invariant measure for the random system. Let $0 < p < 1$. The vector $(p, 1-p)$ represents the probabilities with which we apply the maps $T_{0,c}$ and $T_{1,c}$ respectively. One easily checks that the probability measure $m_p \times \lambda$, where m_p is the $(p,1-p)$-Bernoulli measure on $\{0,1\}^{\mathbb{N}}$ and λ is the one-dimensional Lebesgue measure on $[0,1]$ is invariant and ergodic for L_0. Therefore, in this section we focus on $c = 0$, fix a p and drop the subscripts c, p, so we write $L = L_{0,p}$. We first prove a result on the number of different generalised Lüroth expansions that L produces for Lebesgue almost all $x \in [0,1]$ and then investigate two ways of quantifying the approximation properties of all these expansions.

§2.3.1 Universal generalised Lüroth expansions

The random dynamical system $L = L_{0,p}$ is capable of producing for each number $x \in [0,1]$ essentially all expansions generated by all the members of the family of GLS transformations studied in [BBDK94], i.e., GLS transformations with *standard Lüroth partition*, given by $\mathcal{P}_L = \left\{\left[\frac{1}{n}, \frac{1}{n-1}\right)\right\}_{n\geq 2}$. In the previous section we mentioned that Lebesgue almost every x has uncountably many different 0-Lüroth expansions and thus uncountably many different generalised Lüroth expansions. Here we prove an even stronger statement.

Let $\mathcal{A} = \{(s,d) : s \in \{0,1\}, d \in \mathbb{N}_{\geq 2}\}$ be the alphabet of possible digits for generalised Lüroth expansions and for any $(s,d) \in \mathcal{A}$, set

$$\bar{\Delta}(s,d) = [s] \times \left[\frac{1}{d}, \frac{1}{d-1}\right],$$

for the cylinder set $[s] \subseteq \{0,1\}^{\mathbb{N}}$ and the interval $\left[\frac{1}{d}, \frac{1}{d-1}\right] \subseteq [0,1]$. Then $m_p \times \lambda(\bar{\Delta}(s,d)) = \frac{(-1)^s(p-s)}{d(d-1)} > 0$. Take any sequence $((s_n,d_n))_n \in \mathcal{A}^{\mathbb{N}}$ that does not end in $((0,d+1),(0,2)^\infty)$ for some $d \geq 2$. The set

$$\lim_{n\to\infty} \bar{\Delta}(s_1,d_1) \cap L^{-1}\bar{\Delta}(s_2,d_2) \cap \cdots \cap L^{-(n-1)}\bar{\Delta}(s_n,d_n) \subseteq \{0,1\}^{\mathbb{N}} \times [0,1] \tag{2.13}$$

is non-empty as a countable intersection of closed sets. Moreover,

$$\lim_{n\to\infty} m_p \times \lambda\big(\bar{\Delta}(s_1,d_1) \cap L^{-1}\bar{\Delta}(s_2,d_2) \cap \cdots \cap L^{-(n-1)}\bar{\Delta}(s_n,d_n)\big) = \lim_{n\to\infty} \frac{p^{n-\sum_{i=1}^n s_i}(1-p)^{\sum_{i=1}^n s_i}}{\prod_{i=1}^n d_i(d_i-1)} = 0,$$

so the set from (2.13) consists of precisely one point, call it (ω, x). Then $\omega = (s_n)_n \in \{0,1\}^{\mathbb{N}}$ and by the assumption that $((s_n,d_n))_n$ does not end in $((0,d+1),(0,2)^\infty)$ it follows that $s_n(\omega,x) = s_n$ and $d_n(\omega,x) = d_n$ for all $n \geq 1$, so that

$$x = \sum_{n=1}^{\infty} (-1)^{\sum_{i=1}^{n-1} s_i} \frac{d_n - 1 + s_n}{\prod_{i=1}^n d_i(d_i-1)} \in [0,1].$$

Note that if there is a $k \geq 1$, such that $(s_k,d_k) = (0,d+1)$ and $(s_n,d_n) = (0,2)$ for all $n \geq k+1$, then

$$\begin{aligned}
\sum_{n=1}^{\infty} (-1)^{\sum_{i=1}^{n-1} s_i} \frac{d_n - 1 + s_n}{\prod_{i=1}^n d_i(d_i-1)} &= \sum_{n=1}^{k-1} (-1)^{\sum_{i=1}^{n-1} s_i} \frac{d_n - 1 + s_n}{\prod_{i=1}^n d_i(d_i-1)} \\
&\quad + \frac{(-1)^{\sum_{i=1}^{k-1} s_i}}{\prod_{i=1}^{k-1} d_i(d_i-1)} \Big(\frac{d}{d+1} + \frac{1}{(d+1)d} \sum_{j\geq 1} \frac{1}{\prod_{i=1}^j 2 \cdot 1} \Big) \\
&= \sum_{n=1}^{k-1} (-1)^{\sum_{i=1}^{n-1} s_i} \frac{d_n - 1 + s_n}{\prod_{i=1}^n d_i(d_i-1)} + \frac{(-1)^{\sum_{i=1}^{k-1} s_i}}{d \prod_{i=1}^{k-1} d_i(d_i-1)} \\
&= \sum_{n=1}^{k-1} (-1)^{\sum_{i=1}^{n-1} s_i} \frac{d_n - 1 + s_n}{\prod_{i=1}^n d_i(d_i-1)} \\
&\quad + \frac{(-1)^{\sum_{i=1}^{k-1} s_i}}{\prod_{i=1}^{k-1} d_i(d_i-1)} \Big(\frac{d}{d-1} - \frac{1}{d(d-1)} \sum_{j\geq 1} \frac{1}{\prod_{i=1}^j 2 \cdot 1} \Big).
\end{aligned}$$

In other words, the sequences

$$\psi\big(((0,d+1),(0,2)^\infty)\big) = \psi\big(((1,d),(0,2)^\infty)\big)$$

correspond to the same number x. Therefore, the map $\psi : \mathcal{A}^{\mathbb{N}} \to (0,1]$ given by

$$\psi\big(((s_n,d_n))_{n\geq 1}\big) = \sum_{n=1}^{\infty} (-1)^{\sum_{i=1}^{n-1} s_i} \frac{d_n - 1 + s_n}{\prod_{i=1}^n d_i(d_i-1)} \tag{2.14}$$

is well defined and surjective and by the way we defined the maps $T_{0,0}$ and $T_{1,0}$ on any of the points z_n we get that any sequence $((s_n,d_n))_n \in \mathcal{A}^{\mathbb{N}}$ that does not end

in $((0,d+1),(0,2)^\infty)$ corresponds to a 0-Lüroth expansion of a number $x \in (0,1]$. We call a sequence $((s_n,d_n))_n \in \mathcal{A}^{\mathbb{N}}$ with $\psi\big(((s_n,d_n))_{n\geq 1}\big) = x$ a *universal* 0-Lüroth expansion of x if every finite block $(t_1,b_1),\dots,(t_j,b_j) \in \mathcal{A}^j$ occurs in the sequence $((s_n,d_n))_n$. Note that the sequences ending in $((0,d+1),(0,2)^\infty)$ and $((1,d),(0,2)^\infty)$ can not be universal.

2.3.1 Proposition. *Lebesgue almost all $x \in [0,1]$ have uncountably many universal 0-Lüroth expansions.*

Proof. Define the set

$$N = \{x \in (0,1] \,:\, \exists\, \omega \in \{0,1\}^{\mathbb{N}},\, k \geq 0,\, n \geq 2,\; T_\omega^k(x) = z_n\}.$$

Then by Theorem 1, $N \subseteq \mathbb{Q}$, so $m_p \times \lambda(\{0,1\}^{\mathbb{N}} \times N) = 0$. For any $(i,j) \in \mathcal{A}$ set

$$\Delta(s,d) = [s] \times \Big(\frac{1}{d}, \frac{1}{d-1}\Big).$$

For any $(\omega,x) \in \{0,1\}^{\mathbb{N}} \times (0,1] \setminus N$ the block $(t_1,b_1),\dots,(t_j,b_j)$ occurs in position $k \geq 1$ of the sequence $((s_n(\omega,x),d_n(\omega,x)))_n$ precisely if

$$L^{k-1}(\omega,x) \in \Delta(t_1,b_1) \cap L^{-1}\Delta(t_2,b_2) \cap \cdots \cap L^{-(j-1)}\Delta(t_j,b_j).$$

Since $m_p \times \lambda\big(\Delta(t_1,b_1) \cap L^{-1}\Delta(t_2,b_2) \cap \cdots \cap L^{-(j-1)}\Delta(t_j,b_j)\big) > 0$ it follows from the ergodicity of L that $m_p \times \lambda$-a.e. (ω,x) eventually enters this set, so the set of points (ω,x) for which the 0-Lüroth expansion does not contain $(t_1,b_1),\dots,(t_j,b_j)$ is a $m_p \times \lambda$-null set. There are only countably many different blocks $(t_1,b_1),\dots,(t_j,b_j)$, so the set of points (ω,x) that have a non-universal 0-Lüroth expansion also has measure 0. From Fubini's Theorem we get the existence of a set $B \subseteq (0,1] \setminus N$ with $\lambda(B) = 1$ with the property that for each $x \in B$ a set $A_x \subseteq \{0,1\}^{\mathbb{N}}$ exists with $m_p(A_x) = 1$ and such that for any $(\omega,x) \in A_x \times \{x\}$ the sequence $(s_n(\omega,x),d_n(\omega,x))_n$ is a universal 0-Lüroth expansion of x. The set A_x has full measure, so contains uncountably many ω's. Let $x \in B$ and $\omega,\tilde{\omega} \in A_x$. Then $x \notin N$ and thus if $\omega_n \neq \tilde{\omega}_n$ for some $n \geq 1$, then $s_n(\omega,x) \neq s_n(\tilde{\omega},x)$. Hence, the sequences $\omega \in A_x$ all give different universal 0-Lüroth expansions for x. □

Note that if x has a universal 0-Lüroth expansion, then $x \notin \big\{\frac{2n-1}{2n(n-1)} \,:\, n \geq 1\big\}$. By Remark 2.2.1(iii) the fact that $s_n(\omega,x) \neq s_n(\tilde{\omega},x)$ for some n then implies that $d_{n+1}(\omega,x) \neq d_{n+1}(\tilde{\omega},x)$.

§2.3.2 Speed of convergence

Recall that for each $n \geq 1$ the n-th convergent $\frac{p_n}{q_n}(\omega,x)$ of (ω,x) is given by

$$\frac{p_n}{q_n} = \frac{p_n}{q_n}(\omega,x) = \sum_{k=1}^{n}(-1)^{\sum_{i=1}^{n-1}\omega_i} \prod_{i=1}^{k} \frac{d_k - 1 + \omega_k}{d_i(d_i-1)},$$

so that the study of the quantity $\left|x - \frac{p_n}{q_n}\right|$ provides information on the quality of the approximations of x obtained from L. In the following, we take two different perspectives. In this section we compute the pointwise Lyapunov exponent Λ and in the next section we consider the approximation coefficients $\theta_n = \theta_n^{0,p}$ similar to the ones defined in (2.3).

Let I be an interval of the real line, $\Omega \subseteq \mathbb{N}$ and $\pi : \Omega^{\mathbb{N}} \times I \to I$ the canonical projection onto the second coordinate. The following definition can be found in [GBL97, B13], for example.

2.3.2 Definition (Lyapunov exponent). For any random interval map $R : \Omega^{\mathbb{N}} \times I \to \Omega^{\mathbb{N}} \times I$ and for any point $(\omega, x) \in \Omega^{\mathbb{N}} \times I$, the *pointwise Lyapunov exponent* is defined as

$$\Lambda(\omega, x) := \lim_{n\to\infty} \frac{1}{n} \log\left|\frac{\mathrm{d}}{\mathrm{d}\, x}\pi(R^n(\omega, x))\right|, \tag{2.15}$$

whenever the limit exists.

We use this to determine the speed of convergence of the sequence $\left(\frac{p_n}{q_n}(\omega, x)\right)_n$ to a number $x \in [0, 1]$ for which $T_\omega^n(x) \neq 0$ for all $n \geq 0$. By Proposition 2.2.5 and Proposition 2.2.6 this includes all irrational x and part of the rationals. For any such x and each sequence $(s_n(\omega, x), d_n(\omega, x))_{n\geq 1} \in \mathcal{A}^{\mathbb{N}}$ of signs and digits for a point (ω, x) the rational $\frac{p_n}{q_n}(\omega, x)$ is one of the endpoints of the projection onto the unit interval of the cylinder $[(s_1, d_1), \ldots, (s_n, d_n)]$, i.e., one of the endpoints of the interval

$$\psi([(s_1, d_1), \ldots, (s_n, d_n)]).$$

Since the map T_ω^n is surjective and linear on this interval and $\frac{\mathrm{d}}{\mathrm{d}\, x} T_\omega^n(x) = \prod_{k=1}^n d_k(d_k - 1)$, the Lebesgue measure of this interval is $\left(\prod_{k=1}^n d_k(d_k - 1)\right)^{-1}$. The following result compares to Theorem 2.1.2 from [BI09].

2.3.3 Proposition. *For any $(\omega, x) \in \{0, 1\}^{\mathbb{N}} \times [0, 1]$ with $T_\omega^n(x) \neq 0$ for all $n \geq 0$ the speed of approximation of the random Lüroth map L is given by*

$$\left|x - \frac{p_n}{q_n}(\omega, x)\right| \asymp \exp(-n\Lambda(\omega, x)).$$

In particular, for $m_p \times \lambda$-a.e. (ω, x)

$$\Lambda(\omega, x) = \sum_{d=2}^{\infty} \frac{\log(d(d-1))}{d(d-1)} \cong 1.98329\ldots\ .$$

Furthermore, the map $\Lambda : \{0, 1\}^{\mathbb{N}} \times [0, 1] \to [\log 2, \infty)$ is onto.

Proof. Let (ω, x) be such that $T_\omega^n(x) \neq 0$ for each n. Write $s_n = s_n(\omega, x)$ and $d_n = d_n(\omega, x)$ for each $n \geq 1$. Since $\frac{p_n}{q_n}(\omega, x)$ is one of the endpoints of the interval $\psi([(s_1, d_1), \ldots, (s_n, d_n)])$ it follows that

$$\left|x - \frac{p_n}{q_n}(\omega, x)\right| \leq \prod_{k=1}^{n} \frac{1}{d_k(\omega, x)(d_k(\omega, x) - 1)}.$$

We claim that

$$\left| x - \frac{p_n}{q_n}(\omega, x) \right| \geq \prod_{k=1}^{n+1} \frac{1}{d_k(\omega, x)(d_k(\omega, x) - 1)}. \tag{2.16}$$

To see this, note that for $n = 1$ we have

$$\frac{1}{d_2} \leq T_\omega^1(x) = (-1)^{s_1} d_1(d_1 - 1)x + (-1)^{s_1+1}(d_1 - 1 + s_1) \leq \frac{1}{d_2 - 1}.$$

Hence,

$$\frac{1}{d_1(d_1 - 1)d_2} \leq (-1)^{s_1}\Big(x - \frac{d_1 - 1 + s_1}{d_1(d_1 - 1)}\Big) \leq \frac{1}{d_1(d_1 - 1)(d_2 - 1)}.$$

Since $\frac{p_1}{q_1} = \frac{d_1 - 1 + s_1}{d_1(d_1 - 1)}$ it follows that

$$\left| x - \frac{p_1}{q_1} \right| \geq \frac{1}{d_1(d_1 - 1)d_2(d_2 - 1)}.$$

In the same way, from

$$\frac{1}{d_{n+1}} \leq T_\omega^n(x) = (-1)^{s_n} d_n(d_n - 1)T_\omega^{n-1}(x) + (-1)^{s_n+1}(d_n - 1 + s_n) \leq \frac{1}{d_{n+1} - 1},$$

we obtain

$$\left| T_\omega^{n-1}(x) - \frac{d_n - 1 + s_n}{d_n(d_n - 1)} \right| \geq \frac{1}{d_n(d_n - 1)d_{n+1}(d_{n+1} - 1)}.$$

By the definition of the convergents, we then have

$$\begin{aligned}
\left| x - \frac{p_n}{q_n}(\omega, x) \right| &= \frac{T_\omega^n(x)}{\prod_{i=1}^n d_i(d_i - 1)} \\
&= \frac{(-1)^{s_n} d_n(d_n - 1)T_\omega^{n-1}(x) + (-1)^{s_n+1}(d_n - 1 + s_n)}{\prod_{k=1}^n d_i(d_i - 1)} \\
&= \left| T_\omega^{n-1}(x) - \frac{(d_n - 1 + s_n)}{d_n(d_n - 1)} \right| \cdot \prod_{k=1}^{n-1} \frac{1}{d_i(d_i - 1)} \\
&\geq \prod_{k=1}^{n+1} \frac{1}{d_i(d_i - 1)}.
\end{aligned}$$

This gives the claim. It follows that

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\log\left|x-\frac{p_n}{q_n}(\omega,x)\right| &= \lim_{n\to\infty}\frac{1}{n}\log\left(\prod_{k=1}^{n}\frac{1}{d_k(\omega,x)(d_k(\omega,x)-1)}\right)\\
&= \lim_{n\to\infty}\frac{1}{n}\log\lambda\big(\psi([(s_1,d_1),\dots,(s_n,d_n)])\big)\\
&= \lim_{n\to\infty}\frac{1}{n}\log\prod_{k=1}^{n}\lambda\big(\psi([s_k,d_k])\big)\\
&= \lim_{n\to\infty}\frac{1}{n}\log\left|\frac{\mathrm{d}}{\mathrm{d}\,x}\pi(L^n(\omega,x))\right|^{-1}\\
&= -\Lambda(\omega,x).
\end{aligned}$$

So Λ measures the asymptotic exponential growth of approximation, i.e.,

$$\left|x-\frac{p_n}{q_n}(\omega,x)\right| \asymp \exp(-n\Lambda(x,\omega)).$$

Recall now that $m_p\times\lambda$ is invariant and ergodic for L. Applying Birkhoff's Ergodic Theorem we get for $(m_p\times\lambda)$-a.e. (ω,x) that

$$\begin{aligned}
\Lambda(\omega,x)=\lim_{n\to\infty}\frac{1}{n}\log\left|\frac{\mathrm{d}}{\mathrm{d}\,x}T_\omega^n(x)\right| &= \lim_{n\to\infty}\frac{1}{n}\log\left|\prod_{k=1}^{n}\frac{\mathrm{d}}{\mathrm{d}\,x}T_{\omega_k}(T_\omega^{k-1}(x))\right|\\
&= \lim_{n\to\infty}\frac{1}{n}\log\left|\prod_{k=0}^{n-1}\frac{\mathrm{d}}{\mathrm{d}\,x}\pi\circ L(\sigma^k(\omega),T_\omega^k(x))\right|\\
&= \int_{\{0,1\}^{\mathbb{N}}\times[0,1]}\log\left|\frac{\mathrm{d}}{\mathrm{d}\,x}\pi(L(\omega,x))\right|d(m_p\times\lambda)(\omega,x)\\
&= \int_{\{0,1\}^{\mathbb{N}}\times[0,1]}\log\left|\frac{\mathrm{d}}{\mathrm{d}\,x}T_\omega(x)\right|d(m_p\times\lambda)(\omega,x)\\
&= \sum_{d=2}^{\infty}\frac{\log(d(d-1))}{d(d-1)} =: \Lambda_{m_p\times\lambda}.
\end{aligned}\tag{2.17}$$

This gives the second part of the proposition. For the last part, it is obvious that $\log 2$ is a lower bound for $\Lambda(\omega,x)$, since $|\frac{\mathrm{d}}{\mathrm{d}\,x}T_j(x)|\geq 2$ for $j=0,1$. The rest follows from Theorem 2.1.2 since the sequence $\omega=(\bar{0})$ reduces L to the standard Lüroth map T_L. □

It follows from (2.17) that the set of points (of zero Lebesgue measure) that present a Lyapunov exponent different from $\Lambda_{m_p\times\lambda}$, includes *fixed points*, i.e., points $(\omega,x)\in\{0,1\}^{\mathbb{N}}\times[0,1]$ for which $T_\omega^k(x)=x$ for all $k\geq 1$. For any such point (ω,x),

$$\prod_{k=1}^{n}\frac{\mathrm{d}}{\mathrm{d}\,x}T_{\omega_{k+1}}(T_\omega^k(x))=\prod_{k=1}^{n}\frac{\mathrm{d}}{\mathrm{d}\,x}T_{\omega_{k+1}}(x)=\left(\frac{\mathrm{d}}{\mathrm{d}\,x}T_\omega(x)\right)^n,$$

for all $n \geq 1$. Hence, if $x \in [z_d, z_{d-1})$ then

$$\begin{aligned}\Lambda(\omega, x) &= \lim_{n\to\infty} \frac{1}{n} \log\left|\prod_{k=1}^{n} \frac{\mathrm{d}}{\mathrm{d}\,x}\pi \circ L(L^k(\omega, x))\right| \\ &= \lim_{n\to\infty} \frac{1}{n} \log|d^n| \\ &= \log d.\end{aligned}$$

Since fixed points can be found in any set $\{0,1\}^{\mathbb{N}} \times [z_d, z_{d-1})$ for $d \geq 2$, it follows that Λ attains arbitrarily large values, starting from $\log 2 \cong 0.69314\ldots$, which corresponds to the Lyapunov exponent of the fixed point given by the sequence $(\overline{(0,2)})$.

§2.3.3 Approximation coefficients

A GLS map $T_{\mathcal{P},\varepsilon} : [0,1] \to [0,1]$ is defined by a partition $\mathcal{P} = \{I_n = (\ell_n, r_n]\}_n$ and a vector $\varepsilon = (\varepsilon_n)_n$. To be specific, the intervals I_n satisfy $\lambda(\bigcup I_n) = 1$ and $\lambda(I_n \cap I_k) = 0$ if $n \neq k$ and on I_n the map $T_{P,s}$ is given by

$$T_{\mathcal{P},\varepsilon}(x) = \varepsilon_n \frac{r_n - x}{\ell_n - r_n} + (1 - \varepsilon_n)\frac{x - \ell_n}{\ell_n - r_n},$$

and $T_{\mathcal{P},\varepsilon}(x) = 0$ if $x \in [0,1] \setminus \bigcup I_n$. Any GLS transformation produces by iteration number expansions similar to the Lüroth expansions from (2.1), called *generalised Lüroth expansions*. In [BBDK94, DK96] the authors studied the approximation coefficients

$$\theta_n^{\mathcal{P}_L,\varepsilon} = q_n\left|x - \frac{p_n}{q_n}\right|, \quad n > 0$$

for GLS maps with the standard Lüroth partition $\mathcal{P}_L = \left\{\left[\frac{1}{n}, \frac{1}{n-1}\right)\right\}_{n\geq 2}$. In particular [BBDK94, Corollary 2] gives that amongst all GLS systems with standard Lüroth partition, T_A presents the best approximation properties. More precisely, it states that for any such GLS map $T_{\mathcal{P}_L,\varepsilon}$ there exists a constant $M_{\mathcal{P}_L,\varepsilon}$ such that for Lebesgue a.e. $x \in [0,1]$, the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \theta_i^{\mathcal{P}_L,\varepsilon}(x) \tag{2.18}$$

exists and equals $M_{\mathcal{P}_L,\varepsilon}$. Furthermore, $M_A \leq M_{\mathcal{P}_L,\varepsilon} \leq M_L$. In Remark 2, below Corollary 2, of [BBDK94], the authors remark that not every value in the interval $[M_A, M_L]$ can be obtained by such GLS maps, and they suggest furthermore that the achievable values of $M_{\mathcal{P}_L,\varepsilon}$ might form a fractal set. The situation changes for the random system L, as the next theorem shows.

For the random Lüroth map $L = L_0$ we define for each $(\omega, x) \in \{0,1\}^{\mathbb{N}} \times [0,1]$ and $n \geq 1$ the *n-th approximation coefficient* by

$$\theta_n(\omega, x) = q_n\left|x - \frac{p_n}{q_n}\right|,$$

where $q_n = (d_n - s_n)\prod_{i=1}^{n-1} d_i(d_i - 1)$.

2.3.4 Theorem. *For the random Lüroth map $L = L_0$ and $0 < p < 1$ the limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \theta_i(\omega, x)$$

exists for $m_p \times \lambda$-a.e. $(\omega, x) \in \{0,1\}^{\mathbb{N}} \times [0,1]$ and equals

$$M_p := \mathbb{E}[\theta_n] = p\frac{2\zeta(2) - 3}{2} + \frac{2 - \zeta(2)}{2},$$

where $\zeta(2)$ is the zeta function evaluated at 2. The function $p \mapsto M_p$ maps the interval $[0,1]$ onto the interval $[M_A, M_L]$.

Proof. Fix $(\omega, x) \in \{0,1\}^{\mathbb{N}} \times [0,1]$. Let $\theta_n = \theta_n(\omega, x)$, $d_n = d_n(\omega, x)$ and $x_n = T_\omega^n(x)$. By definition of the convergents $\frac{p_n}{q_n}$ we have

$$\left| x - \frac{p_n}{q_n} \right| = \frac{x_n}{\prod_{i=1}^{n} d_i(d_i - 1)}$$

and the numbers q_n are such that we can write

$$\theta_n = \begin{cases} \dfrac{x_n}{d_n - 1} = d_n x_{n-1} - 1, & \text{if } x_n = T_L(x_{n-1}), \\ \dfrac{x_n}{d_n} = -(d_n - 1)x_{n-1} + 1, & \text{if } x_n = T_A(x_{n-1}). \end{cases} \tag{2.19}$$

Since the Lebesgue measure is stationary for L_0, it follows from Birkhoff's Ergodic Theorem that each x_n, viewed as a random variable, is uniformly distributed over the interval $[0,1]$. We use this fact together with (2.19) to compute the cumulative distribution function (CDF) F_{θ_n} of θ_n. For $\mathbf{P} = m_p \times \lambda$, by definition of the CDF and by the law of total probability, for $y \in [0,1]$ we have

$$\begin{aligned} F_{\theta_n}(y) = \mathbf{P}(\theta_n \le y) &= \mathbf{P}(\theta_n \le y \,|\, x_n = T_L(x_{n-1}))\mathbf{P}(x_n = T_L(x_{n-1})) + \\ &\quad \mathbf{P}(\theta_n \le y \,|\, x_n = T_A(x_{n-1}))\mathbf{P}(x_n = T_A(x_{n-1})) \\ &= \sum_{d=2}^{\infty} \mathbf{P}\left(x_{n-1} \le \frac{1+y}{d}, \quad x_{n-1} \in [z_d, z_{d-1}) \right) \mathbf{P}(x_n = T_L(x_{n-1})) + \\ &\quad \mathbf{P}\left(x_{n-1} \ge \frac{1-y}{d-1}, \quad x_{n-1} \in [z_d, z_{d-1}) \right) \mathbf{P}(x_n = T_A(x_{n-1})) \\ &= \sum_{d=2}^{\infty} p\mathbf{P}\left(x_{n-1} \le \frac{1+y}{d}, \quad x_{n-1} \in [z_d, z_{d-1}) \right) + \\ &\quad (1-p)\mathbf{P}\left(x_{n-1} \ge \frac{1-y}{d-1}, \quad x_{n-1} \in [z_d, z_{d-1}) \right). \end{aligned}$$

Let $d \geq 2$ be such that $x_{n-1} \in \left[\frac{1}{d}, \frac{1}{d-1}\right) = [z_d, z_{d-1})$, then either $x_n = T_L(x_{n-1})$ or $x_n = T_A(x_{n-1})$. In the first case, since $d \in \mathbb{N}$,

$$\frac{1+y}{d} \in \left[\frac{1}{d}, \frac{1}{d-1}\right) \quad \text{if and only if} \quad d \leq \left\lfloor \frac{1}{y} \right\rfloor + 1,$$

and we obtain

$$\mathbf{P}\Big(x_{n-1} \leq \frac{1+y}{d}, \quad x_{n-1} \in [z_d, z_{d-1})\Big) = \begin{cases} \dfrac{y}{d}, & \text{if } d \leq \left\lfloor \dfrac{1}{y} \right\rfloor + 1, \\ \dfrac{1}{d(d-1)}, & \text{otherwise.} \end{cases}$$

Similarly, in the latter

$$\frac{1-y}{d-1} \in \left[\frac{1}{d}, \frac{1}{d-1}\right) \quad \text{if and only if} \quad d \leq \left\lfloor \frac{1}{y} \right\rfloor,$$

which gives

$$\mathbf{P}\Big(x_{n-1} \geq \frac{1-y}{d-1}, \quad x_{n-1} \in [z_d, z_{d-1})\Big) = \begin{cases} \dfrac{y}{d-1}, & \text{if } d \leq \left\lfloor \dfrac{1}{y} \right\rfloor, \\ \dfrac{1}{d(d-1)}, & \text{otherwise.} \end{cases}$$

Note that

$$\sum_{d > \lfloor 1/y \rfloor + 1} \frac{1}{d(d-1)} = \frac{1}{\lfloor 1/y \rfloor + 1} \quad \text{and} \quad \sum_{d > \lfloor 1/y \rfloor} \frac{1}{d(d-1)} = \frac{1}{\lfloor 1/y \rfloor}.$$

Summing over all $d \geq 2$ gives

$$F_{\theta_n}(y) = p\Big(\sum_{d=2}^{\lfloor 1/y \rfloor + 1} \frac{y}{d} + \frac{1}{\lfloor 1/y \rfloor + 1}\Big) + (1-p)\Big(\sum_{d=2}^{\lfloor 1/y \rfloor} \frac{y}{d-1} + \frac{1}{\lfloor 1/y \rfloor}\Big),$$

so that by (2.4) and Theorem 2.1.4 we obtain

$$F_{\theta_n}(y) = pF_L(y) + (1-p)F_A(y).$$

The expectation $\mathbf{E}[\theta_n]$ can be now computed by $\mathbf{E}[\theta_n] = \int_0^1 (1 - F_{\theta_n}(y))dy$, which with the results from Theorem 2.1.4 gives

$$\begin{aligned} \mathbf{E}[\theta_n] =& 1 - p\int_0^1 F_L(y)dy - (1-p)\int_0^1 F_A(y)dy \\ =& 1 - p\Big(\frac{3}{2} - \frac{\zeta(2)}{2}\Big) - (1-p)\Big(\frac{\zeta(2)}{2}\Big) \\ =& p\frac{2\zeta(2)-3}{2} + \frac{2-\zeta(2)}{2}, \end{aligned}$$

that is

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\theta_i(\omega,x)=p\frac{2\zeta(2)-3}{2}+\frac{2-\zeta(2)}{2}=M_p. \tag{2.20}$$

Note that for $p=0$, $x_n=T_A^n(x)$ for $n\geq 0$ and indeed $M_0=M_A$. In the same way, for $p=1$, $x_n=T_L^n(x)$ for $n\geq 0$ and $M_1=M_L$. Lastly, M_p is an increasing function in $p\in[0,1]$, so it can assume any value between M_A and M_L. □

Theorem 2 is now given by Proposition 2.3.1, Proposition 2.3.3 and Theorem 2.3.4.

§2.4 Generalised Lüroth expansions with bounded digits

Both Proposition 2.3.3 and Theorem 2.3.4 depend on the invariant measure $m_p\times\lambda$ for L_0. In Proposition 2.3.3 it is used to compute the value of $\Lambda(\omega,x)$ and in Theorem 2.3.4 we use the fact that the numbers $T_\omega^n(x)$ are uniformly distributed over the interval $[0,1]$ and it is used to deduce (2.20). For $c>0$, the random map L_c becomes fundamentally different: The maps $T_{j,c}$ for $j=0,1$ present finitely many branches that are not always onto. These variations make the measures $m_p\times\lambda$ no longer L_c-invariant. It still follows from results in e.g. [M85, P84, GB03, I12] that for any $0<p<1$ the random map L_c admits an invariant probability measure of type $m_p\times\mu_{p,c}$, where m_p is the $(p,1-p)$-Bernoulli measure on $\{0,1\}^{\mathbb{N}}$ and $\mu_{p,c}\ll\lambda$ is a probability measure on $[c,1]$ that satisfies

$$\mu_{p,c}(B)=p\mu_{p,c}(T_{0,c}^{-1}(B))+(1-p)\mu_{p,c}(T_{1,c}^{-1}(B)) \tag{2.21}$$

for each Borel measurable set $B\subseteq[c,1]$. We set $f_{p,c}:=\frac{\mathrm{d}\,\mu_{p,c}}{\mathrm{d}\,\lambda}$. Here we call $\mu_{p,c}$ a *stationary measure* and $f_{p,c}$ an *invariant density* for L_c. With [P84, Corollary 7] it follows that since $T_{0,c}$ is expanding and has a unique absolutely continuous invariant measure, the measure $\mu_{p,c}$ is the unique stationary measure for L_c and that L_c is ergodic with respect to $m_p\times\mu_{p,c}$.

2.4.1 Example. For $c=\frac{1}{3}$ and $0<p<1$ consider the measure $\mu_{p,\frac{1}{3}}$ with density

$$f_{p,\frac{1}{3}}(x)=\begin{cases}\frac{9}{8}, & \text{if } x\in\left[\frac{1}{3},\frac{2}{3}\right),\\ \\ \frac{15}{8}, & \text{if } x\in\left[\frac{2}{3},1\right].\end{cases}$$

One can check by direct computation that $\mu_{p,\frac{1}{3}}$ satisfies (2.21) and thus is the unique stationary measure for $L_{\frac{1}{3},p}$.

Since such an invariant measure exists, some of the results from the previous section also hold for the maps $L_{c,p}$ with $c>0$. In particular the Lyapunov exponent from (2.15) is well defined and since the stationary measure $m_p\times\mu_{p,c}$ is still ergodic, we can apply Birkhoff's Ergodic Theorem to obtain for $(m_p\times\mu_{p,c})$-a.e. point (ω,x)

the following expression for the Lyapunov exponent:

$$\Lambda(\omega,x)=\sum_{d=2}^{d_c}\mu_{p,c}\left(\left(\frac{1}{d},\frac{1}{d-1}\right)\right)\log(d(d-1))+\mu_{p,c}\left(\left(c,\frac{1}{d_c}\right)\right)\log(d_c(d_c+1)), \tag{2.22}$$

where d_c is the unique positive integer such that $c\in\left[\frac{1}{d_c+1},\frac{1}{d_c}\right)$.

2.4.2 Example. Consider the map $L_{\frac{1}{3},p}$ from Example 2.4.1 again. The possible digits of the generalised Lüroth expansions produced by $L_{\frac{1}{3},p}$ are $(0,2),(1,2),(0,3),(1,3)$. It follows from Birkhoff's Ergodic Theorem that the frequency of the digit $(0,2)$ is given by

$$\begin{aligned}\pi_{(0,2)}&=\lim_{n\to\infty}\frac{\#\{1\le j\le n:\ s_j(\omega,x)=0\ \text{ and }\ d_j(\omega,x)=2\}}{n}\\&=\int_{\{0,1\}^{\mathbb{N}}\times[\frac{1}{3},1]}\mathbf{1}_{[0]\times[\frac{2}{3},\frac{5}{6}]}+\mathbf{1}_{\{0,1\}^{\mathbb{N}}\times(\frac{5}{6},1]}\mathrm{d}m_p\times\mu_{p,\frac{1}{3}}\\&=\frac{15}{8}p\left(\frac{5}{6}-\frac{2}{3}\right)+\frac{15}{8}\left(1-\frac{5}{6}\right)=\frac{5+5p}{16}.\end{aligned}$$

Similarly,

$$\pi_{(1,2)}=\frac{8-5p}{16},\qquad \pi_{(0,1)}=\frac{1+p}{16},\qquad \pi_{(1,3)}=\frac{2-p}{16}.$$

Note that for

$$\pi_2=\lim_{n\to\infty}\frac{\#\{1\le j\le n:\ d_j(\omega,x)=2\}}{n}=\pi_{(0,2)}+\pi_{(1,2)}$$

we also obtain

$$\pi_2=\mu_{p,\frac{1}{3}}\left(\left[\frac{1}{2},1\right]\right)=\frac{13}{16},\quad\text{and}\quad\pi_3=1-\pi_2=\frac{3}{16}.$$

Moreover for $m_p\times\mu_{p,\frac{1}{3}}$-a.e. (ω,x) we have by (2.22) that

$$\begin{aligned}\Lambda(\omega,x)&=\sum_{d=2}^{3}\mu_{p,\frac{1}{3}}\left(\left[\frac{1}{d},\frac{1}{d-1}\right]\right)\log(d(d-1))\\&=\mu_{p,\frac{1}{3}}\left(\left[\frac{1}{2},1\right]\right)\log 2+\mu_{p,\frac{1}{3}}\left(\left[\frac{1}{3},\frac{1}{2}\right]\right)\log 6\\&=\frac{13}{16}\log 2+\frac{3}{16}\log 6=0.89913\ldots<1.198328\ldots=\Lambda_{m_p\times\lambda}.\end{aligned}$$

From (2.22) and example 2.4.2 it is clear that to obtain results similar to Proposition 2.3.3 for $c>0$ we need a good expression for the density of $\mu_{p,c}$ and to determine the approximation coefficient also an accurate description of the location of the points $\frac{p_n}{q_n}$ relative to x. For $c=0$ we saw that it immediately follows that Lebesgue almost all $x\in[0,1]$ have uncountably many different 0-Lüroth expansions. We start this section by giving some results on the number of different c-Lüroth expansions a number x can have for $c>0$.

§2.4.1 Uncountably many universal expansions

Let $c \in \left(0, \frac{1}{2}\right]$ and consider the corresponding alphabet $\mathcal{A}_c = \{(i,j) : i \in \{0,1\},\ j \in \{2,3,\ldots,\lceil \frac{1}{c} \rceil\}\}$. We call a sequence $((s_n,d_n))_{n\geq 1} \in \mathcal{A}_c^{\mathbb{N}}$ *c-Lüroth admissible* if

$$\sum_{n\geq 1} (-1)^{\sum_{i=1}^{n-1} s_{i+k}} \frac{d_{n+k}-1+s_{n+k}}{\prod_{i=1}^{n} d_{i+k}(d_{i+k}-1)} \in [c,1]$$

for all $k \geq 0$ and if moreover the sequence does not end in $((0,d+1),(0,2)^{\infty})$ for some $2 \leq d < \lceil \frac{1}{c} \rceil$.

The first main result of the number of different c-Lüroth expansions a number $x \in [c,1]$ can have is the following.

2.4.3 Theorem. *Let $0 < c \leq \frac{2}{5}$. Then every $x \in [c,1] \setminus \mathbb{Q}$ has uncountably many different c-Lüroth expansions.*

Before we prove the theorem we prove three lemmata.

2.4.4 Lemma. *Let $c \in \left(0, \frac{1}{2}\right)$ and let $((s_n,d_n))_{n\geq 1} \in \mathcal{A}_c^{\mathbb{N}}$ be c-Lüroth admissible. For*

$$x = \sum_{n\geq 1} (-1)^{\sum_{i=1}^{n-1} s_i} \frac{d_n - 1 + s_n}{\prod_{i=1}^{n} d_i(d_i-1)}$$

the following hold.

(i) If $s_1 = 0$, then $x \in \left[\frac{1}{d_1} + \frac{c}{d_1(d_1-1)}, \frac{1}{d_1-1}\right] = [z_{d_1}^{+}, z_{d_1-1}]$.

(ii) If $s_1 = 1$, then $x \in \left[\frac{1}{d_1}, \frac{1}{d_1-1} - \frac{c}{d_1(d_1-1)}\right] = [z_{d_1}, z_{d_1-1}^{-}]$.

Proof. For x we can write

$$x = \frac{d_1 - 1 + s_1}{d_1(d_1-1)} + \frac{(-1)^{s_1}}{d_1(d_1-1)} \sum_{n\geq 1} (-1)^{\sum_{i=1}^{n-1} s_{i+1}} \frac{d_{n+1}-1+s_{n+1}}{\prod_{i=1}^{n} d_{i+1}(d_{i+1}-1)}.$$

Write

$$x_1 = \sum_{n\geq 1} (-1)^{\sum_{i=1}^{n-1} s_{i+1}} \frac{d_{n+1}-1+s_{n+1}}{\prod_{i=1}^{n} d_{i+1}(d_{i+1}-1)}.$$

Since $((s_n,d_n))_{n\geq 1}$ is c-Lüroth admissible, we have $x_1 \in [c,1]$. If $s_1 = 0$, then

$$\frac{d_1 - 1 + s_1}{d_1(d_1-1)} + \frac{(-1)^{s_1}}{d_1(d_1-1)} x_1 = \frac{1}{d_1} + \frac{1}{d_1(d_1-1)} x_1 \in \left[\frac{1}{d_1} + \frac{c}{d_1(d_1-1)}, \frac{1}{d_1-1}\right].$$

Similarly if $s_1 = 1$, then

$$\frac{d_1 - 1 + s_1}{d_1(d_1-1)} + \frac{(-1)^{s_1}}{d_1(d_1-1)} x_1 = \frac{1}{d_1-1} - \frac{1}{d_1(d_1-1)} x_1 \in \left[\frac{1}{d_1}, \frac{1}{d_1-1} - \frac{c}{d_1(d_1-1)}\right].$$

□

Recall the definition of the switch region S from (2.7), that is

$$S = [c, 1] \cap \bigcup_{n>1} [z_n^+, z_{n-1}^-].$$

2.4.5 Lemma. *A sequence $((s_n, d_n))_{n\geq 1} \in \mathcal{A}_c^{\mathbb{N}}$ is c-Lüroth admissible if and only if there is a point $(\omega, x) \in \{0,1\}^{\mathbb{N}} \times [c, 1]$ such that the sequence $(s_n(\omega, x), d_n(\omega, x))_{n\geq 1}$ generated by L_c satisfies $s_n(\omega, x) = s_n$ and $d_n(\omega, x) = d_n$ for each $n \geq 1$.*

Proof. One direction follows since $L_c^k(\omega, x) \in [c, 1]$ for all k and by the definition of L_c, s_i and d_i on the points z_n. For the other direction, let $((s_n, d_n))_n$ be c-Lüroth admissible. Set

$$x = \sum_{n\geq 1} (-1)^{\sum_{i=1}^{n-1} s_i} \frac{d_n - 1 + s_n}{\prod_{i=1}^{n} d_i(d_i - 1)}.$$

For $k \geq 0$ define the numbers

$$x_k = \sum_{n\geq 1} (-1)^{\sum_{i=1}^{n-1} s_{i+k}} \frac{d_{n+k} - 1 + s_{n+k}}{\prod_{i=1}^{n} d_{i+k}(d_{i+k} - 1)},$$

so that $x_0 = x$. Let $\omega \in \{0,1\}^{\mathbb{N}}$ be such that

$$\omega_k = \begin{cases} 0, & \text{if } 1_S(x_{k-1}) = 1 \text{ and } s_k = 0, \\ 1, & \text{otherwise.} \end{cases}$$

By Lemma 2.4.4 and the fact that $((s_n, d_n))_n$ does not end in $((0, d + 1), (0, 2)^\infty)$ it then follows that $s_1(\omega, x) = s_1$ and $d_1(\omega, x) = d_1$. Moreover, $T_\omega^1(x) = x_1$. It then follows by induction that for each $n \geq 1$, $s_n = s_n(\omega, x)$, $d_n = d_n(\omega, x)$ and $T_\omega^n(x) = x_n$. Hence, $((s_n, d_n))_n$ corresponds to the c-Lüroth expansion of (ω, x). □

2.4.6 Lemma. *Let $0 < c \leq \frac{2}{5}$. For any $x \in [c, 1] \setminus \mathbb{Q}$ and any $\omega \in \{0,1\}^{\mathbb{N}}$ there is an $n \geq 0$, such that $T_\omega^n(x) \in S$.*

Proof. Let $0 < c \leq \frac{2}{5}$, $x \in [c, 1] \setminus \mathbb{Q}$ and $\omega \in \{0,1\}^{\mathbb{N}}$. If $x \in S$, then we are done. If $x \notin S$, then $T_{\omega,c}^1(x) = T_{0,c}(x) = T_{1,c}(x) \in (1 - c, 1)$. Assume that $c \leq \frac{1}{3}$, then $z_2^+ = \frac{1+c}{2} \leq 1 - c < 1 - \frac{c}{2} = z_1^-$. Let $f : x \mapsto 2x - 1$ denote the right most branch of T_L. Since $f(z_1^-) = 1 - c$, it follows that $(1 - c, 1) = \cup_{j\in\mathbb{N}} f^{-j}((1 - c, z_1^-])$ and hence there exists a $j \in \mathbb{N}$ such that $T_{\omega,c}^{j+1}(x) = T_L^j(T_{\omega,c}^1(x)) \in (1 - c, z_1^-] \subseteq S$.

Now assume that $\frac{1}{3} < c \leq \frac{2}{5}$. Then $\frac{1}{2} < 1 - c < z_2^+$ and

$$z_2^+ < 2c = T_{0,c}(1 - c) = T_{1,c}(1 - c) \leq z_1^-.$$

We write

$$(1 - c, 1) = (1 - c, z_2^+) \cup [z_2^+, z_1^-] \cup (z_1^-, 1),$$

and we treat the subintervals separately.

1. For the first subinterval note that $(1-c, z_2^+) = \left(1-c, \frac{3-c}{4}\right] \cup \left(\frac{3-c}{4}, \frac{2}{3}\right) \cup \left[\frac{2}{3}, z_2^+\right)$. Here $\frac{2}{3}$ is a repelling fixed point of T_A and this subdivision is such that $T_A\left(\left(1-c, \frac{3-c}{4}\right]\right) = [z_2^+, 2c) \subseteq S$, $T_A\left(\left(\frac{3-c}{4}, \frac{2}{3}\right)\right) = \left(\frac{2}{3}, z_2^+\right)$ and $T_A\left(\left(\frac{2}{3}, z_2^+\right)\right) = \left(1-c, \frac{2}{3}\right)$. This gives the following.

   - If $T^1_{\omega,c}(x) \in \left(1-c, \frac{3-c}{4}\right]$, then $T^2_{\omega,c}(x) \in [z_2^+, 2c) \subseteq S$.

   - If $T^1_{\omega,c}(x) \in \left(\frac{3-c}{4}, \frac{2}{3}\right)$, then $T^2_{\omega,c}(x) = T_A(T^1_{\omega,c}(x))$ and since $\frac{2}{3}$ is a repelling fixed point for T_A there must then exist a j such that $T^j_{\omega,c}(x) \in \left(1-c, \frac{3-c}{4}\right)$, so $T^{j+1}_{\omega,c}(x) \in (z_2^+, 2c) \subseteq S$.

   - If $T^1_{\omega,c}(x) \in \left[\frac{2}{3}, z_2^+\right)$, then either $T^2_{\omega,c}(x) \in \left(1-c, \frac{3-c}{4}\right]$ and $T^3_{\omega,c}(x) \in [z_2^+, 2c) \subseteq S$; or $T^2_{\omega,c}(x) \in \left(\frac{3-c}{4}, \frac{2}{3}\right]$, and again we can find a suitable j as above.

2. If $T^1_{\omega,c}(x) \in [z_2^+, z_1^-]$, then $T^1_{\omega,c}(x) \in S$.

3. If $T^1_{\omega,c}(x) \in (z_1^-, 1)$, since $f(z_1^-) = 1-c$ we can write $(z_1^-, 1)$ as the disjoint union
$$(z_1^-, 1) = \cup_{j\geq 1} f^{-j}([z_2^+, z_1^-]) \cup f^{-j}((1-c, z_2^+)).$$
   Hence, there is a j such that either $T^{j+1}_{\omega,c}(x) = T_L^j \circ T^1_{\omega,c}(x) \in S$ and we are done or $T^{j+1}_{\omega,c}(x) = T_L^j \circ T^1_{\omega,c}(x) \in (1-c, z_2^+)$ and we are in the situation of case 1.

This finishes the proof. □

Proof of Theorem 2.4.3. Let $0 < c \leq \frac{2}{5}$ and $x \in [c,1] \setminus \mathbb{Q}$ be given. To prove the result it is enough to show that for any sequence $((s_n, d_n))_{n\geq 1}$ representing a c-Lüroth expansion of x and any $N \geq 1$ there is an $n \geq N$ and a c-Lüroth expansion

$$((s_1, d_1), \ldots, (s_{N+n}, d_{N+n}), (s'_{N+n+1}, d'_{N+n+1}), (s'_{N+n+2}, d'_{N+n+2}), \ldots)$$

of x with $s'_{N+n+1} \neq s_{N+n+1}$ or $d'_{N+n+1} \neq d_{N+n+1}$.

Let $((s_n, d_n))_{n\geq 1}$ be any c-Lüroth admissible sequence with

$$x = \sum_{n\geq 1} (-1)^{\sum_{i=1}^{n-1} s_i} \frac{d_n - 1 + s_n}{\prod_{i=1}^n d_i(d_i-1)}.$$

By Lemma 2.4.5 there then exists a sequence ω, such that $(s_n(\omega, x), d_n(\omega, x))_n = ((s_n, d_n))_n$. Fix an $N \geq 1$. Lemma 2.4.6 yields the existence of an n such that $T^{N+n}_{\omega,c}(x) \in S$. Take any sequence $\tilde{\omega} \in \{0,1\}^{\mathbb{N}}$ with $\tilde{\omega}_j = \omega_j$ for all $1 \leq j \leq N+n$, $\tilde{\omega}_{N+n+1} = 1 - \omega_{N+n+1}$. Then for each $1 \leq j \leq N+n$,

$$s_j(\tilde{\omega}, x) = s_j(\omega, x) = s_j \quad \text{and} \quad d_j(\tilde{\omega}, x) = d_j(\omega, x) = d_j$$

and

$$s_{N+n+1}(\tilde{\omega}, x) = \tilde{\omega}_{N+n+1} = 1 - \omega_{N+n+1} = 1 - s_{N+n+1}(\omega, x).$$

From $T_{\tilde{\omega},c}^{N+n+1}(x) \notin \mathbb{Q}$ we obtain that either $T_{\tilde{\omega},c}^{N+n+1}(x) > \frac{1}{2}$ and $T_{\omega,c}^{N+n+1}(x) < \frac{1}{2}$ so that $d_{N+n+2}(\tilde{\omega}, x) = 2$ and $d_{N+n+2}(\omega, x) > 2$, or $T_{\tilde{\omega},c}^{N+n+1}(x) < \frac{1}{2}$ and $T_{\omega,c}^{N+n+1}(x) > \frac{1}{2}$ so that $d_{N+n+2}(\tilde{\omega}, x) > 2$ and $d_{N+n+2}(\omega, x) = 2$. In any case we get a c-Lüroth expansion of x that differs from the expansion $((s_n, d_n))_n$ at indices $N+n+1$ and $N+n+2$. □

The second result of this section is on universal expansions. For the remainder of this section we assume that $c = \frac{1}{\ell}$ for some $\ell \in \mathbb{N}_{\geq 3}$. Then $\mathcal{A}_c = \{(i,j) : i \in \{0,1\},\ j \in \{2,3,\ldots,\ell\}\}$ and by Lemma 2.4.5 any sequence in $\mathcal{A}_c^{\mathbb{N}}$ not ending in $((0, d+1), (0,2)^\infty)$ is c-Lüroth admissible. An expansion

$$x = \sum_{n\geq 1} (-1)^{\sum_{i=1}^{n-1} s_i} \frac{d_n - 1 + s_n}{\prod_{i=1}^{n} d_i(d_i - 1)}$$

of a number $x \in [c,1]$ is called a *universal* c-Lüroth expansion if for any block $(t_1, b_1), \ldots, (t_j, b_j) \in \mathcal{A}_c$ there is a $k \geq 1$, such that $s_{k+i} = t_i$ and $d_{k+i} = b_i$ for all $1 \leq i \leq j$, i.e., if each finite block of digits occurs in $((s_n, d_n))_n$.

2.4.7 Theorem. *Let $c = \frac{1}{\ell}$ for some $\ell \in \mathbb{N}_{\geq 3}$. Then Lebesgue almost every $x \in \left[\frac{1}{\ell}, 1\right]$ has uncountably many different universal c-Lüroth expansions.*

The proof of this theorem requires some work and several smaller results. First we prove a property of the measure $m_p \times \mu_{p,c}$.

2.4.8 Proposition. *Let $c = \frac{1}{\ell}$ for some $\ell \in \mathbb{N}_{\geq 3}$. Then for any $0 < p < 1$ the random transformation L_c is mixing and the density of $\mu_{p,c}$ is bounded away from 0.*

Proof. We will show that L_c has the *random covering property*, i.e., that for any non-trivial subinterval $J \subseteq [c,1]$ there is an $n \geq 1$ and an $\omega \in \{0,1\}^{\mathbb{N}}$ such that $T_\omega^n(J) = [c,1]$. The result then follows from [Proposition 2.6, [ANV15]].

Let $J \subseteq [c,1]$ be any interval of positive Lebesgue measure. Since $|\frac{\mathrm{d}}{\mathrm{d}x} T_{\omega,c}^1(x)| \geq 2$, for any $\omega \in \{0,1\}^{\mathbb{N}}$ there is an m such that at least one of the points z_n, z_n^+, z_n^- is in the interior of $T_\omega^m(J)$ and hence $T_\omega^{m+2}(J)$ will contain an interval of the form $(a,1]$ for some a. Since 1 is a fixed point, this implies that there is a k such that $(1 - \frac{1}{\ell}, 1] \subseteq T_\omega^k((a,1])$ for each $\omega \in \{0,1\}^{\mathbb{N}}$. For the smallest $i \geq \log_2(\ell - 1) - 1$ it holds that $T_{0^i,c}\big(1 - \frac{1}{\ell}\big) = 1 - \frac{2^i}{\ell} < \frac{1}{2} + \frac{1}{2\ell} = z_2^+$. Hence

$$[c,1] \subseteq T_{0^{i+1},c}\Big(\Big(1 - \frac{1}{\ell}, 1\Big]\Big),$$

giving the random covering property and the result. □

The difference between Theorem 2.4.7 and Proposition 2.3.1 is that in case $c = 0$ for almost all (ω, x) the sequence $(s_n(\omega,x))_n$ equals the sequence ω, so that it is immediately clear that different sequences ω lead to different expansions. For $c > 0$ typically many sequences ω lead to the same sequence $(s_n(\omega,x))_n$. In other words, the map $g : \{0,1\}^{\mathbb{N}} \times [c,1] \to \mathcal{A}_c^{\mathbb{N}}$ defined by

$$g((\omega,x)) = \big((s_1(\omega,x), d_1(\omega,x)), (s_2(\omega,x), d_2(\omega,x)), (s_3(\omega,x), d_3(\omega,x)), \ldots\big) \quad (2.23)$$

is far from injective. To solve this issue we define another random dynamical system $K_c : \{0,1\}^{\mathbb{N}} \times [c,1] \to \{0,1\}^{\mathbb{N}} \times [c,1]$ given by

$$K_c(\omega, x) = \begin{cases} (\omega, T_A(x)), & \text{if } x \in \cup_{n=2}^{\ell}[z_n, z_n^+), \\ (\omega, T_L(x)), & \text{if } x \in \cup_{n=2}^{\ell}(z_{n-1}^-, z_{n-1}) \cup \{1\}, \\ (\sigma(\omega), T^1_{\omega,c}(x)), & \text{if } x \in S. \end{cases}$$

The difference between K_c and L_c is that the map K_c only shifts in the first coordinate if the point x lies in S, so only if $T_{0,c}(x) \neq T_{1,c}(x)$. For K_c define the function $h : \{0,1\}^{\mathbb{N}} \times [c,1] \to \mathcal{A}_c^{\mathbb{N}}$ by

$$h((\omega, x)) = \big((s_1(\omega, x), d_1(\omega, x)), (s_1(K_c(\omega, x)), d_1(K_c(\omega, x))), \\ (s_1(K_c^2(\omega, x)), d_1(K_c^2(\omega, x))), \dots\big).$$

In the other direction set $\psi_c = \psi|_{\mathcal{A}_c^{\mathbb{N}}}$, so that

$$\psi_c : \mathcal{A}_c^{\mathbb{N}} \to [c,1],\ ((s_n, d_n))_{n\geq 1} \mapsto \sum_{n\geq 1} (-1)^{\sum_{i=1}^{n-1} s_i} \frac{d_n - 1 + s_n}{\prod_{i=1}^{n} d_i(d_i - 1)}.$$

Neither g nor h is surjective, since both maps L_c and K_c do not produce sequences ending in $((0, d+1), (0,2)^{\infty})$ as we saw before. To solve this and also to make h injective, define

$$Z_L = \{(\omega, x) \in \{0,1\}^{\mathbb{N}} \times [c,1] : L_c^n(\omega, x) \in \{0,1\}^{\mathbb{N}} \times S \text{ for infinitely many } n \geq 0\},$$
$$Z_K = \{(\omega, x) \in \{0,1\}^{\mathbb{N}} \times [c,1] : K_c^n(\omega, x) \in \{0,1\}^{\mathbb{N}} \times S \text{ for infinitely many } n \geq 0\},$$
$$D = \{\mathbf{a} \in \mathcal{A}_c^{\mathbb{N}} : \psi_c(\sigma^n \mathbf{a}) \in S \text{ for infinitely many } n \geq 0\}.$$

Then $g : Z_L \to D$ is surjective and $h : Z_K \to D$ is bijective and by Lemma 2.4.6 $m_p \times \mu_{p,c}(Z_L) = 1$. The proof of Theorem 2.4.7 is based on the proofs of [DdV07, Theorem 7 and Lemma 4] and uses the following result on the map K_c.

2.4.9 Proposition. *Let $c = \frac{1}{\ell}$ for some $\ell \geq 3$ and $0 < p < 1$. The measure $m_p \times \mu_{p,c}$ is invariant and ergodic for K_c.*

Proof. First note that $\sigma \circ g = g \circ L_c$. Define a measure ν on $\mathcal{A}_c^{\mathbb{N}}$ with the σ-algebra generated by the cylinders by setting $\nu = m_p \times \mu_{p,c} \circ g^{-1}$. Then by Lemma 2.4.6 we get $\nu(\mathcal{A}_c^{\mathbb{N}} \setminus D) = 0$, that is ν is concentrated on D. So g is a factor map and σ is ergodic and measure preserving with respect to ν. By construction it holds that $K_c^{-1}(Z_K) = Z_K$ and $\sigma^{-1}(D) = D$ and moreover, $\sigma \circ h = h \circ K_c$. Define a measure $\tilde{\nu}$ on $\{0,1\}^{\mathbb{N}} \times [c,1]$ by setting

$$\tilde{\nu}(A) = \nu\big(h(A \cap Z_K)\big).$$

Since h is a bijection from Z_K to D we find that $h : \{0,1\}^{\mathbb{N}} \times [c,1] \to \mathcal{A}_c^{\mathbb{N}}$ is an isomorphism and K_c is measure preserving and ergodic with respect to $\tilde{\nu}$. What is left is to prove that $\tilde{\nu} = m_p \times \mu_{p,c}$, which is what we do now.

Sets of the form

$$h^{-1}([(k_1,i_1),\dots,(k_n,i_n)])$$

generate the product σ-algebra on $\{0,1\}^{\mathbb{N}}\times[c,1]$ given by the product σ-algebra on $\{0,1\}^{\mathbb{N}}$ and the Borel σ-algebra on $[c,1]$. Therefore it is enough to check that

$$\tilde{\nu}(h^{-1}([(k_1,i_1),\dots,(k_n,i_n)])) = m_p\times\mu_{p,c}(h^{-1}([(k_1,i_1),\dots,(k_n,i_n)]))$$

for any cylinder $[(k_1,i_1),\dots,(k_n,i_n)]\subseteq\mathcal{A}_c^{\mathbb{N}}$.

For $i\in\{2,3,\dots,\ell\}$, let

$$\begin{aligned} A_{0i} &= [0]\times\left[z_i^+,z_{i-1}^-\right], \quad A_{22} = \{0,1\}^{\mathbb{N}}\times\left(z_1^-,1\right], \ A_{2i} = \{0,1\}^{\mathbb{N}}\times\left(z_{i-1}^-,z_{i-1}\right),\ i\geq 3,\\ A_{1i} &= [1]\times\left[z_i^+,z_{i-1}^-\right], \quad A_{3i} = \{0,1\}^{\mathbb{N}}\times\left[z_i,z_i^+\right). \end{aligned}$$

For any cylinder $[(k,i)]\subseteq\mathcal{A}_c^{\mathbb{N}}$ we get $h^{-1}([(k,i)]) = A_{ki}\cup A_{(k+2)i}$ and

$$h^{-1}([(k_1,i_1),(k_1,i_2),\dots,(k_n,i_n)]) = \bigcup_{j_1,\dots,j_n} A_{j_1i_1}\cap K_c^{-1}(A_{j_2i_2})\cap\dots\cap K_c^{-(n-1)}(A_{j_ni_n}), \tag{2.24}$$

where the union is disjoint and is taken over all blocks $j_1,\dots,j_n$ that have $j_t\in\{k_t,k_t+2\}\subset\{0,1,2,3\}$ for each t. Any set $A_{j_1i_1}\cap K_c^{-1}A_{j_2i_2}\cap\dots\cap K_c^{-(n-1)}(A_{j_ni_n})$ is a product set. Denote its projection on the second coordinate by $I_{j_1i_1\dots j_ni_n}$. Define the set

$$\{t_1,\dots,t_m\} = \{t : (j_t,i_t)\in\mathcal{A}_c\},$$

where we assume that $1\leq t_1<t_2<\dots<t_m\leq n$. These are the indices t such that $j_t=k_t$ and thus the projection of $A_{j_ti_t}$ to the first coordinate does not equal $\{0,1\}^{\mathbb{N}}$. This implies that we can write

$$A_{j_1i_1}\cap K_c^{-1}(A_{j_2i_2})\cap\dots\cap K_c^{-(n-1)}(A_{j_ni_n}) = [k_{t_1},k_{t_2},\dots,k_{t_m}]\times I_{j_1i_1\dots j_ni_n}$$

and

$$m_p\times\mu_{p,c}(h^{-1}([(k_1,i_1),(k_1,i_2),\dots,(k_n,i_n)])) = \sum_{j_1,\dots,j_n} m_p\times\mu_{p,c}([k_{t_1},k_{t_2},\dots,k_{t_m}]\times I_{j_1i_1\dots j_ni_n}).$$

To compute $\tilde{\nu}(h^{-1}([(k_1,i_1),(k_1,i_2),\dots,(k_n,i_n)]))$, let $\mathcal{S}\subseteq\{0,1\}^n$ denote the set

of blocks $s_1, s_2, \ldots, s_n$ for which $s_t = k_t$ for all $t \in \{t_1, \ldots, t_m\}$. Then

$$\begin{aligned}
\tilde{\nu}(h^{-1}([(k_1,i_1),(k_1,i_2),\ldots,(k_n,i_n)])) & \\
&= \nu(h(Z_K \cap h^{-1}([(k_1,i_1),(k_1,i_2),\ldots,(k_n,i_n)]))) \\
&= \nu(D \cap [(k_1,i_1),(k_1,i_2),\ldots,(k_n,i_n)]) \\
&= \nu([(k_1,i_1),(k_1,i_2),\ldots,(k_n,i_n)]) \\
&= m_p \times \mu_{p,c}\Big(\bigcup_{j_1,\ldots,j_n} \bigcup_{s_1,s_2,\ldots,s_n \in \mathcal{S}} [s_1,s_2,\ldots,s_n] \times I_{j_1 i_1 \ldots j_n i_n} \Big) \\
&= \sum_{j_1,\ldots,j_n} \sum_{s_1,s_2,\ldots,s_n \in \mathcal{S}} m_p \times \mu_{p,c}([s_1,s_2,\ldots,s_n] \times I_{j_1 i_1 \ldots j_n i_n}) \\
&= \sum_{j_1,\ldots,j_n} m_p \times \mu_{p,c}([k_{t_1},k_{t_2},\ldots,k_{t_m}] \times I_{j_1 i_1 \ldots j_n i_n}).
\end{aligned}$$

Hence, $\tilde{\nu} = m_p \times \mu_{p,c}$ and the statement follows. □

Proof of Theorem 2.4.7. Let $0 < p < 1$ be given. For each $s \in \{0,1\}$ and $d \in \{2,3,\ldots,\ell\}$ define

$$\hat{\Delta}_c(s,d) = \begin{cases} [0] \times [z_d^+, z_{d-1}^-] \cup \{0,1\}^{\mathbb{N}} \times (z_{d-1}^-, z_{d-1}], & \text{if } s=0,\ d=2, \\ [0] \times [z_d^+, z_{d-1}^-] \cup \{0,1\}^{\mathbb{N}} \times (z_{d-1}^-, z_{d-1}), & \text{if } s=0,\ d \geq 3 \\ [1] \times [z_d^+, z_{d-1}^1] \cup \{0,1\}^{\mathbb{N}} \times \big[z_d, z_d^+\big), & \text{if } s=1. \end{cases}$$

Fix $(s_1,d_1),\ldots,(s_j,d_j) \in \mathcal{A}_c$. Then the set

$$E = \hat{\Delta}_c(s_1,d_1) \cap K_c^{-1}\hat{\Delta}_c(s_2,d_2) \cap \cdots \cap K_c^{-(j-1)}\hat{\Delta}_c(s_j,d_j) \subseteq \{0,1\}^{\mathbb{N}} \times [c,1]$$

contains precisely those points (ω,x) for which $s_i(\omega,x) = s_i$ and $d_i(\omega,x) = d_i$ for all $1 \leq i \leq j$. Since for each s,d, $m_p \times \lambda(\hat{\Delta}_c(s,d)) = \frac{c}{d(d-1)} + (p-s)(-1)^s \frac{1-2c}{d(d-1)} > 0$ and $K_c(\hat{\Delta}_c(s,d)) = \{0,1\}^{\mathbb{N}} \times [c,1]$, it also follows that $m_p \times \lambda(E) > 0$. The map K_c is ergodic with respect to $m_p \times \mu_{p,c}$ by Lemma 2.4.9 and the measures $\mu_{p,c}$ and λ are equivalent by Proposition 2.4.8. By Birkhoff's Ergodic Theorem it then follows that for $m_p \times \lambda$-a.e. (ω,x) the block $(s_1,d_1),\ldots,(s_j,d_j)$ occurs with positive frequency in the sequence $(s_1(K_c^n(\omega,x)),d_1(K_c^n(\omega,x)))_n$. Since there are only countably many blocks $(s_1,d_1),\ldots,(s_j,d_j)$ it follows that the c-Lüroth expansion of $m_p \times \lambda$-a.e. (ω,x) is universal. Let

$$\tilde{Z} := \{x \in [c,1] : \forall \omega \in \{0,1\}^{\mathbb{N}}\, K_c^n(\omega,x) \in \{0,1\}^{\mathbb{N}} \times S \text{ for infinitely many } n \geq 0\}.$$

Then $\lambda([c,1] \setminus \tilde{Z}) = 0$ by Lemma 2.4.6. From Fubini's Theorem we get the existence of a set $B \subseteq \tilde{Z}$ with $\lambda(B) = 1-c$ and for each $x \in B$ a set $A_x \subseteq \{0,1\}^{\mathbb{N}}$ with $m_p(A_x) = 1$ and such that for any $(\omega,x) \in A_x \times \{x\}$ the sequence $((s_1(K_c^n(\omega,x)),d_1(K_c^n(\omega,x))))_n$ is universal. Since the set A_x has full measure, it contains uncountably many sequences. For any $x \in \tilde{Z}$ different sequences ω define different sequences $((s_1(K_c^n(\omega,x)),d_1(K_c^n(\omega,x))))_{n\geq 1}$. Hence, we obtain for Lebesgue almost every x uncountably many universal c-Lüroth expansions. □

§2.4.2 Explicit expressions for invariant densities

Explicit expressions for the probability density functions $f_{p,c} = \frac{\mathrm{d}\,\mu_{p,c}}{\mathrm{d}\,\lambda}$ can be obtained from the procedure from Theorem 3.4.1 in case $p \neq \frac{1}{2}$ (since otherwise condition (A5) is violated). From this result it follows that

$$f_{p,c} = c_1 + c_2 \sum_{t\geq 0} \sum_{\omega\in\{0,1\}^t} \frac{p_\omega}{T'_{\omega,c}(1-c)} 1_{[c,T_{\omega,c}(1-c))} + c_3 \sum_{t\geq 1} \sum_{\omega\in\{0,1\}^t} \frac{p_\omega}{T'_{\omega,c}(c)} 1_{[c,T_{\omega,c}(c))}, \tag{2.25}$$

where c_1, c_2, c_3 are constants and p_ω is an abbreviation for the product $p_{\omega_1} \cdots p_{\omega_t}$. The sums in this expression have finitely many terms if the random orbits $T_{\omega,c}(1-c)$ and $T_{\omega,c}(c)$ take values in a finite set. This happens for example if the random map $L_{c,p}$ is Markov, which is the case for any $c \in \mathbb{Q} \cap (0, \frac{1}{2}]$. Before proving this result, we recall the notion of a Markov map.

An interval map $T : I \to I$, where I is an interval, is called *Markov* if there exists a finite collection of open non-empty disjoint subintervals of I defined by a set of points $\{z_i\}_{1\leq i\leq N}$ such that, for every i, $T((z_i, z_{i+1}))$ is a homeomorphism onto (z_j, z_ℓ) for some j, ℓ. The corresponding partition is called a *Markov partition*. A piecewise affine interval map $T : I \to I$ admitting a Markov partition is isomorphic to a Markov shift and the results from [FB81] show that the matrix $P = (p_{i,j})_{N\times N}$ given by

$$p_{ij} := \frac{\delta_{ij}}{|T'_j|}, \tag{2.26}$$

for $T_j = T\,|_{(z_{j-1},z_j)}$ and

$$\delta_{ij} = \begin{cases} 1, & \text{if } T((z_{j-1}, z_j)) \supseteq (z_{i-1}, z_i), \\ 0, & \text{otherwise,} \end{cases}$$

represents the Perron-Frobenius operator from (1.2). From [FB81, Theorem 1] a T-invariant density is recovered from a non-trivial eigenvector v of eigenvalue 1, i.e., $Pv = v$.

The construction extends straightforwardly to the random context. Let $\{T_j : I \to I\}_{j\in\Omega}$ be a finite family of interval maps and let $\mathbf{p} = (p_j)_{j\in\Omega}$ be a positive probability vector representing the probabilities with which we choose the maps T_j. The random interval map $R : \Omega^{\mathbb{N}} \times I \to \Omega^{\mathbb{N}} \times I$ is said to be *Markov* if each map $\{T_j\}_{j\in\Omega}$ is Markov. The matrix form P of the random Perron-Frobenius operator for R is then given by

$$P = \sum_{j\in\Omega} p_j P_j,$$

for P_j the transition matrix (2.26) of T_j, and any non-trivial vector v satisfying $Pv = v$ identifies an invariant density for the system. For Markov maps the problem of finding an invariant density function reduces therefore to finding a Markov partition and solving a matrix equation.

2.4.10 Proposition. *For any $c \in \mathbb{Q} \cap (0, \frac{1}{2}]$ the random c-Lüroth transformation L_c is Markov.*

Proof. Let $\mathcal{S}_c = \{s_i\}_i$ be the finite set of points given by

$$\{c, 1\} \cup \{z_n, z_n^+, z_{n-1}^-\}_{n=2}^{\infty} \cap [c, 1],$$

and such that $s_0 = c < s_1 < \ldots < s_k = 1$. These are the critical points in $[c, 1]$ of $L_{c,p}$. For $j = 0, 1$,

$$T_{j,c}(s_i, s_{i+1}) \in \{(1-c, 1), (c, 1-c), (T_{j,c}(c), 1), (c, T_{j,c}(c)), (T_{j,c}(c), 1-c), (1-c, T_{j,c}(c))\},$$

so that, to determine a Markov partition, it is enough to study the orbit of c and $1 - c$. Since $c \in \mathbb{Q}$, Proposition 2.2.3 implies that the set

$$\mathcal{O}_c = \{T_{\omega,c}^n(c) : \omega \in \Omega^{\mathbb{N}}, n \in \mathbb{N}\} \cup \{T_{\omega,c}^n(1-c) : \omega \in \Omega^{\mathbb{N}}, n \in \mathbb{N}\}, \tag{2.27}$$

is finite. By construction, the partition obtained by the points in $\mathcal{S}_c \cup \mathcal{O}_c$ is Markov.

□

The procedure described at the beginning of this section can therefore be applied to any random map L_c with rational cutting point c, yielding an explicit expression for the invariant density function $f_{p,c}$. As we see from Proposition 2.2.6 the rationality of c is essential here, since for $c \in \left[0, \frac{1}{2}\right] \setminus \mathbb{Q}$ the set $\mathcal{O}_c$ from (2.27) contains infinitely many elements. In the following we show that the density $f_{p,c}$ is actually continuous in c for any fixed $0 < p < 1$, and that therefore the density of each map L_c with $c \in \mathbb{R} \setminus \mathbb{Q}$ can be approximated by the densities of maps $L_{\hat{c}}$ for $\hat{c} \in \mathbb{Q}$ sufficiently close to c. The result can be proven in a similar way as [DK17, Theorem 4.1], by paying attention to the fact that now the space of definition of the transformations depends on the parameter.

2.4.11 Proposition. *Fix $0 < p < 1$. Let $\hat{c} \in (0, \frac{1}{2})$, such that $\hat{c} \neq z_n, z_n^+, z_{n-1}^-$ for all $n \geq 3$. Let $(c_k)_k \subseteq (0, \frac{1}{2})$ be a sequence converging to $\hat{c}$. Then $f_{p,c_k} \mapsto f_{p,\hat{c}}$ in $L^1(\lambda)$. If $\hat{c} = z_n$, $\hat{c} = z_n^+$ or $\hat{c} = z_{n-1}^-$ for some $n \geq 3$, then the same is true for any sequence $(c_k)_k$ that converges to $\hat{c}$ from the right.*

Proof. In this proof we use $f_c = f_{p,c}$ to denote the density of the unique stationary measure $\mu_{p,c}$, for any $c \in (0, \frac{1}{2})$. Note that the domain of each of these functions is the interval $I_c = [c, 1]$, that depends on the parameter c. For this reason, we extend these densities to the whole interval $[0, 1]$ by considering, with abuse of notation, $f_c : [0, 1] \to \mathbb{R}$, such that, for any $x \in [0, 1]$, $f_c(x) = f_c(x)\mathbf{1}_{[c,1]}(x)$.

The proof now goes along the following lines.

1. We show that there is a uniform bound, i.e., independent of k, on the total variation and supremum norm of the densities f_{c_k}. This is the point in which we need stronger assumptions on the sequence $(c_k)_k$ in case $\hat{c}$ is one of the critical points of the random map. It then follows from Helly's Selection Theorem that there is some subsequence of (f_{c_k}) for which an a.e. and $L^1(\lambda)$ limit $\hat{f}$ exist.

2. We show that $\hat{f} = f_{\hat{c}}$, which by the same proof implies that any subsequence of (f_{c_k}) has a further subsequence converging a.e. to the same limit $f_{\hat{c}}$. Hence, (f_{c_k}) converges to $f_{\hat{c}}$ in measure.

3. By the uniform integrability of (f_{c_k}), it then follows from Vitali's Convergence Theorem that the convergence of (f_{c_k}) to $f_{\hat{c}}$ is in $L^1(\lambda)$.

Step 1. and 2. use Perron-Frobenius operators. For $j = 0, 1$ the Perron-Frobenius operator $P_{c,j}$ of $T_{c,j}$ is uniquely defined by the equation

$$\int (P_{c,j} f) g \, d\lambda = \int f(g \circ T_{c,j}) \, d\lambda \quad \forall f \in L^1(\lambda),\, g \in L^\infty(\lambda),$$

and the Perron-Frobenius operator P_c of L_c is then defined by

$$P_c f = p P_{c,0} f + (1-p) P_{c,1} f.$$

Equivalently, P_c is uniquely defined by the equation

$$\int (P_c f) g \, d\lambda = p \int f(g \circ T_{c,0}) \, d\lambda + (1-p) \int f(g \circ T_{c,1}) \, d\lambda \quad \forall f \in L^1(\lambda),\, g \in L^\infty(\lambda). \tag{2.28}$$

Since each L_c has a unique probability density f_c it follows from [P84, Theorem 1] that f_c is the L^1 limit of $(\frac{1}{n} \sum_{j=0}^{n-1} P_c^j 1)_{n \geq 1}$ and that it is the unique probability density that satisfies $P_c f_c = f_c$. From [I12, Theorem 5.2] each f_c is a function of bounded variation. We proceed by finding uniform bounds on the total variation and supremum norm of these densities.

For the second iterates of the Perron-Frobenius operators we have

$$P_c^2 f = \sum_{i,j=0}^{1} p_i p_j P_{c,j}(P_{c,i} f).$$

For each $\hat{c} \neq z_n, z_n^+, z_{n-1}^-$ for $n \geq 3$, we can find a uniform lower bound δ on the length of the intervals of monotonicity of any map $T_{c,\mathbf{u}}$, $\mathbf{u} \in \Omega^2$, for all values c that are close enough to $\hat{c}$. Indeed, for any map $T_{c,j}$, $j \in \Omega$, the shortest interval is either the first or the second left most subinterval of the partition. This is due to the fact that the length of any interval that belongs to the switch region can be larger that the length of the intervals to the right or to the left of the interval itself. Note that $\inf_{x \in [0,1]} |T'_{c,j}(x)| = 2$. Applying [BG97, Lemma 5.2.1] to $T_{c,j}$, $j = 0, 1$, and any of the second iterates $T_{c,\mathbf{u}}$, $\mathbf{u} \in \Omega^2$, gives that

$$Var(P_{c,j} f) \leq Var(f) + \frac{1}{\delta} \|f\|_1 \quad \text{and} \quad Var(P_{c,\mathbf{u}} f) \leq \frac{1}{2} Var(f) + \frac{1}{2\delta} \|f\|_1,$$

where Var denotes the total variation over the interval $[0, 1]$. Since these bounds do not depend on $c, j, \mathbf{u}$, the same estimates hold for P_c, so that for any function $f : [0, 1] \to \mathbb{R}$ of bounded variation and any $n \geq 1$,

$$Var(P_c^n f) \leq \frac{1}{2^{\lfloor n/2 \rfloor}} Var(f) + \left(2 + \frac{1}{\delta}\right) \|f\|_1. \tag{2.29}$$

Let $(c_k)_{k\geq 1}$ with $c_k \to \hat{c}$ be a sequence for which the lower bound δ holds for each k. This means in particular that if $\hat{c}$ is a critical point, then we consider sequences $(c_k)_k$ that converge to $\hat{c}$ from the right. For each k and n, denote by $f_{k,n} = \frac{1}{n}\sum_{i=0}^{n-1} P_{c_k} 1$. Since

$$\sup |f_{k,n}| \leq Var(f_{k,n}) + \int f_{k,n}\, d\lambda,$$

it follows from (2.29) that there is a uniform constant $C > 0$, independent of k, n, such that $Var(f_{k,n}), \sup |f_{k,n}| < C$. The same then holds for the limits f_{c_k}. Helly's Selection Theorem then gives the existence of a subsequence (k_i) and a function $\hat{f}$ of bounded variation, such that $f_{c_{k_i}} \to \hat{f}$ in $L^1(\lambda)$ and λ-a.e. and with $Var(\hat{f}), \sup |\hat{f}| < C$. This proves part 1.

By 2. and 3. above, what remains to finish the proof is to show that $P_{\hat{c}}\hat{f} = \hat{f}$. By (2.28) it is enough to show that for any compactly supported C^1 function $g : [0,1] \to \mathbb{R}$ it holds that

$$\left| \int (P_{\hat{c}}\hat{f}) g\, d\lambda - \int \hat{f} g\, d\lambda \right| = 0.$$

Note that

$$\begin{aligned}\left| \int (P_{\hat{c}}\hat{f}) g\, d\lambda - \int \hat{f} g\, d\lambda \right| \leq & p \left| \int \hat{f}(g \circ T_{\hat{c},0})\, d\lambda - \int \hat{f} g\, d\lambda \right| + \\ & (1-p) \left| \int \hat{f}(g \circ T_{\hat{c},1})\, d\lambda - \int \hat{f} g\, d\lambda \right|.\end{aligned}$$

For $j = 0, 1$ we can write

$$\begin{aligned}\left| \int \hat{f}(g \circ T_{\hat{c},j})\, d\lambda - \int \hat{f} g\, d\lambda \right| \leq & \left| \int \hat{f}(g \circ T_{\hat{c},j})\, d\lambda - \int f_{c_{k_i}}(g \circ T_{\hat{c},j})\, d\lambda \right| \\ & + \left| \int f_{c_{k_i}}(g \circ T_{\hat{c},j})\, d\lambda - \int f_{c_{k_i}}(g \circ T_{c_{k_i},j})\, d\lambda \right| \\ & + \left| \int f_{c_{k_i}}(g \circ T_{c_{k_i},j})\, d\lambda - \int \hat{f} g\, d\lambda \right|.\end{aligned}$$

The first and third integral on the right hand side can be bounded by $\|g\|_\infty \|\hat{f} - f_{c_{k_i}}\|_1 \to 0$. For the second integral, $\|f_{c_{k_i}}\|_\infty < C$ and $\int |g \circ T_{\hat{c},j} - g \circ T_{c_{k_i},j}|\, d\lambda \to 0$ by the Dominated Convergence Theorem. Hence, $\hat{f} = f_{\hat{c}}$ and $f_{c_k} \to f_{\hat{c}}$ in $L^1(\lambda)$. □

For each $\hat{c} \in \{z_n, z_n^+, z_{n-1}^-\}$ for some $n \geq 3$, a uniform lower bound δ exists if and only if the sequence of $(c_k)_k$ converges to $\hat{c}$ from the right. Indeed, if we approach $\hat{c}$ from the left, any map $T_{c,j}$, $j = 0, 1$, presents an interval of monotonicity of length $\hat{c} - c_k$, which becomes arbitrarily small for c_k approaching $\hat{c}$. This is the reason we excluded these points from the theorem.

2.4.12 Example. Consider the random Lüroth map L_c for $c = \frac{12}{25}$. By Proposition 2.4.10 we follow the random orbits of $c = \frac{12}{25}$ and $1 - c = \frac{13}{25}$.

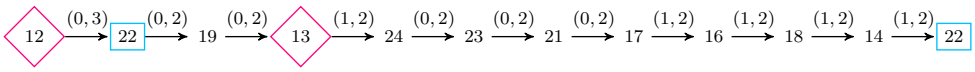


Figure 2.4: The random orbit of $c = \frac{12}{25}$ under the random map L_c. The numbers (s, d) above the arrows indicate which sign and digit are assigned at each iteration. The pink node indicates the points c and $1-c$ respectively, while the blue one the periodicity of the orbit. The orbit points are represented through their numerator only, since the common denominator is 25.

By Figure 2.4 it is clear that for any $\omega \in \{0,1\}^{\mathbb{N}}$

$$T_\omega^{1+j}\left(\frac{12}{25}\right) = T_\omega^{11+j}\left(\frac{12}{25}\right),$$

for all $j \geq 0$, so that the c-Lüroth expansion of c is ultimately periodic of period of length 10. A Markov partition of L_c can be given by

$$\left\{\frac{12}{25}, \frac{1}{2}, \frac{13}{25}, \frac{14}{25}, \frac{16}{25}, \frac{17}{25}, \frac{18}{25}, \frac{37}{50}, \frac{38}{50}, \frac{21}{25}, \frac{22}{25}, \frac{23}{25}, \frac{24}{25}, 1\right\},$$

and the corresponding Perron-Frobenius matrix P is the 13×13 square matrix

$$P = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2^{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2^{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2^{-1} & 0 & 2^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2^{-1} & 0 & 0 & 2^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2^{-1} & 0 & 0 & 0 & 2^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2^{-1} & 0 & 0 & 0 & 0 & 2^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 2^{-1} & 0 & 0 & 0 & 0 & 0 & 2^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 2^{-1} & 0 & 0 & 0 & 0 & 0 & 2^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 2^{-1} & 0 & 0 & 0 & 0 & 0 & & 2^{-1} & 0 & 0 \\
0 & 0 & 0 & 2^{-1} & 0 & 0 & 0 & 0 & 0 & & 0 & 2^{-1} & 0 \\
6^{-1} & 0 & 2^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2^{-1} & 0 \\
6^{-1} & 0 & 2^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2^{-1} \\
6^{-1} & 2^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2^{-1}
\end{pmatrix}.$$

We can also immediately determine a Markov partition for L_c in case $c = \frac{1}{2^k}$.

2.4.13 Proposition. *For any integer $k \geq 1$, the Markov partition of L_c for $c = \frac{1}{2^k}$ is given by*

$$\mathcal{S}_{\frac{1}{2^k}} \cup \bigcup_{i=2}^{k} \left\{1 - \frac{1}{2^i}\right\}.$$

Proof. First, note that for any integer $\ell > 3$, the map $T_{1,\frac{1}{\ell}}$ is Markov, and its Markov partition can be given by

$$\mathcal{S}_{\frac{1}{\ell}} \cup \left\{1 - \frac{1}{\ell}, \frac{2}{\ell}\right\}.$$

Indeed, since $\frac{1}{2} < 1 - \frac{1}{\ell} < 1 - \frac{1}{2\ell}$, then $T_{1,\frac{1}{\ell}}(1 - \frac{1}{\ell}) = \frac{2}{\ell}$. If ℓ is even, the point $z_{\frac{\ell}{2}} = \frac{2}{\ell}$ is already in $\mathcal{S}_{\frac{1}{\ell}}$, otherwise $\frac{2}{\ell+1} < \frac{2}{\ell} < \frac{2}{\ell-1}$ and moreover $\frac{2}{\ell} < z^-_{\frac{\ell-1}{2}} = \frac{2}{\ell-1} - \frac{2}{\ell-1}\frac{2}{\ell+1}\frac{1}{\ell}$. Thus, $T_{1,\frac{1}{\ell}}\left(\frac{2}{\ell}\right) = \frac{1}{2} + \frac{1}{2\ell} \in \mathcal{S}_{\frac{1}{\ell}}$.

We now follow the random orbit of $1-c=1-\frac{1}{2^k}$. Since

$$z_2^+=\frac{1}{2}+\frac{1}{2^{k+1}}<1-\frac{1}{2^i}<1-\frac{1}{2^{k+1}}=z_1^-,$$

for $i=2,3,\dots,k$ then

$$\mathcal{O}_{\frac{1}{2^k}}=\{T_{\omega_1^n}(1-c)\,:\,\omega\in\Omega^{\mathbb{N}},n\in\mathbb{N}\}=\bigcup_{i=2}^{k}\left\{1-\frac{1}{2^i},\frac{1}{2^{i-1}},1\right\}. \qquad \square$$

From Proposition 2.4.13, the Markov partition obtained for $L_{\frac{1}{2^k}}$ consists of $3\cdot(2^k-1)+(k-1)$ intervals. Indeed for $n\in\{3,\dots,2^k\}$ the subintervals $[z_n,z_{n-1})$ are split into 3 parts, while $[z_2^+,z_1^-]$ is divided into k pieces, due to the points $\{1-\frac{1}{2^i}\}_{i=2,\dots,k}$. The Perron-Frobenius matrix P can be computed by considering that

$$T_L\left(\left(1-\frac{1}{2^{i-1}},1-\frac{1}{2^i}\right)\right)=\left(1-\frac{1}{2^{i-2}},1-\frac{1}{2^{i-1}}\right)$$

and

$$T_A\left(\left(1-\frac{1}{2^{i-1}},1-\frac{1}{2^i}\right)\right)=\left(\frac{1}{2^{i-1}},\frac{1}{2^{i-2}}\right),$$

for $i=3,\dots,k$ and

$$T_L\left(\left(z_2^+,1-\frac{1}{2^2}\right)\right)=\left(\frac{1}{2^k},z_2\right)\text{ and }T_A\left(\left(z_2^+,1-\frac{1}{2^2}\right)\right)=\left(\frac{1}{2},\frac{1}{2^k}\right).$$

We obtain

$$P=\begin{pmatrix}
0 & (2^k(2^k-1))^{-1} & 0 & \cdots & 0 & p_02^{-1} & 0 & 0 & \cdots & p_12^{-1} & 0\\
0 & (2^k(2^k-1))^{-1} & 0 & \cdots & 0 & p_02^{-1} & 0 & 0 & \cdots & p_12^{-1} & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & (2^k(2^k-1))^{-1} & 0 & \cdots & 0 & p_02^{-1} & p_12^{-1} & 0 & \cdots & 0 & 0\\
0 & (2^k(2^k-1))^{-1} & 0 & \cdots & 0 & p_12^{-1} & p_02^{-1} & 0 & \cdots & 0 & 0\\
0 & (2^k(2^k-1))^{-1} & 0 & \cdots & 0 & p_12^{-1} & p_02^{-1} & 0 & \cdots & 0 & 0\\
0 & (2^k(2^k-1))^{-1} & 0 & \cdots & 0 & p_12^{-1} & 0 & p_02^{-1} & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & (2^k(2^k-1))^{-1} & 0 & \cdots & 0 & p_12^{-1} & 0 & 0 & \cdots & p_02^{-1} & 0\\
(2^k(2^k-1))^{-1} & 0 & (2^k(2^k-1))^{-1} & \cdots & 2^{-1} & 0 & 0 & 0 & \cdots & 0 & 2^{-1}\\
(2^k(2^k-1))^{-1} & 0 & (2^k(2^k-1))^{-1} & \cdots & 2^{-1} & 0 & 0 & 0 & \cdots & 0 & 2^{-1}
\end{pmatrix},$$

where $p_0=p$ and $p_1=1-p$. Any non-trivial eigenvector $v=(v_1,\dots,v_{3(2^k-1)+(k-1)})$ associated to the eigenvalue 1 of P defines an L_c-invariant density step function that takes value v_j on the subintervals (s_{j-1},s_j) of the chosen Markov partition. The vector v can be picked in such a way that it defines a probability measure $\mu_{p,\frac{1}{2^k}}$. In the following, we give a couple of examples for $k=2$ and $k=3$ that illustrate the procedure used to compute the density function and show the pattern of P.

2.4.14 Example. For $k=2$ a Markov partition of $L_{\frac{1}{4}}$ is given by the points

$$\left\{\frac{1}{4},\frac{13}{48},\frac{11}{36},\frac{1}{3},\frac{3}{8},\frac{11}{24},\frac{1}{2},\frac{5}{8},\frac{3}{4},\frac{7}{8},1\right\}.$$

This generates the matrix

$$P = \begin{pmatrix} 0 & 12^{-1} & 0 & 0 & 6^{-1} & 0 & 0 & p_0 2^{-1} & p_1 2^{-1} & 0 \\ 0 & 12^{-1} & 0 & 0 & 6^{-1} & 0 & 0 & p_0 2^{-1} & p_1 2^{-1} & 0 \\ 0 & 12^{-1} & 0 & 0 & 6^{-1} & 0 & 0 & p_0 2^{-1} & p_1 2^{-1} & 0 \\ 0 & 12^{-1} & 0 & 0 & 6^{-1} & 0 & 0 & p_0 2^{-1} & p_1 2^{-1} & 0 \\ 0 & 12^{-1} & 0 & 0 & 6^{-1} & 0 & 0 & p_0 2^{-1} & p_1 2^{-1} & 0 \\ 0 & 12^{-1} & 0 & 0 & 6^{-1} & 0 & 0 & p_0 2^{-1} & p_1 2^{-1} & 0 \\ 0 & 12^{-1} & 0 & 0 & 6^{-1} & 0 & 0 & p_1 2^{-1} & p_0 2^{-1} & 0 \\ 0 & 12^{-1} & 0 & 0 & 6^{-1} & 0 & 0 & p_1 2^{-1} & p_0 2^{-1} & 0 \\ 12^{-1} & 0 & 12^{-1} & 6^{-1} & 0 & 6^{-1} & 2^{-1} & 0 & 0 & 2^{-1} \\ 12^{-1} & 0 & 12^{-1} & 6^{-1} & 0 & 6^{-1} & 2^{-1} & 0 & 0 & 2^{-1} \end{pmatrix}.$$

The eigenspace associated to the eigenvalue 1 is

$$V = v \begin{pmatrix} 2/(2p+3) \\ 2/(2p+3) \\ 2/(2p+3) \\ 2/(2p+3) \\ 2/(2p+3) \\ 2/(2p+3) \\ (2p+1)/(2p+3) \\ (2p+1)/(2p+3) \\ 1 \\ 1 \end{pmatrix}, \quad v \in \mathbb{R} \setminus \{0\},$$

and the corresponding normalised density function is

$$v(x) = \begin{cases} \dfrac{4}{2p+3} & \text{if } x \in \left[\dfrac{1}{4}, \dfrac{1}{2}\right), \\ \dfrac{4p+2}{2p+3} & \text{if } x \in \left[\dfrac{1}{2}, \dfrac{3}{4}\right), \\ 2 & \text{if } x \in \left[\dfrac{3}{4}, 1\right]. \end{cases}$$

The explicit formula for the unique invariant density allows to say more on the digits frequency and the Lyapunov exponent. Recall from Example 2.4.2 that the frequency of a digit $d \in \{2, 3, 4\}$ is given by

$$\pi_d = \lim_{n\to\infty} \frac{\#\{1 \le j \le n : \, d_j(\omega, x) = d\}}{n} = \pi_{(0,d)} + \pi_{(1,d)}.$$

It then follows by Birkhoff's Ergodic Theorem that, for $m_p \times \mu_{p,\frac{1}{4}}$-a.e. point (ω, x),

$$\pi_2 = \frac{2p+2}{2p+3}, \quad \pi_3 = \frac{2}{3(2p+3)}, \quad \pi_4 = \frac{1}{3(2p+3)}.$$

Recall the definition of Λ from (2.15). We obtain

$$\Lambda_{m_p \times \mu_{p,\frac{1}{4}}} = \frac{p \log 64 + \log 27648}{6p+9} < \Lambda_{m_p \times \lambda}.$$

That is, for $m_p \times \mu_{p,\frac{1}{4}}$-a.e. point, the approximants $\frac{p_n}{q_n}$ obtained by the iteration of the random $\frac{1}{4}$-Lüroth map are in general worse than the corresponding ones obtained via the random 0-Lüroth map with countably many branches.

2.4.15 Example. For $k = 3$ the Markov partition of $L_{\frac{1}{8}}$ is obtained by adding to the set $\mathcal{S}_{\frac{1}{8}}$ the points $\frac{3}{4}$ and $\frac{7}{8}$, resulting in the matrix

$$P = \begin{pmatrix}
0 & 56^{-1} & 0 & 0 & 42^{-1} & \dots & 0 & p_0 2^{-1} & 0 & p_1 2^{-1} & 0 \\
0 & 56^{-1} & 0 & 0 & 42^{-1} & \dots & 0 & p_0 2^{-1} & 0 & p_1 2^{-1} & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 56^{-1} & 0 & 0 & 42^{-1} & \dots & 0 & p_0 2^{-1} & 0 & p_1 2^{-1} & 0 \\
0 & 56^{-1} & 0 & 0 & 42^{-1} & \dots & 0 & p_0 2^{-1} & p_1 2^{-1} & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 56^{-1} & 0 & 0 & 42^{-1} & \dots & 0 & p_0 2^{-1} & p_1 2^{-1} & 0 & 0 \\
0 & 56^{-1} & 0 & 0 & 42^{-1} & \dots & 0 & p_1 2^{-1} & p_0 2^{-1} & 0 & 0 \\
0 & 56^{-1} & 0 & 0 & 42^{-1} & \dots & 0 & p_1 2^{-1} & p_0 2^{-1} & 0 & 0 \\
0 & 56^{-1} & 0 & 0 & 42^{-1} & \dots & 0 & p_1 2^{-1} & 0 & p_0 2^{-1} & 0 \\
0 & 56^{-1} & 0 & 0 & 42^{-1} & \dots & 2^{-1} & 0 & 0 & 0 & 2^{-1} \\
0 & 56^{-1} & 0 & 0 & 42^{-1} & \dots & 2^{-1} & 0 & 0 & 0 & 2^{-1}
\end{pmatrix}.$$

The associated $L_{\frac{1}{8}}$-invariant probability density of the measure $\mu_{p,\frac{1}{8}}$ is

$$v(x) = \begin{cases}
\dfrac{8}{2p^2+3p+5} & \text{if } x \in \left[\dfrac{1}{8}, \dfrac{1}{4}\right), \\
\dfrac{4p+4}{2p^2+3p+5} & \text{if } x \in \left[\dfrac{1}{4}, \dfrac{1}{2}\right), \\
\dfrac{4p^2+2p+4}{2p^2+3p+5} & \text{if } x \in \left[\dfrac{1}{2}, \dfrac{3}{4}\right), \\
\dfrac{4p^2+6p+4}{2p^2+3p+5} & \text{if } x \in \left[\dfrac{3}{4}, \dfrac{7}{8}\right), \\
\dfrac{4p^2+6p+12}{2p^2+3p+5} & \text{if } x \in \left[\dfrac{7}{8}, 1\right].
\end{cases}$$

The frequency of the digits $d \in \{2, 3, \dots, 8\}$ is given by

$$\begin{aligned}
&\pi_2 = \tfrac{2p^2+2p+3}{2p^2+3p+5}, \quad \pi_3 = \tfrac{2(p+1)}{3(2p^2+3p+5)}, \quad \pi_4 = \tfrac{p+1}{3(2p^2+3p+5)}, \\
&\pi_5 = \tfrac{2}{5(2p^2+3p+5)}, \quad \pi_6 = \tfrac{4}{15(2p^2+3p+5)}, \quad \pi_7 = \tfrac{2}{21(2p^2+3p+5)}, \\
&\pi_8 = \tfrac{1}{7(2p^2+3p+5)}.
\end{aligned}$$

Moreover for $m_p \times \mu_{p,\frac{1}{8}}$-a.e. (ω, x) we have by (2.22) that

$$\begin{aligned}\Lambda(\omega,x) &= \sum_{d=2}^{8} \log(d(d-1))\mu_{p,\frac{1}{8}}\left(\left[\frac{1}{d},\frac{1}{d-1}\right)\right)\\ &= \frac{8}{2p^2+3p+5}\left(\frac{\log 56}{56}+\frac{\log 42}{42}+\frac{\log 30}{30}+\frac{\log 20}{20}\right)\\ &\quad+\frac{4p+4}{2p^2+3p+5}\left(\frac{\log 12}{12}+\frac{\log 6}{6}\right)+\frac{2p^2+2p+3}{2p^2+3p+5}\log 2\\ &\cong \frac{1.38628p^2+3.40908p+7.49448}{2p^2+3p+5} < \frac{3}{2} < \Lambda_{m_p\times\lambda}.\end{aligned}$$

§2.5 Remarks

1. Finding an explicit formula for the density of an absolutely continuous invariant measure is not straightfoward. The statement holds for deterministic as well as for random interval transformations, including e.g. the Ito-Tanaka continued fraction transformations and the affine random map L_c with finitely many branches. Also in the Markov case, even though a priori it is possible to represent the Perron-Frobenius operator in matrix form, it might be that the matrix is very heavy to compute due to the high number of points of the Markov partition. Chapter 3 provides a procedure to explicitly determine the density for general random piecewise affine interval maps.

2. Just as the alternating Lüroth map T_A can be seen as a linearisation of the Gauss map G, the maps $T_{0,c}$ and $T_{1,c}$ can be seen as a linearised version of the flipped α-continued fractions maps $T_{1-\alpha}$ and T_α, that will be introduced and examined in Chapter 4. When talking about approximation theory, it is indeed inevitable to refer to continued fractions, since the best approximants of an irrational number x are obtained by considering the convergents of the point in the standard continued fraction expansion.

3. For piecewise affine expanding interval maps, the Markov property makes any invariant density a step function, i.e., a function that is constant on each element of the Markov partition. The same is true when replacing the Markov property with a matching condition, as shown in [BCMP18, Theorem 1.2]. The notion of the dynamical phenomenon of matching, defined for deterministic systems in (1.2.7), will be extended for random interval maps in Chapter 5. Furthermore, an analogous results on the structure of the density function will be derived using the algebraic procedure of Chapter 3.