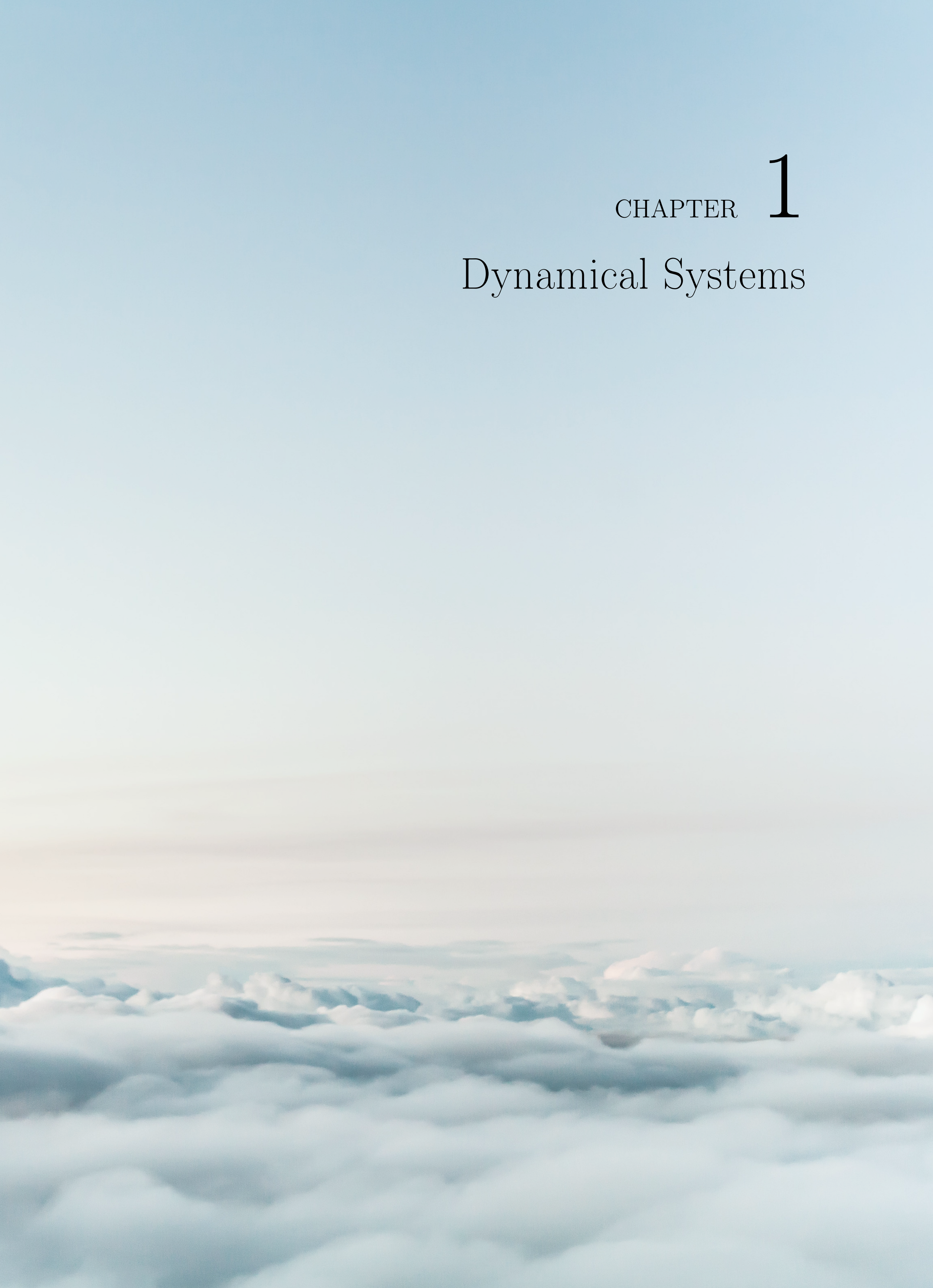
Measures and matching for number systems

Maggioni, M.

CHAPTER 1

Dynamical Systems



§1.1 Motivation and context

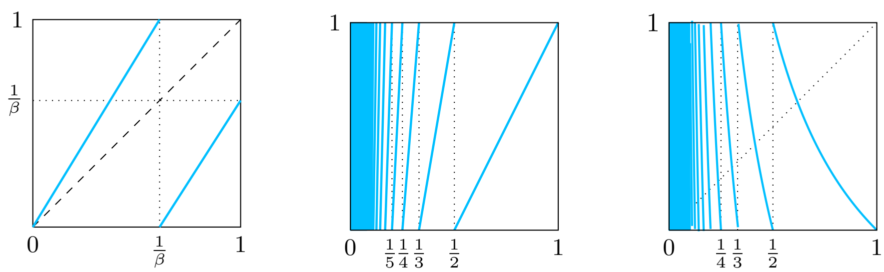
Number expansions are ways of representing numbers with specific symbols and consistent rules. The algorithms, that combine the digits to code numbers, can be obtained through the repeated iteration of particular interval maps, called *number systems*. The advantage of this description is that all tools coming from Ergodic Theory are available, and they can be used to describe general properties of the expansions. This thesis adopts this dynamical approach to study new number expansions.

The *interval maps* of interest are special instances of discrete-time dynamical systems. A discrete-time dynamical system models the evolution of a phenomenon over time through a transformation T acting on a set X. Morally, X represents all possible states and T is the law that rules the evolution, so that, if a system is at state $x \in X$ at time zero, then it will be at state $T(x)$ after one unit of time, and in general at state

$$T^n(x) = \underbrace{T \circ T \circ \cdots \circ T}_{n}(x),$$

after n units of time, for any $n \in \mathbb{N}$. For interval maps, the state space X is an interval in $\mathbb{R}$ and the evolution of any number $x \in X$ is described in terms of its orbit $\{T^n(x)\}_{n\in\mathbb{N}}$.

The procedure that codes numbers through the iteration of an interval map $T : X \to X$ is the following. The interval X is divided into finitely or countably many subintervals and to each of them a *digit* d is assigned. A number expansion of a point $x \in X$ is obtained by considering the sequence of digits $(d_n(x))_{n\in\mathbb{N}}$ realised by following the orbit of x. Specifically, the first digit of the sequence is determined by the symbol associated to the subinterval in which x lies, the second by the one corresponding to the position of $T(x)$ and in general the n-th digit by the position of $T^{n-1}(x)$. Figure 1.1 shows examples of classical number systems: the β-transformation T_β for $\beta = \frac{1+\sqrt{5}}{2}$, the Lüroth map T_L and the Gauss map G.



(a) T_β (b) T_L (c) G

Figure 1.1: Classical examples of interval maps producing number expansions.

This strong connection between the orbits of a point x, $\{T^n(x)\}_{n\in\mathbb{N}}$, and its corresponding digits sequence $(d_n(x))_{n\in\mathbb{N}}$ is the reason why the study of the former reveals properties of the latter. To investigate the quantities $T^n(x)$, rather than computing all orbits of all points, one looks at the probability that the orbits of typical points pass by certain areas of the space X, which corresponds to the probability that specific digits are assigned. A set of high probability corresponds then to a set that is often visited by most of the points of X. In Ergodic Theory, this information is encoded in the so-called invariant measures.

An *invariant measure* is a measure that is preserved by the action of T, in the sense that the set of points that T maps to a set in one unit of time has the same measure as the set itself. For number systems, explicitly knowing such measures is extremely helpful for, e.g., the computation of the *frequency* of the digits in the associated number expansions, which expresses the percentage of seeing a digit d in typical expansions, or the *entropy*, which estimates the possible number of different blocks of digits of length n that can be found by the repeated iteration of T. Ideally, meaningful invariant measures should describe the long-term behaviour of the orbits for a large set of points. Here large is intended with respect to the Lebesgue measure, which is used as a reference measure. For this reason, of particular relevance are invariant measures that are *absolutely continuous* with respect to Lebesgue and are given in terms of density functions. While there exist various results on the existence of such measures for interval maps, finding explicit formulas for the corresponding density functions is still a delicate problem for very many dynamical systems.

This thesis provides explicit expressions for the density functions of absolutely continuous invariant measures for general families of interval maps, that include *random maps* and *infinite measure transformations*, not necessarily number systems. In the random setting, at each time step, instead of a single transformation, a set of maps is available and one of them is applied according to a probabilistic regime. In the infinite configuration, the measure of the state space is infinite and the tools coming from probability theory are no longer available. Natural extensions, the Perron-Frobenius operator and the dynamical phenomenon of *matching* are some of the techniques exploited to obtain such results. In particular, in this thesis the notion of matching is for the first time recognised in an infinite measure system and the definition, known so far for deterministic transformations only, is extended to cover random interval maps as well.

This thesis also presents new developments in the area of number expansions, by introducing new representations of numbers obtained through the iterations of random maps and infinite measure transformations. These include random c-Lüroth expansions, flipped α-continued fractions and random signed binary expansions. The properties of these number expansions are analysed by applying the results obtained previously on the density functions. In particular, explicit expressions for the measures are used to investigate the digit frequency, the Krengel entropy, the Hamming weight and the quality of the approximations.

The remaining part of this chapter introduces all the necessary mathematical tools in more details.

§1.2 Discrete dynamical systems

Ergodic theory is the branch of dynamical systems that studies measure-preserving transformations defined on measure spaces. Here we introduce the basic concepts.

§1.2.1 Deterministic

Let $T : X \to X$ be a transformation. To study the long term behaviour of the system we determine the probabilities to observe typical trajectories within certain areas of the space. This information is provided by the invariant measure.

1.2.1 Definition (Measure preserving). A *measure preserving dynamical system* is defined as the quadruple $(X, \mathcal{B}, \mu, T)$ where the triple $(X, \mathcal{B}, \mu)$ is a measure space and the measurable map $T : X \to X$ *preserves the measure* μ, i.e. $\mu(T^{-1}(B)) = \mu(B)$ for all $B \in \mathcal{B}$. The system is also said to be *T-invariant* with respect to μ.

Equivalently, T is measure-preserving if $T_*\mu = \mu$, for $T_*\mu$ the push-forward of μ with respect to T, i.e.,

$$T_*\mu(B) = \mu(T^{-1}(B)), \quad B \in \mathcal{B}.$$

$(X, \mathcal{B}, \mu)$ is said to be a *finite* or *infinite* measure space if $\mu(X) < \infty$ or $\mu(X) = \infty$, respectively. In this dissertation, for the infinite case, we still assume X to be a countable union of sets of finite measure, i.e., we ask the space to be σ-finite. Two dynamical systems that present the same dynamics, that is for which the long-term and average behaviours are essentially the same, are called isomorphic. We make this precise in the following definition.

1.2.2 Definition (Isomorphic). $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{F}, \nu, S)$ are said to be *isomorphic* if there exist sets $B \in \mathcal{B}$ and $F \in \mathcal{F}$ and a map $\theta : B \to F$ such that

1. $\mu(X \setminus B) = \nu(Y \setminus F) = 0$,
2. $T(B) \subseteq B$ and $S(F) \subseteq F$,
3. θ is invertible and bi-measurable,
4. $S \circ \theta = \theta \circ T$,
5. $\nu = \mu \circ \theta^{-1}$.

In the following, let $(X, \mathcal{B}, \mu, T)$ be a measure preserving dynamical system. A subset $B \in \mathcal{B}$ is said to be invariant for T if $T^{-1}(B) = B$. Clearly, if the space X is the union of two or more disjoint invariant subsets of positive measure, then the study of the properties of T on X reduces to the study of its properties on each of these invariant subsets. It is therefore natural to study the transformations defined on spaces that do not decompose into such subsets. Such a property is called ergodicity.

1.2.3 Definition (Ergodicity). $(X, \mathcal{B}, \mu, T)$ is said to be *ergodic* with respect to μ if for every $B \in \mathcal{B}$, such that $T^{-1}(B) = B$, either $\mu(B) = 0$ or $\mu(X \setminus B) = 0$, i.e., if the only invariant sets are trivial.

A property is said to hold almost everywhere (a.e.), if the set of points for which it does not hold is contained in a set of measure zero. In a probability space, ergodicity implies Birkhoff's Ergodic Theorem, which relates spatial averages to temporal averages.

1.2.4 Theorem (Birkhoff's Ergodic Theorem). *If $(X, T, \mu, \mathcal{B})$ is ergodic and $\mu(X) < \infty$, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(x) = \int_X f d\mu, \quad \mu \text{ a.e.}, \forall f \in L^1(X, \mathbb{R}).$$

Without the assumption $\mu(X) < \infty$, Birkhoff's Ergodic Theorem does not hold. More generally, for infinite measure preserving dynamical systems many classical results of ergodic theory fail, and a new approach is required, see Section 1.2.3.

When the dynamical system under consideration is an interval map $T : I \to I$, that leads to number expansions, Birkhoff's Ergodic Theorem can be used to obtain the average number of occurrences, called *frequency*, of specific digits in typical expansions. Given $x \in I$, the number of visits of x to a measurable set B of positive measure is given by

$$\frac{\#\{0 \le k \le n-1 : T^k(x) \in B\}}{n} = \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_B(T^k(x)) \to \mu(B).$$

However, to apply Theorem 1.2.4, it is necessary to know the invariant measure μ and finding an invariant measure for a transformation is not an easy task. The measures in which we are interested are equivalent to the Lebesgue measure λ.

1.2.5 Definition (Absolutely continuous). A measure μ on Borel subsets of the real line is said to be *absolutely continuous* with respect to λ if for every measurable set B, $\lambda(B) = 0$ implies $\mu(B) = 0$. Equivalently, there exists a Lebesgue integrable non-negative function f, called *density*, on the real line such that

$$\mu(B) = \int_B f d\lambda, \tag{1.1}$$

for all Borel subsets B of the real line. The density, also known as the Radon-Nikodym derivative, of the absolutely continuous measure μ is only defined up to a.e. equivalence. If also λ is absolutely continuous with respect to μ, then the measures are said to be *equivalent*.

Most of the results on the existence of such measures are proven by studying the transfer operator P, known as Perron-Frobenius operator, of the system. The operator P of an interval map $T : I \to I$ is defined by

$$\int_B P_T f d\lambda = \int_{T^{-1}(B)} f d\lambda, \tag{1.2}$$

for any $f \in \mathcal{L}^1(I, \mathbb{R})$ and $B \in \mathcal{B}$. If the map T is also piecewise affine, then P_T can be written as

$$P_T f(x) = \sum_{y \in T^{-1}(x)} \frac{f(y)}{|T'(y)|}.$$

We refer to the book of [BG97] for a classical introduction to the subject and the properties of this operator. An interval map $T : I \to I$ is said to be *expanding* if $|T'(x)| > 1$, for any point x in which the derivative is defined. For one-dimensional piecewise monotonic and expanding transformations $T \in \mathcal{C}^2$, the existence of absolutely continuous invariant measures (acim) is by now pretty well understood. Indeed the seminal paper of [LY73] shows that a fixed point of the Perron-Frobenius operator P of such a transformation T exists and it is the density of an absolutely continuous invariant measure μ. For these transformations, [LY78] shows that the number of acims that a map admits is strictly connected to the number of discontinuities. Furthermore, [K90, G09] propose two similar procedures to obtain formulas for the densities of such measures, by connecting them to the solution vectors of a matrix equation. In Chapter 3 it is shown how to obtain the formulas for the densities when the dynamics of the system is not deterministic, but countably many transformations are available at each iteration.

Another way to possibly obtain the formulas for the densities of acims is via the construction of a natural extension. Roughly speaking, a natural extension of a system is the minimal invertible dynamical system that contains the original system as a subsystem. Invertibility is obtained by extending the dimensions of the space of the original system. The existence of such a construction is obtained in [R61]. In the same article, it is also shown that any two natural extensions of the same dynamical system are isomorphic.

1.2.6 Definition (Natural extension). Let $(X, \mathcal{B}, \mu, T)$ be a dynamical system with T a non-invertible transformation. An invertible dynamical system $(Y, \mathcal{F}, \nu, S)$ is a *natural extension* of $(X, \mathcal{B}, \mu, T)$, if there exist two sets $B \in \mathcal{B}$ and $F \in \mathcal{F}$ and a function $\theta : F \to B$ such that:

1. $\mu(X \setminus B) = \nu(Y \setminus F) = 0$,
2. $T(B) \subseteq B$ and $S(F) \subseteq F$,
3. θ is measurable, measure preserving and surjective,
4. $\theta \circ S = T \circ \theta$,
5. $\bigvee_{n=0}^{\infty} S^n \theta^{-1}(\mathcal{B}) = \mathcal{F}$, where $\bigvee_{n=0}^{\infty} S^n \theta^{-1}(\mathcal{B})$ is the smallest σ-algebra containing the σ-algebras $S^n \theta^{-1}(\mathcal{B})$ for all $n \geq 0$.

This approach has been shown to be very successful in the non-affine case, especially in the field of continued fraction transformations, see Section 1.3.3 and [N81, K91, KSS12, AS13], for example. Chapter 4 exhibits a natural extension for a class of continued fraction transformations with infinite measure.

For some families of transformations, e.g. continued fraction maps, the natural extension map is quite canonical, and the difficulty of the approach lies in finding the appropriate domain on which the map acts. See Section 1.3.3 and Figure 1.8 for an example. For one-dimensional maps, it is often the case that the natural extension is a planar map and the function θ from Definition 1.2.6 corresponds to the projection on one of the two components. What has been proved to be fundamental in such instances, in order to recover the domain of the natural extension, is the property of matching, or synchronization, of the original system T.

1.2.7 Definition (Matching). A piecewise smooth interval map T is said to have *matching* if for any discontinuity point c, of T or its derivative T', its orbits of the left and right limits eventually meet. That is, there exist non-negative integers M and N, called *matching exponents*, such that

$$T^M(c^-) = T^N(c^+), \tag{1.3}$$

for

$$c^- = \lim_{x \uparrow c} T(x) \quad \text{and} \quad c^+ = \lim_{x \downarrow c} T(x).$$

For specific families of interval maps defined on a finite measure space, and in particular for β-transformations and continued fraction type maps, the property of matching has been thoroughly analysed in order to find expressions for the invariant densities. See for instance [NN08, DKS09, KSS10, KS12, KSS12, DK17, BCK17, BCMP18, KLMM20]. Differently from these results, Chapter 4 considers matching for a class of infinite dynamical systems and Chapter 5 introduces the notion of random matching for random dynamical systems. To this aim, in the following sections we give some background on random and infinite (measure) dynamical systems.

§1.2.2 Random

The qualitative analysis of iterations of a single map can be extended to a more general setting where, at each step, an element is chosen from a set of transformations according to a stationary process. See [K86, A98] for a basic introduction. This generalisation is quite natural when considering that, in most physical applications, at every iteration it is usually the case that not the same map, but a slightly modified version of it, is applied. This phenomenon is usually referred to as a stochastic perturbation. Such systems have been recently used also to study interference effects in quantum mechanics, fractals and particle systems on lattices see [BG92, B93, KY07], for example. In general, there is a rich and quite recent literature on random maps, both considered as position independent random perturbations of transformations, and as position dependent ones in the context of iterated function systems, see [G84, BL92, BK93], for example. In all these situations, the dynamics changes from deterministic to random. Roughly speaking, a random map describes a system evolving in discrete time in which at each time step one of a number of transformations is chosen according to a probabilistic regime and applied. One way to describe a random map, in which the process for choosing the individual maps is i.i.d., is with a pseudo-skew product transformation.

For an at most countable set of symbols Ω, denote by $\Omega^{\mathbb{N}}$ the set of *one-sided sequences*. The *left shift* $\sigma : \Omega^{\mathbb{N}} \to \Omega^{\mathbb{N}}$ maps a sequence $(\omega_n)_{n\in\mathbb{N}}$ to a sequence $(u_n)_{n\in\mathbb{N}}$ such that $u_n = \omega_{n+1}$ for all $n \geq 1$.

1.2.8 Definition (Pseudo-skew product map). Let $\{T_j : I \to I\}_{j\in\Omega}$ be a collection of transformations defined on the same interval I and let $\Omega \subseteq \mathbb{N}$ be the index set of these available maps. Let $\sigma : \Omega^{\mathbb{N}} \to \Omega^{\mathbb{N}}$ be the left shift on one-sided sequences. The *random* or *pseudo-skew product map* $R : \Omega^{\mathbb{N}} \times I \to \Omega^{\mathbb{N}} \times I$ is defined by

$$R(\omega, x) = (\sigma(\omega), T_{\omega_1} x),$$

such that the coordinates of ω determine which of the maps T_j is applied at each step.

1.2.9 Definition (Stationary measure for pseudo-skew product maps). Let $\mathbf{p} = (p_j)_{j\in\Omega}$ be a positive probability vector, i.e., $p_j > 0$ for all $j \in \Omega$ and $\sum_{j\in\Omega} p_j = 1$. Each p_j represents the probability with which we choose the map T_j. Denote by $m_{\mathbf{p}}$ the $\mathbf{p}$-Bernoulli measure on $\Omega^{\mathbb{N}}$ and let $\mu_{\mathbf{p}}$ be a probability measure on I that is absolutely continuous with respect to the one-dimensional Lebesgue measure λ. Denote its density by $\frac{d\mu_{\mathbf{p}}}{d\lambda} = f_{\mathbf{p}}$. If $\mu_{\mathbf{p}}$ satisfies for each Borel set $B \subseteq I$ that

$$\mu_{\mathbf{p}}(B) = \int_B f_{\mathbf{p}}\, d\lambda = \sum_{j\in\Omega} p_j \mu_{\mathbf{p}}(T_j^{-1} B), \tag{1.4}$$

then the product measure $m_{\mathbf{p}} \times \mu_{\mathbf{p}}$ is an invariant probability measure for R. $\mu_{\mathbf{p}}$ is called a *stationary measure* and $f_{\mathbf{p}}$ an *invariant density* for the pseudo-skew product R.

There exist various sets of conditions under which the existence of such an invariant measure is guaranteed, see for example [M85, P84, GB03, BG05, I12]. In particular, for piecewise random interval maps see the results of [P84] for a finite family of transformations and of [M85] and [I12] for the countable case.

The introduction of randomness in systems defining number expansions has some quite remarkable consequences. For instance, a single random map produces many more expansions per number than a deterministic transformation, allowing the study of the properties of many number expansions simultaneously. In Chapter 5 this property is used to compute the frequency of the digit 0 in signed binary expansions for Lebesgue almost every point $x \in [-1, 1]$.

§1.2.3 Infinite

Infinite ergodic theory studies dynamical systems with an *infinite* invariant measure. These systems differ from transformations admitting a finite invariant measure, because for them most of the tools coming from probability theory are not applicable and, as a consequence, classic results from (finite) ergodic theory do not hold. We refer to the books of [A97, KMS16] for an introduction to the subject.

A first, crucial example is presented by Birkhoff's Ergodic Theorem. For a finite measure system $T : X \to X$, $\mu(X) < \infty$, Theorem 1.2.4 describes the limiting behaviour of the number of times the orbit of a typical point enters a specific region of the space. Precisely, for any measurable set B and $x \in X$, let

$$S_n^B(x) = \sum_{k=0}^{n-1} \mathbf{1}_B \circ T^k(x), \quad n \geq 1.$$

$S_n^B(x)$ counts how often the orbit of x visits the set B before time n. Birkhoff's Ergodic Theorem expresses the rate at which the occupation time of B diverges as being proportional to n, asymptotically the same for typical points and dependent on the set B only through its measure, i.e.,

$$\frac{1}{n} S_n^B(x) \xrightarrow[n\to\infty]{} \frac{\mu(B)}{\mu(X)} \quad \mu\text{-a.e. } x \in X.$$

For an infinite measure preserving transformation, this is no longer true, as

$$\frac{1}{n} S_n^B(x) \xrightarrow[n\to\infty]{} 0 \quad \mu\text{-a.e. } x \in X,$$

not revealing any dependence on the set. This is just the first of the many substantial differences between finite and infinite dynamical systems. Another one involves Poincaré's Recurrence Theorem. For a finite measure preserving system T, the theorem says that for every measurable set B of positive measure, almost every point of the set will return to the set itself under iterations of T. For the infinite scenario, this is not always the case, since only *conservative* systems have this property.

1.2.10 Definition (Conservative). A measure preserving dynamical system $(X, \mathcal{B}, \mu, T)$ is said to be *conservative* if

$$\mu\left(\bigcup_{n=1}^{\infty} T^{-n}B \setminus B\right) = 0 \quad \text{for all } B \in \mathcal{B},\ \mu(B) > 0.$$

1.2.11 Definition (Wandering sets). A set $W \in \mathcal{B}$ is said to be a *wandering set* for T if $\{T^{-n}(W) : n \geq 0\}$ is a collection of pairwise disjoint sets.

In other words, a measure preserving dynamical system is conservative if every wandering set has measure 0. A handy way of determining if a system is conservative is given by the existence of sweep-out sets.

1.2.12 Definition (Sweep-out sets). A set $Y \in \mathcal{B}$ is said to be a *sweep-out* set for T if $0 < \mu(Y) < \infty$ and

$$\mu\left(X \setminus \bigcup_{n=0}^{\infty} T^{-n}(Y)\right) = 0.$$

1.2.13 Theorem (Maharam's Recurrence Theorem). *If $(X, T, \mu, \mathcal{B})$ is measure preserving and has a sweep-out set, then it is conservative.*

1.2.14 Theorem (Aaronson's Ergodic Theorem). *Let $(X, T, \mu, \mathcal{B})$ be a conservative, ergodic, measure preserving infinite system, and let $(a_n)_{n\geq 1}$ be any positive sequence. Then*

$$\overline{\lim}_{n\to\infty} \frac{1}{a_n} S_n^B(x) = \infty \quad \mu\text{-a.e. } x \in X,$$

or

$$\underline{\lim}_{n\to\infty} \frac{1}{a_n} S_n^B(x) = 0 \quad \mu\text{-a.e. } x \in X.$$

The theorem tells that the pointwise behaviour of occupation times is extremely chaotic and that an analogous version of Birkhoff's Ergodic Theorem for infinite measure systems can't provide the same amount of information. One powerful way to obtain information on such systems is by looking at the dynamics that happens only in specific subsets of finite measure. More precisely, let Y be a sweep-out set of a conservative system T with measure μ and define the *hitting time* of Y, by

$$\varphi : X \to \mathbb{N}, \quad \varphi(x) = \inf\{n \geq 1 : T^n(x) \in Y\}.$$

When restricting the function φ to the set Y, the map φ is called the *return time* to Y and it counts the number of steps the orbit of $x \in Y$ needs to come back to Y. Note that the conservativity of the map T ensures that $\varphi(x) < \infty$ for μ-a.e. point $x \in Y$.

1.2.15 Definition (Inducing). The map $T_Y : Y \to Y$ given by

$$T_Y(x) = T^{\varphi(x)}(x)$$

is called the *induced map* of T on Y.

T_Y is an acceleration of T, achieved by applying as many iterates of T as is required to come back to Y. Basic properties of T, such as invariant measures, ergodicity and conservativity can be recovered from T_Y by a proper choice of a sweep-out set Y. An example of an induced map is given in Section 1.3.3.

We now discuss in more detail some examples of classic interval maps producing number expansions: β-transformations, Lüroth maps and continued fraction transformations. We introduce them separately, in the next sections.

§1.3 Number systems

Number systems offer algorithmic ways of coding numbers with a specific set of symbols, called *digits*. This set, called *alphabet*, can consist of finitely or countably many elements. For example, in the canonical decimal system, we represent numbers as sequences of digits $0, 1, \ldots, 9$, with each position in the sequence corresponding to a specific power of 10. For computer hardware and software, we usually use the binary system, so that we write numbers as strings of digits 0 and 1, where the position of each digit corresponds this time to a specific power of 2. For example, the number $\frac{37}{2}$, can be written as

$$\frac{37}{2} = 1 \cdot 10^1 + 8 \cdot 10^0 + 5 \cdot 10^{-1} = 1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 + 1 \cdot 2^{-1}.$$

The first expression leads to the sequence 18.5, the second to 10010.1, which is the representation of the number in the binary system. The sequences of digits can be obtained through the iteration of one-dimensional discrete time dynamical systems $T : X \to X$. The idea is to discretize the space, by dividing up the state space X into finitely or countably many subintervals and keep track of which piece the system visits at each time step. This is done by assigning a digit to each subinterval: in this way, the evolution of a point $\{T^n(x)\}_n$ is given in terms of an infinite sequence of symbols. Infinite sequences are the main objects of study in symbolic dynamics. In the following paragraph we give some basic definitions that will be used in the coming chapters.

For an alphabet $\mathcal{A}$, the set of one-sided sequences $(a_n)_{n\in\mathbb{N}}$ of symbols from $\mathcal{A}$ is denoted by $\mathcal{A}^{\mathbb{N}}$. A *word*, or a block, over the alphabet $\mathcal{A}$ is a finite string of symbols from $\mathcal{A}$. The *empty word* is denoted by ϵ and it represents the sequence of no symbols. The *length* of a block u corresponds to the number of symbols it contains, and it is denoted by $|u|$. For each $n \in \mathbb{N}$, the set $\mathcal{A}^n$ is the set of all blocks of length n of symbols from $\mathcal{A}$, and we set $\mathcal{A}^0 = \{\epsilon\}$. We use square brackets to denote cylinder sets, i.e., for any block u,

$$[u] = \{(a_n)_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}} : a_1 \cdots a_{|u|} = u\}.$$

The *concatenation* of a pair of words u, v is given by the word uv, of length $|uv| = |u| + |v|$. For each $n \in \mathbb{N}$, u^n corresponds to the concatenation of n copies of u, and we also define $u^\infty = uuu\ldots$ and $u^0 = \epsilon$. Recall that the *left shift* $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ maps each sequence $(a_n)_{n\in\mathbb{N}}$ to a sequence $(b_n)_{n\in\mathbb{N}}$ such that $b_n = a_{n+1}$. We refer to the book [LM95] for a basic introduction on the topic.

The next example shows an interval map producing number expansion with alphabet $\mathcal{A} = \{0, 1\}$.

1.3.1 Example. Let

$$D(x) = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}), \\ 2x - 1 & \text{if } x \in [\frac{1}{2}, 1], \end{cases}$$

be the *doubling map*, see Figure 1.2. Consider the partition of the interval given by $I_0 = [0, \frac{1}{2})$ and $I_1 = [\frac{1}{2}, 1]$, and assign the digit 0 to I_0 and the digit 1 to I_1.

The iteration of the map D to each point $x \in [0, 1]$ produces *binary expansions*, i.e., sequences $(d_n(x))_{n\in\mathbb{N}}$ of 0 and 1 such that

$$x = \sum_{n=1}^{\infty} \frac{d_n(x)}{2^n}.$$

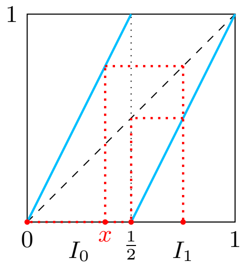


Figure 1.2: The doubling map D and in red the orbit of $x = 3/8$.

For example, let $x = 3/8$ and consider its orbit under the doubling map D. Since $x \in I_0$, the first digit in its binary expansion is $d_1 = 0$. $D(x) = 3/4 \in I_1$, so $d_2 = 1$. $D^2(x) = 1/2 \in I_1$ so $d_3 = 1$. Finally, $D^n(x) = 0 \in I_0$ for $n \geq 3$, since 0 is a fixed point of the map D, i.e., $D(0) = 0$, and so $d_n = 0$ for $n \geq 4$, obtaining the sequence 0110^∞, such that

$$\frac{3}{8} = \frac{0}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \sum_{n \geq 4} \frac{0}{2^n}.$$

See Figure 1.2 for a visualisation of the orbit of $x = 3/8$.

The doubling map is a member of a family of piecewise affine maps called β-transformations, that produce representations of numbers as series of powers of $\beta \in \mathbb{R}_{>1}$. We discuss them in the next section.

§1.3.1 β-expansions

For $\beta > 1$, any real number $x \in [0, 1)$ can be written as

$$x = \sum_{n=1}^{\infty} \frac{d_n(x)}{\beta^n}, \tag{1.5}$$

where the digits $d_n(x)$ are elements of the set $\{0, 1, \ldots, \lfloor \beta \rfloor\}$ and $\lfloor \beta \rfloor$ is the largest integer not exceeding β. The expression from (1.5) is called a *β-expansion* of x and it is produced by the iteration of the β-transformation $T_\beta : [0, 1] \to [0, 1]$, defined by

$$T_\beta(x) = \beta x \mod 1. \tag{1.6}$$

The digits $d_n(x)$ are recovered by setting $d_1(x) = \lfloor \beta x \rfloor$ and

$$d_n(x) = d_1(T_\beta^{n-1}(x)), \quad n \geq 1.$$

For an integer base β, the corresponding β-transformation has full branches, i.e., the map is piecewise surjective. See Figure 1.2 and 1.3(a) for an example in base 2 and base 3 respectively. In this situation the iteration of the transformation gives rise to basically unique β-expansions. That is, Lebesgue almost all numbers in the unit interval have a unique β-expansion and the ones that do not have a unique one, have

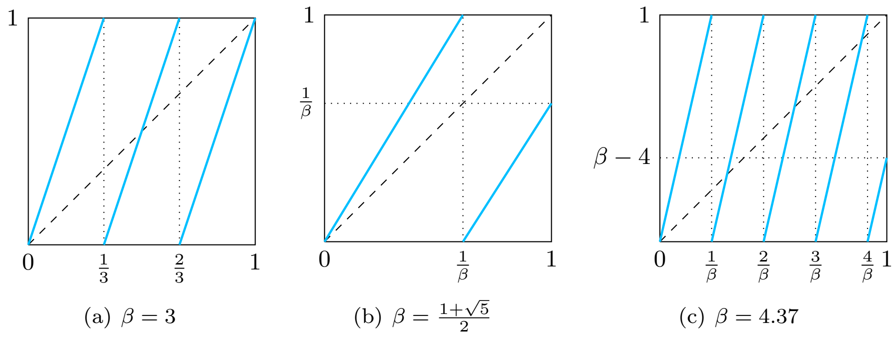


Figure 1.3: Examples of β-transformations T_β.

two.

Expansions in non-integer base have been introduced in [R57]. For a non-integer base, not all branches of the associated β-transformation are full, see Figure 1.3(b) and (c) for an example. In this context, the situation is quite different, as almost all numbers have infinitely many different β-expansions, see [EJK90, S03, DdV07], for example. One way to simultaneously obtain all possible β-expansions of a point x, is described in [DK03]. The construction requires two generalizations of the map T_β, given by H_β and L_β, both defined from the interval $[0, \frac{\lfloor\beta\rfloor}{\beta-1}]$ to itself. More precisely, let

$$H_\beta(x) = \begin{cases} \beta x \mod 1 & \text{if } x \in [0,1), \\ \beta x - \lfloor\beta\rfloor & \text{if } x \in [1, \frac{\lfloor\beta\rfloor}{\beta-1}], \end{cases}$$

and

$$L_\beta(x) = \begin{cases} \beta x & \text{if } x \in [0, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)}], \\ \beta x - i & \text{if } x \in (\frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{i-1}{\beta}, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{i}{\beta}] \text{ for } i \in \{1, 2, \ldots, \lfloor\beta\rfloor\}. \end{cases}$$

Both maps generate β-expansions, with digits

$$d_n(x) = \begin{cases} i & \text{if } H_\beta^{n-1}(x) \in [\frac{i}{\beta}, \frac{i+1}{\beta}) \text{ for } i \in \{0, \ldots, \lfloor\beta\rfloor - 1\}, \\ \lfloor\beta\rfloor & \text{if } H_\beta^{n-1}(x) \in [\frac{\lfloor\beta\rfloor}{\beta}, \frac{\lfloor\beta\rfloor}{\beta-1}], \end{cases}$$

or

$$d_n(x) = \begin{cases} 0 & \text{if } L_\beta^{n-1}(x) \in [0, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)}], \\ i & \text{if } L_\beta^{n-1}(x) \in (\frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{i-1}{\beta}, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{i}{\beta}]. \end{cases}$$

The expansions generated by H_β are called *greedy β-expansions*, since at each iteration the map assigns the largest digit possible. On the other hand, the expansions induced by L_β are called *lazy*, since this time at each iteration the map assigns the smallest digit possible. Superimposing the greedy and the lazy β-transformations on the same state space breaks down the interval into overlapping regions, called *switch* regions, of the form

$$S_i = \left[\frac{i}{\beta}, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{i-1}{\beta}\right], \text{ for } i = \{1, 2, \ldots, \lfloor\beta\rfloor\},$$

and an equaliser region,

$$E = \left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right] \setminus \bigcup_{1\le i\le\lfloor\beta\rfloor} S_i,$$

where H_β and L_β coincide. On the switch region S_i, the map H_β assigns the digit i and L_β the digit $i-1$, while on E both maps assign the same digit. As a consequence, the iteration of the pseudo-skew product map $R = \{H_\beta, L_\beta\}$, also called random β-transformation, can produce for the same point x uncountably many different β-expansions, depending on which map (and therefore on which digit) is chosen in the switch regions. See Figure 1.4 for a visualisation of the maps H_β and L_β, and Figure 1.5 for the associated pseudo-skew product system.

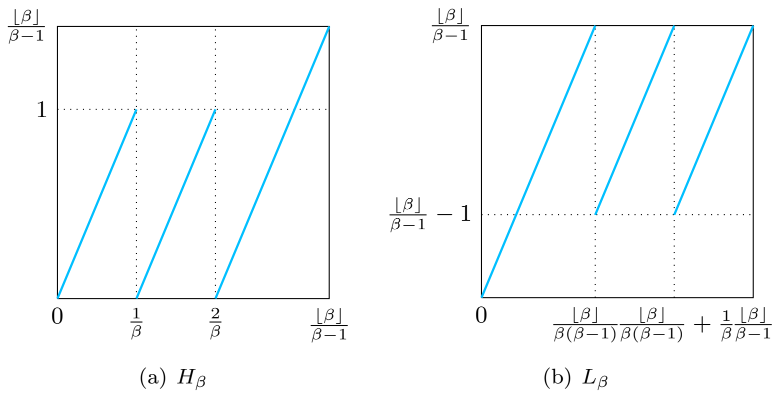


Figure 1.4: Example of the greedy and lazy β-transformations, for $\beta = 2.39$.

For further generalisation on the set of digits and more on random β-transformations, see [DdV05, DHK09, DK10, DK13, K14], for example. In Chapter 3 we develop an algebraic procedure to explicitly compute the invariant measure and the ergodic properties of a class of systems that include random β-transformations.

Differently from β-transformations, that represent numbers using a finite alphabet, Lüroth series use an infinite one, given by all positive integers greater than 1. We introduce them in the next section.

§1.3.2 Lüroth series

Any real number $x \in (0, 1]$ can be written in the form

$$x = \frac{1}{\ell_1(x)} + \frac{1}{\ell_1(x)(\ell_1(x)-1)\ell_2(x)} + \ldots + \frac{1}{\ell_1(x)\cdots\ell_{n-1}(x)(\ell_{n-1}(x)-1)\ell_n(x)} + \ldots, \tag{1.7}$$

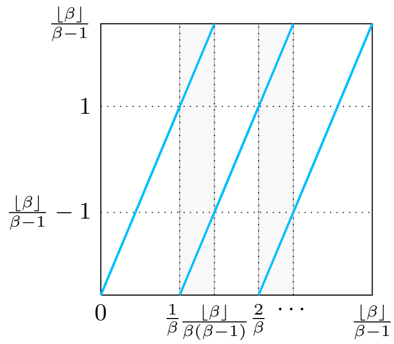


Figure 1.5: The random β-transformation for $\beta = 2.39$.

for some positive integers $\ell_n(x) \geq 2$. The series from (1.7) is called the *Lüroth expansion* of the point x and the digits $\ell_n(x)$ are obtained through the iteration of the Lüroth map $T_L : [0,1] \to [0,1]$ defined by

$$T_L(x) = \begin{cases} n(n-1)x - (n-1) & \text{if } x \in \left(\frac{1}{n}, \frac{1}{n-1}\right] \text{ for } n \geq 2, \\ 0 & \text{if } x = 0, \end{cases}$$

see Figure 1.6(a).

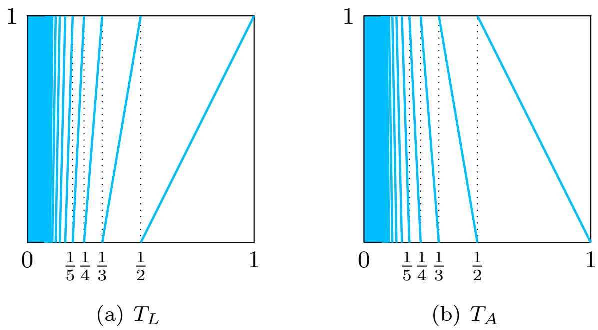


Figure 1.6: The standard and alternating Lüroth maps, respectively.

Specifically, for $n \geq 1$, the digits are obtained by setting

$$\ell_n(x) = d, \quad \text{if } T_L^{n-1}(x) \in \left(\frac{1}{d}, \frac{1}{d-1}\right], \ d \geq 2.$$

Lüroth series have been introduced in 1883 in [L83], and studied in a more general context in [BBDK94, KKK90, KKK91, BI09], for example. Chapter 2 introduces a random version of Lüroth maps, given by the standard Lüroth map T_L and its flipped version, the *alternating Lüroth map* $T_A = 1 - T_L$, see Figure 1.6(b). The system is then further analysed at the end of Chapter 3.

The alternating Lüroth map T_A can also be seen as a linearised version of the Gauss map G that produces continued fraction expansions and it is introduced in the next section.

§1.3.3 Continued fractions

Any real number $x \in (0,1)$ can be written as a *continued fraction* of the form

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots}}},$$

with $a_i(x) \in \mathbb{N}$. The continued fraction is usually denoted by $x = [0; a_1, a_2, a_3, \ldots]$, for $a_i = a_i(x)$, and it can be dynamically generated by iteration of the Gauss map $G : [0,1] \to [0,1]$ defined by

$$G(x) = \begin{cases} \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases} \tag{1.8}$$

see Figure 1.7.

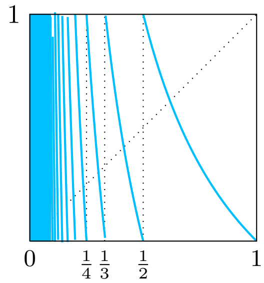


Figure 1.7: The Gauss map G.

The iteration of the Gauss map produces indeed continued fraction expansions with digits chosen in an infinite alphabet, given by $\mathbb{N}$, and defined by

$$a_i(x) = n \quad \text{if } G^{i-1}(x) \in \left(\frac{1}{n+1}, \frac{1}{n} \right], \quad n \geq 1.$$

It follows from Euclid's algorithm that every rational number $\frac{p}{q}$ presents a finite continued fraction, while irrational numbers have an infinite one. For rational numbers, there exists a unique representation $[0; a_1, a_2, \ldots, a_n]$, with $a_n \geq 2$. Without the constraint on the last digit the representation is not unique. Indeed, if $a_n = 1$, then

$$[0; a_1, a_2, \ldots, a_{n-1} + 1] = [0; a_1, a_2 \ldots, a_{n-1}, a_n].$$

For irrational numbers the representation is unique. Furthermore, for any irrational number $x = [0; a_1, a_2, \ldots] \in (0,1)$, it is possible to define a sequence of rational numbers $\left(\frac{p_n}{q_n}(x)\right)_{n\geq 0}$, that provides increasingly good approximations, alternately from above and below, of the irrational number x. For $n \geq 0$, the *n-th convergent* $\frac{p_n}{q_n}(x)$, is defined by

$$\frac{p_n}{q_n}(x) = [0; a_1, a_2, \ldots, a_n],$$

with $\frac{p_0}{q_0}(x) = [0]$, $\frac{p_1}{q_1}(x) = [0; a_1]$, and

$$x = \lim_{n\to\infty} \frac{p_n}{q_n}(x).$$

For more details, see [DK02], for example.

Another way to obtain the Gauss map G is through an induced system. Let $X = (0, \infty)$ and consider $Y = (0,1) \subseteq X$ and the transformation

$$T(x) = \begin{cases} \frac{1}{x} - 1 & \text{if } x \in Y, \\ x - 1 & \text{if } x \in Y^c. \end{cases}$$

Note that T determines the digits $a_i(x)$ of the continued fraction expansions of a point x after having successively subtracting 1 until the iterate of the point lies in Y. The first return time map to Y, $\varphi : Y \to Y$, is indeed given by

$$\varphi(x) = n, \quad \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right],$$

and the induced map from Definition 1.2.15 corresponds to

$$T_Y(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor = G(x).$$

For more details, see [Z09].

G admits an absolutely continuous invariant probability measure with density function

$$h(x) = \frac{1}{\log 2} \frac{1}{x+1},$$

which can be recovered by projecting the density function of the measure of its natural extension. The canonical natural extension of the transformation G is given in [NIT77] by the map $S : [0,1]^2 \to [0,1]^2$, and defined by

$$S(x,y) = \left(G(x), \frac{1}{d_1(x) + y}\right), \quad (x,y) \in [0,1]^2. \tag{1.9}$$

The transformation S admits an acim of density function $\frac{1}{(1+xy)^2}$. See Figure 1.8 for a visualisation of the 2-dimensional domain of the natural extension and the action of S on it. Chapter 4 uses the natural extension construction to obtain infinite invariant measures for a class of continued fraction maps.

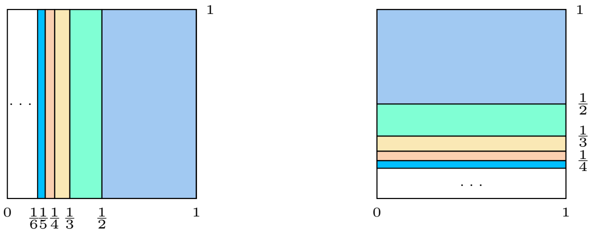


Figure 1.8: The action of the map S on the domain $[0,1]^2$ of the natural extension for the Gauss map G. Areas on the left are mapped to areas on the right with the same colour.

There exists a variety of continued fraction maps that provide continued fractions expansions with different alphabets. For instance, there exist continued fraction maps that provide expansions with negative numerators, or the odd (and even) continued fractions that use only odd (respectively even) digits. See for example [K91, DK00, HK02, KSS10, BCIT13, KKV17] for further generalisations.

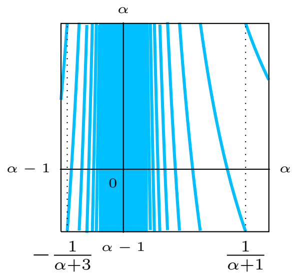


Figure 1.9: Example of T_α for $\alpha = \frac{7}{10}$.

In particualr, in 1981 Nakada formalised in [N81] the family of α-continued fractions $T_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha]$ depending on the parameter $\alpha \in [0, 1]$ and defined by

$$T_\alpha(x) = \frac{1}{|x|} - \left\lfloor \frac{1}{|x|} + 1 - \alpha \right\rfloor,$$

see Figure 1.9. Since then, lots of research has focused on the study of the invariant measure and the matching property of the maps T_α. See [LM08, CMPT10, N11, CT12, CT13, T14], for instance. Note that one can recover the Gauss map G from a map T_α, by setting $\alpha = 1$, i.e., $T_1 = G$.

§1.4 Outline

This work investigates absolutely continuous measures for several dynamical systems. The analysis is performed in the deterministic and random setting, as well as in the frame of finite and infinite ergodic theory.

Chapter 2 presents a family of random maps given by the combination of generalised Lüroth maps defined on a subset X of the unit interval $[0, 1]$. Each random system produces for almost all $x \in X$ uncountably many different generalised Lüroth expansions, that can be studied simultaneously. The chapter suggests the need of an algorithmic procedure to construct the density of an absolutely continuous invariant measure for random piecewise affine systems of the interval.

Chapter 3 answers the urgency of the previous chapter. It offers an algebraic algorithm that receives as an input a random piecewise affine system T that is expanding on average, and gives a formula for a physical T-invariant measure as output. Previously, the density was known only for very few specific cases. The heavy procedure is shown to be efficient for specific classes of transformations, such as random β-transformations, Lüroth maps and more general for systems that present the dynamical features of matching, which is further analysed in Chapter 5.

Chapter 4 looks into the consequences of matching for an infinite class $\{T_\alpha\}_\alpha$ of continued fraction maps. More specifically, the phenomenon of synchronization is used to find the 2-dimensional domain of a planar natural extension. Such information is used to find explicit expressions of the density for a large part of the parameter space. Additionally, matching is proved to hold for Lebesgue almost every parameter α, and it divides the parameter space into intervals of constant matching exponents, called matching intervals. It is the first family of infinite measure systems in which matching is recognized. Lastly, the chapter relates these matching intervals to the corresponding sets of Nakada's α-continued fraction maps.

Chapter 5 extends the notion of matching for deterministic transformations to random matching for random interval maps. Random matching is then studied for a variety of families of random dynamical systems, that includes generalised β-transformations and continued fraction maps. Furthermore, for a large class of piecewise affine random systems of the interval, the property of random matching is proved to imply that any invariant density of a stationary measure is piecewise constant. Lastly, the chapter introduces a family of random maps producing signed binary expansions. The property of random matching and its consequences on the structure of the density function are then applied to study the frequency of the digit 0 in such expansions.