Measures and matching for number systems

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Title: Measures and matching for number systems

Issue Date: 2021-03-04

Measures and Matching for Number Systems

Proefschrift

ter verkrijging van
de graad van Doctor aan de Universiteit Leiden,
op gezag van Rector Magnificus Prof. Dr. Ir. H. Bijl,
volgens besluit van het College voor Promoties
te verdedigen op donderdag 4 maart 2021
klokke 10.00 uur

door

Marta Maggioni
geboren te Calco, Italië
in 1992

Samenstelling van de promotiecommissie:

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This thesis has been financed by the Dutch Research Council (NWO) through the NWO TOP-Grant No. 614.001.509.

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“Sarebbe questa la libertà?
Esistere, lentamente, dolcemente,
come questi alberi,
come una pozza d’acqua,
come il sedile rosso del tram.”
Jean-Paul Sartre, La nausea

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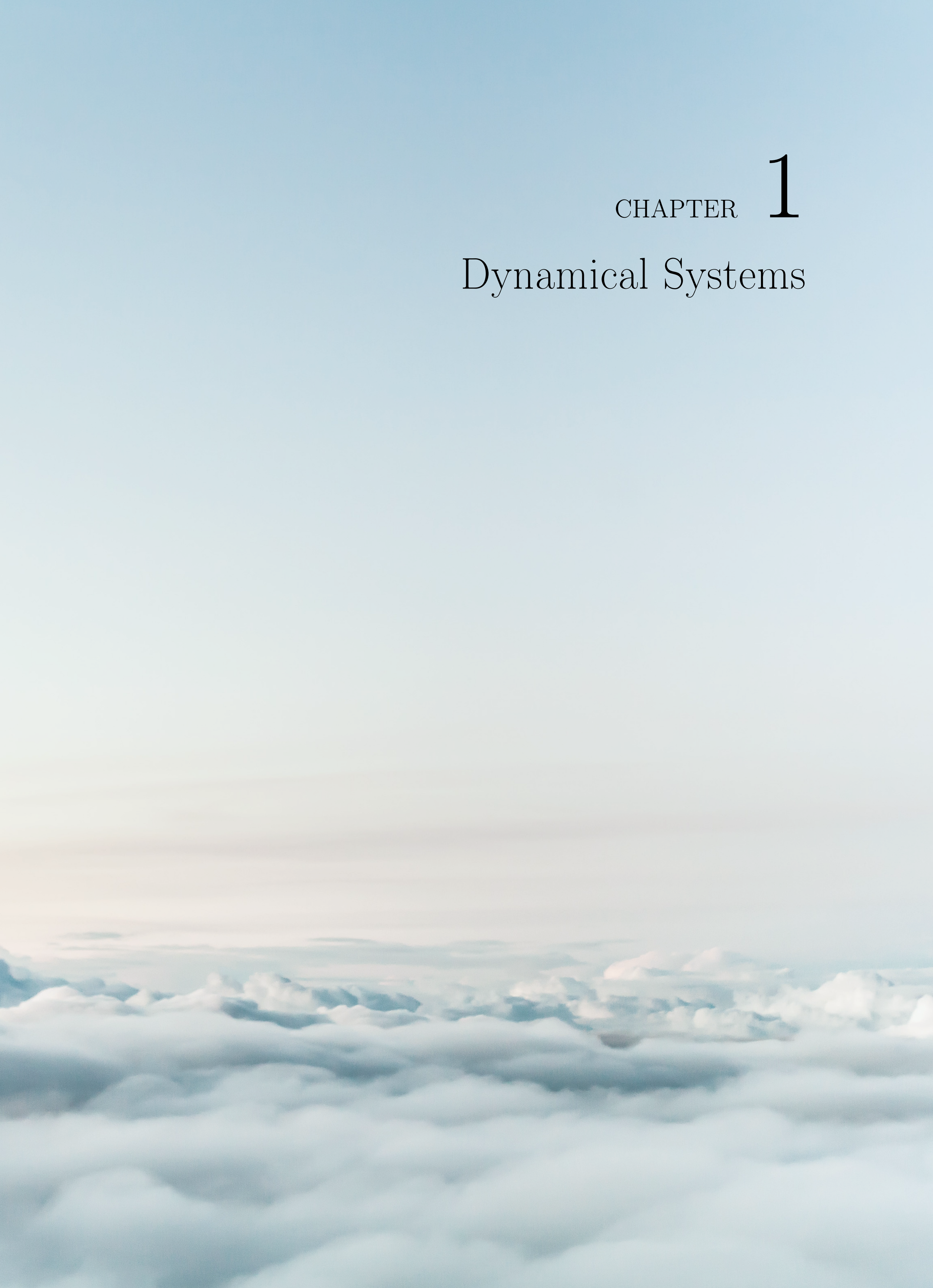


CHAPTER 1

Dynamical Systems

§1.1 Motivation and context

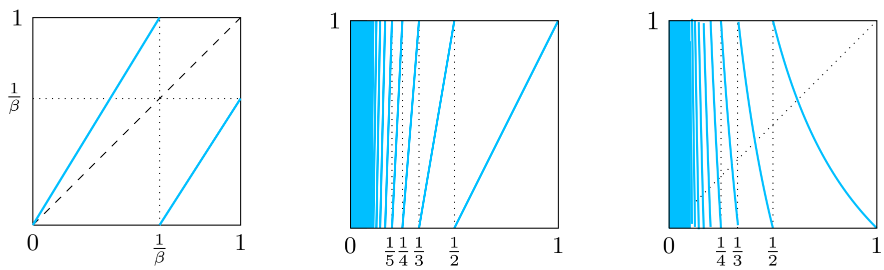
Number expansions are ways of representing numbers with specific symbols and consistent rules. The algorithms, that combine the digits to code numbers, can be obtained through the repeated iteration of particular interval maps, called *number systems*. The advantage of this description is that all tools coming from Ergodic Theory are available, and they can be used to describe general properties of the expansions. This thesis adopts this dynamical approach to study new number expansions.

The *interval maps* of interest are special instances of discrete-time dynamical systems. A discrete-time dynamical system models the evolution of a phenomenon over time through a transformation T acting on a set X. Morally, X represents all possible states and T is the law that rules the evolution, so that, if a system is at state $x \in X$ at time zero, then it will be at state $T(x)$ after one unit of time, and in general at state

$$T^n(x) = \underbrace{T \circ T \circ \cdots \circ T}_{n}(x),$$

after n units of time, for any $n \in \mathbb{N}$. For interval maps, the state space X is an interval in $\mathbb{R}$ and the evolution of any number $x \in X$ is described in terms of its orbit $\{T^n(x)\}_{n\in\mathbb{N}}$.

The procedure that codes numbers through the iteration of an interval map $T : X \to X$ is the following. The interval X is divided into finitely or countably many subintervals and to each of them a *digit* d is assigned. A number expansion of a point $x \in X$ is obtained by considering the sequence of digits $(d_n(x))_{n\in\mathbb{N}}$ realised by following the orbit of x. Specifically, the first digit of the sequence is determined by the symbol associated to the subinterval in which x lies, the second by the one corresponding to the position of $T(x)$ and in general the n-th digit by the position of $T^{n-1}(x)$. Figure 1.1 shows examples of classical number systems: the β-transformation T_β for $\beta = \frac{1+\sqrt{5}}{2}$, the Lüroth map T_L and the Gauss map G.



(a) T_β (b) T_L (c) G

Figure 1.1: Classical examples of interval maps producing number expansions.

This strong connection between the orbits of a point x, $\{T^n(x)\}_{n\in\mathbb{N}}$, and its corresponding digits sequence $(d_n(x))_{n\in\mathbb{N}}$ is the reason why the study of the former reveals properties of the latter. To investigate the quantities $T^n(x)$, rather than computing all orbits of all points, one looks at the probability that the orbits of typical points pass by certain areas of the space X, which corresponds to the probability that specific digits are assigned. A set of high probability corresponds then to a set that is often visited by most of the points of X. In Ergodic Theory, this information is encoded in the so-called invariant measures.

An *invariant measure* is a measure that is preserved by the action of T, in the sense that the set of points that T maps to a set in one unit of time has the same measure as the set itself. For number systems, explicitly knowing such measures is extremely helpful for, e.g., the computation of the *frequency* of the digits in the associated number expansions, which expresses the percentage of seeing a digit d in typical expansions, or the *entropy*, which estimates the possible number of different blocks of digits of length n that can be found by the repeated iteration of T. Ideally, meaningful invariant measures should describe the long-term behaviour of the orbits for a large set of points. Here large is intended with respect to the Lebesgue measure, which is used as a reference measure. For this reason, of particular relevance are invariant measures that are *absolutely continuous* with respect to Lebesgue and are given in terms of density functions. While there exist various results on the existence of such measures for interval maps, finding explicit formulas for the corresponding density functions is still a delicate problem for very many dynamical systems.

This thesis provides explicit expressions for the density functions of absolutely continuous invariant measures for general families of interval maps, that include *random maps* and *infinite measure transformations*, not necessarily number systems. In the random setting, at each time step, instead of a single transformation, a set of maps is available and one of them is applied according to a probabilistic regime. In the infinite configuration, the measure of the state space is infinite and the tools coming from probability theory are no longer available. Natural extensions, the Perron-Frobenius operator and the dynamical phenomenon of *matching* are some of the techniques exploited to obtain such results. In particular, in this thesis the notion of matching is for the first time recognised in an infinite measure system and the definition, known so far for deterministic transformations only, is extended to cover random interval maps as well.

This thesis also presents new developments in the area of number expansions, by introducing new representations of numbers obtained through the iterations of random maps and infinite measure transformations. These include random c-Lüroth expansions, flipped α-continued fractions and random signed binary expansions. The properties of these number expansions are analysed by applying the results obtained previously on the density functions. In particular, explicit expressions for the measures are used to investigate the digit frequency, the Krengel entropy, the Hamming weight and the quality of the approximations.

The remaining part of this chapter introduces all the necessary mathematical tools in more details.

§1.2 Discrete dynamical systems

Ergodic theory is the branch of dynamical systems that studies measure-preserving transformations defined on measure spaces. Here we introduce the basic concepts.

§1.2.1 Deterministic

Let $T : X \to X$ be a transformation. To study the long term behaviour of the system we determine the probabilities to observe typical trajectories within certain areas of the space. This information is provided by the invariant measure.

1.2.1 Definition (Measure preserving). A *measure preserving dynamical system* is defined as the quadruple $(X, \mathcal{B}, \mu, T)$ where the triple $(X, \mathcal{B}, \mu)$ is a measure space and the measurable map $T : X \to X$ *preserves the measure* μ, i.e. $\mu(T^{-1}(B)) = \mu(B)$ for all $B \in \mathcal{B}$. The system is also said to be T*-invariant* with respect to μ.

Equivalently, T is measure-preserving if $T_*\mu = \mu$, for $T_*\mu$ the push-forward of μ with respect to T, i.e.,

$$T_*\mu(B) = \mu(T^{-1}(B)), \quad B \in \mathcal{B}.$$

$(X, \mathcal{B}, \mu)$ is said to be a *finite* or *infinite* measure space if $\mu(X) < \infty$ or $\mu(X) = \infty$, respectively. In this dissertation, for the infinite case, we still assume X to be a countable union of sets of finite measure, i.e., we ask the space to be σ-finite. Two dynamical systems that present the same dynamics, that is for which the long-term and average behaviours are essentially the same, are called isomorphic. We make this precise in the following definition.

1.2.2 Definition (Isomorphic). $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{F}, \nu, S)$ are said to be *isomorphic* if there exist sets $B \in \mathcal{B}$ and $F \in \mathcal{F}$ and a map $\theta : B \to F$ such that

1. $\mu(X \setminus B) = \nu(Y \setminus F) = 0$,
2. $T(B) \subseteq B$ and $S(F) \subseteq F$,
3. θ is invertible and bi-measurable,
4. $S \circ \theta = \theta \circ T$,
5. $\nu = \mu \circ \theta^{-1}$.

In the following, let $(X, \mathcal{B}, \mu, T)$ be a measure preserving dynamical system. A subset $B \in \mathcal{B}$ is said to be invariant for T if $T^{-1}(B) = B$. Clearly, if the space X is the union of two or more disjoint invariant subsets of positive measure, then the study of the properties of T on X reduces to the study of its properties on each of these invariant subsets. It is therefore natural to study the transformations defined on spaces that do not decompose into such subsets. Such a property is called ergodicity.

1.2.3 Definition (Ergodicity). $(X, \mathcal{B}, \mu, T)$ is said to be *ergodic* with respect to μ if for every $B \in \mathcal{B}$, such that $T^{-1}(B) = B$, either $\mu(B) = 0$ or $\mu(X \setminus B) = 0$, i.e., if the only invariant sets are trivial.

A property is said to hold almost everywhere (a.e.), if the set of points for which it does not hold is contained in a set of measure zero. In a probability space, ergodicity implies Birkhoff's Ergodic Theorem, which relates spatial averages to temporal averages.

1.2.4 Theorem (Birkhoff's Ergodic Theorem). *If $(X, T, \mu, \mathcal{B})$ is ergodic and $\mu(X) < \infty$, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(x) = \int_X f d\mu, \quad \mu \text{ a.e.}, \forall f \in L^1(X, \mathbb{R}).$$

Without the assumption $\mu(X) < \infty$, Birkhoff's Ergodic Theorem does not hold. More generally, for infinite measure preserving dynamical systems many classical results of ergodic theory fail, and a new approach is required, see Section 1.2.3.

When the dynamical system under consideration is an interval map $T : I \to I$, that leads to number expansions, Birkhoff's Ergodic Theorem can be used to obtain the average number of occurrences, called *frequency*, of specific digits in typical expansions. Given $x \in I$, the number of visits of x to a measurable set B of positive measure is given by

$$\frac{\#\{0 \le k \le n-1 : T^k(x) \in B\}}{n} = \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_B(T^k(x)) \to \mu(B).$$

However, to apply Theorem 1.2.4, it is necessary to know the invariant measure μ and finding an invariant measure for a transformation is not an easy task. The measures in which we are interested are equivalent to the Lebesgue measure λ.

1.2.5 Definition (Absolutely continuous). A measure μ on Borel subsets of the real line is said to be *absolutely continuous* with respect to λ if for every measurable set B, $\lambda(B) = 0$ implies $\mu(B) = 0$. Equivalently, there exists a Lebesgue integrable non-negative function f, called *density*, on the real line such that

$$\mu(B) = \int_B f d\lambda, \tag{1.1}$$

for all Borel subsets B of the real line. The density, also known as the Radon-Nikodym derivative, of the absolutely continuous measure μ is only defined up to a.e. equivalence. If also λ is absolutely continuous with respect to μ, then the measures are said to be *equivalent*.

Most of the results on the existence of such measures are proven by studying the transfer operator P, known as Perron-Frobenius operator, of the system. The operator P of an interval map $T : I \to I$ is defined by

$$\int_B P_T f d\lambda = \int_{T^{-1}(B)} f d\lambda, \tag{1.2}$$

for any $f \in \mathcal{L}^1(I, \mathbb{R})$ and $B \in \mathcal{B}$. If the map T is also piecewise affine, then P_T can be written as

$$P_T f(x) = \sum_{y \in T^{-1}(x)} \frac{f(y)}{|T'(y)|}.$$

We refer to the book of [BG97] for a classical introduction to the subject and the properties of this operator. An interval map $T : I \to I$ is said to be *expanding* if $|T'(x)| > 1$, for any point x in which the derivative is defined. For one-dimensional piecewise monotonic and expanding transformations $T \in \mathcal{C}^2$, the existence of absolutely continuous invariant measures (acim) is by now pretty well understood. Indeed the seminal paper of [LY73] shows that a fixed point of the Perron-Frobenius operator P of such a transformation T exists and it is the density of an absolutely continuous invariant measure μ. For these transformations, [LY78] shows that the number of acims that a map admits is strictly connected to the number of discontinuities. Furthermore, [K90, G09] propose two similar procedures to obtain formulas for the densities of such measures, by connecting them to the solution vectors of a matrix equation. In Chapter 3 it is shown how to obtain the formulas for the densities when the dynamics of the system is not deterministic, but countably many transformations are available at each iteration.

Another way to possibly obtain the formulas for the densities of acims is via the construction of a natural extension. Roughly speaking, a natural extension of a system is the minimal invertible dynamical system that contains the original system as a subsystem. Invertibility is obtained by extending the dimensions of the space of the original system. The existence of such a construction is obtained in [R61]. In the same article, it is also shown that any two natural extensions of the same dynamical system are isomorphic.

1.2.6 Definition (Natural extension). Let $(X, \mathcal{B}, \mu, T)$ be a dynamical system with T a non-invertible transformation. An invertible dynamical system $(Y, \mathcal{F}, \nu, S)$ is a *natural extension* of $(X, \mathcal{B}, \mu, T)$, if there exist two sets $B \in \mathcal{B}$ and $F \in \mathcal{F}$ and a function $\theta : F \to B$ such that:

1. $\mu(X \setminus B) = \nu(Y \setminus F) = 0$,
2. $T(B) \subseteq B$ and $S(F) \subseteq F$,
3. θ is measurable, measure preserving and surjective,
4. $\theta \circ S = T \circ \theta$,
5. $\bigvee_{n=0}^{\infty} S^n \theta^{-1}(\mathcal{B}) = \mathcal{F}$, where $\bigvee_{n=0}^{\infty} S^n \theta^{-1}(\mathcal{B})$ is the smallest σ-algebra containing the σ-algebras $S^n \theta^{-1}(\mathcal{B})$ for all $n \geq 0$.

This approach has been shown to be very successful in the non-affine case, especially in the field of continued fraction transformations, see Section 1.3.3 and [N81, K91, KSS12, AS13], for example. Chapter 4 exhibits a natural extension for a class of continued fraction transformations with infinite measure.

For some families of transformations, e.g. continued fraction maps, the natural extension map is quite canonical, and the difficulty of the approach lies in finding the appropriate domain on which the map acts. See Section 1.3.3 and Figure 1.8 for an example. For one-dimensional maps, it is often the case that the natural extension is a planar map and the function θ from Definition 1.2.6 corresponds to the projection on one of the two components. What has been proved to be fundamental in such instances, in order to recover the domain of the natural extension, is the property of matching, or synchronization, of the original system T.

1.2.7 Definition (Matching). A piecewise smooth interval map T is said to have *matching* if for any discontinuity point c, of T or its derivative T', its orbits of the left and right limits eventually meet. That is, there exist non-negative integers M and N, called *matching exponents*, such that

$$T^M(c^-) = T^N(c^+), \tag{1.3}$$

for

$$c^- = \lim_{x \uparrow c} T(x) \quad \text{and} \quad c^+ = \lim_{x \downarrow c} T(x).$$

For specific families of interval maps defined on a finite measure space, and in particular for β-transformations and continued fraction type maps, the property of matching has been thoroughly analysed in order to find expressions for the invariant densities. See for instance [NN08, DKS09, KSS10, KS12, KSS12, DK17, BCK17, BCMP18, KLMM20]. Differently from these results, Chapter 4 considers matching for a class of infinite dynamical systems and Chapter 5 introduces the notion of random matching for random dynamical systems. To this aim, in the following sections we give some background on random and infinite (measure) dynamical systems.

§1.2.2 Random

The qualitative analysis of iterations of a single map can be extended to a more general setting where, at each step, an element is chosen from a set of transformations according to a stationary process. See [K86, A98] for a basic introduction. This generalisation is quite natural when considering that, in most physical applications, at every iteration it is usually the case that not the same map, but a slightly modified version of it, is applied. This phenomenon is usually referred to as a stochastic perturbation. Such systems have been recently used also to study interference effects in quantum mechanics, fractals and particle systems on lattices see [BG92, B93, KY07], for example. In general, there is a rich and quite recent literature on random maps, both considered as position independent random perturbations of transformations, and as position dependent ones in the context of iterated function systems, see [G84, BL92, BK93], for example. In all these situations, the dynamics changes from deterministic to random. Roughly speaking, a random map describes a system evolving in discrete time in which at each time step one of a number of transformations is chosen according to a probabilistic regime and applied. One way to describe a random map, in which the process for choosing the individual maps is i.i.d., is with a pseudo-skew product transformation.

For an at most countable set of symbols Ω, denote by $\Omega^{\mathbb{N}}$ the set of *one-sided sequences*. The *left shift* $\sigma : \Omega^{\mathbb{N}} \to \Omega^{\mathbb{N}}$ maps a sequence $(\omega_n)_{n\in\mathbb{N}}$ to a sequence $(u_n)_{n\in\mathbb{N}}$ such that $u_n = \omega_{n+1}$ for all $n \geq 1$.

1.2.8 Definition (Pseudo-skew product map). Let $\{T_j : I \to I\}_{j\in\Omega}$ be a collection of transformations defined on the same interval I and let $\Omega \subseteq \mathbb{N}$ be the index set of these available maps. Let $\sigma : \Omega^{\mathbb{N}} \to \Omega^{\mathbb{N}}$ be the left shift on one-sided sequences. The *random* or *pseudo-skew product map* $R : \Omega^{\mathbb{N}} \times I \to \Omega^{\mathbb{N}} \times I$ is defined by

$$R(\omega, x) = (\sigma(\omega), T_{\omega_1} x),$$

such that the coordinates of ω determine which of the maps T_j is applied at each step.

1.2.9 Definition (Stationary measure for pseudo-skew product maps). Let $\mathbf{p} = (p_j)_{j\in\Omega}$ be a positive probability vector, i.e., $p_j > 0$ for all $j \in \Omega$ and $\sum_{j\in\Omega} p_j = 1$. Each p_j represents the probability with which we choose the map T_j. Denote by $m_{\mathbf{p}}$ the $\mathbf{p}$-Bernoulli measure on $\Omega^{\mathbb{N}}$ and let $\mu_{\mathbf{p}}$ be a probability measure on I that is absolutely continuous with respect to the one-dimensional Lebesgue measure λ. Denote its density by $\frac{d\mu_{\mathbf{p}}}{d\lambda} = f_{\mathbf{p}}$. If $\mu_{\mathbf{p}}$ satisfies for each Borel set $B \subseteq I$ that

$$\mu_{\mathbf{p}}(B) = \int_B f_{\mathbf{p}}\, d\lambda = \sum_{j\in\Omega} p_j \mu_{\mathbf{p}}(T_j^{-1} B), \tag{1.4}$$

then the product measure $m_{\mathbf{p}} \times \mu_{\mathbf{p}}$ is an invariant probability measure for R. $\mu_{\mathbf{p}}$ is called a *stationary measure* and $f_{\mathbf{p}}$ an *invariant density* for the pseudo-skew product R.

There exist various sets of conditions under which the existence of such an invariant measure is guaranteed, see for example [M85, P84, GB03, BG05, I12]. In particular, for piecewise random interval maps see the results of [P84] for a finite family of transformations and of [M85] and [I12] for the countable case.

The introduction of randomness in systems defining number expansions has some quite remarkable consequences. For instance, a single random map produces many more expansions per number than a deterministic transformation, allowing the study of the properties of many number expansions simultaneously. In Chapter 5 this property is used to compute the frequency of the digit 0 in signed binary expansions for Lebesgue almost every point $x \in [-1, 1]$.

§1.2.3 Infinite

Infinite ergodic theory studies dynamical systems with an *infinite* invariant measure. These systems differ from transformations admitting a finite invariant measure, because for them most of the tools coming from probability theory are not applicable and, as a consequence, classic results from (finite) ergodic theory do not hold. We refer to the books of [A97, KMS16] for an introduction to the subject.

A first, crucial example is presented by Birkhoff's Ergodic Theorem. For a finite measure system $T : X \to X$, $\mu(X) < \infty$, Theorem 1.2.4 describes the limiting behaviour of the number of times the orbit of a typical point enters a specific region of the space. Precisely, for any measurable set B and $x \in X$, let

$$S_n^B(x) = \sum_{k=0}^{n-1} \mathbf{1}_B \circ T^k(x), \quad n \geq 1.$$

$S_n^B(x)$ counts how often the orbit of x visits the set B before time n. Birkhoff's Ergodic Theorem expresses the rate at which the occupation time of B diverges as being proportional to n, asymptotically the same for typical points and dependent on the set B only through its measure, i.e.,

$$\frac{1}{n} S_n^B(x) \xrightarrow[n\to\infty]{} \frac{\mu(B)}{\mu(X)} \quad \mu\text{-a.e. } x \in X.$$

For an infinite measure preserving transformation, this is no longer true, as

$$\frac{1}{n} S_n^B(x) \xrightarrow[n\to\infty]{} 0 \quad \mu\text{-a.e. } x \in X,$$

not revealing any dependence on the set. This is just the first of the many substantial differences between finite and infinite dynamical systems. Another one involves Poincaré's Recurrence Theorem. For a finite measure preserving system T, the theorem says that for every measurable set B of positive measure, almost every point of the set will return to the set itself under iterations of T. For the infinite scenario, this is not always the case, since only *conservative* systems have this property.

1.2.10 Definition (Conservative). A measure preserving dynamical system $(X, \mathcal{B}, \mu, T)$ is said to be *conservative* if

$$\mu\left(\bigcup_{n=1}^{\infty} T^{-n}B \setminus B\right) = 0 \quad \text{for all } B \in \mathcal{B},\ \mu(B) > 0.$$

1.2.11 Definition (Wandering sets). A set $W \in \mathcal{B}$ is said to be a *wandering set* for T if $\{T^{-n}(W) : n \geq 0\}$ is a collection of pairwise disjoint sets.

In other words, a measure preserving dynamical system is conservative if every wandering set has measure 0. A handy way of determining if a system is conservative is given by the existence of sweep-out sets.

1.2.12 Definition (Sweep-out sets). A set $Y \in \mathcal{B}$ is said to be a *sweep-out* set for T if $0 < \mu(Y) < \infty$ and

$$\mu\left(X \setminus \bigcup_{n=0}^{\infty} T^{-n}(Y)\right) = 0.$$

1.2.13 Theorem (Maharam's Recurrence Theorem). *If $(X, T, \mu, \mathcal{B})$ is measure preserving and has a sweep-out set, then it is conservative.*

1.2.14 Theorem (Aaronson's Ergodic Theorem). *Let $(X, T, \mu, \mathcal{B})$ be a conservative, ergodic, measure preserving infinite system, and let $(a_n)_{n\geq 1}$ be any positive sequence. Then*

$$\overline{\lim}_{n\to\infty} \frac{1}{a_n} S_n^B(x) = \infty \quad \mu\text{-a.e. } x \in X,$$

or

$$\underline{\lim}_{n\to\infty} \frac{1}{a_n} S_n^B(x) = 0 \quad \mu\text{-a.e. } x \in X.$$

The theorem tells that the pointwise behaviour of occupation times is extremely chaotic and that an analogous version of Birkhoff's Ergodic Theorem for infinite measure systems can't provide the same amount of information. One powerful way to obtain information on such systems is by looking at the dynamics that happens only in specific subsets of finite measure. More precisely, let Y be a sweep-out set of a conservative system T with measure μ and define the *hitting time* of Y, by

$$\varphi : X \to \mathbb{N}, \quad \varphi(x) = \inf\{n \geq 1 : T^n(x) \in Y\}.$$

When restricting the function φ to the set Y, the map φ is called the *return time* to Y and it counts the number of steps the orbit of $x \in Y$ needs to come back to Y. Note that the conservativity of the map T ensures that $\varphi(x) < \infty$ for μ-a.e. point $x \in Y$.

1.2.15 Definition (Inducing). The map $T_Y : Y \to Y$ given by

$$T_Y(x) = T^{\varphi(x)}(x)$$

is called the *induced map* of T on Y.

T_Y is an acceleration of T, achieved by applying as many iterates of T as is required to come back to Y. Basic properties of T, such as invariant measures, ergodicity and conservativity can be recovered from T_Y by a proper choice of a sweep-out set Y. An example of an induced map is given in Section 1.3.3.

We now discuss in more detail some examples of classic interval maps producing number expansions: β-transformations, Lüroth maps and continued fraction transformations. We introduce them separately, in the next sections.

§1.3 Number systems

Number systems offer algorithmic ways of coding numbers with a specific set of symbols, called *digits*. This set, called *alphabet*, can consist of finitely or countably many elements. For example, in the canonical decimal system, we represent numbers as sequences of digits $0, 1, \ldots, 9$, with each position in the sequence corresponding to a specific power of 10. For computer hardware and software, we usually use the binary system, so that we write numbers as strings of digits 0 and 1, where the position of each digit corresponds this time to a specific power of 2. For example, the number $\frac{37}{2}$, can be written as

$$\frac{37}{2} = 1 \cdot 10^1 + 8 \cdot 10^0 + 5 \cdot 10^{-1} = 1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 + 1 \cdot 2^{-1}.$$

The first expression leads to the sequence 18.5, the second to 10010.1, which is the representation of the number in the binary system. The sequences of digits can be obtained through the iteration of one-dimensional discrete time dynamical systems $T : X \to X$. The idea is to discretize the space, by dividing up the state space X into finitely or countably many subintervals and keep track of which piece the system visits at each time step. This is done by assigning a digit to each subinterval: in this way, the evolution of a point $\{T^n(x)\}_n$ is given in terms of an infinite sequence of symbols. Infinite sequences are the main objects of study in symbolic dynamics. In the following paragraph we give some basic definitions that will be used in the coming chapters.

For an alphabet $\mathcal{A}$, the set of one-sided sequences $(a_n)_{n\in\mathbb{N}}$ of symbols from $\mathcal{A}$ is denoted by $\mathcal{A}^{\mathbb{N}}$. A *word*, or a block, over the alphabet $\mathcal{A}$ is a finite string of symbols from $\mathcal{A}$. The *empty word* is denoted by ϵ and it represents the sequence of no symbols. The *length* of a block u corresponds to the number of symbols it contains, and it is denoted by $|u|$. For each $n \in \mathbb{N}$, the set $\mathcal{A}^n$ is the set of all blocks of length n of symbols from $\mathcal{A}$, and we set $\mathcal{A}^0 = \{\epsilon\}$. We use square brackets to denote cylinder sets, i.e., for any block u,

$$[u] = \{(a_n)_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}} : a_1 \cdots a_{|u|} = u\}.$$

The *concatenation* of a pair of words u, v is given by the word uv, of length $|uv| = |u| + |v|$. For each $n \in \mathbb{N}$, u^n corresponds to the concatenation of n copies of u, and we also define $u^\infty = uuu\ldots$ and $u^0 = \epsilon$. Recall that the *left shift* $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ maps each sequence $(a_n)_{n\in\mathbb{N}}$ to a sequence $(b_n)_{n\in\mathbb{N}}$ such that $b_n = a_{n+1}$. We refer to the book [LM95] for a basic introduction on the topic.

The next example shows an interval map producing number expansion with alphabet $\mathcal{A} = \{0, 1\}$.

1.3.1 Example. Let

$$D(x) = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}), \\ 2x - 1 & \text{if } x \in [\frac{1}{2}, 1], \end{cases}$$

be the *doubling map*, see Figure 1.2. Consider the partition of the interval given by $I_0 = [0, \frac{1}{2})$ and $I_1 = [\frac{1}{2}, 1]$, and assign the digit 0 to I_0 and the digit 1 to I_1.

The iteration of the map D to each point $x \in [0, 1]$ produces *binary expansions*, i.e., sequences $(d_n(x))_{n\in\mathbb{N}}$ of 0 and 1 such that

$$x = \sum_{n=1}^{\infty} \frac{d_n(x)}{2^n}.$$

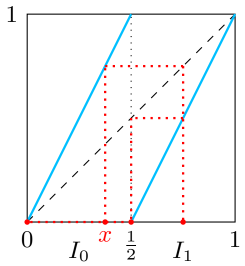


Figure 1.2: The doubling map D and in red the orbit of $x = 3/8$.

For example, let $x = 3/8$ and consider its orbit under the doubling map D. Since $x \in I_0$, the first digit in its binary expansion is $d_1 = 0$. $D(x) = 3/4 \in I_1$, so $d_2 = 1$. $D^2(x) = 1/2 \in I_1$ so $d_3 = 1$. Finally, $D^n(x) = 0 \in I_0$ for $n \geq 3$, since 0 is a fixed point of the map D, i.e., $D(0) = 0$, and so $d_n = 0$ for $n \geq 4$, obtaining the sequence 0110^∞, such that

$$\frac{3}{8} = \frac{0}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \sum_{n \geq 4} \frac{0}{2^n}.$$

See Figure 1.2 for a visualisation of the orbit of $x = 3/8$.

The doubling map is a member of a family of piecewise affine maps called β-transformations, that produce representations of numbers as series of powers of $\beta \in \mathbb{R}_{>1}$. We discuss them in the next section.

§1.3.1 β-expansions

For $\beta > 1$, any real number $x \in [0, 1)$ can be written as

$$x = \sum_{n=1}^{\infty} \frac{d_n(x)}{\beta^n}, \tag{1.5}$$

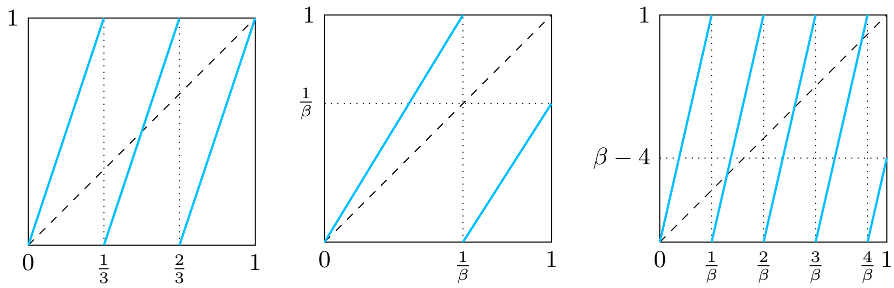
where the digits $d_n(x)$ are elements of the set $\{0, 1, \ldots, \lfloor \beta \rfloor\}$ and $\lfloor \beta \rfloor$ is the largest integer not exceeding β. The expression from (1.5) is called a *β-expansion* of x and it is produced by the iteration of the β-transformation $T_\beta : [0, 1] \to [0, 1]$, defined by

$$T_\beta(x) = \beta x \mod 1. \tag{1.6}$$

The digits $d_n(x)$ are recovered by setting $d_1(x) = \lfloor \beta x \rfloor$ and

$$d_n(x) = d_1(T_\beta^{n-1}(x)), \quad n \geq 1.$$

For an integer base β, the corresponding β-transformation has full branches, i.e., the map is piecewise surjective. See Figure 1.2 and 1.3(a) for an example in base 2 and base 3 respectively. In this situation the iteration of the transformation gives rise to basically unique β-expansions. That is, Lebesgue almost all numbers in the unit interval have a unique β-expansion and the ones that do not have a unique one, have



(a) $\beta = 3$ (b) $\beta = \frac{1+\sqrt{5}}{2}$ (c) $\beta = 4.37$

Figure 1.3: Examples of β-transformations T_β.

two.

Expansions in non-integer base have been introduced in [R57]. For a non-integer base, not all branches of the associated β-transformation are full, see Figure 1.3(b) and (c) for an example. In this context, the situation is quite different, as almost all numbers have infinitely many different β-expansions, see [EJK90, S03, DdV07], for example. One way to simultaneously obtain all possible β-expansions of a point x, is described in [DK03]. The construction requires two generalizations of the map T_β, given by H_β and L_β, both defined from the interval $[0, \frac{\lfloor\beta\rfloor}{\beta-1}]$ to itself. More precisely, let

$$H_\beta(x) = \begin{cases} \beta x \mod 1 & \text{if } x \in [0,1), \\ \beta x - \lfloor\beta\rfloor & \text{if } x \in [1, \frac{\lfloor\beta\rfloor}{\beta-1}], \end{cases}$$

and

$$L_\beta(x) = \begin{cases} \beta x & \text{if } x \in [0, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)}], \\ \beta x - i & \text{if } x \in (\frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{i-1}{\beta}, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{i}{\beta}] \text{ for } i \in \{1, 2, \dots, \lfloor\beta\rfloor\}. \end{cases}$$

Both maps generate β-expansions, with digits

$$d_n(x) = \begin{cases} i & \text{if } H_\beta^{n-1}(x) \in [\frac{i}{\beta}, \frac{i+1}{\beta}) \text{ for } i \in \{0, \dots, \lfloor\beta\rfloor - 1\}, \\ \lfloor\beta\rfloor & \text{if } H_\beta^{n-1}(x) \in [\frac{\lfloor\beta\rfloor}{\beta}, \frac{\lfloor\beta\rfloor}{\beta-1}], \end{cases}$$

or

$$d_n(x) = \begin{cases} 0 & \text{if } L_\beta^{n-1}(x) \in [0, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)}], \\ i & \text{if } L_\beta^{n-1}(x) \in (\frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{i-1}{\beta}, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{i}{\beta}]. \end{cases}$$

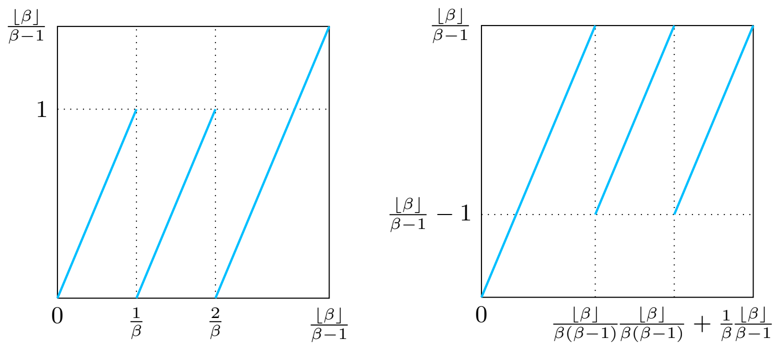
The expansions generated by H_β are called *greedy β-expansions*, since at each iteration the map assigns the largest digit possible. On the other hand, the expansions induced by L_β are called *lazy*, since this time at each iteration the map assigns the smallest digit possible. Superimposing the greedy and the lazy β-transformations on the same state space breaks down the interval into overlapping regions, called *switch* regions, of the form

$$S_i = \left[\frac{i}{\beta}, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{i-1}{\beta}\right], \text{ for } i = \{1, 2, \ldots, \lfloor\beta\rfloor\},$$

and an equaliser region,

$$E = \left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right] \setminus \bigcup_{1\le i\le\lfloor\beta\rfloor} S_i,$$

where H_β and L_β coincide. On the switch region S_i, the map H_β assigns the digit i and L_β the digit $i-1$, while on E both maps assign the same digit. As a consequence, the iteration of the pseudo-skew product map $R = \{H_\beta, L_\beta\}$, also called random β-transformation, can produce for the same point x uncountably many different β-expansions, depending on which map (and therefore on which digit) is chosen in the switch regions. See Figure 1.4 for a visualisation of the maps H_β and L_β, and Figure 1.5 for the associated pseudo-skew product system.



(a) H_β (b) L_β

Figure 1.4: Example of the greedy and lazy β-transformations, for $\beta = 2.39$.

For further generalisation on the set of digits and more on random β-transformations, see [DdV05, DHK09, DK10, DK13, K14], for example. In Chapter 3 we develop an algebraic procedure to explicitly compute the invariant measure and the ergodic properties of a class of systems that include random β-transformations.

Differently from β-transformations, that represent numbers using a finite alphabet, Lüroth series use an infinite one, given by all positive integers greater than 1. We introduce them in the next section.

§1.3.2 Lüroth series

Any real number $x \in (0,1]$ can be written in the form

$$x = \frac{1}{\ell_1(x)} + \frac{1}{\ell_1(x)(\ell_1(x)-1)\ell_2(x)} + \ldots + \frac{1}{\ell_1(x)\cdots\ell_{n-1}(x)(\ell_{n-1}(x)-1)\ell_n(x)} + \ldots, \tag{1.7}$$

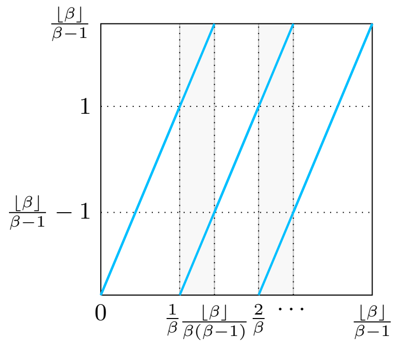


Figure 1.5: The random β-transformation for $\beta = 2.39$.

for some positive integers $\ell_n(x) \geq 2$. The series from (1.7) is called the *Lüroth expansion* of the point x and the digits $\ell_n(x)$ are obtained through the iteration of the Lüroth map $T_L : [0,1] \to [0,1]$ defined by

$$T_L(x) = \begin{cases} n(n-1)x - (n-1) & \text{if } x \in \left(\frac{1}{n}, \frac{1}{n-1}\right] \text{ for } n \geq 2, \\ 0 & \text{if } x = 0, \end{cases}$$

see Figure 1.6(a).

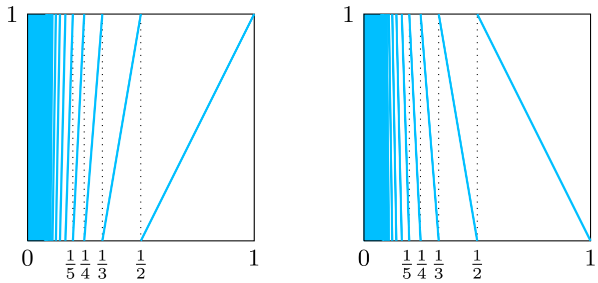


Figure 1.6: The standard and alternating Lüroth maps, respectively.

Specifically, for $n \geq 1$, the digits are obtained by setting

$$\ell_n(x) = d, \quad \text{if } T_L^{n-1}(x) \in \left(\frac{1}{d}, \frac{1}{d-1}\right], \; d \geq 2.$$

Lüroth series have been introduced in 1883 in [L83], and studied in a more general context in [BBDK94, KKK90, KKK91, BI09], for example. Chapter 2 introduces a random version of Lüroth maps, given by the standard Lüroth map T_L and its flipped version, the *alternating Lüroth map* $T_A = 1 - T_L$, see Figure 1.6(b). The system is then further analysed at the end of Chapter 3.

The alternating Lüroth map T_A can also be seen as a linearised version of the Gauss map G that produces continued fraction expansions and it is introduced in the next section.

§1.3.3 Continued fractions

Any real number $x \in (0,1)$ can be written as a *continued fraction* of the form

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots}}},$$

with $a_i(x) \in \mathbb{N}$. The continued fraction is usually denoted by $x = [0; a_1, a_2, a_3, \ldots]$, for $a_i = a_i(x)$, and it can be dynamically generated by iteration of the Gauss map $G : [0,1] \to [0,1]$ defined by

$$G(x) = \begin{cases} \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases} \tag{1.8}$$

see Figure 1.7.

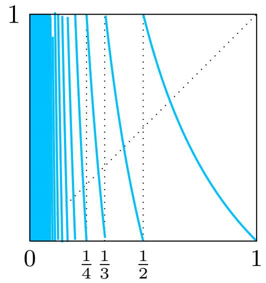


Figure 1.7: The Gauss map G.

The iteration of the Gauss map produces indeed continued fraction expansions with digits chosen in an infinite alphabet, given by $\mathbb{N}$, and defined by

$$a_i(x) = n \quad \text{if } G^{i-1}(x) \in \left(\frac{1}{n+1}, \frac{1}{n} \right], \quad n \geq 1.$$

It follows from Euclid's algorithm that every rational number $\frac{p}{q}$ presents a finite continued fraction, while irrational numbers have an infinite one. For rational numbers, there exists a unique representation $[0; a_1, a_2, \ldots, a_n]$, with $a_n \geq 2$. Without the constraint on the last digit the representation is not unique. Indeed, if $a_n = 1$, then

$$[0; a_1, a_2, \ldots, a_{n-1} + 1] = [0; a_1, a_2 \ldots, a_{n-1}, a_n].$$

For irrational numbers the representation is unique. Furthermore, for any irrational number $x = [0; a_1, a_2, \ldots] \in (0,1)$, it is possible to define a sequence of rational numbers $\left(\frac{p_n}{q_n}(x)\right)_{n\geq 0}$, that provides increasingly good approximations, alternately from above and below, of the irrational number x. For $n \geq 0$, the *n-th convergent* $\frac{p_n}{q_n}(x)$, is defined by

$$\frac{p_n}{q_n}(x) = [0; a_1, a_2, \ldots, a_n],$$

with $\frac{p_0}{q_0}(x) = [0]$, $\frac{p_1}{q_1}(x) = [0; a_1]$, and

$$x = \lim_{n\to\infty} \frac{p_n}{q_n}(x).$$

For more details, see [DK02], for example.

Another way to obtain the Gauss map G is through an induced system. Let $X = (0, \infty)$ and consider $Y = (0,1) \subseteq X$ and the transformation

$$T(x) = \begin{cases} \frac{1}{x} - 1 & \text{if } x \in Y, \\ x - 1 & \text{if } x \in Y^c. \end{cases}$$

Note that T determines the digits $a_i(x)$ of the continued fraction expansions of a point x after having successively subtracting 1 until the iterate of the point lies in Y. The first return time map to Y, $\varphi : Y \to Y$, is indeed given by

$$\varphi(x) = n, \quad \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right],$$

and the induced map from Definition 1.2.15 corresponds to

$$T_Y(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor = G(x).$$

For more details, see [Z09].

G admits an absolutely continuous invariant probability measure with density function

$$h(x) = \frac{1}{\log 2} \frac{1}{x+1},$$

which can be recovered by projecting the density function of the measure of its natural extension. The canonical natural extension of the transformation G is given in [NIT77] by the map $S : [0,1]^2 \to [0,1]^2$, and defined by

$$S(x,y) = \left(G(x), \frac{1}{d_1(x) + y}\right), \quad (x,y) \in [0,1]^2. \tag{1.9}$$

The transformation S admits an acim of density function $\frac{1}{(1+xy)^2}$. See Figure 1.8 for a visualisation of the 2-dimensional domain of the natural extension and the action of S on it. Chapter 4 uses the natural extension construction to obtain infinite invariant measures for a class of continued fraction maps.

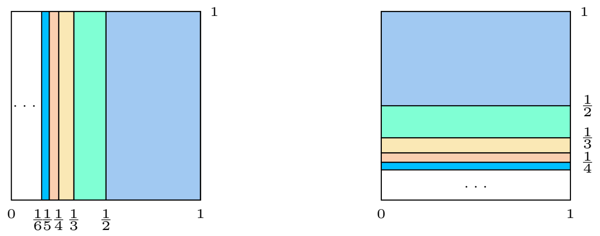


Figure 1.8: The action of the map S on the domain $[0,1]^2$ of the natural extension for the Gauss map G. Areas on the left are mapped to areas on the right with the same colour.

There exists a variety of continued fraction maps that provide continued fractions expansions with different alphabets. For instance, there exist continued fraction maps that provide expansions with negative numerators, or the odd (and even) continued fractions that use only odd (respectively even) digits. See for example [K91, DK00, HK02, KSS10, BCIT13, KKV17] for further generalisations.

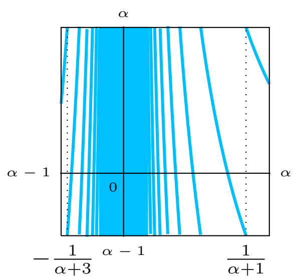


Figure 1.9: Example of T_α for $\alpha = \frac{7}{10}$.

In particualr, in 1981 Nakada formalised in [N81] the family of α-continued fractions $T_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha]$ depending on the parameter $\alpha \in [0,1]$ and defined by

$$T_\alpha(x) = \frac{1}{|x|} - \left\lfloor \frac{1}{|x|} + 1 - \alpha \right\rfloor,$$

see Figure 1.9. Since then, lots of research has focused on the study of the invariant measure and the matching property of the maps T_α. See [LM08, CMPT10, N11, CT12, CT13, T14], for instance. Note that one can recover the Gauss map G from a map T_α, by setting $\alpha = 1$, i.e., $T_1 = G$.

§1.4 Outline

This work investigates absolutely continuous measures for several dynamical systems. The analysis is performed in the deterministic and random setting, as well as in the frame of finite and infinite ergodic theory.

Chapter 2 presents a family of random maps given by the combination of generalised Lüroth maps defined on a subset X of the unit interval $[0, 1]$. Each random system produces for almost all $x \in X$ uncountably many different generalised Lüroth expansions, that can be studied simultaneously. The chapter suggests the need of an algorithmic procedure to construct the density of an absolutely continuous invariant measure for random piecewise affine systems of the interval.

Chapter 3 answers the urgency of the previous chapter. It offers an algebraic algorithm that receives as an input a random piecewise affine system T that is expanding on average, and gives a formula for a physical T-invariant measure as output. Previously, the density was known only for very few specific cases. The heavy procedure is shown to be efficient for specific classes of transformations, such as random β-transformations, Lüroth maps and more general for systems that present the dynamical features of matching, which is further analysed in Chapter 5.

Chapter 4 looks into the consequences of matching for an infinite class $\{T_\alpha\}_\alpha$ of continued fraction maps. More specifically, the phenomenon of synchronization is used to find the 2-dimensional domain of a planar natural extension. Such information is used to find explicit expressions of the density for a large part of the parameter space. Additionally, matching is proved to hold for Lebesgue almost every parameter α, and it divides the parameter space into intervals of constant matching exponents, called matching intervals. It is the first family of infinite measure systems in which matching is recognized. Lastly, the chapter relates these matching intervals to the corresponding sets of Nakada's α-continued fraction maps.

Chapter 5 extends the notion of matching for deterministic transformations to random matching for random interval maps. Random matching is then studied for a variety of families of random dynamical systems, that includes generalised β-transformations and continued fraction maps. Furthermore, for a large class of piecewise affine random systems of the interval, the property of random matching is proved to imply that any invariant density of a stationary measure is piecewise constant. Lastly, the chapter introduces a family of random maps producing signed binary expansions. The property of random matching and its consequences on the structure of the density function are then applied to study the frequency of the digit 0 in such expansions.

CHAPTER 2

Random c-Lüroth Expansions

This chapter is based on: [KM21].

Abstract

We introduce a family of random c-Lüroth transformations $\{L_c\}_{c\in[0,\frac{1}{2}]}$, obtained by randomly combining the standard and alternating Lüroth maps with probabilities p and $1-p$, $0<p<1$. We prove that the pseudo-skew product map L_c produces for each $c \le \frac{2}{5}$ and for Lebesgue almost all $x \in [c,1]$ uncountably many different generalised Lüroth expansions that can be investigated simultaneously. Moreover, for $c = \frac{1}{\ell}$, for some $\ell \in \mathbb{N}_{\ge 3} \cup \{\infty\}$, Lebesgue almost all x have uncountably many universal generalised Lüroth expansions with digits less than or equal to ℓ. For $c = 0$ we show that typically the speed of convergence to an irrational number x, of the sequence of Lüroth approximants generated by L_0, is equal to that of the standard Lüroth approximants; and that the quality of the approximation coefficients depends on p and varies continuously between the values for the alternating and the standard Lüroth map. Furthermore, we show that for each $c \in \mathbb{Q}$ the map L_c admits a Markov partition. For specific values of $c > 0$, we compute the density of the stationary measure and we use it to study the typical speed of convergence of the approximants and the digit frequencies.

§2.1 Motivation and context

In 1883 Lüroth showed in [L83] that each $x \in [0,1]$ can be expressed in the form

$$x = \frac{1}{d_1} + \frac{1}{d_1(d_1-1)d_2} + \ldots = \sum_{m \geq 1} (d_m - 1) \prod_{j=1}^{m} \frac{1}{d_j(d_j-1)}, \tag{2.1}$$

where $d_m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ for each m (and with $\frac{1}{\infty} = 0$). Such expressions are now called *Lüroth expansions*. By considering the numbers

$$\frac{p_n}{q_n} := \sum_{m=1}^{n} (d_m - 1) \prod_{j=1}^{m} \frac{1}{d_j(d_j-1)}, \quad n \geq 1, \tag{2.2}$$

one obtains a sequence of rationals converging to the number x, making Lüroth expansions suitable for finding rational approximations of irrational numbers. Since their introduction in 1883 much research has been done on the approximation properties of Lüroth expansions from various perspectives. In this article we address these questions by adopting a random dynamical systems approach. It turns out that this yields for each x many different number expansions similar to the Lüroth expansion from (2.1), without compromising the quality of approximation. Before we state our results, we first give a brief summary of a selection of the known results.

A Lüroth expansion is called *ultimately periodic* if there exist $n \geq 0$ and $r \geq 1$ such that $d_{n+j} = d_{n+r+j}$ for all $j \geq 1$ (and *periodic* if $n = 0$). One of the most basic results on Lüroth expansions, obtained in [L83], is on periodicity.

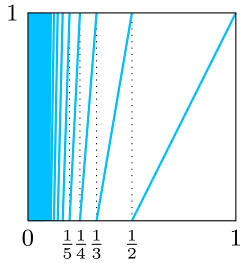
2.1.1 Theorem (page 416, [L83]). *A real number $x \in (0,1)$ has an ultimately periodic Lüroth expansion if and only if $x \in \mathbb{Q}$.*

Many other properties of Lüroth expansions were obtained using a dynamical system. Lüroth expansions can be obtained dynamically by iterating the *Lüroth transformation* $T_L : [0,1] \to [0,1]$ given by $T_L(0) = 0$, $T_L(1) = 1$ and

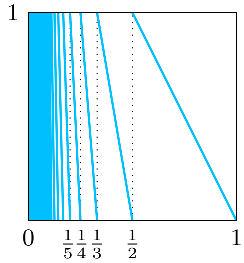
$$T_L(x) = \left\lceil \frac{1}{x} \right\rceil \left(\left\lceil \frac{1}{x} \right\rceil - 1 \right) x - \left(\left\lceil \frac{1}{x} \right\rceil - 1 \right)$$

for $x \neq 0, 1$, where $\lceil x \rceil$ denotes the smallest integer not less than x. See Figure 2.1(a) for the graph.

The digits d_n, $n \geq 1$, are obtained by setting $d_n(x) = k$ if $T_L^{n-1}(x) \in \left[\frac{1}{k}, \frac{1}{k-1}\right)$, $k \geq 2$, $d_n(x) = 2$ if $T_L^{n-1}(x) = 1$ and $d_n(x) = \infty$ if $T_L^{n-1}(x) = 0$. Hence, the map T_L produces for each $x \in [0,1]$ a Lüroth expansion as in (2.1). From the graph of T_L one sees immediately that Lebesgue almost all numbers $x \in [0,1]$ have a unique Lüroth expansion and if x does not have a unique expansion, then it has exactly two different ones, one with $d_n = d$ and $d_{n+j} = \infty$ and one with $d_n = d+1$ and $d_{n+j} = 2$ for some n, d and all $j \geq 1$. This holds for any number $x \in [0,1]$ for which there is an



(a) T_L



(b) T_A

Figure 2.1: The standard and the alternating Lüroth maps in (a) and (b), respectively.

$n \geq 1$ such that $T_L^n(x) = 0$. By identifying these two expansions we can speak of the unique Lüroth expansion of any number $x \in [0,1]$.

From the dynamics of T_L we get information on the digit frequencies in Lüroth expansions. The map T_L is measure preserving and ergodic with respect to the Lebesgue measure λ on $[0,1]$. It is then a straightforward application of Birkhoff's Ergodic Theorem that, in the Lüroth expansion of Lebesgue almost every x, the frequency of the digit d equals $\frac{1}{d(d-1)}$, which corresponds to the length of the interval $\left[\frac{1}{d}, \frac{1}{d-1}\right)$. It was proven by Šalát in [S68] that for any $D \geq 2$ the set of points $x \in (0,1)$ for which all Lüroth expansion digits are bounded by D has Hausdorff dimension < 1 with the dimension approaching 1 as $D \to \infty$. The articles [S68, BBDK94, BI09, SF11, CWZ13, MT13, SFM17] all consider Lüroth expansions with certain restrictions on the digits d_n.

The quality of approximation by Lüroth expansions depends on the *approximants* or *convergents* $\frac{p_n}{q_n}$ given in (2.2). In [BI09] the authors give a multifractal analysis of the speed with which the sequence $\left(\frac{p_n}{q_n}\right)_n$ converges to the corresponding x using the Lyapunov exponent. The Lyapunov exponent of T_L at $x \in (0,1)$ is defined by

$$\Lambda_L(x) = \lim_{n\to\infty} \frac{1}{n} \log \prod_{k=0}^{n-1} |T_L'(T_L^k(x))|,$$

whenever this limit exists. It follows from another application of Birkhoff's Ergodic Theorem that for λ-a.e. $x \in (0,1)$,

$$\Lambda(x) = \sum_{d=2}^{\infty} \frac{\log(d(d-1))}{d(d-1)}.$$

One of the results from [BI09], which we mention here for further reference, is the following.

2.1.2 Theorem ([BI09]). *For λ-a.e. $x \in (0,1)$,*

$$\lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{p_n}{q_n} \right| = -\sum_{d=2}^{\infty} \frac{\log(d(d-1))}{d(d-1)}.$$

Moreover, the range of possible values of this rate is $(-\infty, -\log 2]$.

Another way to express the quality of the approximations is via the limiting behaviour of the *approximation coefficients*

$$\theta_n^L(x) := q_n \left| x - \frac{p_n}{q_n} \right|, \tag{2.3}$$

where $q_n = d_n \prod_{i=1}^{n-1} d_i(d_i - 1)$. In [DK96] the authors proved the following result.

2.1.3 Theorem (Theorem 2, [DK96]). *For λ-a.e. $x \in [0,1]$ and for every $z \in (0,1]$ the limit*

$$\lim_{N\to\infty} \frac{\#\{1 \le j \le N \,:\, \theta_j^L(x) < z\}}{N}$$

exists and equals

$$F_L(z) := \sum_{k=2}^{\lfloor \frac{1}{z} \rfloor + 1} \frac{z}{k} + \frac{1}{\lfloor \frac{1}{z} \rfloor + 1}. \tag{2.4}$$

We refer to e.g. [SYZ14, V14, G16, GL16, ZC16, S17, LCTW18, SX18, TW18] for results on other properties of Lüroth expansions.

In [BBDK94] the authors placed the map T_L in the larger framework of *generalised Lüroth series transformations* (GLS). A GLS transformation is a piecewise affine onto map $T_{\mathcal{P},\varepsilon} : [0,1] \to [0,1]$ given by an at most countable interval partition $\mathcal{P}$ on $[0,1]$ and a vector $\varepsilon = (\varepsilon_n)_n \in \{0,1\}^{\#\mathcal{P}}$ specifying for each partition element the orientation of $T_{\mathcal{P},\varepsilon}$ on that interval. The Lüroth transformation can be obtained by taking the partition $\mathcal{P}_L = \left\{ \left[\frac{1}{n}, \frac{1}{n-1}\right) \right\}_{n \ge 2}$ and orientation vector $\varepsilon = (0)_{n\ge1}$, i.e., all branches are orientation preserving. In [BBDK94] the authors considered all GLS transformations $T_{\mathcal{P}_L,\varepsilon}$ with partition $\mathcal{P}_L$. Besides the Lüroth transformation, another specific instance of this family is the *alternating Lüroth map* $T_A : [0,1] \to [0,1]$ given by $T_A(x) = 1 - T_L(x)$, see Figure 2.1(b), which has $\varepsilon = (1)_{n\ge1}$, so that all branches orientation reversing. Similar to the Lüroth expansion from (2.1), iterations of any GLS transformation $T_{\mathcal{P}_L,\varepsilon}$ yield number expansions for $x \in [0,1]$ of the form

$$x = \sum_{n=1}^{\infty} (-1)^{\sum_{i=1}^{n-1} s_i} \frac{d_n - 1 + s_n}{\prod_{i=1}^{n} d_i(d_i - 1)}, \tag{2.5}$$

where $s_n \in \{0,1\}$ and $d_n \ge 2$, called *generalised Lüroth expansions*. Here we let $\sum_{i=1}^{0} s_i = 0$. For each map $T_{\mathcal{P}_L,\varepsilon}$ the authors of [BBDK94] considered the approximation coefficients $\theta_n^{\mathcal{P}_L,\varepsilon}$ and the corresponding distribution function $F_{\mathcal{P}_L,\varepsilon}$ and they found the following.

2.1.4 Theorem (Theorem 4, [BBDK94]). *The distribution function of θ_n^A for the map T_A is given for $0 < z \leq 1$ by*

$$F_A(z) = \sum_{k=2}^{\lfloor \frac{1}{z} \rfloor} \frac{z}{k-1} + \frac{1}{\lfloor \frac{1}{z} \rfloor}.$$

For any GLS transformation $T_{\mathcal{P}_L,\varepsilon}$ it holds that

$$F_A \leq F_{\mathcal{P}_L,\varepsilon} \leq F_L,$$

Furthermore, the first moments of F_L and F_A are given respectively by

$$M_L := \int_{[0,1]} 1 - F_L \, d\lambda = \frac{\zeta(2)}{2} - \frac{1}{2} \quad \text{and} \quad M_A := \int_{[0,1]} 1 - F_A \, d\lambda = 1 - \frac{\zeta(2)}{2},$$

where $\zeta(2)$ is the zeta function evaluated at 2.

The authors of [BBDK94] remark that they suspect that the set of values that the limit $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \theta_i^{\mathcal{P}_L,\varepsilon}$ can take is a fractal subset of the interval $[M_A, M_L]$. Other results on the map T_A can be found e.g. in [KKK90, KKK91]. For results on different families of GLS transformations, see e.g. [KMS11, M11, CWY14, CW14].

In this chapter we adopt a random approach to Lüroth expansions. We introduce a family of random Lüroth systems $\{L_{c,p}\}_{c\in[0,\frac{1}{2}],0\leq p\leq 1}$ that are obtained from randomly combining the maps T_L and T_A. The parameter c is the cutting point, that defines the interval $[c,1]$ on which each $L_{c,p}$ is defined. More precisely, we overlap T_L and T_A on the interval $[c,1]$ and remove from both maps the pieces that map points into $[0,c)$. The parameter p reflects the probability with which we apply the map T_L. To be precise, let $T_0 := T_L$ and $T_1 := T_A$ and let σ denote the left shift on sequences. Then the *random c-Lüroth transformation* $L_{c,p} : \{0,1\}^{\mathbb{N}} \times [c,1] \to \{0,1\}^{\mathbb{N}} \times [c,1]$ is defined by

$$L_{c,p}(\omega, x) = \big(\sigma(\omega), T_{\omega_1}(x) 1_{[c,1]}(T_{\omega_1}(x)) + T_{1-\omega_1}(x) 1_{[0,c)}(T_{\omega_1}(x))\big).$$

By iteration, each map $L_{c,p}$ produces for each pair (ω, x) a generalised Lüroth expansion for x as in (2.5). So for typical $x \in [c,1]$ multiple generalised Lüroth expansions of the form (2.5) are obtained. If $c = 0$ the corresponding random Lüroth expansions have digits in the set $\mathbb{N}_{\geq 2} \cup \{\infty\}$, but for $c > 0$ the available set of digits is bounded from above. This makes the two cases inherently different. We summarise our main results in the following three theorems.

1 Theorem. *Let $c \in \left[0, \frac{1}{2}\right]$.*

(i) If $x \in [c,1] \setminus \mathbb{Q}$, then no generalised Lüroth expansion of x produced by $L_{c,p}$ is ultimately periodic.

(ii) If $x \in [c,1] \cap \mathbb{Q}$ then, depending on the values of x and c, the map $L_{c,p}$ can produce any of the following number of different generalised Lüroth expansions:

- *a unique expansion,*
- *a finite or countable number of ultimately periodic expansions,*
- *countably many ultimately periodic expansions and uncountably many expansions that are not ultimately periodic.*

We also give a characterisation on when each of these cases occurs. This result does not depend on the value of p. The following results further explore the properties of the produced generalised Lüroth expansions in case $c = 0$ and $c > 0$. Note that generalised Lüroth expansions are given by a sequence of pairs $((s_n, d_n))_n$ with $s_n \in \{0, 1\}$ and $d_n \geq 2$, rather than just the sequence $(d_n)_n$. We call a generalised Lüroth expansion generated by a map $L_{c,p}$ *universal* if any possible block of digits $(s_1, d_1), \ldots, (s_n, d_n)$ of any length n from the alphabet associated to $L_{c,p}$ occurs in the expansion.

2 Theorem. *Let $c = 0$ and $0 < p < 1$.*

(i) The map $L_{0,p}$ generates for Lebesgue almost every $x \in [0, 1]$ uncountably many universal generalised Lüroth expansions.

(ii) The speed of convergence of the sequence $\left(\frac{p_n}{q_n}\right)_n$ to x for any generalised Lüroth expansion produced by $L_{0,p}$ typically satisfies

$$\lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{p_n}{q_n} \right| = -\sum_{d=2}^{\infty} \frac{\log(d(d-1))}{d(d-1)}$$

and the range of possible values of this rate is $(-\infty, -\log 2]$. In particular, this rate does not depend on p.

(iii) Typically the approximation coefficients generated by $L_{0,p}$ satisfy

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \theta_i^{0,p} = p\frac{2\zeta(2) - 3}{2} + \frac{2 - \zeta(2)}{2},$$

where $\zeta(2)$ is the zeta function evaluated at 2. In particular, this limit can attain any value in the interval $[M_A, M_L]$.

The result in (ii) is given by considering the Lyapunov exponent of the random system $L_{0,p}$ as was done for Theorem 2.1.2. We see that the speed of convergence is not compromised by adding randomness to the system. For (iii) we note that instead of a fractal set inside $[M_A, M_L]$ we can obtain the full interval by adding randomness.

3 Theorem. *Let $c > 0$ and $0 < p < 1$.*

(i) If $0 < c \leq \frac{2}{5}$, then the map $L_{c,p}$ generates for every irrational $x \in [c, 1]$ uncountably many different generalised Lüroth expansions.

(ii) If $c = \frac{1}{\ell}$ for some $\ell \in \mathbb{N}_{\geq 3}$, then $L_{c,p}$ generates for every irrational $x \in [c, 1]$ uncountably many universal generalised Lüroth expansions.

Note that results corresponding to (ii) and (iii) from Theorem 2 in case $c > 0$ are missing. For $c > 0$ the speed of convergence of the sequence $\left(\frac{p_n}{q_n}\right)_n$ is still governed by the Lyapunov exponent of the map $L_{c,p}$, but this is not so easily computed. For $c = 0$ we are in the lucky circumstance that $m_p \times \lambda$ is an invariant measure for $L_{0,p}$, where m_p is the $(p, 1-p)$-Bernoulli measure on $\{0,1\}^{\mathbb{N}}$ and λ is the one-dimensional Lebesgue measure. General results give the existence of an invariant measure of the form $m_p \times \mu_{p,c}$ where $\mu_{p,c} \ll \lambda$. In most cases the random systems $L_{c,p}$ satisfy the conditions from Theorem 3.4.1, which gives an expression for the density $\frac{\mathrm{d}\,\mu_{p,c}}{\mathrm{d}\,\lambda}$ in individual cases. In the last section we discuss some values of c for which we can determine a nice formula for this density. We then get a result similar to Theorem 2(ii) and compute the frequency of the digits d in the generalised Lüroth expansions.

The chapter is organised as follows. In Section 2.2 we describe how to obtain generalised Lüroth expansions from the random maps $L_{c,p}$ and we characterise numbers with ultimately periodic expansions. Here we prove Theorem 1. Section 2.3 is dedicated to the case $c = 0$. Theorem 2(i) is proved in Proposition 2.3.1, part (ii) is covered by Proposition 2.3.3 and part (iii) is done in Proposition 2.3.4. In Section 2.4 we focus on $c > 0$. Theorem 3 corresponds to the content of Theorem 2.4.3 and Theorem 2.4.7. Proposition 2.4.10 contains the result that $L_{c,p}$ admits a Markov partition for any $c \in \left(0, \frac{1}{2}\right] \cap \mathbb{Q}$ and is followed by various examples in which we explicitly compute the density of the measure $\mu_{c,p}$ and in some cases also the typical speed of convergence of the convergents $\frac{p_n}{q_n}$ as well as the frequency of the digits. The last section stresses the need for computable procedures for obtaining the densities for absolutely continuous invariant measures of random interval maps and reveals links with all the remaining chapters of the dissertation.

§2.2 Random c-Lüroth transformations

In this section we introduce the family $\{L_{c,p}\}_{c\in[0,\frac{1}{2}],0\le p\le 1}$ of random (c,p)-Lüroth transformations and show how these maps produce generalised Lüroth expansions for all $x \in [c,1]$. Since the probability p does not play a role in this section, we drop the subscript for now and refer to the map L_c as the random c-Lüroth transformation instead. First recall some notation for sequences. Let $\mathcal{A}$ be an at most countable set of symbols, called *alphabet*. Let $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ denote the left shift on sequences, so for a sequence $a = (a_i)_{i\ge 1} \in \mathcal{A}^{\mathbb{N}}$ we have $\sigma(a) = a'$, where $a'_i = a_{i+1}$ for all i. (With slight abuse of notation we will always use σ to denote the left shift on sequences, regardless of the underlying alphabet.) For a finite string $a = a_1 \dots a_k \in \mathcal{A}^k$ and $1 \le n \le k$ or an infinite sequence $\mathbf{a} = (a_n)_{n\ge 1} \in \mathcal{A}^{\mathbb{N}}$ and $n \ge 1$ we denote by a_1^n the initial part $a_1^n = a_1 \cdots a_n$. We call a sequence $\mathbf{a} = (a_i)_{i\ge 1}$ *ultimately periodic* if there exist an $n \ge 0$ and an $r \ge 1$ such that $a_{n+j} = a_{n+r+j}$ for all $j \ge 1$ and *periodic* if $n = 0$. Finally, we denote cylinder sets in $\mathcal{A}^{\mathbb{N}}$ using square brackets, i.e.,

$$[b_1, \dots, b_k] = \{\mathbf{a} \in \mathcal{A}^{\mathbb{N}} : a_j = b_j, \text{ for all } 1 \le j \le k\}.$$

For $c \in [0, \frac{1}{2}]$ and $n \geq 1$ let $z_n = \frac{1}{n}$,

$$z_n^+ := z_n + c z_n z_{n-1} \quad \text{and} \quad z_n^- := z_n - c z_n z_{n+1}. \tag{2.6}$$

Then $z_n \leq z_n^+ \leq z_{n-1}^- \leq z_{n-1}$. Define

$$T_{0,c}(x) = \begin{cases} T_A(x), & \text{if } x \in \{0\} \cup \bigcup_{n=2}^{\infty} [z_n, z_n^+), \\ T_L(x), & \text{if } x \in \{1\} \cup \bigcup_{n=2}^{\infty} [z_n^+, z_{n-1}), \end{cases}$$

and

$$T_{1,c}(x) = \begin{cases} T_A(x), & \text{if } x \in \{0\} \cup \bigcup_{n=2}^{\infty} [z_n, z_{n-1}^-], \\ T_L(x), & \text{if } x \in \{1\} \cup \bigcup_{n=2}^{\infty} (z_{n-1}^-, z_{n-1}). \end{cases}$$

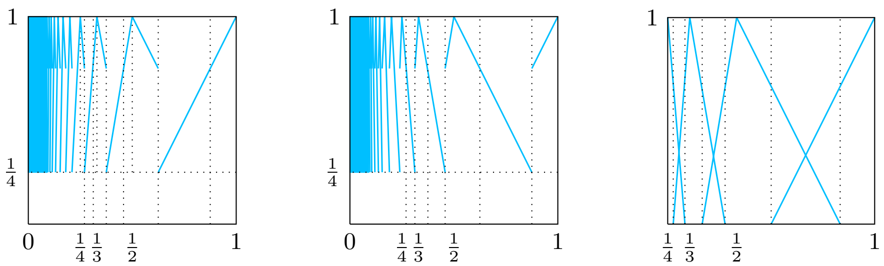
As can be seen from Figure 2.2, the interval $[c, 1]$ is an attractor for the dynamics of both $T_{0,c}$ and $T_{1,c}$, so we choose $[c, 1]$ as their domains. Note that $T_{0,c}$ and $T_{1,c}$ assume the same values on the intervals $[z_n, z_n^+)$ and $(z_n^-, z_n]$ and differ on $[z_n^+, z_{n-1}^-]$ for $n > 1$. We denote the union of the subintervals on which $T_{0,c}$ and $T_{1,c}$ assume different values by S, i.e.,

$$S = [c, 1] \cap \bigcup_{n>1} [z_n^+, z_{n-1}^-], \tag{2.7}$$

and call this the *switch region*. For $c = 0$ we see that $S = (0, 1] \setminus \{z_n : n \geq 1\}$ and $T_{0,0} = T_L$ and $T_{1,0} = T_A$ except for the points z_n where $T_{0,0}(z_n) = T_{1,0}(z_n) = 1$. We combine these two maps to make a random dynamical system by defining the *random c-Lüroth transformation* $L_c : \{0,1\}^{\mathbb{N}} \times [c, 1] \to \{0,1\}^{\mathbb{N}} \times [c, 1]$ by

$$L_c(\omega, x) = (\sigma(\omega), T_{\omega_1,c}(x)).$$

See Figure 2.2(c), for an example with $c = \frac{1}{4}$.



(a) $T_{0,\frac{1}{4}}$ (b) $T_{1,\frac{1}{4}}$ (c) $L_{\frac{1}{4}}$

Figure 2.2: The maps $T_{0,\frac{1}{4}}$, $T_{1,\frac{1}{4}}$, and the random c-Lüroth map $L_{\frac{1}{4}}$ on $\left[\frac{1}{4}, 1\right]$.

To denote the orbit along a specific path $\omega \in \{0,1\}^{\mathbb{N}}$ we use the following notation. For a finite string $\omega \in \{0,1\}^k$ and $0 \le n \le k$, let

$$T_{\omega,c} = T_{\omega_k,c} \circ T_{\omega_{k-1},c} \circ \cdots \circ T_{\omega_1,c} \quad \text{and} \quad T^n_{\omega,c} = T_{\omega_1^n,c} = T_{\omega_n,c} \circ T_{\omega_{n-1},c} \circ \cdots \circ T_{\omega_1,c}. \tag{2.8}$$

Similarly, for an infinite path $\omega \in \{0,1\}^{\mathbb{N}}$ and $n \ge 1$ we let

$$T^n_{\omega,c} = T_{\omega_1^n,c} = T_{\omega_n,c} \circ T_{\omega_{n-1},c} \circ \cdots \circ T_{\omega_1,c}. \tag{2.9}$$

For any ω we also set $T^0_{\omega,c} = T_{\omega_1^0,c} = id$. Note that we have defined $T_{0,c}$ and $T_{1,c}$ in such a way that $T_{j,c}(x) \ne 0$ for any j, c, x. With this notation we can obtain *c-Lüroth expansions* of points in $[c,1]$ by defining for each $(\omega, x) \in \{0,1\}^{\mathbb{N}} \times [c,1]$ two sequences $(s_i)_{i\ge1}$ (the signs) and $(d_i)_{i\ge1}$ (the digits) as follows. For $(\omega,x) \in \{0,1\}^{\mathbb{N}} \times [c,1]$ set for $c > 0$ and for $i \ge 1$,

$$s_i = s_i(\omega,x) = \begin{cases} 0, & \text{if } L_c^{i-1}(\omega,x) \in [0] \times S \cup \{0,1\}^{\mathbb{N}} \times \big(\cup_n (z_n^-, z_n) \cup \{1\} \big), \\ 1, & \text{if } L_c^{i-1}(\omega,x) \in [1] \times S \cup \{0,1\}^{\mathbb{N}} \times \big(\cup_n [z_n, z_n^+) \cup \{0\} \big). \end{cases}$$

For $c = 0$, set $s_i = \omega_i$ for each i. For $c \ge 0$, $x \ne 0$ and for $i \ge 1$ set

$$d_i = d_i(\omega,x) = \begin{cases} 2, & \text{if } T^{i-1}_{\omega,c}(x) = 1, \\ n, & \text{if } T^{i-1}_{\omega,c}(x) \in [z_n, z_{n-1}),\ n \ge 2. \end{cases}$$

Then one can write for $x \ne 0$ and each $i \ge 1$ that

$$T^i_{\omega,c}(x) = (-1)^{s_i} d_i(d_i - 1) T^{i-1}_{\omega,c}(x) + (-1)^{s_i+1}(d_i - 1 + s_i).$$

By inversion and iteration we obtain what we call the *c-Lüroth expansion* of (ω, x):

$$x = \sum_{n=1}^{\infty} (-1)^{\sum_{i=1}^{n-1} s_i(\omega,x)} \frac{d_n(\omega,x) - 1 + s_n(\omega,x)}{\prod_{i=1}^n d_i(\omega,x)(d_i(\omega,x) - 1)}, \tag{2.10}$$

where $\sum_{i=1}^0 s_i(\omega,x) = 0$ and the sum converges since $d_i \ge 2$ and $T^i_{\omega,c}(x) \in (0,1]$ for all ω, x, i. Note that this expansion of x is of the form (2.5), i.e., it is a generalised Lüroth expansion.

2.2.1 Remark. (i) Due to the fact that $T_{j,c}(x) \ne 0$ for all j, c and $x \ne 0$, the maps L_c do not produce finite generalised Lüroth expansions. That is, L_c assigns to each (ω, x) an infinite sequence $((s_n(\omega,x), d_n(\omega,x)))_n$ with $s_n \in \{0,1\}$ and $d_n \ge 2$.

(ii) As an immediate consequence of the above, we see that for each $D \ge 2$ *every* $x \in \left[\frac{1}{D}, 1\right]$ has a generalised Lüroth expansion that only uses digits $d_n \le D$ and that is generated by any random c-Lüroth system with $c \ge \frac{1}{D}$. This is in contrast to the deterministic case, where by the result of Šalát in [S68] for any D the set of points x that has D as an upper bound for the Lüroth digits has Hausdorff dimension strictly less than 1.

(iii) If $x \in S$ and $x \notin \left\{ \frac{2n-1}{2n(n-1)} : n \ge 1 \right\}$ (so $T_L(x) \ne T_A(x)$), then $d_2(\omega, x) = 2$ for all ω with $T_{\omega_1}(x) > T_{1-\omega_1}(x)$ and $d_2(\omega,x) > 2$ otherwise. For $c = 0$ this implies, since $(s_n)_{n\geq 1} = (\omega_n)_{n\geq 1}$ and $S = (0,1] \setminus \{\frac{1}{n} : n \geq 1\}$, that Lebesgue almost all $x \in [0,1]$ have uncountably many different random Lüroth expansions. In Section 2.4 we see that a similar statement holds for $c > 0$ and we get that most points even have uncountably many different generalised Lüroth expansions with all $d_n \leq D$.

Similar to the deterministic case, we call a c-Lüroth expansion of (ω, x) *(ultimately) periodic* if the corresponding sequence $((s_i, d_i))_{i\geq 1}$ is (ultimately) periodic. From the expression (2.10) it is clear that the c-Lüroth expansion of (ω, x) is ultimately periodic if and only if there are $n \geq 0$ and $r \geq 1$ such that

$$T^{n+j}_{\omega,c}(x) = T^{n+r+j}_{\omega,c}(x),$$

for all $j \geq 0$, implying that $x \in \mathbb{Q}$. We define the following weaker notion.

2.2.2 Definition (Returning points). Let $c \in \left[0, \frac{1}{2}\right]$. We call a number $x \in [c,1]$ *returning* for L_c if for every $\omega \in \{0,1\}^{\mathbb{N}}$ there exist an $n = n(\omega) \geq 0$ and an $r = r(\omega) \geq 1$ such that $T^n_{\omega,c}(x) = T^{n+r}_{\omega,c}(x)$.

2.2.3 Lemma. *Let $c \in \left[0, \frac{1}{2}\right]$. If $x \in \mathbb{Q} \cap [c,1]$, then x is a returning point for L_c. Hence, the set of returning points is dense in $[c,1]$.*

Proof. Let $\frac{N}{Q}$ be the reduced rational representing x, for $N, Q \in \mathbb{N}$. Then for any $\omega \in \{0,1\}^{\mathbb{N}}$ and any $t \in \mathbb{N}$ there exist $a, b \in \mathbb{Z}$ such that $T^t_{\omega,c}(x) = a\frac{N}{Q} + b \in [c,1]$. This implies that $aN + bQ \in \{0, 1, \dots, Q\}$. It then follows by Dirichlet's Box Principle that for some $n \in \mathbb{N}$ there exists an $r \geq 1$ such that $T^n_{\omega,c}(x) = T^{n+r}_{\omega,c}(x)$. ☐

Contrarily to the deterministic case, the fact that there are n, r with $T^n_{\omega,c}(x) = T^{n+r}_{\omega,c}(x)$ does not necessarily imply that $T^{n+j}_{\omega,c}(x) = T^{n+r+j}_{\omega,c}(x)$ for all $j \geq 0$, since one can make a different choice when arriving at a point in S for the second time. To characterise the ultimately periodic c-Lüroth expansions we define *loops*.

2.2.4 Definition (Loop). A string $\mathbf{u}$ of symbols in $\{0,1\}$ is called a *loop* for $x \in [c,1]$ at the point $y \in [c,1]$ if there exist $\omega \in \{0,1\}^{\mathbb{N}}$ and $n = n(\omega) \geq 0$, $r = r(\omega) \geq 1$ such that

$$\omega^{n+r}_{n+1} = \mathbf{u}, \quad T^n_{\omega,c}(x) = y = T^{n+r}_{\omega,c}(x) \quad \text{and} \quad T^{n+j}_{\omega,c}(x) \neq y \quad \text{for } 1 \leq j < r. \tag{2.11}$$

We say that x admits the loop $\mathbf{u}$ at y.

For each $x, y \in [c,1]$ we define an equivalence relation on the collection of loops $\{\mathbf{u}\}$ of x at y by setting $\mathbf{u}_1 \sim \mathbf{u}_2$ if the corresponding paths $\omega_1, \omega_2 \in \{0,1\}^{\mathbb{N}}$ satisfying (2.11) both assign the same strings of signs and digits, i.e., if

$$(s(\omega_1,x), d(\omega_1,x))^{n(\omega_1)+r}_{n(\omega_1)+1} = (s(\omega_2,x), d(\omega_2,x))^{n(\omega_2)+r}_{n(\omega_2)+1}, \tag{2.12}$$

where $r = r(\omega_1) = r(\omega_2)$. We need this definition since for $x \in [c,1] \setminus S$, $T_{\omega,c}(x)$ is independent of the choice of $\omega \in \{0,1\}$, i.e., $T_{0,c}(x) = T_{1,c}(x)$ and the corresponding sign and digit only depend on the position of x, and not on ω. As a consequence, it is necessary that $T^n_{\omega,c}(x) \in S$ for some $\omega \in \{0,1\}^{\mathbb{N}}$ and $n \geq 0$, to have more than one loop (that is, more than one equivalence class).

2.2.5 Proposition. *Let $c \in \left[0, \frac{1}{2}\right]$ and $x \in \mathbb{Q} \cap [c, 1]$.*

(i) Suppose that there exists an $\omega \in \{0,1\}^{\mathbb{N}}$ such that $T^n_{\omega,c}(x) \notin S$ for all $n \geq 0$. Then x has a unique and ultimately periodic c-Lüroth expansion.

(ii) Suppose that for each $y \in [c, 1]$ the point x admits at most one loop at y. Then all c-Lüroth expansions of x are ultimately periodic, so there are at most countably many of them.

(iii) Suppose there is a $y \in [c, 1]$, such that x admits at least two loops $\mathbf{u}_1 \nsim \mathbf{u}_2$ at y. Then x has uncountably many c-Lüroth expansions that are not ultimately periodic, and countably many ultimately periodic c-Lüroth expansions.

Proof. For (i) note that if there exists an $\omega \in \{0,1\}^{\mathbb{N}}$ such that $T^n_{\omega,c}(x) \notin S$ for all $n \geq 0$, then $T^n_{\omega,c}(x)$ is independent of the choice of ω for any $n \geq 0$ and any path $\omega \in \{0,1\}^{\mathbb{N}}$ yields the same c-Lüroth expansion. The result then follows from Lemma 2.2.3.

For (ii) suppose by contradiction that there exists an $\omega \in \{0,1\}^{\mathbb{N}}$ such that (ω, x) presents a c-Lüroth expansion that is not ultimately periodic. Since $x \in \mathbb{Q}$, the set $\{T^n_{\omega,c}(x)\}_{n \geq 0}$ consists of finitely many points, and so, in particular, there exists a point $y = T^j_{\omega,c}(x)$ for some $j \geq 0$, that is visited infinitely often. Let $\{j_i\}_{i \in \mathbb{N}}$ be the sequence such that $T^{j_i}_{\omega,c} = y$ for every j_i. Since (ω, x) does not have an ultimately periodic expansion, there exists a k such that

$$(s(\omega, x), d(\omega, x))^{j_k}_{j_{k-1}+1} \neq (s(\omega, x), d(\omega, x))^{j_{k+1}}_{j_k+1},$$

which means in particular that the loops $\omega^{j_k}_{j_{k-1}+1}$ and $\omega^{j_{k+1}}_{j_k+1}$ are not in the same equivalence class, contradicting the assumption on the number of admissible loops at y. The second part follows since the sequence $((s_n(\omega, x), d_n(\omega, x)))_n$ takes its digits in an at most countable alphabet.

For (iii) let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two loops of x at y with $\mathbf{u}_1 \nsim \mathbf{u}_2$. Consider $\omega \in \{0,1\}^{\mathbb{N}}$ satisfying (2.11), i.e., such that

$$\omega^{n+r}_{n+1} = \mathbf{u}_1, \quad \text{and} \quad T^n_{\omega,c}(x) = T^{n+r}_{\omega,c}(x) = y,$$

for some $n, r \in \mathbb{N}$. Now take any sequence $(\ell_j)_{j \geq 1} \subseteq \mathbb{N}^{\mathbb{N}}$ and consider the path $\tilde{\omega} \in \{0,1\}^{\mathbb{N}}$ defined by the concatenation

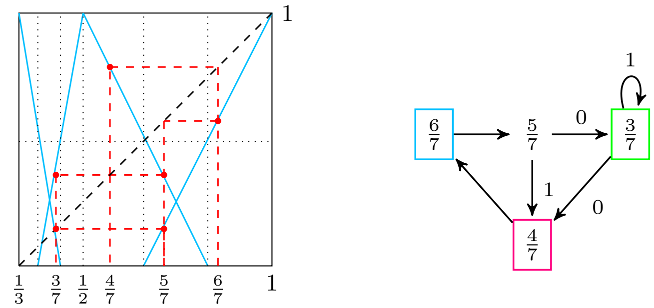
$$\tilde{\omega} = \omega^n_1 \mathbf{u}_1^{\ell_1} \mathbf{u}_2^{\ell_2} \mathbf{u}_1^{\ell_3} \mathbf{u}_2^{\ell_4} \mathbf{u}_1^{\ell_5} \mathbf{u}_2^{\ell_6} \dots .$$

If (ℓ_j) is not ultimately periodic, then it is guaranteed by (2.12) that the sequence $(s_i(\tilde{\omega}, x), d_i(\tilde{\omega}, x))_{i \in \mathbb{N}}$ is not ultimately periodic and as a result $(\tilde{\omega}, x)$ presents a c-Lüroth expansion that is not ultimately periodic. Since there are uncountably many such sequences (ℓ_j) each yielding a different corresponding sequence of signs and digits, the first part of the statement follows. Taking any ultimately periodic sequence (ℓ_j) instead will yield an ultimately periodic c-Lüroth expansion. □

2.2.6 Proposition. *Let $c \in \left[0, \frac{1}{2}\right]$. If $x \in [c,1] \setminus \mathbb{Q}$, then any c-Lüroth expansion of x is infinite and not ultimately periodic.*

Proof. Let $x \in [c,1] \setminus \mathbb{Q}$ be given and assume that there exists an $\omega \in \{0,1\}^{\mathbb{N}}$ such that the corresponding c-Lüroth expansion of (ω, x) is ultimately periodic. Then there is an $n \geq 0$ and an $r \geq 1$ such that $T^n_{\omega,c}(x) = T^{n+r}_{\omega,c}(x)$. Hence, there are $a, b, c, d \in \mathbb{Z}$ such that $ax + b = cx + d$, implying that $x \in \mathbb{Q}$, contradicting the choice of x. $\square$

2.2.7 Example. To illustrate Proposition 2.2.5 we give an example of the various possibilities for periodicity of expansions of rational numbers. Let $c = \frac{1}{3}$ and first consider the rational number $\frac{6}{7}$. See Figure 2.3(a) for the random map $L_{\frac{1}{3}}$ with the possible orbits of $\frac{6}{7}$. Figure 2.3(b) is a visualisation of the random orbits of $\frac{6}{7}$. We explicitly identify paths $\omega \in \{0,1\}^{\mathbb{N}}$ that produce c-Lüroth expansions of $\frac{6}{7}$ that are periodic, ultimately periodic and not ultimately periodic and list them in Table 2.1.



(a) (b)

Figure 2.3: The random Lüroth map L_c for $c = \frac{1}{3}$ with the random orbits of $\frac{6}{7}$ in red in (a) and another visualisation of the orbits of $\frac{6}{7}$ in (b). The digits with the arrows indicate which one of the maps $T_{0,c}$ or $T_{1,c}$ is applied. If there is no digit, then both maps yield the same orbit point.

ω	**Expansion of** $\frac{6}{7}$
$(011)^{\infty}$	$((0,2),(1,2)^2)^{\infty}$ is periodic
001^{∞}	$((0,2)^2,(1,3)^{\infty})$ is ultimately periodic
$0^2 1 0^4 1^2 0^4 1^3 0^4 1^4 \dots$	$((0,2)^2,(1,3),(0,3),(1,2),(0,2)^2,(1,3)^2,(0,3),(1,2),\dots)$ is not ultimately periodic

Table 2.1: Examples of ω's and the corresponding type of the c-Lüroth expansions.

For the point $\frac{3}{4}$ it holds that for any $\omega \in \{0,1\}^{\mathbb{N}}$, $T^1_{\omega,c}\left(\frac{3}{4}\right) = \frac{1}{2}$ and $T^n_{\omega,c}\left(\frac{3}{4}\right) = 1$ for any $n \geq 2$. Hence, $\frac{3}{4}$ has precisely two c-Lüroth expansions (for any c) that are given by the sequences

$$((0,2),(1,2),(0,2)^\infty) \quad \text{and} \quad ((1,2)^2,(0,2)^\infty).$$

Hence in case (ii) of Proposition 2.2.5 there are rational x that have only a finite number of ultimately periodic expansions and we cannot improve on the statement without giving a further description of the specific positions of the random orbit points.

2.2.8 Remark. Lemma 2.2.3 gives a further bound on the number of admissible digits $(d_n)_{n\in\mathbb{N}}$ in the c-Lüroth expansions of a rational number x. More precisely, if $c=0$, any c-Lüroth expansion of $x=\frac{N}{Q}\in\mathbb{Q}\cap[0,1]$ has at most $Q+1$ different digits. Differently, by Proposition 2.2.6, for irrational numbers the set of admissible digits is $\mathbb{N}_{\geq 2}$. Note that for $\frac{1}{\ell+1}\leq c<\frac{1}{\ell}$, the bound is given by the minimum between the previous quantities and ℓ.

The proof of Theorem 1 is now given by Proposition 2.2.5 and Proposition 2.2.6.

§2.3 Approximations of irrationals

The approximation properties of generalised c-Lüroth expansions can be studied via the dynamical properties of the associated random system. For this one needs to have an accurate description of an invariant measure for the random system. Let $0<p<1$. The vector $(p,1-p)$ represents the probabilities with which we apply the maps $T_{0,c}$ and $T_{1,c}$ respectively. One easily checks that the probability measure $m_p\times\lambda$, where m_p is the $(p,1-p)$-Bernoulli measure on $\{0,1\}^{\mathbb{N}}$ and λ is the one-dimensional Lebesgue measure on $[0,1]$ is invariant and ergodic for L_0. Therefore, in this section we focus on $c=0$, fix a p and drop the subscripts c,p, so we write $L=L_{0,p}$. We first prove a result on the number of different generalised Lüroth expansions that L produces for Lebesgue almost all $x\in[0,1]$ and then investigate two ways of quantifying the approximation properties of all these expansions.

§2.3.1 Universal generalised Lüroth expansions

The random dynamical system $L=L_{0,p}$ is capable of producing for each number $x\in[0,1]$ essentially all expansions generated by all the members of the family of GLS transformations studied in [BBDK94], i.e., GLS transformations with *standard Lüroth partition*, given by $\mathcal{P}_L=\left\{\left[\frac{1}{n},\frac{1}{n-1}\right)\right\}_{n\geq 2}$. In the previous section we mentioned that Lebesgue almost every x has uncountably many different 0-Lüroth expansions and thus uncountably many different generalised Lüroth expansions. Here we prove an even stronger statement.

Let $\mathcal{A}=\{(s,d)\,:\,s\in\{0,1\},\,d\in\mathbb{N}_{\geq 2}\}$ be the alphabet of possible digits for generalised Lüroth expansions and for any $(s,d)\in\mathcal{A}$, set

$$\bar{\Delta}(s,d)=[s]\times\left[\frac{1}{d},\frac{1}{d-1}\right],$$

for the cylinder set $[s] \subseteq \{0,1\}^{\mathbb{N}}$ and the interval $\left[\frac{1}{d}, \frac{1}{d-1}\right] \subseteq [0,1]$. Then $m_p \times \lambda(\bar{\Delta}(s,d)) = \frac{(-1)^s(p-s)}{d(d-1)} > 0$. Take any sequence $((s_n,d_n))_n \in \mathcal{A}^{\mathbb{N}}$ that does not end in $((0,d+1),(0,2)^\infty)$ for some $d \geq 2$. The set

$$\lim_{n\to\infty} \bar{\Delta}(s_1,d_1) \cap L^{-1}\bar{\Delta}(s_2,d_2) \cap \cdots \cap L^{-(n-1)}\bar{\Delta}(s_n,d_n) \subseteq \{0,1\}^{\mathbb{N}} \times [0,1] \tag{2.13}$$

is non-empty as a countable intersection of closed sets. Moreover,

$$\lim_{n\to\infty} m_p \times \lambda\big(\bar{\Delta}(s_1,d_1) \cap L^{-1}\bar{\Delta}(s_2,d_2) \cap \cdots \cap L^{-(n-1)}\bar{\Delta}(s_n,d_n)\big) = \lim_{n\to\infty} \frac{p^{n-\sum_{i=1}^n s_i}(1-p)^{\sum_{i=1}^n s_i}}{\prod_{i=1}^n d_i(d_i-1)} = 0,$$

so the set from (2.13) consists of precisely one point, call it (ω,x). Then $\omega = (s_n)_n \in \{0,1\}^{\mathbb{N}}$ and by the assumption that $((s_n,d_n))_n$ does not end in $((0,d+1),(0,2)^\infty)$ it follows that $s_n(\omega,x) = s_n$ and $d_n(\omega,x) = d_n$ for all $n \geq 1$, so that

$$x = \sum_{n=1}^\infty (-1)^{\sum_{i=1}^{n-1} s_i} \frac{d_n - 1 + s_n}{\prod_{i=1}^n d_i(d_i-1)} \in [0,1].$$

Note that if there is a $k \geq 1$, such that $(s_k,d_k) = (0,d+1)$ and $(s_n,d_n) = (0,2)$ for all $n \geq k+1$, then

$$\begin{aligned}
\sum_{n=1}^\infty (-1)^{\sum_{i=1}^{n-1} s_i} \frac{d_n - 1 + s_n}{\prod_{i=1}^n d_i(d_i-1)} &= \sum_{n=1}^{k-1} (-1)^{\sum_{i=1}^{n-1} s_i} \frac{d_n - 1 + s_n}{\prod_{i=1}^n d_i(d_i-1)} \\
&\quad + \frac{(-1)^{\sum_{i=1}^{k-1} s_i}}{\prod_{i=1}^{k-1} d_i(d_i-1)} \Big(\frac{d}{d+1} + \frac{1}{(d+1)d} \sum_{j\geq 1} \frac{1}{\prod_{i=1}^j 2\cdot 1} \Big) \\
&= \sum_{n=1}^{k-1} (-1)^{\sum_{i=1}^{n-1} s_i} \frac{d_n - 1 + s_n}{\prod_{i=1}^n d_i(d_i-1)} + \frac{(-1)^{\sum_{i=1}^{k-1} s_i}}{d\prod_{i=1}^{k-1} d_i(d_i-1)} \\
&= \sum_{n=1}^{k-1} (-1)^{\sum_{i=1}^{n-1} s_i} \frac{d_n - 1 + s_n}{\prod_{i=1}^n d_i(d_i-1)} \\
&\quad + \frac{(-1)^{\sum_{i=1}^{k-1} s_i}}{\prod_{i=1}^{k-1} d_i(d_i-1)} \Big(\frac{d}{d-1} - \frac{1}{d(d-1)} \sum_{j\geq 1} \frac{1}{\prod_{i=1}^j 2\cdot 1} \Big).
\end{aligned}$$

In other words, the sequences

$$\psi\big(((0,d+1),(0,2)^\infty)\big) = \psi\big(((1,d),(0,2)^\infty)\big)$$

correspond to the same number x. Therefore, the map $\psi : \mathcal{A}^{\mathbb{N}} \to (0,1]$ given by

$$\psi\big(((s_n,d_n))_{n\geq 1}\big) = \sum_{n=1}^\infty (-1)^{\sum_{i=1}^{n-1} s_i} \frac{d_n - 1 + s_n}{\prod_{i=1}^n d_i(d_i-1)} \tag{2.14}$$

is well defined and surjective and by the way we defined the maps $T_{0,0}$ and $T_{1,0}$ on any of the points z_n we get that any sequence $((s_n,d_n))_n \in \mathcal{A}^{\mathbb{N}}$ that does not end in $((0, d+1), (0,2)^\infty)$ corresponds to a 0-Lüroth expansion of a number $x \in (0,1]$. We call a sequence $((s_n, d_n))_n \in \mathcal{A}^{\mathbb{N}}$ with $\psi\big(((s_n, d_n))_{n\geq 1}\big) = x$ a *universal* 0-Lüroth expansion of x if every finite block $(t_1, b_1), \dots, (t_j, b_j) \in \mathcal{A}^j$ occurs in the sequence $((s_n, d_n))_n$. Note that the sequences ending in $((0, d+1), (0,2)^\infty)$ and $((1, d), (0,2)^\infty)$ can not be universal.

2.3.1 Proposition. *Lebesgue almost all $x \in [0,1]$ have uncountably many universal 0-Lüroth expansions.*

Proof. Define the set

$$N = \{x \in (0,1] \,:\, \exists\, \omega \in \{0,1\}^{\mathbb{N}},\, k \geq 0,\, n \geq 2,\; T_\omega^k(x) = z_n\}.$$

Then by Theorem 1, $N \subseteq \mathbb{Q}$, so $m_p \times \lambda(\{0,1\}^{\mathbb{N}} \times N) = 0$. For any $(i,j) \in \mathcal{A}$ set

$$\Delta(s,d) = [s] \times \Big(\frac{1}{d}, \frac{1}{d-1}\Big).$$

For any $(\omega, x) \in \{0,1\}^{\mathbb{N}} \times (0,1] \setminus N$ the block $(t_1, b_1), \dots, (t_j, b_j)$ occurs in position $k \geq 1$ of the sequence $((s_n(\omega,x), d_n(\omega,x)))_n$ precisely if

$$L^{k-1}(\omega, x) \in \Delta(t_1, b_1) \cap L^{-1}\Delta(t_2, b_2) \cap \cdots \cap L^{-(j-1)}\Delta(t_j, b_j).$$

Since $m_p \times \lambda\big(\Delta(t_1, b_1) \cap L^{-1}\Delta(t_2, b_2) \cap \cdots \cap L^{-(j-1)}\Delta(t_j, b_j)\big) > 0$ it follows from the ergodicity of L that $m_p \times \lambda$-a.e. (ω, x) eventually enters this set, so the set of points (ω, x) for which the 0-Lüroth expansion does not contain $(t_1, b_1), \dots, (t_j, b_j)$ is a $m_p \times \lambda$-null set. There are only countably many different blocks $(t_1, b_1), \dots, (t_j, b_j)$, so the set of points (ω, x) that have a non-universal 0-Lüroth expansion also has measure 0. From Fubini's Theorem we get the existence of a set $B \subseteq (0,1] \setminus N$ with $\lambda(B) = 1$ with the property that for each $x \in B$ a set $A_x \subseteq \{0,1\}^{\mathbb{N}}$ exists with $m_p(A_x) = 1$ and such that for any $(\omega, x) \in A_x \times \{x\}$ the sequence $(s_n(\omega,x), d_n(\omega,x))_n$ is a universal 0-Lüroth expansion of x. The set A_x has full measure, so contains uncountably many ω's. Let $x \in B$ and $\omega, \tilde{\omega} \in A_x$. Then $x \notin N$ and thus if $\omega_n \neq \tilde{\omega}_n$ for some $n \geq 1$, then $s_n(\omega, x) \neq s_n(\tilde{\omega}, x)$. Hence, the sequences $\omega \in A_x$ all give different universal 0-Lüroth expansions for x. □

Note that if x has a universal 0-Lüroth expansion, then $x \notin \big\{\frac{2n-1}{2n(n-1)} \,:\, n \geq 1\big\}$. By Remark 2.2.1(iii) the fact that $s_n(\omega, x) \neq s_n(\tilde{\omega}, x)$ for some n then implies that $d_{n+1}(\omega, x) \neq d_{n+1}(\tilde{\omega}, x)$.

§2.3.2 Speed of convergence

Recall that for each $n \geq 1$ the n-th convergent $\frac{p_n}{q_n}(\omega, x)$ of (ω, x) is given by

$$\frac{p_n}{q_n} = \frac{p_n}{q_n}(\omega, x) = \sum_{k=1}^{n} (-1)^{\sum_{i=1}^{n-1} \omega_i} \prod_{i=1}^{k} \frac{d_k - 1 + \omega_k}{d_i(d_i - 1)},$$

so that the study of the quantity $\left|x - \frac{p_n}{q_n}\right|$ provides information on the quality of the approximations of x obtained from L. In the following, we take two different perspectives. In this section we compute the pointwise Lyapunov exponent Λ and in the next section we consider the approximation coefficients $\theta_n = \theta_n^{0,p}$ similar to the ones defined in (2.3).

Let I be an interval of the real line, $\Omega \subseteq \mathbb{N}$ and $\pi : \Omega^{\mathbb{N}} \times I \to I$ the canonical projection onto the second coordinate. The following definition can be found in [GBL97, B13], for example.

2.3.2 Definition (Lyapunov exponent). For any random interval map $R : \Omega^{\mathbb{N}} \times I \to \Omega^{\mathbb{N}} \times I$ and for any point $(\omega, x) \in \Omega^{\mathbb{N}} \times I$, the *pointwise Lyapunov exponent* is defined as

$$\Lambda(\omega, x) := \lim_{n\to\infty} \frac{1}{n} \log\left|\frac{\mathrm{d}}{\mathrm{d}\,x}\pi(R^n(\omega, x))\right|, \tag{2.15}$$

whenever the limit exists.

We use this to determine the speed of convergence of the sequence $\left(\frac{p_n}{q_n}(\omega, x)\right)_n$ to a number $x \in [0, 1]$ for which $T_\omega^n(x) \neq 0$ for all $n \geq 0$. By Proposition 2.2.5 and Proposition 2.2.6 this includes all irrational x and part of the rationals. For any such x and each sequence $(s_n(\omega, x), d_n(\omega, x))_{n\geq 1} \in \mathcal{A}^{\mathbb{N}}$ of signs and digits for a point (ω, x) the rational $\frac{p_n}{q_n}(\omega, x)$ is one of the endpoints of the projection onto the unit interval of the cylinder $[(s_1, d_1), \ldots, (s_n, d_n)]$, i.e., one of the endpoints of the interval

$$\psi([(s_1, d_1), \ldots, (s_n, d_n)]).$$

Since the map T_ω^n is surjective and linear on this interval and $\frac{\mathrm{d}}{\mathrm{d}\,x} T_\omega^n(x) = \prod_{k=1}^n d_k(d_k-1)$, the Lebesgue measure of this interval is $\left(\prod_{k=1}^n d_k(d_k-1)\right)^{-1}$. The following result compares to Theorem 2.1.2 from [BI09].

2.3.3 Proposition. *For any $(\omega, x) \in \{0, 1\}^{\mathbb{N}} \times [0, 1]$ with $T_\omega^n(x) \neq 0$ for all $n \geq 0$ the speed of approximation of the random Lüroth map L is given by*

$$\left|x - \frac{p_n}{q_n}(\omega, x)\right| \asymp \exp(-n\Lambda(\omega, x)).$$

In particular, for $m_p \times \lambda$-a.e. (ω, x)

$$\Lambda(\omega, x) = \sum_{d=2}^{\infty} \frac{\log(d(d-1))}{d(d-1)} \cong 1.98329\ldots\ .$$

Furthermore, the map $\Lambda : \{0, 1\}^{\mathbb{N}} \times [0, 1] \to [\log 2, \infty)$ is onto.

Proof. Let (ω, x) be such that $T_\omega^n(x) \neq 0$ for each n. Write $s_n = s_n(\omega, x)$ and $d_n = d_n(\omega, x)$ for each $n \geq 1$. Since $\frac{p_n}{q_n}(\omega, x)$ is one of the endpoints of the interval $\psi([(s_1, d_1), \ldots, (s_n, d_n)])$ it follows that

$$\left|x - \frac{p_n}{q_n}(\omega, x)\right| \leq \prod_{k=1}^{n} \frac{1}{d_k(\omega, x)(d_k(\omega, x) - 1)}.$$

We claim that

$$\left| x - \frac{p_n}{q_n}(\omega, x) \right| \geq \prod_{k=1}^{n+1} \frac{1}{d_k(\omega, x)(d_k(\omega, x) - 1)}. \tag{2.16}$$

To see this, note that for $n = 1$ we have

$$\frac{1}{d_2} \leq T_\omega^1(x) = (-1)^{s_1} d_1(d_1 - 1)x + (-1)^{s_1+1}(d_1 - 1 + s_1) \leq \frac{1}{d_2 - 1}.$$

Hence,

$$\frac{1}{d_1(d_1 - 1)d_2} \leq (-1)^{s_1} \left(x - \frac{d_1 - 1 + s_1}{d_1(d_1 - 1)} \right) \leq \frac{1}{d_1(d_1 - 1)(d_2 - 1)}.$$

Since $\frac{p_1}{q_1} = \frac{d_1 - 1 + s_1}{d_1(d_1 - 1)}$ it follows that

$$\left| x - \frac{p_1}{q_1} \right| \geq \frac{1}{d_1(d_1 - 1)d_2(d_2 - 1)}.$$

In the same way, from

$$\frac{1}{d_{n+1}} \leq T_\omega^n(x) = (-1)^{s_n} d_n(d_n - 1)T_\omega^{n-1}(x) + (-1)^{s_n+1}(d_n - 1 + s_n) \leq \frac{1}{d_{n+1} - 1},$$

we obtain

$$\left| T_\omega^{n-1}(x) - \frac{d_n - 1 + s_n}{d_n(d_n - 1)} \right| \geq \frac{1}{d_n(d_n - 1)d_{n+1}(d_{n+1} - 1)}.$$

By the definition of the convergents, we then have

$$\begin{aligned}
\left| x - \frac{p_n}{q_n}(\omega, x) \right| &= \frac{T_\omega^n(x)}{\prod_{i=1}^n d_i(d_i - 1)} \\
&= \frac{(-1)^{s_n} d_n(d_n - 1)T_\omega^{n-1}(x) + (-1)^{s_n+1}(d_n - 1 + s_n)}{\prod_{k=1}^n d_i(d_i - 1)} \\
&= \left| T_\omega^{n-1}(x) - \frac{(d_n - 1 + s_n)}{d_n(d_n - 1)} \right| \cdot \prod_{k=1}^{n-1} \frac{1}{d_i(d_i - 1)} \\
&\geq \prod_{k=1}^{n+1} \frac{1}{d_i(d_i - 1)}.
\end{aligned}$$

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This gives the claim. It follows that

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\log\left|x-\frac{p_n}{q_n}(\omega,x)\right| &= \lim_{n\to\infty}\frac{1}{n}\log\left(\prod_{k=1}^{n}\frac{1}{d_k(\omega,x)(d_k(\omega,x)-1)}\right)\\
&= \lim_{n\to\infty}\frac{1}{n}\log\lambda\big(\psi([(s_1,d_1),\dots,(s_n,d_n)])\big)\\
&= \lim_{n\to\infty}\frac{1}{n}\log\prod_{k=1}^{n}\lambda\big(\psi([s_k,d_k])\big)\\
&= \lim_{n\to\infty}\frac{1}{n}\log\left|\frac{\mathrm{d}}{\mathrm{d}\,x}\pi(L^n(\omega,x))\right|^{-1}\\
&= -\Lambda(\omega,x).
\end{aligned}$$

So Λ measures the asymptotic exponential growth of approximation, i.e.,

$$\left|x-\frac{p_n}{q_n}(\omega,x)\right| \asymp \exp(-n\Lambda(x,\omega)).$$

Recall now that $m_p\times\lambda$ is invariant and ergodic for L. Applying Birkhoff's Ergodic Theorem we get for $(m_p\times\lambda)$-a.e. (ω,x) that

$$\begin{aligned}
\Lambda(\omega,x) = \lim_{n\to\infty}\frac{1}{n}\log\left|\frac{\mathrm{d}}{\mathrm{d}\,x}T_\omega^n(x)\right| &= \lim_{n\to\infty}\frac{1}{n}\log\left|\prod_{k=1}^{n}\frac{\mathrm{d}}{\mathrm{d}\,x}T_{\omega_k}(T_\omega^{k-1}(x))\right|\\
&= \lim_{n\to\infty}\frac{1}{n}\log\left|\prod_{k=0}^{n-1}\frac{\mathrm{d}}{\mathrm{d}\,x}\pi\circ L(\sigma^k(\omega),T_\omega^k(x))\right|\\
&= \int_{\{0,1\}^{\mathbb{N}}\times[0,1]}\log\left|\frac{\mathrm{d}}{\mathrm{d}\,x}\pi(L(\omega,x))\right|d(m_p\times\lambda)(\omega,x)\\
&= \int_{\{0,1\}^{\mathbb{N}}\times[0,1]}\log\left|\frac{\mathrm{d}}{\mathrm{d}\,x}T_\omega(x)\right|d(m_p\times\lambda)(\omega,x)\\
&= \sum_{d=2}^{\infty}\frac{\log(d(d-1))}{d(d-1)} =: \Lambda_{m_p\times\lambda}.
\end{aligned}\tag{2.17}$$

This gives the second part of the proposition. For the last part, it is obvious that $\log 2$ is a lower bound for $\Lambda(\omega,x)$, since $|\frac{\mathrm{d}}{\mathrm{d}\,x}T_j(x)|\geq 2$ for $j=0,1$. The rest follows from Theorem 2.1.2 since the sequence $\omega=(\overline{0})$ reduces L to the standard Lüroth map T_L. □

It follows from (2.17) that the set of points (of zero Lebesgue measure) that present a Lyapunov exponent different from $\Lambda_{m_p\times\lambda}$, includes *fixed points*, i.e., points $(\omega,x)\in\{0,1\}^{\mathbb{N}}\times[0,1]$ for which $T_\omega^k(x)=x$ for all $k\geq 1$. For any such point (ω,x),

$$\prod_{k=1}^{n}\frac{\mathrm{d}}{\mathrm{d}\,x}T_{\omega_{k+1}}(T_\omega^k(x)) = \prod_{k=1}^{n}\frac{\mathrm{d}}{\mathrm{d}\,x}T_{\omega_{k+1}}(x) = \left(\frac{\mathrm{d}}{\mathrm{d}\,x}T_\omega(x)\right)^n,$$

for all $n \geq 1$. Hence, if $x \in [z_d, z_{d-1})$ then

$$\begin{aligned}\Lambda(\omega, x) &= \lim_{n\to\infty} \frac{1}{n} \log\left|\prod_{k=1}^{n} \frac{\mathrm{d}}{\mathrm{d}\,x}\pi \circ L(L^k(\omega, x))\right| \\ &= \lim_{n\to\infty} \frac{1}{n} \log|d^n| \\ &= \log d.\end{aligned}$$

Since fixed points can be found in any set $\{0,1\}^{\mathbb{N}} \times [z_d, z_{d-1})$ for $d \geq 2$, it follows that Λ attains arbitrarily large values, starting from $\log 2 \cong 0.69314\ldots$, which corresponds to the Lyapunov exponent of the fixed point given by the sequence $(\,\overline{(0,2)}\,)$.

§2.3.3 Approximation coefficients

A GLS map $T_{\mathcal{P},\varepsilon} : [0,1] \to [0,1]$ is defined by a partition $\mathcal{P} = \{I_n = (\ell_n, r_n]\}_n$ and a vector $\varepsilon = (\varepsilon_n)_n$. To be specific, the intervals I_n satisfy $\lambda(\bigcup I_n) = 1$ and $\lambda(I_n \cap I_k) = 0$ if $n \neq k$ and on I_n the map $T_{P,s}$ is given by

$$T_{\mathcal{P},\varepsilon}(x) = \varepsilon_n \frac{r_n - x}{\ell_n - r_n} + (1 - \varepsilon_n)\frac{x - \ell_n}{\ell_n - r_n},$$

and $T_{\mathcal{P},\varepsilon}(x) = 0$ if $x \in [0,1] \setminus \bigcup I_n$. Any GLS transformation produces by iteration number expansions similar to the Lüroth expansions from (2.1), called *generalised Lüroth expansions*. In [BBDK94, DK96] the authors studied the approximation coefficients

$$\theta_n^{\mathcal{P}_L,\varepsilon} = q_n\left|x - \frac{p_n}{q_n}\right|, \quad n > 0$$

for GLS maps with the standard Lüroth partition $\mathcal{P}_L = \left\{\left[\frac{1}{n}, \frac{1}{n-1}\right)\right\}_{n\geq 2}$. In particular [BBDK94, Corollary 2] gives that amongst all GLS systems with standard Lüroth partition, T_A presents the best approximation properties. More precisely, it states that for any such GLS map $T_{\mathcal{P}_L,\varepsilon}$ there exists a constant $M_{\mathcal{P}_L,\varepsilon}$ such that for Lebesgue a.e. $x \in [0,1]$, the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \theta_i^{\mathcal{P}_L,\varepsilon}(x) \tag{2.18}$$

exists and equals $M_{\mathcal{P}_L,\varepsilon}$. Furthermore, $M_A \leq M_{\mathcal{P}_L,\varepsilon} \leq M_L$. In Remark 2, below Corollary 2, of [BBDK94], the authors remark that not every value in the interval $[M_A, M_L]$ can be obtained by such GLS maps, and they suggest furthermore that the achievable values of $M_{\mathcal{P}_L,\varepsilon}$ might form a fractal set. The situation changes for the random system L, as the next theorem shows.

For the random Lüroth map $L = L_0$ we define for each $(\omega, x) \in \{0,1\}^{\mathbb{N}} \times [0,1]$ and $n \geq 1$ the *n-th approximation coefficient* by

$$\theta_n(\omega, x) = q_n\left|x - \frac{p_n}{q_n}\right|,$$

where $q_n = (d_n - s_n)\prod_{i=1}^{n-1} d_i(d_i - 1)$.

2.3.4 Theorem. *For the random Lüroth map $L = L_0$ and $0 < p < 1$ the limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \theta_i(\omega, x)$$

exists for $m_p \times \lambda$-a.e. $(\omega, x) \in \{0,1\}^{\mathbb{N}} \times [0,1]$ and equals

$$M_p := \mathbb{E}[\theta_n] = p\frac{2\zeta(2) - 3}{2} + \frac{2 - \zeta(2)}{2},$$

where $\zeta(2)$ is the zeta function evaluated at 2. The function $p \mapsto M_p$ maps the interval $[0,1]$ onto the interval $[M_A, M_L]$.

Proof. Fix $(\omega, x) \in \{0,1\}^{\mathbb{N}} \times [0,1]$. Let $\theta_n = \theta_n(\omega, x)$, $d_n = d_n(\omega, x)$ and $x_n = T_\omega^n(x)$. By definition of the convergents $\frac{p_n}{q_n}$ we have

$$\left| x - \frac{p_n}{q_n} \right| = \frac{x_n}{\prod_{i=1}^{n} d_i(d_i - 1)}$$

and the numbers q_n are such that we can write

$$\theta_n = \begin{cases} \dfrac{x_n}{d_n - 1} = d_n x_{n-1} - 1, & \text{if } x_n = T_L(x_{n-1}), \\ \dfrac{x_n}{d_n} = -(d_n - 1)x_{n-1} + 1, & \text{if } x_n = T_A(x_{n-1}). \end{cases} \tag{2.19}$$

Since the Lebesgue measure is stationary for L_0, it follows from Birkhoff's Ergodic Theorem that each x_n, viewed as a random variable, is uniformly distributed over the interval $[0,1]$. We use this fact together with (2.19) to compute the cumulative distribution function (CDF) F_{θ_n} of θ_n. For $\mathbf{P} = m_p \times \lambda$, by definition of the CDF and by the law of total probability, for $y \in [0,1]$ we have

$$\begin{aligned} F_{\theta_n}(y) = \mathbf{P}(\theta_n \le y) = &\mathbf{P}(\theta_n \le y \,|\, x_n = T_L(x_{n-1}))\mathbf{P}(x_n = T_L(x_{n-1})) + \\ &\mathbf{P}(\theta_n \le y \,|\, x_n = T_A(x_{n-1}))\mathbf{P}(x_n = T_A(x_{n-1})) \\ = &\sum_{d=2}^{\infty} \mathbf{P}\left(x_{n-1} \le \frac{1+y}{d}, \quad x_{n-1} \in [z_d, z_{d-1}) \right)\mathbf{P}(x_n = T_L(x_{n-1})) + \\ &\mathbf{P}\left(x_{n-1} \ge \frac{1-y}{d-1}, \quad x_{n-1} \in [z_d, z_{d-1}) \right)\mathbf{P}(x_n = T_A(x_{n-1})) \\ = &\sum_{d=2}^{\infty} p\mathbf{P}\left(x_{n-1} \le \frac{1+y}{d}, \quad x_{n-1} \in [z_d, z_{d-1}) \right) + \\ &(1-p)\mathbf{P}\left(x_{n-1} \ge \frac{1-y}{d-1}, \quad x_{n-1} \in [z_d, z_{d-1}) \right). \end{aligned}$$

Let $d \geq 2$ be such that $x_{n-1} \in \left[\frac{1}{d}, \frac{1}{d-1}\right) = [z_d, z_{d-1})$, then either $x_n = T_L(x_{n-1})$ or $x_n = T_A(x_{n-1})$. In the first case, since $d \in \mathbb{N}$,

$$\frac{1+y}{d} \in \left[\frac{1}{d}, \frac{1}{d-1}\right) \quad \text{if and only if} \quad d \leq \left\lfloor \frac{1}{y} \right\rfloor + 1,$$

and we obtain

$$\mathbf{P}\left(x_{n-1} \leq \frac{1+y}{d}, \quad x_{n-1} \in [z_d, z_{d-1})\right) = \begin{cases} \dfrac{y}{d}, & \text{if } d \leq \left\lfloor \dfrac{1}{y} \right\rfloor + 1, \\ \dfrac{1}{d(d-1)}, & \text{otherwise.} \end{cases}$$

Similarly, in the latter

$$\frac{1-y}{d-1} \in \left[\frac{1}{d}, \frac{1}{d-1}\right) \quad \text{if and only if} \quad d \leq \left\lfloor \frac{1}{y} \right\rfloor,$$

which gives

$$\mathbf{P}\left(x_{n-1} \geq \frac{1-y}{d-1}, \quad x_{n-1} \in [z_d, z_{d-1})\right) = \begin{cases} \dfrac{y}{d-1}, & \text{if } d \leq \left\lfloor \dfrac{1}{y} \right\rfloor, \\ \dfrac{1}{d(d-1)}, & \text{otherwise.} \end{cases}$$

Note that

$$\sum_{d > \lfloor 1/y \rfloor + 1} \frac{1}{d(d-1)} = \frac{1}{\lfloor 1/y \rfloor + 1} \quad \text{and} \quad \sum_{d > \lfloor 1/y \rfloor} \frac{1}{d(d-1)} = \frac{1}{\lfloor 1/y \rfloor}.$$

Summing over all $d \geq 2$ gives

$$F_{\theta_n}(y) = p\left(\sum_{d=2}^{\lfloor 1/y \rfloor + 1} \frac{y}{d} + \frac{1}{\lfloor 1/y \rfloor + 1}\right) + (1-p)\left(\sum_{d=2}^{\lfloor 1/y \rfloor} \frac{y}{d-1} + \frac{1}{\lfloor 1/y \rfloor}\right),$$

so that by (2.4) and Theorem 2.1.4 we obtain

$$F_{\theta_n}(y) = pF_L(y) + (1-p)F_A(y).$$

The expectation $\mathbf{E}[\theta_n]$ can be now computed by $\mathbf{E}[\theta_n] = \int_0^1 (1 - F_{\theta_n}(y))dy$, which with the results from Theorem 2.1.4 gives

$$\begin{aligned} \mathbf{E}[\theta_n] =& 1 - p\int_0^1 F_L(y)dy - (1-p)\int_0^1 F_A(y)dy \\ =& 1 - p\left(\frac{3}{2} - \frac{\zeta(2)}{2}\right) - (1-p)\left(\frac{\zeta(2)}{2}\right) \\ =& p\frac{2\zeta(2) - 3}{2} + \frac{2 - \zeta(2)}{2}, \end{aligned}$$

that is

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \theta_i(\omega, x) = p\frac{2\zeta(2)-3}{2} + \frac{2-\zeta(2)}{2} = M_p. \tag{2.20}$$

Note that for $p = 0$, $x_n = T_A^n(x)$ for $n \geq 0$ and indeed $M_0 = M_A$. In the same way, for $p = 1$, $x_n = T_L^n(x)$ for $n \geq 0$ and $M_1 = M_L$. Lastly, M_p is an increasing function in $p \in [0,1]$, so it can assume any value between M_A and M_L. □

Theorem 2 is now given by Proposition 2.3.1, Proposition 2.3.3 and Theorem 2.3.4.

§2.4 Generalised Lüroth expansions with bounded digits

Both Proposition 2.3.3 and Theorem 2.3.4 depend on the invariant measure $m_p \times \lambda$ for L_0. In Proposition 2.3.3 it is used to compute the value of $\Lambda(\omega, x)$ and in Theorem 2.3.4 we use the fact that the numbers $T_\omega^n(x)$ are uniformly distributed over the interval $[0,1]$ and it is used to deduce (2.20). For $c > 0$, the random map L_c becomes fundamentally different: The maps $T_{j,c}$ for $j = 0,1$ present finitely many branches that are not always onto. These variations make the measures $m_p \times \lambda$ no longer L_c-invariant. It still follows from results in e.g. [M85, P84, GB03, I12] that for any $0 < p < 1$ the random map L_c admits an invariant probability measure of type $m_p \times \mu_{p,c}$, where m_p is the $(p, 1-p)$-Bernoulli measure on $\{0,1\}^{\mathbb{N}}$ and $\mu_{p,c} \ll \lambda$ is a probability measure on $[c,1]$ that satisfies

$$\mu_{p,c}(B) = p\mu_{p,c}(T_{0,c}^{-1}(B)) + (1-p)\mu_{p,c}(T_{1,c}^{-1}(B)) \tag{2.21}$$

for each Borel measurable set $B \subseteq [c,1]$. We set $f_{p,c} := \frac{\mathrm{d}\,\mu_{p,c}}{\mathrm{d}\,\lambda}$. Here we call $\mu_{p,c}$ a *stationary measure* and $f_{p,c}$ an *invariant density* for L_c. With [P84, Corollary 7] it follows that since $T_{0,c}$ is expanding and has a unique absolutely continuous invariant measure, the measure $\mu_{p,c}$ is the unique stationary measure for L_c and that L_c is ergodic with respect to $m_p \times \mu_{p,c}$.

2.4.1 Example. For $c = \frac{1}{3}$ and $0 < p < 1$ consider the measure $\mu_{p,\frac{1}{3}}$ with density

$$f_{p,\frac{1}{3}}(x) = \begin{cases} \frac{9}{8}, & \text{if } x \in \left[\frac{1}{3}, \frac{2}{3}\right), \\ \frac{15}{8}, & \text{if } x \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

One can check by direct computation that $\mu_{p,\frac{1}{3}}$ satisfies (2.21) and thus is the unique stationary measure for $L_{\frac{1}{3},p}$.

Since such an invariant measure exists, some of the results from the previous section also hold for the maps $L_{c,p}$ with $c > 0$. In particular the Lyapunov exponent from (2.15) is well defined and since the stationary measure $m_p \times \mu_{p,c}$ is still ergodic, we can apply Birkhoff's Ergodic Theorem to obtain for $(m_p \times \mu_{p,c})$-a.e. point (ω, x)

the following expression for the Lyapunov exponent:

$$\Lambda(\omega, x) = \sum_{d=2}^{d_c} \mu_{p,c}\left(\left(\frac{1}{d}, \frac{1}{d-1}\right)\right) \log(d(d-1)) + \mu_{p,c}\left(\left(c, \frac{1}{d_c}\right)\right) \log(d_c(d_c+1)), \tag{2.22}$$

where d_c is the unique positive integer such that $c \in \left[\frac{1}{d_c+1}, \frac{1}{d_c}\right)$.

2.4.2 Example. Consider the map $L_{\frac{1}{3},p}$ from Example 2.4.1 again. The possible digits of the generalised Lüroth expansions produced by $L_{\frac{1}{3},p}$ are $(0,2),(1,2),(0,3),(1,3)$. It follows from Birkhoff's Ergodic Theorem that the frequency of the digit $(0,2)$ is given by

$$\begin{aligned}
\pi_{(0,2)} &= \lim_{n\to\infty} \frac{\#\{1 \le j \le n : s_j(\omega,x) = 0 \text{ and } d_j(\omega,x) = 2\}}{n} \\
&= \int_{\{0,1\}^{\mathbb{N}}\times[\frac{1}{3},1]} \mathbf{1}_{[0]\times[\frac{2}{3},\frac{5}{6}]} + \mathbf{1}_{\{0,1\}^{\mathbb{N}}\times(\frac{5}{6},1]} \mathrm{d}m_p \times \mu_{p,\frac{1}{3}} \\
&= \frac{15}{8}p\left(\frac{5}{6} - \frac{2}{3}\right) + \frac{15}{8}\left(1 - \frac{5}{6}\right) = \frac{5+5p}{16}.
\end{aligned}$$

Similarly,

$$\pi_{(1,2)} = \frac{8-5p}{16}, \qquad \pi_{(0,1)} = \frac{1+p}{16}, \qquad \pi_{(1,3)} = \frac{2-p}{16}.$$

Note that for

$$\pi_2 = \lim_{n\to\infty} \frac{\#\{1 \le j \le n : d_j(\omega,x) = 2\}}{n} = \pi_{(0,2)} + \pi_{(1,2)}$$

we also obtain

$$\pi_2 = \mu_{p,\frac{1}{3}}\left(\left[\frac{1}{2}, 1\right]\right) = \frac{13}{16}, \quad \text{and} \quad \pi_3 = 1 - \pi_2 = \frac{3}{16}.$$

Moreover for $m_p \times \mu_{p,\frac{1}{3}}$-a.e. (ω, x) we have by (2.22) that

$$\begin{aligned}
\Lambda(\omega, x) &= \sum_{d=2}^{3} \mu_{p,\frac{1}{3}}\left(\left[\frac{1}{d}, \frac{1}{d-1}\right)\right) \log(d(d-1)) \\
&= \mu_{p,\frac{1}{3}}\left(\left[\frac{1}{2}, 1\right]\right) \log 2 + \mu_{p,\frac{1}{3}}\left(\left[\frac{1}{3}, \frac{1}{2}\right)\right) \log 6 \\
&= \frac{13}{16}\log 2 + \frac{3}{16}\log 6 = 0.89913\ldots < 1.198328\ldots = \Lambda_{m_p\times\lambda}.
\end{aligned}$$

From (2.22) and example 2.4.2 it is clear that to obtain results similar to Proposition 2.3.3 for $c > 0$ we need a good expression for the density of $\mu_{p,c}$ and to determine the approximation coefficient also an accurate description of the location of the points $\frac{p_n}{q_n}$ relative to x. For $c = 0$ we saw that it immediately follows that Lebesgue almost all $x \in [0,1]$ have uncountably many different 0-Lüroth expansions. We start this section by giving some results on the number of different c-Lüroth expansions a number x can have for $c > 0$.

§2.4.1 Uncountably many universal expansions

Let $c \in \left(0, \frac{1}{2}\right]$ and consider the corresponding alphabet $\mathcal{A}_c = \{(i,j) : i \in \{0,1\}, j \in \{2,3,\dots,\lceil \frac{1}{c} \rceil\}\}$. We call a sequence $((s_n,d_n))_{n\geq 1} \in \mathcal{A}_c^{\mathbb{N}}$ *c-Lüroth admissible* if

$$\sum_{n\geq 1}(-1)^{\sum_{i=1}^{n-1} s_{i+k}} \frac{d_{n+k}-1+s_{n+k}}{\prod_{i=1}^{n} d_{i+k}(d_{i+k}-1)} \in [c,1]$$

for all $k \geq 0$ and if moreover the sequence does not end in $((0,d+1),(0,2)^\infty)$ for some $2 \leq d < \lceil \frac{1}{c} \rceil$.

The first main result of the number of different c-Lüroth expansions a number $x \in [c,1]$ can have is the following.

2.4.3 Theorem. *Let $0 < c \leq \frac{2}{5}$. Then every $x \in [c,1] \setminus \mathbb{Q}$ has uncountably many different c-Lüroth expansions.*

Before we prove the theorem we prove three lemmata.

2.4.4 Lemma. *Let $c \in \left(0, \frac{1}{2}\right)$ and let $((s_n,d_n))_{n\geq 1} \in \mathcal{A}_c^{\mathbb{N}}$ be c-Lüroth admissible. For*

$$x = \sum_{n\geq 1}(-1)^{\sum_{i=1}^{n-1} s_i} \frac{d_n-1+s_n}{\prod_{i=1}^{n} d_i(d_i-1)}$$

the following hold.

(i) If $s_1 = 0$, then $x \in \left[\frac{1}{d_1} + \frac{c}{d_1(d_1-1)}, \frac{1}{d_1-1}\right] = [z^+_{d_1}, z_{d_1-1}]$.

(ii) If $s_1 = 1$, then $x \in \left[\frac{1}{d_1}, \frac{1}{d_1-1} - \frac{c}{d_1(d_1-1)}\right] = [z_{d_1}, z^-_{d_1-1}]$.

Proof. For x we can write

$$x = \frac{d_1-1+s_1}{d_1(d_1-1)} + \frac{(-1)^{s_1}}{d_1(d_1-1)} \sum_{n\geq 1}(-1)^{\sum_{i=1}^{n-1} s_{i+1}} \frac{d_{n+1}-1+s_{n+1}}{\prod_{i=1}^{n} d_{i+1}(d_{i+1}-1)}.$$

Write

$$x_1 = \sum_{n\geq 1}(-1)^{\sum_{i=1}^{n-1} s_{i+1}} \frac{d_{n+1}-1+s_{n+1}}{\prod_{i=1}^{n} d_{i+1}(d_{i+1}-1)}.$$

Since $((s_n,d_n))_{n\geq 1}$ is c-Lüroth admissible, we have $x_1 \in [c,1]$. If $s_1 = 0$, then

$$\frac{d_1-1+s_1}{d_1(d_1-1)} + \frac{(-1)^{s_1}}{d_1(d_1-1)}x_1 = \frac{1}{d_1} + \frac{1}{d_1(d_1-1)}x_1 \in \left[\frac{1}{d_1} + \frac{c}{d_1(d_1-1)}, \frac{1}{d_1-1}\right].$$

Similarly if $s_1 = 1$, then

$$\frac{d_1-1+s_1}{d_1(d_1-1)} + \frac{(-1)^{s_1}}{d_1(d_1-1)}x_1 = \frac{1}{d_1-1} - \frac{1}{d_1(d_1-1)}x_1 \in \left[\frac{1}{d_1}, \frac{1}{d_1-1} - \frac{c}{d_1(d_1-1)}\right].$$

$\square$

Recall the definition of the switch region S from (2.7), that is

$$S = [c, 1] \cap \bigcup_{n>1} [z_n^+, z_{n-1}^-].$$

2.4.5 Lemma. *A sequence $((s_n, d_n))_{n\geq 1} \in \mathcal{A}_c^{\mathbb{N}}$ is c-Lüroth admissible if and only if there is a point $(\omega, x) \in \{0,1\}^{\mathbb{N}} \times [c, 1]$ such that the sequence $(s_n(\omega, x), d_n(\omega, x))_{n\geq 1}$ generated by L_c satisfies $s_n(\omega, x) = s_n$ and $d_n(\omega, x) = d_n$ for each $n \geq 1$.*

Proof. One direction follows since $L_c^k(\omega, x) \in [c, 1]$ for all k and by the definition of L_c, s_i and d_i on the points z_n. For the other direction, let $((s_n, d_n))_n$ be c-Lüroth admissible. Set

$$x = \sum_{n\geq 1} (-1)^{\sum_{i=1}^{n-1} s_i} \frac{d_n - 1 + s_n}{\prod_{i=1}^{n} d_i(d_i - 1)}.$$

For $k \geq 0$ define the numbers

$$x_k = \sum_{n\geq 1} (-1)^{\sum_{i=1}^{n-1} s_{i+k}} \frac{d_{n+k} - 1 + s_{n+k}}{\prod_{i=1}^{n} d_{i+k}(d_{i+k} - 1)},$$

so that $x_0 = x$. Let $\omega \in \{0,1\}^{\mathbb{N}}$ be such that

$$\omega_k = \begin{cases} 0, & \text{if } 1_S(x_{k-1}) = 1 \text{ and } s_k = 0, \\ 1, & \text{otherwise.} \end{cases}$$

By Lemma 2.4.4 and the fact that $((s_n, d_n))_n$ does not end in $((0, d+1), (0, 2)^\infty)$ it then follows that $s_1(\omega, x) = s_1$ and $d_1(\omega, x) = d_1$. Moreover, $T_\omega^1(x) = x_1$. It then follows by induction that for each $n \geq 1$, $s_n = s_n(\omega, x)$, $d_n = d_n(\omega, x)$ and $T_\omega^n(x) = x_n$. Hence, $((s_n, d_n))_n$ corresponds to the c-Lüroth expansion of (ω, x). □

2.4.6 Lemma. *Let $0 < c \leq \frac{2}{5}$. For any $x \in [c, 1] \setminus \mathbb{Q}$ and any $\omega \in \{0,1\}^{\mathbb{N}}$ there is an $n \geq 0$, such that $T_\omega^n(x) \in S$.*

Proof. Let $0 < c \leq \frac{2}{5}$, $x \in [c, 1] \setminus \mathbb{Q}$ and $\omega \in \{0,1\}^{\mathbb{N}}$. If $x \in S$, then we are done. If $x \notin S$, then $T_{\omega,c}^1(x) = T_{0,c}(x) = T_{1,c}(x) \in (1 - c, 1)$. Assume that $c \leq \frac{1}{3}$, then $z_2^+ = \frac{1+c}{2} \leq 1 - c < 1 - \frac{c}{2} = z_1^-$. Let $f : x \mapsto 2x - 1$ denote the right most branch of T_L. Since $f(z_1^-) = 1 - c$, it follows that $(1 - c, 1) = \cup_{j\in\mathbb{N}} f^{-j}((1 - c, z_1^-])$ and hence there exists a $j \in \mathbb{N}$ such that $T_{\omega,c}^{j+1}(x) = T_L^j(T_{\omega,c}^1(x)) \in (1 - c, z_1^-] \subseteq S$.

Now assume that $\frac{1}{3} < c \leq \frac{2}{5}$. Then $\frac{1}{2} < 1 - c < z_2^+$ and

$$z_2^+ < 2c = T_{0,c}(1 - c) = T_{1,c}(1 - c) \leq z_1^-.$$

We write

$$(1 - c, 1) = (1 - c, z_2^+) \cup [z_2^+, z_1^-] \cup (z_1^-, 1),$$

and we treat the subintervals separately.

1. For the first subinterval note that $(1-c, z_2^+) = \big(1-c, \frac{3-c}{4}\big] \cup \big(\frac{3-c}{4}, \frac{2}{3}\big) \cup \big[\frac{2}{3}, z_2^+\big)$. Here $\frac{2}{3}$ is a repelling fixed point of T_A and this subdivision is such that $T_A\big(\big(1-c, \frac{3-c}{4}\big]\big) = [z_2^+, 2c) \subseteq S$, $T_A\big(\big(\frac{3-c}{4}, \frac{2}{3}\big)\big) = \big(\frac{2}{3}, z_2^+\big)$ and $T_A\big(\big(\frac{2}{3}, z_2^+\big)\big) = \big(1-c, \frac{2}{3}\big)$. This gives the following.

   - If $T^1_{\omega,c}(x) \in \big(1-c, \frac{3-c}{4}\big]$, then $T^2_{\omega,c}(x) \in [z_2^+, 2c) \subseteq S$.

   - If $T^1_{\omega,c}(x) \in \big(\frac{3-c}{4}, \frac{2}{3}\big)$, then $T^2_{\omega,c}(x) = T_A(T^1_{\omega,c}(x))$ and since $\frac{2}{3}$ is a repelling fixed point for T_A there must then exist a j such that $T^j_{\omega,c}(x) \in \big(1-c, \frac{3-c}{4}\big)$, so $T^{j+1}_{\omega,c}(x) \in (z_2^+, 2c) \subseteq S$.

   - If $T^1_{\omega,c}(x) \in \big[\frac{2}{3}, z_2^+\big)$, then either $T^2_{\omega,c}(x) \in \big(1-c, \frac{3-c}{4}\big]$ and $T^3_{\omega,c}(x) \in [z_2^+, 2c) \subseteq S$; or $T^2_{\omega,c}(x) \in \big(\frac{3-c}{4}, \frac{2}{3}\big]$, and again we can find a suitable j as above.

2. If $T^1_{\omega,c}(x) \in [z_2^+, z_1^-]$, then $T^1_{\omega,c}(x) \in S$.

3. If $T^1_{\omega,c}(x) \in (z_1^-, 1)$, since $f(z_1^-) = 1-c$ we can write $(z_1^-, 1)$ as the disjoint union
$$(z_1^-, 1) = \cup_{j\geq 1} f^{-j}([z_2^+, z_1^-]) \cup f^{-j}((1-c, z_2^+)).$$
Hence, there is a j such that either $T^{j+1}_{\omega,c}(x) = T^j_L \circ T^1_{\omega,c}(x) \in S$ and we are done or $T^{j+1}_{\omega,c}(x) = T^j_L \circ T^1_{\omega,c}(x) \in (1-c, z_2^+)$ and we are in the situation of case 1.

This finishes the proof. □

Proof of Theorem 2.4.3. Let $0 < c \leq \frac{2}{5}$ and $x \in [c, 1] \setminus \mathbb{Q}$ be given. To prove the result it is enough to show that for any sequence $((s_n, d_n))_{n\geq 1}$ representing a c-Lüroth expansion of x and any $N \geq 1$ there is an $n \geq N$ and a c-Lüroth expansion

$$((s_1, d_1), \dots, (s_{N+n}, d_{N+n}), (s'_{N+n+1}, d'_{N+n+1}), (s'_{N+n+2}, d'_{N+n+2}), \dots)$$

of x with $s'_{N+n+1} \neq s_{N+n+1}$ or $d'_{N+n+1} \neq d_{N+n+1}$.

Let $((s_n, d_n))_{n\geq 1}$ be any c-Lüroth admissible sequence with

$$x = \sum_{n\geq 1} (-1)^{\sum_{i=1}^{n-1} s_i} \frac{d_n - 1 + s_n}{\prod_{i=1}^{n} d_i(d_i - 1)}.$$

By Lemma 2.4.5 there then exists a sequence ω, such that $(s_n(\omega, x), d_n(\omega, x))_n = ((s_n, d_n))_n$. Fix an $N \geq 1$. Lemma 2.4.6 yields the existence of an n such that $T^{N+n}_{\omega,c}(x) \in S$. Take any sequence $\tilde{\omega} \in \{0,1\}^{\mathbb{N}}$ with $\tilde{\omega}_j = \omega_j$ for all $1 \leq j \leq N+n$, $\tilde{\omega}_{N+n+1} = 1 - \omega_{N+n+1}$. Then for each $1 \leq j \leq N+n$,

$$s_j(\tilde{\omega}, x) = s_j(\omega, x) = s_j \quad \text{and} \quad d_j(\tilde{\omega}, x) = d_j(\omega, x) = d_j$$

and

$$s_{N+n+1}(\tilde{\omega}, x) = \tilde{\omega}_{N+n+1} = 1 - \omega_{N+n+1} = 1 - s_{N+n+1}(\omega, x).$$

From $T^{N+n+1}_{\tilde{\omega},c}(x) \notin \mathbb{Q}$ we obtain that either $T^{N+n+1}_{\tilde{\omega},c}(x) > \frac{1}{2}$ and $T^{N+n+1}_{\omega,c}(x) < \frac{1}{2}$ so that $d_{N+n+2}(\tilde{\omega},x) = 2$ and $d_{N+n+2}(\omega,x) > 2$, or $T^{N+n+1}_{\tilde{\omega},c}(x) < \frac{1}{2}$ and $T^{N+n+1}_{\omega,c}(x) > \frac{1}{2}$ so that $d_{N+n+2}(\tilde{\omega},x) > 2$ and $d_{N+n+2}(\omega,x) = 2$. In any case we get a c-Lüroth expansion of x that differs from the expansion $((s_n,d_n))_n$ at indices $N+n+1$ and $N+n+2$. □

The second result of this section is on universal expansions. For the remainder of this section we assume that $c = \frac{1}{\ell}$ for some $\ell \in \mathbb{N}_{\geq 3}$. Then $\mathcal{A}_c = \{(i,j) : i \in \{0,1\}, j \in \{2,3,\ldots,\ell\}\}$ and by Lemma 2.4.5 any sequence in $\mathcal{A}_c^{\mathbb{N}}$ not ending in $((0,d+1),(0,2)^\infty)$ is c-Lüroth admissible. An expansion

$$x = \sum_{n\geq 1} (-1)^{\sum_{i=1}^{n-1} s_i} \frac{d_n - 1 + s_n}{\prod_{i=1}^{n} d_i(d_i-1)}$$

of a number $x \in [c,1]$ is called a *universal* c-Lüroth expansion if for any block $(t_1,b_1),\ldots,(t_j,b_j) \in \mathcal{A}_c$ there is a $k \geq 1$, such that $s_{k+i} = t_i$ and $d_{k+i} = b_i$ for all $1 \leq i \leq j$, i.e., if each finite block of digits occurs in $((s_n,d_n))_n$.

2.4.7 Theorem. *Let $c = \frac{1}{\ell}$ for some $\ell \in \mathbb{N}_{\geq 3}$. Then Lebesgue almost every $x \in \left[\frac{1}{\ell}, 1\right]$ has uncountably many different universal c-Lüroth expansions.*

The proof of this theorem requires some work and several smaller results. First we prove a property of the measure $m_p \times \mu_{p,c}$.

2.4.8 Proposition. *Let $c = \frac{1}{\ell}$ for some $\ell \in \mathbb{N}_{\geq 3}$. Then for any $0 < p < 1$ the random transformation L_c is mixing and the density of $\mu_{p,c}$ is bounded away from 0.*

Proof. We will show that L_c has the *random covering property*, i.e., that for any non-trivial subinterval $J \subseteq [c,1]$ there is an $n \geq 1$ and an $\omega \in \{0,1\}^{\mathbb{N}}$ such that $T^n_\omega(J) = [c,1]$. The result then follows from [Proposition 2.6, [ANV15]].

Let $J \subseteq [c,1]$ be any interval of positive Lebesgue measure. Since $|\frac{\mathrm{d}}{\mathrm{d}x} T^1_{\omega,c}(x)| \geq 2$, for any $\omega \in \{0,1\}^{\mathbb{N}}$ there is an m such that at least one of the points z_n, z_n^+, z_n^- is in the interior of $T^m_\omega(J)$ and hence $T^{m+2}_\omega(J)$ will contain an interval of the form $(a,1]$ for some a. Since 1 is a fixed point, this implies that there is a k such that $(1-\frac{1}{\ell},1] \subseteq T^k_\omega((a,1])$ for each $\omega \in \{0,1\}^{\mathbb{N}}$. For the smallest $i \geq \log_2(\ell-1)-1$ it holds that $T_{0^i,c}\big(1-\frac{1}{\ell}\big) = 1 - \frac{2^i}{\ell} < \frac{1}{2} + \frac{1}{2\ell} = z_2^+$. Hence

$$[c,1] \subseteq T_{0^{i+1},c}\Big(\Big(1-\frac{1}{\ell},1\Big]\Big),$$

giving the random covering property and the result. □

The difference between Theorem 2.4.7 and Proposition 2.3.1 is that in case $c=0$ for almost all (ω,x) the sequence $(s_n(\omega,x))_n$ equals the sequence ω, so that it is immediately clear that different sequences ω lead to different expansions. For $c>0$ typically many sequences ω lead to the same sequence $(s_n(\omega,x))_n$. In other words, the map $g : \{0,1\}^{\mathbb{N}} \times [c,1] \to \mathcal{A}_c^{\mathbb{N}}$ defined by

$$g((\omega,x)) = \big((s_1(\omega,x),d_1(\omega,x)),(s_2(\omega,x),d_2(\omega,x)),(s_3(\omega,x),d_3(\omega,x)),\ldots\big) \quad (2.23)$$

is far from injective. To solve this issue we define another random dynamical system $K_c : \{0,1\}^{\mathbb{N}} \times [c,1] \to \{0,1\}^{\mathbb{N}} \times [c,1]$ given by

$$K_c(\omega, x) = \begin{cases} (\omega, T_A(x)), & \text{if } x \in \cup_{n=2}^{\ell} [z_n, z_n^+), \\ (\omega, T_L(x)), & \text{if } x \in \cup_{n=2}^{\ell} (z_{n-1}^-, z_{n-1}) \cup \{1\}, \\ (\sigma(\omega), T^1_{\omega,c}(x)), & \text{if } x \in S. \end{cases}$$

The difference between K_c and L_c is that the map K_c only shifts in the first coordinate if the point x lies in S, so only if $T_{0,c}(x) \neq T_{1,c}(x)$. For K_c define the function $h : \{0,1\}^{\mathbb{N}} \times [c,1] \to \mathcal{A}_c^{\mathbb{N}}$ by

$$\begin{aligned} h((\omega, x)) = \big((s_1(\omega,x), d_1(\omega,x)),&(s_1(K_c(\omega,x)), d_1(K_c(\omega,x))), \\ &(s_1(K_c^2(\omega,x)), d_1(K_c^2(\omega,x))), \dots\big). \end{aligned}$$

In the other direction set $\psi_c = \psi|_{\mathcal{A}_c^{\mathbb{N}}}$, so that

$$\psi_c : \mathcal{A}_c^{\mathbb{N}} \to [c,1],\ ((s_n, d_n))_{n\geq 1} \mapsto \sum_{n\geq 1} (-1)^{\sum_{i=1}^{n-1} s_i} \frac{d_n - 1 + s_n}{\prod_{i=1}^n d_i(d_i-1)}.$$

Neither g nor h is surjective, since both maps L_c and K_c do not produce sequences ending in $((0, d+1), (0,2)^\infty)$ as we saw before. To solve this and also to make h injective, define

$$\begin{aligned} Z_L &= \{(\omega, x) \in \{0,1\}^{\mathbb{N}} \times [c,1] : L_c^n(\omega,x) \in \{0,1\}^{\mathbb{N}} \times S \text{ for infinitely many } n \geq 0\}, \\ Z_K &= \{(\omega, x) \in \{0,1\}^{\mathbb{N}} \times [c,1] : K_c^n(\omega,x) \in \{0,1\}^{\mathbb{N}} \times S \text{ for infinitely many } n \geq 0\}, \\ D &= \{\mathbf{a} \in \mathcal{A}_c^{\mathbb{N}} : \psi_c(\sigma^n \mathbf{a}) \in S \text{ for infinitely many } n \geq 0\}. \end{aligned}$$

Then $g : Z_L \to D$ is surjective and $h : Z_K \to D$ is bijective and by Lemma 2.4.6 $m_p \times \mu_{p,c}(Z_L) = 1$. The proof of Theorem 2.4.7 is based on the proofs of [DdV07, Theorem 7 and Lemma 4] and uses the following result on the map K_c.

2.4.9 Proposition. *Let $c = \frac{1}{\ell}$ for some $\ell \geq 3$ and $0 < p < 1$. The measure $m_p \times \mu_{p,c}$ is invariant and ergodic for K_c.*

Proof. First note that $\sigma \circ g = g \circ L_c$. Define a measure ν on $\mathcal{A}_c^{\mathbb{N}}$ with the σ-algebra generated by the cylinders by setting $\nu = m_p \times \mu_{p,c} \circ g^{-1}$. Then by Lemma 2.4.6 we get $\nu(\mathcal{A}_c^{\mathbb{N}} \setminus D) = 0$, that is ν is concentrated on D. So g is a factor map and σ is ergodic and measure preserving with respect to ν. By construction it holds that $K_c^{-1}(Z_K) = Z_K$ and $\sigma^{-1}(D) = D$ and moreover, $\sigma \circ h = h \circ K_c$. Define a measure $\tilde{\nu}$ on $\{0,1\}^{\mathbb{N}} \times [c,1]$ by setting

$$\tilde{\nu}(A) = \nu\big(h(A \cap Z_K)\big).$$

Since h is a bijection from Z_K to D we find that $h : \{0,1\}^{\mathbb{N}} \times [c,1] \to \mathcal{A}_c^{\mathbb{N}}$ is an isomorphism and K_c is measure preserving and ergodic with respect to $\tilde{\nu}$. What is left is to prove that $\tilde{\nu} = m_p \times \mu_{p,c}$, which is what we do now.

Sets of the form

$$h^{-1}([(k_1, i_1), \ldots, (k_n, i_n)])$$

generate the product σ-algebra on $\{0,1\}^{\mathbb{N}} \times [c, 1]$ given by the product σ-algebra on $\{0,1\}^{\mathbb{N}}$ and the Borel σ-algebra on $[c, 1]$. Therefore it is enough to check that

$$\tilde{\nu}(h^{-1}([(k_1, i_1), \ldots, (k_n, i_n)])) = m_p \times \mu_{p,c}(h^{-1}([(k_1, i_1), \ldots, (k_n, i_n)]))$$

for any cylinder $[(k_1, i_1), \ldots, (k_n, i_n)] \subseteq \mathcal{A}_c^{\mathbb{N}}$.

For $i \in \{2, 3, \ldots, \ell\}$, let

$$\begin{aligned} &A_{0i} = [0] \times \left[z_i^+, z_{i-1}^-\right], \quad A_{22} = \{0,1\}^{\mathbb{N}} \times \left(z_1^-, 1\right], \; A_{2i} = \{0,1\}^{\mathbb{N}} \times \left(z_{i-1}^-, z_{i-1}\right), \; i \geq 3, \\ &A_{1i} = [1] \times \left[z_i^+, z_{i-1}^-\right], \quad A_{3i} = \{0,1\}^{\mathbb{N}} \times \left[z_i, z_i^+\right). \end{aligned}$$

For any cylinder $[(k, i)] \subseteq \mathcal{A}_c^{\mathbb{N}}$ we get $h^{-1}([(k, i)]) = A_{ki} \cup A_{(k+2)i}$ and

$$h^{-1}([(k_1, i_1), (k_1, i_2), \ldots, (k_n, i_n)]) = \bigcup_{j_1, \ldots, j_n} A_{j_1 i_1} \cap K_c^{-1}(A_{j_2 i_2}) \cap \cdots \cap K_c^{-(n-1)}(A_{j_n i_n}), \tag{2.24}$$

where the union is disjoint and is taken over all blocks $j_1, \ldots, j_n$ that have $j_t \in \{k_t, k_t + 2\} \subset \{0, 1, 2, 3\}$ for each t. Any set $A_{j_1 i_1} \cap K_c^{-1} A_{j_2 i_2} \cap \cdots \cap K_c^{-(n-1)}(A_{j_n i_n})$ is a product set. Denote its projection on the second coordinate by $I_{j_1 i_1 \ldots j_n i_n}$. Define the set

$$\{t_1, \ldots, t_m\} = \{t : (j_t, i_t) \in \mathcal{A}_c\},$$

where we assume that $1 \leq t_1 < t_2 < \cdots < t_m \leq n$. These are the indices t such that $j_t = k_t$ and thus the projection of $A_{j_t i_t}$ to the first coordinate does not equal $\{0,1\}^{\mathbb{N}}$. This implies that we can write

$$A_{j_1 i_1} \cap K_c^{-1}(A_{j_2 i_2}) \cap \cdots \cap K_c^{-(n-1)}(A_{j_n i_n}) = [k_{t_1}, k_{t_2}, \ldots, k_{t_m}] \times I_{j_1 i_1 \ldots j_n i_n}$$

and

$$\begin{aligned} m_p \times \mu_{p,c}(h^{-1}([(k_1, i_1), (k_1, i_2), \ldots, (k_n, i_n)])) = \\ \sum_{j_1, \ldots, j_n} m_p \times \mu_{p,c}([k_{t_1}, k_{t_2}, \ldots, k_{t_m}] \times I_{j_1 i_1 \ldots j_n i_n}). \end{aligned}$$

To compute $\tilde{\nu}(h^{-1}([(k_1, i_1), (k_1, i_2), \ldots, (k_n, i_n)]))$, let $\mathcal{S} \subseteq \{0,1\}^n$ denote the set of blocks $s_1, s_2, \ldots, s_n$ for which $s_t = k_t$ for all $t \in \{t_1, \ldots, t_m\}$. Then

$$\begin{aligned}
\tilde{\nu}(h^{-1}&([(k_1, i_1), (k_1, i_2), \ldots, (k_n, i_n)])) \\
&= \nu(h(Z_K \cap h^{-1}([(k_1, i_1), (k_1, i_2), \ldots, (k_n, i_n)]))) \\
&= \nu(D \cap [(k_1, i_1), (k_1, i_2), \ldots, (k_n, i_n)]) \\
&= \nu([(k_1, i_1), (k_1, i_2), \ldots, (k_n, i_n)]) \\
&= m_p \times \mu_{p,c}\Bigg(\bigcup_{j_1,\ldots,j_n} \bigcup_{s_1,s_2,\ldots,s_n \in \mathcal{S}} [s_1, s_2, \ldots, s_n] \times I_{j_1 i_1 \ldots j_n i_n} \Bigg) \\
&= \sum_{j_1,\ldots,j_n} \sum_{s_1,s_2,\ldots,s_n \in \mathcal{S}} m_p \times \mu_{p,c}([s_1, s_2, \ldots, s_n] \times I_{j_1 i_1 \ldots j_n i_n}) \\
&= \sum_{j_1,\ldots,j_n} m_p \times \mu_{p,c}([k_{t_1}, k_{t_2}, \ldots, k_{t_m}] \times I_{j_1 i_1 \ldots j_n i_n}).
\end{aligned}$$

Hence, $\tilde{\nu} = m_p \times \mu_{p,c}$ and the statement follows. $\square$

Proof of Theorem 2.4.7. Let $0 < p < 1$ be given. For each $s \in \{0, 1\}$ and $d \in \{2, 3, \ldots, \ell\}$ define

$$\hat{\Delta}_c(s, d) = \begin{cases} [0] \times [z_d^+, z_{d-1}^-] \cup \{0,1\}^{\mathbb{N}} \times (z_{d-1}^-, z_{d-1}], & \text{if } s = 0,\ d = 2, \\ [0] \times [z_d^+, z_{d-1}^-] \cup \{0,1\}^{\mathbb{N}} \times (z_{d-1}^-, z_{d-1}), & \text{if } s = 0,\ d \geq 3 \\ [1] \times [z_d^+, z_{d-1}^1] \cup \{0,1\}^{\mathbb{N}} \times \left[z_d, z_d^+\right), & \text{if } s = 1. \end{cases}$$

Fix $(s_1, d_1), \ldots, (s_j, d_j) \in \mathcal{A}_c$. Then the set

$$E = \hat{\Delta}_c(s_1, d_1) \cap K_c^{-1} \hat{\Delta}_c(s_2, d_2) \cap \cdots \cap K_c^{-(j-1)} \hat{\Delta}_c(s_j, d_j) \subseteq \{0,1\}^{\mathbb{N}} \times [c, 1]$$

contains precisely those points (ω, x) for which $s_i(\omega, x) = s_i$ and $d_i(\omega, x) = d_i$ for all $1 \leq i \leq j$. Since for each s, d, $m_p \times \lambda(\hat{\Delta}_c(s, d)) = \frac{c}{d(d-1)} + (p - s)(-1)^s \frac{1-2c}{d(d-1)} > 0$ and $K_c(\hat{\Delta}_c(s, d)) = \{0,1\}^{\mathbb{N}} \times [c, 1]$, it also follows that $m_p \times \lambda(E) > 0$. The map K_c is ergodic with respect to $m_p \times \mu_{p,c}$ by Lemma 2.4.9 and the measures $\mu_{p,c}$ and λ are equivalent by Proposition 2.4.8. By Birkhoff's Ergodic Theorem it then follows that for $m_p \times \lambda$-a.e. (ω, x) the block $(s_1, d_1), \ldots, (s_j, d_j)$ occurs with positive frequency in the sequence $(s_1(K_c^n(\omega, x)), d_1(K_c^n(\omega, x)))_n$. Since there are only countably many blocks $(s_1, d_1), \ldots, (s_j, d_j)$ it follows that the c-Lüroth expansion of $m_p \times \lambda$-a.e. (ω, x) is universal. Let

$$\tilde{Z} := \{x \in [c, 1] \,:\, \forall \omega \in \{0,1\}^{\mathbb{N}}\ K_c^n(\omega, x) \in \{0,1\}^{\mathbb{N}} \times S \text{ for infinitely many } n \geq 0\}.$$

Then $\lambda([c, 1] \setminus \tilde{Z}) = 0$ by Lemma 2.4.6. From Fubini's Theorem we get the existence of a set $B \subseteq \tilde{Z}$ with $\lambda(B) = 1 - c$ and for each $x \in B$ a set $A_x \subseteq \{0,1\}^{\mathbb{N}}$ with $m_p(A_x) = 1$ and such that for any $(\omega, x) \in A_x \times \{x\}$ the sequence $((s_1(K_c^n(\omega, x)), d_1(K_c^n(\omega, x))))_n$ is universal. Since the set A_x has full measure, it contains uncountably many sequences. For any $x \in \tilde{Z}$ different sequences ω define different sequences $((s_1(K_c^n(\omega, x)), d_1(K_c^n(\omega, x))))_{n \geq 1}$. Hence, we obtain for Lebesgue almost every x uncountably many universal c-Lüroth expansions. $\square$

§2.4.2 Explicit expressions for invariant densities

Explicit expressions for the probability density functions $f_{p,c} = \frac{\mathrm{d}\,\mu_{p,c}}{\mathrm{d}\,\lambda}$ can be obtained from the procedure from Theorem 3.4.1 in case $p \neq \frac{1}{2}$ (since otherwise condition (A5) is violated). From this result it follows that

$$f_{p,c} = c_1 + c_2 \sum_{t\geq 0} \sum_{\omega\in\{0,1\}^t} \frac{p_\omega}{T'_{\omega,c}(1-c)} 1_{[c,T_{\omega,c}(1-c))} + c_3 \sum_{t\geq 1} \sum_{\omega\in\{0,1\}^t} \frac{p_\omega}{T'_{\omega,c}(c)} 1_{[c,T_{\omega,c}(c))}, \tag{2.25}$$

where c_1, c_2, c_3 are constants and p_ω is an abbreviation for the product $p_{\omega_1} \cdots p_{\omega_t}$. The sums in this expression have finitely many terms if the random orbits $T_{\omega,c}(1-c)$ and $T_{\omega,c}(c)$ take values in a finite set. This happens for example if the random map $L_{c,p}$ is Markov, which is the case for any $c \in \mathbb{Q} \cap (0, \frac{1}{2}]$. Before proving this result, we recall the notion of a Markov map.

An interval map $T : I \to I$, where I is an interval, is called *Markov* if there exists a finite collection of open non-empty disjoint subintervals of I defined by a set of points $\{z_i\}_{1\leq i\leq N}$ such that, for every i, $T((z_i, z_{i+1}))$ is a homeomorphism onto (z_j, z_ℓ) for some j, ℓ. The corresponding partition is called a *Markov partition.* A piecewise affine interval map $T : I \to I$ admitting a Markov partition is isomorphic to a Markov shift and the results from [FB81] show that the matrix $P = (p_{i,j})_{N\times N}$ given by

$$p_{ij} := \frac{\delta_{ij}}{|T'_j|}, \tag{2.26}$$

for $T_j = T\,|_{(z_{j-1},z_j)}$ and

$$\delta_{ij} = \begin{cases} 1, & \text{if } T((z_{j-1}, z_j)) \supseteq (z_{i-1}, z_i), \\ 0, & \text{otherwise,} \end{cases}$$

represents the Perron-Frobenius operator from (1.2). From [FB81, Theorem 1] a T-invariant density is recovered from a non-trivial eigenvector v of eigenvalue 1, i.e., $Pv = v$.

The construction extends straightforwardly to the random context. Let $\{T_j : I \to I\}_{j\in\Omega}$ be a finite family of interval maps and let $\mathbf{p} = (p_j)_{j\in\Omega}$ be a positive probability vector representing the probabilities with which we choose the maps T_j. The random interval map $R : \Omega^{\mathbb{N}} \times I \to \Omega^{\mathbb{N}} \times I$ is said to be *Markov* if each map $\{T_j\}_{j\in\Omega}$ is Markov. The matrix form P of the random Perron-Frobenius operator for R is then given by

$$P = \sum_{j\in\Omega} p_j P_j,$$

for P_j the transition matrix (2.26) of T_j, and any non-trivial vector v satisfying $Pv = v$ identifies an invariant density for the system. For Markov maps the problem of finding an invariant density function reduces therefore to finding a Markov partition and solving a matrix equation.

2.4.10 Proposition. *For any $c \in \mathbb{Q} \cap (0, \frac{1}{2}]$ the random c-Lüroth transformation L_c is Markov.*

Proof. Let $\mathcal{S}_c = \{s_i\}_i$ be the finite set of points given by

$$\{c, 1\} \cup \{z_n, z_n^+, z_{n-1}^-\}_{n=2}^{\infty} \cap [c, 1],$$

and such that $s_0 = c < s_1 < \ldots < s_k = 1$. These are the critical points in $[c, 1]$ of $L_{c,p}$. For $j = 0, 1$,

$$T_{j,c}(s_i, s_{i+1}) \in \{(1-c, 1), (c, 1-c), (T_{j,c}(c), 1), (c, T_{j,c}(c)), (T_{j,c}(c), 1-c), (1-c, T_{j,c}(c))\},$$

so that, to determine a Markov partition, it is enough to study the orbit of c and $1 - c$. Since $c \in \mathbb{Q}$, Proposition 2.2.3 implies that the set

$$\mathcal{O}_c = \{T^n_{\omega,c}(c) \, : \, \omega \in \Omega^{\mathbb{N}}, \, n \in \mathbb{N}\} \cup \{T^n_{\omega,c}(1-c) \, : \, \omega \in \Omega^{\mathbb{N}}, \, n \in \mathbb{N}\}, \tag{2.27}$$

is finite. By construction, the partition obtained by the points in $\mathcal{S}_c \cup \mathcal{O}_c$ is Markov.

□

The procedure described at the beginning of this section can therefore be applied to any random map L_c with rational cutting point c, yielding an explicit expression for the invariant density function $f_{p,c}$. As we see from Proposition 2.2.6 the rationality of c is essential here, since for $c \in \left[0, \frac{1}{2}\right] \setminus \mathbb{Q}$ the set $\mathcal{O}_c$ from (2.27) contains infinitely many elements. In the following we show that the density $f_{p,c}$ is actually continuous in c for any fixed $0 < p < 1$, and that therefore the density of each map L_c with $c \in \mathbb{R} \setminus \mathbb{Q}$ can be approximated by the densities of maps $L_{\hat{c}}$ for $\hat{c} \in \mathbb{Q}$ sufficiently close to c. The result can be proven in a similar way as [DK17, Theorem 4.1], by paying attention to the fact that now the space of definition of the transformations depends on the parameter.

2.4.11 Proposition. *Fix $0 < p < 1$. Let $\hat{c} \in (0, \frac{1}{2})$, such that $\hat{c} \neq z_n, z_n^+, z_{n-1}^-$ for all $n \geq 3$. Let $(c_k)_k \subseteq (0, \frac{1}{2})$ be a sequence converging to $\hat{c}$. Then $f_{p,c_k} \mapsto f_{p,\hat{c}}$ in $L^1(\lambda)$. If $\hat{c} = z_n$, $\hat{c} = z_n^+$ or $\hat{c} = z_{n-1}^-$ for some $n \geq 3$, then the same is true for any sequence $(c_k)_k$ that converges to $\hat{c}$ from the right.*

Proof. In this proof we use $f_c = f_{p,c}$ to denote the density of the unique stationary measure $\mu_{p,c}$, for any $c \in (0, \frac{1}{2})$. Note that the domain of each of these functions is the interval $I_c = [c, 1]$, that depends on the parameter c. For this reason, we extend these densities to the whole interval $[0, 1]$ by considering, with abuse of notation, $f_c : [0, 1] \to \mathbb{R}$, such that, for any $x \in [0, 1]$, $f_c(x) = f_c(x)\mathbf{1}_{[c,1]}(x)$.

The proof now goes along the following lines.

1. We show that there is a uniform bound, i.e., independent of k, on the total variation and supremum norm of the densities f_{c_k}. This is the point in which we need stronger assumptions on the sequence $(c_k)_k$ in case $\hat{c}$ is one of the critical points of the random map. It then follows from Helly's Selection Theorem that there is some subsequence of (f_{c_k}) for which an a.e. and $L^1(\lambda)$ limit $\hat{f}$ exist.

2. We show that $\hat{f} = f_{\hat{c}}$, which by the same proof implies that any subsequence of (f_{c_k}) has a further subsequence converging a.e. to the same limit $f_{\hat{c}}$. Hence, (f_{c_k}) converges to $f_{\hat{c}}$ in measure.

3. By the uniform integrability of (f_{c_k}), it then follows from Vitali's Convergence Theorem that the convergence of (f_{c_k}) to $f_{\hat{c}}$ is in $L^1(\lambda)$.

Step 1. and 2. use Perron-Frobenius operators. For $j = 0, 1$ the Perron-Frobenius operator $P_{c,j}$ of $T_{c,j}$ is uniquely defined by the equation

$$\int (P_{c,j} f) g \, d\lambda = \int f(g \circ T_{c,j}) \, d\lambda \quad \forall f \in L^1(\lambda),\ g \in L^\infty(\lambda),$$

and the Perron-Frobenius operator P_c of L_c is then defined by

$$P_c f = p P_{c,0} f + (1-p) P_{c,1} f.$$

Equivalently, P_c is uniquely defined by the equation

$$\int (P_c f) g \, d\lambda = p \int f(g \circ T_{c,0}) \, d\lambda + (1-p) \int f(g \circ T_{c,1}) \, d\lambda \quad \forall f \in L^1(\lambda),\ g \in L^\infty(\lambda). \tag{2.28}$$

Since each L_c has a unique probability density f_c it follows from [P84, Theorem 1] that f_c is the L^1 limit of $(\frac{1}{n} \sum_{j=0}^{n-1} P_c^j 1)_{n \geq 1}$ and that it is the unique probability density that satisfies $P_c f_c = f_c$. From [I12, Theorem 5.2] each f_c is a function of bounded variation. We proceed by finding uniform bounds on the total variation and supremum norm of these densities.

For the second iterates of the Perron-Frobenius operators we have

$$P_c^2 f = \sum_{i,j=0}^{1} p_i p_j P_{c,j}(P_{c,i} f).$$

For each $\hat{c} \neq z_n, z_n^+, z_{n-1}^-$ for $n \geq 3$, we can find a uniform lower bound δ on the length of the intervals of monotonicity of any map $T_{c,\mathbf{u}}$, $\mathbf{u} \in \Omega^2$, for all values c that are close enough to $\hat{c}$. Indeed, for any map $T_{c,j}$, $j \in \Omega$, the shortest interval is either the first or the second left most subinterval of the partition. This is due to the fact that the length of any interval that belongs to the switch region can be larger that the length of the intervals to the right or to the left of the interval itself. Note that $\inf_{x \in [0,1]} |T'_{c,j}(x)| = 2$. Applying [BG97, Lemma 5.2.1] to $T_{c,j}$, $j = 0, 1$, and any of the second iterates $T_{c,\mathbf{u}}$, $\mathbf{u} \in \Omega^2$, gives that

$$Var(P_{c,j} f) \leq Var(f) + \frac{1}{\delta} \|f\|_1 \quad \text{and} \quad Var(P_{c,\mathbf{u}} f) \leq \frac{1}{2} Var(f) + \frac{1}{2\delta} \|f\|_1,$$

where Var denotes the total variation over the interval $[0, 1]$. Since these bounds do not depend on $c, j, \mathbf{u}$, the same estimates hold for P_c, so that for any function $f : [0, 1] \to \mathbb{R}$ of bounded variation and any $n \geq 1$,

$$Var(P_c^n f) \leq \frac{1}{2^{\lfloor n/2 \rfloor}} Var(f) + \left(2 + \frac{1}{\delta}\right) \|f\|_1. \tag{2.29}$$

Let $(c_k)_{k\geq 1}$ with $c_k \to \hat{c}$ be a sequence for which the lower bound δ holds for each k. This means in particular that if $\hat{c}$ is a critical point, then we consider sequences $(c_k)_k$ that converge to $\hat{c}$ from the right. For each k and n, denote by $f_{k,n} = \frac{1}{n}\sum_{i=0}^{n-1} P_{c_k}1$. Since

$$\sup |f_{k,n}| \leq Var(f_{k,n}) + \int f_{k,n}\, d\lambda,$$

it follows from (2.29) that there is a uniform constant $C > 0$, independent of k, n, such that $Var(f_{k,n}), \sup|f_{k,n}| < C$. The same then holds for the limits f_{c_k}. Helly's Selection Theorem then gives the existence of a subsequence (k_i) and a function $\hat{f}$ of bounded variation, such that $f_{c_{k_i}} \to \hat{f}$ in $L^1(\lambda)$ and λ-a.e. and with $Var(\hat{f}), \sup|\hat{f}| < C$. This proves part 1.

By 2. and 3. above, what remains to finish the proof is to show that $P_{\hat{c}}\hat{f} = \hat{f}$. By (2.28) it is enough to show that for any compactly supported C^1 function $g : [0,1] \to \mathbb{R}$ it holds that

$$\left| \int (P_{\hat{c}}\hat{f}) g\, d\lambda - \int \hat{f} g\, d\lambda \right| = 0.$$

Note that

$$\begin{aligned}\left| \int (P_{\hat{c}}\hat{f}) g\, d\lambda - \int \hat{f} g\, d\lambda \right| \leq & p \left| \int \hat{f}(g \circ T_{\hat{c},0})\, d\lambda - \int \hat{f} g\, d\lambda \right| + \\ & (1-p) \left| \int \hat{f}(g \circ T_{\hat{c},1})\, d\lambda - \int \hat{f} g\, d\lambda \right|.\end{aligned}$$

For $j = 0, 1$ we can write

$$\begin{aligned}\left| \int \hat{f}(g \circ T_{\hat{c},j})\, d\lambda - \int \hat{f} g\, d\lambda \right| \leq & \left| \int \hat{f}(g \circ T_{\hat{c},j})\, d\lambda - \int f_{c_{k_i}}(g \circ T_{\hat{c},j})\, d\lambda \right| \\ & + \left| \int f_{c_{k_i}}(g \circ T_{\hat{c},j})\, d\lambda - \int f_{c_{k_i}}(g \circ T_{c_{k_i},j})\, d\lambda \right| \\ & + \left| \int f_{c_{k_i}}(g \circ T_{c_{k_i},j})\, d\lambda - \int \hat{f} g\, d\lambda \right|.\end{aligned}$$

The first and third integral on the right hand side can be bounded by $\|g\|_\infty \|\hat{f} - f_{c_{k_i}}\|_1 \to 0$. For the second integral, $\|f_{c_{k_i}}\|_\infty < C$ and $\int |g \circ T_{\hat{c},j} - g \circ T_{c_{k_i},j}|\, d\lambda \to 0$ by the Dominated Convergence Theorem. Hence, $\hat{f} = f_{\hat{c}}$ and $f_{c_k} \to f_{\hat{c}}$ in $L^1(\lambda)$. □

For each $\hat{c} \in \{z_n, z_n^+, z_{n-1}^-\}$ for some $n \geq 3$, a uniform lower bound δ exists if and only if the sequence of $(c_k)_k$ converges to $\hat{c}$ from the right. Indeed, if we approach $\hat{c}$ from the left, any map $T_{c,j}$, $j = 0, 1$, presents an interval of monotonicity of length $\hat{c} - c_k$, which becomes arbitrarily small for c_k approaching $\hat{c}$. This is the reason we excluded these points from the theorem.

2.4.12 Example. Consider the random Lüroth map L_c for $c = \frac{12}{25}$. By Proposition 2.4.10 we follow the random orbits of $c = \frac{12}{25}$ and $1 - c = \frac{13}{25}$.

$$12 \xrightarrow{(0,3)} \boxed{22} \xrightarrow{(0,2)} 19 \xrightarrow{(0,2)} 13 \xrightarrow{(1,2)} 24 \xrightarrow{(0,2)} 23 \xrightarrow{(0,2)} 21 \xrightarrow{(0,2)} 17 \xrightarrow{(1,2)} 16 \xrightarrow{(1,2)} 18 \xrightarrow{(1,2)} 14 \xrightarrow{(1,2)} \boxed{22}$$

Figure 2.4: The random orbit of $c = \frac{12}{25}$ *under the random map* L_c. *The numbers* (s, d) *above the arrows indicate which sign and digit are assigned at each iteration. The pink node indicates the points* c *and* $1-c$ *respectively, while the blue one the periodicity of the orbit. The orbit points are represented through their numerator only, since the common denominator is* 25.

By Figure 2.4 it is clear that for any $\omega \in \{0,1\}^{\mathbb{N}}$

$$T_\omega^{1+j}\left(\frac{12}{25}\right) = T_\omega^{11+j}\left(\frac{12}{25}\right),$$

for all $j \geq 0$, so that the c-Lüroth expansion of c is ultimately periodic of period of length 10. A Markov partition of L_c can be given by

$$\left\{\frac{12}{25}, \frac{1}{2}, \frac{13}{25}, \frac{14}{25}, \frac{16}{25}, \frac{17}{25}, \frac{18}{25}, \frac{37}{50}, \frac{38}{50}, \frac{21}{25}, \frac{22}{25}, \frac{23}{25}, \frac{24}{25}, 1\right\},$$

and the corresponding Perron-Frobenius matrix P is the 13×13 square matrix

$$P = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2^{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2^{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2^{-1} & 0 & 2^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2^{-1} & 0 & 0 & 2^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2^{-1} & 0 & 0 & 0 & 2^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2^{-1} & 0 & 0 & 0 & 0 & 2^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 2^{-1} & 0 & 0 & 0 & 0 & 0 & 2^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 2^{-1} & 0 & 0 & 0 & 0 & 0 & 2^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 2^{-1} & 0 & 0 & 0 & 0 & 0 & & 2^{-1} & 0 & 0 \\
0 & 0 & 0 & 2^{-1} & 0 & 0 & 0 & 0 & 0 & & 0 & 2^{-1} & 0 \\
6^{-1} & 0 & 2^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2^{-1} & 0 \\
6^{-1} & 0 & 2^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2^{-1} \\
6^{-1} & 2^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2^{-1}
\end{pmatrix}.$$

We can also immediately determine a Markov partition for L_c in case $c = \frac{1}{2^k}$.

2.4.13 Proposition. *For any integer* $k \geq 1$, *the Markov partition of* L_c *for* $c = \frac{1}{2^k}$ *is given by*

$$\mathcal{S}_{\frac{1}{2^k}} \cup \bigcup_{i=2}^{k} \left\{1 - \frac{1}{2^i}\right\}.$$

Proof. First, note that for any integer $\ell > 3$, the map $T_{1,\frac{1}{\ell}}$ is Markov, and its Markov partition can be given by

$$\mathcal{S}_{\frac{1}{\ell}} \cup \left\{1 - \frac{1}{\ell}, \frac{2}{\ell}\right\}.$$

Indeed, since $\frac{1}{2} < 1 - \frac{1}{\ell} < 1 - \frac{1}{2\ell}$, then $T_{1,\frac{1}{\ell}}(1 - \frac{1}{\ell}) = \frac{2}{\ell}$. If ℓ is even, the point $z_{\frac{\ell}{2}} = \frac{2}{\ell}$ is already in $\mathcal{S}_{\frac{1}{\ell}}$, otherwise $\frac{2}{\ell+1} < \frac{2}{\ell} < \frac{2}{\ell-1}$ and moreover $\frac{2}{\ell} < z^{-}_{\frac{\ell-1}{2}} = \frac{2}{\ell-1} - \frac{2}{\ell-1}\frac{2}{\ell+1}\frac{1}{\ell}$. Thus, $T_{1,\frac{1}{\ell}}\left(\frac{2}{\ell}\right) = \frac{1}{2} + \frac{1}{2\ell} \in \mathcal{S}_{\frac{1}{\ell}}$.

We now follow the random orbit of $1-c = 1-\frac{1}{2^k}$. Since

$$z_2^+ = \frac{1}{2} + \frac{1}{2^{k+1}} < 1 - \frac{1}{2^i} < 1 - \frac{1}{2^{k+1}} = z_1^-,$$

for $i = 2, 3, \ldots, k$ then

$$\mathcal{O}_{\frac{1}{2^k}} = \{T_{\omega_1^n}(1-c) \,:\, \omega \in \Omega^{\mathbb{N}}, n \in \mathbb{N}\} = \bigcup_{i=2}^{k} \left\{1 - \frac{1}{2^i}, \frac{1}{2^{i-1}}, 1\right\}. \qquad \square$$

From Proposition 2.4.13, the Markov partition obtained for $L_{\frac{1}{2^k}}$ consists of $3\cdot(2^k - 1) + (k-1)$ intervals. Indeed for $n \in \{3, \ldots, 2^k\}$ the subintervals $[z_n, z_{n-1})$ are split into 3 parts, while $[z_2^+, z_1^-]$ is divided into k pieces, due to the points $\{1-\frac{1}{2^i}\}_{i=2,\ldots,k}$. The Perron-Frobenius matrix P can be computed by considering that

$$T_L\left(\left(1 - \frac{1}{2^{i-1}}, 1 - \frac{1}{2^i}\right)\right) = \left(1 - \frac{1}{2^{i-2}}, 1 - \frac{1}{2^{i-1}}\right)$$

and

$$T_A\left(\left(1 - \frac{1}{2^{i-1}}, 1 - \frac{1}{2^i}\right)\right) = \left(\frac{1}{2^{i-1}}, \frac{1}{2^{i-2}}\right),$$

for $i = 3, \ldots, k$ and

$$T_L\left(\left(z_2^+, 1 - \frac{1}{2^2}\right)\right) = \left(\frac{1}{2^k}, z_2\right) \text{ and } T_A\left(\left(z_2^+, 1 - \frac{1}{2^2}\right)\right) = \left(\frac{1}{2}, \frac{1}{2^k}\right).$$

We obtain

$$P = \begin{pmatrix}
0 & (2^k(2^k-1))^{-1} & 0 & \cdots & 0 & p_0 2^{-1} & 0 & 0 & \cdots & p_1 2^{-1} & 0 \\
0 & (2^k(2^k-1))^{-1} & 0 & \cdots & 0 & p_0 2^{-1} & 0 & 0 & \cdots & p_1 2^{-1} & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & (2^k(2^k-1))^{-1} & 0 & \cdots & 0 & p_0 2^{-1} & p_1 2^{-1} & 0 & \cdots & 0 & 0 \\
0 & (2^k(2^k-1))^{-1} & 0 & \cdots & 0 & p_1 2^{-1} & p_0 2^{-1} & 0 & \cdots & 0 & 0 \\
0 & (2^k(2^k-1))^{-1} & 0 & \cdots & 0 & p_1 2^{-1} & p_0 2^{-1} & 0 & \cdots & 0 & 0 \\
0 & (2^k(2^k-1))^{-1} & 0 & \cdots & 0 & p_1 2^{-1} & 0 & p_0 2^{-1} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & (2^k(2^k-1))^{-1} & 0 & \cdots & 0 & p_1 2^{-1} & 0 & 0 & \cdots & p_0 2^{-1} & 0 \\
(2^k(2^k-1))^{-1} & 0 & (2^k(2^k-1))^{-1} & \cdots & 2^{-1} & 0 & 0 & 0 & \cdots & 0 & 2^{-1} \\
(2^k(2^k-1))^{-1} & 0 & (2^k(2^k-1))^{-1} & \cdots & 2^{-1} & 0 & 0 & 0 & \cdots & 0 & 2^{-1}
\end{pmatrix},$$

where $p_0 = p$ and $p_1 = 1-p$. Any non-trivial eigenvector $v = (v_1, \ldots, v_{3(2^k-1)+(k-1)})$ associated to the eigenvalue 1 of P defines an L_c-invariant density step function that takes value v_j on the subintervals (s_{j-1}, s_j) of the chosen Markov partition. The vector v can be picked in such a way that it defines a probability measure $\mu_{p,\frac{1}{2^k}}$. In the following, we give a couple of examples for $k=2$ and $k=3$ that illustrate the procedure used to compute the density function and show the pattern of P.

2.4.14 Example. For $k=2$ a Markov partition of $L_{\frac{1}{4}}$ is given by the points

$$\left\{\frac{1}{4}, \frac{13}{48}, \frac{11}{36}, \frac{1}{3}, \frac{3}{8}, \frac{11}{24}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}, 1\right\}.$$

This generates the matrix

$$P = \begin{pmatrix} 0 & 12^{-1} & 0 & 0 & 6^{-1} & 0 & 0 & p_0 2^{-1} & p_1 2^{-1} & 0 \\ 0 & 12^{-1} & 0 & 0 & 6^{-1} & 0 & 0 & p_0 2^{-1} & p_1 2^{-1} & 0 \\ 0 & 12^{-1} & 0 & 0 & 6^{-1} & 0 & 0 & p_0 2^{-1} & p_1 2^{-1} & 0 \\ 0 & 12^{-1} & 0 & 0 & 6^{-1} & 0 & 0 & p_0 2^{-1} & p_1 2^{-1} & 0 \\ 0 & 12^{-1} & 0 & 0 & 6^{-1} & 0 & 0 & p_0 2^{-1} & p_1 2^{-1} & 0 \\ 0 & 12^{-1} & 0 & 0 & 6^{-1} & 0 & 0 & p_0 2^{-1} & p_1 2^{-1} & 0 \\ 0 & 12^{-1} & 0 & 0 & 6^{-1} & 0 & 0 & p_1 2^{-1} & p_0 2^{-1} & 0 \\ 0 & 12^{-1} & 0 & 0 & 6^{-1} & 0 & 0 & p_1 2^{-1} & p_0 2^{-1} & 0 \\ 12^{-1} & 0 & 12^{-1} & 6^{-1} & 0 & 6^{-1} & 2^{-1} & 0 & 0 & 2^{-1} \\ 12^{-1} & 0 & 12^{-1} & 6^{-1} & 0 & 6^{-1} & 2^{-1} & 0 & 0 & 2^{-1} \end{pmatrix}.$$

The eigenspace associated to the eigenvalue 1 is

$$V = v \begin{pmatrix} 2/(2p+3) \\ 2/(2p+3) \\ 2/(2p+3) \\ 2/(2p+3) \\ 2/(2p+3) \\ 2/(2p+3) \\ (2p+1)/(2p+3) \\ (2p+1)/(2p+3) \\ 1 \\ 1 \end{pmatrix}, \qquad v \in \mathbb{R} \setminus \{0\},$$

and the corresponding normalised density function is

$$v(x) = \begin{cases} \dfrac{4}{2p+3} & \text{if } x \in \left[\dfrac{1}{4}, \dfrac{1}{2}\right), \\ \dfrac{4p+2}{2p+3} & \text{if } x \in \left[\dfrac{1}{2}, \dfrac{3}{4}\right), \\ 2 & \text{if } x \in \left[\dfrac{3}{4}, 1\right]. \end{cases}$$

The explicit formula for the unique invariant density allows to say more on the digits frequency and the Lyapunov exponent. Recall from Example 2.4.2 that the frequency of a digit $d \in \{2, 3, 4\}$ is given by

$$\pi_d = \lim_{n\to\infty} \frac{\#\{1 \le j \le n : d_j(\omega, x) = d\}}{n} = \pi_{(0,d)} + \pi_{(1,d)}.$$

It then follows by Birkhoff's Ergodic Theorem that, for $m_p \times \mu_{p,\frac{1}{4}}$-a.e. point (ω, x),

$$\pi_2 = \frac{2p+2}{2p+3}, \qquad \pi_3 = \frac{2}{3(2p+3)}, \qquad \pi_4 = \frac{1}{3(2p+3)}.$$

Recall the definition of Λ from (2.15). We obtain

$$\Lambda_{m_p \times \mu_{p,\frac{1}{4}}} = \frac{p \log 64 + \log 27648}{6p+9} < \Lambda_{m_p \times \lambda}.$$

That is, for $m_p \times \mu_{p,\frac{1}{4}}$-a.e. point, the approximants $\frac{p_n}{q_n}$ obtained by the iteration of the random $\frac{1}{4}$-Lüroth map are in general worse than the corresponding ones obtained via the random 0-Lüroth map with countably many branches.

2.4.15 Example. For $k = 3$ the Markov partition of $L_{\frac{1}{8}}$ is obtained by adding to the set $\mathcal{S}_{\frac{1}{8}}$ the points $\frac{3}{4}$ and $\frac{7}{8}$, resulting in the matrix

$$P = \begin{pmatrix}
0 & 56^{-1} & 0 & 0 & 42^{-1} & \dots & 0 & p_0 2^{-1} & 0 & p_1 2^{-1} & 0 \\
0 & 56^{-1} & 0 & 0 & 42^{-1} & \dots & 0 & p_0 2^{-1} & 0 & p_1 2^{-1} & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 56^{-1} & 0 & 0 & 42^{-1} & \dots & 0 & p_0 2^{-1} & 0 & p_1 2^{-1} & 0 \\
0 & 56^{-1} & 0 & 0 & 42^{-1} & \dots & 0 & p_0 2^{-1} & p_1 2^{-1} & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 56^{-1} & 0 & 0 & 42^{-1} & \dots & 0 & p_0 2^{-1} & p_1 2^{-1} & 0 & 0 \\
0 & 56^{-1} & 0 & 0 & 42^{-1} & \dots & 0 & p_1 2^{-1} & p_0 2^{-1} & 0 & 0 \\
0 & 56^{-1} & 0 & 0 & 42^{-1} & \dots & 0 & p_1 2^{-1} & p_0 2^{-1} & 0 & 0 \\
0 & 56^{-1} & 0 & 0 & 42^{-1} & \dots & 0 & p_1 2^{-1} & 0 & p_0 2^{-1} & 0 \\
0 & 56^{-1} & 0 & 0 & 42^{-1} & \dots & 2^{-1} & 0 & 0 & 0 & 2^{-1} \\
0 & 56^{-1} & 0 & 0 & 42^{-1} & \dots & 2^{-1} & 0 & 0 & 0 & 2^{-1}
\end{pmatrix}.$$

The associated $L_{\frac{1}{8}}$-invariant probability density of the measure $\mu_{p,\frac{1}{8}}$ is

$$v(x) = \begin{cases}
\dfrac{8}{2p^2+3p+5} & \text{if } x \in \left[\dfrac{1}{8}, \dfrac{1}{4}\right), \\
\dfrac{4p+4}{2p^2+3p+5} & \text{if } x \in \left[\dfrac{1}{4}, \dfrac{1}{2}\right), \\
\dfrac{4p^2+2p+4}{2p^2+3p+5} & \text{if } x \in \left[\dfrac{1}{2}, \dfrac{3}{4}\right), \\
\dfrac{4p^2+6p+4}{2p^2+3p+5} & \text{if } x \in \left[\dfrac{3}{4}, \dfrac{7}{8}\right), \\
\dfrac{4p^2+6p+12}{2p^2+3p+5} & \text{if } x \in \left[\dfrac{7}{8}, 1\right].
\end{cases}$$

The frequency of the digits $d \in \{2, 3, \dots, 8\}$ is given by

$$\pi_2 = \tfrac{2p^2+2p+3}{2p^2+3p+5}, \qquad \pi_3 = \tfrac{2(p+1)}{3(2p^2+3p+5)}, \qquad \pi_4 = \tfrac{p+1}{3(2p^2+3p+5)},$$
$$\pi_5 = \tfrac{2}{5(2p^2+3p+5)}, \qquad \pi_6 = \tfrac{4}{15(2p^2+3p+5)}, \qquad \pi_7 = \tfrac{2}{21(2p^2+3p+5)},$$
$$\pi_8 = \tfrac{1}{7(2p^2+3p+5)}.$$

Moreover for $m_p \times \mu_{p,\frac{1}{8}}$-a.e. (ω, x) we have by (2.22) that

$$\begin{aligned}
\Lambda(\omega, x) &= \sum_{d=2}^{8} \log(d(d-1))\mu_{p,\frac{1}{8}}\left(\left[\frac{1}{d}, \frac{1}{d-1}\right)\right) \\
&= \frac{8}{2p^2+3p+5}\left(\frac{\log 56}{56} + \frac{\log 42}{42} + \frac{\log 30}{30} + \frac{\log 20}{20}\right) \\
&+ \frac{4p+4}{2p^2+3p+5}\left(\frac{\log 12}{12} + \frac{\log 6}{6}\right) + \frac{2p^2+2p+3}{2p^2+3p+5}\log 2 \\
&\cong \frac{1.38628p^2 + 3.40908p + 7.49448}{2p^2+3p+5} < \frac{3}{2} < \Lambda_{m_p \times \lambda}.
\end{aligned}$$

§2.5 Remarks

1. Finding an explicit formula for the density of an absolutely continuous invariant measure is not straightfoward. The statement holds for deterministic as well as for random interval transformations, including e.g. the Ito-Tanaka continued fraction transformations and the affine random map L_c with finitely many branches. Also in the Markov case, even though a priori it is possible to represent the Perron-Frobenius operator in matrix form, it might be that the matrix is very heavy to compute due to the high number of points of the Markov partition. Chapter 3 provides a procedure to explicitly determine the density for general random piecewise affine interval maps.

2. Just as the alternating Lüroth map T_A can be seen as a linearisation of the Gauss map G, the maps $T_{0,c}$ and $T_{1,c}$ can be seen as a linearised version of the flipped α-continued fractions maps $T_{1-\alpha}$ and T_α, that will be introduced and examined in Chapter 4. When talking about approximation theory, it is indeed inevitable to refer to continued fractions, since the best approximants of an irrational number x are obtained by considering the convergents of the point in the standard continued fraction expansion.

3. For piecewise affine expanding interval maps, the Markov property makes any invariant density a step function, i.e., a function that is constant on each element of the Markov partition. The same is true when replacing the Markov property with a matching condition, as shown in [BCMP18, Theorem 1.2]. The notion of the dynamical phenomenon of matching, defined for deterministic systems in (1.2.7), will be extended for random interval maps in Chapter 5. Furthermore, an analogous results on the structure of the density function will be derived using the algebraic procedure of Chapter 3.

CHAPTER 3

Measures for Random Systems

This chapter is based on: [KM18].

Abstract

For random systems T, that are expanding on average and given by piecewise affine interval maps, we explicitly construct the density functions of absolutely continuous T-invariant measures. In case the random transformation uses only expanding maps our procedure produces all invariant densities of the system. Examples include random tent maps, random W-shaped maps, random β-transformations and random c-Lüroth maps.

§3.1 Motivation and context

The Perron-Frobenius operator has been used since the seminal paper [LY73] of Lasota and Yorke to establish the existence of absolutely continuous invariant measures for deterministic dynamical systems. Later on, the same approach was successfully used in the setting of random systems. In this context, instead of a single map, a family of transformarions is considered from which at each iteration one is selected according to a probabilistic regime and applied. In [P84] Pelikan gave sufficient conditions under which a random system with a finite number of piecewise C^2-transformations on the interval has absolutely continuous invariant measures, and he discussed the possible number of ergodic components. Around the same time, a similar result was obtained by Morita in [M85], allowing for the possibility to choose from an infinite family of maps. In recent years these results have been generalised in various ways, see for example [B00, GB03, BG05, I12].

As shown in Chapter 2, finding an explicit formula for the density functions of these absolutely continuous invariant measures is not simple. Unless in the scenario of Markov maps, the Perron-Frobenius operator can only help if one can make an educated guess. An explicit expression for the invariant density is therefore available only for specific families of maps. In 1957 Rényi gave in [R57] an expression for the invariant density of the β-transformation $x \mapsto \beta x \pmod 1$ in case $\beta = \frac{1+\sqrt{5}}{2}$, the golden mean. Later Parry and Gel'fond gave a general formula for the invariant density of the β-transformation in [P60, G59]. In [DK10] generalisations of the β-transformation were considered. A more general set-up allowing different slopes was proposed in [K90] by Kopf. He introduced for any piecewise affine, expanding interval map, satisfying some minor restraints, a matrix M and associated each absolutely continuous invariant measure of the system to a vector from the null space of M. Twenty years later, Góra developed in [G09] a similar procedure for deterministic piecewise affine eventually expanding interval maps. Unless the map in question has many onto branches, the matrix involved in the procedure from [G09] is of higher dimension than the one used in [K90].

This chapter concerns finding explicit expressions for the invariant densities of random systems. We consider any finite or countable family $\{T_j : [0,1] \to [0,1]\}_{j\in\Omega}$ of piecewise affine maps that are expanding on average. The random system T is given by choosing at each step one of these maps according to a probability vector $\mathbf{p} = (p_j)_{j\in\Omega}$. We provide a procedure to construct explicit formulae for invariant probability densities of T. This is the content of Theorem 3.4.1. The results from Theorem 3.4.1 cover those from [K14] and [S19] regarding the expression for the invariant density for random β-transformations. In case we assume that all maps T_j are expanding, we obtain the stronger result that the procedure leading to Theorem 3.4.1 actually produces all absolutely continuous invariant measures of T. We prove this in Theorem 3.5.3.

The chapter is outlined as follows. In the second section we specify our set-up and introduce the necessary assumptions and notation. The third section is devoted to the definition of a matrix M and to the proof that the null space of M is non-trivial. In the fourth section we prove Theorem 3.4.1, relating each non-trivial vector γ from the null space of M to the density h_γ of an absolutely continuous invariant measure of the system T. In the fifth section we prove Theorem 3.5.3 on when we get all invariant densities. It is in this section that the extra difficulties that we had to overcome for dealing with random systems instead of deterministic ones, are most visible. In the sixth section we apply the results to some examples, that include random tent maps, random W-shaped maps and random β-transformations. In the last section we apply the results for the random c-Lüroth maps introduced in Chapter 2.

§3.2 Affine random interval systems

Let $R : \Omega^{\mathbb{N}} \times [0,1] \to \Omega^{\mathbb{N}} \times [0,1]$ be a pseudo-skew product as defined in Definition 1.2.8, with associated probability vector $\mathbf{p} = (p_j)_{j\in\Omega}$ and piecewise affine maps $\{T_j : [0,1] \to [0,1]\}_{j\in\Omega}$. Let $\pi_2 : \Omega^{\mathbb{N}} \times [0,1] \to [0,1]$ be the canonical projection on the second component, i.e., $\pi_2(\omega, x) = x$ and let T be the random system $T = \pi_2 \circ R$, such that

$$T(\omega, x) = T_{\omega_1}(x) \text{ with probability } p_{\omega_1}.$$

We put some assumptions on the systems T we consider.

(A1) Assume that the set of all the critical points of the maps T_j is *finite*.

Call these critical points $0 = z_0 < z_1 < \cdots < z_N = 1$. The points z_i together specify a common partition $\{I_i\}_{1\le i\le N}$ of subintervals of $[0,1]$, such that all maps T_j are monotone on each of the intervals I_i. Hence, there exist $k_{i,j}, d_{i,j} \in \mathbb{R}$ such that the maps $T_{i,j} := T_j|_{I_i}$ are given by

$$T_{i,j}(x) = k_{i,j}x + d_{i,j}.$$

(A2) Assume that T is *expanding on average* with respect to $\mathbf{p}$, i.e., assume that there is a constant $0 < \rho < 1$, such that for all $x \in [0,1]$, $\sum_{j\in\Omega} \frac{p_j}{|T_j'(x)|} \le \rho < 1$. This is equivalent to assuming that for each $1 \le i \le N$,

$$\sum_{j\in\Omega} \frac{p_j}{|k_{i,j}|} \le \rho < 1.$$

Recall that a measure $\mu_{\mathbf{p}}$ on $[0,1]$ is an absolutely continuous stationary measure for T and $\mathbf{p}$ if there is a density function h, such that for each Borel set $B \subseteq [0,1]$ we have

$$\mu_{\mathbf{p}}(B) = \int_B h\, d\lambda = \sum_{j\in\Omega} p_j \mu_{\mathbf{p}}(T_j^{-1}B), \tag{3.1}$$

where λ denotes the one-dimensional Lebesgue measure. Under these conditions the random system T satisfies the conditions (a) and (b) from [I12], which studies the existence of invariant densities h satisfying the random Perron-Frobenius equation

$$P_T h = \sum_{j\in\Omega} p_j P_{T_j} h. \tag{3.2}$$

for P_{T_j} the Perron-Frobenius operator defined in 1.2. The operator P_T is linear and positive. We call an $L^1(\lambda)$-function h *T-invariant* for the random system T if it is a fixed point of P_T, i.e., if it satisfies $P_T h = h$ Lebesgue almost everywhere. A density function h is the density of an absolutely continuous stationary measure $\mu_{\mathbf{p}}$ satisfying (3.1) if and only if it is a fixed point of P_T. From [I12, Theorem 5.2] it follows that a stationary measure $\mu_{\mathbf{p}}$ of the form (3.1), and hence a T-invariant function h, exists. Inoue obtained this result by showing that the operator P_T, applied to functions of bounded variation, satisfies a Lasota-Yorke type inequality. From the famous Ionescu-Tulcea and Marinescu Theorem one can then deduce much more than mere existence of an absolutely continuous invariant measure, it says that P_T as an operator on the space of functions of bounded variation is quasi-compact. The specific implications of the quasi-compactness of P_T that we use in this paper are the following. The eigenvalue 1 of P_T has a finite dimensional eigenspace. In other words, the subspace of $L^1(\lambda)$ of T-invariant functions is a finite-dimensional sublattice of the space of functions of bounded variation. As such, it has a finite base $H = \{v_1, \dots, v_r\}$ of T-invariant density functions of bounded variation, each corresponding to an ergodic measure, so that any other T-invariant $L^1(\lambda)$-function h can be written as a linear combination of the v_i: $h = \sum_{i=1}^r c_i v_i$ for some constants $c_i \in \mathbb{R}$. Furthermore, if we set $U_i := \{x : v_i(x) > 0\}$ for the *support* of the function v_i, then each U_i is forward invariant under T in the sense that

$$\lambda\Big(U_i \triangle \bigcup_{j\in\Omega} T_j(U_i)\Big) = 0, \tag{3.3}$$

where $\triangle$ denotes the symmetric difference. Also, the sets U_i are mutually disjoint and none of the sets U_i can properly contain another forward invariant set. We will use these properties in the proofs from Section 3.5. An account of these implications on the operator P_T can be found in [P84, M85, I12], for example. For more information, we also refer to standard textbooks like [BG97] and [LM94].

In this article we find T-invariant functions $h : [0,1] \to \mathbb{R}$ by linking them to the vectors from the null space of a matrix M. To guarantee that this null space is non-trivial, we need to assume that not all the lines $x \mapsto k_{i,j}x + d_{i,j}$, $1 \le i \le N$, with respective weights p_j, have a common intersection point with the diagonal. More precisely, consider for each interval I_i the weighted intersection point with the diagonal

$$x = \sum_{j\in\Omega} p_j \Big(\frac{x}{k_{i,j}} - \frac{d_{i,j}}{k_{i,j}}\Big).$$

Our third assumption states that for each i there is an n, such that these points do not coincide.

(A3) Assume that for each $1 \le i \le N$, there is an $1 \le n \le N$, such that

$$\frac{\sum_{j\in\Omega} \frac{p_j}{k_{i,j}} d_{i,j}}{1 - \sum_{j\in\Omega} \frac{p_j}{k_{i,j}}} \neq \frac{\sum_{j\in\Omega} \frac{p_j}{k_{n,j}} d_{n,j}}{1 - \sum_{j\in\Omega} \frac{p_j}{k_{n,j}}}.$$

Note that if $d_{i,j} < 0$, then $k_{i,j} > -d_{i,j}$ and if $d_{i,j} > 1$, then $k_{i,j} < 1 - d_{i,j}$. Hence, in all cases $|d_{i,j}| < |k_{i,j}| + 1$ and by (A2),

$$\sum_{j\in\Omega} \frac{p_j}{|k_{i,j}|}|d_{i,j}| \le 1+\rho. \tag{3.4}$$

So, the quantities in (A3) are all finite. Our last assumption is on the orbits of the points 0 and 1.

(A4) For each j, assume that

$$d_{1,j} = \begin{cases} 0, & \text{if } k_{1,j} > 0, \\ 1, & \text{if } k_{1,j} < 0, \end{cases} \qquad \text{and} \qquad d_{N,j} = \begin{cases} 1 - k_{N,j}, & \text{if } k_{N,j} > 0, \\ -k_{N,j}, & \text{if } k_{N,j} < 0. \end{cases}$$

In other words, the points 0 and 1 are mapped to 0 or 1 under all maps T_j, making the system continuous at the origin, when we consider it as acting on the circle $\mathbb{R}/\mathbb{Z}$ with the points 0 and 1 identified. Since we can deal with finitely many discontinuities, there is no actual need for these last assumptions, but they make computations easier. Any system not satisfying it can be extended to a system that does satisfy this condition and for which no absolutely continuous invariant measure puts weight on the added pieces. See Figure 3.1 for an illustration and see Section 3.6.4 for a concrete example, given by the random (α, β)-transformation.

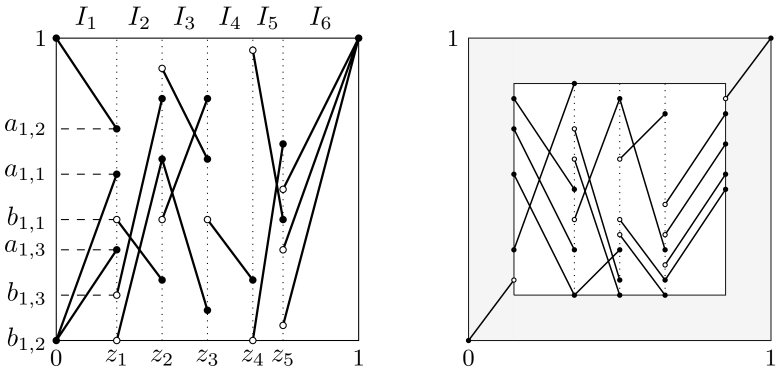

Figure 3.1: On the left is an arbitrary map T satisfying the above conditions. On the right we see a random map T in the white box that does not satisfy (A4). By adding the branches in the grey part and rescaling, we obtain a system that does satisfy these conditions. Note that any point in the grey part (except for 0 and 1) moves to the white part after a finite number of iterations and stays there. Hence, any invariant density will equal 0 on the grey part.

Finally, we include an assumption stating that the weighted inverse derivative cannot be 0 anywhere.

(A5) Assume that for any $x \in [0,1]$, the weighted inverse derivative satisfies $\sum_{j\in\Omega} \frac{p_j}{T_j'(x)} \neq 0$. This is equivalent to assuming that for each $1 \le i \le N$,

$$\sum_{j\in\Omega} \frac{p_j}{k_{i,j}} \neq 0.$$

Conditions (A3) and (A5) are sufficient to get our main results, but probably not necessary. Note that (A5) is automatically fulfilled for any deterministic Lasota-Yorke map (and in particular for any deterministic piecewise linear map) and also for any random system for which on each interval I_i the derivatives of all maps T_j have the same sign. The last section contains an example that does not satisfy (A5) for a specific choice of $\mathbf{p}$. We will see that the procedure which leads to our main results still gives all invariant densities in that case. Moreover, if (A5) is not satisfied for some probability vector $\mathbf{p}$, then changing $\mathbf{p}$ slightly already lifts this restriction.

§3.3 The matrix equation

An invariant measure reflects the dynamics of a system. For the maps T_j, $j \in \Omega$, the dynamics is determined by the orbits of the critical points, which are the endpoints of the lines $x \mapsto k_{i,j}x + d_{i,j}$, $1 \le i \le N$. We start this section by defining some quantities that keep track of their orbits.

Let Ω^* be the set of all finite strings of elements from Ω together with the empty string ε. For $t \ge 0$, let $\Omega^t \subseteq \Omega^*$ denote the subset of those strings that have length t. So in particular, $\Omega^0 = \{\varepsilon\}$. Let $|\omega|$ denote the length of the string ω. For any string $\omega \in \Omega^*$ with $|\omega| \ge t$, we let ω_1^t denote the starting block of length t. For two strings ω, $\omega' \in \Omega^*$ we simply write $\omega\omega'$ for their concatenation. Each element $\omega \in \Omega^t$ defines a possible start of an orbit of a point in $[0,1]$ by composition of maps: for $x \in [0,1]$ and $\omega = \omega_1 \cdots \omega_t \in \Omega^t$, define

$$T_\omega(x) = T_{\omega_t} \circ T_{\omega_{t-1}} \circ \cdots \circ T_{\omega_1}(x)$$

and set $T_\varepsilon(x) = x$. For $\omega \in \Omega^*$, set $\tau_\omega(y,0) = 1$ and for $1 \le t \le |\omega|$, set

$$\tau_\omega(y,t) := \frac{p_{\omega_t}}{k_{i,\omega_t}}, \qquad \text{if } T_{\omega_1^{t-1}}(y) \in I_i.$$

Define

$$\delta_\omega(y,t) := \prod_{n=0}^{t} \tau_\omega(y,n). \tag{3.5}$$

Then $\delta_\omega(y,t)$ is the weighted slope of the map $T_{\omega_1^t}$ at the point y. Note that $\tau_\omega(y,t)$ and $\delta_\omega(y,t)$ only depend on the block ω_1^t and not on what comes after. Moreover, for a concatenation ωj, given by any block ω with $|\omega| = t-1$ and any $j \in \Omega$, it holds that $\tau_{\omega j}(y,t) = \tau_j(T_\omega(y),1)$ and $\delta_{\omega j}(y,t) = \tau_{\omega j}(y,t)\delta_\omega(y,t-1)$. By assumption (A2) we have that for any $y \in [0,1]$,

$$\begin{aligned} \Big|\sum_{t\ge 0}\sum_{\omega\in\Omega^t} \delta_\omega(y,t)\Big| &\le 1 + \sum_{t\ge1}\sum_{\omega\in\Omega^{t-1}}\sum_{j\in\Omega} |\delta_\omega(y,t-1)||\tau_{\omega j}(y,t)| \\ &\le 1 + \sum_{t\ge1}\sum_{\omega\in\Omega^{t-1}} |\delta_\omega(y,t-1)|\rho \le \frac{1}{1-\rho}. \end{aligned} \tag{3.6}$$

Let $\mathbf{1}_A$ denote the characteristic function of the set A and set

$$\mathrm{KI}_n(y) := \sum_{t\ge1}\sum_{\omega\in\Omega^t} \delta_\omega(y,t)\mathbf{1}_{I_n}(T_{\omega_1^{t-1}}(y)) \qquad \text{for } 1 \le n \le N.$$

$\mathrm{KI}_n(y)$ keeps track of all the number of visits of the random orbit of y to the interval I_n and adds the corresponding weighted slopes. For $1 \le i \le N-1$, set $A_i := I_1 \cup ... \cup I_i$ and $B_i := I_{i+1} \cup ... \cup I_N$. We define

$$\begin{aligned} \mathrm{KA}_i(y) &:= \sum_{t\ge 0} \sum_{\omega\in\Omega^t} \delta_\omega(y,t) \mathbf{1}_{A_i}(T_\omega(y)), \\ \mathrm{KB}_i(y) &:= \sum_{t\ge 0} \sum_{\omega\in\Omega^t} \delta_\omega(y,t) \mathbf{1}_{B_i}(T_\omega(y)). \end{aligned} \tag{3.7}$$

By (3.6) $|\mathrm{KI}_n|$, $|\mathrm{KA}_i|$ and $|\mathrm{KB}_i|$ are finite for all $y \in [0,1]$. For each $1 \le n \le N$, let S_n be the average inverse of the slope:

$$S_n := \sum_{j\in\Omega} \frac{p_j}{k_{n,j}},$$

which is non-zero by (A5), so that S_n^{-1} is well defined. The next two lemmata give some identities that we will use later.

3.3.1 Lemma. *For each $y \in [0,1]$ and $1 \le i \le N-1$ we have*

$$\mathrm{KA}_i(y) = \sum_{n=1}^{i} S_n^{-1} \,\mathrm{KI}_n(y) \qquad \textit{and} \qquad \mathrm{KB}_i(y) = \sum_{n=i+1}^{N} S_n^{-1}\,\mathrm{KI}_n(y).$$

Proof. For any $1 \le n \le N$ we have

$$\begin{aligned} \sum_{t\ge 0} \sum_{\omega\in\Omega^t} \delta_\omega(y,t)\mathbf{1}_{I_n}(T_\omega(y)) &= \sum_{t\ge 0} \sum_{\omega\in\Omega^t} \Big(\sum_{j\in\Omega} \frac{p_j}{k_{n,j}}\Big)^{-1} \Big(\sum_{j\in\Omega}\frac{p_j}{k_{n,j}}\Big)\delta_\omega(y,t)\mathbf{1}_{I_n}(T_\omega(y)) \\ &= \sum_{t\ge 0} \sum_{\omega\in\Omega^t} S_n^{-1} \sum_{j\in\Omega} \tau_{\omega j}(y,t+1)\delta_\omega(y,t)\mathbf{1}_{I_n}(T_\omega(y)) \\ &= S_n^{-1} \sum_{t\ge 0} \sum_{\omega\in\Omega^{t+1}} \delta_\omega(y,t+1)\mathbf{1}_{I_n}(T_{\omega_1^t}(y)) = S_n^{-1}\,\mathrm{KI}_n(y). \end{aligned} \tag{3.8}$$

Putting this in the definition of $\mathrm{KA}_i(y)$ from (3.7) gives the first part of the lemma. Using (3.8), we also get that

$$\mathrm{KA}_i(y) + \mathrm{KB}_i(y) = \sum_{t\ge 0} \sum_{\omega\in\Omega^t} \delta_\omega(y,t) = \sum_{n=1}^{N} S_n^{-1}\,\mathrm{KI}_n(y). \tag{3.9}$$

The result for KB_i follows. □

Define

$$K_n := S_n^{-1} - 1 \qquad \text{and} \qquad D_n := S_n^{-1}\Big(\sum_{j\in\Omega}\frac{p_j}{k_{n,j}} d_{n,j}\Big),$$

So that

$$\frac{D_n}{K_n} = \frac{\sum_{j\in\Omega}\frac{p_j}{k_{n,j}} d_{n,j}}{1 - \sum_{j\in\Omega}\frac{p_j}{k_{n,j}}}.$$

Assumption (A3) now implies that for each $1 \leq i \leq N$, there is an $1 \leq n \leq N$, such that $\frac{D_i}{K_i} \neq \frac{D_n}{K_n}$. We have the following properties for K_n and D_n.

3.3.2 Lemma. *Let $y \in [0,1]$. Then*

$$\sum_{n=1}^{N} K_n \,\mathrm{KI}_n(y) = 1 \qquad \text{and} \qquad -\sum_{n=1}^{N} D_n \,\mathrm{KI}_n(y) = y.$$

Proof. For the first part, note that by (3.9) we have

$$\sum_{n=1}^{N} S_n^{-1} \,\mathrm{KI}_n(y) = 1 + \sum_{t\geq 1} \sum_{\omega\in\Omega^t} \delta_\omega(y,t) = 1 + \sum_{n=1}^{N} \mathrm{KI}_n(y). \tag{3.10}$$

For the second part, let $1 \leq i \leq N$ be such that $y \in I_i$. Then for $j \in \Omega$ we get $T_{i,j}(y) = k_{i,j}y + d_{i,j}$, and thus

$$y = \sum_{j\in\Omega} \Big(\frac{p_j}{k_{i,j}} T_{i,j}(y) - \frac{p_j}{k_{i,j}} d_{i,j} \Big).$$

For $t \geq 1$ and $\omega \in \Omega^*$ with $|\omega| \geq t$, set

$$\theta_\omega(y,t) := -\frac{p_{\omega_t}}{k_{n,\omega_t}} d_{n,\omega_t} \qquad \text{if } \; T_{\omega_1^{t-1}}(y) \in I_n. \tag{3.11}$$

Then

$$y = \sum_{\omega\in\Omega} \tau_\omega(y,1) T_\omega(y) + \theta_\omega(y,1). \tag{3.12}$$

Since $\tau_j(T_\omega(y),1) = \tau_{\omega j}(y,2)$ and $\theta_j(T_\omega(y),1) = \theta_{\omega j}(y,2)$, we obtain for $\omega \in \Omega$ that

$$T_\omega(y) = \sum_{j\in\Omega} \tau_{\omega j}(y,2) T_{\omega j}(y) + \theta_{\omega j}(y,2). \tag{3.13}$$

Repeated application of (3.13) in (3.12), together with the definition of δ_ω from (3.5), yields after n steps,

$$y = \sum_{t=1}^{n+1} \sum_{\omega\in\Omega^t} \delta_\omega(y,t-1)\theta_\omega(y,t) + \sum_{\omega\in\Omega^{n+1}} \delta_\omega(y,n+1) T_\omega(y).$$

From (3.6) we obtain that $\lim\limits_{n\to\infty} \sum\limits_{\omega\in\Omega^{n+1}} \big|\delta_\omega(y,n+1)T_\omega(y)\big| = 0$. Hence, by (A2), (3.4) and (3.6),

$$
\begin{aligned}
y &= \sum_{t\geq 0}\sum_{\omega\in\Omega^{t+1}} \delta_\omega(y,t)\theta_\omega(y,t+1) &(3.14)\\
&= -\sum_{n=1}^{N}\sum_{t\geq 0}\sum_{\omega\in\Omega^{t}} \delta_\omega(y,t)\mathbf{1}_{I_n}(T_\omega(y))\Big(\sum_{j\in\Omega}\frac{p_j}{k_{n,j}}d_{n,j}\Big)\\
&= -\sum_{n=1}^{N} S_n^{-1}\Big(\sum_{j\in\Omega}\frac{p_j}{k_{n,j}}d_{n,j}\Big)\sum_{t\geq 0}\sum_{\omega\in\Omega^{t}} \delta_\omega(y,t)\Big(\sum_{j\in\Omega}\frac{p_j}{k_{n,j}}\Big)\mathbf{1}_{I_n}(T_\omega(y))\\
&= -\sum_{n=1}^{N} D_n\sum_{t\geq 0}\sum_{\omega\in\Omega^{t}} \delta_\omega(y,t)\Big(\sum_{j\in\Omega}\tau_{\omega j}(y,t+1)\Big)\mathbf{1}_{I_n}(T_\omega(y))\\
&= -\sum_{n=1}^{N} D_n\sum_{t\geq 0}\sum_{\omega\in\Omega^{t+1}} \delta_\omega(y,t+1)\mathbf{1}_{I_n}(T_{\omega_1^t}(y)) = -\sum_{n=1}^{N} D_n\,\mathrm{KI}_n(y). &(3.15)
\end{aligned}
$$

□

For the invariant densities, we need to keep track of the orbits of the limits from the left and from the right of each partition point. Set, for $1\leq i\leq N-1$ and $j\in\Omega$,

$$a_{i,j} := k_{i,j}z_i + d_{i,j} = \lim_{x\uparrow z_i} T_j(x), \qquad \text{and} \qquad b_{i,j} := k_{i+1,j}z_i + d_{i+1,j} = \lim_{x\downarrow z_i} T_j(x).$$

See also Figure 3.1.

3.3.3 Definition. The $N\times(N-1)$-matrix $M=(\mu_{n,i})$ given by

$$
\mu_{n,i} := \begin{cases}
\displaystyle\sum_{j\in\Omega}\Big[\frac{p_j}{k_{i,j}} + \frac{p_j}{k_{i,j}}\,\mathrm{KI}_n(a_{i,j}) - \frac{p_j}{k_{i+1,j}}\,\mathrm{KI}_n(b_{i,j})\Big], & \text{for } n=i,\\
\displaystyle\sum_{j\in\Omega}\Big[\frac{p_j}{k_{i,j}}\,\mathrm{KI}_n(a_{i,j}) - \frac{p_j}{k_{i+1,j}} - \frac{p_j}{k_{i+1,j}}\,\mathrm{KI}_n(b_{i,j})\Big], & \text{for } n=i+1,\\
\displaystyle\sum_{j\in\Omega}\Big[\frac{p_j}{k_{i,j}}\,\mathrm{KI}_n(a_{i,j}) - \frac{p_j}{k_{i+1,j}}\,\mathrm{KI}_n(b_{i,j})\Big], & \text{else,}
\end{cases}
$$

is called the *fundamental matrix* of the random piecewise affine system T.

Note that assumption (A2) together with the fact that $|\mathrm{KI}_n(y)|<\infty$ for all $y\in[0,1]$ implies that all entries of M are finite. In the next section we associate invariant functions h_γ to vectors $\gamma\in\mathbb{R}^{N-1}$ in the null space of M. Here we prove that the null space of M is non-trivial.

3.3.4 Lemma. *The system $M\gamma=0$ admits at least one non-trivial solution.*

Proof. Since M has dimension $N\times(N-1)$, by the Rouché-Capelli Theorem the associated homogeneous system admits a non-trivial solution if and only if the rank of M is at most $N-2$. Below we will give non-trivial linear dependence relations between all combinations of $N-1$ out of N rows. It follows that any minor of order $N-1$ of M is zero and thus that the rank of M is at most $N-2$. We first show that for every $1 \le i \le N-1$,

$$\sum_{n=1}^{N} K_n \mu_{n,i} = 0 \qquad \text{and} \qquad \sum_{n=1}^{N} D_n \mu_{n,i} = 0.$$

Indeed by Lemma 3.3.2,

$$\begin{aligned}
&\sum_{n=1}^{N} K_n \mu_{n,i} = \\
&\quad = \sum_{j\in\Omega} \left[\frac{p_j}{k_{i,j}} K_i - \frac{p_j}{k_{i+1,j}} K_{i+1} + \frac{p_j}{k_{i,j}} \sum_{n=1}^{N} K_n \, \mathrm{KI}_n(a_{i,j}) - \frac{p_j}{k_{i+1,j}} \sum_{n=1}^{N} K_n \, \mathrm{KI}_n(b_{i,j}) \right] \\
&\quad = S_i(S_i^{-1} - 1) - S_{i+1}(S_{i+1}^{-1} - 1) + S_i - S_{i+1} = 0.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\sum_{n=1}^{N} D_n \mu_{n,i} = \\
&\quad = \sum_{j\in\Omega} \left[\frac{p_j}{k_{i,j}} D_i - \frac{p_j}{k_{i+1,j}} D_{i+1} + \frac{p_j}{k_{i,j}} \sum_{n=1}^{N} D_n \, \mathrm{KI}_n(a_{i,j}) - \frac{p_j}{k_{i+1,j}} \sum_{n=1}^{N} D_n \, \mathrm{KI}_n(b_{i,j}) \right] \\
&\quad = \sum_{j\in\Omega} \left(S_i S_i^{-1} \frac{p_j}{k_{i,j}} d_{i,j} - S_{i+1} S_{i+1}^{-1} \frac{p_j}{k_{i+1,j}} d_{i+1,j} - \frac{p_j}{k_{i,j}} a_{i,j} + \frac{p_j}{k_{i+1,j}} b_{i,j} \right) = 0.
\end{aligned}$$

Consequently, for every $1 \le l \le N$ and every $1 \le i \le N-1$,

$$\sum_{n=1, n\neq l}^{N} (D_l K_n - D_n K_l) \mu_{n,i} = 0.$$

By assumption (A3) this gives non-trivial linear dependence relations between all combinations of $N-1$ out of N rows, giving the result. □

3.3.5 Remark. Note that if $S_n = 0$ for some $1 \le n \le N$, then the quantities K_n and D_n are not well defined. In this case $\mu_{n,i} = \sum_{j\in\Omega} \frac{p_j}{k_{i,j}} \mathrm{KI}_n(a_{i,j}) - \frac{p_j}{k_{i+1,j}} \mathrm{KI}_n(b_{i,j})$ for each $1 \le i \le N-1$ and by the definition of KI_n we can write for any $y \in [0,1]$ that

$$\begin{aligned}
\mathrm{KI}_n(y) &= \sum_{t\ge 1} \sum_{\omega\in\Omega^{t-1}} \sum_{j\in\Omega} \delta_\omega(y, t-1) \frac{p_j}{k_{n,j}} 1_{I_n}(T_{\omega_1^{t-1}}(y)) \\
&= \sum_{t\ge 1} \sum_{\omega\in\Omega^{t-1}} \delta_\omega(y, t-1) 1_{I_n}(T_{\omega_1^{t-1}}(y)) S_n = 0.
\end{aligned}$$

Hence, $\mu_{n,i} = 0$ for each i. From this, it is clear that if $S_n = 0$ for at least two indices n, then a non-trivial vector γ such that $M\gamma = 0$ still exists. If there is a unique ℓ with $S_\ell = 0$, then to obtain a non-trivial solution one still needs to find suitable constants c_n such that $\sum_{n=1, n\neq\ell}^{N} c_n \mu_{n,i} = 0$ for each i.

Any vector γ from the null space of M satisfies the following orthogonal relations, linking γ to the functions KA_i and KB_i.

3.3.6 Lemma. *For all $1 \le i \le N-1$ we have the following orthogonal relations:*

$$\gamma_i + \sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \left[\frac{p_j}{k_{m,j}} \mathrm{KA}_i(a_{m,j}) - \frac{p_j}{k_{m+1,j}} \mathrm{KA}_i(b_{m,j}) \right] = 0;$$

and

$$\gamma_i - \sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \left[\frac{p_j}{k_{m,j}} \mathrm{KB}_i(a_{m,j}) - \frac{p_j}{k_{m+1,j}} \mathrm{KB}_i(b_{m,j}) \right] = 0.$$

Proof. If γ is a solution of the system $M\gamma = 0$, then $\sum_{m=1}^{N-1} \gamma_m \mu_{n,m} = 0$ for all n. Lemma 3.3.1 gives for $n = 1$,

$$\begin{aligned} 0 &= S_1^{-1} \sum_{m=1}^{N-1} \gamma_m \mu_{1,m} \\ &= S_1^{-1}\gamma_1 \sum_{j\in\Omega} \frac{p_j}{k_{1,j}} + S_1^{-1} \sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \left(\frac{p_j}{k_{m,j}} \mathrm{KI}_1(a_{m,j}) - \frac{p_j}{k_{m+1,j}} \mathrm{KI}_1(b_{m,j}) \right) \\ &= \gamma_1 + \sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \left(\frac{p_j}{k_{m,j}} \mathrm{KA}_1(a_{m,j}) - \frac{p_j}{k_{m+1,j}} \mathrm{KA}_1(b_{m,j}) \right). \end{aligned}$$

For $2 \le n \le N-1$ we obtain similarly

$$\begin{aligned} 0 = S_n^{-1} \sum_{m=1}^{N-1} \gamma_m \mu_{n,m} = {}& S_n^{-1} \sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \left(\frac{p_j}{k_{m,j}} \mathrm{KI}_n(a_{m,j}) - \frac{p_j}{k_{m+1,j}} \mathrm{KI}_n(b_{m,j}) \right) \\ &+ S_n^{-1}\left(\gamma_n \sum_{j\in\Omega} \frac{p_j}{k_{n,j}} - \gamma_{n-1} \sum_{j\in\Omega} \frac{p_j}{k_{n,j}} \right) \\ = {}& S_n^{-1} \sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \left(\frac{p_j}{k_{m,j}} \mathrm{KI}_n(a_{m,j}) - \frac{p_j}{k_{m+1,j}} \mathrm{KI}_n(b_{m,j}) \right) + \gamma_n - \gamma_{n-1}. \end{aligned} \tag{3.16}$$

Then summing over all $1 \le n \le i$ and using (3.16) and Lemma 3.3.1 gives

$$\begin{aligned} 0 &= \sum_{n=1}^{i} S_n^{-1} \sum_{m=1}^{N-1} \gamma_m \mu_{n,m} \\ &= \gamma_i + \sum_{n=1}^{i} S_n^{-1} \sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \left(\frac{p_j}{k_{m,j}} \mathrm{KI}_n(a_{m,j}) - \frac{p_j}{k_{m+1,j}} \mathrm{KI}_n(b_{m,j}) \right) \\ &= \gamma_i + \sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \left(\frac{p_j}{k_{m,j}} \mathrm{KA}_i(a_{m,j}) - \frac{p_j}{k_{m+1,j}} \mathrm{KA}_i(b_{m,j}) \right). \end{aligned}$$

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This gives the relations for KA_i.

From $\sum_{m=1}^{N-1} \gamma_m \mu_{n,m} = 0$ for all n it also follows that $\sum_{m=1}^{N-1} \gamma_m \sum_{n=1}^{N} \mu_{n,m} = 0$. From this we obtain that

$$\sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \frac{p_j}{k_{m,j}} \Big(1 + \sum_{n=1}^{N} \mathrm{KI}_n(a_{m,j})\Big) = \sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \frac{p_j}{k_{m+1,j}} \Big(1 + \sum_{n=1}^{N} \mathrm{KI}_n(b_{m,j})\Big).$$

Then (3.10) from the proof of Lemma 3.3.2 gives that

$$\sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \frac{p_j}{k_{m,j}} \sum_{n=1}^{N} S_n^{-1} \mathrm{KI}_n(a_{m,j}) = \sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \frac{p_j}{k_{m+1,j}} \sum_{n=1}^{N} S_n^{-1} \mathrm{KI}_n(b_{m,j}).$$

Hence, by Lemma 3.3.1 we get for each i that

$$\sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \frac{p_j}{k_{m,j}} (\mathrm{KA}_i(a_{m,j}) + \mathrm{KB}_i(a_{m,j})) =$$
$$= \sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \frac{p_j}{k_{m+1,j}} (\mathrm{KA}_i(b_{m,j}) + \mathrm{KB}_i(b_{m,j})).$$

This gives the orthogonal relations for KB_i. □

In the proofs of our main results we only use the second part of Lemma 3.3.6, i.e., the orthogonal relations for KB_i, but since we obtain the orthogonal relations for KA_i and KB_i more or less simultaneously, we have listed them both.

§3.4 An explicit formula for invariant measures

We now state our main result. For $y \in [0,1]$, define the $L^1(\lambda)$-function $L_y : [0,1] \to \mathbb{R}$ by

$$L_y(x) = \sum_{t\geq 0} \sum_{\omega\in\Omega^t} \delta_\omega(y,t) \mathbf{1}_{[0,T_\omega(y))}(x). \tag{3.17}$$

3.4.1 Theorem. *Let T be a random piecewise affine system on the unit interval $[0,1]$ that satisfies the assumptions (A1) to (A5) from Section 3.2. Let M be the corresponding fundamental matrix and let $\gamma = (\gamma_1, \dots, \gamma_{N-1})^\intercal$ be a non-trivial solution of the system $M\gamma = 0$. For each $1 \leq m \leq N-1$, define the function $h_m : [0,1] \to \mathbb{R}$ by*

$$h_m(x) := \sum_{\ell\in\Omega} \left[\frac{p_\ell}{k_{m,\ell}} L_{a_{m,\ell}}(x) - \frac{p_\ell}{k_{m+1,\ell}} L_{b_{m,\ell}}(x) \right]. \tag{3.18}$$

Then a T-invariant function is given by

$$h_\gamma : [0,1] \to \mathbb{R}, x \mapsto \sum_{m=1}^{N-1} \gamma_m h_m(x), \tag{3.19}$$

and $h_\gamma \neq 0$.

To show that $P_T h_\gamma = h_\gamma$ λ-a.e. we have to determine for each $x \in [0,1]$ and each branch $T_{i,j}$, whether or not x has an inverse image in the branch $T_{i,j}$. Let

$$x_{i,j} := \frac{x - d_{i,j}}{k_{i,j}}$$

be the inverse of x under the map $T_{i,j} : \mathbb{R} \to \mathbb{R}$. By the definitions in (3.18) and (3.19), we have to show that

$$\begin{aligned} h_\gamma(x) &= \sum_{j\in\Omega}\sum_{i=1}^{N} \frac{p_j}{|k_{i,j}|} h_\gamma(x_{i,j}) \mathbf{1}_{I_i}(x_{i,j}) \\ &= \sum_{j\in\Omega}\sum_{i=1}^{N} \frac{p_j}{|k_{i,j}|} \mathbf{1}_{I_i}(x_{i,j}) \sum_{m=1}^{N-1} \gamma_m \sum_{\ell\in\Omega}\left(\frac{p_\ell}{k_{m,\ell}} L_{a_{m,\ell}}(x_{i,j}) - \frac{p_\ell}{k_{m+1,\ell}} L_{b_{m,\ell}}(x_{i,j})\right). \end{aligned} \tag{3.20}$$

The parts for $L_{a_{m,\ell}}$ and $L_{b_{m,\ell}}$ behave similarly. That is why we first study

$$\sum_{j\in\Omega}\sum_{i=1}^{N} \frac{p_j}{|k_{i,j}|} \mathbf{1}_{I_i}(x_{i,j}) L_y(x_{i,j})$$

for general $y \in [0,1]$ through several lemmas. We introduce some notation to manage the long expressions. For $1 \le i \le N-1$, let

$$\eta_i := \sum_{j\in\Omega} \frac{p_j(\mathbf{1}_{(0,\infty)}(k_{i,j}) - a_{i,j})}{k_{i,j}} \quad \text{and} \quad \phi_i := \sum_{j\in\Omega} \frac{p_j(-\mathbf{1}_{(-\infty,0)}(k_{i+1,j}) + b_{i,j})}{k_{i+1,j}}.$$

For $y \in [0,1]$ let $1 \le n \le N$ be the index such that $y \in I_n$ and set

$$C(y) := \sum_{j\in\Omega}\left(\sum_{i=1}^{n-1} \frac{p_j}{|k_{i,j}|} + \frac{p_j}{|k_{n,j}|}\mathbf{1}_{(-\infty,0)}(k_{n,j})\right). \tag{3.21}$$

3.4.2 Lemma. *Let $y \in [0,1]$. Then*

$$y = \sum_{t\ge 0}\sum_{\omega\in\Omega^t} \delta_\omega(y,t) C(T_\omega(y)) - \sum_{i=1}^{N-1} (\eta_i + \phi_i)\, \mathrm{KB}_i(y).$$

Proof. Let $y \in [0,1]$ be given and recall the definition of $\theta_\omega(z,t)$ from (3.11). If $y \in I_n$, then

$$\begin{aligned}
&C\ (y) - \sum_{i=1}^{N-1} (\eta_i + \phi_i)\mathbf{1}_{B_i}(y) \\
&= \sum_{j\in\Omega} \frac{p_j}{|k_{n,j}|}\mathbf{1}_{(-\infty,0)}(k_{n,j}) \\
&\quad + \sum_{j\in\Omega}\sum_{i=1}^{n-1}\left(\frac{p_j}{|k_{i,j}|} - \frac{p_j(\mathbf{1}_{(0,\infty)}(k_{i,j}) - a_{i,j})}{k_{i,j}} - \frac{p_j(-\mathbf{1}_{(-\infty,0)}(k_{i+1,j}) + b_{i,j})}{k_{i+1,j}}\right) \\
&= \sum_{j\in\Omega}\left(- \frac{p_j}{k_{n,j}}b_{n-1,j} + \frac{p_j}{|k_{1,j}|} - \frac{p_j}{k_{1,j}}\mathbf{1}_{(0,\infty)}(k_{1,j}) + \frac{p_j}{k_{1,j}}a_{1,j} + \sum_{i=2}^{n-1}\frac{p_j}{k_{i,j}}(a_{i,j} - b_{i-1,j})\right) \\
&= -\sum_{j\in\Omega}\frac{p_j}{k_{n,j}}d_{n,j} = \sum_{j\in\Omega}\theta_j(y,1),
\end{aligned}$$

where we have used the assumptions from (A4) in the second to last step. So, for any $t \geq 0$ and $\omega \in \Omega^t$, we get that

$$C(T_\omega(y)) - \sum_{i=1}^{N-1}(\eta_i + \phi_i)\mathbf{1}_{B_i}(T_\omega(y)) = \sum_{j\in\Omega}\theta_{\omega j}(y, t+1), \tag{3.22}$$

where ωj denotes the concatenation of ω with $j \in \Omega$. Recall from the first line of (3.14) that

$$y = \sum_{t\geq 0}\sum_{\omega\in\Omega^t}\delta_\omega(y,t)\sum_{j\in\Omega}\theta_{\omega j}(y,t+1).$$

Combining this with (3.22) and the definition of KB_i from (3.7) then gives the result. □

For each $1 \leq i \leq N-1$, define the functions $E_i, F_i : [0,1] \to \mathbb{R}$ by

$$\begin{aligned}
E_i(x) &:= \sum_{j\in\Omega}\frac{p_j}{k_{i,j}}\Big(- \mathbf{1}_{[a_{i,j},1]}(x)\mathbf{1}_{(0,\infty)}(k_{i,j}) + \mathbf{1}_{[0,a_{i,j})}(x)\mathbf{1}_{(-\infty,0)}(k_{i,j})\Big), \\
F_i(x) &:= \sum_{j\in\Omega}\frac{p_j}{k_{i+1,j}}\Big(- \mathbf{1}_{[0,b_{i,j})}(x)\mathbf{1}_{(0,\infty)}(k_{i+1,j}) + \mathbf{1}_{[b_{i,j},1]}(x)\mathbf{1}_{(-\infty,0)}(k_{i+1,j})\Big),
\end{aligned}$$

and let $E_N, F_0 : [0,1] \to \mathbb{R}$ be the zero functions. Then for each $2 \leq i \leq N-1$, we have that for Lebesgue almost every $x \in [0,1]$,

$$E_i(x) + F_{i-1}(x) = \sum_{j\in\Omega}\frac{p_j}{|k_{i,j}|}(\mathbf{1}_{I_i}(x_{i,j}) - 1),$$

where we have used (A4) for $i = 1, N$. In fact, equality holds for all but countably many points.

3.4.3 Lemma. *For $y \in [0,1]$ we have that for Lebesgue almost every $x \in [0,1]$,*

$$\sum_{j\in\Omega}\sum_{i=1}^{N}\frac{p_j}{|k_{i,j}|}\mathbf{1}_{I_i}(x_{i,j})L_y(x_{i,j}) =$$
$$= \sum_{i=1}^{N-1}(E_i(x)+\eta_i+F_i(x)+\phi_i)\,\mathrm{KB}_i(y)+y+L_y(x)-\mathbf{1}_{[0,y)}(x).$$

Proof. For $y \in [0,1]$, let $1 \le n \le N$ be the index such that $y \in I_n$. By Fubini's Theorem, we get

$$\sum_{j\in\Omega}\sum_{i=1}^{N}\frac{p_j}{|k_{i,j}|}\mathbf{1}_{I_i}(x_{i,j})L_y(x_{i,j}) = \sum_{t\ge 0}\sum_{\omega\in\Omega^t}\delta_\omega(y,t)\sum_{i=1}^{N}\sum_{j\in\Omega}\frac{p_j}{|k_{i,j}|}\mathbf{1}_{I_i\cap[0,T_\omega(y))}(x_{i,j}). \tag{3.23}$$

For Lebesgue almost every $x \in [0,1]$ it holds that

$$\begin{aligned}
\sum_{j\in\Omega}\frac{p_j}{|k_{n,j}|}\mathbf{1}_{(-\infty,0)}(k_{n,j})+&\sum_{j\in\Omega}\frac{p_j}{k_{n,j}}\mathbf{1}_{[0,T_j(y))}(x)+F_{n-1}(x)\\
&=\sum_{j\in\Omega}\Bigg(\frac{p_j}{|k_{n,j}|}\mathbf{1}_{(-\infty,0)}(k_{n,j})(1-\mathbf{1}_{[0,T_j(y))}(x)-\mathbf{1}_{[b_{n-1,j},1]}(x))\\
&\quad+\frac{p_j}{|k_{n,j}|}\mathbf{1}_{(0,\infty)}(k_{n,j})(\mathbf{1}_{[0,T_j(y))}(x)-\mathbf{1}_{[0,b_{n-1,j})}(x))\Bigg)\\
&=\sum_{j\in\Omega}\frac{p_j}{|k_{n,j}|}\mathbf{1}_{I_n\cap[0,y)}(x_{n,j}).
\end{aligned} \tag{3.24}$$

Since $y \in I_n$ we have for Lebesgue almost every $x \in [0,1]$ that

$$\sum_{i=1}^{N-1}(E_i(x)+F_i(x))\mathbf{1}_{B_i}(y) = \sum_{i=1}^{n-1}\sum_{j\in\Omega}\frac{p_j}{|k_{i,j}|}(\mathbf{1}_{I_i}(x_{i,j})-1)+F_{n-1}(x).$$

Combining this with (3.24) and the definition of $C(y)$ from (3.21) we obtain that for each $y \in [0,1]$, there is a set of $x \in [0,1]$ of full Lebesgue measure, for which

$$\begin{aligned}
&\sum_{j\in\Omega}\sum_{i=1}^{N}\frac{p_j}{|k_{i,j}|}\mathbf{1}_{I_i\cap[0,y)}(x_{i,j})\\
&=\sum_{j\in\Omega}\sum_{i=1}^{n-1}\frac{p_j}{|k_{i,j}|}\mathbf{1}_{I_i}(x_{i,j})+\sum_{j\in\Omega}\frac{p_j}{|k_{n,j}|}\mathbf{1}_{(-\infty,0)}(k_{n,j})+\sum_{j\in\Omega}\frac{p_j}{k_{n,j}}\mathbf{1}_{[0,T_j(y))}(x)+F_{n-1}(x)\\
&=\sum_{i=1}^{N-1}(E_i(x)+F_i(x))\mathbf{1}_{B_i}(y)+C(y)+\sum_{j\in\Omega}\tau_j(y,1)\mathbf{1}_{[0,T_j(y))}(x).
\end{aligned}$$

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Hence, by (3.23) we also have that for Lebesgue almost every $x \in [0,1]$,

$$\sum_{j\in\Omega}\sum_{i=1}^{N}\frac{p_j}{|k_{i,j}|}\mathbf{1}_{I_i}(x_{i,j})L_y(x_{i,j}) = \sum_{i=1}^{N-1}(E_i(x)+F_i(x))\sum_{t\geq 0}\sum_{\omega\in\Omega^t}\delta_\omega(y,t)\mathbf{1}_{B_i}(T_\omega(y))$$
$$+\sum_{t\geq 0}\sum_{\omega\in\Omega^t}\delta_\omega(y,t)C(T_\omega(y))+\sum_{t\geq 1}\sum_{\omega\in\Omega^t}\delta_\omega(y,t)\mathbf{1}_{[0,T_\omega(y))}(x).$$

The statement now follows from the definition of KB_i from (3.7) and Lemma 3.4.2. □

Proof of Theorem 3.4.1. First note that for all $1 \leq i \leq N-1$ and all $x \in [0,1]$,

$$E_i(x)+\eta_i = \sum_{j\in\Omega}\frac{p_j}{k_{i,j}}\Big(\mathbf{1}_{[0,a_{i,j})}(x)-a_{i,j}\Big)$$

and

$$F_i(x)+\phi_i = \sum_{j\in\Omega}\frac{p_j}{k_{i+1,j}}\Big(-\mathbf{1}_{[0,b_{i,j})}(x)+b_{i,j}\Big).$$

Together they give that

$$\sum_{\ell\in\Omega}\Big(\frac{p_\ell}{k_{m,\ell}}(-\mathbf{1}_{[0,a_{m,\ell})}(x)+a_{m,\ell})-\frac{p_\ell}{k_{m+1,\ell}}(-\mathbf{1}_{[0,b_{m,\ell})}(x)+b_{m,\ell})\Big)$$
$$= -(E_m(x)+\eta_m+F_m(x)+\phi_m).$$

Using this together with Lemma 3.4.3 and Fubini's Theorem, we get by (3.20) that for Lebesgue almost every $x \in [0,1]$,

$$P_T h_\gamma(x) =$$
$$= \sum_{m=1}^{N-1}\gamma_m\sum_{i=1}^{N-1}(E_i(x)+\eta_i+F_i(x)+\phi_i)\sum_{\ell\in\Omega}\Big(\frac{p_\ell}{k_{m,\ell}}\mathrm{KB}_i(a_{m,\ell})-\frac{p_\ell}{k_{m+1,\ell}}\mathrm{KB}_i(b_{m,\ell})\Big)$$
$$-\sum_{m=1}^{N-1}\gamma_m(E_m(x)+\eta_m+F_m(x)+\phi_m)+h_\gamma(x).$$

From the second part of Lemma 3.3.6 we can deduce by multiplying with $E_i(x)+\eta_i+F_i(x)+\phi_i$ and summing over all i that

$$\sum_{i=1}^{N-1}(E_i(x)+\eta_i+F_i(x)+\phi_i)\gamma_i$$
$$= \sum_{i=1}^{N-1}(E_i(x)+\eta_i+F_i(x)+\phi_i)\sum_{m=1}^{N-1}\gamma_m\sum_{j\in\Omega}\Big(\frac{p_j}{k_{m,j}}\mathrm{KB}_i(a_{m,j})-\frac{p_j}{k_{m+1,j}}\mathrm{KB}_i(b_{m,j})\Big).$$

Hence, we have obtained that h_γ is a T-invariant function in $L^1(\lambda)$.

It remains to show that $h_\gamma \neq 0$. Recall from Section 3.2 that any T-invariant $L^1(\lambda)$-function is of bounded variation. So, at any point $y \in [0,1]$ the limits $\lim_{x\uparrow y} h_\gamma(x)$ and $\lim_{x\downarrow y} h_\gamma(x)$ exist. Consider $1 \leq \ell \leq N-1$ and assume $z_\ell \in I_\ell$. Then for all $y \in [0,1]$, by (3.6) and (3.7), we obtain by the Dominated Convergence Theorem,

$$\lim_{x\downarrow z_\ell} L_y(x) = \sum_{t\geq 0}\sum_{\omega\in\Omega^t} \delta_\omega(y,t) \lim_{x\downarrow z_\ell} \mathbf{1}_{[0,T_\omega(y))}(x) = \sum_{t\geq 0}\sum_{\omega\in\Omega^t} \delta_\omega(y,t)\mathbf{1}_{B_\ell}(T_\omega(y)) = \mathrm{KB}_\ell(y).$$

From this, Lemma 3.3.6 and the Dominated Convergence Theorem again we then get

$$\begin{aligned}
\lim_{x\downarrow z_\ell} h_\gamma(x) &= \sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \lim_{x\downarrow z_\ell} \left[\frac{p_j}{k_{m,j}} L_{a_{m,j}}(x) - \frac{p_j}{k_{m+1,j}} L_{b_{m,j}}(x)\right] \\
&= \sum_{m=1}^{N-1} \gamma_m \sum_{j\in\Omega} \left[\frac{p_j}{k_{m,j}} \mathrm{KB}_\ell(a_{m,j}) - \frac{p_j}{k_{m+1,j}} \mathrm{KB}_\ell(b_{m,j})\right] \\
&= \gamma_\ell.
\end{aligned} \tag{3.25}$$

If, on the other hand, $z_\ell \in I_{\ell+1}$, then we obtain similarly that $\lim_{x\uparrow z_\ell} L_y(x) = \mathrm{KB}_\ell(y)$ and thus that $\lim_{x\uparrow z_\ell} h_\gamma(x) = \gamma_\ell$. Hence, $h_\gamma = 0$ implies $\gamma = 0$. This proves the theorem. □

3.4.4 Remark. Theorem 3.4.1 assigns to each solution γ of $M\gamma = 0$ a T-invariant $L^1(\lambda)$-function $h_\gamma \neq 0$. If $\gamma \neq 0$, then from h_γ we can get invariant densities for T as follows. If h_γ is positive or negative, then we can scale h_γ to an invariant density function. If not, then we can write $h_\gamma = h^+ - h^-$ for two positive functions $h^+ : [0,1] \to [0,\infty)$ and $h^- : [0,1] \to [0,\infty)$ and by the linearity and the positivity of P_T it follows that

$$h^+ - h^- = h_\gamma = P_T h_\gamma = P_T h^+ - P_T h^-.$$

Hence, h^+ and h^- can both be normalised to obtain invariant densities for T.

3.4.5 Remark. In order to compute h_γ, one needs to compute the fundamental matrix M and a vector γ first. Lemma 3.3.4 implies that when N is small, the computation of γ is straightforward. Indeed, for $N = 2$, M is the null-vector, and we can take $\gamma = 1$. This is illustrated by the example of the random tent maps from Section 3.6.1. For $N = 3$, it is enough to compute only one row of M and take $\gamma = \begin{pmatrix} -\mu_{i,2} & \mu_{i,1} \end{pmatrix}^\intercal$. We see an illustration of this fact in Sections 3.6.3 and 3.6.4 on random β-transformations. For larger N, the computation of M can still be simplified by using the relations from Lemma 3.3.2.

To end this section we give a small example to show that condition (A5) is not necessary for Theorem 3.4.1 to hold. Consider the random system with $\Omega = \{0,1\}$, $T_0(x) = 2x \pmod 1$ the doubling map, $T_1(x) = 1 - T_0(x)$ and $p_0 = p_1 = \frac{1}{2}$. Then $N = 2$ and for both $n = 1,2$ we have $S_n = \frac{1}{2}\cdot\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2} = 0$. Hence $M = \begin{pmatrix} 0 & 0 \end{pmatrix}^\intercal$ and any $\gamma = \gamma_1 \in \mathbb{R}\setminus\{0\}$ is a non-trivial solution to $M\gamma = 0$. Since all critical points of T_0 and T_1 are mapped to 0 or 1, the function h_1 from (3.18) will be of the form $c \cdot \mathbf{1}_{[0,1)}$ for some $c \neq 0$ and the function $h_\gamma = \frac{\gamma}{c} \cdot \mathbf{1}_{[0,1)}$ is indeed invariant for T.

§3.5 All invariant measures

The aim of this section is twofold. Firstly, we prove that the way T is defined on the partition points z_ℓ does not influence the final result. In other words, the set of invariant functions we obtain from Theorem 3.4.1 if $z_\ell \in I_\ell$ is equal to the set of invariant functions we obtain if we choose $z_\ell \in I_{\ell+1}$. This is the content of Proposition 3.5.1. The amount of work it takes to compute the matrix M and the invariant functions h_γ depend on whether $z_\ell \in I_\ell$ or $z_\ell \in I_{\ell+1}$. Proposition 3.5.1 tells us that we are free to choose the most convenient option. We shall see several examples below. Next we will use Proposition 3.5.1 to prove that, under the additional assumption that all maps T_j are expanding, Theorem 3.4.1 actually produces all absolutely continuous invariant measures of T. We do this by proving in Theorem 3.5.3 that the map $\gamma \mapsto h_\gamma$ is a bijection between the null space of M and the subspace of $L^1(\lambda)$ of all T-invariant functions.

3.5.1 Proposition. Let T be a random system with partition $\{I_i\}_{1\le i\le N}$ and corresponding partition points $z_0,\dots,z_N$. Let $\{\hat{I}_i\}_{1\le i\le N}$ be another partition of $[0,1]$ given by $z_0,\dots,z_N$ and differing from $\{I_i\}_{1\le i\le N}$ only on one or more of the points $z_1,\dots,z_{N-1}$. Let $\hat{T}$ be the corresponding random system, i.e., $\hat{T}(x) = T(x)$ for all $x \neq z_i$, $1 \le i \le N-1$. Let $\hat{M}$ be the fundamental matrix of $\hat{T}$. There is a 1-to-1 correspondence between the solutions γ of $M\gamma = 0$ and the solutions $\hat{\gamma}$ of $\hat{M}\hat{\gamma} = 0$. Moreover, the functions h_γ and $\hat{h}_{\hat{\gamma}}$ coincide.

Proof. First assume that there is only one point z_ℓ on which $\{I_i\}_{1\le i\le N}$ and $\{\hat{I}_i\}_{1\le i\le N}$ differ. We show that any column of $\hat{M}$ is a linear combination of columns of M. More precisely, we show that the i-th column of $\hat{M}$ is a linear combination of the i-th and the ℓ-th column of M. Assume without loss of generality that $z_\ell \in I_\ell$ and therefore $z_\ell \in \hat{I}_{\ell+1}$. This implies that $T_j(z_\ell) = a_{\ell,j}$, whereas $\hat{T}_j(z_\ell) = b_{\ell,j}$. This difference is reflected in the values of the quantities $\mathrm{KI}_n(a_{i,s})$ and $\mathrm{KI}_n(b_{i,s})$ appearing in the matrix M in case $a_{i,s}$ or $b_{i,s}$ enters z_ℓ under some iteration of T. We will describe these changes, but first we define some quantities.

For any $y \in \{a_{i,j}, b_{i,j} \,:\, 1 \le i \le N-1,\, j \in \Omega\}$ let $\Omega_y \subseteq \Omega^*$ be the collection of paths that lead y to z_ℓ, i.e., $\omega \in \Omega_y$ if and only if there is a $0 \le t < |\omega|$, such that $T_{\omega_1^t}(y) = z_\ell$. Let

$$\Omega_y^t := \{\omega \in \Omega^* \,|\, \exists\, \eta \in \Omega_y \,:\, \omega = \eta_1^t,\, T_\omega(y) = z_\ell \text{ and } T_{\omega_1^s}(y) \neq z_\ell \text{ for } s < t\}.$$

Then Ω_y^t is the collection of words of length t that lead y to z_ℓ via a path that does not lead y to z_ℓ before time t. We are interested in the difference between the quantities $\mathrm{KI}_n(y)$ and $\hat{\mathrm{KI}}_n(y)$ and we let C_n^y denote the part that they have in common, i.e., set

$$C_n^y := \sum_{t\ge 1} \sum_{\omega \in \Omega_y^t \cup \Omega^t \setminus \Omega_y} \delta_\omega(y,t) \mathbf{1}_{I_n}(T_{\omega_1^{t-1}}(y)).$$

Then for $n \neq \ell$, we get

$$\begin{aligned}
\mathrm{KI}_n(y) &= C_n^y + \sum_{t\geq 0}\sum_{\omega\in\Omega_y^t}\sum_{u\geq 1}\sum_{\eta\in\Omega^u} \delta_\omega(y,t)\delta_\eta(z_\ell,u)\mathbf{1}_{I_n}(T_{\eta_1^{u-1}}(z_\ell)) \\
&= C_n^y + \sum_{t\geq 0}\sum_{\omega\in\Omega_y^t}\sum_{u\geq 1}\sum_{\eta\in\Omega^u}\sum_{j\in\Omega} \delta_\omega(y,t)\frac{p_j}{k_{\ell,j}}\delta_\eta(a_{\ell,j},u)\mathbf{1}_{I_n}(T_{\eta_1^{u-1}}(a_{\ell,j})) \\
&= C_n^y + \sum_{t\geq 0}\sum_{\omega\in\Omega_y^t} \delta_\omega(y,t)\sum_{j\in\Omega}\frac{p_j}{k_{\ell,j}}\mathrm{KI}_n(a_{\ell,j}),
\end{aligned}$$

and similarly, for $n = \ell$ we obtain

$$\mathrm{KI}_\ell(y) = C_\ell^y + \sum_{t\geq 0}\sum_{\omega\in\Omega_y^t} \delta_\omega(y,t)\sum_{j\in\Omega}\frac{p_j}{k_{\ell,j}}(1+\mathrm{KI}_\ell(a_{\ell,j})).$$

If we set $Q(y) = \sum_{t\geq 0}\sum_{\omega\in\Omega_y^t}\delta_\omega(y,t)$ as the constant that keeps track of all the paths that lead y to z_ℓ for the first time, then we can write

$$\begin{aligned}
\mathrm{KI}_n(y) &= C_n^y + Q(y)\sum_{j\in\Omega}\frac{p_j}{k_{\ell,j}}\mathrm{KI}_n(a_{\ell,j}), \text{ for } n\neq\ell, \\
\mathrm{KI}_\ell(y) &= C_\ell^y + Q(y)\sum_{j\in\Omega}\frac{p_j}{k_{\ell,j}}(1+\mathrm{KI}_\ell(a_{\ell,j})).
\end{aligned} \tag{3.26}$$

On the other hand, for $\hat{\mathrm{KI}}_n(y)$ we get

$$\begin{aligned}
\hat{\mathrm{KI}}_n(y) &= C_n^y + Q(y)\sum_{j\in\Omega}\frac{p_j}{k_{\ell+1,j}}\hat{\mathrm{KI}}_n(b_{\ell,j}), \text{ for } n\neq\ell+1, \\
\hat{\mathrm{KI}}_{\ell+1}(y) &= C_{\ell+1}^y + Q(y)\sum_{j\in\Omega}\frac{p_j}{k_{\ell+1,j}}(1+\hat{\mathrm{KI}}_{\ell+1}(b_{\ell,j})).
\end{aligned} \tag{3.27}$$

If $b_{\ell,j}$ does not return to z_ℓ, then $\mathrm{KI}_n(b_{\ell,j}) = \hat{\mathrm{KI}}_n(b_{\ell,j})$. Set

$$B := \{j\in\Omega \,:\, \Omega_{b_{\ell,j}} \neq \emptyset\}.$$

Then

$$\hat{\mathrm{KI}}_n(y) = C_n^y + Q(y)\sum_{j\notin B}\frac{p_j}{k_{\ell+1,j}}\mathrm{KI}_n(b_{\ell,j}) + Q(y)\sum_{j\in B}\frac{p_j}{k_{\ell+1,j}}\hat{\mathrm{KI}}_n(b_{\ell,j}), \text{ for } n\neq\ell+1,$$

$$\begin{aligned}
&\hat{\mathrm{KI}}_{\ell+1}(y) = \\
&= C_{\ell+1}^y + Q(y)\sum_{j\notin B}\frac{p_j}{k_{\ell+1,j}}(1+\mathrm{KI}_{\ell+1}(b_{\ell,j})) + Q(y)\sum_{j\in B}\frac{p_j}{k_{\ell+1,j}}(1+\hat{\mathrm{KI}}_{\ell+1}(b_{\ell,j})).
\end{aligned}$$

To determine the difference between $\mathrm{KI}_n(y)$ and $\hat{\mathrm{KI}}_n(y)$, we would like an expression of $\hat{\mathrm{KI}}_n(b_{\ell,j})$ in terms of $\mathrm{KI}_n(b_{\ell,j})$ for $j\in B$. Fix $n\neq\ell+1$ for a moment and set for each $j\in B$,

$$A_j = C_n^{b_{\ell,j}} + Q(b_{\ell,j})\sum_{i\notin B}\frac{p_i}{k_{\ell+1,i}}\mathrm{KI}_n(b_{\ell,i}).$$

Then we can find expressions of $\mathrm{K\hat{I}}_n(b_{\ell,j})$ in terms of the values $\mathrm{KI}_n(b_{\ell,i})$ by solving the following system of linear equations:

$$\mathrm{K\hat{I}}_n(b_{\ell,j}) = A_j + Q(b_{\ell,j}) \sum_{i\in B} \frac{p_i}{k_{\ell+1,i}} \mathrm{K\hat{I}}_n(b_{\ell,i}), \quad j \in B.$$

A solution is easily computed through Cramer's method, which gives for $j \in B$

$$\mathrm{K\hat{I}}_n(b_{\ell,j}) = \frac{A_j\left(1 - \sum_{u\in B\setminus\{j\}} Q(b_{\ell,u})\frac{p_u}{k_{\ell+1,u}}\right) + Q(b_{\ell,j}) \sum_{u\in B\setminus\{j\}} \frac{p_u}{k_{\ell+1,u}} A_u}{1 - \sum_{i\in B} Q(b_{\ell,i})\frac{p_i}{k_{\ell+1,i}}}. \tag{3.28}$$

Set

$$B_\ell := 1 - \sum_{j\in\Omega} Q(b_{\ell,j})\frac{p_j}{k_{\ell+1,j}}.$$

Below we will use B_ℓ^{-1}. If $|Q(b_{\ell,j})| \le 1$, then

$$\left|\sum_{j\in\Omega} Q(b_{\ell,j})\frac{p_j}{k_{\ell+1,j}}\right| \le \sum_{j\in\Omega} |Q(b_{\ell,j})|\frac{p_j}{|k_{\ell+1,j}|} \le \sum_{j\in\Omega} \frac{p_j}{|k_{\ell+1,j}|} \le \rho < 1,$$

so in this case $B_\ell \neq 0$ and B_ℓ^{-1} is well defined. We now show that $|Q(b_{\ell,j})| \le 1$. If $b_{\ell,j} = z_\ell$, then $\Omega^t_{b_{\ell,j}} = \emptyset$ for any $t \ge 1$, and so $Q(b_{\ell,j}) = 1$. If $b_{\ell,j} \neq z_\ell$, then $Q(b_{\ell,j}) = \sum_{t\ge1}\sum_{\omega\in\Omega^t_{b_{\ell,j}}} \delta_\omega(b_{\ell,j},t)$. By the expanding on average property (A2), for any $y \in I$, any $t \ge 0$ and any $\omega \in \Omega^{\mathbb{N}}$,

$$|\delta_\omega(y,t)| > \sum_{j\in\Omega} |\delta_\omega(y,t)\tau_j(T_{\omega_1^t}(y),1)| = \sum_{j\in\Omega} |\delta_{\omega j}(y,t+1)|. \tag{3.29}$$

Note that by the definition of $Q(b_{\ell,j})$ the union

$$\bigcup_{t\ge1}\bigcup_{\omega\in\Omega^t_{b_{\ell,j}}} [\omega] \subseteq \Omega^{\mathbb{N}} \tag{3.30}$$

is a disjoint union of cylinder sets. Hence, by repeated application of (3.29) we obtain for each $n \ge 1$ that

$$\begin{aligned}
1 = |\delta_\epsilon(b_{\ell,j},0)| &> \sum_{i_1\in\Omega} |\delta_{i_1}(b_{\ell,j},1)| = \sum_{i_1\in\Omega_{b_{\ell,j}}} |\delta_{i_1}(b_{\ell,j},1)| + \sum_{i_1\in\Omega^c_{b_{\ell,j}}} |\delta_{i_1}(b_{\ell,j},1)| \\
&> \sum_{i_1\in\Omega_{b_{\ell,j}}} |\delta_{i_1}(b_{\ell,j},1)| + \sum_{i_1\in\Omega^c_{b_{\ell,j}}} \sum_{i_2\in\Omega} |\delta_{i_1i_2}(b_{\ell,j},2)| \\
&= \sum_{t=1}^{2} \sum_{\omega\in\Omega^t_{b_{\ell,j}}} |\delta_\omega(b_{\ell,j},t)| + \sum_{\omega\in(\Omega_{b_{\ell,j}}\cup\Omega^2_{b_{\ell,j}})^c} |\delta_\omega(b_{\ell,j},2)| \\
&> \cdots > \sum_{t=1}^{n} \sum_{\omega\in\Omega^t_{b_{\ell,j}}} |\delta_\omega(b_{\ell,j},t)| + \sum_{\omega\in(\cup_{t=1}^n\Omega^t_{b_{\ell,j}})^c} |\delta_\omega(b_{\ell,j},n)|.
\end{aligned}$$

Since this holds for each n, we get $|Q(b_{\ell,j})| \le 1$ and $B_\ell \ne 0$.

For $i \notin B$ it holds that $\mathrm{KI}_n(b_{\ell,i}) = C_n^{b_{\ell,i}}$. Then by the definition of B_ℓ, we get

$$\begin{aligned}
\sum_{j\in B} \frac{p_j}{k_{\ell+1,j}} \hat{\mathrm{KI}}_n(b_{\ell,j}) &= B_\ell^{-1} \sum_{j\in B} \frac{p_j}{k_{\ell+1,j}} A_j \\
&= B_\ell^{-1} \sum_{j\in B} \frac{p_j}{k_{\ell+1,j}} \Big(C_n^{b_{\ell,j}} + Q(b_{\ell,j}) \sum_{i\notin B} \frac{p_i}{k_{\ell+1,i}} C_n^{b_{\ell,i}} \Big) \\
&= B_\ell^{-1} \sum_{j\in B} \frac{p_j}{k_{\ell+1,j}} C_n^{b_{\ell,j}} + B_\ell^{-1}(1-B_\ell) \sum_{i\notin B} \frac{p_i}{k_{\ell+1,i}} C_n^{b_{\ell,i}} \\
&= B_\ell^{-1} \sum_{j\in\Omega} \frac{p_j}{k_{\ell+1,j}} C_n^{b_{\ell,j}} - \sum_{i\notin B} \frac{p_i}{k_{\ell+1,i}} C_n^{b_{\ell,i}}.
\end{aligned} \tag{3.31}$$

We obtain similar expressions for $n = \ell + 1$. For each $1 \le i \le N-1$, let

$$Q_i := \sum_{j\in\Omega} \Big(\frac{p_j}{k_{i,j}} Q(a_{i,j}) - \frac{p_j}{k_{i+1,j}} Q(b_{i,j}) \Big).$$

We show that for each $1 \le n \le N$ and $1 \le i \le N-1$ we have

$$\hat{\mu}_{n,i} = \mu_{n,i} - Q_i B_\ell^{-1} \mu_{n,\ell},$$

i.e., the i-th column of $\hat{M}$ is a linear combination of the i-th and the ℓ-th column of M. We give the proof only for $n \notin \{\ell, \ell+1, i, i+1\}$, since the other cases are very similar. To prove this, we first rewrite $\mu_{n,i} - Q_i B_\ell^{-1} \mu_{n,\ell}$. Therefore, note that

$$\begin{aligned}
&\sum_{j\in\Omega} \frac{p_j}{k_{\ell,j}} \mathrm{KI}_n(a_{\ell,j}) - B_\ell^{-1} \Big(\sum_{j\in\Omega} \frac{p_j}{k_{\ell,j}} \mathrm{KI}_n(a_{\ell,j}) - \sum_{j\in B} \frac{p_j}{k_{\ell+1,j}} Q(b_{\ell,j}) \sum_{i\in\Omega} \frac{p_i}{k_{\ell,i}} \mathrm{KI}_n(a_{\ell,i}) \Big) \\
&\qquad = \sum_{j\in\Omega} \frac{p_j}{k_{\ell,j}} \mathrm{KI}_n(a_{\ell,j}) (1 - B_\ell^{-1} B_\ell) = 0.
\end{aligned}$$

Then we obtain from the definition of M, (3.26) and the above equation that

$$\begin{aligned}
\mu_{n,i} - Q_i B_\ell^{-1} \mu_{n,\ell} = &\sum_{j\in\Omega} \Big(\frac{p_j}{k_{i,j}} C_n^{a_{i,j}} - \frac{p_j}{k_{i+1,j}} C_n^{b_{i,j}} \Big) + Q_i \sum_{j\in\Omega} \frac{p_j}{k_{\ell,j}} \mathrm{KI}_n(a_{\ell,j}) \\
&- Q_i B_\ell^{-1} \sum_{j\in\Omega} \frac{p_j}{k_{\ell,j}} \mathrm{KI}_n(a_{\ell,j}) + Q_i B_\ell^{-1} \sum_{j\notin B} \frac{p_j}{k_{\ell+1,j}} \mathrm{KI}_n(b_{\ell,j}) \\
&+ Q_i B_\ell^{-1} \sum_{j\in B} \frac{p_j}{k_{\ell+1,j}} \Big(C_n^{b_{\ell,j}} + Q(b_{\ell,j}) \sum_{u\in\Omega} \frac{p_u}{k_{\ell,u}} \mathrm{KI}_n(a_{\ell,u}) \Big) \\
= &\sum_{j\in\Omega} \Big(\frac{p_j}{k_{i,j}} C_n^{a_{i,j}} - \frac{p_j}{k_{i+1,j}} C_n^{b_{i,s}} \Big) + Q_i B_\ell^{-1} \sum_{j\in\Omega} \frac{p_j}{k_{\ell+1,j}} C_n^{b_{\ell,j}}.
\end{aligned}$$

For $\hat{\mu}_{n,i}$ we get by combining (3.27) and (3.31) that

$$\begin{aligned}
\hat{\mu}_{n,i} = &\sum_{j\in\Omega} \Big(\frac{p_j}{k_{i,j}} C_n^{a_{i,j}} + \frac{p_j}{k_{i+1,j}} C_n^{b_{i,j}} \Big) + Q_i \sum_{j\notin B} \frac{p_j}{k_{\ell+1,j}} \mathrm{KI}_n(b_{\ell,j}) \\
&+ Q_i B_\ell^{-1} \sum_{j\in\Omega} \frac{p_j}{k_{\ell+1,j}} C_n^{b_{\ell,j}} - Q_i \sum_{j\notin B} \frac{p_j}{k_{\ell+1,j}} C_n^{b_{\ell,j}} = \mu_{n,i} - Q_i B_\ell^{-1} \mu_{n,\ell}.
\end{aligned}$$

One now easily checks that if $\gamma = (\gamma_1, \dots, \gamma_{N-1})^\intercal$ is a solution of $M\gamma = 0$, then the vector $\hat{\gamma} = (\hat{\gamma}_1, \dots, \hat{\gamma}_{N-1})^\intercal$ given by

$$\hat{\gamma}_\ell = \gamma_\ell + \sum_{i=1}^{N-1} \frac{Q_i}{B_\ell - Q_\ell} \gamma_i \tag{3.32}$$

and $\hat{\gamma}_i = \gamma_i$ if $i \neq \ell$, satisfies $\hat{M}\hat{\gamma} = 0$. The fact that $B_\ell - Q_\ell \neq 0$ follows in the same way as that $B_\ell \neq 0$. Hence, there is a 1-to-1 relation between the solutions γ of $M\gamma = 0$ and $\hat{\gamma}$ of $\hat{M}\hat{\gamma} = 0$.

It remains to prove that the functions h_γ and $\hat{h}_{\hat{\gamma}}$ coincide. For that we need to consider the functions L_y. As we did for KI_n, let L^y denote the parts that L_y and $\hat{L}_y$ have in common, i.e., set

$$L^y = \sum_{t \geq 0} \sum_{\omega \in \Omega_y^t \cup \Omega^t \setminus \Omega_y} \delta_\omega(y,t) \mathbf{1}_{[0, T_\omega(y))}.$$

Set $A := \{j \in \Omega \, : \, \Omega_{a_{\ell,j}} \neq \emptyset\}$. Then

$$\begin{aligned}
L_y &= L^y + Q(y) \sum_{t \geq 1} \sum_{\omega \in \Omega^t} \delta_\omega(z_\ell, t) \mathbf{1}_{[0, \hat{T}_\omega(z_\ell))} \\
&= L^y + Q(y) \Big(\sum_{j \in \Omega} \mathbf{1}_{[0, a_{\ell,j})} + \sum_{t \geq 1} \sum_{\omega \in \Omega^t} \frac{p_j}{k_{\ell,j}} \delta_\omega(b_{\ell,j}, u) \mathbf{1}_{[0, \hat{T}_\omega(a_{\ell,j}))} \Big) \\
&= L^y + Q(y) \sum_{j \notin A} \frac{p_j}{k_{\ell,j}} L_{a_{\ell,j}} + Q(y) \sum_{j \in A} \frac{p_j}{k_{\ell,j}} L_{a_{\ell,j}}.
\end{aligned}$$

By Cramer's rule we obtain for each $j \in A$, that (compare (3.31))

$$\sum_{j \in A} \frac{p_j}{k_{\ell,j}} L_{a_{\ell,j}} = (B_\ell - Q_\ell)^{-1} \sum_{j \in \Omega} \frac{p_j}{k_{\ell,j}} L^{a_{\ell,j}} - \sum_{j \notin A} \frac{p_j}{k_{\ell,j}} L_{a_{\ell,j}}. \tag{3.33}$$

Similarly, we obtain that

$$\hat{L}_y = L^y + Q(y) \sum_{j \notin B} \frac{p_j}{k_{\ell+1,j}} L_{b_{\ell,j}} + Q(y) \sum_{j \in B} \frac{p_j}{k_{\ell+1,j}} \hat{L}_{b_{\ell,j}} \tag{3.34}$$

and

$$\sum_{j \in B} \frac{p_j}{k_{\ell+1,j}} \hat{L}_{b_{\ell,j}} = B_\ell^{-1} \sum_{j \in \Omega} \frac{p_j}{k_{\ell+1,j}} L^{b_{\ell,j}} - \sum_{j \notin B} \frac{p_j}{k_{\ell+1,j}} L^{b_{\ell,j}}. \tag{3.35}$$

To prove that $h_\gamma = \hat{h}_{\hat{\gamma}}$, note that on the one hand,

$$h_\gamma = \sum_{m=1}^{N-1} \gamma_m \sum_{j \in \Omega} \Big(\frac{p_j}{k_{m,j}} L^{a_{m,j}} - \frac{p_j}{k_{m+1,j}} L^{b_{m,j}} \Big) + \sum_{m=1}^{N-1} \gamma_m Q_m \sum_{j \in \Omega} \frac{p_j}{k_{\ell,j}} L_{a_{\ell,j}}.$$

On the other hand, using equations (3.32), (3.34) and (3.35) we obtain for $\hat{h}_{\hat{\gamma}}$ that

$$\begin{aligned}
\hat{h}_{\hat{\gamma}} = {} & \sum_{m=1}^{N-1} \gamma_m \sum_{s\in\Omega} \Big(\frac{p_s}{k_{m,s}} L^{a_{m,s}} - \frac{p_s}{k_{m+1,s}} L^{b_{m,s}} \Big) \\
& + \sum_{m=1}^{N-1} \gamma_m Q_m \Big(1 + \frac{Q_\ell}{B_\ell - Q_\ell} \Big) \sum_{s\in\Omega} \frac{p_s}{k_{\ell+1,s}} \hat{L}_{b_{\ell,s}} \\
& + \sum_{m=1}^{N-1} \gamma_m \frac{Q_m}{B_\ell - Q_\ell} \sum_{s\in\Omega} \Big(\frac{p_s}{k_{\ell,s}} L^{a_{\ell,s}} - \frac{p_s}{k_{\ell+1,s}} L^{b_{\ell,s}} \Big) \\
= {} & \sum_{m=1}^{N-1} \gamma_m \sum_{s\in\Omega} \Big(\frac{p_s}{k_{m,s}} L^{a_{m,s}} - \frac{p_s}{k_{m+1,s}} L^{b_{m,s}} \Big) \\
& + \sum_{m=1}^{N-1} \gamma_m Q_m \frac{B_\ell}{B_\ell - Q_\ell} B_\ell^{-1} \sum_{s\in\Omega} \frac{p_s}{k_{\ell+1,s}} L^{b_{\ell,s}} \\
& + \sum_{m=1}^{N-1} \gamma_m \frac{Q_m}{B_\ell - Q_\ell} \sum_{s\in\Omega} \Big(\frac{p_s}{k_{\ell,s}} L^{a_{\ell,s}} - \frac{p_s}{k_{\ell+1,s}} L^{b_{\ell,s}} \Big) \\
= {} & \sum_{m=1}^{N-1} \gamma_m \sum_{s\in\Omega} \Big(\frac{p_s}{k_{m,s}} L^{a_{m,s}} - \frac{p_s}{k_{m+1,s}} L^{b_{m,s}} \Big) + \sum_{m=1}^{N-1} \gamma_m \frac{Q_m}{B_\ell - Q_\ell} \sum_{s\in\Omega} \frac{p_s}{k_{\ell,s}} L^{a_{\ell,s}}.
\end{aligned}$$

By (3.33) this implies that $h_\gamma = \hat{h}_{\hat{\gamma}}$.

If the partitions $\{I_n\}_{1\le n\le N}$ and $\{\hat{I}_n\}_{1\le n\le N}$ differ in more than one partition point z_ℓ, we can obtain the results from the above by changing one partition point at a time. □

The next lemma states that adding extra points to the set $z_0, \ldots, z_N$ does not influence the set of densities obtained from Theorem 3.4.1. This lemma is one of the ingredients of the proof of Theorem 3.5.3 below.

3.5.2 Lemma. *Let T be a random system with partition $\{I_i\}_{1\le i\le N}$ and corresponding partition points $z_0, \ldots, z_N$. Consider a refinement of the partition, given by adding extra points $z_1^\dagger, \ldots, z_s^\dagger$, for some $s \in \mathbb{N}$. Let $T^\dagger$ be the corresponding random system, i.e., $T^\dagger(x) = T(x)$ for all $x \in [0,1]$, and let $M^\dagger$ be the fundamental matrix of $T^\dagger$. There is a 1-to-1 correspondence between the solutions γ of $M\gamma = 0$ and the solutions $\gamma^\dagger$ of $M^\dagger\gamma^\dagger = 0$. Moreover, the functions h_γ and $h^\dagger_{\gamma^\dagger}$ coincide.*

Proof. Let $Z^\dagger := \{z_1^\dagger, \ldots, z_s^\dagger\}$. By introducing these extra points the fundamental matrix $M^\dagger$ of $T^\dagger$ becomes an $(N+s)\times(N+s-1)$ matrix. It is possible to construct this matrix from M in s steps

$$M \to M_1^\dagger \to M_2^\dagger \to \cdots \to M_s^\dagger = M^\dagger,$$

by adding one of the points from $Z^\dagger$ to the partition of T at a time. All of these steps work in exactly the same way, so it is enough to prove the result for $s = 1$. Therefore, assume $Z^\dagger = \{z^\dagger\}$. There is an $1 \leq i \leq N$ such that $z^\dagger$ splits the interval I_i into two subintervals, say I_i^L and I_i^R. By Proposition 3.5.1, it is irrelevant whether $z^\dagger \in I_i^L$ or $z^\dagger \in I_i^R$. By construction, $z^\dagger$ is a continuity point of $T^\dagger = T$, so

$$a_{i,j}^\dagger = b_{i,j}^\dagger = k_{i,j} z^\dagger + d_{i,j},$$

and for each n we have

$$\sum_{j \in \Omega} \left[\frac{p_j}{k_{i,j}} \mathrm{KI}_n(a_{i,j}^\dagger) - \frac{p_j}{k_{i,j}} \mathrm{KI}_n(b_{i,j}^\dagger) \right] = 0.$$

Therefore $M^\dagger$ has, with respect to M, an extra column at the ith position, whose entries are all zeroes except for the diagonal and subdiagonal entries, which are given by $\sum_{j\in\Omega} \frac{p_j}{k_{i,j}}$ and $-\sum_{j\in\Omega} \frac{p_j}{k_{i,j}}$, respectively. Moreover, the ith and $(i+1)$th row of $M^\dagger$ are obtained by splitting the ith row of M into two, such that $\mathrm{KI}_i(a_{n,j}) = \mathrm{KI}_i^\dagger(a_{n,j}) + \mathrm{KI}_{i+1}^\dagger(a_{n,j})$ for all n, and analogously for $b_{n,j}$.

The null space of $M^\dagger$ equals the null space of the $(N+1) \times N$ matrix A obtained from $M^\dagger$ by replacing the $(i+1)$th row by the sum of the ith and the $(i+1)$th row. Then all the entries of the ith column of A are 0 except for the diagonal entry, and the matrix M appears as a submatrix of A, by deleting the ith column and the ith row. Hence, any solution γ of $M\gamma = 0$ can be transformed in a solution $\gamma^\dagger$ of $M^\dagger\gamma^\dagger = 0$ by setting $\gamma_j^\dagger = \gamma_j$ for $j \neq i$ and by using the relation $\sum_{j=1}^N A_{i,j}\gamma_j^\dagger = 0$ for $\gamma_i^\dagger$. This gives the first part of the lemma.

Finally, for corresponding solutions γ and $\gamma^\dagger$ the associated densities h_γ and $h_{\gamma^\dagger}^\dagger$ coincide, since

$$\sum_{j \in \Omega} \left[\frac{p_j}{k_{i,j}} L_{a_{i,j}^\dagger}(x) - \frac{p_j}{k_{i,j}} L_{b_{i,j}^\dagger}(x) \right] = 0.$$

□

The next theorem says that in case all maps T_j are expanding, Theorem 3.4.1 in fact produces all absolutely continuous invariant measures for the system T.

3.5.3 Theorem. *Let $\Omega \subseteq \mathbb{N}$ and let T be a random piecewise affine system satisfying assumptions (A1), (A3) and (A4). Assume furthermore that $|k_{i,j}| > 1$ for each $j \in \Omega$ and $1 \leq i \leq N$. An $L^1(\lambda)$-function h is an invariant function for the random system T if and only if $h = h_\gamma$ for some solution γ of the system $M\gamma = 0$.*

An essential ingredient in the proof of this theorem is the extension of a result by Boyarksy, Góra and Islam from [GBI06] given in the next lemma. [GBI06, Theorem 3.6] states that in case we have a random system consisting of two maps that are both expanding, the supports of the invariant densities of T are a finite union of intervals. As the next lemma shows, this result in fact goes through for any finite or countable number of maps with only a small change in the proof. In case of piecewise affine maps, some small steps can be simplified a bit. We have included the proof for the convenience of the reader.

3.5.4 Lemma (cf. Lemma 3.4 and Theorem 3.6 from [GBI06]). *Let $\Omega \subseteq \mathbb{N}$ and let T be a random system of piecewise affine maps satisfying (A1) and such that for each $j \in \Omega$ the map T_j is expanding, i.e., it satisfies $|k_{i,j}| > 1$ for all $1 \leq i \leq N$. If h is a T-invariant density, then the support of h is a finite union of open intervals.*

Proof. Let $H = \{v_1, \dots, v_r\}$ be the base of the subspace of $L^1(\lambda)$ of T-invariant functions, consisting of density functions of bounded variation, mentioned in Section 3.2. Since any invariant function h for T can be written as $h = \sum_{n=1}^{r} c_n v_n$ for some constants $c_n \in \mathbb{R}$, it is enough to prove the result for elements in H. Therefore, let $h \in H$ and let $U := \operatorname{supp}(h)$ denote the support of h. Since h is a function of bounded variation, we can take h to be lower semicontinuous and U can be written as a countable union of open intervals, each separated by an interval of positive length: $U = \bigcup_{k \geq 1} U_k$. Assume without loss of generality that $\lambda(U_{k+1}) \leq \lambda(U_k)$ for each $k \geq 1$. Let $Z := \{z_1, \dots, z_{N-1}\}$ and let $\mathcal{D}$ be the set of indices k, such that U_k contains one of the points $z \in Z$, i.e.,

$$\mathcal{D} = \{k \geq 1 \,|\, \exists\, z \in Z \,:\, z \in U_k\}.$$

We first show that $\mathcal{D} \neq \emptyset$ by proving that $Z \cap U_1 \neq \emptyset$. Suppose on the contrary that U_1 does not contain a point z, then for each $j \in \Omega$, $T_j(U_1)$ is an interval and since each T_j is expanding, we have $\lambda(T_j(U_1)) > \lambda(U_1)$. By the property from (3.3) that U is forward invariant, we know that $T_j(U_1) \subseteq U$ for each j, so it must be contained in one of the intervals U_k. This gives a contradiction.

Now, let J be the smallest interval in the set

$$\{U_k \cap I_n \,:\, k \in \mathcal{D},\, 1 \leq n \leq N\}.$$

Note that this is a finite set, since Z and $\mathcal{D}$ are both finite. Moreover, by the above this set is not empty, so J exists. Since each U_k is an open interval, we have $\lambda(J) > 0$. Let $\mathcal{F} = \{k \geq 1 \,:\, \lambda(U_k) \geq \lambda(J)\}$, where k is not necessarily in J, and let $S = \bigcup_{k \in \mathcal{F}} U_k$. Since any connected component U_k of S has Lebesgue measure bigger than $\lambda(J)$, S is a finite union of open intervals. We first prove that $T_j(S) \subseteq S$ for any $j \in \Omega$. Let $U_k \subseteq S$ and suppose first that $k \notin \mathcal{D}$. Then for each $j \in \Omega$, as above $T_j(U_k)$ is an interval with $\lambda(T_j(U_k)) > \lambda(U_k) \geq \lambda(J)$. So, $T_j(U_k)$ is contained in another interval U_i that satisfies $\lambda(U_i) > \lambda(J)$ and thus satisfies $U_i \subseteq S$. Hence, $T_j(U_k) \subseteq S$. If, on the other hand, $k \in \mathcal{D}$, then $T_j(U_k)$ consists of a finite union of intervals and since T_j is expanding, the Lebesgue measure of each of these intervals exceeds $\lambda(J)$. Hence, each of the connected components of $T_j(U_k)$ is contained in some interval U_i that satisfies $\lambda(U_i) > \lambda(J)$ and therefore $U_i \subseteq S$. Hence, also in this case $T_j(U_k) \subseteq S$, implying that $T_j(S) \subseteq S$ for all $j \in \Omega$.

Obviously, $S \subseteq U$. Using the fact that $T_j(S) \subseteq S$ for all $j \in \Omega$, we will now show that $U \subseteq S$. Suppose this is not the case and let U_s be the largest interval in $U \setminus S$. Since $U_k \subseteq S$ for any $k \in \mathcal{D}$, we have $s \notin \mathcal{D}$. So, again, for each $j \in \Omega$ the set $T_j(U_s)$ is an interval with $\lambda(T_j(U_s)) > \lambda(U_s)$ and hence, $T_j(U_s) \subseteq S$. Thus $U_s \subseteq T_j^{-1}(S)$ and since $U_s \not\subseteq S$, we have $U_s \subseteq T_j^{-1}(S) \setminus S$. Let $\mu_{\mathbf{p}}$ be the absolutely continuous

T-invariant measure with density h. We show that $\mu_{\mathbf{p}}(T_j^{-1}(S) \setminus S) = 0$. Since for each $j \in \Omega$ we have

$$S \subseteq T_j^{-1}(T_j(S)) \subseteq T_j^{-1}(S),$$

we obtain from (3.1) that

$$\begin{aligned} 0 = \mu_{\mathbf{p}}(S) - \mu_{\mathbf{p}}(S) &= \sum_{j\in\Omega} p_j \mu_{\mathbf{p}}(T_j^{-1}(S)) - \sum_{j\in\Omega} p_j \mu_{\mathbf{p}}(S) \\ &= \sum_{j\in\Omega} p_j(\mu_{\mathbf{p}}(T_j^{-1}(S)) - \mu_{\mathbf{p}}(S)) = \sum_{j\in\Omega} p_j \mu_{\mathbf{p}}(T_j^{-1}(S) \setminus S). \end{aligned}$$

Since $p_j > 0$ for all j, we have that $\mu_{\mathbf{p}}(T_j^{-1}(S) \setminus S) = 0$ for each j. Hence, $\mu_{\mathbf{p}}(U_s) = 0$, which contradicts the fact that $U_s \subseteq U$. □

3.5.5 Remark. The article [GBI06] contains an example that shows that the previous lemma is not necessarily true if we drop the assumption that all maps T_j are expanding. In [GBI06, Example 3.7] the authors describe a random system T using an expanding and a non-expanding map, of which for a certain probability vector $\mathbf{p}$ the support of the invariant density is a countable union of intervals. The fact that the supports of the elements from H are finite unions of open intervals plays an essential role in the proof of Theorem 3.5.3 as we shall see now.

Proof of Theorem 3.5.3. We will show that the linear mapping from the null space of M to the subspace of $L^1(\lambda)$ of all T-invariant functions is a linear isomorphism. Let $H = \{v_1, \dots, v_r\}$ again be the basis of density functions of bounded variation, whose corresponding measures are ergodic, for the subspace of T-invariant $L^1(\lambda)$-functions mentioned in Section 3.2. Recall that any invariant function h for T can be written as $h = \sum_{n=1}^r c_n v_n$ for some constants $c_n \in \mathbb{R}$.

The injectivity follows from the proof of Theorem 3.4.1, where we showed that $h_\gamma = 0$ implies $\gamma = 0$. We prove surjectivity by providing for each $h \in H$ a vector γ such that $h_\gamma = h$. We will do this by altering T in several steps, so that we finally obtain a system T_U that has a vector γ_U associated to it for which the corresponding density $h^U_{\gamma_U}$ vanishes outside the support U of h. Then, using Proposition 3.5.1 and Lemma 3.5.2 we transform the solution γ_U to a solution γ for T that produces the original density h.

Fix $h \in H$, and let $U := \mathrm{supp}(h)$. Let $Z = \{z_1, \dots, z_{N-1}\}$ again be the set of critical points of the system. Following [K90, Theorem 2], we classify the points in Z as follows:

$$\begin{aligned} Z_1 &= \{z_i \in Z \,|\, z_i \text{ is in the interior of } U\}, \\ Z_2 &= \{z_i \in Z \,|\, z_i \text{ is a left/right endpoint of a subinterval of } U \text{ and } z_i \in I_{i+1}\,(z_i \in I_i)\}, \\ Z_3 &= \{z_i \in Z \,|\, z_i \text{ is a left/right endpoint of a subinterval of } U \text{ and } z_i \in I_i\,(z_i \in I_{i+1})\}, \\ Z_4 &= \{z_i \in Z \,|\, z_i \text{ is an exterior point for } U\}. \end{aligned}$$

We now modify the partition $\{I_i\}_{1\le i\le N}$ on the points in Z_3, so that it corresponds better to the set U. Let $\{\hat{I}_i\}_{1\le i\le N}$ be a partition of $[0,1]$ given by $z_0, \dots, z_N$ and

$$\boxed{\begin{array}{c}\{I_i\}_{1\le i\le N}\\ T\\ M\gamma=0\end{array}} \longrightarrow \boxed{\begin{array}{c}\{\hat{I}_i\}_{1\le i\le N}\\ \hat{T}\\ \hat{M}\hat{\gamma}=0\end{array}} \longrightarrow \boxed{\begin{array}{c}\{\hat{I}_i^\dagger\}_{1\le i\le N}\\ \hat{T}^\dagger\\ \hat{M}^\dagger\hat{\gamma}^\dagger=0\end{array}} \longrightarrow \boxed{\begin{array}{c}\{\hat{I}_i^\dagger\}_{1\le i\le N}\\ T_U\\ M_U\gamma_U=0\end{array}}$$

Figure 3.2: The steps we take in transforming T to T_U.

differing from $\{I_i\}_{1\le i\le N}$ only for $z_i \in Z_3$, i.e., $z_i \in \hat{I}_i$ if and only if $z_i \notin I_i$. Let $\hat{T}$ be the corresponding random system, i.e., $\hat{T}(x) = T(x)$ for all $x \notin Z_3$. By Proposition 3.5.1, the corresponding matrices M and $\hat{M}$ have vectors in their null spaces that differ only on the entries i for which $z_i \in Z_3$, but such that they define the same density.

There might be boundary points of U that are not in Z. Let $Z^\dagger$ be the set of such points. From Lemma 3.5.4 it follows that U is a finite union of open intervals, so the set $Z^\dagger$ is finite. Consider the partition $\{\hat{I}_i^\dagger\}$ given by the points in $Z \cup Z^\dagger$ and let $\hat{T}^\dagger$ be the system with this partition and given by $\hat{T}^\dagger(x) = \hat{T}(x)$ for all x. By Lemma 3.5.2, the corresponding matrices $\hat{M}$ and $\hat{M}^\dagger$ have vectors in their null spaces that differ only on the extra entries corresponding to points $z^\dagger \in Z^\dagger$, but such that they define the same density.

Define a new piecewise affine random system T_U by modifying $\hat{T}^\dagger$ outside of U. To be more precise, we let $T_U(x) = \hat{T}^\dagger(x)$ for all $x \in U$ and on each connected component of $[0,1] \setminus U$ we assume all maps $T_{U,j}$ to be equal and onto, i.e., mapping the interval onto $[0,1]$. Recall from (3.3) that the set U is forward invariant under T. Then any invariant function of T_U vanishes on $[0,1] \setminus U$ λ-almost everywhere, since the set of points $x \in [0,1] \setminus U$, such that $T^n(x) \in [0,1] \setminus U$ for all $n \ge 0$ is a self-similar set of Hausdorff dimension less than 1. From Theorem 3.4.1 we get a non-trivial solution γ_U of $M_U\gamma_U = 0$ with a corresponding function h_U that vanishes on $[0,1]\setminus U$. Since $\hat{T}$ and T_U coincide on U, the function h_U is also invariant for $\hat{T}$ and hence for T. From the fact that U is the support of one of the densities in the basis H and $\operatorname{supp}(h_U) \subseteq U$, we then conclude that $h_U = h$, up to possibly a set of Lebesgue measure 0.

It remains to show that γ_U can be transformed into a vector from the null space of M, leading to the same density h_U. We first show that $\hat{M}^\dagger\gamma_U = 0$. Note that for $z_i \in Z_4$, since h_U is of bounded variation,

$$\lim_{x\uparrow z_i} h_U(x) = 0 = \lim_{x\downarrow z_i} h_U(x).$$

Hence, by the calculations in (3.25) $\gamma_{U,i} = 0$. Similarly, for $z_i \in Z_2 \cup Z_3$ we have that either $\lim_{x\uparrow z_i} h_U(x) = 0$ or $\lim_{x\downarrow z_i} h_U(x) = 0$, which again by the calculations in (3.25) gives $\gamma_{U,i} = 0$. Hence, $\gamma_{U,i} = 0$ for each i such that $z_i \in Z_2 \cup Z_3 \cup Z_4$. Similarly, $\gamma_{U,i} = 0$ for each i such that $z_i \in Z^\dagger$. In the multiplication $\hat{M}^\dagger\gamma_U$ the orbits of the points $a_{i,j}$ and $b_{i,j}$ which are different under $\hat{T}^\dagger$ and T_U are multiplied by 0. Since U is forward invariant, all orbits of points $a_{i,j}$ and $b_{i,j}$ corresponding to i such that $z_i \in Z_1$ will stay in U and will thus be equal under $\hat{T}^\dagger$ and T_U. These facts imply that also $\hat{M}^\dagger\gamma_U = 0$ and that the corresponding invariant density for $\hat{T}^\dagger$ is again h_U.

From Lemma 3.5.2 it follows that there is a vector $\hat{\gamma}$ in the null space of $\hat{M}$ with $\hat{h}_{\hat{\gamma}} = h_U$. Finally, Proposition 3.5.1 then tells us how we can modify $\hat{\gamma}$ to get a vector γ in the null space of M with $h_\gamma = \hat{h}_{\hat{\gamma}} = h_U = h$. $\square$

§3.6 Examples

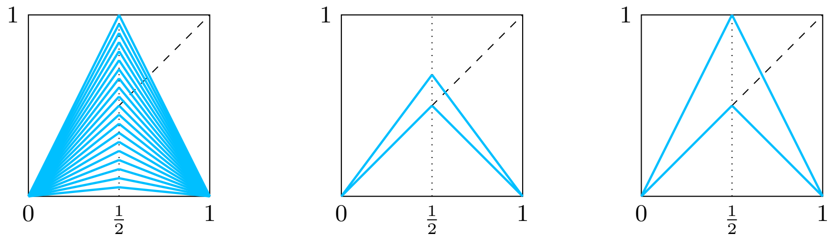
We apply Theorems 3.4.1 and 3.5.3 to various examples.

§3.6.1 Random tent maps

For any countable set of slopes $\{k_j\}_{j\in\Omega}$ with $k_j \in (0,2)$ for each j, consider the family $T := \{T_j\}_{j\in\Omega}$, where each T_j is a tent map of slope k_j, i.e., $T_j : [0,1] \to [0,1]$ is given by

$$T_j(x) = \begin{cases} k_j x, & \text{if } x \in [0,1/2], \\ k_j - k_j x, & \text{if } x \in (1/2,1], \end{cases}$$

see Figure 3.3(a). Let $\mathbf{p} = (p_j)_{j\geq 0}$ be a probability vector such that T is expanding



(a) Countably many tent maps. (b) Two tent maps. (c) Linear logistic maps.

Figure 3.3: Random families of tent maps.

on average, i.e. $\sum_{j\in\mathbb{N}} \frac{p_j}{k_j} < 1$, so (A2) holds. One easily verifies that then conditions (A3) and (A5) hold as well. For $N = 2$ set

$$z_0 = 0, \quad z_1 = \frac{1}{2}, \quad z_2 = 1,$$

and $I_1 = [z_0, z_1]$, $I_2 = (z_1, z_2]$. Since z_1 is the only discontinuity point, the fundamental matrix M is the null vector. As a consequence, we can choose $\gamma = 1$, to obtain the invariant density

$$h_\gamma = c \sum_{j\in\Omega} \frac{2p_j}{k_j} L_{k_j/2},$$

for some normalising constant c. If we set for each $j \in \mathbb{N}$ and $w \in \Omega^t$, $t \geq 0$,

$$\ell_{\omega,j} = \#\Big\{1 \leq n \leq t \,:\, T_{\omega_1^{n-1}}\Big(\frac{k_j}{2}\Big) \in \Big(\frac{1}{2}, 1\Big]\Big\},$$

then this becomes

$$h_\gamma = c \sum_{j\in\Omega} \frac{2p_j}{k_j} \sum_{t\geq 0} \sum_{\omega\in\Omega^t} (-1)^{\ell_{\omega,j}} \prod_{n=0}^{t} \frac{p_{\omega_n}}{k_{\omega_n}} \mathbf{1}_{[0,T_\omega(\frac{k_j}{2}))}. \tag{3.36}$$

If we assume that $k_j > 1$ for all j, then it follows from Theorem 3.5.3 that the density from (3.36) is the unique invariant density for $(T, \mathbf{p})$. If we do not assume this, then we can still draw the same conclusion in case there are only finitely many maps. Namely, to satisfy the condition (A2) there has to be at least one j such that $k_j > 1$. The existence and uniqueness of an absolutely continuous invariant measure for the map T_j is then guaranteed by the results from [LY73, LY78]. In case the set $\{k_j\}_{j\in\mathbb{N}}$ is finite, it then follows from [P84, Corollary 7] that there is only one invariant density for $(T, \mathbf{p})$.

In [AGH18] the authors considered random combinations of logistic maps. In [AGH18, Theorem 4.2] they proved that the random system $\{f_0, f_1\}$ with $f_0(x) = 2x(1-x)$ and $f_1(x) = 4x(1-x)$ has a σ-finite absolutely continuous invariant measure that is infinite in case the map f_0 is chosen with probability $p_0 > \frac{1}{2}$. The linear analogue of this system shows a different picture. Fix $a \in (1,2]$ and consider the random system with two maps $T_0(x) = \min\{x, 1-x\}$ and $T_{a,1}(x) = \min\{ax, a-ax\}$. See Figure 3.3(b) for an example with $a = \frac{4}{3}$. For any $p \in (0,1)$, set $p_0 = p$ and $p_1 = 1-p$ and note that $p_0 + \frac{p_1}{a} < 1$. The assumptions (A1)-(A5) are then met and the random system $T = \{T_0, T_{a,1}\}$ has a finite absolutely continuous invariant measure for any such p. A straightforward computation yields $L_{\frac{1}{2}} = \frac{1}{1-p}\mathbf{1}_{[0,\frac{1}{2})} + \frac{1}{a}L_{\frac{a}{2}}$, so that up to a normalising constant, the unique absolutely continuous invariant density is then

$$h_{\gamma,a} = \frac{2p}{1-p}\mathbf{1}_{[0,\frac{1}{2})} + \frac{2}{a}L_{\frac{a}{2}}. \tag{3.37}$$

In particular, for $a = 2$ as shown in Figure 3.3(c) we get

$$h_{\gamma,2} = (1+p)\mathbf{1}_{[0,\frac{1}{2}]} + (1-p)\mathbf{1}_{(\frac{1}{2},1]}.$$

Note that for $p = 1$ we have a deterministic, non-expanding interval map that does not satisfy the requirements from [K90]. However, the limit $\lim_{p\to 1} h_{\gamma,2} = 2\cdot\mathbf{1}_{[0,\frac{1}{2}]}$ is an invariant density for the system. On the other hand, for a fixed $p \in (0,1)$ the limit $\lim_{a\to 1} h_{\gamma,a}$ is not an absolutely continuous measure. To see this, note that $h_{\gamma,a}$ is determined by the random orbits of $\frac{a}{2}$ and that $1 - \frac{a}{2} \leq T_\omega(\frac{a}{2}) \leq \frac{a}{2}$ for any ω. Hence, by (3.37) and the definition of the L-functions in (3.17) it follows that $h_{\gamma,a} = 0$ on $(\frac{a}{2}, 1]$, while on $[0, 1-\frac{a}{2})$ we have $h_{\gamma,a} = v$ on $[0, 1-\frac{a}{2})$ for some constant $v \in \mathbb{R}$. For any point in $x \in [0, 1-\frac{a}{2})$, the random Perron-Frobenius operator from (3.2) now yields

$$v = h_{\gamma,a}(x) = P_T h_{\gamma,a}(x) = pv + (1-p)\frac{v}{a},$$

which holds if and only if $v = 0$. It follows that for any $a \in (1,2]$ and any $p \in (0,1)$, $\mathrm{supp}(h_{\gamma,a}) \subseteq [1-\frac{a}{2}, \frac{a}{2}]$. As a consequence $\lim_{a\to 1} h_\gamma = \delta_{\frac{1}{2}}$, where $\delta_{\frac{1}{2}}$ is the Dirac delta function at $\frac{1}{2}$.

§3.6.2 A random family of W-shaped maps

Keller introduced in [K82] a family of piecewise expanding W-maps to study the phenomenon of instability of absolutely continuous invariant measures. Later the stability of W-shaped maps was studied in other papers as well, see for example [LGB+13, EM12]. Here we construct a random family of W-shaped maps, where each element of the collection is an expanding on average random map $W_a := \{W_{a,0}, W_{a,1}\}$ defined on the unit interval. We give an absolutely continuous invariant probability measure.

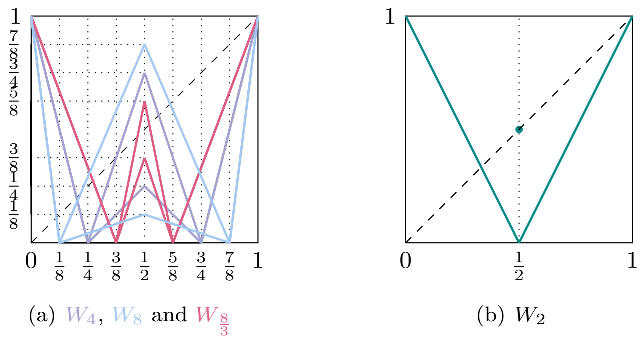


Figure 3.4: Examples of random systems W_a for various values of a.

For $a > 2$, let $\Omega = \{0, 1\}$ and $N = 4$. Set

$$z_0 = 0, \quad z_1 = 1/a, \quad z_2 = 1/2, \quad z_3 = (a-1)/a, \quad z_4 = 1$$

and

$$I_1 = [z_0, z_1], \quad I_2 = (z_1, z_2], \quad I_3 = (z_2, z_3), \quad I_4 = [z_3, z_4].$$

Let

$$W_{a,0}(x) = \begin{cases} 1 - ax, & \text{if } x \in I_1, \\ \frac{2}{a-2}x - \frac{2}{(a-2)a}, & \text{if } x \in I_2, \\ W_{a,0}(1-x), & \text{otherwise,} \end{cases}$$

and

$$W_{a,1}(x) = \begin{cases} 1 - ax, & \text{if } x \in I_1, \\ \frac{2(a-1)}{a-2}x - \frac{2(a-1)}{(a-2)a}, & \text{if } x \in I_2, \\ W_{a,1}(1-x), & \text{otherwise.} \end{cases}$$

For $a > 4$ the map $W_{a,0}$ presents two contractive branches. Let $1 > p > \frac{(a-4)(a-1)}{(a-2)^2}$ be arbitrary, and let $p_{a,0} = 1 - p$ and $p_{a,1} = p$. With this choice of probability vector the random map W_a satisfies (A1)-(A5). The fundamental matrix M is given by

$$M = \begin{pmatrix} \frac{1-a}{a^2} - \frac{C}{a} & \frac{p_{a0}(2-a)(a-1)}{a^2} + \frac{p_{a1}(2-a)}{a^2(a-1)} & -\frac{C}{a} + \frac{1}{a^2} \\ C & -C & 0 \\ 0 & -C & C \\ \frac{1}{a^2(a-1)} - \frac{C}{a(a-1)} & \frac{p_{a0}(2-a)}{a^2(a-1)} - \frac{p_{a1}(2-a)(a^2-a-1)}{a^2(a-1)^2} & -\frac{C}{a(a-1)} + \frac{1+a-a^2}{a^2(a-1)} \end{pmatrix}$$

for some constant C. Its null space consists of all vectors of the form

$$s\begin{pmatrix}1 & 1 & 1\end{pmatrix}^{\intercal}, \quad s \in \mathbb{R}.$$

From

$$L_0 = \frac{1}{1-a}, \qquad L_{\frac{1}{a}} = \frac{1}{a(a-1)} + \mathbf{1}_{[0,\frac{1}{a}]} \quad \text{and} \quad L_{\frac{a-1}{a}} = -\frac{1}{a(a-1)} + \mathbf{1}_{[0,\frac{a-1}{a}]},$$

we get the invariant density

$$h_{a,p} = c\left[((a-1) - p(a-2)) \cdot \mathbf{1}_{[0,\frac{1}{a})} + \mathbf{1}_{[\frac{1}{a},\frac{a-1}{a}]} + \left(1 - p\frac{a-2}{a-1}\right) \cdot \mathbf{1}_{(\frac{a-1}{a},1]}\right],$$

for the normalising constant

$$c = \frac{a(a-1)}{2(a-1)^2 - pa(a-2)}.$$

Theorem 3.5.3 implies that if $a < 4$, then this is the unique absolutely continuous invariant density for W_a. Note that

$$\lim_{a\to 2} h_{a,p}(x) = \frac{1}{2}\mathbf{1}_{[0,1]}(x) + \frac{1}{2}\delta_{\frac{1}{2}}(x).$$

On the other hand, for the limit map W_2 shown in Figure 3.4(b) Lebesgue measure is the only absolutely continuous invariant measure.

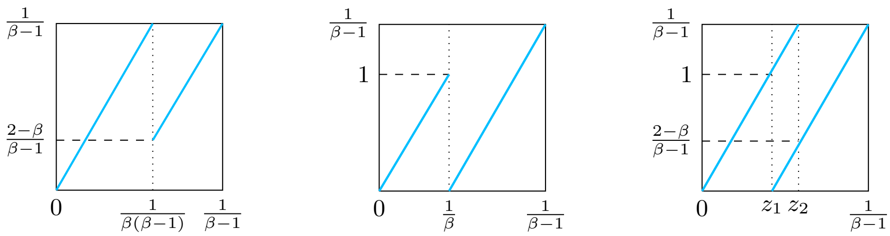
§3.6.3 Random β-transformations

Recall from 1.3.1 the definition of β-expansions. One of the more striking results is that Lebesgue almost all $x \in \left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right]$ have uncountably many different β-expansions (see [EJK90, S03, DdV07]). In [DK03] Dajani and Kraaikamp introduced a random system that produces for each $x \in \left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right]$ all its possible β-expansions. We will define this system for $1 < \beta < 2$ for simplicity, but everything easily extends to $\beta > 2$. Set

$$z_0 = 0, \quad z_1 = \frac{1}{\beta}, \quad z_2 = \frac{1}{\beta(\beta-1)}, \quad z_3 = \frac{1}{\beta-1},$$

and let

$$T_0(x) = \begin{cases} \beta x, & \text{if } x \in [z_0, z_2], \\ \beta x - 1, & \text{if } x \in (z_2, z_3], \end{cases} \quad \text{and} \quad T_1(x) = \begin{cases} \beta x, & \text{if } x \in [0, z_1), \\ \beta x - 1, & \text{if } x \in [z_1, z_3], \end{cases}$$



(a) T_0 (b) T_1 (c) T

Figure 3.5: In (a) we see the lazy β-transformation T_0, in (b) the greedy β-transformation T_1 and in (c) we see them combined. Whether or not $1 > \frac{2-\beta}{\beta-1}$ depends on the chosen value of β.

see Figure 5.1. The map T_0 is called the *lazy β-transformation* and the map T_1 is the *greedy β-transformation*. We do not bother to rescale the system to the unit interval $[0, 1]$, since this has no effect on the computations.

One of the reasons why people are interested in the random β-transformation is for its relation to the infinite Bernoulli convolution, see [DdV05, DK13, K14]. The density of the absolutely continuous invariant measures has been the subject of several papers. For a special class of values β an explicit expression of the density of $\mu_{\mathbf{p}}$ was found in [DdV07] using a Markov chain. In [K14] Kempton produced an explicit formula for the invariant density for all $1 < \beta < 2$ in case $p_0 = p_1 = \frac{1}{2}$ by constructing a natural extension of the system. He states that there is a straightforward extension of this method to $\beta > 2$. Recently Suzuki obtained a formula for the density of $\mu_{\mathbf{p}}$ for all $\beta > 1$ and any $\mathbf{p}$ in [S19]. Since the random β-transformation satisfies the assumptions (A1)-(A5) for any probability vector $\mathbf{p} = (p_0, p_1)$, we can also obtain the invariant density from Theorem 3.4.1. To illustrate our method we calculate the density for $\beta \in (1, 2)$ and $p_0 = p_1 = \frac{1}{2}$.

Let $\Omega = \{0, 1\}$, $N = 3$ and set

$$I_1 = [z_0, z_1), \quad I_2 = [z_1, z_2], \quad I_3 = (z_2, z_3].$$

Define the left and right limits at each point of discontinuity:

$$\begin{array}{llll} a_{1,0} = 1, & b_{1,0} = 1, & a_{2,0} = \frac{1}{\beta-1}, & b_{2,0} = \frac{2-\beta}{\beta-1}, \\ a_{1,1} = 1, & b_{1,1} = 0, & a_{2,1} = \frac{2-\beta}{\beta-1}, & b_{2,1} = \frac{2-\beta}{\beta-1}. \end{array}$$

As pointed out in Remark 3.4.5, to determine γ it would suffice to compute only one row of M, but for the sake of completeness we give M below. Let $\mathrm{KI}_n(1) = c_n$. By the symmetry of the system, for each $x \in [z_0, z_3]$ and all $(i, j) \in \{1, 2, 3\} \times \{0, 1\}$,

$$T_{i,j}(z_3 - x) = z_3 - T_{4-i,1-j}(x). \tag{3.38}$$

If for any $\omega = \omega_1 \dots \omega_t \in \{0, 1\}^*$, we let $\bar{\omega} \in \{0, 1\}^*$ denote the string $\bar{\omega} = (1 - \omega_1) \dots (1 - \omega_t)$, then (3.38) implies that $T_\omega(1) \in I_n$ if and only if $T_{\bar{\omega}}\big(\frac{2-\beta}{\beta-1}\big) \in I_{4-n}$ and so $\mathrm{KI}_n\left(\frac{2-\beta}{\beta-1}\right) = c_{4-n}$.

We obtain

$$M = \begin{pmatrix} \frac{1}{\beta} + \frac{1}{2\beta}(c_1 - \frac{1}{\beta-1}) & -\frac{1}{2\beta}c_3 \\ -\frac{1}{\beta} + \frac{1}{2\beta}c_2 & \frac{1}{\beta} - \frac{1}{2\beta}c_2 \\ \frac{1}{2\beta}c_3 & -\frac{1}{\beta} - \frac{1}{2\beta}(c_1 - \frac{1}{\beta-1}) \end{pmatrix}.$$

The null space consists of all vectors of the form

$$s\begin{pmatrix}1 & 1\end{pmatrix}^\intercal, \quad s \in \mathbb{R}.$$

From Theorem 3.5.3 we then know that the system T has a unique invariant density. We obtain

$$h_\gamma = \frac{c}{2\beta}\sum_{t\geq 0}\sum_{\omega\in\{0,1\}^t}\left(\frac{1}{2\beta}\right)^t\left(\mathbf{1}_{[0,T_\omega(1))} + \mathbf{1}_{[T_\omega(\frac{2-\beta}{\beta-1}),\frac{1}{\beta-1}]}\right),$$

for some normalising constant c. This matches the density found in [K14, Theorem 2.1] except for possibly countably many points.

If we set $p_0 \neq \frac{1}{2}$, the computations are less straightforward. Nevertheless, we can obtain a nice closed formula for the density in specific instances. Let $p_0 = p \in [0,1]$ be arbitrary and consider $\beta = \frac{1+\sqrt{5}}{2}$, the golden mean. Then β satisfies $\beta^2 - \beta - 1 = 0$ and the system has the nice property that $T_{2,0}(z_1) = z_2$ and $T_{2,1}(z_2) = z_1$ for $z_1 = \frac{1}{\beta}$ and $z_2 = 1$. Also note that $\frac{1}{\beta-1} = \beta$. This specific case has also been studied in [DdV07, Example 1]. The resulting matrix M is given by

$$M = \frac{\beta}{\beta^2 - p(1-p)}\begin{pmatrix} p^2 & -p(1-p) \\ -p & (1-p) \\ (1-p)p & -(1-p)^2 \end{pmatrix},$$

and its null space consists of all vectors of the form

$$s\begin{pmatrix}1-p & p\end{pmatrix}^\intercal, \quad s \in \mathbb{R}.$$

For the functions L_y we obtain $L_0 = 0$, $L_\beta = \beta^2$ and

$$\begin{aligned} L_{\frac{1}{\beta}} &= \frac{p^2\beta^2}{\beta^2 - p(1-p)} + \frac{\beta^2}{\beta^2 - p(1-p)}\mathbf{1}_{[0,\frac{1}{\beta})} + \frac{p\beta}{\beta^2 - p(1-p)}\mathbf{1}_{[0,1)}, \\ L_1 &= \frac{p\beta^3}{\beta^2 - p(1-p)} + \frac{(1-p)\beta}{\beta^2 - p(1-p)}\mathbf{1}_{[0,\frac{1}{\beta})} + \frac{\beta^2}{\beta^2 - p(1-p)}\mathbf{1}_{[0,1)}. \end{aligned}$$

The unique invariant density turns out to be

$$h_\gamma = \frac{\beta^2}{1+\beta^2}\big((1-p)\beta\cdot\mathbf{1}_{[0,\beta-1]} + \mathbf{1}_{(\beta-1,1)} + p\beta\cdot\mathbf{1}_{[1,\beta]}\big),$$

which for $p = \frac{1}{2}$ corresponds to

$$h_\gamma = \frac{\beta^2}{2(1+\beta^2)}(\beta\cdot\mathbf{1}_{[0,\beta-1]} + 2\cdot\mathbf{1}_{(\beta-1,1)} + \beta\cdot\mathbf{1}_{[1,\beta]}).$$

CHAPTER 3

§3.6.4 The random (α, β)-transformation

As an example of a system that is not everywhere expanding, but is expanding on average, we consider a random combination of the greedy β-transformation and the non-expanding (α, β)-transformation introduced in [DHK09]. More specifically, let $0 < \alpha < 1$ and $1 < \beta < 2$ be given and

$$z_0 = 0, \quad z_1 = 1/\beta, \quad z_2 = 1.$$

Define the (α, β)-transformation T_0 on the interval $[0, 1]$ by

$$T_0(x) = \begin{cases} \beta x, & \text{if } x \in [0, z_1), \\ \frac{\alpha}{\beta}(\beta x - 1), & \text{if } x \in [z_1, z_2]. \end{cases}$$

Let $T_1 : [0, 1] \to [0, 1]$ be the greedy β-transformation again, given by $T_1(x) = \beta x \pmod 1$. For any $0 < p < \frac{\alpha(\beta-1)}{\beta-\alpha}$ the random system T with probability vector $\mathbf{p} = (p, 1-p)$ satisfies the conditions (A1), (A2), (A3) and (A5). The assumptions on the boundary points from (A4) do not hold, but this is easily solved by adding an extra interval $(z_2, z_3]$ for $z_3 = \frac{1}{\beta-1}$ and extending T_0 and T_1 to it by setting $T_0(x) = T_1(x) = \beta x - 1$.

This random system T does not satisfy the conditions of Theorem 3.5.3 and we can therefore not conclude directly that Theorem 3.4.1 produces all invariant densities for T. However, the set $\Omega = \{0, 1\}$ is finite and the map T_1 is expanding with $T_1'(x) = \beta > 1$ for all x and therefore T satisfies the conditions from [P84, Corollary 7] on the number of ergodic components of the pseudo skew-product R. Since the greedy β-transformation T_1 has a unique absolutely continuous invariant measure, this corollary implies that also the random system T has a unique invariant density. We use Theorem 3.4.1 to get this density.

Let $0 < p < \frac{\alpha(\beta-1)}{\beta-\alpha}$ be arbitrary and set

$$I_1 = [z_0, z_1), \quad I_2 = [z_1, z_2], \quad I_3 = (z_2, z_3].$$

The left and right limits at each point of discontinuity are given by:

$$\begin{array}{llll} a_{1,0} = 1, & b_{1,0} = 0, & a_{2,0} = \alpha - \frac{\alpha}{\beta}, & b_{2,0} = \beta - 1, \\ a_{1,1} = 1, & b_{1,1} = 0, & a_{2,1} = \beta - 1, & b_{2,1} = \beta - 1. \end{array}$$

By construction, none of the points in $[0, 1]$ will ever enter the interval I_3, therefore $\mathrm{KI}_3(y) = 0$ for all $y \in [0, 1]$. As a consequence, the last row of the 3×2 fundamental matrix M is given by $\mu_{3,1} = 0$ and $\mu_{3,2} = -\frac{1}{\beta}$. This fact, together with the fact that we know from Lemma 3.3.4 that the null space of M is non-trivial, forces the first column of M to be zero, i.e., $\mu_{1,1} = \mu_{2,1} = \mu_{3,1} = 0$. Hence, the null space of M consists of all vectors of the form

$$s \begin{pmatrix} 1 & 0 \end{pmatrix}^{\mathsf{T}}, \quad s \in \mathbb{R},$$

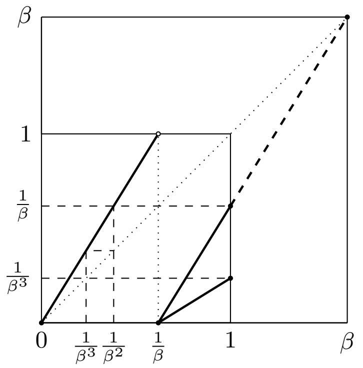


Figure 3.6: The random (α, β)-transformation for $\beta = \frac{1+\sqrt{5}}{2}$ and $\alpha = \frac{1}{\beta}$.

and the unique invariant density of the system T is

$$h_\gamma = \frac{c}{\beta} L_1 = \frac{c}{\beta} \sum_{t \geq 0} \sum_{\omega \in \Omega^t} \delta_\omega(1,t) \mathbf{1}_{[0, T_\omega(1))},$$

for some normalising constant c. In case we choose $\beta = \frac{1+\sqrt{5}}{2}$ and $\alpha = \frac{1}{\beta}$ as in Figure 3.6, we can compute further to get

$$h_\gamma = \frac{\beta^2}{\beta^2 + 1 + 2p} \left(p\beta \mathbf{1}_{[0,1/\beta^3]} + p\mathbf{1}_{[0,1/\beta^2]} + \frac{1}{\beta} \mathbf{1}_{[0,1/\beta]} + \mathbf{1}_{[0,1]} \right).$$

§3.7 c-Lüroth expansions

Recall the definition of Lüroth maps given 1.3.2 in Chapter 1 and then used in 2.2 in Chapter 2. Over the years, many people have considered digit properties of Lüroth expansions, such as digit frequencies and the sizes of sets of numbers for which the digit sequence $(d_n)_{n \geq 1}$ is bounded. See for example [BI09, FLMW10, SF11, MT13, GL16]. The set of points that have all Lüroth digits bounded by some integer D corresponds to the set of points that avoid the set $[0, \frac{1}{D}]$ under all iterations of the map T_L. For the deterministic system T_L, such a set is usually a fractal no matter how large we take the upper bound D. This situation can be modified by dealing with a random setting. More specifically, recall the c-Lüroth maps and expansions defined in Section **??**. The pseudo-skew product L_c, for $c > 0$, is constructed in such a way that the combination of T_L and T_A prevent any point of the interval to visit the subinterval $[0, c)$, giving c-Lüroth expansions with bounded digits.

We give an example for $c = \frac{1}{3}$, in which all $x \in \left[\frac{1}{3}, 1\right]$ have a random $\frac{1}{3}$-Lüroth expansion that uses only digits 2 and 3. Using the density given by Theorem 3.4.1 we can compute the frequency of each of these digits for any typical point $x \in \left[\frac{1}{3}, 1\right]$.

Let $T_0 = T_{0,\frac{1}{3}}$ and $T_1 = T_{1,\frac{1}{3}}$. Consider the partition of $I_{\frac{1}{3}}$ by setting

$$I_1 = \left[\frac{1}{3}, \frac{7}{18}\right] \quad I_2 = \left(\frac{7}{18}, \frac{4}{9}\right] \quad I_3 = \left(\frac{4}{9}, \frac{1}{2}\right]$$
$$I_4 = \left(\frac{1}{2}, \frac{2}{3}\right] \quad I_5 = \left(\frac{2}{3}, \frac{5}{6}\right] \quad I_6 = \left(\frac{5}{6}, 1\right].$$

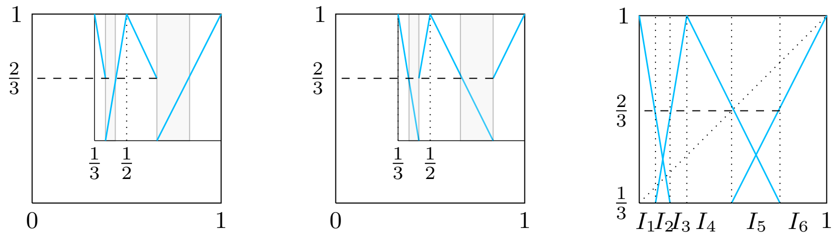
Note that

$$T_0(x) = \begin{cases} T_L(x) & \text{if } x \in I_2 \cup I_3 \cup I_5 \cup I_6, \\ \\ T_A(x) & \text{if } x \in I_1 \cup I_4, \end{cases}$$

and

$$T_1(x) = \begin{cases} T_A(x) & \text{if } x \in I_1 \cup I_2 \cup I_4 \cup I_5, \\ \\ T_L(x) & \text{if } x \in I_3 \cup I_6. \end{cases}$$

See Figure 3.7. Let $\mathbf{p} = (p, 1-p)$, for some $0 < p < 1$.



(a) T_0 (b) T_1 (c) T

Figure 3.7: The systems T_0, T_1 and $L_{\frac{1}{3}}$ on the interval $I_{\frac{1}{3}} = [\frac{1}{3}, 1]$.

To use Theorem 3.4.1, we need to determine the orbits of all the points $a_{n,j}$ and $b_{n,j}$, which in this case are $\frac{1}{3}$, $\frac{2}{3}$ and 1. One easily checks that all $\mathrm{KI}_n(a_{i,j})$ and $\mathrm{KI}_n(b_{i,j})$ are zero, except for

$$\mathrm{KI}_1\left(\frac{1}{3}\right) = -\frac{1}{6}, \qquad \mathrm{KI}_6\left(\frac{1}{3}\right) = -\frac{1}{6}, \qquad \mathrm{KI}_6(1) = 1 \quad \text{and} \quad \mathrm{KI}_4\left(\frac{2}{3}\right) = -\frac{1}{3}.$$

The fundamental matrix M of the system is therefore given by

$$M = \begin{pmatrix} \frac{p-6}{36} & \frac{1-p}{36} & 0 & \frac{p}{12} & \frac{1-p}{12} \\ \frac{1-2p}{6} & \frac{2p-1}{6} & 0 & 0 & 0 \\ 0 & -\frac{1}{6} & \frac{1}{6} & 0 & 0 \\ \frac{p}{18} & \frac{1-p}{18} & \frac{1}{2} & \frac{p-3}{6} & \frac{1-p}{6} \\ 0 & 0 & 0 & \frac{1-2p}{2} & \frac{2p-1}{2} \\ \frac{p}{36} & \frac{1-p}{36} & \frac{2}{3} & \frac{p}{12} & -\frac{p+5}{12} \end{pmatrix},$$

and its null space consists of all vectors of the form

$$s\begin{pmatrix}3 & 3 & 3 & 5 & 5\end{pmatrix}^{\mathsf{T}}, \quad s \in \mathbb{R}.$$

Again this is a one-dimensional space, so by Theorem 3.5.3 T has a unique invariant density. The corresponding measure $m_{\mathbf{p}} \times \mu_{\mathbf{p}}$ is necessarily ergodic for $L_{\frac{1}{3}}$. In the following we denote by $\boldsymbol{L}$ the L functions from (3.17) to distinguish them from the pseudo-skew product map $L_{\frac{1}{3}}$. From

$$\boldsymbol{L}_{\frac{1}{3}} = -\frac{1}{3}, \qquad \boldsymbol{L}_{\frac{2}{3}} = \frac{2}{3} \cdot \mathbf{1}_{[\frac{1}{3},\frac{2}{3}]} \quad \text{and} \quad \boldsymbol{L}_1 = 2$$

we get the invariant density

$$h_\gamma = \frac{3}{8}(3 \cdot \mathbf{1}_{[\frac{1}{3},\frac{2}{3}]} + 5 \cdot \mathbf{1}_{(\frac{2}{3},1]}).$$

For any point $(\omega, x) \in \{0,1\}^{\mathbb{N}} \times \left[\frac{1}{3}, 1\right]$ the frequency of the digit 2 in its random Lüroth expansion is given by

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_{\{0,1\}^{\mathbb{N}} \times (\frac{1}{2},1]}(L^k_{\frac{1}{3}}(\omega, x)).$$

Since $m_{\mathbf{p}} \times \mu_{\mathbf{p}}$ is ergodic, by Birkhoff's Ergodic Theorem we have that for $m_{\mathbf{p}} \times \mu_{\mathbf{p}}$-a.e. $(\omega, x) \in \{0,1\}^{\mathbb{N}} \times \left[\frac{1}{3}, 1\right]$ the frequency of 2 in the associated random Lüroth expansion is

$$\int_{(\frac{1}{2},1]} h_\gamma d\lambda = \frac{13}{16},$$

giving also that the frequency of the digit 3 is $\frac{3}{16}$.

Even though condition (A5) is not satisfied for $p = \frac{1}{2}$, the fundamental matrix M can still be computed and its null space is still given by $s\begin{pmatrix}3 & 3 & 3 & 5 & 5\end{pmatrix}^{\mathsf{T}}$, $s \in \mathbb{R}$. Moreover, the function $h_\gamma = \frac{3}{8}(3 \cdot \mathbf{1}_{[\frac{1}{3},\frac{2}{3}]} + 5 \cdot \mathbf{1}_{(\frac{2}{3},1]})$ is still the unique invariant density. We believe that Theorem 3.4.1 and Theorem 3.5.3 should still hold without the assumption (A5)

3.7.1 Remark. Note that for any rational c, the density of $\mu_{\mathbf{p}}$ can also be recovered using the matrix form P of the Perron-Frobenius operator, which is the approach used in 2.4. However, it is often the case that the matrix P is much larger than our fundamental matrix M. For instance, consider again Example 2.4.12 from Chapter 2. The Perron-Frobenius matrix P for the c-Lüroth transformation L_c for $c = \frac{12}{25}$ is a 13×13 square matrix. The corresponding fundamental matrix M is the 4×3 matrix

$$M = \begin{pmatrix} \frac{1}{6} & -\frac{p}{12} & -\frac{1-p}{12} \\ \frac{1}{2} & -\frac{1}{2} + \frac{p}{4} & -\frac{1-p}{4} \\ 0 & \frac{1-2p}{2} & -\frac{1-2p}{2} \\ \frac{2}{3} & \frac{p}{6} & -\frac{1}{2} + \frac{1-p}{6}. \end{pmatrix}.$$

For
$$\begin{array}{lll} \mathrm{KI}_5(1) = 1 & \mathrm{KI}_2(c) = \frac{1}{6} & \mathrm{KI}_3(c) = -\frac{1}{50} \\ \mathrm{KI}_5(c) = \frac{8}{75} & \mathrm{KI}_3(1-c) = -\frac{492}{1025} & \mathrm{KI}_5(1-c) = -\frac{451}{1025}, \end{array}$$
and $\mathrm{KI}_n(y) = 0$ for any other combination of $n \in \{1,2,3,4,5\}$ and $y \in \{c, 1-c\}$ not listed. Note that, due to the periodicity of the random orbit of c, the computation of the quantities KI uses the equality
$$\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2^{10}}\right)^n = \frac{1024}{1025}.$$
Furthermore, from Proposition 2.2.6 one finds that for irrational cutting points L_c does not admit a Markov partition so, while the aforementioned method, using the Perron-Frobenius operator in matrix form, does not apply anymore, Theorem 3.4.1 is also capable of handling these situations.

§3.8 Remarks

The procedure proposed in Section 3.4, and in particular the computation of the quantities KI_n and the functions L_y, seems at first glance quite complicated. However, this is not the case for an extensive class of transformations. This includes the random β-transformations studied in Sections 3.6.3, 3.6.4, the c-Lüroth maps introduced in Chapter 2 and the other families of examples proposed in Section 3.6. Moreover, for Markov maps the computation becomes pretty straightforward. We will see in Chapter 5 that this is also true for random interval maps having random matching. Furthermore, we will show how the entire procedure can be actually even implemented in Python, see Chapter 5 Section 5.6.

CHAPTER 4

Matching for flipped α-CF

This chapter is based on: [KLMM20].

Abstract

As a natural counterpart to Nakada's α-continued fraction maps, we study a one-parameter family of continued fraction transformations with an indifferent fixed point. We prove that matching holds for Lebesgue almost every parameter in this family and that the exceptional set has Hausdorff dimension 1. Due to this matching property, we can construct a planar version of the natural extension. We use this construction to obtain an explicit expression for the density of the unique infinite σ-finite absolutely continuous invariant measure, and we also compute the Krengel entropy, the return sequence and the wandering rate of the maps for a large part of the parameter space.

§4.1 Motivation and context

Over the past decades the dynamical phenomenon of matching, or synchronisation as described in Definition 1.2.7, has surfaced increasingly often in the study of the dynamics of interval maps. Recall that a map T is said to have matching if for any discontinuity point c of the map T or its derivative T' the orbits of the left and right limits of c eventually meet. That is, there exist non-negative integers M and N, called *matching exponents*, such that

$$T^M(c^-) = T^N(c^+), \tag{4.1}$$

where

$$c^- = \lim_{x \uparrow c} T(x) \quad \text{and} \quad c^+ = \lim_{x \downarrow c} T(x).$$

General results on the implications of matching are scarce. There are many results however on the consequences of matching for specific families of interval maps. In [KS12, BSORG13, BCK17, BCMP18, CM18, DK17] matching was considered for various families of piecewise linear maps in relation to expressions for the invariant densities, entropy and multiple tilings. Another type of transformation for which matching has proven to be convenient is for continued fraction maps, most notably for Nakada's α-continued fraction maps. This family was introduced in [N81] by defining for each $\alpha \in \left[\frac{1}{2}, 1\right]$ the map $S_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha]$ by $S_\alpha(0) = 0$ and for $x \neq 0$,

$$S_\alpha(x) = \frac{1}{|x|} - \left\lfloor \frac{1}{|x|} + 1 - \alpha \right\rfloor. \tag{4.2}$$

In [N81] Nakada constructed a planar natural extension of S_α and proved the existence of a unique absolutely continuous invariant probability measure. In [LM08] the family was extended to include the parameters $\alpha \in \left[0, \frac{1}{2}\right)$. On this part of the parameter space the planar natural extension strongly depends on the matching property, and it is much more complicated. This also affects the behaviour of the metric entropy as a function of α, which is described in detail in [LM08, NN08, CMPT10, KSS12, CT13, T14]. In [DKS09, KU10, CIT18] matching was successfully considered for other families of continued fraction transformations.

The matching behaviour of these different families has some striking similarities. The parameter space usually breaks down into maximal intervals on which the exponents M and N from (4.1) are constant, called *matching intervals*. These matching intervals usually cover most of the space, leaving a Lebesgue null set. The set where matching fails, called the *exceptional set*, is often of positive Haussdorff dimension, see [CT12, KSS12, BCIT13, BCK17, DK17] for example.

So far, matching has been considered only for dynamical systems with a finite absolutely continuous invariant measure. In this article, we introduce and study the matching behaviour and its consequences for a one-parameter family of continued fraction transformations on the interval that have a unique absolutely continuous, σ-finite invariant measure that is infinite. This family of *flipped α-continued fraction transformations* we introduce arises naturally as a counterpart to Nakada's α-continued fraction maps. Due to matching we obtain a nice planar natural extension on a large part of the parameter space, which allows us to explicitly compute dynamical features of the maps, such as the invariant density, Krengel entropy and wandering rate.

The family of maps $\{T_\alpha\}_{\alpha\in(0,1)}$ we consider is defined as follows. For each $\alpha \in (0,1)$ let

$$D_\alpha = \bigcup_{n\geq 1}\left[\frac{1}{n+\alpha}, \frac{1}{n}\right] \subseteq [0,1], \tag{4.3}$$

and $I_\alpha := [\min\{\alpha, 1-\alpha\}, 1]$, and define the map $T_\alpha : I_\alpha \to I_\alpha$ by

$$T_\alpha(x) = \begin{cases} G(x) = \dfrac{1}{x} - \left\lfloor \dfrac{1}{x} \right\rfloor, & \text{if } x \in D_\alpha^{\mathsf{c}}, \\[2ex] 1 - G(x) = -\dfrac{1}{x} + \left(1 + \left\lfloor \dfrac{1}{x} \right\rfloor\right), & \text{if } x \in D_\alpha, \end{cases}$$

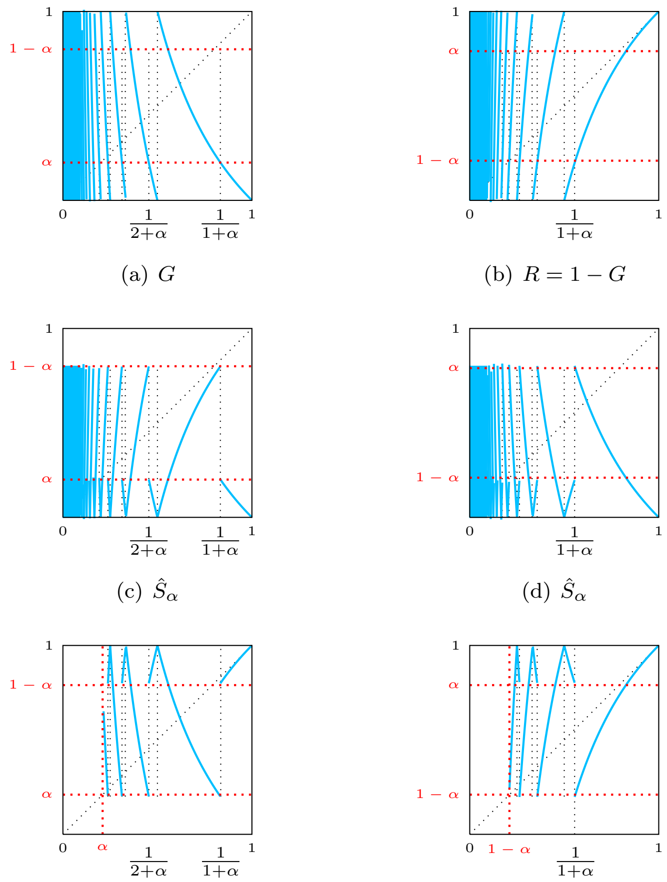
where $G(x) = \frac{1}{x} \pmod 1$ is the *Gauss map* and D_α^{c} denotes the complement of D_α in $[0,1]$. Note that for $\alpha = 0$ one recovers the Gauss map G, and $\alpha = 1$ gives $1 - G$, which is a shifted version of the *Rényi map* or *backwards continued fraction map*. Since these transformations have already been studied extensively, we omit them from our analysis. Figures 4.1(c) and 4.1(f) show the graphs of the maps T_α for a parameter $\alpha < \frac{1}{2}$ and a parameter $\alpha > \frac{1}{2}$, respectively. We could define T_α on the whole interval $[0,1]$, but since the dynamics of T_α is attracted to the interval I_α we just take that as the domain. Since I_α is bounded away from 0, any map T_α has only a finite number of branches. Note also that each map T_α has an indifferent fixed point at 1.

We call these transformations flipped α-continued fraction maps, due to their relation to the family of maps described in [MMY97]. The authors defined for each $\alpha \in [0,1]$ the *folded α-continued fraction map* $\hat{S}_\alpha : [0, \max\{\alpha, 1-\alpha\}] \to [0, \max\{\alpha, 1-\alpha\}]$ by $\hat{S}_\alpha(0) = 0$ and for $x \neq 0$,

$$\hat{S}_\alpha(x) = \left|\frac{1}{x} - \left\lfloor \frac{1}{x} + 1 - \alpha \right\rfloor\right| = \begin{cases} G(x) = \dfrac{1}{x} - \left\lfloor \dfrac{1}{x} \right\rfloor, & \text{if } x \in D_\alpha, \\[2ex] 1 - G(x) = -\dfrac{1}{x} + \left(1 + \left\lfloor \dfrac{1}{x} \right\rfloor\right), & \text{if } x \in D_\alpha^{\mathsf{c}}. \end{cases}$$

The dynamical properties of the folded α-continued fraction maps are essentially equal to those of Nakada's α-continued fraction maps. The name represents the idea that these maps 'fold' the interval $[\alpha-1, \alpha]$ onto $[0, \max\{\alpha, 1-\alpha\}]$. As shown in Figure 4.1 the families $\{T_\alpha\}$ and $\{\hat{S}_\alpha\}$ are obtained by flipping the Gauss map on complementary parts of the unit interval and, as such, both families are particular instances of what are called *D-continued fraction maps* in [DHKM12]. Furthermore, for $\alpha = \frac{1}{2}$ the transformation T_α seems to be closely related, but not isomorphic, to the object of study of [DK00].

CHAPTER 4



(a) G (b) $R = 1 - G$

(c) $\hat{S}_\alpha$ (d) $\hat{S}_\alpha$

(e) T_α (f) T_α

Figure 4.1: The Gauss map G and the flipped map $R = 1 - G$ in (a) and (b). The folded α-continued fraction map $\hat{S}_\alpha$ and the flipped α-continued fraction map T_α for $\alpha < \frac{1}{2}$ in (c) and (e) and for $\alpha > \frac{1}{2}$ in (d) and (f).

The first main result of this article is on the matching behaviour of the family T_α.

4.1.1 Theorem. *The set of parameters $\alpha \in (0, 1)$ for which the transformation T_α does not have matching is a Lebesgue null set of full Hausdorff dimension.*

We also give an explicit description of the matching intervals by relating them to the matching intervals of Nakada's α-continued fraction transformations. The matching behaviour allows us to construct a planar version of the natural extension for $\alpha \in [0, \frac{1}{2}\sqrt{2}]$ leading to the following result.

4.1.2 Theorem. *Let $0 \leq \alpha \leq \frac{1}{2}\sqrt{2}$, let $\mathcal{B}_\alpha$ be the Borel σ-algebra on I_α and let*

$g = \frac{\sqrt{5}-1}{2}$. *The absolutely continuous measure* μ_α *on* $(I_\alpha, \mathcal{B}_\alpha)$ *with density*

$$f_\alpha(x) = \begin{cases} \frac{1}{x}\mathbf{1}_{[\alpha,\frac{\alpha}{1-\alpha}]}(x) + \frac{1}{1+x}\mathbf{1}_{[\frac{\alpha}{1-\alpha},1-\alpha]}(x) + \frac{2}{1-x^2}\mathbf{1}_{[1-\alpha,1]}(x), & \text{for } \alpha \in [0,\frac{1}{2}), \\ \frac{1}{1-x}\mathbf{1}_{[1-\alpha,\alpha]}(x) + \frac{1}{x(1-x)}\mathbf{1}_{[\alpha,\frac{1-\alpha}{\alpha}]}(x) + \frac{x^2+1}{x(1-x^2)}\mathbf{1}_{[\frac{1-\alpha}{\alpha},1]}(x), & \text{for } \alpha \in [\frac{1}{2}, g), \\ (\frac{1}{1-x} + \frac{1}{x+\frac{1}{g-1}})\mathbf{1}_{[1-\alpha,\frac{2\alpha-1}{\alpha}]}(x) + \frac{1}{1-x}\mathbf{1}_{[\frac{2\alpha-1}{\alpha},\alpha]}(x) + & \\ +(\frac{1}{1-x} + \frac{1}{x} - \frac{1}{x+\frac{1}{g}})\mathbf{1}_{[\alpha,\frac{2\alpha-1}{1-\alpha}]}(x) + \frac{x^2+1}{x(1-x^2)}\mathbf{1}_{[\frac{2\alpha-1}{1-\alpha},1]}(x), & \text{for } \alpha \in [g, \frac{2}{3}), \\ (\frac{1}{1-x} + \frac{1}{x+\frac{1}{g-1}})\mathbf{1}_{[1-\alpha,\frac{2\alpha-1}{\alpha}]}(x) + \frac{1}{1-x}\mathbf{1}_{[\frac{2\alpha-1}{\alpha},\alpha]}(x) + & \\ +(\frac{1}{1-x} + \frac{1}{x} - \frac{1}{x+\frac{1}{g}})\mathbf{1}_{[\alpha,\frac{1-\alpha}{2\alpha-1}]}(x) + & \\ +(\frac{1}{1-x} + \frac{1}{x+1} - \frac{1}{x+\frac{1}{g}} + \frac{1}{x} - \frac{1}{x+\frac{1}{g+1}})\mathbf{1}_{[\frac{1-\alpha}{2\alpha-1},1]}(x), & \text{for } \alpha \in [\frac{2}{3}, \frac{1}{2}\sqrt{2}], \end{cases}$$

is the unique (up to scalar multiplication) σ*-finite, infinite absolutely continuous invariant measure for* T_α*. Furthermore, for* $\alpha \in (0, g]$ *the Krengel entropy equals* $\frac{\pi^2}{6}$*. For* $\alpha \in (0, \frac{1}{2}\sqrt{2})$ *the wandering rate is given by* $w_n(T_\alpha) \sim \log n$ *and the return sequence by* $a_n(T_\alpha) \sim \frac{n}{\log n}$*.*

The value $\frac{1}{2}\sqrt{2}$ is the endpoint of the fourth matching interval. As α grows beyond $\frac{1}{2}\sqrt{2}$ the matching intervals get smaller and smaller with higher matching exponents and the natural extension domain develops a more fractal structure, see Figure 4.4. As a consequence, it is in principle still possible to obtain results similar to those from Theorem 4.1.2 for bigger values of α, but the natural extension and the computations involved become increasingly complicated.

The chapter is outlined as follows. In the next section we give some preliminaries on continued fractions and explain how the maps T_α can be used to generate them for numbers in the interval I_α. We also prove that the maps T_α fall into the family of what are called AFN-maps in [Z98]. In the third section we study the phenomenon of matching, leading to Theorem 4.1.1, and we give an explicit description of the matching intervals. The fourth section is devoted to defining a planar natural extension for the maps T_α for $\alpha \leq \frac{1}{2}\sqrt{2}$. This is then used to obtain the invariant densities appearing in Theorem 4.1.2. In the last section we compute the Krengel entropy, the wandering rate and the return sequence for T_α, giving the last part of Theorem 4.1.2.

§4.2 More CF-maps

§4.2.1 Semi-regular CF-expansions

In 1913, Perron introduced the notion of *semi-regular* continued fraction expansions, which are finite or infinite expressions for real numbers of the following form:

$$x = d_0 + \cfrac{\epsilon_0}{d_1 + \cfrac{\epsilon_1}{d_2 + \ddots + \cfrac{\epsilon_{n-1}}{d_n + \ddots}}},$$

where $d_0 \in \mathbb{Z}$ and for each $n \geq 1$, $\epsilon_{n-1} \in \{-1, 1\}$, $d_n \in \mathbb{N}$ and $d_n + \epsilon_n \geq 1$; see for example [P57]. We denote the semi-regular continued fraction expansion of a number x by

$$x = [d_0; \epsilon_0/d_1, \epsilon_1/d_2, \epsilon_2/d_3, \ldots].$$

The maps T_α generate semi-regular continued fraction expansions of real numbers by iteration. Define for any $\alpha \in (0,1)$ and any $x \in I_\alpha$ the partial quotients $d_k = d_k(x) = d_1(T_\alpha^{k-1}(x))$ and the signs $\epsilon_k = \epsilon_k(x) = \epsilon_1(T_\alpha^{k-1}(x))$ by setting

$$d_1(x) := \begin{cases} \lfloor \frac{1}{x} \rfloor, & \text{if } x \in D_\alpha^{\mathsf{c}}, \\ \lfloor \frac{1}{x} \rfloor + 1, & \text{otherwise;} \end{cases} \quad \text{and} \quad \epsilon_1(x) := \begin{cases} 1, & \text{if } x \in D_\alpha^{\mathsf{c}}, \\ -1, & \text{otherwise.} \end{cases}$$

With this notation the map T_α can be written as $T_\alpha(x) = \epsilon_1(x)\big(\frac{1}{x} - d_1(x)\big)$, implying

$$x = \frac{1}{d_1 + \epsilon_1 T_\alpha(x)} = \cfrac{1}{d_1 + \cfrac{\epsilon_1}{d_2 + \ddots + \cfrac{\epsilon_{n-1}}{d_n + \epsilon_n T_\alpha^n(x)}}}. \tag{4.4}$$

Denote by $(p_n/q_n)_{n \geq 1}$ the sequence of convergents of such an expansion, that is,

$$p_n/q_n = [0; 1/d_1, \epsilon_1/d_2, \ldots, \epsilon_{n-1}/d_n].$$

Since we obtained T_α from the Gauss map, by flipping on the domain D_α from (4.3), it follows from [DHKM12, Theorem 1] that for any $x \in I_\alpha$ we have: $\lim\limits_{n\to\infty} \frac{p_n}{q_n} = x$. Therefore, we can write

$$x = \cfrac{1}{d_1 + \cfrac{\epsilon_1}{d_2 + \ddots + \cfrac{\epsilon_{n-1}}{d_n + \ddots}}} =: [0; 1/d_1, \epsilon_1/d_2, \epsilon_2/d_3, \ldots]_\alpha,$$

which we call the *flipped α-continued fraction expansion* of x.

In case $\epsilon_n = 1$ for all $n \geq 1$ the continued fraction expansion is called *regular* and we use the common notation $[a_1, a_2, a_3, \ldots]$ for them. Regular continued fraction expansions are generated by the Gauss map $G : [0,1] \to [0,1]$ given by $G(0) = 0$ and $G(x) = \frac{1}{x} \pmod 1$ if $x \neq 0$. Therefore, G acts as a shift on the regular continued fraction expansions:

$$x = [a_1, a_2, a_3, \ldots] \quad \Rightarrow \quad G(x) = [a_2, a_3, a_4, \ldots].$$

It is well known that the regular continued fraction expansion of a number x is finite if and only if $x \in \mathbb{Q}$. For any $x \in [0, \frac{1}{2}]$ the following correspondence between the regular continued fraction expansions of x and $1 - x$ holds:

$$x = [a_1, a_2, a_3, \ldots] \quad \Leftrightarrow \quad 1 - x = [1, a_1 - 1, a_2, a_3, \ldots]. \tag{4.5}$$

We will need this property later.

On sequences of digits $(a_n)_{n\geq 1} \in \mathbb{N}^{\mathbb{N}}$ the *alternating ordering* is defined by setting $(a_n)_{n\geq 1} \prec (b_n)_{n\geq 1}$ if and only if for the smallest index $m \geq 1$ such that $a_m \neq b_m$ it holds that $(-1)^m a_m < (-1)^m b_m$. The same definition holds for finite strings of digits of the same length. The alternating ordering on continued fraction expansions is consistent with standard ordering on the real line, i.e.,

$$(a_n)_{n\geq 1} \prec (b_n)_{n\geq 1} \quad \Leftrightarrow \quad [a_1, a_2, a_3, \ldots] < [b_1, b_2, b_3, \ldots].$$

The next proposition will be needed in the following section.

4.2.1 Proposition. Let $\alpha \in (0,1)$ and $x \in I_\alpha$ be given. Then $x \in \mathbb{Q}$ if and only if there is an $N \geq 0$ such that $T_\alpha^N(x) = 1$.

Proof. If there is an $N \geq 0$ such that $T_\alpha^N(x) = 1$, then it follows immediately from (4.4) that $x \in \mathbb{Q}$. Suppose $x \in \mathbb{Q}$. Note that $T_\alpha^n(x) \in \mathbb{Q} \cap I_\alpha$ for all $n \geq 0$ and write $T_\alpha^n(x) = \frac{s_n}{t_n}$ with $s_n, t_n \in \mathbb{N}$ and t_n as small as possible. Assume for a contradiction that $T_\alpha^n(x) \neq 1$ for all $n \geq 1$. Then $s_n < t_n$ and since either $T_\alpha^{n+1}(x) = \frac{t_n - k s_n}{s_n}$ or $T_\alpha^{n+1}(x) = \frac{(k+1)s_n - t_n}{s_n}$, we get $0 < t_{n+1} < t_n$. This gives a contradiction. $\square$

§4.2.2 AFN-maps

We start our investigation into the dynamical properties of the maps T_α by showing that they fall into the category of AFN-maps considered in [Z98, Z00]. Let λ denote the one-dimensional Lebesgue measure and let X be a finite union of bounded intervals. A map $T : X \to X$ is called an *AFN-map* if there is a finite partition $\mathcal{P}$ of X consisting of non-empty, open intervals I_i, such that the restriction $T\,|_{I_i}$ is continuous, strictly monotone and twice differentiable. Moreover, T has to satisfy the following three properties:

(A) Adler's condition: $\frac{T''}{(T')^2}$ is bounded on $\cup_i I_i$;

(F) The finite image condition: $T(\mathcal{P}) := \{T(I_i) : I_i \in \mathcal{P}\}$ is finite;

(N) The repelling indifferent fixed point condition: there exists a finite set $\mathcal{Z} \subseteq \mathcal{P}$, such that each $Z_i \in \mathcal{Z}$ has an indifferent fixed point x_{Z_i}, that is,

$$\lim_{x \to x_{Z_i}, x \in Z_i} T_\alpha(x) = x_{Z_i} \quad \text{and} \quad \lim_{x \to x_{Z_i}, x \in Z_i} T'(x) = 1,$$

T' decreases on $(-\infty, x_{Z_i}) \cap Z_i$ and increases on $(x_{Z_i}, \infty) \cap Z_i$. Lastly, T is assumed to be uniformly expanding on sets bounded away from $\{x_{Z_i} : Z_i \in \mathcal{Z}\}$.

For the maps T_α we can take $\mathcal{P}$ to be the collection of intervals of monotonicity (or cylinder sets) of T_α, defined for each $\epsilon \in \{-1, 1\}$ and $d \geq 1$ by

$$\Delta(\epsilon, d) = int\{x \in I_\alpha : \epsilon_1(x) = \epsilon \text{ and } d_1(x) = d\}, \tag{4.6}$$

where we use *int* to denote the interior of the set.

4.2.2 Lemma. *For each $\alpha \in (0,1)$ the map T_α is an AFN-map.*

Proof. Let $\mathcal{P} = \{\Delta(\epsilon, d)\}$. Then T_α is continuous, strictly monotone and twice differentiable on each of the intervals in $\mathcal{P}$. We check the three other conditions. For **(A)** note that $T'_\alpha(x) = \pm\frac{1}{x^2}$, so that $\left|\frac{T''_\alpha(x)}{(T'_\alpha(x))^2}\right| = \left|\frac{2x^4}{x^3}\right| = \pm 2x \leq 2$ for any x for which T'_α is defined. Also, for any $J \in \mathcal{P}$ we have

$$T_\alpha(J) \in \big\{(\alpha, 1), (1-\alpha, 1), (\alpha, T_\alpha(\alpha)), (1-\alpha, T_\alpha(1-\alpha)), (T_\alpha(\alpha), 1), (T_\alpha(1-\alpha), 1)\big\},$$

giving **(F)**. Finally, T_α has only 1 as an indifferent fixed point. Since $T'_\alpha(x) = 1/x^2 > 1$ for any $x \in I_\alpha \setminus \{1\}$ where $T'_\alpha(x)$ is defined, we see that T'_α decreases near 1 and also **(N)** holds. □

Using [Z98, Theorem 1] we then obtain the following result.

4.2.3 Proposition. For each $\alpha \in \big(0, 1\big)$ there exists a unique absolutely continuous, infinite, σ-finite T_α-invariant measure μ_α that is ergodic and conservative for T_α.

Proof. Since T_α is an AFN-map, [Z98, Theorem 1] immediately implies that there are finitely many disjoint open sets $X_1, \ldots, X_N \subseteq I_\alpha$, such that $T_\alpha(X_i) = X_i \pmod{\lambda}$ and $T|_{X_i}$ is conservative and ergodic with respect to λ. Each X_i is a finite union of open intervals and supports a unique (up to a constant factor) absolutely continuous T_α-invariant measure. Moreover, this invariant measure is infinite if and only if X_i contains an interval $(1-\delta, 1)$ for some $\delta > 0$. Since each open interval contains a rational point in its interior, Proposition 4.2.1 together with the forward invariance of the sets X_i implies that there can only be one set X_i and that this set contains an interval of the form $(1-\delta, 1)$. Hence, there is a unique (up to a constant factor) absolutely continuous invariant measure μ_α that is infinite, σ-finite, ergodic and conservative for T_α. □

From Proposition 4.2.3 and [Z00, Theorem 1] it follows that each map T_α is *pointwise dual-ergodic*, i.e., there are positive constants $a_n(T_\alpha)$, $n \geq 1$, such that for each $f \in L^1(I_\alpha, \mathcal{B}_\alpha, \mu_\alpha)$, where $\mathcal{B}_\alpha$ is the Borel $\sigma-$algebra on I_α,

$$\lim_{n\to\infty} \frac{1}{a_n(T_\alpha)} \sum_{k=0}^{n-1} P^k_{T_\alpha} f = \int_{I_\alpha} f \, d\mu_\alpha \quad \mu_\alpha\text{-a.e.}, \tag{4.7}$$

where P_{T_α} denotes the Perron-Frobenius operator of the map T_α, given in Definition 1.2. The sequence $(a_n(T_\alpha))_{n\geq 1}$ is called the *return sequence* of T_α and will be given for $\alpha \in \left(0, \frac{1}{2}\sqrt{2}\right)$ in Section 4.5.

§4.3 Matching almost everywhere

In this section we prove that matching holds for almost every $\alpha \in (0,1)$. The discontinuity points of the map T_α are of the form $\frac{1}{k+\alpha}$ for some positive integer k. For any such point,

$$c^- = \lim_{x \uparrow \frac{1}{k+\alpha}} T_\alpha(x) = \alpha \quad \text{and} \quad c^+ = \lim_{x \downarrow \frac{1}{k+\alpha}} T_\alpha(x) = -(k+\alpha) + k + 1 = 1 - \alpha.$$

Recall the definition of matching from equation (4.1): matching for T_α holds if there exist non-negative integers M, N such that

$$T_\alpha^M(\alpha) = T_\alpha^N(1-\alpha). \tag{4.8}$$

Some authors also require the evaluation of the derivative of the iterates in the left and right limits of the critical points to coincide. In our case, we do not need this constraint, since we prove that matching is a local property.

In the next proposition we show that the first half of the parameter space consists of a single matching interval.

4.3.1 Proposition. For $\alpha \in \left(0, \frac{1}{2}\right)$ it holds that $T_\alpha(\alpha) = T_\alpha^2(1-\alpha)$.

Proof. Fix $\alpha = [a_1, a_2, \ldots] \in \left(0, \frac{1}{2}\right)$. First note that $\frac{1}{2} < 1 - \alpha < \frac{1}{1+\alpha}$, so that by (4.5) we obtain that

$$T_\alpha(1-\alpha) = G(1-\alpha) = \frac{\alpha}{1-\alpha} = [a_1 - 1, a_2, a_3, \ldots].$$

Hence

$$\frac{1}{a_1+1} < \alpha < \frac{1}{a_1+\alpha} \Leftrightarrow \frac{1}{a_1} < \frac{\alpha}{1-\alpha} < \frac{1}{a_1 - 1 + \alpha},$$

which gives that either α and $T_\alpha(1-\alpha)$ are both in D_α or in D_α^{c}. In both cases,

$$T_\alpha(\alpha) = T_\alpha^2(1-\alpha).$$

□

For $\alpha > \frac{1}{2}$ the situation is much more complicated. One explanation for this difference comes from two operations that convert one semi-regular continued fraction expansion of a number into another: singularisation and insertion. Both operations were introduced in [P57] and later appeared in many other places in the literature, see e.g. [K91, DK00, HK02, S04, DHKM12]. *Singularisation* deletes one of the convergents $\frac{p_n}{q_n}$ from the sequence while altering the ones before and after; *insertion* inserts the mediant $\frac{p_n+p_{n+1}}{q_n+q_{n+1}}$ of $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$ into the sequence. It follows from [DHKM12, Section

2.1] that for $\alpha < \frac{1}{2}$ the flipped α-continued fraction expansions of numbers in I_α can be obtained from their regular continued fraction expansions by insertions only, while for $\alpha > \frac{1}{2}$ one needs singularisations as well.

Define the map $R : [0,1] \to [0,1]$ by $R(x) = 1 - G(x)$, see Figure 4.1(b). Before we prove that matching holds Lebesgue almost everywhere, we describe the effect of R on the regular continued fraction expansions of numbers in $(0,1)$.

4.3.2 Lemma. *Let $x \in (0,1)$ have regular continued fraction expansion $x = [x_1, x_2, x_3, \dots]$. Then for each $j \geq 1$,*

$$R^{x_2+x_4+\dots+x_{2j}}(x) = [x_{2j+1}+1, x_{2j+2}, x_{2j+3}, \dots]$$

and if $0 < \ell < x_{2j}$, then

$$R^{x_2+x_4+\dots+x_{2j-2}+\ell}(x) = [1, x_{2j}-\ell, x_{2j+1}, x_{2j+2}, \dots].$$

Proof. By (4.5) it holds that

$$R(x) = 1 - G(x) = 1 - [x_2, x_3, x_4, \dots] = \begin{cases} [x_3+1, x_4, x_5, \dots] = R^{x_2}(x), & \text{if } x_2 = 1, \\ [1, x_2-1, x_3, x_4, \dots], & \text{if } x_2 > 1. \end{cases}$$

The statement then easily follows by induction. ☐

Chapter 4

4.3.3 Remark. The previous lemma implies that R preserves the parity of the regular continued fraction digits. More precisely, if $x \in (0,1)$, then (except for possibly the first two digits) the regular continued fraction expansion of $R(x)$ has regular continued fraction digits of x with even indices in even positions and regular continued fraction digits with odd indices in odd positions.

The map T_α equals the map R on D_α and G on D_α^{c}. The next lemma specifies the times n at which the orbit of α (or $1-\alpha$) can enter D_α^{c} for the first time.

4.3.4 Lemma. *Let $\alpha = [1, a_1, a_2, a_3, \dots] \in \left(\frac{1}{2}, 1\right)$. If $m := \min\{i \geq 0 : T_\alpha^i(\alpha) \in D_\alpha^{\mathsf{c}}\}$ exists, then $m = a_1 + a_3 + \dots + a_{2j+1} - 1$ where j is the unique integer such that*

$$a_1 + a_3 + \dots + a_{2j-1} - 1 < m \leq a_1 + a_3 + \dots + a_{2j+1} - 1.$$

Similarly, if $k := \min\{i \geq 0 : T_\alpha^i(1-\alpha) \in D_\alpha^{\mathsf{c}}\}$ exists, then $k = a_2 + a_4 + \dots + a_{2j} - 1$ where j is the unique integer such that

$$a_2 + a_4 + \dots + a_{2j-2} - 1 < k \leq a_2 + a_4 + \dots + a_{2j} - 1.$$

Proof. For the first statement, by the definition of m we know that $T_\alpha^i(\alpha) = R^i(\alpha)$ for all $i \leq m$. From Lemma 4.3.2 it then follows that if $m = a_1 + a_3 + \dots + a_{2j-1}$, then

$$T_\alpha^m(\alpha) = [a_{2j}+1, a_{2j+1}, a_{2j+2}, \dots],$$

and if $m = a_1 + a_3 + \dots + a_{2j-1} + \ell$ for some $0 < \ell \leq a_{2j+1} - 1$, then

$$T_\alpha^m(\alpha) = [1, a_{2j+1} - \ell, a_{2j+2}, a_{2j+3}, \dots].$$

Recall that $D_\alpha^{\mathsf{c}} = \bigcup_d \Delta(1,d)$. The right boundary point of any cylinder $\Delta(1,d) = (\frac{1}{d+1}, \frac{1}{d+\alpha})$ has regular continued fraction expansion $[d,1,a_1,a_2,\ldots]$. Since the regular continued fraction expansion of the left boundary point also starts with the digits $d,1$, any $x \in D_\alpha^{\mathsf{c}}$ has a regular continued fraction expansion of the form $[x_1,1,x_3,\ldots]$. In particular this holds for $T_\alpha^m(\alpha)$, which implies that either $a_{2j+1} = 1$ or $\ell = a_{2j+1} - 1$. In both cases, $m = a_1 + a_3 + \cdots + a_{2j+1} - 1$. For the second part of the lemma, recall from (4.5) that $1 - \alpha = [a_1 + 1, a_2, a_3, \ldots]$. The proof of the second part then goes along the same lines as above. $\square$

Recall from the introduction the definition of matching intervals as the maximal parameter intervals on which the matching exponents M, N from (4.1) are constant. We can obtain a complete description of the matching intervals by relating them to the matching intervals of Nakada's α-continued fraction maps from (4.2). First we recall some notation and results on matching for the maps from (4.2). Any rational number $a \in \mathbb{Q} \cap (0,1)$ has two regular continued fraction expansions:

$$a = [a_1, \ldots, a_n] = [a_1, \ldots, a_n - 1, 1], \quad a_n \geq 2.$$

The *quadratic interval* I_a associated to a is the interval with endpoints

$$[\overline{a_1, \ldots, a_n}] \quad \text{and} \quad [\overline{a_1, \ldots, a_n - 1, 1}].$$

The quadratic interval I_1 is defined separately by $I_1 = (g, 1)$, where $g = \frac{\sqrt{5}-1}{2}$. A quadratic interval I_a is called *maximal* if it is not properly contained in any other quadratic interval. By [CT12, Theorem 1.3] maximal intervals correspond to matching intervals for Nakada's α-continued fraction maps.

Let $\mathcal{R} = \{a \in \mathbb{Q} \cap (0,1] : I_a \text{ is maximal}\}$ and $a = [a_1, \ldots, a_n] \in \mathcal{R}$ with $a_n \geq 2$. The map $x \mapsto \frac{1}{1+x}$ is the inverse of the right most branch of the Gauss map. Therefore, $\frac{1}{1+a} = [1, a_1, a_2, \ldots, a_n - 1, 1] = [1, a_1, a_2, \ldots, a_n]$. Write

$$\begin{aligned} J_a^L &= \Big([1, \overline{a_1, a_2, \ldots, a_n - 1, 1}], [1, a_1, a_2, \ldots, a_n - 1, 1]\Big), \\ J_a^R &= \Big([1, a_1, a_2, \ldots, a_n], [1, \overline{a_1, a_2, \ldots, a_n}]\Big), \end{aligned}$$

if n is odd and

$$\begin{aligned} J_a^L &= \Big([1, \overline{a_1, a_2, \ldots, a_n}], [1, a_1, a_2, \ldots, a_n]\Big), \\ J_a^R &= \Big([1, a_1, a_2, \ldots, a_n - 1, 1], [1, \overline{a_1, a_2, \ldots, a_n - 1, 1}]\Big), \end{aligned}$$

if n is even, so that $\frac{1}{1+I_a} = J_a^L \cup J_a^R \cup \big\{\frac{1}{1+a}\big\}$. Finally, let

$$M = a_1 + a_3 + \cdots + a_n \quad \text{and} \quad N = a_2 + a_4 + \cdots + a_{n-1} + 2 \tag{4.9}$$

if n is odd and

$$M = a_1 + a_3 + \cdots + a_{n-1} + 1 \quad \text{and} \quad N = a_2 + a_4 + \cdots + a_n + 1 \tag{4.10}$$

if n is even. The next theorem states that the intervals J_a^L and J_a^R are matching intervals for the flipped α-continued fraction maps with matching exponents that depend on M and N.

4.3.5 Theorem. *Let $a \in \mathcal{R}$ and let M and N be as in* (4.9) *and* (4.10). *For each $\alpha \in J_a^L$ the map T_α satisfies $T_\alpha^M(\alpha) = T_\alpha^N(1-\alpha)$ and for each $\alpha \in J_a^R$ the map T_α satisfies $T_\alpha^{M+1}(\alpha) = T_\alpha^{N-1}(1-\alpha)$.*

Proof. First we consider the special maximal quadratic interval $I_1 = (g, 1)$ separately, for which n is odd and $J_1^L = \emptyset$ and $J_1^R = \left(\frac{1}{2}, g\right)$. Let $\alpha \in J_1^R$. Then $\alpha = [1, 1, a_2, a_3, \ldots]$ and $1 - \alpha = [2, a_2, a_3, \ldots]$. Note that $M = 1$ and $N = 2$. From $\alpha < g$ it follows that $\alpha^2 + \alpha - 1 < 0$. This implies that $1 - \alpha > \frac{1}{2+\alpha}$, so that $T_\alpha^{N-1}(1-\alpha) = T_\alpha(1-\alpha) = R(1-\alpha)$. It also implies that $\frac{1}{2} < \alpha < \frac{1}{1+\alpha}$ and that

$$T_\alpha(\alpha) = G(\alpha) = \frac{1}{\alpha} - 1 > \frac{1}{1+\alpha},$$

so that $T_\alpha^{M+1}(\alpha) = T_\alpha^2(\alpha) = R \circ G(\alpha)$, which by Lemma 4.3.2 equals $T_\alpha^{N-1}(1-\alpha)$.

Fix $a \in \mathcal{R} \setminus \{1\}$ and write $a = [a_1, a_2, \ldots, a_n] = [a_1, a_2, \ldots, a_n - 1, 1]$ for its regular continued fraction expansions. We only prove the statement for J_a^L, since the proof for J_a^R is similar. Assume without loss of generality that n is odd. The proof is analogous for n even and the parity is fixed only to determine the endpoints of J_a^L. We start by proving that matching cannot occur for indices smaller than M and N.

Write $\mathbf{a} = a_1, a_2, \ldots, a_{n-1}, a_n - 1, 1 \in \mathbb{N}^{n+1}$ and let $\alpha \in J_a^L = ([1, \bar{\mathbf{a}}], [1, \mathbf{a}])$. Then there is some finite or infinite string of positive integers $\mathbf{w} = a_{n+2}, a_{n+3}, \ldots$, such that $\alpha = [1, \mathbf{a}, \mathbf{w}]$. The assumption that n is odd together with the fact that the Gauss map preserves the alternating ordering imply that

$$\bar{\mathbf{a}} \succ \mathbf{w}. \tag{4.11}$$

Assume that $m = \min\{i \geq 0 \, : \, T_\alpha^i(\alpha) \in D_\alpha^{\mathsf{c}}\}$ exists. By Lemma 4.3.4 there is a j, such that $m = a_1 + a_3 + \cdots + a_{2j-1} - 1$. Assume that $2j - 1 < n$. By the definition of m it holds that $T_\alpha^m(\alpha) \in D_\alpha^{\mathsf{c}}$. So, using Lemma 4.3.2 we obtain that

$$[1, a_{2j}, a_{2j+1}, \ldots] = G(T_\alpha^m(\alpha)) = T_\alpha^{m+1}(\alpha) > \alpha = [1, \mathbf{a}, \mathbf{w}].$$

Since $a \in \mathcal{R}$, the result from [CT12, Proposition 4.5.2] implies that for any two non-empty strings $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{a} = \mathbf{uv}$, the inequality

$$\mathbf{v} \succ \mathbf{uv} \tag{4.12}$$

holds. Thus, if we take $\mathbf{v} = a_{2j}, a_{2j+1}, \ldots, a_n - 1, 1$ and $\mathbf{u} = a_1, a_2, \ldots, a_{2j-1}$, then we find $\mathbf{v} \preceq \mathbf{uv}$, which contradicts (4.12). Hence, if m exists, then $m \geq M - 1$. In a similar way we can deduce that if $k = \min\{i \geq 0 \, : \, T_\alpha^i(1-\alpha) \in D_\alpha^{\mathsf{c}}\}$ exists, then $k \geq N - 2$.

Now assume that there exist $\ell < M - 1$ and $i < N - 2$, such that

$$T_\alpha^\ell(\alpha) = R^\ell(\alpha) = R^i(1-\alpha) = T_\alpha^i(1-\alpha). \tag{4.13}$$

Recall from Remark 4.3.3 that R preserves the parity of the regular continued fraction digits. Since $\alpha = [1, a_1, a_2, \ldots]$ and $1 - \alpha = [a_1 + 1, a_2, a_3, \ldots]$, the assumption (4.13) then implies the existence of an even index $2 \le j \le n-1$ and an odd index $1 \le \ell < n$, such that

$$a_j, a_{j+1}, a_{j+2}, \ldots = a_\ell, a_{\ell+1}, a_{\ell+2}, \ldots.$$

This implies that a has an ultimately periodic regular continued fraction expansion, which contradicts the fact that $a \in \mathbb{Q}$. Hence, matching cannot occur with indices $\ell < M - 1$ and $i < N - 2$.

Next consider $T_\alpha^M(\alpha)$ and $T_\alpha^{N-2}(1-\alpha)$. From Lemma 4.3.2 applied to $\alpha = [1, a_1, a_2, \ldots]$, i.e., $x_i = a_{i-1}$, we get that

$$T_\alpha^{M-1}(\alpha) = R^{a_1 + a_3 + \cdots + a_n - 1}(\alpha) = [2, \mathbf{w}].$$

From $\alpha = [1, \mathbf{a}, \mathbf{w}] > g$, it follows that $G(\alpha) = [\mathbf{a}, \mathbf{w}] < g$. Combining this with the fact that the property from (4.11) implies $\mathbf{w} \prec \mathbf{aw} \prec 1\mathbf{aw}$. Hence, $T_\alpha^{M-1}(\alpha) > [2, 1, \mathbf{a}, \mathbf{w}] = \frac{1}{2+\alpha}$ which gives $T_\alpha^{M-1}(\alpha) \in (\frac{1}{2+\alpha}, \frac{1}{2})$. This implies that $T_\alpha^M(\alpha) = R(T_\alpha^{M-1}(\alpha)) = R([2, \mathbf{w}])$. For $1 - \alpha = [a_1 + 1, a_2, a_3, \ldots, a_{n-1}, a_n - 1, 1, \mathbf{w}]$ we get from Lemma 4.3.2 that

$$T_\alpha^{N-2}(1-\alpha) = R^{N-2}(1-\alpha) = [a_n, 1, \mathbf{w}].$$

Again using that $\mathbf{w} \prec \mathbf{aw}$ gives $T_\alpha^{N-2}(1-\alpha) \in \Delta(1, a_n) = ([a_n, 1], [a_n, 1, \mathbf{a}, \mathbf{w}])$. Since $\alpha > g$, it follows that $T_\alpha^N(1-\alpha) = R \circ G(T_\alpha^{N-2}(1-\alpha))$. Then, again by using Lemma 4.3.2, we obtain

$$T_\alpha^N(1-\alpha) = R([1, \mathbf{w}]) = R([2, \mathbf{w}]) = T_\alpha^M(\alpha).$$

For $\alpha \in J_a^R$, one can show similarly that $T_\alpha^{M-1}(\alpha) = [1, 1, \mathbf{w}] \in \Delta(1, 1)$. Since $\alpha > g$, this gives $T_\alpha^{M+1}(\alpha) = R \circ G(T_\alpha^{M-1}(\alpha)) = R([1, \mathbf{w}])$. On the other hand, $T_\alpha^{N-2}(1-\alpha) = R_\alpha^{N-2}(1-\alpha) = [a_n + 1, \mathbf{w}] > \frac{1}{a_n + 1 + \alpha}$. So,

$$T_\alpha^{N-1}(1-\alpha) = R_\alpha^{N-1}(1-\alpha) = R([a_n + 1, \mathbf{w}]) = R([1, \mathbf{w}]) = T_\alpha^{M+1}(\alpha).$$

□

From this theorem we obtain the result from Theorem 4.1.1 on the size of the set of non-matching parameters. We use $\dim_H(A)$ to denote the Hausdorff dimension of a set A and let $\mathcal{E}$ denote the non-matching set, that is,

$$\mathcal{E} = \{\alpha \in (0, 1) : T_\alpha \text{ does not have the matching property}\}.$$

Proof of Theorem 4.1.1. We use known results on the exceptional set $\mathcal{N}$ of non-matching parameters for Nakada's α-continued fraction maps from (4.2). It is proven in [CT12] and [KSS12] that $\lambda(\mathcal{N}) = 0$ and in [CT12, Theorem 1.2] that $\dim_H(\mathcal{N}) = 1$. Since the bi-Lipschitz map $x \mapsto \frac{1}{1+x}$ on $(0, 1)$ preserves Lebesgue null sets and Hausdorff dimension, the same properties hold for the set $E := \frac{1}{1+\mathcal{N}}$. Note that T_α has matching for all $\alpha \in E^{\mathsf{c}}$, since according to Theorem 4.3.5 either α is in a matching interval or it is of the form $\frac{1}{1+a}$ for some rational number a and then both α and $1-\alpha$ eventually get mapped to 1. Hence, $\mathcal{E} \subseteq E$ and it follows that $\lambda(\mathcal{E}) = 0$.

Now consider $E \setminus \mathcal{E}$. Let $a \in \mathcal{N}$. By [KSS12, Section 4], this is equivalent to $G^n(a) \geq a$ for all $n \geq 1$. Let $\alpha := \frac{1}{1+a}$ and write $[1, a_1, a_2, \ldots]$ for its regular continued fraction expansion. Suppose there exists a minimal $m \geq 0$ such that $T_\alpha^m(\alpha) \in D_\alpha^{\mathsf{c}}$. Then there exists a positive integer d such that

$$\frac{1}{d+1} < T_\alpha^m(\alpha) < \frac{1}{d+\alpha}.$$

By Lemma 4.3.2 the inequality implies in particular that for some $j > 2$

$$[a_j, a_{j+1}, \ldots] < [a_1, a_2, \ldots],$$

i.e., $G^{j-1}(a) < a$, which contradicts the assumption on a. Hence, $T_\alpha^k(\alpha) \notin D_\alpha^{\mathsf{c}}$ for all $k \geq 0$. Since the regular continued fraction expansion of $1-\alpha$ is given by $1-\alpha = [a_1+1, a_2, \ldots]$, the same conclusion holds for $1-\alpha$, that is, $T_\alpha^k(\alpha) \notin D_\alpha^{\mathsf{c}}$ for all $k \geq 0$. Hence, $T_\alpha^k(\alpha) = R^k(\alpha)$ and $T_\alpha^k(1-\alpha) = R^k(1-\alpha)$ for all k. Assume that $\alpha \notin \mathcal{E}$, so there are positive integers M, N such that $T_\alpha^M(\alpha) = T_\alpha^N(1-\alpha)$. By Remark 4.3.3 there is an odd index $\ell \geq 1$ and an even index $k \geq 2$ such that

$$a_\ell, a_{\ell+1}, \ldots = a_k, a_{k+1}, \ldots.$$

Therefore α is ultimately periodic and thus a preimage of a quadratic irrational. This implies that $\dim_H(E \setminus \mathcal{E}) = 0$ and hence $\dim_H(\mathcal{E}) = 1$. $\square$

These matching results are the main reason for the existence of the nice geometric versions of the natural extensions that we investigate in the next section.

§4.4 Natural extensions

For non-invertible dynamical systems, especially for continued fraction transformations, the natural extension is a very useful tool to obtain dynamical properties of the system, see Definition 1.2.6. Canonical constructions of the natural extension were first studied by Rohlin in [R61]. Based on these results it was shown in [S88, ST91] that for infinite measure systems like T_α a natural extension always exists and that any two natural extensions of the same system are necessarily isomorphic. Moreover, many ergodic properties carry over from the natural extension to the original map. The amount of information on the original system that can be gained from the natural extension, depends to a large extent on the version of the natural extension one considers. For continued fraction maps, there is a canonical construction, described in Section 1.9, that has led to many useful observations; see for example [N81, K91, KSS12, AS13, H02]. It turns out that a similar construction also works for the family $\{T_\alpha\}_{\alpha \in (0,1)}$.

In this section we construct a natural extension for the system $(I_\alpha, \mathcal{B}_\alpha, \mu_\alpha, T_\alpha)$, where $\mathcal{B}_\alpha$ is the Borel $\sigma-$algebra on I_α and μ_α is the measure from Proposition 4.2.3.

This natural extension is given by the dynamical system $(\mathcal{D}_\alpha, \mathcal{B}(\mathcal{D}_\alpha), \nu_\alpha, \mathcal{T}_\alpha)$, where $\mathcal{D}_\alpha$ is some domain in $\mathbb{R}^2$ that needs to be determined, $\mathcal{B}(\mathcal{D}_\alpha)$ is the Borel σ-algebra on $\mathcal{D}_\alpha$, ν_α is the measure defined by

$$\nu_\alpha(A) = \iint_A \frac{1}{(1+xy)^2}\, d\lambda^2(x,y) \quad \text{for any } A \in \mathcal{B}(\mathcal{D}_\alpha), \tag{4.14}$$

where λ^2 is the two-dimensional Lebesgue measure, and $\mathcal{T}_\alpha : \mathcal{D}_\alpha \to \mathcal{D}_\alpha$ is given by

$$\mathcal{T}_\alpha(x,y) = \left(T_\alpha(x), \frac{\epsilon_1(x)}{d_1(x)+y}\right).$$

To prove that $(\mathcal{D}_\alpha, \mathcal{B}(\mathcal{D}_\alpha), \nu_\alpha, \mathcal{T}_\alpha)$ is the natural extension of $(I_\alpha, \mathcal{B}_\alpha, \mu_\alpha, T_\alpha)$ we need to show that ν_α is $\mathcal{T}_\alpha$-invariant and that all of the following properties hold ν_α-almost everywhere:

(ne1) $\mathcal{T}_\alpha$ is invertible;

(ne2) the projection map $\pi : \mathcal{D}_\alpha \to I_\alpha$ is measurable and surjective;

(ne3) $\pi \circ \mathcal{T}_\alpha = T_\alpha \circ \pi$, where π is the projection onto the first coordinate;

(ne4) $\bigvee_{n=0}^{\infty} \mathcal{T}_\alpha^n \pi^{-1}(\mathcal{B}_\alpha) = \mathcal{B}(\mathcal{D}_\alpha)$, where $\bigvee_{n=0}^{\infty} \mathcal{T}_\alpha^n \pi^{-1}(\mathcal{B}_\alpha)$ is the smallest σ-algebra containing the σ-algebras $\mathcal{T}_\alpha^n \pi^{-1}(\mathcal{B}_\alpha)$ for all $n \geq 0$.

The shape of $\mathcal{D}_\alpha$ will depend on the orbits of α and $1-\alpha$ up to the moment of matching. As might be imagined in light of Proposition 4.3.1 and Theorem 4.3.5, the situation for $0 < \alpha < \frac{1}{2}$ is simpler than for $\frac{1}{2} < \alpha < 1$. We will provide a detailed description and proof for $0 < \alpha < \frac{1}{2}$ and list some analytical and numerical results for $\frac{1}{2} < \alpha < 1$.

§4.4.1 For $\alpha < \frac{1}{2}$

We claim that for $\alpha < \frac{1}{2}$ the domain of the natural extension is given by

$$\mathcal{D}_\alpha := \left[\alpha, \frac{\alpha}{1-\alpha}\right] \times [0,\infty) \cup \left(\frac{\alpha}{1-\alpha}, 1-\alpha\right] \times [0,1] \cup \left(1-\alpha, 1\right] \times [-1,1],$$

see Figure 4.2. Before we check (ne1)–(ne4), we introduce some notation. Partition D_α according to the cylinder sets of T_α described in (4.6). Let

$$\tilde{\Delta}(-1,2) = \Delta(-1,2)\times(-1,1), \quad \tilde{\Delta}(1,1) = \left(\frac{1}{2}, 1-\alpha\right)\times(0,1)\cup\left(1-\alpha, \frac{1}{1+\alpha}\right)\times(-1,1),$$

and for $(\epsilon, d) \notin \{(1,1),(-1,2)\}$,

$$\tilde{\Delta}(\epsilon,d) = \begin{cases} \Delta(\epsilon,d)\times(0,1), & \text{if } \Delta(\epsilon,d) \subseteq \left[\frac{\alpha}{1-\alpha}, 1-\alpha\right], \\ \Delta(\epsilon,d)\times(0,\infty), & \text{if } \Delta(\epsilon,d) \subseteq \left[\alpha, \frac{\alpha}{1-\alpha}\right], \end{cases}$$

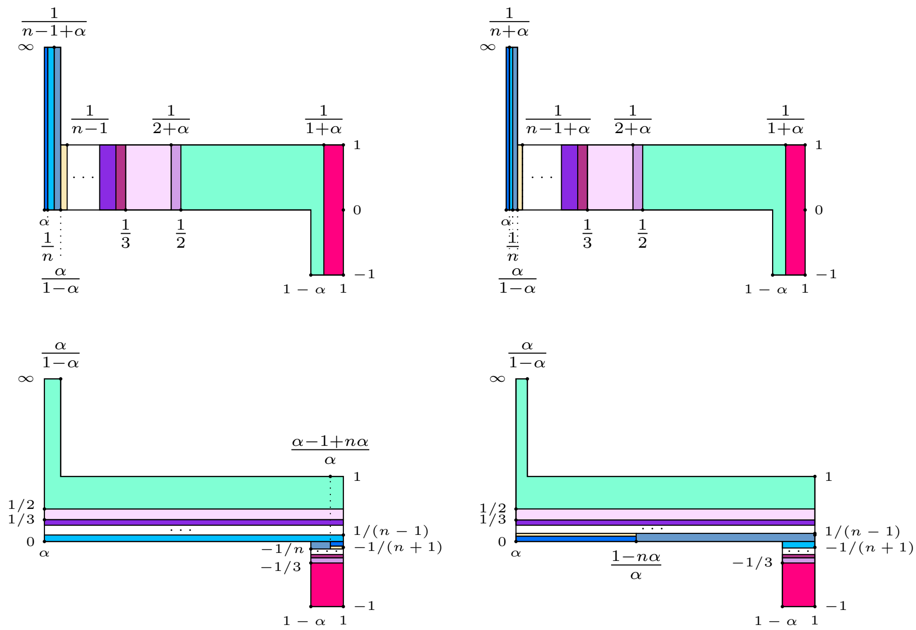
and for the ϵ and d such that $\frac{\alpha}{1-\alpha} \in \Delta(\epsilon, d)$,

$$\tilde{\Delta}_L(\epsilon, d) = \left(\Delta(\epsilon, d) \cap \left[\alpha, \frac{\alpha}{1-\alpha}\right]\right) \times (0, \infty), \quad \tilde{\Delta}_R(\epsilon, d) = \left(\Delta(\epsilon, d) \cap \left[\frac{\alpha}{1-\alpha}, 1\right]\right) \times (0, 1).$$

Due to the matching property described in Proposition 4.3.1 we have, up to a Lebesgue measure zero set,

- $\mathcal{T}_\alpha(\tilde{\Delta}(1,1)) = \left(\alpha, \frac{\alpha}{1-\alpha}\right) \times \left(\frac{1}{2}, \infty\right) \cup \left(\frac{\alpha}{1-\alpha}, 1\right) \times \left(\frac{1}{2}, 1\right)$,
- $\bigcup_{d \geq 2} \mathcal{T}_\alpha(\tilde{\Delta}(-1, d)) = (1-\alpha, 1) \times (-1, 0)$,
- $\bigcup_{d \geq 1} \mathcal{T}_\alpha(\tilde{\Delta}(1, d)) = (\alpha, 1) \times (0, 1)$,

where we have included the sets $\tilde{\Delta}_L(\epsilon, d)$ and $\tilde{\Delta}_R(\epsilon, d)$ in the appropriate union. Hence, $\mathcal{T}_\alpha$ is Lebesgue almost everywhere invertible, which gives (ne1).



(a) $\mathcal{D}_\alpha$ (above) and $\mathcal{T}_\alpha(\mathcal{D}_\alpha)$ (below) for $\frac{1}{n+\alpha} < \alpha < \frac{1}{n}$, $n \geq 2$

(b) $\mathcal{D}_\alpha$ (above) and $\mathcal{T}_\alpha(\mathcal{D}_\alpha)$ (below) for $\frac{1}{n+1} < \alpha < \frac{1}{n+\alpha}$, $n \geq 2$

Figure 4.2: The transformation $\mathcal{T}_\alpha$ maps areas on the top to areas on the bottom with the same colours, respectively.

The properties (ne2) and (ne3) follow immediately. Left to prove are (ne4) and the fact that ν_α is $\mathcal{T}_\alpha$-invariant. To prove that ν_α is invariant for $\mathcal{T}_\alpha$, it suffices to check that $\nu_\alpha(A) = \nu_\alpha(\mathcal{T}_\alpha^{-1}(A))$ for any rectangle $A = [a,b] \times [c,d] \subseteq \mathcal{T}_\alpha(D)$ for any $D = \tilde{\Delta}(\epsilon, d)$, $D = \tilde{\Delta}_L(\epsilon, d)$ or $D = \tilde{\Delta}_R(\epsilon, d)$. This computation is very similar to the corresponding ones for natural extensions of other continued fraction maps that can be found in literature, see e.g. [N81, Theorem 1]. We reproduce it here for the convenience of the reader. For any such rectangle A we have on the one hand,

$$\begin{aligned}\nu_\alpha(A) = &\iint_A \frac{1}{(1+xy)^2}\, d\lambda^2(x,y) = \int_{[c,d]} \frac{b}{1+by} - \frac{a}{1+ay}\, d\lambda(y)\\ &= \ln\left(\frac{1+bd}{1+bc}\right) - \ln\left(\frac{1+ad}{1+ac}\right) = \ln\left(\frac{1+ac+bd+abcd}{1+bc+ad+abcd}\right).\end{aligned}$$

If there is a $k \geq 1$ such that $\pi(D) \subseteq \Delta(1,k)$, then

$$\begin{aligned}\nu_\alpha\left(\mathcal{T}_\alpha^{-1}(A)\right) &= \iint_{\mathcal{T}_\alpha^{-1}(A)} \frac{1}{(1+xy)^2}\, d\lambda^2(x,y)\\ &= \int_{[\frac{1}{d}-k,\frac{1}{c}-k]} \frac{1}{k+a+y} - \frac{1}{k+b+y} d\lambda(y)\\ &= \ln\left(\frac{k+a+\frac{1}{c}-k}{k+a+\frac{1}{d}-k}\right) - \ln\left(\frac{k+b+\frac{1}{c}-k}{k+b+\frac{1}{d}-k}\right)\\ &= \ln\left(\frac{1+ac+bd+abcd}{1+bc+ad+abcd}\right).\end{aligned}$$

If there is a $k \geq 2$ such that $\pi(D) \subseteq \Delta(-1,k)$, then

$$\begin{aligned}\nu_\alpha\left(\mathcal{T}_\alpha^{-1}(A)\right) &= \iint_{\mathcal{T}_\alpha^{-1}(A)} \frac{1}{(1+xy)^2} d\lambda^2(x,y)\\ &= \int_{[-k-\frac{1}{c},-k-\frac{1}{d}]} \frac{1}{k-b+y} - \frac{1}{k-a+y} d\lambda(y)\\ &= \ln\left(\frac{k-b-k-\frac{1}{d}}{k-b-k-\frac{1}{c}}\right) - \ln\left(\frac{k-a-k-\frac{1}{d}}{k-a-k-\frac{1}{c}}\right)\\ &= \ln\left(\frac{1+ac+bd+abcd}{1+bc+ad+abcd}\right).\end{aligned}$$

In both cases $\nu_\alpha(A) = \nu_\alpha(\mathcal{T}_\alpha^{-1}(A))$ proving that ν_α is a $\mathcal{T}_\alpha$-invariant measure.

To prove that (ne4) holds, it is enough to show that $\bigvee_{n=0}^\infty \mathcal{T}_\alpha^n \pi^{-1}(\mathcal{B}_\alpha)$ separates points, i.e., that for λ^2-almost all $(x,y),(x',y') \in \mathcal{D}_\alpha$ with $(x,y) \neq (x',y')$ there are disjoint sets $A, B \in \bigvee_{n=0}^\infty \mathcal{T}_\alpha^n \pi^{-1}(\mathcal{B}_\alpha)$ with $(x,y) \in A$ and $(x',y') \in B$. Since $\mathcal{B}_\alpha$ is separating, the property is clear if $x \neq x'$. Furthermore, note that for λ-almost all values of y there is an ε and a d, such that on a neighbourhood of (x,y), the inverse of $\mathcal{T}_\alpha$ is given by

$$\mathcal{T}_\alpha^{-1}(x,y) = \left(\frac{1}{d+\varepsilon x}, \frac{\varepsilon}{y} - d\right).$$

The map $\frac{\varepsilon}{y} - d$ is expanding and $\mathcal{T}_\alpha^{-1}$ maps horizontal strips to vertical strips. Hence, we can also separate points that agree on the x-coordinate, giving (ne4). Therefore, we have obtained the following result.

4.4.1 Theorem. *Let $\alpha \in \left(0, \frac{1}{2}\right)$. The dynamical system $(\mathcal{D}_\alpha, \mathcal{B}(\mathcal{D}_\alpha), \nu_\alpha, \mathcal{T}_\alpha)$ is a version of the natural extension of the dynamical system $(I_\alpha, \mathcal{B}_\alpha, \mu_\alpha, T_\alpha)$ where $\mu_\alpha := \nu_\alpha \circ \pi^{-1}$.*

The measure $\mu_\alpha = \nu_\alpha \circ \pi^{-1}$ is the unique invariant measure for T_α that is absolutely continuous with respect to λ from Proposition 4.2.3. Projecting on the first coordinate gives the following explicit expression for the density f_α of μ_α:

$$\begin{aligned} f_\alpha(x) &= \frac{1}{x}\mathbf{1}_{[\alpha, \frac{\alpha}{1-\alpha}]}(x) + \frac{1}{1+x}\mathbf{1}_{[\frac{\alpha}{1-\alpha}, 1]}(x) + \frac{1}{1-x}\mathbf{1}_{[1-\alpha, 1]}(x) \\ &= \frac{1}{x}\mathbf{1}_{[\alpha, \frac{\alpha}{1-\alpha}]}(x) + \frac{1}{1+x}\mathbf{1}_{[\frac{\alpha}{1-\alpha}, 1-\alpha]}(x) + \frac{2}{1-x^2}\mathbf{1}_{[1-\alpha, 1]}(x). \end{aligned} \tag{4.15}$$

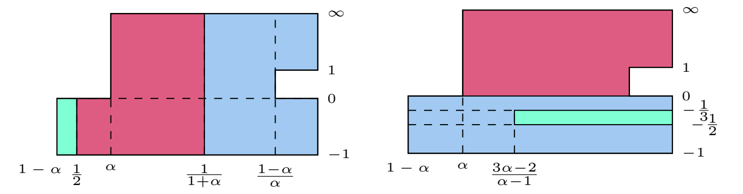
Here, by “unique”, we of course mean unique up to scalar multiples. We choose to work with the above expression, because it comes from projecting the canonical measure (4.14) for the natural extension, and is thus a natural choice.

§4.4.2 For $\alpha \geq \frac{1}{2}$

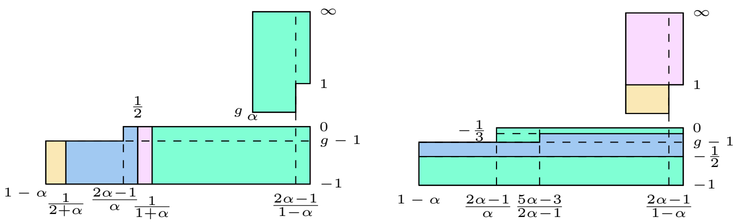
As indicated by Theorem 4.3.5 the situation for $\alpha \geq \frac{1}{2}$ becomes increasingly complicated. Figure 4.3 shows the natural extension domain $\mathcal{D}_\alpha$ for $\alpha \in \left[\frac{1}{2}, \frac{1}{2}\sqrt{2}\right)$ with the action of $\mathcal{T}_\alpha$ and Table 4.1 provides the corresponding densities. We do not provide further details as the proofs are exactly like the one for $0 < \alpha < \frac{1}{2}$.

α	**Density** f_α
$[\frac{1}{2}, g)$	$\frac{1}{1-x}\mathbf{1}_{[1-\alpha,\alpha]}(x) + \frac{1}{x(1-x)}\mathbf{1}_{[\alpha,\frac{1-\alpha}{\alpha}]}(x) + \frac{x^2+1}{x(1-x^2)}\mathbf{1}_{[\frac{1-\alpha}{\alpha},1]}(x)$
$[g, \frac{2}{3})$	$\left(\frac{1}{1-x} + \frac{1}{x+\frac{1}{g-1}}\right)\mathbf{1}_{[1-\alpha,\frac{2\alpha-1}{\alpha}]}(x) + \frac{1}{1-x}\mathbf{1}_{[\frac{2\alpha-1}{\alpha},\alpha]}(x)+$
	$+\left(\frac{1}{1-x} + \frac{1}{x} - \frac{1}{x+\frac{1}{g}}\right)\mathbf{1}_{[\alpha,\frac{2\alpha-1}{1-\alpha}]}(x) + \frac{x^2+1}{x(1-x^2)}\mathbf{1}_{[\frac{2\alpha-1}{1-\alpha},1]}(x)$
$[\frac{2}{3}, \frac{1}{2}\sqrt{2}]$	$\left(\frac{1}{1-x} + \frac{1}{x+\frac{1}{g-1}}\right)\mathbf{1}_{[1-\alpha,\frac{2\alpha-1}{\alpha}]}(x) + \frac{1}{1-x}\mathbf{1}_{[\frac{2\alpha-1}{\alpha},\alpha]}(x)+$
	$+\left(\frac{1}{1-x} + \frac{1}{x} - \frac{1}{x+\frac{1}{g}}\right)\mathbf{1}_{[\alpha,\frac{1-\alpha}{2\alpha-1}]}(x)+$
	$+\left(\frac{1}{1-x} + \frac{1}{x+1} - \frac{1}{x+\frac{1}{g}} + \frac{1}{x} - \frac{1}{x+\frac{1}{g+1}}\right)\mathbf{1}_{[\frac{1-\alpha}{2\alpha-1},1]}(x)$

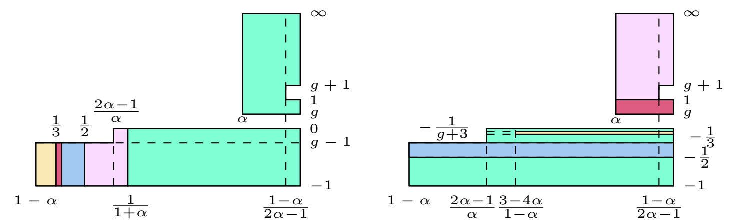
Table 4.1: Invariant densities for $\alpha \in \left[\frac{1}{2}, \frac{1}{2}\sqrt{2}\right]$.



(a) $\alpha \in [\frac{1}{2}, g)$



(b) $\alpha \in [g, \frac{2}{3})$



(c) $\alpha \in [\frac{2}{3}, \frac{1}{2}\sqrt{2})$

Figure 4.3: The maps $\mathcal{T}_\alpha$ for various values of α. Areas on the left are mapped to areas on the right with the same color.

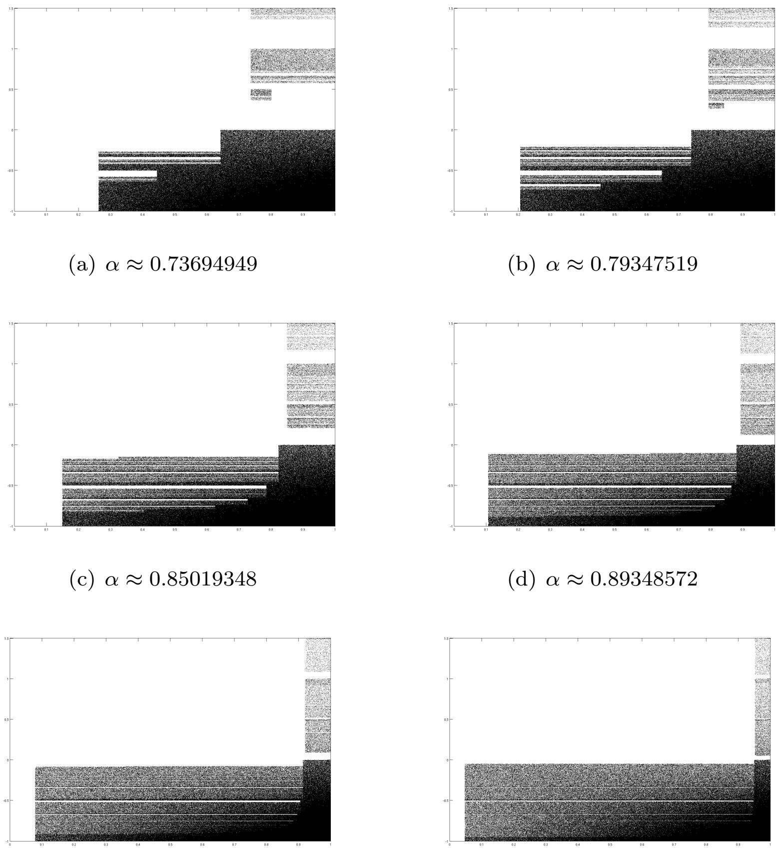
As α increases even further, the domain $\mathcal{D}_\alpha$ starts to exhibit a fractal structure. Figure 4.4 shows numerical simulations for various values of $\alpha > \frac{1}{2}\sqrt{2}$.

§4.5 Entropy, wandering rate and isomorphisms

With an explicit expression for the density of μ_α at hand, we can compute several dynamical quantities associated to the systems T_α. In this section we compute the Krengel entropy, return sequence and wandering rate of T_α for a large part of the parameter space $(0, 1)$.

In [K67] Krengel extended the notion of metric entropy to infinite, measure preserving and conservative systems $(X, \mathcal{B}, \mu, T)$ by considering the metric entropy on finite measure induced systems. More precisely, if A is a sweep-out set for T with $\mu(A) < \infty$, T_A the induced transformation of T on A and μ_A the restriction of μ to A, then the *Krengel entropy* of T is defined to be

$$h_{\mathrm{Kr},\mu}(T) = \mu(A) h_{\mu_A}(T_A),$$



(a) $\alpha \approx 0.73694949$

(b) $\alpha \approx 0.79347519$

(c) $\alpha \approx 0.85019348$

(d) $\alpha \approx 0.89348572$

(e) $\alpha \approx 0.92087668$

(f) $\alpha \approx 0.95234649$

Figure 4.4: Numerical simulations of $\mathcal{D}_\alpha$ for $\alpha > \frac{1}{2}\sqrt{2}$.

where $h_{\mu_A}(T_A)$ is the metric entropy of the system $(A, \mathcal{B} \cap A, T_A, \mu_A)$. Krengel proved in [K67] that this quantity is independent of the choice of A. In [Z00, Theorem 6] it is shown that if T is an AFN-map the Krengel entropy can be computed using Rohlin's formula:

$$h_{\mathrm{Kr},\mu}(T) = \int_X \log(|T'|)d\mu. \tag{4.16}$$

The following theorem follows from Lemma 4.2.2, (4.15) and Table 4.1.

4.5.1 Theorem. *For any $\alpha \in \big(0, g\big]$ the system $(I_\alpha, \mathcal{B}_\alpha, \mu_\alpha, T_\alpha)$ has $h_{Kr,\mu_\alpha}(T_\alpha) = \frac{\pi^2}{6}$.*

Proof. First fix $\alpha \in \big(0, \frac{1}{2}\big)$. By Lemma 4.2.2 we can use formula (4.16) to compute the Krengel entropy of T_α. For this computation we use some properties of the dilogarithm function, which is defined by

$$\mathrm{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2} \qquad \text{for } |x| \leq 1,$$

and satisfies (see [L81] for more information)

- $\mathrm{Li}_2(0) = 0$;
- $\mathrm{Li}_2(-1) = -\pi^2/12$;
- $\mathrm{Li}_2(x) + \mathrm{Li}_2(-\frac{x}{1-x}) = -\frac{1}{2}\log^2(1-x)$.

Using the density from (4.15) and these three properties of Li_2 we get

$$\begin{aligned}
\int_{I_\alpha} \log(|T_\alpha'|)d\mu_\alpha &= -2\Big(\int_\alpha^{\frac{\alpha}{1-\alpha}} \frac{\log x}{x} dx + \int_{\frac{\alpha}{1-\alpha}}^{1-\alpha} \frac{\log x}{1+x} dx + 2\int_{1-\alpha}^{1} \frac{\log x}{1-x^2} dx \Big) \\
&= [-\log^2 x]_\alpha^{\frac{\alpha}{1-\alpha}} - 2[\mathrm{Li}_2(-x) + \log x \log(x+1)]_{\frac{\alpha}{1-\alpha}}^{1-\alpha} \\
&\quad - 2[\mathrm{Li}_2(1-x) + \mathrm{Li}_2(-x) + \log x \log(x+1)]_{1-\alpha}^{1} \\
&= -\log^2\Big(\frac{\alpha}{1-\alpha}\Big) + \log^2(\alpha) + 2\,\mathrm{Li}_2\Big(\frac{-\alpha}{1-\alpha}\Big) + 2\log\Big(\frac{\alpha}{1-\alpha}\Big)\log\Big(\frac{1}{1-\alpha}\Big) \\
&\quad - 2\,\mathrm{Li}_2(-1) + 2\,\mathrm{Li}_2(\alpha) \\
&= \log^2(\alpha) - \log^2\big(\frac{\alpha}{1-\alpha}\big) - \log^2(1-\alpha) - 2\log\big(\frac{\alpha}{1-\alpha}\big)\log(1-\alpha) + \frac{\pi^2}{6} \\
&= \log^2(\alpha) - \log^2(\alpha) + 2\log(\alpha)\log(1-\alpha) - 2\log^2(1-\alpha) + \\
&\quad - 2\log(\alpha)\log(1-\alpha) + 2\log^2(1-\alpha) + \frac{\pi^2}{6} \\
&= \frac{\pi^2}{6}.
\end{aligned}$$

A similar computation yields $h_{\mathrm{Kr},\mu_\alpha}(T_\alpha) = \frac{\pi^2}{6}$ for $\alpha \in [\frac{1}{2}, g]$. □

4.5.2 Remark. Numerical evidence using the densities from Table 4.1 suggests that $h_{\mathrm{Kr},\mu_\alpha}(T_\alpha) = \frac{\pi^2}{6}$ for $\alpha \in (g, \frac{1}{2}\sqrt{2})$ as well. Even though we were not able to calculate the Krengel entropy for $\alpha \in (g, \frac{1}{2}\sqrt{2})$ explicitly, we conjecture that in fact $h_{\mathrm{Kr},\mu_\alpha}(T_\alpha) = \frac{\pi^2}{6}$ for all $\alpha \in (0,1)$. This claim is supported by the fact that the Krengel entropy for Nakada's α-continued fraction maps S_α from (4.2) is $\frac{\pi^2}{6}$ as well, see [KSS12, Theorem 2].

The *return sequence* of T_α is the sequence $(a_n(T_\alpha))_{n\geq 1}$ of positive real numbers satisfying (4.7). The pointwise dual ergodicity of each map T_α implies that such a sequence, which is unique up to asymptotic equivalence, exists. The *asymptotic type* of T_α corresponds to the family of all sequences asymptotically equivalent to some positive multiple of $(a_n(T_\alpha))_{n\geq 1}$. The return sequence of a system is related to its wandering rate, which quantifies how big the system is in relation to its subsets of finite measure. To be more precise, if $(X, \mathcal{B}, \mu, T)$ is a conservative, ergodic, measure preserving system and $A \in \mathcal{B}$ a set of finite positive measure, then the wandering rate of A with respect to T is the sequence $(w_n(A))_{n\geq 1}$ given by

$$w_n(A) := \mu\Big(\bigcup_{k=0}^{n-1} T^{-k}A\Big).$$

It follows from [Z00, Theorem 2] that for each of the maps T_α there exists a positive sequence $(w_n(T_\alpha))$ such that $w_n(T_\alpha) \uparrow \infty$ and $w_n(T_\alpha) \sim w_n(A)$ as $n \to \infty$ for all sets A that have positive, finite measure and are bounded away from 1. The asymptotic equivalence class of $(w_n(T_\alpha))$ defines the *wandering rate* of T_α. Using the machinery from [Z00] and the explicit formula of the density we compute both the return sequence and the wandering rate of the maps T_α.

4.5.3 Proposition. For all $\alpha \in (0,1)$ there is a constant $c_\alpha > 0$ such that

$$w_n(T_\alpha) \sim c_\alpha \log n \quad \textit{and} \quad a_n(T_\alpha) \sim \frac{n}{c_\alpha \log n}.$$

If $\alpha \in (0, \frac{1}{2}\sqrt{2})$, then $c_\alpha = 1$.

Proof. Using the Taylor expansion of the maps T_α, one sees that for $x \to 1$ we have $T_\alpha(x) = x - (x-1)^2 + o((x-1)^2)$. Hence, T_α admits what are called *nice expansions* in [Z00]. For $x \in \big(\frac{1}{1+\alpha}, 1\big]$ we can write $f_\alpha(x) = \frac{x-2}{x-1}H(x)$, where the function $x \mapsto \frac{x-2}{x-1}$ corresponds to the map called G in [Z00, Theorem A]. It then follows by [Z00, Theorems 3 and 4] that the wandering rate is

$$w_n(T_\alpha) \sim c_\alpha \log n \tag{4.17}$$

and the return sequence is

$$a_n(T_\alpha) \sim \frac{n}{c_\alpha \log n}, \tag{4.18}$$

for $c_\alpha = \lim_{x\uparrow 1} H(x)$. For $\alpha \in \big(0, \frac{1}{2}\sqrt{2}\big)$ the explicit formula for the densities from (4.15) and Table 4.1 gives $c_\alpha = 1$. $\square$

We have now established all parts of Theorem 4.1.2.

Proof of Theorem 4.1.2. The densities are given by (4.15) and listed in Table 4.1. The entropy is given by Theorem 4.5.1 and the wandering rate and return sequence in Proposition 4.5.3. $\square$

4.5.4 Remark. (i) As in Remark 4.5.2 we suspect that in fact $c_\alpha = 1$ for all $\alpha \in (0,1)$.

(ii) Since all the results from [Z00] apply to our family, we can use these to get an even more detailed description of the ergodic behaviour of the maps T_α. We briefly mention a few more results for $\alpha \in (0, 1/2\sqrt{2}]$. Since the return sequence $(a_n(T_\alpha))_{n\geq 1}$

is regularly varying with index 1, by [Z00, Theorem 5] and [A97, Corollary 3.7.3], we have

$$\frac{\log n}{n}\sum_{k=0}^{n-1} f\circ T_\alpha^k \xrightarrow{\mu_\alpha} \int_{I_\alpha} f\,d\mu_\alpha, \qquad \text{for } f\in L^1(I_\alpha,\mathcal{B}_\alpha,\mu_\alpha) \text{ and } \int_{I_\alpha} f\,d\mu_\alpha \neq 0. \quad (4.19)$$

where the convergence is in measure since the regularly varying index 1 of the return sequence $(a_n(T_\alpha))_{n\geq 1}$ turns the right hand side of the Darling-Kac Theorem into a constant. In other words, a weak law of large numbers holds for T_α.

In addition, we can obtain asymptotics for the excursion times to the interval $\left[\frac{1}{1+\alpha},1\right]$, corresponding to the rightmost branch of T_α. Let Y be a sweep-out set, T_Y the induced map on Y and $\varphi : x\mapsto \min\{n\geq 1 : T^n(x)\in Y\}$ the first return map. Write $\varphi_n^Y := \sum_{k=0}^{n-1}\varphi\circ T_Y^k$ and note that the asymptotic inverse of the sequence $(a_n(T_\alpha))_{n\geq 1}$ is $(n\log n)_{n\geq 1}$, so that the statement from (4.19) is equivalent to the following dual:

$$\frac{1}{n\log n}\varphi_n^Y \xrightarrow{\mu_\alpha} \frac{1}{\mu_\alpha(Y)}.$$

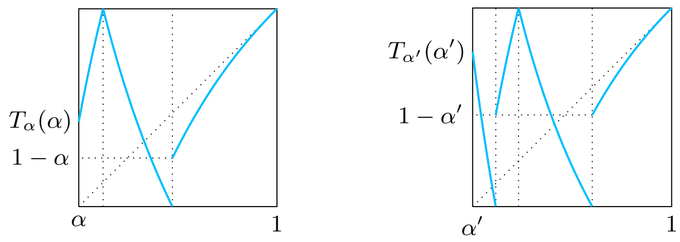
If we induce on $Y := \left[\min\{\alpha,1-\alpha\},\frac{1}{1+\alpha}\right]$, then φ_n^Y sums the lengths (increased by n) of the first n blocks of consecutive digits $(\epsilon,d)=(-1,2)$, and we obtain

$$\frac{1}{n\log n}\varphi_n^Y - \frac{1}{\log n} \xrightarrow{\mu_\alpha} \frac{1}{\mu_\alpha(Y)}.$$

From Theorem 4.1.2, it follows that for $\alpha<\frac{1}{2}$, $\mu_\alpha(Y)=\log(2+\alpha)$. Note that for α decreasing the right hand side is increasing, meaning we spend on average more time in $\Delta(-1,2)$. Intuitively, for a smaller α, every time we enter $\Delta(-1,2)$ we are closer to the indifferent fixed point, and it takes longer before we manage to escape from it.

Note that the Krengel entropy, return sequence and wandering rate we found do not display any dependence on α. These quantities give isomorphism invariants for dynamical systems with infinite invariant measures. Two measure preserving dynamical systems $(X,\mathcal{B},\mu,T)$ and $(Y,\mathcal{C},\nu,S)$ on σ-finite measure spaces are called *c-isomorphic* for $c\in(0,\infty]$ if there are sets $N\in\mathcal{B}$, $M\in\mathcal{C}$ with $\mu(N)=0=\nu(M)$ and $T(X\backslash N)\subseteq X\backslash N$ and $S(Y\backslash M)\subseteq Y\backslash M$ and if there is a map $\phi : X\backslash N\to Y\backslash M$ that is invertible, bi-measurable and satisfies $\phi\circ T=S\circ\phi$ and $\mu\circ\phi^{-1}=c\cdot\nu$. Invariants for c-isomorphisms are the asymptotic proportionality classes of the return sequence (see [A97, Propositions 3.7.1 and 3.3.2] and [Z00, Remark 8]) and the normalised wandering rates, which combine the Krengel entropy with the wandering rates (see e.g. [T83, Z00]). It follows from Theorem 4.1.2 that all these quantities are equal for all T_α, $\alpha\in(0,\frac{1}{2}\sqrt{2})$. Using the idea from [K14], however, we find many pairs α and α' such that T_α and $T_{\alpha'}$ are not c-isomorphic for any $c\in(0,\infty]$. Consider for example any $\alpha\in\left(\sqrt{2}-1,\frac{1}{2}\right)$, so that $\alpha\in\left(\frac{1}{2+\alpha},\frac{1}{2}\right)$, and any $\alpha'\in\left(\frac{1}{3},\frac{3-\sqrt{5}}{2}\right)$, so that $T_{\alpha'}(\alpha')>1-\alpha'$, see Figure 4.5. For a contradiction, suppose that there is a c-isomorphism $\phi : I_\alpha\to I_{\alpha'}$ for some $c\in(0,\infty]$. Let $J=[\alpha,1-\alpha]$ and note that any $x\in J$ has precisely one pre-image under T_α. Since $\phi\circ T_\alpha = T_{\alpha'}\circ\phi$ and ϕ is invertible, any element of the set $\phi(J)$ must also have precisely one pre-image. Since

$T_{\alpha'}(\alpha') > 1-\alpha'$, there are no such points, so $\mu_{\alpha'}(\phi(J)) = 0$. On the other hand, since J is bounded away from 1, it follows that $0 < \mu_\alpha(J) < \infty$. Hence, there can be no c, such that $\mu_{\alpha'} \circ \phi^{-1} = c \cdot \mu_\alpha$. Obviously a similar argument holds for many other combinations of α and α', even for $\alpha > \frac{1}{2}$, and in case the argument does not work for T_α and $T_{\alpha'}$, one can also consider iterates T_α^n, $T_{\alpha'}^n$. Hence, even though the above discussed isomorphism invariants are equal for all $\alpha \in \left(0, \frac{1}{2}\right)$, it is not generally the case that any two maps T_α are c-isomorphic. We conjecture that for almost all pairs (α, α'), the maps T_α and $T_{\alpha'}$ are not c-isomorphic.



(a) $\alpha \in \left(\sqrt{2}-1, \frac{1}{2}\right)$ (b) $\alpha' \in \left(\frac{1}{3}, \frac{3-\sqrt{5}}{2}\right)$

Figure 4.5: Maps T_α and $T_{\alpha'}$ that are not c-isomorphic for any $c \in (0, \infty]$.

§4.6 Remarks

1. The theory of piecewise affine interval maps is richer than the one available for smooth transformations. This is in part due to the fact that the former present a piecewise constant derivatives, leading for instance to a convenient representation of the Perron-Frobenius operator. However, for the specific case of flipped α-continued fraction maps, the contrary is true. Indeed, as remarked at the end of Chapter 2, the c-Lüroth map $T_{1,\alpha}$ can be seen as a linearisation of the flipped α-continued fraction map T_α for $\alpha \in (0, \frac{1}{2})$. While the density, of the absolutely continuous invariant measure, is explicitely given for T_α in Theorem 4.1.2 as a piecewise smooth function, for $T_{1,\alpha}$ we only know from [K90] that the density is a step function, and from Corollary 2.4.13 we can deduce that it is defined on a finite partition for rational parameters α and on a countable one for irrationals.

2. The consequences of matching on the structure of the density of an absolutely continuous invariant measure are given in [BCMP18, Theorem 1.2] for piecewise affine (or smooth) interval maps admitting a probability measure. This chapter provides an analogous result for a specific class of continued fraction maps with a σ-finite infinite invariant measure. In Chapter 5, we extend the notion of matching and explore its consequences for random interval maps.

CHAPTER 5

Matching and Measure for Random Systems

This chapter is based on: [DKM20].

Abstract

We extend the notion of matching for interval maps to random matching for pseudo-skew products on the interval. For a certain family of piecewise affine interval random systems the property of random matching implies that any invariant density is piecewise constant. Furthermore, we introduce a one-parameter family of random dynamical systems that produce signed binary expansions of numbers in the interval $[-1, 1]$. We provide results on minimal weight expansions by proving that the frequency of the digit 0 in the associated signed binary expansions never exceeds $1/2$. We do this by showing that the family has random matching for Lebesgue almost every parameter.

§5.1 Motivation and context

The research on optimal algorithms that raise elements of a group into some power has a long and rich history. It is a matter of fact that the computation of powers of elements in a group is the basis of many public key cryptosystems, where the group chosen is either the multiplicative group of a finite field $\mathbb{F}_q$ or the group of points on an elliptic curve. The optimality of the algorithms refers to the ability of computing high powers in a short amount of time. One way to reduce the time complexity is given by the so-called binary method, introduced in [K69], and based on the binary expansion of the power. More precisely, if x is an element of a given group, and $a = \sum_{i=0}^{n} d_i 2^i \in \mathbb{N}$ for some digits $d_i \in \{0,1\}$, then

$$x^a = \prod_{i=0}^{n} x^{d_i 2^i},$$

and the power x^a is computed by taking the product of the repeated squarings. While the number of squarings is given by the length n of the binary expansion of a, the number of multiplications equals the number of non-zero bits d_i in the expansion or its *Hamming weight*. Clearly, the lower it is, the less multiplications are required and the faster the algorithm is. To increase the number of zero bits, [B51] introduced a *signed binary representation*, i.e., a binary representation with digits in the set $\{-1,0,1\}$. This signed binary method was later adopted in several methods in elliptic cryptosystems, see e.g. [CMO98, HP06] and the references therein.

For any fixed integer a, its ordinary binary representation with digits $\{0,1\}$ is uniquely determined, but this is not the case for the signed one, with digits in $\{-1,0,1\}$. In fact, each integer has infinitely many signed binary representations, which led to the study of algorithms that choose the ones with minimal weight (see e.g. [MO90, KT93, LK97]). The authors of [GH06] showed that typically a number has several signed binary representations with minimal weight, but already in the 1960's Reitwiesner proved in [R60] that the signed representation is unique when adding the constraint $d_i d_{i+1} = 0$. Such representations are usually known as *signed separated binary expansions*, or SSB for short. In [DKL06] it is shown how to obtain SSB expansions through the binary odometer and a three state Markov chain. Furthermore, in [DKL06] the set $K := \{d_1 d_2 \ldots \in \{-1,0,1\}^{\mathbb{N}} : \forall i \in \mathbb{N}, d_i d_{i+1} = 0\}$ is introduced as a compactification of $\mathbb{Z}$. The authors identify the set K, endowed with the left shift σ, with the map $S(x) = 2x \mod \mathbb{Z}$ on the interval $[-\frac{2}{3}, \frac{2}{3}]$, through the conjugation

$$\psi(d_1 d_2 \ldots) = \sum_{k=1}^{\infty} \frac{d_k}{2^k}. \tag{5.1}$$

The dynamical viewpoint allows them to obtain metric properties of the system (K,σ), such as a σ-invariant measure, the maximal entropy and the frequency of 0 in typical expansions.

In [DK17] this dynamical approach was further developed by considering a family of *symmetric doubling maps* $\{S_\alpha : [-1,1] \to [-1,1]\}_{\alpha \in [1,2]}$ defined by $S_\alpha(x) = 2x - d\alpha$

and

$$d = \begin{cases} -1, & \text{if } x \in [-1, -\frac{1}{2}), \\ 0, & \text{if } x \in [-\frac{1}{2}, \frac{1}{2}], \\ 1, & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

The map S from [DKL06], producing SSB expansions, is then easily identified with the map $S_{\frac{3}{2}}$, through the conjugacy $x \mapsto \frac{3}{2}x$. For each $\alpha \in [1,2]$ iterations of S_α produce signed binary expansions of the form $x = \sum_{k=1}^{\infty} \frac{d_k}{2^k}$ with $d_k \in \{-1, 0, 1\}$ for each number $x \in [-1,1]$. The authors of [DK17] showed that the frequency of 0 in such expansions depends continuously on the parameter α and it takes its maximum value $\frac{2}{3}$, corresponding to the minimum Hamming weight of $\frac{1}{3}$, precisely for α in the subinterval $[\frac{6}{5}, \frac{3}{2}]$. It follows that the SSB expansion of integers produces sequences where typically only $\frac{1}{3}$ of the digits is different from zero. The results from [DK17] are achieved by finding a detailed description of the invariant probability density f_α of S_α for each value of α and then explicitly computing the frequency of the digit 0 using Birkhoff's Ergodic Theorem. The fact that the family $\{S_\alpha\}$ exhibits the dynamical phenomenon of matching was essential for these results.

In this chapter we consider signed binary expansions in the framework of random dynamical systems. The advantage of random systems in this context is that a single random system produces many more number expansions per number than a deterministic map, allowing one to study the properties of many expansions simultaneously. See e.g. [DK03, DdV05, DdV07, DK07, KKV17, DO18] for the use of random systems in the study of different types of number expansions. We will introduce a family of random systems $\{R_\alpha\}_{\alpha\in[1,2]}$, called *random symmetric doubling maps*, such that each element R_α produces for typical numbers in the interval $[-1,1]$ infinitely many different signed binary expansions. This is contrary to the map S_α, which produces a unique signed binary expansion for each number in $[-1,1]$. Our main result for the family $\{R_\alpha\}_{\alpha\in[1,2]}$ is that the frequency of the digit 0 in typical signed binary expansions produced by any of the maps R_α is at most $\frac{1}{2}$, and therefore the Hamming weight is at least $\frac{1}{2}$. This reinforces the result from [DK17] that the maps S_α with $\alpha \in \left[\frac{6}{5}, \frac{3}{2}\right]$ perform best in terms of minimal weight.

We obtain this result from Birkhoff's Ergodic Theorem after gathering detailed knowledge on the invariant probability densities of the random maps R_α. We first express these densities as infinite sums of indicator functions using the algebraic procedure from [KM18]. To compute the frequency of 0 we need to evaluate the Lebesgue integral of these densities over part of the domain and therefore we convert the infinite sums into finite sums. For this we introduce a random version of the dynamical concept of matching that is available for one-dimensional systems (see e.g. [NN08, DKS09, BCIT13, BSORG13, BCK17, BCMP18, CIT18, CM18, KLMM20]). Our definition of random matching properly extends the one-dimensional notion of matching and we illustrate the concept with examples of random continued fraction maps and random generalised β-transformations. We show that under certain mild conditions, if a random system of piecewise affine maps defined on the same interval has random matching, then any invariant probability density of the system is piecewise constant. The precise formulation of this statement and the conditions are given in the next section. Finally, we use this random matching property to show that for Lebesgue almost all parameters α the invariant density of the random systems R_α, producing signed binary expansions, is in fact piecewise constant.

This chapter is outlined as follows. The second section is devoted to random matching for random systems defined on an interval. We first recall some preliminaries on invariant measures for random interval maps. We then define the notion of random matching and state and prove the result about densities of random systems of piecewise affine maps with matching. We also discuss the examples of random continued fraction transformations and random generalised β-transformations. In the third section we introduce and discuss the family $\{R_\alpha\}$ of random symmetric doubling maps and the corresponding signed binary expansions. We prove that R_α has random matching for Lebesgue almost all $\alpha \in [1,2]$. We also provide a full description of the *matching intervals*, i.e., intervals of parameters that exhibit comparable matching behaviour, and describe the invariant densities of the maps R_α. Finally we prove that typically the frequency of the digit 0 in the signed binary expansions produced by R_α does not exceed $\frac{1}{2}$ for any parameter α.

§5.2 Random matching

Matching is a dynamical phenomenon observed in certain families of piecewise smooth interval maps. Recall that, if $T : I \to I$ is such a map on an interval I of real numbers, then we say that T has *matching* if for every discontinuity point c of T or of the derivative T' the orbits of the left and right limits $T^k(c^-) = \lim_{x\uparrow c} T^k(x)$ and $T^k(c^+) = \lim_{x\downarrow c} T^k(x)$ eventually meet, i.e., if for each c there exist positive constants $M = M_c$ and $Q = Q_c$, such that

$$T^M(c^-) = T^Q(c^+). \tag{5.2}$$

T is then said to have *strong matching* if, moreover, the orbits of the left and right limits have equal one-sided derivatives at the moment they meet, i.e., if besides (5.2) it also holds that

$$(T^M)'(c^-) = (T^Q)'(c^+). \tag{5.3}$$

It was proven in [BCMP18, Theorem 1.2] (see also Remark 1.3 in [BCMP18]) that for any piecewise smooth T with strong matching, any invariant probability measure μ that is absolutely continuous with respect to the Lebesgue measure has a piecewise smooth density. For continued fraction transformations (as in [NN08, DKS09, KLMM20] for example) it seems that matching is sufficient to guarantee the existence of a piecewise smooth density (since this is sufficient to construct a natural extension with finitely many pieces). The strong matching condition then enforces some stability in the matching behaviour of certain one-parameter families of continued fraction maps, which becomes visible in the appearance of so called *matching intervals* in the parameter space: If such a family has strong matching for one parameter, then one can find an interval of parameters around it, such that all the corresponding transformations have matching in the same number of steps and with comparable orbits.

In this section we extend the above definitions of matching and strong matching to random dynamical systems. Let $\Omega \subseteq \mathbb{N}$ be the index set of the available maps, so we have a collection of transformations $\{T_j : I \to I\}_{j \in \Omega}$ defined on the same interval $I \subseteq \mathbb{R}$ at our disposal. Let $\sigma : \Omega^{\mathbb{N}} \to \Omega^{\mathbb{N}}$ be the left shift on one-sided sequences. Recall by Definition 1.2.8 that the *random map* or *pseudo-skew product* $R : \Omega^{\mathbb{N}} \times I \to \Omega^{\mathbb{N}} \times I$ is defined by

$$R(\omega, x) = (\sigma(\omega), T_{\omega_1} x).$$

Let $\mathbf{p} = (p_j)_{j \in \Omega}$ be a positive probability vector, representing the probabilities with which we choose the maps T_j. Consider the product measure $m_{\mathbf{p}} \times \mu_{\mathbf{p}}$ from Definition 1.2.9, for $m_{\mathbf{p}}$ the $\mathbf{p}$-Bernoulli measure on $\Omega^{\mathbb{N}}$ and $\mu_{\mathbf{p}}$ a probability measure on I that is absolutely continuoues with resepct to the Lebesgue measure λ, and denote its density by $f_{\mathbf{p}}$. Here we call $\mu_{\mathbf{p}}$ a *stationary measure* and $f_{\mathbf{p}}$ an *invariant density* for R.

In the literature there exist various sets of conditions under which the existence of such an invariant measure is guaranteed. See for example [M85, P84, B00, GB03, BG05, I12]. Here we explicitly mention a special case of the conditions by Inoue from [I12] which are simple to state and suit our purposes in the next sections. Assume that the following three conditions hold:

(a1) There is a finite or countable interval partition $\{I_i\}_{i \in \Lambda}$, such that each map T_j is C^1 and monotone on the interior of each interval I_i.

Let C denote the set of all boundary points of the intervals I_i that are in the interior of I. We choose the collection $\{I_i\}$ as small as possible, so that C contains precisely those points that are a critical point of T_j or T_j' for at least one $j \in \Omega$. We call elements $c \in C$ *critical points* for the corresponding random system R.

(a2) The random system R is *expanding on average*, i.e., there exists a constant $0 < \rho < 1$, such that $\sum_{j \in \Omega} \frac{p_j}{|T_j'(x)|} \leq \rho$ holds for each $x \in I \setminus C$.

(a3) For each $j \in \Omega$ the map

$$x \mapsto \begin{cases} \frac{p_j}{|T_j'(x)|} & \text{if } x \neq c, \\ 0 & \text{otherwise,} \end{cases}$$

is of bounded variation.

It then follows from [I12, Theorem 5.2] that an invariant measure for R of the form $m_{\mathbf{p}} \times \mu_{\mathbf{p}}$ exists. Let $\mathcal{R}$ denote the class of random maps R that satisfy these conditions. We will define random matching for maps in $\mathcal{R}$, but first we recall some notation on sequences and strings.

For each $k > 0$ the set $\Omega^k = \{\mathbf{u} = u_1 \cdots u_k : u_i \in \Omega, 1 \leq i \leq k\}$ is the set of all k-strings of elements in Ω. We let $\Omega^0 = \{\epsilon\}$, with ϵ the empty string. For a finite string $\mathbf{u}$ let $|\mathbf{u}|$ denote its length, i.e., $|\mathbf{u}| = k$ if $\mathbf{u} \in \Omega^k$. Also, for $1 \leq n \leq k$ we let $\mathbf{u}_1^n := u_1 \cdots u_n$ and we set $\mathbf{u}_1^0 = \epsilon$. Similarly, for an infinite sequence $\omega \in \Omega^{\mathbb{N}}$ and $n \geq 1$ we use the notation $\omega_1^n := \omega_1 \cdots \omega_n$ with $\omega_1^0 = \epsilon$. Finally, we use square brackets to denote cylinder sets, so

$$[\mathbf{u}] = \{\omega \in \Omega^{\mathbb{N}} : \omega_1 \cdots \omega_{|\mathbf{u}|} = \mathbf{u}\}. \tag{5.4}$$

For $\mathbf{u} \in \Omega^k$ and $0 \leq n \leq k$, let

$$T_{\mathbf{u}} = T_{u_k} \circ T_{u_{k-1}} \circ \cdots \circ T_{u_1} \quad \text{and} \quad T_{\mathbf{u}}^n = T_{\mathbf{u}_1^n} = T_{u_n} \circ T_{u_{n-1}} \circ \cdots \circ T_{u_1}.$$

Note that $T_{\mathbf{u}}^0 = T_{\mathbf{u}_1^0} = T_\epsilon = id$. Similarly if $\omega \in \Omega^{\mathbb{N}}$, we let $T_\omega^n = T_{\omega_1^n} = T_{\omega_n} \circ T_{\omega_{n-1}} \circ \cdots \circ T_{\omega_1}$ for any $n \geq 0$. For $\mathbf{u} \in \Omega^k$ the left and right random orbits of the critical points $c \in C$ are

$$T_{\mathbf{u}}(c^-) = \lim_{x \uparrow c} T_{\mathbf{u}}(x) \quad \text{and} \quad T_{\mathbf{u}}(c^+) = \lim_{x \downarrow c} T_{\mathbf{u}}(x).$$

The one-sided derivatives along $\mathbf{u}$ are given by

$$T'_{\mathbf{u}}(c^-) = \lim_{x \uparrow c} \prod_{n=1}^{k} T'_{u_n}(T_{\mathbf{u}_1^{n-1}}(x)) \quad \text{and} \quad T'_{\mathbf{u}}(c^+) = \lim_{x \downarrow c} \prod_{n=1}^{k} T'_{u_n}(T_{\mathbf{u}_1^{n-1}}(x)).$$

We use the abbreviation $p_{\mathbf{u}} := p_{u_1} \cdots p_{u_k}$ with $p_\epsilon = 1$.

5.2.1 Definition. (Random matching) A random map $R \in \mathcal{R}$ has *random matching* if for every $c \in C$ there exists an $M = M_c \in \mathbb{N}$ and a set

$$Y = Y_c \subseteq \Big\{T_\omega^k(c^-) : \omega \in \Omega^{\mathbb{N}},\, 1 \leq k \leq M\Big\} \cap \Big\{T_\omega^k(c^+) : \omega \in \Omega^{\mathbb{N}},\, 1 \leq k \leq M\Big\}$$

such that for every $\omega \in \Omega^{\mathbb{N}}$ there exist $k = k_c(\omega), \ell = \ell_c(\omega) \leq M$ with $T_\omega^k(c^-), T_\omega^\ell(c^+) \in Y$.

The main difference with one-dimensional matching as in (5.2) and (5.3) is that in a random system R the critical points have many different random orbits. Definition 5.2.1 states that any random orbit of the left or the right limit of any critical point c passes through the set Y_c at the latest at time M. The indices k, ℓ are introduced to cater for the possibility that these orbits pass through the set Y_c at different moments. Since all points in Y_c are in the orbit of both c^- and c^+, this implies that all random orbits of the left limit meet with some random orbit of the right limit and vice versa. This corresponds to the statement in (5.2). Note that we do not ask $T_\omega^k(c^-) = T_\omega^\ell(c^+)$.

5.2.2 Definition. (Strong random matching) A random map $R \in \mathcal{R}$ has *strong random matching* if it has random matching and if for each $c \in C$ and $y \in Y_c$ the following holds. Set

$$\Omega(y)^- = \Big\{\mathbf{u} \in \bigcup_{k=1}^{M} \Omega^k : \exists \omega \in \Omega^{\mathbb{N}} \text{ with } \mathbf{u} = \omega_1 \cdots \omega_{k_c(\omega)} \text{ and } T_{\mathbf{u}}(c^-) = y\Big\},$$

$$\Omega(y)^+ = \Big\{\mathbf{u} \in \bigcup_{k=1}^{M} \Omega^k : \exists \omega \in \Omega^{\mathbb{N}} \text{ with } \mathbf{u} = \omega_1 \cdots \omega_{\ell_c(\omega)} \text{ and } T_{\mathbf{u}}(c^+) = y\Big\}.$$

Then,

$$\sum_{\mathbf{u}\in\Omega(y)^-} \frac{p_{\mathbf{u}}}{T'_{\mathbf{u}}(c^-)} = \sum_{\mathbf{u}\in\Omega(y)^+} \frac{p_{\mathbf{u}}}{T'_{\mathbf{u}}(c^+)}. \tag{5.5}$$

Definition 5.2.2 guarantees that one can choose the times k, ℓ such that at those times orbits enter the set Y with the same weighted derivative. This is comparable to (5.3). Note that

$$\bigcup_{y\in Y_c} \bigcup_{\mathbf{u}\in\Omega(y)^-} [\mathbf{u}] = \Omega^{\mathbb{N}} = \bigcup_{y\in Y_c} \bigcup_{\mathbf{u}\in\Omega(y)^+} [\mathbf{u}],$$

where $[\mathbf{u}]$ is a cylinder as defined in (5.4), so we have indeed captured all random orbits of c. Note that Definition 5.2.2 depends on the choices of $k_c(\omega)$ and $\ell_c(\omega)$ for each c in Definition 5.2.1.

If Ω consists of one element only, then the random map is actually a deterministic map. In this case Definition 5.2.1 and Definition 5.2.2 reduce to the definitions of one-dimensional matching and strong matching given in (5.2) and (5.3), so the random definitions extend the deterministic ones.

In case each map $T_j : I \to I$ is piecewise affine on a finite partition $c_0 < c_1 < \ldots < c_N$ the conditions (a1) and (a3) are automatically satisfied and under some additional assumptions strong random matching has consequences for invariant densities. For this result we consider a subset of the collection of random maps $\mathcal{R}$. We define the subset $\mathcal{R}_A \subset \mathcal{R}$ to be the set of random systems in $\mathcal{R}$ that satisfy the following additional assumptions:

(c1) There exists a finite interval partition $\{I_i\}_{1\le i\le N}$ of $I = [c_0, c_N]$ given by the points $c_0 < c_1 < \ldots < c_N$, such that each map $T_j : I \to I$, $j \in \Omega$, is piecewise affine with respect to this partition. In other words, for each $j \in \Omega$ and $1 \le i \le N$ we can write $T_j|_{(c_{i-1},c_i)}(x) = k_{i,j}x + d_{i,j}$ for some constants $k_{i,j}, d_{i,j}$.

(c2) For each $1 \le i \le N$ there is an $1 \le n \le N$, such that

$$\frac{\sum_{j\in\Omega} \frac{p_j}{k_{i,j}} d_{i,j}}{1 - \sum_{j\in\Omega} \frac{p_j}{k_{i,j}}} \neq \frac{\sum_{j\in\Omega} \frac{p_j}{k_{n,j}} d_{n,j}}{1 - \sum_{j\in\Omega} \frac{p_j}{k_{n,j}}}. \tag{5.6}$$

(c3) For each $1 \le i \le N$, $\sum_{j\in\Omega} \frac{p_j}{k_{i,j}} \neq 0$.

Using the results from Chapter 3, we will show that for $R \in \mathcal{R}_A$ the following holds.

5.2.3 Theorem. *Let $R \in \mathcal{R}_A$. If R has strong random matching, then there exists an invariant probability measure $m_{\mathbf{p}} \times \mu_{\mathbf{p}}$ for R with $\mu_{\mathbf{p}}$ absolutely continuous with respect to Lebesgue and such that its density $f_{\mathbf{p}}$ is piecewise constant. If moreover every map T_j is expanding, i.e., if $|k_{i,j}| > 1$ for each $1 \le i \le N$ and $j \in \Omega$, then any invariant probability density $f_{\mathbf{p}}$ of R is piecewise constant.*

Assumptions (c2) and (c3) are used in Chapter 3 to prove that for systems in $\mathcal{R}_A$ there exists an invariant probability density function that can be written as an infinite sum of indicator functions. We use this fact in the proof below. These conditions, which are not very restrictive, guarantee that the method from Chapter 3 works, but they might not be necessary for the results from Theorem 3.4.1 and Theorem 5.2.3. In fact, the deterministic analog of Theorem 5.2.3, which can be found in [BCMP18, Theorem 1.2], does not have a condition like (5.6). Their proof uses an induced system with a full branched return map instead. One could try to transfer the proof of [BCMP18, Theorem 1.2] to the setting of random interval maps to avoid (c2) and (c3). Then, the recent results from Inoue in [I20] on first return time functions for random systems seem relevant. These results show, however, that an induced system for a random interval map will become position dependent instead of i.i.d., which might make such an extension not so straightforward.

Proof. The set of critical points of R is given by $C = \{c_1, \dots, c_{N-1}\}$. Any random map $R \in \mathcal{R}_A$ satisfies the conditions of Theorem 3.4.1. Thus, there exists an invariant probability measure $m_{\mathbf{p}} \times \mu_{\mathbf{p}}$ for R with a probability density $f_{\mathbf{p}}$ for $\mu_{\mathbf{p}}$ of the form

$$f_{\mathbf{p}} = \sum_{i=1}^{N-1} \gamma_i \sum_{k\geq 1} \sum_{\mathbf{u}\in\Omega^k} \Big(\frac{p_{\mathbf{u}}}{T'_{\mathbf{u}}(c_i^-)} 1_{[c_0, T_{\mathbf{u}}(c_i^-))} - \frac{p_{\mathbf{u}}}{T'_{\mathbf{u}}(c_i^+)} 1_{[c_0, T_{\mathbf{u}}(c_i^+))} \Big), \tag{5.7}$$

for some constants γ_i depending only on the critical points c_i. Fix an i and let M, Y be such that R satisfies the conditions of Definition 5.2.1 and Definition 5.2.2 for c_i. Then by (5.5)

$$\sum_{y\in Y} \Big(\sum_{\mathbf{u}\in\Omega(y)^-} \frac{p_{\mathbf{u}}}{T'_{\mathbf{u}}(c_i^-)} 1_{[c_0, T_{\mathbf{u}}(c_i^-))} - \sum_{\mathbf{u}\in\Omega(y)^+} \frac{p_{\mathbf{u}}}{T'_{\mathbf{u}}(c_i^+)} 1_{[c_0, T_{\mathbf{u}}(c_i^+))} \Big) = 0.$$

For each $1 \leq i \leq N-1$ and each $1 \leq k \leq M$, let

$$\Omega_-^{i,k} = \{\mathbf{u} \in \Omega^k \,:\, \exists \omega \in \Omega^{\mathbb{N}} \text{ with } \mathbf{u} = \omega_1 \cdots \omega_k \text{ and } k < k_{c_i}(\omega)\}$$

and similarly

$$\Omega_+^{i,k} = \{\mathbf{u} \in \Omega^k \,:\, \exists \omega \in \Omega^{\mathbb{N}} \text{ with } \mathbf{u} = \omega_1 \cdots \omega_k \text{ and } k < \ell_{c_i}(\omega)\}.$$

Then $f_{\mathbf{p}}$ can be written as

$$f_{\mathbf{p}} = \sum_{i=1}^{N-1} \gamma_i \sum_{k=1}^{M} \Big(\sum_{\mathbf{u}\in\Omega_-^{i,k}} \frac{p_{\mathbf{u}}}{T'_{\mathbf{u}}(c_i^-)} 1_{[c_0, T_{\mathbf{u}}(c_i^-))} - \sum_{\mathbf{u}\in\Omega_+^{i,k}} \frac{p_{\mathbf{u}}}{T'_{\mathbf{u}}(c_i^+)} 1_{[c_0, T_{\mathbf{u}}(c_i^+))} \Big).$$

Hence $f_{\mathbf{p}}$ is constant on each interval in the finite partition of I specified by the orbit points in the set

$$\bigcup_{i=1}^{N-1} \bigcup_{k=1}^{M} \Big(\{T_{\mathbf{u}}(c_i^-) \,:\, \mathbf{u} \in \Omega_-^{i,k}\} \cup \{T_{\mathbf{u}}(c_i^+) \,:\, \mathbf{u} \in \Omega_+^{i,k}\} \Big).$$

This gives the first part of the result.

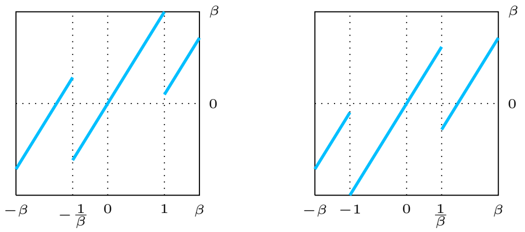
For the second part, note that under the additional assumption that $|k_{i,j}| > 1$ for all i, j the map R satisfies the conditions of Theorem 3.5.3. As a consequence, any invariant density $f_{\mathbf{p}}$ of R can be written as in (5.7) for some values γ_i. This proves the theorem.

§5.3 Examples

We give two examples of families of random interval maps depending on one parameter. We show that for each of these families there exist parameter intervals such that the systems have strong random matching for every parameter within these intervals. Moreover, within such an interval matching happens in a comparable way, i.e., with the same M and similar sets Y. As in the deterministic case, we call these intervals *matching intervals*. To ease the notation we use the symbol $\star$ to indicate the set of strings obtained by replacing $\star$ with any $j \in \Omega$. E.g., if $\Omega = \{0,1,2\}$, then $0\star = \{00, 01, 02\}$.

§5.3.1 Random signed β-transformations

Let $\beta = \frac{1+\sqrt{5}}{2}$ be the golden mean, so $\beta^2 = \beta + 1$. For each $\alpha \in (1, \beta^2)$, let $\mathbf{p} = (p_0, p_1)$ be a positive probability vector and consider the random system R_α given by two generalised β-transformations $T_{\alpha,j} : [-\beta, \beta] \to [-\beta, \beta]$, $j = 0, 1$, defined as follows.



(a) $T_{\alpha,0}$ (b) $T_{\alpha,1}$

Figure 5.1: The maps $T_{\alpha,0}$ and $T_{\alpha,1}$ for $\alpha \in \left(\frac{3\beta-2}{2}, 4\beta - 5\right)$.

Consider the points

$$z_0 = -\beta, \quad z_1 = -1, \quad z_2 = -\frac{1}{\beta}, \quad z_3 = \frac{1}{\beta}, \quad z_4 = 1, \quad z_5 = \beta,$$

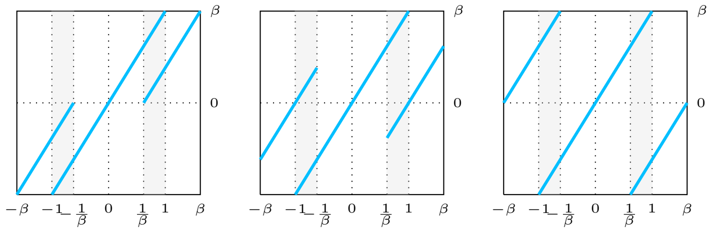
and let

$$T_{\alpha,0}(x) = \begin{cases} \beta x + \alpha & \text{if } x \in [z_0, z_2), \\ \beta x & \text{if } x \in [z_2, z_4), \\ \beta x - \alpha & \text{if } x \in [z_4, z_5], \end{cases}$$

and

$$T_{\alpha,1}(x) = \begin{cases} \beta x + \alpha & \text{if } x \in [z_0, z_1], \\ \beta x & \text{if } x \in (z_1, z_3], \\ \beta x - \alpha & \text{if } x \in (z_3, z_5]. \end{cases}$$

See Figure (5.1) for the graphs of $T_{\alpha,0}$ and $T_{\alpha,1}$, and Figure (5.2) for some examples of R_α, for different values of α.



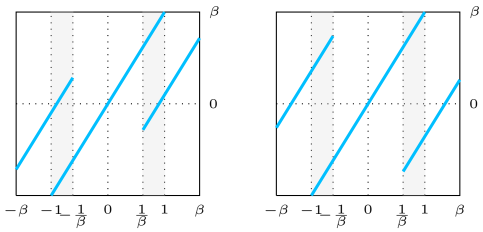
(a) R_1 (b) R_β (c) R_{β^2}

Figure 5.2: R_α for $\alpha = 1$ in (a), $\alpha = \beta$ in (b), and $\alpha = \beta^2$ in (c).

Let

$$J_1 = \left(\frac{3\beta - 2}{2}, 4\beta - 5\right) \quad \text{and} \quad J_2 = \left(\frac{6-\beta}{2}, \frac{\beta+5}{3}\right).$$

We will show that strong random matching happens for any $\alpha \in J_1 \cup J_2$. Note that $C = \{-1, -\frac{1}{\beta}, \frac{1}{\beta}, 1\}$ so that by the symmetry of the maps, to show that R_α has random matching, we only need to consider the points $\frac{1}{\beta}$ and 1.



(a) J_1 (b) J_2

Figure 5.3: R_α for $\alpha \in J_1,\ J_2$ respectively.

We start by considering $\alpha \in J_1$. See Figure (5.3)(a) for the corresponding random system R_α. The parameter interval is constructed in such a way that for any $\alpha \in J_1$ the initial parts of the random orbits of the left and right limits to $\frac{1}{\beta}$ and 1 are determined in the following way. For $j = 0, 1$ and any $\omega \in \{0,1\}^{\mathbb{N}}$,

$$\begin{aligned} &T_{\alpha,0}(1^-) = \beta, && T_{\alpha,\omega}(\beta) = \beta^2 - \alpha, && T^2_{\alpha,\omega}(\beta) = \beta^2(\beta - \alpha), \\ &T_{\alpha,1}(1^-) = T_{\alpha,j}(1^+) = \beta - \alpha, && T_{\alpha,\omega}(\beta - \alpha) = \beta(\beta - \alpha), && T^2_{\alpha,\omega}(\beta - \alpha) = \beta^2(\beta - \alpha). \end{aligned}$$

Hence, for $1 \in C$ we can take $M = k_1(\omega) = \ell_1(\omega) = 3$ for each ω, $Y = \{\beta^2(\beta - \alpha)\}$ and one easily checks the conditions of both Definition 5.2.1 and Definition 5.2.2.

For $\frac{1}{\beta}$ the orbits are more complicated. Firstly, $T_{\alpha,j}\big(\frac{1}{\beta}^-\big) = 1 = T_{\alpha,0}\big(\frac{1}{\beta}^+\big)$ and $T_{\alpha,1}\big(\frac{1}{\beta}^+\big) = 1 - \alpha$. We saw the orbit of 1 above, so we concentrate on the orbit of $1 - \alpha$. We have $T_{\alpha,j}(1-\alpha) = \beta(1-\alpha) \in \big(-1, -\frac{1}{\beta}\big)$, so $T_{\alpha,0}(\beta(1-\alpha)) = \beta(\beta - \alpha)$ and

$T_{\alpha,1}(\beta(1-\alpha)) = \beta^2(1-\alpha)$. The next couple of iterations are depicted in Figure 5.4, where we have used the property that $\beta^2 = \beta + 1$ to compute the orbit points.

$$\begin{tikzcd}
1-\alpha \arrow[r] & \beta(1-\alpha) \arrow[r, "0"] \arrow[d, "1"] & \beta(\beta-\alpha) \arrow[r] & \boxed{\beta^2(\beta-\alpha)} \arrow[r] & \beta^3(\beta-\alpha) \arrow[r, "0"] \arrow[d, "1"] & \beta^4(\beta-\alpha) \arrow[r] & \boxed{\beta^5(\beta-\alpha)-\alpha} \\
& \beta^2(1-\alpha) \arrow[d] & & & \beta^4(\beta-\alpha)-\alpha \arrow[d] & & \\
& \beta^3(1-\alpha)+\alpha \arrow[d] & & & \boxed{\beta^5(\beta-\alpha)-\beta\alpha} & & \\
& \beta^4(1-\alpha)+\beta\alpha \arrow[r, "0"] \arrow[d, "1"] & \beta^5(1-\alpha)+\beta^2\alpha+\alpha \arrow[r] & \boxed{\beta^6(1-\alpha)+\beta^3\alpha+\beta\alpha = \beta^5(\beta-\alpha)-\alpha} & & & \\
& \beta^5(1-\alpha)+\beta^2\alpha \arrow[d] & & & & & \\
& \boxed{\beta^6(1-\alpha)+\beta^3\alpha+\alpha = \beta^5(\beta-\alpha)-\beta\alpha} & & & & &
\end{tikzcd}$$

Figure 5.4: The first couple of points in the orbit of $1-\alpha$ under the random generalised β-transformation. We have boxed $\beta^2(\beta-\alpha)$, since this point also appears in all random orbits of 1.

Take $M = k_{\frac{1}{\beta}}(\omega) = \ell_{\frac{1}{\beta}}(\omega) = 7$ for each ω and set $Y = \{\beta^5(\beta-\alpha)-\alpha, \beta^5(\beta-\alpha)-\beta\alpha = \beta^6 - 3\beta^3\alpha\}$. Then,

$$\Omega(\beta^5(\beta-\alpha)-\alpha)^+ = 0\star\star\star\star 0\star \cup 1\star 0\star\star 0\star \cup 1\star 1\star\star 0\star$$

and $\Omega(\beta^5(\beta-\alpha)-\alpha)^- = \star\star\star\star\star 0\star$. Hence,

$$\sum_{\mathbf{u}\in\Omega(\beta^5(\beta-\alpha)-\alpha)^+} \frac{p_{\mathbf{u}}}{T'_{\mathbf{u}}(\frac{1}{\beta}^+)} = \frac{p_0^2 + p_1 p_0^2 + p_1^2 p_0}{\beta^7} = \frac{p_0}{\beta^7} = \sum_{\mathbf{u}\in\Omega(\beta^5(\beta-\alpha)-\alpha)^-} \frac{p_{\mathbf{u}}}{T'_{\mathbf{u}}(\frac{1}{\beta}^-)}.$$

A similar computation gives (5.5) for $\beta^5(\beta-\alpha)-\beta\alpha$, so R_α has strong random matching.

Note that also in this example the orbits of 1^+ meet with some of the orbits of 1^- earlier, in this case already after one step. Hence, we could also take $Y_1 = \{\beta-\alpha, \beta^2(\beta-\alpha)\}$ and split the random orbits as follows:

$$\Omega(\beta-\alpha)^+ = \{1\} = \Omega(\beta-\alpha)^- \quad \text{and} \quad \Omega(\beta^2(\beta-\alpha))^+ = 0\star\star = \Omega(\beta^2(\beta-\alpha))^-.$$

Then for some ω the values $k_1(\omega), \ell_1(\omega)$ are lower, but we have to check condition (5.5) for two points instead of one. For the critical point $\frac{1}{\beta}$ we could use $Y = \{\beta-\alpha, \beta^2(\beta-\alpha), \beta^5(\beta-\alpha)-\alpha, \beta^5(\beta-\alpha)-\beta\alpha\}$ or also $Y = \{\beta^2(\beta-\alpha), \beta^5(\beta-\alpha)-\alpha, \beta^5(\beta-\alpha)-\beta\alpha\}$. By the flexibility in the choice of Y given by Definition 5.2.1 one can choose the set Y that is most convenient. Theorem 5.2.3 explains the need for condition (5.5) in Definition 5.2.2.

We now consider $\alpha \in J_2$; see Figure (5.3)(b) for the corresponding random system R_α. For $j = 0, 1$, the initial parts of the random orbits of the left and right limits to $\frac{1}{\beta}$ and 1 are given by

$$T_{\alpha,j}(\tfrac{1}{\beta}^-) = T_{\alpha,0}(\tfrac{1}{\beta}^+) = 1, \quad T_{\alpha,1}(\tfrac{1}{\beta}^+) = 1 - \alpha.$$

and

$$T_{\alpha,j}(1^+) = T_{\alpha,1}(1^-) = \beta - \alpha, \quad T_{\alpha,0}(1^+) = \beta.$$

Moreover, for any $\omega \in \{0,1\}^{\mathbb{N}}$

$$\begin{aligned}
&T_{\alpha,\omega}(1) = \beta - \alpha, \\
&T_{\alpha,\omega}(\beta) = \beta^2 - \alpha, && T^2_{\alpha,\omega}(\beta) = \beta(\beta^2 - \alpha), \\
&T^1_{\alpha,\omega}(1-\alpha) = \beta - \alpha(\beta - 1), && T^2_{\alpha,\omega}(1-\alpha) = \beta^2 - \alpha, \quad T^3_{\alpha,\omega}(1-\alpha) = \beta(\beta^2 - \alpha).
\end{aligned}$$

Note also that $T_{\alpha,j1}(\beta - \alpha) = \beta(\beta^2 - \alpha)$. Since all possible random orbits of $\frac{1}{\beta}$ and 1 pass by the point $\beta - \alpha$ or by one of its iterates, $\beta(\beta^2 - \alpha)$, to study the orbits of these critical points is enough to follow the one of $\beta - \alpha$. This is depicted in Figure 5.5.

$$\begin{tikzcd}
\beta-\alpha \arrow[r] & \beta(\beta-\alpha) \arrow[r, "0"] \arrow[d, "1"] & \boxed{\beta(\beta^2-\alpha)} \arrow[r, "1"] \arrow[d, "0"] & \beta^4-\alpha(\beta^2+1) \arrow[r] & \beta(\beta^4-3\alpha) \arrow[r] & \beta^2(\beta^4-3\alpha) \arrow[r, "1"] \arrow[d, "0"] & \boxed{\beta^3(\beta^4-3\alpha)-\alpha} \\
& \beta^2(\beta-\alpha) \arrow[d] & \beta^2(\beta^2-\alpha) \arrow[d] & & & \boxed{\beta^3(\beta^4-3\alpha)} & \\
& \beta(\beta^3-2\alpha) \arrow[d] & \boxed{\beta^2(\beta^3-2\alpha)} & & & & \\
& \boxed{\beta^2(\beta^3-2\alpha)} \arrow[d] & & & & & \\
& \beta^3(\beta^3-2\alpha) \arrow[r, "0"] \arrow[d, "1"] & \boxed{\beta^7-2\beta^4\alpha+\alpha} & & & & \\
& \boxed{\beta^4(\beta^3-2\alpha)} & & & & &
\end{tikzcd}$$

Figure 5.5: The first couple of points in the orbit of $\beta - \alpha$ under R_α for $\alpha \in J_2$. We have boxed $\beta(\beta^2 - \alpha)$, since this point appears in the orbits of 1, β and $1 - \alpha$.

Take $M = k_{\frac{1}{\beta}}(\omega) = \ell_{\frac{1}{\beta}}(\omega) = 8$ for each ω and set

$$Y = \{\beta^4(\beta^3 - 2\alpha) = \beta^3(\beta^4 - 3\alpha) - \alpha, \beta^3(\beta^4 - 3\alpha) = \beta^7 - 2\beta^4\alpha + \alpha\}.$$

Then,

$$\Omega(\beta^4(\beta^3 - 2\alpha))^- = \star\star\star 01 \star\star 1 \cup \star\star\star 00 \star\star 1 \cup \star\star\star 1 \star\star\star 1,$$

and

$$\begin{aligned}
\Omega(\beta^4(\beta^3 - 2\alpha))^+ =& 0 \star\star 01 \star\star 1 \cup 0 \star\star 00 \star\star 1 \cup 0 \star\star 1 \star\star\star 1 \cup \\
& 1 \star\star\star 0 \star\star 1 \cup 1 \star\star\star 1 \star\star 1.
\end{aligned}$$

Hence,

$$\sum_{\mathbf{u}\in\Omega(\beta^4(\beta^3-2\alpha))^-} \frac{p_{\mathbf{u}}}{T'_{\mathbf{u}}(\frac{1}{\beta}^-)} = \frac{p_1^2p_0 + p_1p_0^2 + p_1^2}{\beta^8} = \frac{p_1}{\beta^8} = \sum_{\mathbf{u}\in\Omega(\beta^4(\beta^3-2\alpha))^+} \frac{p_{\mathbf{u}}}{T'_{\mathbf{u}}(\frac{1}{\beta}^+)}.$$

A similar computation gives (5.5) for $\beta^3(\beta^4-3\alpha)$, so R_α has strong random matching for any $\alpha \in J_2$.

Note that also in this example the orbits of $\frac{1}{\beta}^-$ meet with some of the orbits of $\frac{1}{\beta}^+$ earlier, in this case already after one step. Hence, we could also take the points 1, and $\beta(\beta^2-\alpha)\}$ in the set Y and split the random orbits accordingly.

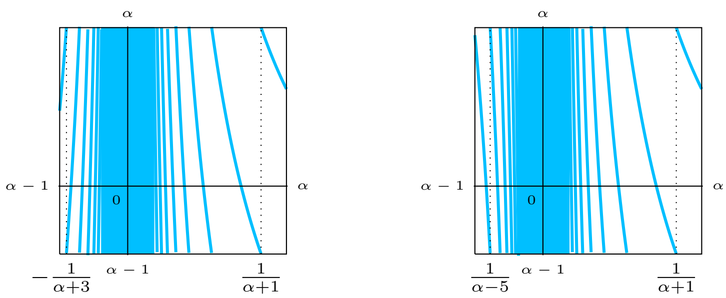
5.3.1 Remark. The random generalised β-transformations R_α from Example 5.3.1 satisfy all conditions of Theorem 5.2.3. Hence, for any $\alpha \in J_1 \cup J_2$ any invariant density of the random system R_α is piecewise constant.

§5.3.2 Random CF-maps

For $\alpha \in (0,1)$ let $T_{\alpha,0}, T_{\alpha,1} : [\alpha-1,\alpha] \to [\alpha-1,\alpha]$ be the Nakada and Ito-Tanaka α-continued fraction transformations, introduced in [N81] and in [TI81] respectively, and given by

$$T_{\alpha,0}(x) = \frac{1}{|x|} - \left\lfloor \frac{1}{|x|} + 1 - \alpha \right\rfloor \quad \text{and} \quad T_{\alpha,1}(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} + 1 - \alpha \right\rfloor,$$

for $x \neq 0$ and $T_{\alpha,0}(0) = 0 = T_{\alpha,1}(0)$. The graphs are shown in Figure 5.6.



(a) $T_{\alpha,0}$ (b) $T_{\alpha,1}$

Figure 5.6: The Nakada α-continued fraction map $T_{\alpha,0}$ in (a) and the Ito-Tanaka α-continued fraction map $T_{\alpha,1}$ in (b) for $\alpha = \frac{7}{10} \in \left(\frac{5-\sqrt{13}}{2}, \frac{\sqrt{2}}{2}\right)$.

Let R_α denote the corresponding pseudo-skew product on $\{0,1\}^{\mathbb{N}} \times [\alpha-1,\alpha]$. For $x \in [0,\alpha]$, the two maps coincide and

$$T_{\alpha,0}(x) = T_{\alpha,1}(x) = \frac{1}{x} - n \quad \text{for} \quad x \in \left(\frac{1}{\alpha+n}, \frac{1}{\alpha+n-1}\right].$$

For $x \in [\alpha - 1, 0)$, we have

$$T_{\alpha,0}(x) = -\frac{1}{x} - n \quad \text{for} \quad x \in \Big[-\frac{1}{\alpha+n-1}, -\frac{1}{\alpha+n}\Big),$$
$$T_{\alpha,1}(x) = \frac{1}{x} + n \qquad \text{for} \quad x \in \Big[\frac{1}{\alpha-n}, \frac{1}{\alpha-(n+1)}\Big).$$

We first show that for any $\alpha \in \big(\frac{\sqrt{10}-2}{2}, 2-\sqrt{2}\big)$ the map R_α has random matching. For this note that the critical points c are all in the set $\{\frac{1}{\alpha+n}, -\frac{1}{\alpha+n}, \frac{1}{\alpha-n} : n \in \mathbb{N}\}$. For any positive critical point $c > 0$ and any $j \in \{0,1\}$, $T_j(c^-), T_j(c^+) \in \{\alpha-1, \alpha\}$. For $c < 0$, c is either a critical point for T_0 and a continuity point for T_1, or a critical point for T_1 and a continuity point for T_0. Specifically, since $\alpha > \frac{1}{2}$, for $c = -\frac{1}{\alpha+n}$ we have

$$T_0(c^-) = \alpha, \quad T_0(c^+) = \alpha - 1, \quad \text{and} \quad T_1(c^-) = T_1(c^+) = 1-\alpha,$$

and for $c = \frac{1}{\alpha-n}$

$$T_1(c^-) = \alpha - 1, \quad T_1(c^+) = \alpha, \quad \text{and} \quad T_0(c^-) = T_0(c^+) = 1-\alpha.$$

As a consequence, to show that R_α has random matching we only need to consider the orbits of $\alpha - 1$ and α. Due to the choice of endpoints of the parameter interval $\big(\frac{\sqrt{10}-2}{2}, 2-\sqrt{2}\big)$, the first three orbit points of α and $\alpha - 1$ are easily determined. They are given in Figure 5.7. Hence, if we take $M = 3$ and

$$Y = \Big\{\frac{5\alpha-3}{1-2\alpha}, \frac{4-7\alpha}{1-2\alpha}\Big\}, \quad \text{if } c > 0$$

and

$$Y = \Big\{\frac{5\alpha-3}{1-2\alpha}, \frac{4-7\alpha}{1-2\alpha}, 1-\alpha\Big\}, \quad \text{if } c < 0,$$

then R_α has random matching according to Definition 5.2.1.

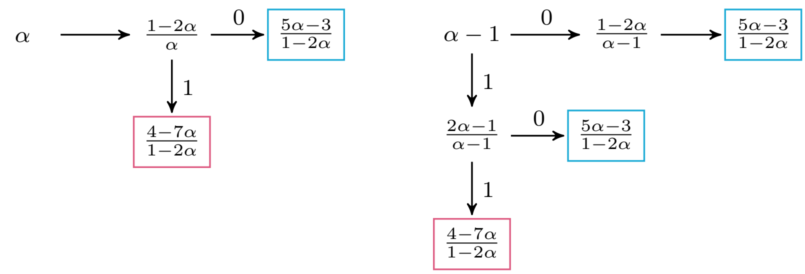


Figure 5.7: The first three elements in the orbits of α and $\alpha-1$ under the random continued fraction map R_α for $\alpha \in \big(\frac{\sqrt{10}-2}{2}, 2-\sqrt{2}\big)$. The digits above the arrows indicate which one of the maps $T_{\alpha,0}$ or $T_{\alpha,1}$ is applied. If there is no digit, then both maps yield the same orbit point. Orbit points in boxes with the same colour are equal.

R_α does not satisfy strong random matching with this choice of Y. To see this, note that $T'_{\alpha,1}(x) = -\frac{1}{x^2}$ for all x where the derivative exists, while $T'_{\alpha,0}(x) = -\frac{1}{x^2}$ if

$x > 0$ and $T'_{\alpha,0}(x) = \frac{1}{x^2}$ if $x < 0$. Now take for example $c = \frac{1}{\alpha+n} > 0$ and $y = \frac{4-7\alpha}{1-2\alpha}$. Then $\Omega(y)^- = \star 11 = \{011, 111\}$ and $\Omega(y)^+ = \star\star 1$. For the quantities from (5.5), we obtain

$$\sum_{\mathbf{u}\in\Omega(y)^-} \frac{p_{\mathbf{u}}}{T'_{\mathbf{u}}}(c^-) = -p_1^2 c^2 (2\alpha-1)^2 \quad \text{and} \quad \sum_{\mathbf{u}\in\Omega(y)^+} \frac{p_{\mathbf{u}}}{T'_{\mathbf{u}}}(c^+) = -p_1 c^2 (2\alpha-1)^2,$$

which are not equal for any $p_1 \in (0,1)$.

We now identify a countable number of parameter intervals on which the maps R_α have strong matching with the same exponent $M = 4$, i.e., we identify a countable number of *matching intervals* for the family R_α. For $n \geq 4$ let the interval $J_n := (\ell_n, r_n)$ be defined by the left and right endpoints

$$\ell_n = \frac{n+1-\sqrt{n^2-2n+5}}{2} \quad \text{and} \quad r_n = \sqrt{\frac{n-2}{n}}, \tag{5.8}$$

respectively. Set $g := \frac{\sqrt{5}-1}{2}$ for the small golden mean and note that $g < \ell_n < r_n$ for all $n \geq 4$ and that $\lim_{n\to\infty} \ell_n = \lim_{n\to\infty} r_n = 1$. See Figure 5.8 for an illustration of the location of these intervals.

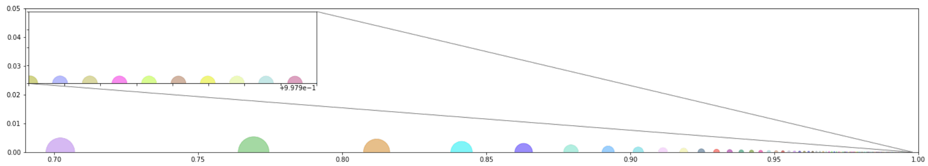


Figure 5.8: The semicircles indicate the locations of the intervals J_n.

The intervals J_n are chosen in such a way that we can determine the first three orbit points of α and $\alpha - 1$. Let $n \geq 4$ and $\alpha \in J_n$. In particular $\alpha > g$ and for $j = 0, 1$,

$$T_{\alpha,j}(\alpha) = \frac{1-\alpha}{\alpha} > 0.$$

The point ℓ_n is chosen so that $\alpha - 1 \in (\frac{1}{\alpha-n}, \frac{\alpha+1}{1-n(\alpha+1)}) \subseteq (-\frac{1}{\alpha+n-2}, -\frac{1}{\alpha+n-1})$. Since $\frac{\alpha+1}{1-n(\alpha+1)} < \frac{1}{\alpha-n-1}$ we get

$$T_{\alpha,1}(\alpha-1) = \frac{n(\alpha-1)+1}{\alpha-1} \quad \text{and} \quad T_{\alpha,0}(\alpha-1) = \frac{\alpha(n-1)+2-n}{1-\alpha}.$$

It also implies $\frac{1-\alpha}{\alpha} \in (\frac{1}{\alpha+n-2}, \frac{1}{\alpha+n-3})$. As a consequence, for $l = 0, 1$,

$$T_{\alpha,jl}(\alpha) = \frac{\alpha(n-1)+2-n}{1-\alpha} = T_{\alpha,0}(\alpha-1) > 0.$$

We further divide the interval J_n. For $k \in \{2, 3, \dots, n\}$, let

$$i_{n,k} = \frac{-4+2n-kn+k+\sqrt{k^2n^2-2k^2n+k^2+4}}{2(n-1)},$$

and note that $J_n \subseteq \cup_{k=2}^{n-1}(i_{n,k+1}, i_{n,k}]$. Therefore, for each $\alpha \in J_n$ there exists a $k \in \{2, 3, \dots, n-1\}$ such that $\alpha \in (i_{n,k+1}, i_{n,k}]$. The last condition is equivalent to

$$\frac{1}{\alpha + k} < \frac{\alpha(n-1) + 2 - n}{1 - \alpha} \leq \frac{1}{\alpha + k - 1}, \tag{5.9}$$

so that for $\mathbf{u} \in \Omega^3$ it holds that

$$T_{\alpha,\mathbf{u}}(\alpha) = \frac{1 - 2k + kn - \alpha(kn - k + 1)}{\alpha(n-1) + 2 - n}.$$

On the other hand, the choice of r_n guarantees that $T_{\alpha,1}(\alpha - 1) = \frac{1+n(\alpha-1)}{n} > \frac{1}{\alpha+1}$. Then for $j = 0, 1$,

$$T_{\alpha,1j}(\alpha - 1) = \frac{\alpha(n-1) + 2 - n}{-1 - n(\alpha - 1)}.$$

Equation (5.9) holds if and only if

$$\frac{1}{\alpha + k - 1} < \frac{\alpha(n-1) + 2 - n}{-1 - n(\alpha - 1)} \leq \frac{1}{\alpha + k - 2}$$

is satisfied. In this case, for $l = 0, 1$

$$T_{\alpha,1jl}(\alpha - 1) = \frac{1 - 2k + kn - \alpha(kn - k + 1)}{\alpha(n-1) + 2 - n}.$$

Figure 5.9 shows all the relevant orbit points of α and $\alpha - 1$.

$$\begin{tikzcd}
\alpha \arrow[r] & \frac{1-\alpha}{\alpha} \arrow[r] & \frac{\alpha(n-1)+2-n}{1-\alpha} \arrow[r] & \boxed{\frac{1-2k+kn-\alpha(kn-k+1)}{\alpha(n-1)+2-n}} \\
\alpha - 1 \arrow[r, "0"] \arrow[d, "1"] & \frac{\alpha(n-1)+2-n}{1-\alpha} \arrow[r] & \boxed{\frac{1-2k+kn-\alpha(kn-k+1)}{\alpha(n-1)+2-n}} & \\
\frac{n(\alpha-1)+1}{\alpha-1} \arrow[r] & \frac{\alpha(n-1)+2-n}{-1-n(\alpha-1)} \arrow[r] & \boxed{\frac{1-2k+kn-\alpha(kn-k+1)}{\alpha(n-1)+2-n}} &
\end{tikzcd}$$

Figure 5.9: The first few points in the orbits of α and $\alpha - 1$ under the random continued fraction map R_α for $\alpha \in J_n \cap (i_{n,k+1}, i_{n,k}]$.

Definition 5.2.1 holds for $\alpha \in J_n \cap (i_{n,k+1}, i_{n,k}]$ with $M = 4$ and

$$Y = \left\{ \frac{1 - 2k + kn - \alpha(kn - k + 1)}{\alpha(n-1) + 2 - n} \right\}$$

for any critical point $c > 0$. For $c < 0$ we add the point $1 - \alpha$ to Y. Here the values $k_c(\omega)$ and $\ell_c(\omega)$ either equal 1, 3 or 4 according to the number of orbit points in the paths in Figure 5.9. For Definition 5.2.2, for $c > 0$ and $y \in Y$ we have

$\Omega(y)^- = \star 0 \star \cup \star 1 \star\star$ and $\Omega(y)^+ = \star\star\star\star$, so that

$$\begin{aligned}\sum_{\mathbf{u}\in\Omega(y)^-} \frac{p_{\mathbf{u}}}{T'_{\mathbf{u}}(c^-)} &= (-c^2)p_0(\alpha-1)^2 \cdot \frac{(\alpha(n-1)+2-n)^2}{-(\alpha-1)^2} \\ &\quad + (-c^2)p_1(-(\alpha-1)^2)\cdot\frac{(1+n(\alpha-1))^2}{-(\alpha-1)^2}\cdot\frac{(\alpha(n-1)+2-n)^2}{-(1+n(\alpha-1))^2} \\ &= c^2(\alpha(n-1)+2-n)^2.\end{aligned}$$

and

$$\sum_{\mathbf{u}\in\Omega(y)^+} \frac{p_{\mathbf{u}}}{T'_{\mathbf{u}}(c^+)} = (-c^2)(-\alpha^2)\frac{(1-\alpha)^2}{-\alpha^2}\cdot\frac{(\alpha(n-1)+2-n)^2}{-(1-\alpha)^2} = c^2(\alpha(n-1)+2-n)^2,$$

implying that also condition (5.5) holds. For $c = -1/(\alpha+n)$ we get $\Omega(1-\alpha)^- = \Omega(1-\alpha)^+ = \{1\}$, $\Omega(y)^- = 0\star\star\star$ and $\Omega(y)^+ = 00\star\cup 01\star\star$, and for $c = 1/(\alpha-n)$ we obtain $\Omega(1-\alpha)^- = \Omega(1-\alpha)^+ = \{0\}$, $\Omega(y)^- = 10\star\cup 11\star\star$ and $\Omega(y)^+ = 1\star\star\star$. In both cases the result follows in a similar fashion. So, the random continued fraction system R_α has strong random matching for any $\mathbf{p}$ and any $\alpha \in J_n$.

Note that in this example the orbits of α meet with some of the orbits of $\alpha-1$ already after two time steps in the point $\frac{\alpha(n-1)+2-n}{1-\alpha}$. Hence,

$$\frac{\alpha(n-1)+2-n}{1-\alpha} \in \Big\{T^k_\omega(c^-) : \omega\in\Omega^{\mathbb{N}},\, k\le M\Big\} \cap \Big\{T^k_\omega(c^+) : \omega\in\Omega^{\mathbb{N}},\, k\le M\Big\}.$$

Therefore, for a critical point $c>0$, we could also take

$$Y = \{\frac{\alpha(n-1)+2-n}{1-\alpha}, \frac{1-2k+kn-\alpha(kn-k+1)}{\alpha(n-1)+2-n}\}$$

and split the random orbits of α for example in the following way:

$$\Omega\Big(\frac{\alpha(n-1)+2-n}{1-\alpha}\Big)^+ = \star\star 0 \quad\text{and}\quad \Omega\Big(\frac{1-2k+kn-\alpha(kn-k+1)}{\alpha(n-1)+2-n}\Big)^+ = \star\star 1\star.$$

For the orbits passing through $\alpha-1$ we have

$$\Omega\Big(\frac{\alpha(n-1)+2-n}{1-\alpha}\Big)^- = \star 0 \quad\text{and}\quad \Omega\Big(\frac{1-2k+kn-\alpha(kn-k+1)}{\alpha(n-1)+2-n}\Big)^- = \star 1\star\star.$$

One can check that condition (5.5) is satisfied and R_α has strong random matching with this choice of Y. Note that in this case many sequences ω have smaller values $k_c(\omega)$ and $\ell_c(\omega)$ than with $Y = \{\frac{1-2k+kn-\alpha(kn-k+1)}{\alpha(n-1)+2-n}\}$ and that for some $\omega\in\Omega^{\mathbb{N}}$ we do not take $k_c(\omega)$ equal to the first time that the random orbit $T^k_\omega(c^-)$ enters Y. For example, for $c>0$ and any ω with $\omega_3 = 1$ we have $T^3_\omega(c^+) = \frac{\alpha(n-1)+2-n}{1-\alpha}\in Y$, but we take $k_c(\omega) = 4$. The flexibility in the choice of Y and the length of the paths $k_c(\omega)$ and $\ell_c(\omega)$ embedded in Definition 5.2.1 allows one to choose the option that is computationally most convenient.

§5.4 Random transformations and expansions

In the second part of this Chapter we use random matching to study the frequency of the digit 0 in the signed binary expansions produced by a family of random system of piecewise affine maps. In this section we define this family and its relation to binary expansions.

§5.4.1 Random symmetric doubling maps

A signed binary expansion of a number $x \in [-1, 1]$ can be obtained by iterating any piecewise affine map $D : [-1, 1] \to [-1, 1]$ that is given by $D(x) = 2x - d$ with $d \in \{-1, 0, 1\}$ on each of its intervals of monotonicity. One can for example take any $a \in \left[\frac{1}{4}, \frac{1}{2}\right]$ and then define the symmetric map

$$D_a(x) = \begin{cases} 2x+1, & \text{if } -1 \leq x < -a, \\ 2x, & \text{if } -a \leq x \leq a, \\ 2x-1, & \text{if } a < x \leq 1. \end{cases}$$

By setting $d_n(x) = d$, $d \in \{-1, 0, 1\}$, if $D_a^n(x) = 2D_a^{n-1}(x) - d$, one obtains

$$x = \frac{d_1(x)}{2} + \frac{D_a(x)}{2} = \cdots = \frac{d_1(x)}{2} + \cdots + \frac{d_n(x)}{2^n} + \frac{D_a^n(x)}{2^n} \to \sum_{n \geq 1} \frac{d_n(x)}{2^n},$$

so this gives a signed binary expansion of x. The family of maps $\{D_a\}_{\frac{1}{4} \leq a \leq \frac{1}{2}}$ is the object of study in [DK17]. As can be seen from Figure 5.10(a) the interval $[-2a, 2a]$ is an attractor for the dynamics of D_a. Since this interval depends on a, in [DK17] the authors decided to work instead with the measurably isomorphic family $\{S_\alpha\}_{1 \leq \alpha \leq 2}$ given by

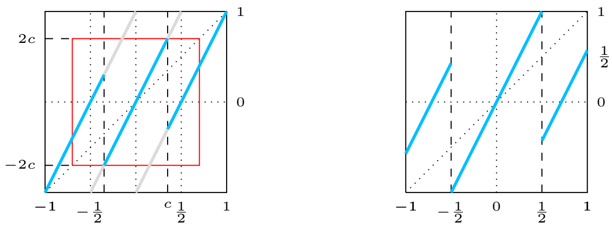
$$S_\alpha(x) = \begin{cases} 2x+\alpha, & \text{if } -1 \leq x < -\frac{1}{2}, \\ 2x, & \text{if } -\frac{1}{2} \leq x \leq \frac{1}{2}, \\ 2x-\alpha, & \text{if } \frac{1}{2} < x \leq 1, \end{cases} \tag{5.10}$$

see Figure 5.10(b), which transfers the dependence on the parameter from the domain $[-1, 1]$ to the branches of the maps.

While each deterministic map produces for each number in its domain a single signed binary expansion, one can define random dynamical systems that produce for Lebesgue almost all numbers uncountably many different signed binary expansions. The family of random maps $\{R_\alpha\}$, which we define next, extends the family of deterministic maps $\{S_\alpha\}$. So the dependence on the parameter is visible in the branches of the maps instead of in the domains.

Let $\Omega = \{0, 1\}$ and define for $j \in \Omega$ and each parameter $\alpha \in [1, 2]$ the maps $T_j = T_{\alpha,j} : [-1, 1] \to [-1, 1]$ by

$$T_{\alpha,0}(x) = \begin{cases} 2x+\alpha, & \text{if } x \in \left[-1, \frac{1-\alpha}{2}\right], \\ 2x, & \text{if } x \in \left(\frac{1-\alpha}{2}, \frac{1}{2}\right], \\ 2x-\alpha, & \text{if } x \in \left(\frac{1}{2}, 1\right], \end{cases} \tag{5.11}$$



(a) D_c (b) $S_{\frac{1}{2c}}$

Figure 5.10: The maps D_c and $S_{\frac{1}{2c}}$ for $c = \frac{7}{20}$. The grey lines indicate the remainder of the maps $x \mapsto 2x+1$, $x \mapsto 2x$ and $x \mapsto 2x-1$. The red box in (a) shows the attractor of the map D_c.

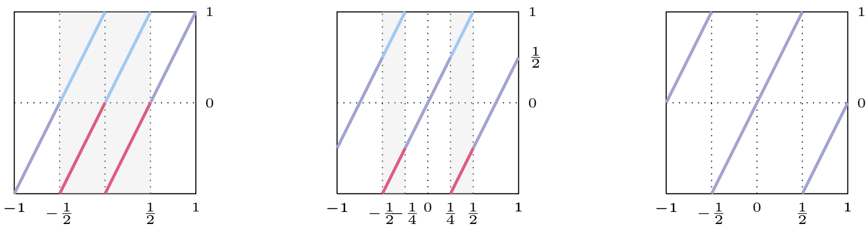
and

$$T_{\alpha,1}(x) = \begin{cases} 2x+\alpha, & \text{if } x \in \big[-1, -\frac{1}{2}\big), \\ 2x, & \text{if } x \in \big[-\frac{1}{2}, \frac{\alpha-1}{2}\big), \\ 2x-\alpha, & \text{if } x \in \big[\frac{\alpha-1}{2}, 1\big]. \end{cases} \tag{5.12}$$

See Figure 5.11 for three examples. The maps $T_{\alpha,0}$ and $T_{\alpha,1}$ differ on the intervals $\big[-\frac{1}{2}, 1-2\alpha\big]$ and $\big[2\alpha-1, \frac{1}{2}\big]$, which are indicated by the grey areas in the pictures. Let $R = R_\alpha : \Omega^{\mathbb{N}} \times [-1,1] \to \Omega^{\mathbb{N}} \times [-1,1]$ be the random system obtained from $T_{\alpha,0}$ and $T_{\alpha,1}$, i.e.,

$$R_\alpha(\omega, x) = \big(\sigma(\omega), T_{\alpha,\omega_1}(x)\big),$$

where σ is the left shift on $\Omega^{\mathbb{N}}$. We call the systems R_α *random symmetric doubling maps* and the subscript α will sometimes be suppressed if it does not lead to confusion.



(a) R_1 (b) $R_{\frac{3}{2}}$ (c) R_2

Figure 5.11: The maps $T_{\alpha,0}$ and $T_{\alpha,1}$ for $\alpha = 1$ in (a), $\alpha = \frac{3}{2}$ in (b), and $\alpha = 2$ in (c). The blue lines correspond to $T_{\alpha,0}$, the pink ones to $T_{\alpha,1}$ and the purple ones to both.

Fix an $\alpha \in [1,2]$. Recall from (5.4) that we use square brackets to denote the cylinder sets in $\Omega^{\mathbb{N}}$. Let $\pi : \Omega^{\mathbb{N}} \times [-1,1] \to [-1,1]$ denote the canonical projection onto the second coordinate and set

$$s_n(\omega,x)=\begin{cases}-1, & \text{if } R^{n-1}(\omega,x)\in\Omega^{\mathbb{N}}\times\big[-1,-\frac{1}{2}\big)\cup[0]\times\big[-\frac{1}{2},\frac{1-\alpha}{2}\big],\\ 0, & \text{if } R^{n-1}(\omega,x)\in[1]\times\big[-\frac{1}{2},\frac{1-\alpha}{2}\big]\\ & \qquad\qquad \cup\,\Omega^{\mathbb{N}}\times\big(\frac{1-\alpha}{2},\frac{\alpha-1}{2}\big)\cup[0]\times\big[\frac{\alpha-1}{2},\frac{1}{2}\big],\\ 1, & \text{if } R^{n-1}(\omega,x)\in[1]\times\big[\frac{\alpha-1}{2},\frac{1}{2}\big]\cup\Omega^{\mathbb{N}}\times\big(\frac{1}{2},1\big].\end{cases} \tag{5.13}$$

Then

$$\pi(R^n(\omega,x))=2\pi(R^{n-1}(\omega,x))-s_n(\omega,x)\alpha,$$

so that just as in the deterministic case by iteration we obtain

$$x=\frac{s_1(\omega,x)\alpha}{2}+\cdots+\frac{s_n(\omega,x)\alpha}{2^n}+\frac{\pi(R^n(\omega,x))}{2^n}\to\alpha\sum_{n\geq 1}\frac{s_n(\omega,x)}{2^n}.$$

In other words, iterations of the random system R give a signed binary expansion for the pair (ω,x).

Note that for each $x\in[-1,1]$ there is an $\omega\in\Omega^{\mathbb{N}}$, such that $\pi(R_\alpha^n(\omega,x))=S_\alpha^n(x)$, where S_α is the map in the family $\{S_\alpha\}$ from [DK17]. In particular, the random signed binary expansions produced by the family $\{R_\alpha\}$ include, among many others, the SSB expansions. The randomness of the system allows us to choose (up to a certain degree) where and when we want to have a digit 0. Below we investigate the frequency of the digit 0 in typical expansions produced by the maps R. We do so by applying Birkhoff's Ergodic Theorem for invariant measures for R of the form $m\times\mu$ with m a Bernoulli measure and μ absolutely continuous with respect to the Lebesgue measure. For that we need to investigate the density of such measures μ.

§5.4.2 Random matching almost everywhere

For any $\alpha\in(1,2]$ the common partition on which T_0 and T_1 are monotone is given by the points

$$c_0=-1,\quad c_1=-\frac{1}{2},\quad c_2=\frac{1-\alpha}{2},\quad c_3=\frac{\alpha-1}{2},\quad c_4=\frac{1}{2},\quad c_5=1.$$

Set, in accordance with (a1),

$$I_1=[c_0,c_1),\quad I_2=[c_1,c_2],\quad I_3=(c_2,c_3),\quad I_4=[c_3,c_4],\quad I_5=(c_4,c_5],$$

then $C=\{c_1,c_2,c_3,c_4\}$. For $0<p<1$, use $\mathbf{p}=(p_0,p_1)$ to denote the probability vector with $p_0=p$ and $p_1=1-p$. Since T_0 and T_1 from (5.11), (5.12) are both piecewise affine with slope 2, we have $\frac{p_j}{|T_j'(x)|}=\frac{p_j}{T_j'(x)}=\frac{p_j}{2}$, $j=0,1$. So the random system R satisfies conditions (a1), (a2), (a3), i.e., $R\in\mathcal{R}$. Due to the symmetry in the map, to verify whether R has strong random matching it is enough to check the conditions of Definitions 5.2.1 and Definitions 5.2.2 for the points $1=T_0(c_4)^-$ and $1-\alpha=T_0(c_4)^+$.

Before we proceed with a description of the matching behaviour of the family of random systems $\{R_\alpha\}$, we first recall the results from [DK17, Propositions 2.1 and 2.3] on matching for the family of deterministic symmetric doubling maps $\{S_\alpha\}$, see (5.10). Let

$$M_\alpha = \inf\left\{n \geq 0 \, : \, \frac{1}{2} < S_\alpha^n(1) < \alpha - \frac{1}{2}\right\} + 1. \tag{5.14}$$

Then according to [DK17, Propositions 2.1 and 2.3] for all $\alpha \in [1,2]$,

$$S_\alpha^k(1-\alpha) = S_\alpha^k(1) - \alpha \quad \text{for } k < M_\alpha \tag{5.15}$$

and for Lebesgue almost all $\alpha \in [1,2]$ in fact $M_\alpha < \infty$ and

$$S_\alpha^{M_\alpha+1}\left(\frac{1}{2}^-\right) = S_\alpha^{M_\alpha}(1) = S_\alpha^{M_\alpha}(1-\alpha) = S_\alpha^{M_\alpha+1}\left(\frac{1}{2}^+\right).$$

In other words, for Lebesgue almost all parameters $\alpha \in [1,2]$ the map S_α has matching with matching exponent $M = M_\alpha + 1$ that is determined by the first time the orbit of 1 enters the interval $\left(\frac{1}{2}, \alpha - \frac{1}{2}\right)$. Moreover, $S_\alpha^{M_\alpha - 1}(1-\alpha) < -\frac{1}{2}$ for all α, $M_\alpha = 1$ for $\alpha \in \left[\frac{3}{2}, 2\right]$ and $M_\alpha > 1$ for $\alpha \in \left(1, \frac{3}{2}\right)$. Due to the constant slope and the same matching exponent M_α of the left and right limits, in this case matching implies strong matching.

5.4.1 Remark. The discrepancy between $M_\alpha+1$ here and M_α as matching exponent in [DK17] comes from the fact that in [DK17] the orbits are considered as starting from 1 and $1-\alpha$, whereas in (5.2) and (5.3) we followed the convention in [BCMP18] and start at the critical point $c = \frac{1}{2}$ instead.

From this we deduce the following small lemma.

5.4.2 Lemma. *For $\alpha \in [1,2]$ and for all $k < M_\alpha - 1$, either $S^k(1) \in I_4$ and $S^k(1) - \alpha \in I_1$ or $S^k(1) \in I_5$ and $S^k(1) - \alpha \in I_2$.*

Proof. From (5.15) it follows for all $k < M_\alpha - 1$ that $S^k(1) - \alpha \geq -1$, implying that

$$\alpha - 1 \leq S^k(1) \leq 1 \quad \text{and} \quad -1 \leq S^k(1) - \alpha \leq 1 - \alpha.$$

The fact that $S^k(1) \in I_4 \cup I_5$ follows since $\frac{\alpha-1}{2} \leq \alpha - 1$. If $S^k(1) \in I_4$, then $S^k(1) - \alpha \leq \frac{1}{2} - \alpha < -\frac{1}{2}$, so $S^k(1) - \alpha \in I_1$. Suppose $S^k(1) \in I_5$. If $S^k(1) - \alpha < -\frac{1}{2}$, this would imply that $S^k(1) \in \left(\frac{1}{2}, \alpha - \frac{1}{2}\right)$, contradicting the definition of M_α in (5.14). Hence, $S^k(1) - \alpha \in I_2$. ☐

The next result states that a random equivalent of (5.15) holds for $\alpha \in \left(1, \frac{3}{2}\right)$.

5.4.3 Proposition. *For all $\alpha \in [1,2]$, $0 \leq k \leq M_\alpha$ and $\mathbf{u} \in \Omega^k$, it holds that $T_{\mathbf{u}}(1), T_{\mathbf{u}}(1-\alpha) \in \{S^k(1), S^k(1) - \alpha\}$.*

Proof. First consider $\alpha \in \left[\frac{3}{2}, 2\right]$. Then $M_\alpha = 1$ and the result trivially holds. Fix an $\alpha \in \left(1, \frac{3}{2}\right)$. Since T_0 and T_1 agree on I_5 we can find a sequence $\widehat{\omega} \in \Omega^{\mathbb{N}}$ with $\widehat{\omega}_1 = 0$ that gives

$$T_{\widehat{\omega}_1^k}(1) = S^k(1) \quad \text{for all } k \geq 0.$$

CHAPTER 5

Note that $1 \in I_5$ and from $\alpha \in \left(1, \frac{3}{2}\right)$ we get $1-\alpha \in I_2$, so

$$T_0(1-\alpha) = T_0(1) = T_1(1) = 2-\alpha = S(1) \quad \text{and} \quad T_1(1-\alpha) = 2-2\alpha = S(1-\alpha). \tag{5.16}$$

Hence, from the first iterate on, the orbits of 1 and $1-\alpha$ under the deterministic map S are contained in the orbit of $1-\alpha$ under the random map R. To prove the statement, we therefore only have to consider $T_\omega^n(1-\alpha)$ for any $\omega \in \Omega^{\mathbb{N}}$ and $n \geq 1$. In particular (5.16) implies that

$$T_{\widehat{\omega}_1^k}(1-\alpha) = S^k(1)$$

for all $k \geq 1$. We prove the statement by induction.

The statement obviously holds for $k = 0$ and by (5.16) also for $k = 1$. Let $1 \leq n < M_\alpha$ and suppose the statement holds for all $k \leq n$. Then

$$T_{\widehat{\omega}_1^n}(1-\alpha) = S^n(1) \quad \text{and} \quad T_{\omega_1^n}(1-\alpha) \in \{S^n(1), S^n(1)-\alpha\} \quad \text{for all } \omega \in \Omega^{\mathbb{N}}.$$

By Lemma 5.4.2 there are three cases.
1. If $S^n(1) \in I_4$, then $S^{n+1}(1) = 2S^n(1)$ and $S^n(1)-\alpha \in I_1$. So for the random images we get

$$T_0(S^n(1)) = 2S^n(1), \quad T_1(S^n(1)) = 2S^n(1)-\alpha,$$

and

$$T_0(S^n(1)-\alpha) = T_1(S^n(1)-\alpha) = 2S^n(1)-\alpha.$$

2. If $S^n(1) \in I_5$ and $S^n(1)-\alpha \in I_2$, then

$$T_0(S^n(1)) = T_1(S^n(1)) = 2S^n(1)-\alpha = S^{n+1}(1)$$

and

$$T_0(S^n(1)-\alpha) = 2S^n(1)-\alpha, \quad T_1(S^n(1)) = 2S^n(1)-2\alpha.$$

3. If $S^n(1) \in I_5$ and $S^n(1) \in I_1$ (so $n = M_\alpha - 1$), then

$$T_0(S^n(1)) = T_1(S^n(1)) = 2S^n - \alpha = S^{n+1}(1)$$

and

$$T_0(S^n(1)-\alpha) = T_1(S^n(1)-\alpha) = 2S^n(1)-\alpha = S^{n+1}(1).$$

Hence, for all $\mathbf{u} \in \Omega^n$ and $j = 0, 1$, $T_{\mathbf{u}j}(1-\alpha) \in \{S^{n+1}(1), S^{n+1}(1)-\alpha\}$, which gives the result. □

From this proposition we can deduce that matching is prevalent for the family $\{R_\alpha\}$ and we can find the precise matching times. We first prove the following lemma, stating that all the orbit points $S^n(1), S^n(1-\alpha)$ up to the moment of matching are different.

5.4.4 Lemma. *For each $k < M_\alpha$ the set $\{S^n(1), S^n(1-\alpha) : 0 \leq n \leq k\}$ has $2(k+1)$ elements.*

Proof. Since $k < M_\alpha$ it follows from (5.15) that $S^n(1) \neq S^n(1-\alpha)$ for each n. It also cannot hold that there are $0 \leq n < k < M_\alpha$ such that $S^n(1) = S^k(1-\alpha)$ or $S^k(1) = S^n(1-\alpha)$, since this would imply that $|S^k(1) - S^n(1)| = \alpha$ and that would contradict the fact that $S^n(1), S^k(1) \in I_4 \cup I_5$. This leaves the possibility that there are $0 \leq n < k < M_\alpha$ such that $S^n(1) = S^k(1)$, i.e., that the orbit of 1 under S is ultimately periodic, or $S^n(1-\alpha) = S^k(1-\alpha)$. Assume $S^n(1) = S^k(1)$ for some $n < k$. It follows that $S^n(1-\alpha) = S^n(1) - \alpha = S^k(1) - \alpha = S^k(1-\alpha)$, so the orbit of $1-\alpha$ is also ultimately periodic and by Proposition 5.4.3 all these orbit points lie at distance α of the corresponding orbit points of 1. This contradicts the fact that α is a matching parameter. Hence, the set $\{S^n(1), S^n(1-\alpha) : 0 \leq n \leq k\}$ has $2(k+1)$ elements. □

5.4.5 Theorem. *For Lebesgue almost all parameters $\alpha \in [1,2]$ the map R_α has strong random matching with $M = M_\alpha + 1$, where M_α is given by (5.14), and $Y = \{S^{M_\alpha}(1)\}$. Moreover, R_α does not satisfy the conditions of strong random matching for any $K < M$.*

Proof. First consider $\alpha \in \left[\frac{3}{2}, 2\right]$. Then $T_j(1-\alpha) = 2 - \alpha = T_j(1)$ for $j = 0, 1$, so random matching occurs for R with $M = 2$ and $Y = \{2 - \alpha\}$ and both parts of the theorem hold.

Now, fix $\alpha \in [1, \frac{3}{2})$ such that $S = S_\alpha$ has matching. Then, $S^k(1) \neq S^k(1-\alpha)$ for $1 \leq k < M_\alpha$ and $S^{M_\alpha - 1}(1) \in \left(\frac{1}{2}, \alpha - \frac{1}{2}\right)$, so that $S^{M_\alpha}(1) = 2S^{M_\alpha - 1}(1) - \alpha$. By Proposition 5.4.3 for each $\mathbf{u} \in \Omega^{M_\alpha - 1}$ either

$$T_{\mathbf{u}}(1-\alpha) = S^{M_\alpha - 1}(1) > \frac{1}{2}$$

or

$$T_{\mathbf{u}}(1-\alpha) = S^{M_\alpha - 1}(1) - \alpha < -\frac{1}{2}.$$

In both cases this leads to $T_{\mathbf{u}j}(1-\alpha) = 2S^{M_\alpha - 1}(1) - \alpha$ for both $j = 0, 1$. The same statement holds for $T_{\mathbf{u}}(1)$, so that for $c = \frac{1}{2}$ we therefore have

$$T_{1\mathbf{u}j}\left(\frac{1}{2}^-\right) = T_{0\mathbf{u}j}\left(\frac{1}{2}^+\right) = T_{1\mathbf{u}j}\left(\frac{1}{2}^+\right) = T_{\mathbf{u}j}(1-\alpha) = S^{M_\alpha}(1)$$

and

$$T_{0\mathbf{u}j}\left(\frac{1}{2}^-\right) = T_{\mathbf{u}j}(1) = S^{M_\alpha}(1).$$

Hence, we can take $Y_{\frac{1}{2}} = \{S^{M_\alpha}(1)\}$. Since this set contains one element only and the maps T_j have the same constant slope, condition (5.5) from Definition 5.2.2 follows immediately. The first part of the theorem now follows since the deterministic maps S_α have matching for Lebesgue almost all parameters α. For the critical points $c \neq \frac{1}{2}$ the statement follows by symmetry.

For the second part we assume for $\alpha \in [1, \frac{3}{2})$ that $S = S_\alpha$ has matching and we proceed by contradiction. Therefore, assume that R_α satisfies the conditions of

Definition 5.2.1 and Definition 5.2.2 for $c = \frac{1}{2}$ for some minimal $1 \leq K < M = M_\alpha + 1$. Suppose that $S^n(1) \in Y_{\frac{1}{2}}$ for some $n < K - 1$. By Lemma 5.4.4 any $\mathbf{u}$ for which $T_{\mathbf{u}}\big(\frac{1}{2}^{\pm}\big) = S^n(1)$ has length $|\mathbf{u}| = n + 1$. Together with (5.5) and the fact that the maps $T_{\alpha,0}$ and $T_{\alpha,1}$ both have constant slope 2, this implies that

$$\sum_{\mathbf{u} \in \Omega(S^n(1))^-} p_{\mathbf{u}} = \sum_{\mathbf{u} \in \Omega(S^n(1))^+} p_{\mathbf{u}}. \tag{5.17}$$

For any $\mathbf{u} \in \Omega^{n+1} \setminus \Omega(S^n(1))^-, \mathbf{u}' \in \Omega^{n+1} \setminus \Omega(S^n(1))^+$ we have by Proposition 5.4.3 that $T_{\mathbf{u}}\big(\frac{1}{2}^-\big) = T_{\mathbf{u}'}\big(\frac{1}{2}^+\big) = S^n(1) - \alpha$. Furthermore,

$$\begin{aligned} 1 = \sum_{\mathbf{u} \in \Omega^{n+1}} p_{\mathbf{u}} &= \sum_{\mathbf{u} \in \Omega(S^n(1))^-} p_{\mathbf{u}} + \sum_{\mathbf{u} \in \Omega^{n+1} \setminus \Omega(S^n(1))^-} p_{\mathbf{u}} \\ &= \sum_{\mathbf{u} \in \Omega(S^n(1))^+} p_{\mathbf{u}} + \sum_{\mathbf{u} \in \Omega^{n+1} \setminus \Omega(S^n(1))^+} p_{\mathbf{u}}. \end{aligned}$$

From (5.17) and Proposition 5.4.3 we see that

$$\sum_{\mathbf{u} \in \Omega(S^n(1-\alpha))^-} p_{\mathbf{u}} = \sum_{\mathbf{u} \in \Omega^{n+1} \setminus \Omega(S^n(1))^-} p_{\mathbf{u}} = \sum_{\mathbf{u} \in \Omega^{n+1} \setminus \Omega(S^n(1))^+} p_{\mathbf{u}} = \sum_{\mathbf{u} \in \Omega(S^n(1-\alpha))^+} p_{\mathbf{u}}.$$

This implies that the conditions of Definition 5.2.1 and Definition 5.2.2 hold with $M_{\frac{1}{2}} = n + 1$ and $Y_{\frac{1}{2}} = \{S^n(1), S^n(1-\alpha)\}$, contradicting the minimality of K. In a similar way we can exclude the possibility that $S^n(1-\alpha) \in Y$ for $n < K - 1$. Since there is an $\widetilde{\omega} \in \Omega^{\mathbb{N}}$ such that for each $k < M - 1$, $T_{\widetilde{\omega}_1^k}(1-\alpha) = S^k(1-\alpha) = S^k(1) - \alpha$, it must hold that

$$Y_{\frac{1}{2}} = \{S^{K-1}(1), S^{K-1}(1) - \alpha\}.$$

To conclude the proof we show that for this set $Y_{\frac{1}{2}}$ condition (5.5) cannot hold. By the constant slope, condition (5.5) can be rephrased as

$$\begin{cases} \displaystyle\sum_{\mathbf{u} \in \Omega(S^{K-1}(1))^-} p_{\mathbf{u}} - \sum_{\mathbf{u} \in \Omega(S^{K-1}(1))^+} p_{\mathbf{u}} & = 0, \\ \displaystyle\sum_{\mathbf{u} \in \Omega(S^{K-1}(1)-\alpha)^-} p_{\mathbf{u}} - \sum_{\mathbf{u} \in \Omega(S^{K-1}(1)-\alpha)^+} p_{\mathbf{u}} & = 0. \end{cases} \tag{5.18}$$

and by Lemma 5.4.4 any $\mathbf{u} \in \Omega(S^{K-1}(1))^{\pm} \cup \Omega(S^{K-1}(1) - \alpha)^{\pm}$ has length K. Since $K < M_\alpha + 1$, so $K - 2 < M_\alpha - 1$, Lemma 5.4.2 tells us that there are only two possibilities:

1. $S^{K-2}(1) \in I_4$ and $S^{K-2}(1) - \alpha \in I_1$;
2. $S^{K-2}(1) \in I_5$ and $S^{K-2}(1) - \alpha \in I_2$.

If case 1. holds, then $T_0(S^{K-2}(1)) = S^{K-1}(1)$ and

$$T_1(S^{K-2}(1)) = T_0(S^{K-2}(1) - \alpha) = T_1(S^{K-2}(1) - \alpha) = S^{K-1}(1) - \alpha,$$

so that (5.18) becomes

$$\begin{cases} \sum\limits_{\mathbf{u}\in\Omega(S^{K-2}(1))^-} p_{\mathbf{u}} p_0 - \sum\limits_{\mathbf{u}\in\Omega(S^{K-2}(1))^+} p_{\mathbf{u}} p_0 = 0, \\ \sum\limits_{\mathbf{u}\in\Omega(S^{K-2}(1))^-} p_{\mathbf{u}} p_1 + \sum\limits_{\mathbf{u}\in(\Omega(S^{K-2}(1)-\alpha)^-} p_{\mathbf{u}} - \sum\limits_{\mathbf{u}\in\Omega(S^{K-2}(1))^+} p_{\mathbf{u}} p_1 - \sum\limits_{\mathbf{u}\in(\Omega(S^{K-2}(1)-\alpha)^+} p_{\mathbf{u}} = 0. \end{cases}$$

The last system of equations implies

$$\begin{cases} \sum\limits_{\mathbf{u}\in\Omega(S^{K-2}(1))^-} p_{\mathbf{u}} - \sum\limits_{\mathbf{u}\in\Omega(S^{K-2}(1))^+} p_{\mathbf{u}} & = 0, \\ \sum\limits_{\mathbf{u}\in(\Omega(S^{K-2}(1)-\alpha)^-} p_{\mathbf{u}} - \sum\limits_{\mathbf{u}\in(\Omega(S^{K-2}(1)-\alpha)^+} p_{\mathbf{u}} & = 0, \end{cases}$$

which contradicts the minimality of K. For the second case, the same contradiction is obtained in a similar way. □

5.4.6 Remark. From the previous result we see that matching occurs for the random systems R_α for the same parameters α and at the same time as for the deterministic systems S_α. [DK17] contains a complete description of the matching intervals of the maps S_α. The interval $[1, 2]$ can be divided into intervals of parameters for which matching of the maps S_α occurs after the same number of steps. By the above, these matching intervals also apply to the systems R_α.

§5.4.3 A formula for the stationary measure

Let λ be the Lebesgue measure on $[-1, 1]$. The existence of an invariant measure of the form $m_{\mathbf{p}} \times \mu_{\mathbf{p}}$ with $\mu_{\mathbf{p}} \ll \lambda$ for the random symmetric doubling maps R_α is guaranteed by the results of [P84, M85]. Furthermore, since T_0 is expanding and has a unique absolutely continuous invariant measure, it follows from [P84, Corollary 7] that also for R_α there is a unique measure $m_{\mathbf{p}} \times \mu_{\mathbf{p}}$ and that R_α is ergodic with respect to this measure. To show that $R_\alpha \in \mathcal{R}_A$, we check conditions (c1), (c2), (c3). (c1) is immediate and (c3) follows from the constant slope 2 of the maps $T_{\alpha,0}$ and $T_{\alpha,1}$. We check condition (5.6). Note that for any $\alpha \neq 2$,

$$\frac{\sum_{j\in\Omega} \frac{p_j}{k_{3,j}} d_{3,j}}{1 - \sum_{j\in\Omega} \frac{p_j}{k_{3,j}}} = \frac{\sum_{j\in\Omega} \frac{p_j}{2} 0}{1 - \sum_{j\in\Omega} \frac{p_j}{2}} = 0$$

and

$$\frac{\sum_{j\in\Omega} \frac{p_j}{k_{1,j}} d_{1,j}}{1 - \sum_{j\in\Omega} \frac{p_j}{k_{1,j}}} = \frac{\sum_{j\in\Omega} \frac{p_j}{2} \alpha}{1 - \sum_{j\in\Omega} \frac{p_j}{2}} = 2\alpha \neq 0.$$

Then Theorem 3.5.3 implies that an explicit formula for the density of this measure can be found via the algebraic procedure in Chapter 3 and from Theorem 5.2.3 and Theorem 5.4.5 we know that for Lebesgue almost all parameters α this density is piecewise constant. We will execute the procedure from Chapter 3 and start by introducing the same notation as in Chapter 3. Since Ω consists of two elements only, from now on we will just use p as an index instead of $\mathbf{p}$ whenever appropriate.

Denote by $a_{i,j}$ and $b_{i,j}$ the left and right limits at each critical point $c_i \in C$, i.e., for $1 \leq i \leq 4$ and $j \in \Omega$:

$$a_{i,j} = T_j(c_i^-) = \lim_{x \uparrow c_i} T_j(x), \qquad \text{and} \qquad b_{i,j} = T_j(c_i^+) = \lim_{x \downarrow c_i} T_j(x).$$

The images of the critical points are then given by

$$\begin{aligned} a_{1,0} &= a_{1,1} = b_{1,0} = \alpha - 1, & b_{1,1} &= -1, \\ a_{2,1} &= b_{2,0} = b_{2,1} = 1 - \alpha, & a_{2,0} &= 1, \\ a_{3,0} &= a_{3,1} = b_{3,0} = \alpha - 1, & b_{3,1} &= -1, \\ a_{4,1} &= b_{4,0} = b_{4,1} = 1 - \alpha, & a_{4,0} &= 1. \end{aligned}$$

For $y \in [-1, 1]$ and $1 \leq n \leq 4$ set

$$\mathrm{KI}_n(y) = \sum_{k \geq 1} \sum_{\mathbf{u} \in \Omega^k} \frac{p_{\mathbf{u}}}{2^k} 1_{I_n}(T_{\mathbf{u}_1^{k-1}}(y)). \tag{5.19}$$

The quantity $\mathrm{KI}_n(y)$ weighs the number of times the random orbits of y enters the interval I_n. The weight depends on the length and probability of each path $\omega \in \Omega^{\mathbb{N}}$ leading the point y to I_n. The *fundamental matrix* $A = (A_{n,i})$ of R is the 5×4 matrix with entries

$$A_{n,i} = \begin{cases} \sum_{j \in \Omega} p_j (1 + \mathrm{KI}_n(a_{i,j}) - \mathrm{KI}_n(b_{i,j})), & \text{for } n = i, \\ \sum_{j \in \Omega} p_j (\mathrm{KI}_n(a_{i,j}) - \mathrm{KI}_n(b_{i,j}) - 1), & \text{for } n = i+1, \\ \sum_{j \in \Omega} p_j (\mathrm{KI}_n(a_{i,j}) - \mathrm{KI}_n(b_{i,j})), & \text{else.} \end{cases}$$

Since for R there is a unique invariant probability measure $m_p \times \mu_p$ with $\mu_p \ll \lambda$, Theorem 3.5.3 implies that the null space of the matrix A is one-dimensional. According to Theorem 3.4.1 there is a unique vector $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \in \mathbb{R}^4 \setminus \{\mathbf{0}\}$ with $A\gamma = \mathbf{0}$ and such that the probability density f_p of μ_p has the form (5.7). Using the values of $a_{i,j}$ and $b_{i,j}$ computed above, we can reduce this to

$$\begin{aligned} f_p = (\gamma_1 + \gamma_3) &\sum_{k \geq 0} \sum_{\mathbf{u} \in \Omega^k} \frac{p_{1\mathbf{u}}}{2^{k+1}} \big(1_{[-1, T_{\mathbf{u}}(\alpha - 1))} - 1_{[-1, T_{\mathbf{u}}(-1))} \big) \\ &+ (\gamma_2 + \gamma_4) \sum_{k \geq 0} \sum_{\mathbf{u} \in \Omega^k} \frac{p_{0\mathbf{u}}}{2^{k+1}} \big(1_{[-1, T_{\mathbf{u}}(1))} - 1_{[-1, T_{\mathbf{u}}(1-\alpha))} \big). \end{aligned} \tag{5.20}$$

By symmetry to determine f_p it is enough to know the random orbits of 1 and $1 - \alpha$ only. From (5.20) we see that the density is piecewise constant when the orbits of 1 and $1 - \alpha$ are finite or when they merge with the same weight. In the former case the map admits a Markov partition, the latter case happens if R exhibits strong random matching. We focus on the second situation, since we know from Theorem 5.4.5 that this holds for Lebesgue almost all parameters.

Fix an $\alpha \in [1,2]$ such that R presents strong random matching. Let M be as in Theorem 5.4.5. Then for each i, j, n,

$$\mathrm{KI}_n(a_{i,j}) - \mathrm{KI}_n(b_{i,j}) = \sum_{k=1}^{M-1} \sum_{\mathbf{u}\in\Omega^k} \frac{p_{\mathbf{u}}}{2^k} \big(1_{I_n}(T_{\mathbf{u}_1^{k-1}}(a_{i,j})) - 1_{I_n}(T_{\mathbf{u}_1^{k-1}}(b_{i,j})) \big).$$

From Lemma 5.4.2 and the symmetry of the map we get

$$\mathrm{KI}_3(1) - \mathrm{KI}_3(1-\alpha) = 0 = \mathrm{KI}_3(-1) - \mathrm{KI}_3(\alpha - 1),$$

implying that $A_{3,1} = A_{3,4} = 0$, $A_{3,2} = -1$ and $A_{3,3} = 1$. Hence, any solution vector $\hat{\gamma}$ for $A\hat{\gamma} = \mathbf{0}$ has the form $\hat{\gamma} = (\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_2, \hat{\gamma}_3)$ and (5.20) becomes

$$\begin{aligned} f_p = (\gamma_1 + \gamma_2)\frac{p_1}{2} \sum_{k=0}^{M-2} \sum_{\mathbf{u}\in\Omega^k} \frac{p_{\mathbf{u}}}{2^k} \big(1_{[-1, T_{\mathbf{u}}(\alpha-1))} - 1_{[-1, T_{\mathbf{u}}(-1))} \big) \\ + (\gamma_2 + \gamma_3)\frac{p_0}{2} \sum_{k=0}^{M-2} \sum_{\mathbf{u}\in\Omega^k} \frac{p_{\mathbf{u}}}{2^k} \big(1_{[-1, T_{\mathbf{u}}(1))} - 1_{[-1, T_{\mathbf{u}}(1-\alpha))} \big), \end{aligned} \tag{5.21}$$

where $\gamma = (\gamma_1, \gamma_2, \gamma_2, \gamma_3)$ is the unique non-trivial vector in the null space of the fundamental matrix A that makes f_p into a probability density function. In the next section we will derive a number of properties of f_p with the goal of determining the frequency of the digit 0 in the signed binary expansions of $m_p \times \mu_p$ typical points.

§5.4.4 Minimal weight expansions

Recall from (5.13) that the random signed binary expansion of a point (ω, x) has a digit 0 in the n-th position precisely if

$$R^{n-1}(\omega, x) \in [1] \times I_2 \cup \Omega^{\mathbb{N}} \times I_3 \cup [0] \times I_4 =: D_0.$$

Since R is ergodic with respect to $m_p \times \mu_p$, it follows from Birkhoff's Ergodic Theorem that the frequency of the digit 0 in $m_p \times \mu_p$-almost all (ω, x) equals

$$\pi_0(\alpha, p) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{D_0}(R^k(\omega, x)) = (1-p)\mu_p(I_2) + \mu_p(I_3) + p\mu_p(I_4). \tag{5.22}$$

To give an example, consider $\alpha = 1$, see Figure 5.11(a). It is straightforward to check that the probability density $f_p = (1-p)1_{[-1,0]} + p1_{[0,1]}$ is invariant. This gives

$$\pi_0(1, p) = p\mu_p\Big(\Big[0, \frac{1}{2}\Big]\Big) + (1-p)\mu_p\Big(\Big[-\frac{1}{2}, 0\Big]\Big) = \frac{p^2 + (1-p)^2}{2} \le \frac{1}{2} \tag{5.23}$$

with equality only for $p = 0$ or $p = 1$.

To estimate $\pi_0(\alpha, p)$ for other values of α we use a few lemmata. For $k \geq 1$ set $E_k = \{\mathbf{u} \in \Omega^k : T_{\mathbf{u}}(1) = S^k(1)\}$ and $F_k = \{\mathbf{u} \in \Omega^k : T_{\mathbf{u}}(1-\alpha) = S^k(1)\}$. Also, use $(b_n)_{n\geq 1}$ to denote the digits in the signed binary expansion of 1 generated by S, i.e.,

$$b_n = \begin{cases} -1, & \text{if } S^{n-1}(1) < -\frac{1}{2}, \\ 0, & \text{if } -\frac{1}{2} \leq S^{n-1}(1) \leq \frac{1}{2}, \\ 1, & \text{if } S^{n-1}(1) > \frac{1}{2}. \end{cases}$$

Write $\mathbf{b}_k = b_1 \cdots b_k$ for any $k \geq 1$.

5.4.7 Lemma. *For all $1 \leq k < M-1$, $F_k \subseteq E_k$ and $E_k \setminus F_k = \{\mathbf{b}_k\}$.*

Proof. First note that the n-th signed binary digit of 1 generated by S, $n < M-1$, equals 0 if $S^{n-1}(1) \in I_4$ and 1 if $S^{n-1}(1) \in I_5$. We prove the statement by induction. For $k=1$ we have $E_1 = \{0,1\}$ and $F_1 = \{0\}$. Assume the statement holds for some $1 \leq k < M-2$. If $S^k(1) = T_{\mathbf{b}_k}(1) \in I_4$, then $b_{k+1} = 0$ and we know from the assumptions and since $S^k(1) - \alpha \in I_1$ that

$$T_{\mathbf{b}_k 0}(1) = S^{k+1}(1),\, T_{\mathbf{b}_k 1}(1) = T_{\mathbf{b}_k 0}(1-\alpha) = T_{\mathbf{b}_k 1}(1-\alpha) = S^{k+1}(1) - \alpha.$$

Hence, $\mathbf{b}_k 0 \in E_{k+1} \setminus F_{k+1}$ and $\mathbf{b}_k 1 \notin E_{k+1} \cup F_{k+1}$. If $S^k(1) = T_{\mathbf{b}_k}(1) \in I_5$, then $b_{k+1} = 1$ and

$$T_{\mathbf{b}_k 0}(1) = T_{\mathbf{b}_k 1}(1) = T_{\mathbf{b}_k 0}(1-\alpha) = S^{k+1}(1),\, T_{\mathbf{b}_k 1}(1-\alpha) = S^{k+1}(1) - \alpha.$$

So, $\mathbf{b}_k 1 \in E_{k+1} \setminus F_{k+1}$ and $\mathbf{b}_k 0 \in E_{k+1} \cap F_{k+1}$. For any other $\mathbf{u} \in \Omega^k$ it holds that $T_{\mathbf{u}}(1) = T_{\mathbf{u}}(1-\alpha)$, so that either $\mathbf{u}j \in E_{k+1} \cap F_{k+1}$ or $\mathbf{u}j \notin E_{k+1} \cup F_{k+1}$, $j = 0, 1$. This gives the statement. □

5.4.8 Lemma. *The density f_p is constant and equal to $\frac{1}{\alpha}$ on the interval* $[1-\alpha, \alpha-1]$.

Proof. For any $\mathbf{u} \in \Omega^k$, write $\bar{\mathbf{u}} = (1-u_1)\cdots(1-u_k)$ and for a subset $E \subseteq \Omega^k$ write $\bar{E} = \{\mathbf{u} \in \Omega^k : \bar{\mathbf{u}} \in E\}$. By Lemma 5.4.7 we have for each $k < M$,

$$\delta_k := \sum_{\mathbf{u} \in E_k} \frac{p_{\mathbf{u}}}{2^k} - \sum_{\mathbf{u} \in F_k} \frac{p_{\mathbf{u}}}{2^k} = \frac{p_{\mathbf{b}_k}}{2^k}, \quad \bar{\delta}_k := \sum_{\mathbf{u} \in \bar{E}_k^c} \frac{p_{\mathbf{u}}}{2^k} - \sum_{\mathbf{u} \in \bar{F}_k^c} \frac{p_{\mathbf{u}}}{2^k} = \frac{p_{\bar{\mathbf{b}}_k}}{2^k},$$

Recall the formula for the density f_p from (5.21). Using Proposition 5.4.3 we get

$$\begin{aligned}
&\frac{p_0}{2}\sum_{k=0}^{M-2}\sum_{\mathbf{u}\in\Omega^k}\frac{p_{\mathbf{u}}}{2^k}\left(1_{[-1,T_{\mathbf{u}}(1))}-1_{[-1,T_{\mathbf{u}}(1-\alpha))}\right)\\
&\quad=\frac{p_0}{2}\sum_{k=0}^{M-2}\sum_{\substack{\mathbf{u}\in\Omega^k:\\ T_{\mathbf{u}}(1)=S^k(1),\\ T_{\mathbf{u}}(1-\alpha)=S^k(1)-\alpha}}\frac{p_{\mathbf{u}}}{2^k}1_{[T_{\mathbf{u}}(1-\alpha),T_{\mathbf{u}}(1))}-\\
&\qquad\frac{p_0}{2}\sum_{k=0}^{M-2}\sum_{\substack{\mathbf{u}\in\Omega^k:\\ T_{\mathbf{u}}(1)=S^k(1)-\alpha,\\ T_{\mathbf{u}}(1-\alpha)=S^k(1)}}\frac{p_{\mathbf{u}}}{2^k}1_{[T_{\mathbf{u}}(1),T_{\mathbf{u}}(1-\alpha))}\\
&\quad=\frac{p_0}{2}\sum_{k=0}^{M-2}\delta_k 1_{[S^k(1)-\alpha,S^k(1))}.
\end{aligned}$$

For the other side it holds similarly using the symmetry of the system that

$$\frac{p_1}{2}\sum_{k=0}^{M-2}\sum_{\mathbf{u}\in\Omega^k}\frac{p_{\mathbf{u}}}{2^k}\left(1_{[-1,T_{\mathbf{u}}(\alpha-1))}-1_{[-1,T_{\mathbf{u}}(-1))}\right)=\frac{p_1}{2}\sum_{k=0}^{M-2}\bar{\delta}_k 1_{[-S^k(1),\alpha-S^k(1))}.$$

By (5.15) we have for all $k<M-1$,

$$S^k(1),\alpha-S^k(1)\in[\alpha-1,1]\quad\text{and}\quad S^k(-1),S^k(\alpha-1)\in[-1,1-\alpha],$$

so that on $[1-\alpha,\alpha-1]$ we obtain

$$f_p\mid_{[1-\alpha,\alpha-1]}(x)=(\gamma_1+\gamma_2)\frac{p_1}{2}\sum_{k=0}^{M-2}\bar{\delta}_k+(\gamma_2+\gamma_3)\frac{p_0}{2}\sum_{k=0}^{M-2}\delta_k.$$

Since f_p is a probability density it follows that

$$1=\int_{[-1,1]}f_p\,d\lambda=(\gamma_1+\gamma_2)\frac{p_1}{2}\sum_{k=0}^{M-2}\bar{\delta}_k\alpha+(\gamma_2+\gamma_3)\frac{p_0}{2}\sum_{k=0}^{M-2}\delta_k\alpha. \tag{5.24}$$

Hence,

$$f_p\mid_{[1-\alpha,\alpha-1]}(x)=\frac{1}{\alpha},$$

which gives the result. □

With this information we can compute $\pi_0(\alpha,p)$ for $\alpha\in\left[\frac{3}{2},2\right]$. Since in this case $\alpha-1\geq\frac{1}{2}$ it follows from Lemma 5.4.8 that

$$\pi_0(\alpha,p)=\frac{\alpha-1}{\alpha}+\frac{2-\alpha}{2\alpha}=\frac{1}{2}. \tag{5.25}$$

That is, for $\alpha\geq\frac{3}{2}$, and any $0<p<1$, the frequency of the digit 0 is equal to $\frac{1}{2}$ in the signed binary expansion of $m_p\times\mu_p$-almost all (ω,x). For the other values of α we need to do some more work.

5.4.9 Lemma. *Let $\gamma = (\gamma_1, \gamma_2, \gamma_2, \gamma_3)$ be the unique vector in the null space of A that makes f_p into a probability density function. Then $\gamma_2 = \frac{1}{\alpha}$.*

Proof. Since $S^k(1) \in I_4 \cup I_5$ for all $k < M-1$ it follows from the definition of the function KI_n in (5.19) and Proposition 5.4.3 that for $y = 1, 1-\alpha$,

$$\mathrm{KI}_4(y) + \mathrm{KI}_5(y) = \sum_{k=0}^{M-2} \sum_{j \in \Omega} \sum_{\mathbf{u} \in \Omega^k} \frac{p_j}{2} \frac{p_\mathbf{u}}{2^k} 1_{I_4 \cup I_5}(T_\mathbf{u}(y)) = \frac{1}{2} \sum_{k=0}^{M-2} \sum_{\substack{\mathbf{u} \in \Omega^k: \\ T_\mathbf{u}(y) = S^k(1)}} \frac{p_\mathbf{u}}{2^k},$$

so that

$$\frac{1}{2} \sum_{k=0}^{M-2} \delta_k = \mathrm{KI}_4(1) - \mathrm{KI}_4(1-\alpha) + \mathrm{KI}_5(1) - \mathrm{KI}_5(1-\alpha).$$

A similar statement holds for -1 and $\alpha - 1$. The fourth and fifth line of the linear system $A\gamma = \mathbf{0}$ read

$$p_1(\mathrm{KI}_4(\alpha-1) - \mathrm{KI}_4(-1))(\gamma_1+\gamma_2) + p_0(\mathrm{KI}_4(1) - \mathrm{KI}_4(1-\alpha))(\gamma_2+\gamma_3) - \gamma_2 + \gamma_3 = 0$$

and

$$p_1(\mathrm{KI}_5(\alpha-1) - \mathrm{KI}_5(-1))(\gamma_1+\gamma_2) + p_0(\mathrm{KI}_5(1) - \mathrm{KI}_5(1-\alpha))(\gamma_2+\gamma_3) - \gamma_3 = 0,$$

respectively. Adding them up gives

$$\begin{aligned}\gamma_2 &= p_1(\gamma_1+\gamma_2)(\mathrm{KI}_4(\alpha-1) - \mathrm{KI}_4(-1) + \mathrm{KI}_5(\alpha-1) - \mathrm{KI}_5(-1)) + \\ &\quad p_0(\gamma_2+\gamma_3)(\mathrm{KI}_4(1) - \mathrm{KI}_4(1-\alpha) + \mathrm{KI}_5(1) - \mathrm{KI}_5(1-\alpha)) \\ &= \frac{p_1}{2}(\gamma_1+\gamma_2) \sum_{k=0}^{M-2} \bar{\delta}_k + \frac{p_0}{2}(\gamma_2+\gamma_3) \sum_{k=0}^{M-2} \delta_k.\end{aligned}$$

The result then follows from (5.24). □

Combining Lemma 5.4.8 and Lemma 5.4.9 gives the following expression for the density f_p:

$$f_p = \left(\gamma_1 + \frac{1}{\alpha}\right) \frac{p_1}{2} \sum_{k=0}^{M-2} \frac{p_{\bar{\mathbf{b}}_k}}{2^k} 1_{[-S^k(1), \alpha - S^k(1))} + \left(\frac{1}{\alpha} + \gamma_3\right) \frac{p_0}{2} \sum_{k=0}^{M-2} \frac{p_{\mathbf{b}_k}}{2^k} 1_{[S^k(1)-\alpha, S^k(1))}, \tag{5.26}$$

where $\mathbf{b}_k = b_1 \cdots b_k$ denote the first k digits in the signed binary expansion of 1 given by S.

5.4.10 Lemma. *Let $\alpha \in \left(1, \frac{3}{2}\right)$ be a parameter for which the random system R has strong random matching. Then both $\gamma_1, \gamma_3 \geq 0$. As a consequence, $f_p > 0$ and μ_p is equivalent to the Lebesgue measure.*

Proof. Let $\gamma = (\gamma_1, \gamma_2, \gamma_2, \gamma_3)$ be the unique vector in the null space of A that makes f_p into a probability density. Set

$$y = \max_{k \in \{1,2,\dots,M-2\}} \{S^k(1), \alpha - S^k(1)\}.$$

By Lemma 5.4.4 we can assume that $y \neq 1$. Then

$$\mu_p([y,1]) = m_p \times \mu_p(R^{-1}(\Omega \times [y,1])) = p\mu_p\Big(\Big[\frac{y-\alpha}{2}, \frac{1-\alpha}{2}\Big] \cup \Big[\frac{y}{2}, \frac{1}{2}\Big]\Big).$$

By the definition of y one can see from (5.26) that

$$\mu_p([y,1]) = \frac{p(\gamma_2+\gamma_3)}{2}(1-y).$$

Furthermore, $T_0(1-\alpha) = 2-\alpha$ and $T_1(1-\alpha) = 2-2\alpha$, so in particular $y \geq \max\{2-\alpha, 2\alpha-2\}$. It follows that $1-\alpha \leq \frac{y-\alpha}{2} < \frac{1-\alpha}{2}$. Thus by Lemma 5.4.8 and Lemma 5.4.9, $f_p\,|_{[\frac{y-\alpha}{2},\frac{1-\alpha}{2}]} = \gamma_2$. We proceed by showing that none of the points $S^k(1)$ or $\alpha - S^k(1)$, $1 \leq k \leq M-2$, lie in the interval $\big[\frac{y}{2}, \frac{1}{2}\big]$, which then by (5.26) implies that the density f_p is also constant on the interval $\big[\frac{y}{2}, \frac{1}{2}\big]$. For $k = M-2 = M_\alpha - 1$, matching for S implies that $\frac{1}{2} < S^{M-2}(1) < \alpha - \frac{1}{2}$ and $\alpha - S^{M-2}(1) > \frac{1}{2}$. Suppose there exists a $k \in \{1,2,\dots,M-3\}$ such that $\frac{y}{2} < S^k(1) < \frac{1}{2}$ (or $\frac{y}{2} < \alpha - S^k(1) < \frac{1}{2}$). Then $S^{k+1}(1) > y$ (or $\alpha - S^{k+1}(1) > y$), which gives a contradiction with the definition of y. The same holds for $\alpha - S^k(1)$. Hence, there is a constant $c \geq 0$ such that

$$\frac{p(\gamma_2+\gamma_3)}{2}(1-y) = \mu_p([y,1]) = p\Big(\gamma_2\frac{(1-y)}{2} + c\frac{(1-y)}{2}\Big).$$

So, $0 \leq \mu_p\big(\big[\frac{y}{2}, \frac{1}{2}\big]\big) = c = \gamma_3$. The proof that $\gamma_1 \geq 0$ goes similarly. The fact that f_p is strictly positive and the equivalence of μ_p and the Lebesgue measure now follow from (5.26). □

The following result can be proven in essentially the same way as [DK17, Theorem 4.1]. We include a proof here for convenience.

5.4.11 Lemma (cf. Theorem 4.1 of [DK17]). *Fix $0 < p < 1$. The map $\alpha \mapsto \pi_0(\alpha, p)$ is continuous on $\big(1, \frac{3}{2}\big)$.*

Proof. In this proof we use $f_\alpha = f_{\alpha,p}$ to denote the unique density from (5.26). By (5.22), for the continuity of $\alpha \mapsto \pi_0(\alpha, p)$ it is sufficient to prove L^1-convergence of the densities f_α; i.e., for any sequence $\{\alpha_k\}_{k\geq 1} \subseteq \big(1, \frac{3}{2}\big)$ converging to a fixed $\hat{\alpha} \in \big(1, \frac{3}{2}\big)$, there is convergence $f_{\alpha_k} \to f_{\hat{\alpha}}$ in $L^1(\lambda)$. The proof of this fact goes along the following lines:

1. First we show that there is a uniform bound, i.e., independent of k, on the total variation and supremum norm of the densities f_{α_k}. It then follows from Helly's Selection Theorem that there is some subsequence of (f_{α_k}) for which an a.e. and L^1 limit $\hat{f}$ exist.

2. We show that $\hat{f} = f_{\hat{\alpha}}$, which by the same proof implies that any subsequence of (f_{α_k}) has a further subsequence converging a.e. to the same limit $f_{\hat{\alpha}}$. Hence, (f_{α_k}) converges to $f_{\hat{\alpha}}$ in measure.

3. By the uniform integrability of (f_{α_k}) it then follows from Vitali's Convergence Theorem that the convergence of (f_{α_k}) to $f_{\hat{\alpha}}$ is in L^1.

Step 1. and 2. use Perron-Frobenius operators. For $j = 0, 1$ the Perron-Frobenius operator $P_{\alpha,j}$ of $T_{\alpha,j}$ is uniquely defined by the equation

$$\int (P_{\alpha,j} f) g \, d\lambda = \int f (g \circ T_{\alpha,j}) \, d\lambda \quad \forall f \in L^1(\lambda), \, g \in L^\infty(\lambda)$$

and the Perron-Frobenius operator P_α of R_α is then defined by $P_\alpha f = p P_{\alpha,0} f + (1 - p) P_{\alpha,1} f$. Equivalently, P_α is uniquely defined by the equation

$$\int (P_\alpha f) g \, d\lambda = p \int f (g \circ T_{\alpha,0}) \, d\lambda + (1-p) \int f (g \circ T_{\alpha,1}) \, d\lambda \quad \forall f \in L^1(\lambda), \, g \in L^\infty(\lambda). \tag{5.27}$$

Since each R_α has a unique probability density f_α it follows from [P84, Theorem 1] that f_α is the L^1 limit of $(\frac{1}{n} \sum_{j=0}^{n-1} P_\alpha^j 1)_{n \geq 1}$ and it is the unique probability density that satisfies $P_\alpha f_\alpha = f_\alpha$. From [I12, Theorem 5.2] each f_α is a function of bounded variation. We proceed by finding uniform bounds on the total variation and supremum norm of these densities.

Fix $\hat{\alpha} \in \big(1, \frac{3}{2}\big)$. For the second iterates of the Perron-Frobenius operators we have

$$P_\alpha^2 f = \sum_{i,j=0}^{1} p_i p_j P_{\alpha,j}(P_{\alpha,i} f).$$

Since the intervals of monotonicity of any of the maps $T_{\alpha,\mathbf{u}}$ for $\mathbf{u} \in \Omega^2$, only become arbitrarily small for α approaching 1 and $\frac{3}{2}$, we can find a uniform lower bound δ on the length of the intervals of monotonicity of any map $T_{\alpha,\mathbf{u}}$, $\mathbf{u} \in \Omega^2$, for all values α that are close enough to $\hat{\alpha}$. Applying [BG97, Lemma 5.2.1] to $T_{\alpha,j}$, $j = 0, 1$, and any of the second iterates $T_{\alpha,\mathbf{u}}$, $\mathbf{u} \in \Omega^2$, gives that

$$Var(P_{\alpha,j} f) \leq Var(f) + \frac{1}{\delta} \|f\|_1 \quad \text{and} \quad Var(P_{\alpha,\mathbf{u}} f) \leq \frac{1}{2} Var(f) + \frac{1}{2\delta} \|f\|_1,$$

where Var denotes the total variation over the interval $[-1, 1]$. Since these bounds do not depend on $\alpha, j, \mathbf{u}$, the same estimates hold for P_α, so that for any function $f : [-1, 1] \to \mathbb{R}$ of bounded variation and any $n \geq 1$,

$$Var(P_\alpha^n f) \leq \frac{1}{2^{\lfloor n/2 \rfloor}} Var(f) + \Big(2 + \frac{1}{\delta}\Big) \|f\|_1. \tag{5.28}$$

Let $\{\alpha_k\}_{k \geq 1}$ with $\alpha_k \to \hat{\alpha}$ be a sequence for which the lower bound δ holds for each k. For each k and n, write $f_{k,n} = \frac{1}{n} \sum_{i=0}^{n-1} P_{\alpha_k} 1$. Since

$$\sup |f_{k,n}| \leq Var(f_{k,n}) + \int f_{k,n} \, d\lambda,$$

it follows from (5.28) that there is a uniform constant $C > 0$ (independent of k, n) such that $Var(f_{k,n}), \sup |f_{k,n}| < C$. The same then holds for the limits f_{α_k}. Helly's Selection Theorem then gives the existence of a subsequence $\{k_i\}$ and a function $\hat{f}$ of bounded variation, such that $f_{\alpha_{k_i}} \to \hat{f}$ in $L^1(\lambda)$ and λ-a.e. and with $Var(\hat{f}), \sup |\hat{f}| < C$. This finishes 1.

By 2. and 3. above, what remains to finish the proof is to show that $P_{\hat{\alpha}}\hat{f} = \hat{f}$. By (5.27) it is enough to show that for any compactly supported C^1 function $g : [-1,1] \to \mathbb{R}$ it holds that

$$\left| \int (P_{\hat{\alpha}}\hat{f}) g \, d\lambda - \int \hat{f} g \, d\lambda \right| = 0.$$

Note that

$$\begin{aligned} \left| \int (P_{\hat{\alpha}}\hat{f}) g \, d\lambda - \int \hat{f} g \, d\lambda \right| \le & p \left| \int \hat{f}(g \circ T_{\hat{\alpha},0}) \, d\lambda - \int \hat{f} g \, d\lambda \right| + \\ & (1-p) \left| \int \hat{f}(g \circ T_{\hat{\alpha},1}) \, d\lambda - \int \hat{f} g \, d\lambda \right|. \end{aligned}$$

For $j = 0, 1$ we can write

$$\begin{aligned} \left| \int \hat{f}(g \circ T_{\hat{\alpha},j}) \, d\lambda - \int \hat{f} g \, d\lambda \right| \le & \left| \int \hat{f}(g \circ T_{\hat{\alpha},j}) \, d\lambda - \int f_{\alpha_{k_i}} (g \circ T_{\hat{\alpha},j}) \, d\lambda \right| \\ & + \left| \int f_{\alpha_{k_i}} (g \circ T_{\hat{\alpha},j}) \, d\lambda - \int f_{\alpha_{k_i}} (g \circ T_{\alpha_{k_i},j}) \, d\lambda \right| \\ & + \left| \int f_{\alpha_{k_i}} (g \circ T_{\alpha_{k_i},j}) \, d\lambda - \int \hat{f} g \, d\lambda \right|. \end{aligned}$$

The first and third integral on the right hand side can be bounded by $\|g\|_\infty \|\hat{f} - f_{\alpha_{k_i}}\|_1 \to 0$. For the second integral, $\|f_{\alpha_{k_i}}\|_\infty < C$ and $\int |g \circ T_{\hat{\alpha},j} - g \circ T_{\alpha_{k_i},j}| \, d\lambda \to 0$ by the Dominated Convergence Theorem. Hence, $\hat{f} = f_{\hat{\alpha}}$ and $f_{\alpha_k} \to f_{\hat{\alpha}}$ in L^1. □

Figure 5.12 shows a numerical approximation of the graph of the function $(\alpha, p) \mapsto \pi_0(\alpha, p)$. We can now prove that the maximal value of the frequency of the digit 0 is in fact $\frac{1}{2}$.

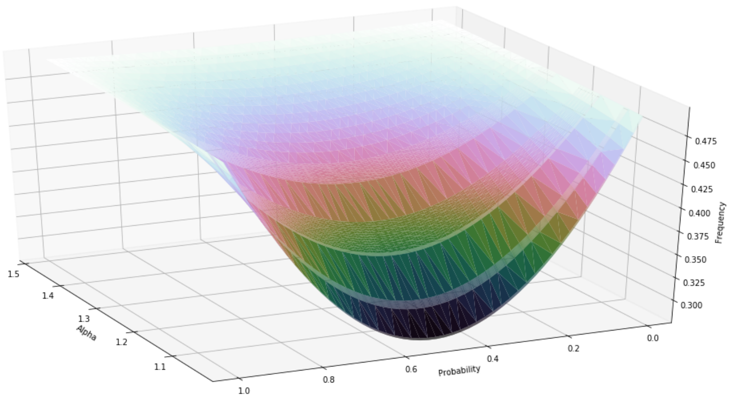


Figure 5.12: The graph of $(\alpha, p) \mapsto \pi_0(\alpha, p)$.

5.4.12 Theorem. *For any $0 < p < 1$ and any $\alpha \in [1,2]$ the frequency $\pi_0(\alpha,p)$ is at most $\frac{1}{2}$ for $m_p \times \lambda$-a.e. $(\omega,x) \in \Omega^{\mathbb{N}} \times [-1,1]$.*

Proof. For $\alpha \in \left[\frac{3}{2},2\right]$ the statement follows from (5.25) and for $\alpha = 1$ from (5.23). Let $\alpha \in \left(1,\frac{3}{2}\right)$. The deterministic map $T_{\alpha,0}$ has density $f_0 = \frac{1}{\alpha}1_{[1-\alpha,1]}$ and $T_{\alpha,1}$ has $f_1 = \frac{1}{\alpha}1_{[-1,\alpha-1]}$. Hence $\pi_0(\alpha,p) = \frac{1}{2}$ for $p = 0,1$. Let $0 < p < 1$ and let α be a parameter satisfying the conditions of Lemma 5.4.10. We know that f_p is constant and equal to $\frac{1}{\alpha}$ on $[1-\alpha,\alpha-1]$. For $x > \alpha - 1$ the density can be written as

$$\begin{aligned} f_p(x) = \frac{1}{\alpha} - \Bigg(&\frac{(1-p)(\gamma_1+\gamma_2)}{2} \sum_{k=0}^{M-2} \frac{p_{\bar{\mathbf{b}}_k}}{2^k} 1_{[\alpha-1,x]}(\alpha - S^k(1)) \\ &+ \frac{p(\gamma_2+\gamma_3)}{2} \sum_{k=0}^{M-2} \frac{p_{\mathbf{b}_k}}{2^k} 1_{[\alpha-1,x]}(S^k(1)) \Bigg) \\ = \frac{1}{\alpha} - \frac{(1-p)(\gamma_1+\gamma_2)}{2} - \Bigg(&\frac{(1-p)(\gamma_1+\gamma_2)}{2} \sum_{k=1}^{M-2} \frac{p_{\bar{\mathbf{b}}_k}}{2^k} 1_{[\alpha-1,x]}(\alpha - S^k(1)) \\ &+ \frac{p(\gamma_2+\gamma_3)}{2} \sum_{k=1}^{M-2} \frac{p_{\mathbf{b}_k}}{2^k} 1_{[\alpha-1,x]}(S^k(1)) \Bigg) \\ \le \frac{1}{\alpha} - \frac{(1-p)(\gamma_1+\gamma_2)}{2}&. \end{aligned}$$

Similarly, for $x < 1 - \alpha$ we get $f_p(x) \le \frac{1}{\alpha} - \frac{p(\gamma_2+\gamma_3)}{2}$. By (5.22) and Lemma 5.4.10,

$$\begin{aligned} \pi_0(\alpha,p) &= \frac{\alpha-1}{\alpha} + \frac{\alpha-1}{2\alpha} + p\mu_{\alpha,p}\left(\left[\alpha-1,\frac{1}{2}\right]\right) + (1-p)\mu_{\alpha,p}\left(\left[-\frac{1}{2},1-\alpha\right]\right) \\ &\le \frac{3(\alpha-1)}{2\alpha} + \frac{3-2\alpha}{2\alpha}\left(1 - \frac{p(1-p)\alpha}{2}\min\{\gamma_1+\gamma_2,\gamma_2+\gamma_3\}\right) \\ &= \frac{1}{2} - \frac{3-2\alpha}{2}\frac{p(1-p)}{2}\min\{\gamma_1+\gamma_2,\gamma_2+\gamma_3\} \\ &< \frac{1}{2}. \end{aligned}$$

Since matching holds for Lebesgue almost all parameters α, the statement now follows from Lemma 5.4.11 and the equivalence of μ_p and the Lebesgue measure. □

§5.5 Final remarks

§5.5.1 On the symmetric doubling maps

The numerical approximation of the graph of $(\alpha,p) \mapsto \pi_0(\alpha,p)$ shown in Figure 5.12 seems to suggest some other features of the map that we have not proved. Firstly, it suggests some symmetry. In fact it can be shown that for each fixed α and any $x \in [0,1]$, it holds that $f_p(x) = f_{1-p}(-x)$. For this one needs to consider the fundamental matrix $\tilde{A}$ corresponding to the random system $\tilde{R}_\alpha$ obtained by switching the roles

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of p and $1-p$. Then using the permutation (12)(45), one can relate various of the quantities involved for $\tilde{A}$ to the fundamental matrix A of R_α.

Secondly, for any matching parameter α and any $0 < p < 1$ the density $f_{\alpha,p}$ is a finite combination of indicator functions, whose supports depend on the position of the points in the set $\{S^k(1), \alpha - S^k(1)\}_{k=0}^{M-2}$ and whose coefficients are polynomials in p. So, for such a fixed α and any $x \in [-1,1]$, the map $p \mapsto f_{\alpha,p}(x)$ is continuous in p.

Thirdly, the graph also suggests that the map presents a minimum at $p = \frac{1}{2}$. Using the above two facts we were only able to show the following:

5.5.1 Proposition. *Let $\alpha \in [1,2]$ be such that R has strong random matching. Then the map $p \mapsto \pi_0(\alpha, p)$ has an extremal value at $p = \frac{1}{2}$.*

Proof. By combining (5.22) and the fact that $f_p(x) = f_{1-p}(-x)$ we obtain

$$\pi_0(\alpha,p) = (1-p)\mu_{1-p}(I_4) + \mu_p(I_3) + p\mu_p(I_4).$$

Computing the derivative with respect to p then gives

$$\partial_p \pi_0(\alpha,p) = -\mu_{1-p}(I_4) - (1-p)\partial_p(\mu_{1-p}(I_4)) + \partial_p(\mu_p(I_3)) + \mu_p(I_4) + \partial_p(\mu_p(I_4)). \tag{5.29}$$

From Lemma 5.4.8 it follows that $\partial_p(\mu_p(I_3)) = -\partial_p(\mu_{1-p}(I_3))$, implying that

$$\partial_p(\mu_p(I_3)) = 0 \quad \text{at} \quad p = \frac{1}{2}.$$

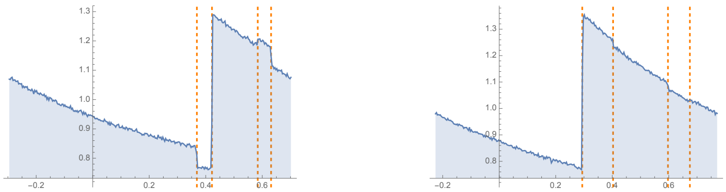
Therefore, by (5.29) $\partial_p \pi_0(\alpha,p) = 0$ at $p = \frac{1}{2}$. □

§5.5.2 On random CF-maps

Theorem 5.2.3 states that for random piecewise affine maps of the interval satisfying (c1), (c2) and (c3) strong random matching implies that there exists a piecewise constant invariant density. Condition (5.5) was sufficient for the theorem to work, which was one of the main motivations for Definition 5.2.2.

Theorem 5.2.3 is a random analogue of [BCMP18, Theorem 1.2], except that there the statement has less assumptions. The authors mention in [BCMP18, Remark 1.3] that for piecewise smooth interval maps with strong matching the corresponding invariant probability densities are piecewise smooth. On the other hand, as we noted before, the natural extension construction which for continued fraction transformations is often used to find invariant densities, seems to suggest that matching alone is sufficient to guarantee the existence of a piecewise smooth density. It would be interesting to investigate this further for the random continued fraction transformation.

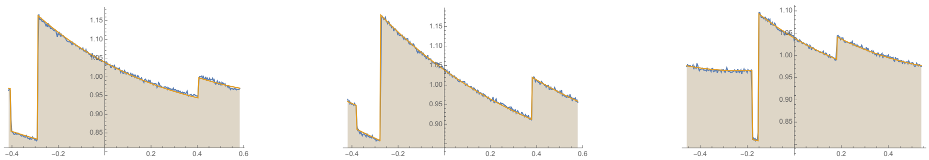
In a first attempt to investigate to what extent Theorem 5.2.3 can be generalised to piecewise smooth random systems on an interval, we include some numerical simulations. Recall from Example 5.3.2 that the random continued fraction maps R_α have strong random matching for α in the intervals J_n with endpoints as in (5.8), see also Figure 5.8. Figure 5.13 shows two simulations of the invariant densities for such



(a) $\alpha = 0.70315\ldots$, $p_0 = 0.3$ (b) $\alpha = 0.77287\ldots$, $p_0 = 0.6$

Figure 5.13: Numerical simulations of the invariant probability densities of the random continued fraction maps R_α from Example 5.3.2. In (a) we take $\alpha \in J_4$ and $p_0 = 0.3$ and in (b) we have $\alpha \in J_5$ and $p_0 = 0.6$. The dashed lines indicate the positions of the prematching points*, i.e., the points in the orbits of α and $\alpha - 1$ before the moment of matching.*

systems R_α. The densities seem to be piecewise smooth with discontinuities precisely at the orbit points of α and $\alpha - 1$ before matching. This seems to support the claim that strong random matching is sufficient to guarantee the existence of a piecewise smooth invariant density.



(a) $\alpha = 0.584\ldots$, $p_0 = 0.3$ (b) $\alpha = 0.579\ldots$, $p_0 = 0.65$ (c) $\alpha = 0.541\ldots$, $p_0 = 0.25$

Figure 5.14: Numerical simulations of the invariant probability densities of the random continued fraction maps R_α from Example 5.3.2 for three values of α between $\frac{1}{2}$ and $2 - \sqrt{2}$. The map in (a) has $\alpha \in \left(\frac{\sqrt{10}-2}{2}, 2 - \sqrt{2}\right)$, which is the matching interval considered in Example 5.3.2. The orange graph is the graph of the weighted average of the densities of $T_{\alpha,0}$ and $T_{\alpha,1}$ with the appropriate values of p.

In Example 5.3.2 we also considered the maps R_α for $\alpha \in \left(\frac{\sqrt{10}-2}{2}, 2 - \sqrt{2}\right)$. We showed that R_α has random matching with $M = 3$, but no strong matching at that moment. With a similar approach it can be shown that R_α has random matching for various other intervals in $\left[\frac{1}{2}, \frac{\sqrt{5}-1}{2}\right]$. For $\alpha \in \left[\frac{1}{2}, 2 - \sqrt{2}\right]$ both deterministic maps $T_{\alpha,0}$ and $T_{\alpha,1}$ have strong matching with $M, Q \leq 2$, as was shown in [N81] and [TI81], and moreover, for both of them the invariant densities are known. In Figure 5.14 we have plotted the weighted average of these densities together with numerical simulations of the densities for various values of $\alpha \in \left[\frac{1}{2}, 2 - \sqrt{2}\right]$ and $0 < p < 1$. This makes us wonder whether we need strong random matching to guarantee the existence of a piecewise smooth invariant density for these random systems or whether random matching is sufficient.

§5.6 Appendix

Figure 5.12 has been produced with the following Phyton program. Note that a computer is indeed able to handle the heavy algebraic procedure proposed in Chapter 3 and compute, from the fundamental matrix M, the density function $f_{\mathbf{p}}$ of the stationary measure $\mu_{\mathbf{p}}$.

```
import numpy as np
import array as arr
import sympy as sym
import random as rd
import scipy.optimize as opt
from random import choices
from fractions import Fraction
from decimal import Decimal
from sympy import pprint, Symbol, init_printing
from numpy.linalg import matrix_rank
from sympy import *
from pylab import *
from scipy import optimize
from matplotlib import pyplot
from sympy.utilities.lambdify import lambdify
sym.init_printing()

aNumerator = 1
aDenominator = 3
print ("Rational a, st I_a is a maximal quadratic interval
        for alpha-CF: ", Fraction(aNumerator,aDenominator))

# compute the regular continued fraction expansion of odd length
def continuedFraction(n, d):
    if d == 0: return []
    q = n//d
    r = n - q*d
    return [q] + continuedFraction(d, r)

print ("Regular continued fraction expansion: ",
        continuedFraction(aNumerator,aDenominator))
m = np.sum(continuedFraction(aNumerator,aDenominator))
print ("Matching exponent: ", m)

vectorContinuedFraction = continuedFraction
        (aNumerator,aDenominator)[1:]
lengthVCF= len(vectorContinuedFraction)
if (lengthVCF % 2) == 0:
    vectorContinuedFraction[lengthVCF-1] =
        vectorContinuedFraction[lengthVCF-1]-1
    vectorContinuedFraction = np.append
```

```
        (vectorContinuedFraction,1)
print ("Regular continued fraction expansion with odd length: ",
        vectorContinuedFraction)

# compute alpha via the valuation function
#        of the associated binary vector
vectorBinary = np.ones((m,), dtype=int)
partialSum = np.zeros((lengthVCF,), dtype=int)
partialSum[0] = vectorContinuedFraction[0]
for i in range(1,lengthVCF):
    partialSum[i] = vectorContinuedFraction[i]+partialSum[i-1]
for i in range (0,lengthVCF-1,2):
    vectorBinary[partialSum[i]:partialSum[i+1]] = 0
print("Associated binary vector: ", vectorBinary)

alphaInverse = 0
for i in range(1, m+1):
    alphaInverse += vectorBinary[i-1]/2**i
alphaInverseFraction = Fraction(alphaInverse)
alphaVar = Fraction(alphaInverseFraction.denominator,
        alphaInverseFraction.numerator)
print("Alpha variable: ", alphaVar)

#Matching interval
startInterval = Fraction(2**m+1,2**m*alphaInverseFraction+1)
endInterval = Fraction(2**m-1,2**m*alphaInverseFraction-1)
print("Matching interval: ", startInterval, endInterval)

#Random parameter alpha in the matching interval
        (startInterval, endInterval) of matching exponent m
alpha = rd.uniform(startInterval, endInterval)
print (Fraction(alpha), "alpha: ", alpha)
p = Symbol('p')
print("Probability vector: ", p, 1-p)

def t(x):
    if x>0.5:
        return Fraction(2*x-alpha)
    elif x<-0.5:
        return Fraction(2*x+alpha)
    else:
        return Fraction(2*x)

# compute the orbits of 1, 1-alpha and their opposites
orbitOne = np.ones((m+1,), dtype=object)
orbitOneWeighted = np.ones((m+1,), dtype=object)
orbitOneWeighted[0] = 1-p
```

```
orbitOneNegative = np.ones((m+1,), dtype=object)
orbitOneAlpha = np.ones((m+1,), dtype=object)
orbitOneAlphaNegative = np.ones((m+1,), dtype=object)

for i in range (1,m+1):
    orbitOne[i]= t(orbitOne[i-1])
    if (alpha-1)/2<orbitOne[i]<0.5:
        orbitOneWeighted[i]=p
    else:
        orbitOneWeighted[i]=1-p

orbitOne = orbitOne[:m]
orbitOneWeighted = orbitOneWeighted[:m]

for i in range (0,m):
    orbitOneAlpha[i]= Fraction(orbitOne[i]-alpha)
    orbitOneNegative[i]= Fraction(-orbitOne[i])
    orbitOneAlphaNegative[i]= Fraction(-orbitOne[i]+alpha)

# compute the partition given by points
#       in the prematching set and the switch regions
partition = np.append( orbitOneAlphaNegative,
        np.append(orbitOneAlpha,
        np.append(orbitOne, orbitOneNegative)))
partitionSorted = np.sort(partition)
partitionSwitch = np.append(partition,
        [-Fraction(1,2), (alpha-1)/2, -(alpha-1)/2,
        Fraction(1,2)])
partitionSortedSwitch = np.sort(partitionSwitch)

functionLOne = np.ones((m,), dtype=object)
functionLOneNegative = np.ones((m,), dtype=object)

for i in range (1,m):
    functionLOne[i] = sym.simplify
        (functionLOne[i-1]*Fraction(1,2)*orbitOneWeighted[i-1])
    functionLOneNegative[i] = sym.simplify
        (functionLOneNegative[i-1]*Fraction(1,2)*
        (1-orbitOneWeighted[i-1]))
print("L_(1)-L_(1-alpha): ", functionLOne)
print("L_(alpha-1)-L_(-1): ", functionLOneNegative)

# compute the quantities KI_n
kiOne = np.zeros((5,), dtype=object)
kiOneWeighted = np.ones((m+1,), dtype=object)
kiOneWeighted[1] = Fraction(1,2)
for i in range (2,m+1):
```

```
    kiOneWeighted[i]= sym.simplify
        (kiOneWeighted[i-1]*Fraction(1,2)*orbitOneWeighted[i-2])

for i in range (1,m):
    if orbitOneWeighted[i-1]==p:
        kiOne[3] += kiOneWeighted[i]
        kiOne[0] += -kiOneWeighted[i]
    else:
        kiOne[4] += kiOneWeighted[i]
        kiOne[1] += -kiOneWeighted[i]
kiOne[4] += kiOneWeighted[m]
kiOne[0] += -kiOneWeighted[m]
print ("KIn(1)-KIn(1-alpha): ", sym.simplify(kiOne))

kiOneNegative=np.zeros((5,), dtype=object)
kiOneWeightedNegative=np.ones((m+1,), dtype=object)
kiOneWeightedNegative[1]=Fraction(1,2)
for i in range (2,m+1):
    kiOneWeightedNegative[i]= sym.simplify
        (kiOneWeightedNegative[i-1]*Fraction(1,2)*
        (1-orbitOneWeighted[i-2]))

for i in range (1,m):
    if orbitOneWeighted[i-1]==p:
        kiOneNegative[4] += kiOneWeightedNegative[i]
        kiOneNegative[1] += -kiOneWeightedNegative[i]
    else:
        kiOneNegative[3] += kiOneWeightedNegative[i]
        kiOneNegative[0] += -kiOneWeightedNegative[i]
kiOneNegative[4] += kiOneWeightedNegative[m]
kiOneNegative[0] += -kiOneWeightedNegative[m]
print ("KIn(alpha-1)-KIn(-1): ", sym.simplify(kiOneNegative))

# compute the fundamental matrix
mFirstRow = np.zeros((3,), dtype=object)
mSecondRow = np.zeros((3,), dtype=object)
mThirdRow = np.zeros((3,), dtype=object)

mFirstRow[0] = -Fraction(1,2)+(1-p)/2*kiOneNegative[1]
mFirstRow[1] = Fraction(1,2)+p/2*kiOne[1]+(1-p)/2*
        kiOneNegative[1]
mFirstRow[2] = p/2*kiOne[1]

mSecondRow[0] = (1-p)/2*kiOneNegative[3]
mSecondRow[1] = p/2*kiOne[3]-Fraction(1,2)+(1-p)/2*
        kiOneNegative[3]
mSecondRow[2] = Fraction(1,2)+p/2*kiOne[3]
```

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```
mThirdRow[0] = (1-p)/2*kiOneNegative[4]
mThirdRow[1] = p/2*kiOne[4]+(1-p)/2*kiOneNegative[4]
mThirdRow[2] = -Fraction(1,2)+p/2*kiOne[4]

matrixM = np.array([mFirstRow, mSecondRow, mThirdRow])
fundamentalMatrix = sym.Matrix(matrixM)

vectorGamma = fundamentalMatrix.nullspace()
print ("Solution vector Gamma: ", sym.factor(vectorGamma))
gamma = sym.Matrix(vectorGamma)
x = sym.factor(gamma[0])
y = sym.factor(gamma[1])
z = sym.factor(gamma[2])

densityCoefficient1 = sym.factor((x+y)*(1-p))
densityCoefficient2 = sym.factor((y+z)*p)

functionLOneNegativeWeighted = sym.factor
        (functionLOneNegative*densityCoefficient1)
functionLOneWeighted = sym.factor
        (functionLOne*densityCoefficient2)

# compute the density
density = np.zeros((len(partitionSortedSwitch),), dtype=object)
for i in range(0,m):
    initialPosition = np.where
        (partitionSortedSwitch==orbitOneAlpha[i])[0][0]+1
    finalPosition = np.where
        (partitionSortedSwitch==orbitOne[i])[0][0]
    for k in range (initialPosition, finalPosition+1):
        density[k] += functionLOneWeighted[i]
    initialPositionNegative = np.where
        (partitionSortedSwitch==orbitOneNegative[i])[0][0]+1
    finalPositionNegative = np.where
        (partitionSortedSwitch==orbitOneAlphaNegative[i])[0][0]
    for k in range
        (initialPositionNegative, finalPositionNegative+1):
        density[k] += functionLOneNegativeWeighted[i]

simplifiedDensity=sym.factor(density)
print ("Density function: ", simplifiedDensity)

normalizingConstant = 0
for i in range(1,len(partitionSortedSwitch)):
    normalizingConstant +=
        (partitionSortedSwitch[i]-partitionSortedSwitch[i-1])*
```

```
        density[i]
simplifiedNormalizingConstant=sym.factor(1/normalizingConstant)
print ("Normalizing constant: ", simplifiedNormalizingConstant)

normalizedDensity= sym.factor
        (simplifiedNormalizingConstant*density)
print ("Normalized density: ", normalizedDensity)

# compute the frequency of the zero digit
frequencyZero = 0
measureI4 = 0
measureI2 = 0
measureI3 = 0

startI4 = np.where(partitionSortedSwitch==(alpha-1)/2)[0][0]+1
endI4 = np.where(partitionSortedSwitch==1/2)[0][0]

for k in range (startI4,endI4+1):
    measureI4 +=
        (partitionSortedSwitch[k]-partitionSortedSwitch[k-1])*
        normalizedDensity[k]
measureI4 = measureI4*p

startI2 = np.where(partitionSortedSwitch==-1/2)[0][0]+1
endI2 = np.where(partitionSortedSwitch==-(alpha-1)/2)[0][0]

for k in range (startI2,endI2+1):
    measureI2 +=
        (partitionSortedSwitch[k]-partitionSortedSwitch[k-1])*
        normalizedDensity[k]
measureI2 = measureI2*(1-p)

startI3 = np.where(partitionSortedSwitch==-(alpha-1)/2)[0][0]+1
endI3 = np.where(partitionSortedSwitch==(alpha-1)/2)[0][0]

for k in range (startI3,endI3+1):
    measureI3 +=
        (partitionSortedSwitch[k]-partitionSortedSwitch[k-1])*
        normalizedDensity[k]

frequencyZero = measureI4 + measureI2 + measureI3
print ("Frequency of the 0 digit:", sym.factor(frequencyZero))

frequencyZeroFunction = lambdify((p,),
        frequencyZero, modules='numpy')

p_scipy = np.linspace(0.0, 1.0)
```

```
y_sympy = frequencyZeroFunction(p_scipy)

pyplot.plot(p_scipy, y_sympy, 'b-', label='Freq(p)')
pyplot.xlabel(r'$p$')
pyplot.ylabel(r'$Frequency$')
pyplot.legend(loc='upper right')
pyplot.show()
```

Bibliography

[A97] J. Aaronson. *An introduction to infinite ergodic theory*, volume 50 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.

[AGH18] N. Abbasi, M. Gharaei, and A. J. Homburg. Iterated function systems of logistic maps: synchronization and intermittency. *Nonlinearity*, 31(8):3880–3913, 2018.

[ANV15] R. Aimino, M. Nicol, and S. Vaienti. Annealed and quenched limit theorems for random expanding dynamical systems. *Probab. Theory Related Fields*, 162(1-2):233–274, 2015.

[A98] L. Arnold. *Random dynamical systems*. Springer Monographs in Mathematics. Springer-Verlag Berlin Heidelberg, 1998.

[AS13] P. Arnoux and T. A. Schmidt. Cross sections for geodesic flows and α-continued fractions. *Nonlinearity*, 26(3):711–726, 2013.

[BG05] W. Bahsoun and P. Góra. Position dependent random maps in one and higher dimensions. *Studia Math.*, 166(3):271–286, 2005.

[B93] M. F. Barnsley. *Fractals everywhere*. Academic Press, 1993.

[BI09] L. Barreira and G. Iommi. Frequency of digits in the Lüroth expansion. *J. Number Theory*, 129(6):1479–1490, 2009.

[BBDK94] J. Barrionuevo, R. M. Burton, K. Dajani, and C. Kraaikamp. Ergodic properties of generalized Lüroth series. *TU Delft Report*, 94-105:1–16, 1994.

[B13] C. Bektur. Lyapunov exponents for position dependent random maps: formulae and applications. *Stoch. Anal. Appl.*, 31(4):600–608, 2013.

[BCIT13] C. Bonanno, C. Carminati, S. Isola, and G. Tiozzo. Dynamics of continued fractions and kneading sequences of unimodal maps. *Discrete Contin. Dyn. Syst.*, 33(4):1313–1332, 2013.

[B51] A. D. Booth. A signed binary multiplication technique. *The Quarterly Journal of Mechanics and Applied Mathematics*, 4(2):236–240, 1951.

[BSORG13] V. Botella-Soler, J. A. Oteo, J. Ros, and P. Glendinning. Lyapunov exponent and topological entropy plateaus in piecewise linear maps. *J. Phys. A*, 46(12):125101, 26, 2013.

[BG92] A. Boyarsky and P. Góra. A dynamical system model for interference effects and the two-slit experiment of quantum physics. *Physics Letters A*, 168(2):103–112, 1992.

[BG97] A. Boyarsky and P. Góra. *Laws of chaos.* Probability and its Applications. Birkhäuser Boston, Inc., Boston, MA, 1997. Invariant measures and dynamical systems in one dimension.

[BL92] A. Boyarsky and Y. S. Lou. Existence of absolutely continuous invariant measures for higher-dimensional random maps. *Dynamics and Stability of Systems*, 7(4):233–244, 1992.

[BCK17] H. Bruin, C. Carminati, and C. Kalle. Matching for generalised β-transformations. *Indag. Math. (N.S.)*, 28(1):55–73, 2017.

[BCMP18] H. Bruin, C. Carminati, S. Marmi, and A. Profeti. Matching in a family of piecewise affine maps. *Nonlinearity*, 32(1):172–208, 2018.

[BK93] R. Burton and G. Keller. Stationary measures for randomly chosen maps. *Journal of Theoretical Probability*, 6:1–16, 1993.

[B00] J. Buzzi. Absolutely continuous s.r.b. measures for random lasota-yorke maps. *Transactions of the American Mathematical Society*, 352(7):3289–3303, 2000.

[CWZ13] C. Cao, J. Wu, and Z. Zhang. The efficiency of approximating real numbers by Lüroth expansion. *Czechoslovak Math. J.*, 63(138)(2):497–513, 2013.

[CIT18] C. Carminati, S. Isola, and G. Tiozzo. Continued fractions with $SL(2,Z)$-branches: combinatorics and entropy. *Trans. Amer. Math. Soc.*, 370(7):4927–4973, 2018.

[CMPT10] C. Carminati, S. Marmi, A. Profeti, and G. Tiozzo. The entropy of α-continued fractions: numerical results. *Nonlinearity*, 23(10):2429–2456, 2010.

[CT12] C. Carminati and G. Tiozzo. A canonical thickening of $\mathbb{Q}$ and the entropy of α-continued fraction transformations. *Ergodic Theory Dynam. Systems*, 32(4):1249–1269, 2012.

[CT13] C. Carminati and G. Tiozzo. Tuning and plateaux for the entropy of α-continued fractions. *Nonlinearity*, 26(4):1049–1070, 2013.

[CWY14] H. Chen, W. Wang, and M. Yu. Hausdorff dimension of the graph of the error-sum function of α-Lüroth series. *Turkish J. Math.*, 38(5):803–811, 2014.

[CW14] H. Chen and Z. Wen. The Hausdorff dimension of certain sets in a class of α-Lüroth expansions. *Sci. China Math.*, 57(2):303–313, 2014.

[CMO98] H. Cohen, A. Miyaji, and T. Ono. Efficient elliptic curve exponentiation using mixed coordinates. pages 51–65. Springer Berlin Heidelberg, 1998.

[CM18] D. Cosper and M. Misiurewicz. Entropy locking. *Fund. Math.*, 241(1):83–96, 2018.

[DHK09] K. Dajani, Y. Hartono, and C. Kraaikamp. Mixing properties of (α, β)-expansions. *Ergodic Theory Dynam. Systems*, 29(4):1119–1140, 2009.

[DHKM12] K. Dajani, D. Hensley, C. Kraaikamp, and V. Masarotto. Arithmetic and ergodic properties of 'flipped' continued fraction algorithms. *Acta Arith.*, 153(1):51–79, 2012.

[DK07] K. Dajani and C. Kalle. Random β-expansions with deleted digits. *Discrete Contin. Dyn. Syst.*, 18(1):199–217, 2007.

[DK10] K. Dajani and C. Kalle. A note on the greedy β-transformation with arbitrary digits. In *École de Théorie Ergodique*, volume 20 of *Sémin. Congr.*, pages 83–104. Soc. Math. France, Paris, 2010.

[DK13] K. Dajani and C. Kalle. Local dimensions for the random β-transformation. *New York J. Math.*, 19:285–303, 2013.

[DK17] K. Dajani and C. Kalle. Invariant measures, matching and the frequency of 0 for signed binary expansions. arXiv: 1703.06335, 2017.

[DKM20] K. Dajani, C. Kalle, and M. Maggioni. Matching for random systems with an application to minimal weight expansions. 2020. Submitted to *Nonlinearity*.

[DK96] K. Dajani and C. Kraaikamp. On approximation by Lüroth series. *J. Théor. Nombres Bordeaux*, 8(2):331–346, 1996.

[DK00] K. Dajani and C. Kraaikamp. The mother of all continued fractions. *Colloq. Math.*, 84/85(part 1):109–123, 2000. Dedicated to the memory of Anzelm Iwanik.

[DK02] K. Dajani and C. Kraaikamp. *Ergodic Theory of Numbers*. Number v. 29 in Carus Mathematical Monographs. Mathematical Association of America, 2002.

[DK03] K. Dajani and C. Kraaikamp. Random β-expansions. *Ergodic Theory Dynam. Systems*, 23(2):461–479, 2003.

[DKS09] K. Dajani, C. Kraaikamp, and W. Steiner. Metrical theory for α-Rosen fractions. *J. Eur. Math. Soc. (JEMS)*, 11(6):1259–1283, 2009.

[DKL06] K. Dajani, C. Kraaikamp, and P. Liardet. Ergodic properties of signed binary expansions. *Discrete Contin. Dyn. Syst.*, 15(1):87–119, 2006.

[DO18] K. Dajani and M. Oomen. Random N-continued fraction expansions. *J. Approx. Theory*, 227:1–26, 2018.

[DdV05] K. Dajani and M. de Vries. Measures of maximal entropy for random β-expansions. *J. Eur. Math. Soc.*, 7(1):51–68, 2005.

[DdV07] K. Dajani and M. de Vries. Invariant densities for random β-expansions. *J. Eur. Math. Soc.*, 9(1):157–176, 2007.

[EJK90] P. Erdös, I. Joó, and V. Komornik. Characterization of the unique expansions $1 = \sum_{i=1}^{\infty} q^{-n_i}$ and related problems. *Bull. Soc. Math. France*, 118(3):377–390, 1990.

[EM12] P. Eslami and M. Misiurewicz. Singular limits of absolutely continuous invariant measures for families of transitive maps. *J. Difference Equ. Appl.*, 18(4):739–750, 2012.

[FLMW10] A. Fan, L. Liao, J. Ma, and B. Wang. Dimension of Besicovitch-Eggleston sets in countable symbolic space. *Nonlinearity*, 23(5):1185–1197, 2010.

[FB81] N. Friedman and A. Boyarsky. Matrices and eigenfunctions induced by Markov maps. *Linear Algebra Appl.*, 38:141–147, 1981.

[G59] A. O. Gel′fond. A common property of number systems. *Izv. Akad. Nauk SSSR. Ser. Mat.*, 23:809–814, 1959.

[G16] R. Giuliano. Convergence results for r-iterated means of the denominators of the Lüroth series. *Unif. Distrib. Theory*, 11(2):179–203, 2016.

[GB03] P. Góra and A. Boyarsky. Absolutely continuous invariant measures for random maps with position dependent probabilities. *J. Math. Anal. Appl.*, 278(1):225–242, 2003.

[GBL97] P. Góra, A. Boyarsky, and Y. S. Lou. Lyapunov exponents for higher-dimensional random maps. *J. Appl. Math. Stochastic Anal.*, 10(3):209–218, 1997.

[G84] P. Góra. On small stochastic perturbations of mappings of the unit interval. *Colloquium Mathematicae*, 49(1):73–85, 1984.

[G09] P. Góra. Invariant densities for piecewise linear maps of the unit interval. *Ergodic Theory Dynam. Systems*, 29(5):1549–1583, 2009.

[GBI06] P. Góra, A. Boyarsky, and S. Islam. Invariant densities of random maps have lower bounds on their supports. *J. Appl. Math. Stoch. Anal.*, pages Art. ID 79175, 13, 2006.

[GH06] P. J. Grabner and C. Heuberger. On the number of optimal base 2 representations of integers. *Des. Codes Cryptogr.*, 40(1):25–39, 2006.

[GL16] Y. Gui and W. Li. Hausdorff dimensions of sets related to Lüroth expansion. *Acta Math. Hungar.*, 150(2):286–302, 2016.

[H02] A. Haas. Invariant measures and natural extensions. *Canad. Math. Bull.*, 45(1):97–108, 2002.

[HK02] Y. Hartono and C. Kraaikamp. On continued fractions with odd partial quotients. *Rev. Roumaine Math. Pures Appl.*, 47(1):43–62 (2003), 2002.

[HP06] C. Heuberger and H. Prodinger. Analysis of alternative digit sets for non-adjacent representations. *Monatsh. Math.*, 147:219–248, 2006.

[I12] T. Inoue. Invariant measures for position dependent random maps with continuous random parameters. *Studia Math.*, 208(1):11–29, 2012.

[I20] T. Inoue. First return maps of random maps and invariant measures. *Nonlinearity*, 33(1):249–275, 2020.

[K14] C. Kalle. Isomorphisms between positive and negative β-transformations. *Ergodic Theory Dynam. Systems*, 34(1):153–170, 2014.

[KKV17] C. Kalle, T. Kempton, and E. Verbitskiy. The random continued fraction transformation. *Nonlinearity*, 30(3):1182–1203, 2017.

[KLMM20] C. Kalle, N. Langeveld, M. Maggioni, and S. Munday. Matching for a family of infinite measure continued fraction transformations. 2020.

[KM18] C. Kalle and M. Maggioni. Invariant densities for random systems of the interval. 2018. To appear in *Ergodic Theory and Dynamical Systems*.

[KM21] C. Kalle and M. Maggioni. On approximation by random Lüroth expansions. 2021. Submitted to *International Journal of Number Theory*.

[KS12] C. Kalle and W. Steiner. β-expansions, natural extensions and multiple tilings associated with Pisot units. *Trans. Amer. Math. Soc.*, 364(5):2281–2318, 2012.

[KKK90] S. Kalpazidou, A. Knopfmacher, and J. Knopfmacher. Lüroth-type alternating series representations for real numbers. *Acta Arith.*, 55(4):311–322, 1990.

[KKK91] S. Kalpazidou, A. Knopfmacher, and J. Knopfmacher. Metric properties of alternating Lüroth series. *Portugal. Math.*, 48(3):319–325, 1991.

[KU10] S. Katok and I. Ugarcovici. Structure of attractors for (a, b)-continued fraction transformations. *J. Mod. Dyn.*, 4(4):637–691, 2010.

[K82] G. Keller. Stochastic stability in some chaotic dynamical systems. *Monatsh. Math.*, 94(4):313–333, 1982.

[K14] K. Kempton. On the invariant density of the random β-transformation. *Acta Math. Hungar.*, 142(2):403–419, 2014.

[KMS11] M. Kesseböhmer, S. Munday, and B. Stratmann. Strong renewal theorems and lyapunov spectra for α-farey and α-lüroth systems. *Ergodic Theory and Dynamical Systems*, 32(3):989–1017, 2011.

[KMS16] M. Kesseböhmer, S. Munday, and B. Stratmann. *Infinite ergodic theory of numbers*. De Gruyter Graduate. De Gruyter, Berlin, 2016.

[K86] Y. Kifer. *Ergodic Theory of Random Transformations*, volume 10 of *Progress in Probability*. Birkhäuser Basel, 1986.

[K69] D. E. Knuth. *The Art of Computer Programming*. Addison-Wesley Professional, 1st edition, 1969.

[KY07] E. Kobre and L. S. Young. Extended systems with deterministic local dynamics and random jumps. *Communications in Mathematical Physics*, 275(3):709–720, 11 2007.

[K90] C. Kopf. Invariant measures for piecewise linear transformations of the interval. *Appl. Math. Comput.*, 39(2, part II):123–144, 1990.

[KT93] K. Koyama and Y. Tsuruoka. Speeding up elliptic cryptosystems by using a signed binary window method. pages 345–357. Springer Berlin Heidelberg, 1993.

[K91] C. Kraaikamp. A new class of continued fraction expansions. *Acta Arith.*, 57(1):1–39, 1991.

[KSS10] C. Kraaikamp, T. Schmidt, and I. Smeets. Natural extensions for α-rosen continued fractions. *J. Math. Soc. Japan*, 62(2):649–671, 2010.

[KSS12] C. Kraaikamp, T. Schmidt, and W. Steiner. Natural extensions and entropy of α-continued fractions. *Nonlinearity*, 25(8):2207–2243, 2012.

[K67] U. Krengel. Entropy of conservative transformations. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 7:161–181, 1967.

[LK97] C. Laih and W. Kuo. Speeding up the computations of elliptic curves cryptoschemes. *Computers and Mathematics with Applications*, 33(5):29–36, 1997.

[LM94] A. Lasota and M. C. Mackey. *Chaos, fractals, and noise*, volume 97 of *Applied Mathematical Sciences*. Springer-Verlag, New York, second edition, 1994. Stochastic aspects of dynamics.

[LY73] A. Lasota and J.A. Yorke. On the existence of invariant measures for piecewise monotonic transformations. *Trans. Amer. Math. Soc.*, 186:481–488 (1974), 1973.

[L81] L. Lewin. *Polylogarithms and associated functions*. North-Holland Publishing Co., New York-Amsterdam, 1981.

[LCTW18] L. Li, F. Cao, S. Tang, and Y. Wu. Lüroth expansion digits and Maclaurin's inequality. *Wuhan Univ. J. Nat. Sci.*, 23(6):471–474, 2018.

[LGB+13] Z. Li, P. Góra, A. Boyarsky, H. Proppe, and P. Eslami. Family of piecewise expanding maps having singular measure as a limit of ACIMs. *Ergodic Theory Dynam. Systems*, 33(1):158–167, 2013.

[LY78] T. Y. Li and J. A. Yorke. Ergodic transformations from an interval into itself. *Trans. Amer. Math. Soc.*, 235:183–192, 1978.

[LM95] D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding.* Cambridge University Press, 1995.

[L83] J. Lüroth. Ueber eine eindeutige Entwickelung von Zahlen in eine unendliche Reihe. *Math. Ann.*, 21(3):411–423, 1883.

[LM08] L. Luzzi and S. Marmi. On the entropy of Japanese continued fractions. *Discrete Contin. Dyn. Syst.*, 20(3):673–711, 2008.

[MT13] B. Mance and J. Tseng. Bounded Lüroth expansions: applying Schmidt games where infinite distortion exists. *Acta Arith.*, 158(1):33–47, 2013.

[MMY97] S. Marmi, P. Moussa, and J. C. Yoccoz. The Brjuno functions and their regularity properties. *Comm. Math. Phys.*, 186(2):265–293, 1997.

[MO90] F. Morain and J. Olivos. Speeding up the computations on an elliptic curve using addition-subtraction chains. *RAIRO - Theoretical Informatics and Applications - Informatique Théorique et Applications*, 24(6):531–543, 1990.

[M85] T. Morita. Random iteration of one-dimensional transformations. *Osaka J. Math.*, 22(3):489–518, 1985.

[M11] S. Munday. A note on Diophantine fractals for α-Lüroth systems. *Integers*, 11B:Paper No. A10, 14, 2011.

[NIT77] H. Nakada, S. Ito, and S. Tanaka. On the invariant measure for the transformations associated with some real continued-fractions. *Keio Engineering Reports*, 30(13):159–175, 1977.

[N81] H. Nakada. Metrical theory for a class of continued fraction transformations and their natural extensions. *Tokyo J. Math.*, 4(2):399–426, 1981.

[NN08] H. Nakada and R. Natsui. The non-monotonicity of the entropy of α-continued fraction transformations. *Nonlinearity*, 21(6):1207–1225, 2008.

[N11] R. Natsui. On the Legendre constant of α-continued fractions. *J. Number Theory*, 131(3):487–507, 2011.

[P60] W. Parry. On the β-expansions of real numbers. *Acta Math. Acad. Sci. Hungar.*, 11:401–416, 1960.

[P84] S. Pelikan. Invariant densities for random maps of the interval. *Trans. Amer. Math. Soc.*, 281(2):813–825, 1984.

[P57] O. Perron. *Die Lehre von den Kettenbrüchen. Dritte, verbesserte und erweiterte Aufl. Bd. II. Analytisch-funktionentheoretische Kettenbrüche.* B. G. Teubner Verlagsgesellschaft, Stuttgart, 1957.

[R60] G. W. Reitwiesner. Binary arithmetic. volume 1 of *Advances in Computers*, pages 231–308. Elsevier, 1960.

[R57] A. Rényi. Representations for real numbers and their ergodic properties. *Acta Math. Acad. Sci. Hungar*, 8:477–493, 1957.

[R61] V. A. Rohlin. Exact endomorphisms of a Lebesgue space. *Izv. Akad. Nauk SSSR Ser. Mat.*, 25:499–530, 1961.

[S68] T. Salát. Zur metrischen Theorie der Lürothschen Entwicklungen der reellen Zahlen. *Czechoslovak Math. J.*, 18 (93):489–522, 1968.

[S04] B. Schratzberger. S-expansions in dimension two. *J. Théor. Nombres Bordeaux*, 16(3):705–732, 2004.

[SF11] L. Shen and K. Fang. The fractional dimensional theory in Lüroth expansion. *Czechoslovak Math. J.*, 61(136)(3):795–807, 2011.

[S17] L. Shen. Hausdorff dimension of the set concerning with Borel-Bernstein theory in Lüroth expansions. *J. Korean Math. Soc.*, 54(4):1301–1316, 2017.

[SYZ14] L. Shen, Y. Yu, and Y. Zhou. A note on the largest digits in Lüroth expansion. *Int. J. Number Theory*, 10(4):1015–1023, 2014.

[S03] N. Sidorov. Almost every number has a continuum of β-expansions. *Amer. Math. Monthly*, 110(9):838–842, 2003.

[S88] C. E. Silva. On μ-recurrent nonsingular endomorphisms. *Israel J. Math.*, 61(1):1–13, 1988.

[ST91] C. E. Silva and P. Thieullen. The subadditive ergodic theorem and recurrence properties of Markovian transformations. *J. Math. Anal. Appl.*, 154(1):83–99, 1991.

[SFM17] K. Song, L. Fang, and J. Ma. Level sets of partial maximal digits for Lüroth expansion. *Int. J. Number Theory*, 13(10):2777–2790, 2017.

[SX18] Y. Sun and J. Xu. On the maximal run-length function in the Lüroth expansion. *Czechoslovak Math. J.*, 68(143)(1):277–291, 2018.

[S19] S. Suzuki. Invariant density functions of random β-transformations. *Ergodic Theory Dynam. Systems*, 39(4):1099–1120, 2019.

[TI81] S. Tanaka and S. Ito. On a family of continued-fraction transformations and their ergodic properties. *Tokyo J. Math.*, 4(1):153–175, 1981.

[TW18] X. Tan and X. Wang. On the fast growth rate of the sum of digits of Lüroth expansion. *Math. Appl. (Wuhan)*, 31(2):300–304, 2018.

[T83] M. Thaler. Transformations on $[0, 1]$ with infinite invariant measures. *Israel J. Math.*, 46(1-2):67–96, 1983.

[T14] G. Tiozzo. The entropy of Nakada's α-continued fractions: analytical results. *Annali della Scuola Normale Superiore di Pisa*, 13(4):1009–1037, 2014.

[V14] J. Vandehey. A simpler normal number construction for simple Lüroth series. *J. Integer Seq.*, 17(6):Article 14.6.1, 18, 2014.

[ZC16] Z. Zhang and C. Cao. On points containing arithmetic progressions in their Lüroth expansion. *Acta Math. Sci. Ser. B (Engl. Ed.)*, 36(1):257–264, 2016.

[Z98] R. Zweimüller. Ergodic structure and invariant densities of non-Markovian interval maps with indifferent fixed points. *Nonlinearity*, 11(5):1263–1276, 1998.

[Z00] R. Zweimüller. Ergodic properties of infinite measure-preserving interval maps with indifferent fixed points. *Ergodic Theory Dynam. Systems*, 20(5):1519–1549, 2000.

[Z09] R. Zweimüller. Surrey notes on infinite ergodic theory. 2009.

Samenvatting

Getalsontwikkelingen zijn een manier om getallen te representeren met specifieke symbolen en consistente regels. Klassieke voorbeelden zijn β-expansies, binaire representaties en kettingbreuken. De algoritmes die de symbolen combineren om getallen te coderen worden verkregen door het herhaald itereren van bepaalde intervalafbeeldingen die getalsystemen worden genoemd. Het voordeel van deze dynamische beschrijving is dat al het gereedschap uit de ergodische theorie beschikbaar is. Dit betreft in het bijzonder het begrip van invariante maten μ die absoluut continu zijn met betrekking tot de Lebesgue maat λ. Om precies te zijn, een maat μ is T-invariant voor een intervalafbeelding T en absoluut continu met betrekking tot λ als

$$\mu(B) = \int_B f \mathrm{d}\lambda = \mu(T^{-1}(B))$$

geldt voor iedere Borel deelverzameling B van $\mathbb{R}$, waarbij f een integreerbare functie is die de dichtheid wordt genoemd. Voor getalsystemen zoals bijvoorbeeld β-transformaties, de standaard Lüroth afbeelding en de Gauss afbeelding hebben expliciete uitdrukkingen voor invariante maten laten zien dat ze nuttig zijn voor het bepalen van eigenschappen van getalsontwikkelingen zoals symboolfrequentie, entropie en de kwaliteit van de benaderingen. Maar hoewel er verschillende resultaten zijn over het bestaan van zulke maten, is het vinden van een expliciete formule nog steeds een ingewikkeld probleem.

In dit proefschrift worden expliciete uitdrukkingen gegeven voor de dichtheidsfuncties van absoluut continue invariante maten voor algemene families van intervalafbeeldingen, waaronder stochastische afbeeldingen en transformaties met een oneindige maat, die niet noodzakelijk getalsystemen zijn. In de stochastische setting is niet één maar een verzameling afbeeldingen $\{T_j : I \to I\}_{j\in\Omega}$ beschikbaar en wordt op ieder tijdstip één van deze afbeeldingen toegepast volgens een bepaalde kansverdeling die wordt gespecificeerd door een positieve kansvector $\mathbf{p} = (p_j)_{j\in\Omega}$. Zo'n stochastische afbeelding neemt onder bepaalde voorwaarden een stationaire maat $\mu_\mathbf{p}$ met dichtheid $f_\mathbf{p}$ aan, dat wil zeggen dat

$$\mu_\mathbf{p}(B) = \int_B f_\mathbf{p}\, d\lambda = \sum_{j\in\Omega} p_j \mu_\mathbf{p}(T_j^{-1}B)$$

geldt voor iedere Borel deelverzameling B van $\mathbb{R}$. In de oneindige configuratie is de maat op de toestandsruimte oneindig en is het gereedschap uit de kansrekenening niet meer beschikbaar. In dit geval vinden we expliciete uitdrukkingen voor de dichtheidsfuncties met andere technieken waarbij de Perron-Frobenius operator, natuurlijke uitbreidingen en het dynamische verschijnsel synchronisatie worden gebruikt. Synchronisatie geldt voor een stuksgewijs gladde intervalafbeelding T als voor elk kritieke punt c de banen van de linker- en rechterlimiet van c elkaar uiteindelijk tegenkomen, dat wil zeggen als

$$\lim_{x \uparrow c} T^{M_c}(x) = \lim_{x \downarrow c} T^{N_c}(x)$$

geldt voor zekere niet-negatieve gehele getallen M_c en N_c. Vervolgens passen we de resultaten toe op de dichtheidsfuncties om eigenschappen van nieuwe getalsontwikkelingen te bestuderen.

Dit proefschrift is als volgt opgebouwd. Hoofdstuk 2 introduceert een familie van stochastische intervalafbeeldingen die gegeven worden door combinaties van gegeneraliseerde Lüroth afbeeldingen gedefinieerd op de intervallen $I_c = [c, 1]$ met $c \geq 0$. Dit stochastische systeem genereert voor bijna alle $x \in I_c$ overaftelbaar veel verschillende getalsontwikkelingen die gelijktijdig bestudeerd kunnen worden. Bovendien kan van elke irrationale $x \in I_c$ iedere mogelijke getalsontwikkeling worden afgekapt om een rationale benadering van het punt te krijgen. De kwaliteit van deze benaderingen wordt bestudeerd met de Lyapunov-exponent en de benaderingscoëfficiënten. Dit hoofdstuk suggereert dat een algoritmische procedure nodig is om voor stochastische intervalsystemen die stuksgewijs affien zijn de dichtheid van een absoluut continue invariante maat te construeren om meer informatie te krijgen over de bijbehorende getalsontwikkelingen.

Hoofdstuk 3 maakt de noodzaak van het vorige hoofdstuk duidelijk. Het bespreekt een algebraïsch algoritme met als invoer een stochastisch stuksgewijs affien systeem T dat gemiddeld genomen expandeert, en met als uitvoer een formule voor de stationaire maat. Deze uitgebreide procedure blijkt efficiënt te zijn voor specifieke klassen van transformaties, zoals de stochastische β-transformaties, Lüroth afbeeldingen en meer in het algemeen voor systemen waarin de dynamische kenmerken van synchronisatie opduiken. Dit wordt verder geanalyseerd in Hoofdstuk 5.

Hoofdstuk 4 onderzoekt de gevolgen van synchronisatie voor een oneindige klasse $\{T_\alpha\}_\alpha$ van kettingbreukafbeeldingen. Meer specifiek wordt het verschijnsel synchronisatie gebruikt om het tweedimensionale domein van de natuurlijke uitbreiding te vinden. Bovendien is voor het eerst op het gebied van transformaties met een oneindige maat bewezen dat synchronisatie geldt voor Lebesgue-bijna elke parameter α, en het verdeelt de parameterruimte in intervallen met constante synchronisatie-exponenten die synchronisatie-intervallen worden genoemd. Tenslotte relateert dit hoofdstuk deze synchronisatie-intervallen aan de corresponderende verzamelingen van Nakada's α-kettingbreukafbeeldingen.

Hoofdstuk 5 breidt het begrip synchronisatie, tot dusver alleen bekend voor determistische transformaties, uit tot het zogeheten begrip stochastische synchronisatie voor stochastische intervalafbeeldingen. Stochastische synchronisatie wordt vervolgens bestudeerd voor een verscheidenheid aan families van stochastische dynamische systemen, waaronder de gegeneraliseerde β-transformaties en kettingbreukafbeeldingen. Bovendien impliceert stochastische synchronisatie voor een grote klasse van stuksgewijs affiene stochastische intervalsystemen dat iedere dichtheidsfunctie van een stationaire maat stuksgewijs constant is. Tenslotte introduceert dit hoofdstuk een familie van stochastische afbeeldingen die ontwikkelingen in basis 2 genereren waarbij de symbolen worden gekozen uit de verzameling $\{-1, 0, 1\}$. De eigenschap van stochastische synchronisatie en de gevolgen daarvan voor de structuur van de dichtheidsfunctie worden vervolgens toegepast om de frequentie van het symbool 0 in zulke ontwikkelingen te bestuderen.

Acknowledgements

This thesis collects the scientific results of the last four years. It is the conclusion of an amazing journey that shaped me into the scientist and the person I am today. Many people have contributed to this, and I would love to thank all of them for their immense support.

Firstly, many sincere thanks go to my supervisor Charlene for always being inspiring and supportive, and for sharing her passion and her enthusiasm for Maths with me. Doing research with her has been exciting and illuminating: I could have not asked for a better guidance. I am also grateful to my promoter Frank for always having the right words and a kind approach in every situation. My gratitude extends to NWO for funding this project.

Secondly, I want to thank all the people from the MI. Their presence makes Leiden University a warm and welcoming place, where doors are always open to everyone. Particularly, to all my co-authors, to all the people that worked in office 203b, to the members of the reading club, to the ones of the iPOD seminars, to the organizers of the NETWORKS goes to school events, to the ones of the PRE-university college, to the companions of coffee breaks and lunch walks: I am very grateful to have shared the PhD journey with you.

Thirdly, huge thanks go to all the people I can call Friends. Thank you for all the laughs, the climbing sessions, the traveling around the world, the complaints about the rain, the countless meals we cooked together, for surviving together through the 2020 quarantine and through 2020 itself. Amongst them, enormous thanks go to my incredible paranymphs.

Fourthly, I am deeply thankful to Pablo for all the life moments we have shared: You are home for me.

Last but not least, I am extremely lucky to have an awesome family, who has been and will always be with me, and for me: Thanks to mum, dad, and my brother.

Curriculum Vitae

Marta Maggioni was born in Italy in 1992. She graduated *cum laude* in 2011 with a scientific degree from the high school M.G. Agnesi. Afterwards she studied Mathematics at the University of Milano Bicocca, where she obtained a Bachelor of Science *cum laude* in Mathematics in 2014. She was part of the Algant Erasmus Mundus fellowship, studying at the University of Leiden, The Netherlands, and the University of Regensburg, Germany. She obtained *cum laude* a double degree as a Master of Science in Mathematics in 2016 in Bordeaux, with a thesis on *Nearly holomorphic modular forms*, supervised by Prof. Dr. G. Kings. In September 2016 she moved back to Leiden to start her PhD research project under the supervision of Dr. C. Kalle. She served as a teaching assistant in several courses, mainly in the area of Algebra, Probability and Statistics. In 2019 she supervised students of the Leiden PRE-University programme, who won the Jan Kijne Onderzoeksprijs. She presented her research at multiple scientific conferences and was invited to local seminars around Europe. She organised the first event *Networks goes to school* in 2018 with Dr. N. Starreveld, by an initiative of B. Groeneveld. The masterclass is now an annual meeting for secondary students and teachers across The Netherlands. She is a member of SAMI - Supporting African Maths Initiative, and she volunteered in various Maths camps in Kenya and Tanzania over the years.

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