Global fields and their L-functions

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Curriculum Vitae

Pavel Solomatin was born on May 14, 1991 in Moscow, Russian Federation. Started from 2004 he attended experimental division of the mathematical high school No. 179, with the curriculum focusing on both computer science and mathematics.

Upon graduating in 2008 he went on to obtain a bachelor diploma in pure mathematics from the National Research University Higher School of Economics (HSE). There he fell in love with number theory and defended his bachelor thesis titled "*On Arithmetic Properties of Abelian Varieties over Finite Fields*" under scientific direction of professor Alexey Zykin.

In 2012 Pavel enrolled in the master program at HSE math department, graduating in 2014 with the thesis titled "*Curves with Many Points over Finite Fields: The Class Field Theory Approach*", which was also written under kind Alexey's guidance. During his master studies Pavel's research received support from the grant "*Arithmetic Properties of Abelian Varieties over Finite and Function Fields*" issued by the Russian Academy of Science. Directly after graduation in 2014 he received an ALGANT scholarship which allowed him to continue his mathematical studies as a Ph.D. candidate under scientific direction of professor Bart de Smit (Leiden University) and professor Karim Belabas (University of Bordeaux). Besides the scientific research Pavel was also devoting efforts to teaching various kinds of mathematical disciplines for graduate and undergraduate students. For this job he received the "*Best teacher 2013–2014*" award according to the choice of students from the HSE-NES joint bachelor program in economics. While staying in the Netherlands his teaching duties were related to assistance for multiple courses of the Dutch "Master in Mathematics" program, such as "Number Theory" and "Elliptic Curves".

Starting from February 2018 he is working full-time as a software engineer at a Dutch IT-company called Ortec. There he is applying his passion for mathematics in order to improve our world.

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