Global fields and their L-functions

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Chapter 6

On Abelianized Absolute Galois groups of Imaginary Quadratic Fields

6.1 Introduction

The main purpose of the present chapter is to use techniques from the previous chapter in order to extend results of the paper [1]. We would like to emphasise that results and proofs in this chapter are parallel to those of the previous chapter. In particular, we use similar notations here.

6.1.1 Results of the Chapter

Let K be an imaginary quadratic field different from $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$. Let $\mathcal{T} = \prod_{n \in \mathbb{N}} \mathbb{Z}/n\mathbb{Z}$ and let $\mathrm{Cl}(K)$ denote the ideal class group of K. Let $\mathcal{G}_K^{ab}$ denote the abelianized absolute Galois group of K. Summarising the results of [1] we have:

Theorem 6.1. *In the above setting the following holds:*

1. *There exists an exact sequence of topological groups:* $0 \to \widehat{\mathbb{Z}}^2 \times \mathcal{T} \to \mathcal{G}_K^{ab} \to \mathrm{Cl}(K) \to 0$;

2. *The topological closure $\overline{\mathcal{G}_K^{ab}[\mathrm{tors}]}$ of the torsion subgroup of $\mathcal{G}_K^{ab}$ is $\mathcal{T}$;*

3. *The torsion subgroup of the quotient $\mathcal{G}_K^{ab}/\mathcal{T}$ is trivial if and only if $\mathcal{G}_K^{ab} \simeq \widehat{\mathbb{Z}}^2 \times \mathcal{T}$;*

4. *There exist an injective map from $(\mathcal{G}_K^{ab}/\mathcal{T})[\mathrm{tors}]$ to $\mathrm{Cl}(K)$ and an algorithm, which on input K decides whether the group $(\mathcal{G}_K^{ab}/\mathcal{T})[\mathrm{tors}]$ is trivial or not.*

Proof. See theorem 3.5, 4.4 and 5.1 from [1]. □

Let us denote the image of $(\mathcal{G}_K^{ab}/\mathcal{T})[\mathrm{tors}]$ in $\mathrm{Cl}(K)$ by $\mathrm{Cl}^{split}(K)$. Roughly speaking our main result states that the isomorphism type of $\mathcal{G}_K^{ab}$ is uniquely determined by the isomorphism type of $\mathrm{Cl}^{split}(K)$. More concretely, first we will prove:

Theorem 6.2. *Given the group $\mathcal{T}$ and a finite abelian group A there exists a unique isomorphism type of a pro-finite abelian group $\mathcal{D}_A$ such that the following holds:*

1. *There exists an exact sequence:* $0 \to \mathcal{T} \to \mathcal{D}_A \to A \to 0$;

2. *All torsion elements of $\mathcal{D}_A$ are in $\mathcal{T}$.*

Proof. See section 6.2. $\qquad\square$

Then the main result of the present chapter could be stated as:

Theorem 6.3. *Let K be an imaginary quadratic field different from $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$. There exists an isomorphism of topological groups $\mathcal{G}_K^{ab} \simeq \mathcal{D}_A \times \widehat{\mathbb{Z}}^2$, with $A \simeq \mathrm{Cl}^{split}(K)$.*

Proof. See section 6.2. $\qquad\square$

The above theorem extends results of Theorem 6.1 as follows:

Corollary 6.4. *For a fixed prime number p and an imaginary quadratic field K with class number $h_K = p$ there are only two isomorphism types of $\mathcal{G}_K^{ab}$ which could occur: either $\mathrm{Cl}^{split}(K) = 0$ or $\mathrm{Cl}^{split}(K) \simeq \mathbb{Z}/p\mathbb{Z}$. In particular, it was shown in [1] that imaginary quadratic fields with the discriminant D_K occurring in the list $\{-35, -51, -91, -115, -123, -187, -235, -267, -403, -427\}$ all have class-number 2 and have non-trivial $\mathrm{Cl}^{split}(K)$, therefore they all share the same isomorphism class of $\mathcal{G}_K^{ab}$.*

Also we will use Theorem 6.3 in order to prove:

Corollary 6.5. *There are infinitely many isomorphism types of pro-finite groups which occur as $\mathcal{G}_K^{ab}$ for some imaginary quadratic fields.*

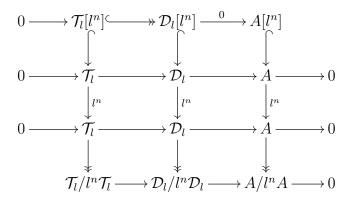*Proof.* See section 6.3. $\qquad\square$

6.2 The Proof of the Theorem

Our goal in this section is to prove Theorem 6.3. We will do this in three steps. First we will prove the group-theoretical Theorem 6.2. Secondly, in lemma 6.8 we will show that given an imaginary quadratic field $K \neq \mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$ there exist a pro-finite group $\mathcal{D}_K$ and an isomorphism $\mathcal{G}_K^{ab} \simeq \mathcal{D}_K \times \widehat{\mathbb{Z}}^2$. Finally, in lemma 6.9 we will show that the group $\mathcal{D}_K$ satisfies conditions of Theorem 6.2 with $A \simeq \mathrm{Cl}^{split}(K)$. Therefore the isomorphism class of $\mathcal{D}_K$ is uniquely determined by the isomorphism class of the abelian group $\mathrm{Cl}^{split}(K)$ and hence we obtain a proof of Theorem 6.3.
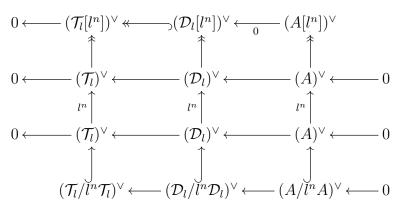
   Remark: Since each pro-finite abelian group is isomorphic to the limit of finite abelian groups, by the Chinese remainder theorem we have that it is also isomorphic to the product over prime numbers of its primary components. We will work with these components separately instead of working with the whole group.

Proof of Theorem 6.2

As in the previous sections for a pro-finite abelian group G and a prime number l we denote by G_l the l-primary component $G \otimes \mathbb{Z}_l$ of G. In the setting of Theorem 6.2 the multiplication by l^n map induces the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{T}_l[l^n] & \lhook\joinrel\twoheadrightarrow & \mathcal{D}_l[l^n] & \xrightarrow{\ 0\ } & A[l^n] & & \\
& & \big\uparrow & & \big\uparrow & & \big\uparrow & & \\
0 & \longrightarrow & \mathcal{T}_l & \longrightarrow & \mathcal{D}_l & \longrightarrow & A & \longrightarrow & 0 \\
& & \big\downarrow{\scriptstyle l^n} & & \big\downarrow{\scriptstyle l^n} & & \big\downarrow{\scriptstyle l^n} & & \\
0 & \longrightarrow & \mathcal{T}_l & \longrightarrow & \mathcal{D}_l & \longrightarrow & A & \longrightarrow & 0 \\
& & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
& & \mathcal{T}_l/l^n\mathcal{T}_l & \longrightarrow & \mathcal{D}_l/l^n\mathcal{D}_l & \longrightarrow & A/l^nA & \longrightarrow & 0
\end{array}
$$

Since any torsion element x of $\mathcal{D}_l$ is in $\mathcal{T}_l$ the map from $\mathcal{D}_l[l^n]$ to $A[l^n]$ is the zero map and the map from $\mathcal{T}_l[l^n]$ to $\mathcal{D}_l[l^n]$ is an isomorphism. Now applying the Pontryagin duality to the above diagram we get:

$$
\begin{array}{ccccccccc}
0 & \longleftarrow & (\mathcal{T}_l[l^n])^\vee & \twoheadleftarrow\joinrel\rhook & (\mathcal{D}_l[l^n])^\vee & \xleftarrow{\ 0\ } & (A[l^n])^\vee & & \\
& & \big\uparrow & & \big\uparrow & & \big\uparrow & & \\
0 & \longleftarrow & (\mathcal{T}_l)^\vee & \longleftarrow & (\mathcal{D}_l)^\vee & \longleftarrow & (A)^\vee & \longleftarrow & 0 \\
& & \big\uparrow{\scriptstyle l^n} & & \big\uparrow{\scriptstyle l^n} & & \big\uparrow{\scriptstyle l^n} & & \\
0 & \longleftarrow & (\mathcal{T}_l)^\vee & \longleftarrow & (\mathcal{D}_l)^\vee & \longleftarrow & (A)^\vee & \longleftarrow & 0 \\
& & \big\uparrow & & \big\uparrow & & \big\uparrow & & \\
& & (\mathcal{T}_l/l^n\mathcal{T}_l)^\vee & \longleftarrow & (\mathcal{D}_l/l^n\mathcal{D}_l)^\vee & \longleftarrow & (A/l^nA)^\vee & \longleftarrow & 0
\end{array}
$$

Note that $(\mathcal{T}_l)^\vee$ is isomorphic to the direct sum of cyclic groups $(\mathcal{T}_l)^\vee \simeq \oplus_{k\in\mathbb{N}}\mathbb{Z}/l^k\mathbb{Z}$ and therefore $\cap_n l^n(\mathcal{T}_l)^\vee = \{0\}$. It means we have $(\cap_n l^n(\mathcal{D}_l)^\vee) \subset (A)^\vee$. Our goal is to show that $(\cap_n l^n(\mathcal{D}_l)^\vee) = (A)^\vee$.

Lemma 6.6. *Given any non-zero element x of $(A)^\vee \subset (\mathcal{D}_l)^\vee$ and any natural number n there exists an element $c_x \in (\mathcal{D}_l)^\vee$ such that $l^n c_x = x$.*

Proof. For fixed n consider the above diagram. Since the second row is exact the image of x in $(\mathcal{T}_l)^\vee$ is zero. Then its image in $(\mathcal{T}_l[l^n])^\vee$ is also zero. Since $(\mathcal{T}_l[l^n])^\vee \simeq (\mathcal{D}_l[l^n])^\vee$ it means that image of the non-zero element x in $(\mathcal{D}_l[l^n])^\vee$ is zero. Since the second column is exact this means that x lies in the image of the multiplication by l^n map from $(\mathcal{D}_l)^\vee$ to $(\mathcal{D}_l)^\vee$ and therefore there exists c_x such that $l^n c_x = x$. $\qquad\square$

It means that we have proved:

Corollary 6.7. *The exact sequence $0 \leftarrow (\mathcal{T}_l)^\vee \leftarrow (\mathcal{D}_l)^\vee \leftarrow (A)^\vee \leftarrow 0$ satisfies conditions of Theorem 5.14.*

and therefore $\mathcal{D}_l$ is uniquely determined since its Pontryagin dual $(\mathcal{D}_l)^\vee$ is uniquely determined
by Theorem 5.14.

6.2.1   Proof of Theorem 6.3

Consider the exact sequence mentioned in Theorem 6.1:

$$0 \to \widehat{\mathbb{Z}}^2 \times \mathcal{T} \to \mathcal{G}_K^{ab} \to \mathrm{Cl}(K) \to 0. \tag{6.1}$$

Taking a prime number l we get the following exact sequence of pro-l abelian groups:

$$0 \to \mathbb{Z}_l{}^2 \times \mathcal{T}_l \to \mathcal{G}_{K,l}^{ab} \to \mathrm{Cl}_l(K) \to 0, \tag{6.2}$$

where $\mathcal{T}_l = \prod_{k \in \mathbb{N}} \mathbb{Z}/l^k\mathbb{Z}$ and $\mathbb{Z}_l$ denotes the group of l-adic integers. If $\mathrm{Cl}_l(K)$ is the trivial
group then obviously $\mathcal{G}_{K,l}^{ab} \simeq \mathbb{Z}_l{}^2 \times \mathcal{T}_l$. Our goal is to describe the isomorphism type of $\mathcal{G}_{K,l}^{ab}$ in
the case when $\mathrm{Cl}_l(K)$ is not trivial.

Lemma 6.8. *There exists a pro-finite abelian group $\mathcal{D}_l$ such that $\mathcal{G}_{K,l}^{ab} \simeq \mathcal{D}_l \times \mathbb{Z}_l^2$.*

Proof. By Theorem 6.1 we know that $\mathcal{T}_l$ is the closure of the torsion subgroup of $\mathcal{G}_{K,l}^{ab}$. Note
that $\mathcal{T}_l$ is a closed subgroup and hence the quotient is also pro-l group. Taking the quotient of
the sequence 6.2 by $\mathcal{T}_l$ we obtain:

$$0 \to \mathbb{Z}_l{}^2 \to \mathcal{G}_{K,l}^{ab}/\mathcal{T}_l \to \mathrm{Cl}_l(K) \to 0.$$

Since $\mathbb{Z}_l$ is torsion free, $(\mathcal{G}_{K,l}^{ab}/\mathcal{T}_l)[\mathrm{tors}]$ maps invectively to $\mathrm{Cl}_l(K)$ which is finite. Denoting the
group $\mathcal{G}_{K,l}^{ab}/\mathcal{T}_l$ by B_l we get isomorphism of topological groups[1]: $B_l \simeq B_l[\mathrm{tors}] \oplus B_l'$, where B_l'
denotes the non-torsion part of B_l. Since $\mathbb{Z}_l$ is torsion free we also have the following exact
sequence:

$$0 \to \mathbb{Z}_l{}^2 \to B_l' \to \mathrm{Cl}_l(K)/\phi(B_l[\mathrm{tors}]) \to 0.$$
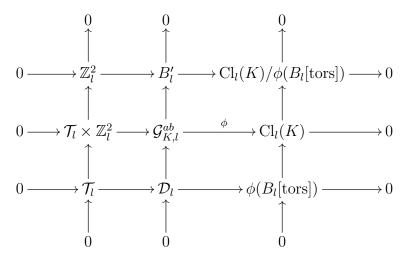
Since B_l' is torsion free this exact sequence implies that B_l' is a free $\mathbb{Z}_l$-module of rank two and
hence $B_l' \simeq \mathbb{Z}_l^2$.

Let us denote the quotient map $\mathcal{G}_{K,l}^{ab} \to \mathrm{Cl}_l(K)$ by ϕ. In notations from the introduction,
$\phi(B_l[\mathrm{tors}]) = \mathrm{Cl}^{split}(K)$. Consider the pre-image $\mathcal{D}_l \subset \mathcal{G}_{K,l}^{ab}$ of the group $\phi(B_l[\mathrm{tors}]) \subset \mathrm{Cl}_l(K)$.
Note that $\mathcal{D}_l$ is a closed subgroup and we have the following exact sequence:

$$0 \to \mathcal{T}_l \to \mathcal{D}_l \to \phi(B_l[\mathrm{tors}]) \to 0.$$

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[1]This is true because $B_l[\mathrm{tors}]$ is finite.

Summing up we have the following commutative diagram of pro-l abelian groups:

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbb{Z}_l^2 & \longrightarrow & B_l' & \longrightarrow & \mathrm{Cl}_l(K)/\phi(B_l[\mathrm{tors}]) & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{T}_l \times \mathbb{Z}_l^2 & \longrightarrow & \mathcal{G}_{K,l}^{ab} & \overset{\phi}{\longrightarrow} & \mathrm{Cl}_l(K) & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{T}_l & \longrightarrow & \mathcal{D}_l & \longrightarrow & \phi(B_l[\mathrm{tors}]) & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

Now consider the exact sequence coming from the middle column of the above diagram:

$$0 \to \mathcal{D}_l \to \mathcal{G}_{K,l}^{ab} \to B_l' \to 0.$$

We know that $B_l' \simeq \mathbb{Z}_l^2$, but $\mathbb{Z}_l$ is a projective module and hence we could split this sequence to obtain an isomorphism $\mathcal{G}_{K,l}^{ab} \simeq \mathcal{D}_l \times B_l' \simeq \mathcal{D}_l \times \mathbb{Z}_l^2$.

$\square$

In order to finish our proof we will show:

Lemma 6.9. *The group $\mathcal{D}_l$ is determined uniquely by the isomorphism type of $\phi(B_l[\mathrm{tors}]) = \mathrm{Cl}^{split}(K)$.*

Proof. Consider the exact sequence:

$$0 \to \mathcal{T}_l \to \mathcal{D}_l \to \mathrm{Cl}^{split}(K) \to 0.$$

We know that the closure of the torsion subgroup of $\mathcal{G}_{K,l}^{ab}$ is $\mathcal{T}_l$, and therefore $\mathcal{D}_l$ contains no torsion elements apart from elements of $\mathcal{T}_l$. I.e. that the group $\mathcal{D}_l$ satisfies both conditions of Theorem 6.2 and hence its isomorphism class is uniquely determined by $\mathrm{Cl}^{split}(K)$. $\square$

6.3 Corollaries

In this section we will prove corollary 6.5. First of all, we already showed that given two imaginary quadratic fields K, K' different from $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$ the following implication holds:

$$\mathrm{Cl}^{split}(K) \not\simeq \mathrm{Cl}^{split}(K') \Rightarrow \mathcal{G}_K^{ab} \not\simeq \mathcal{G}_{K'}^{ab}.$$

This statement allows us to reduce our question to construction of a sequence of imaginary quadratic fields K_i with $\# \mathrm{Cl}^{split}(K_i) \to \infty$ as $i \to \infty$. Given a finite abelian group A we denote by $r_l(A)$ the rank of its l-part i.e. the dimension of the vector space A_l over $\mathbb{Z}/l\mathbb{Z}$, where A_l is the l-primary component of A. Given an imaginary quadratic field K different from $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$ we have:

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Lemma 6.10. *The following inequalities hold:*

$$0 \leq r_l(\mathrm{Cl}(K)) - r_l(\mathrm{Cl}^{split}(K)) \leq 2.$$

Proof. Consider the exact sequence:

$$0 \to \mathrm{Cl}_l^{split}(K) \to \mathrm{Cl}_l(K) \to \mathrm{Cl}_l(K)/\mathrm{Cl}_l^{split}(K) \to 0.$$

Which implies:

$$r_l(\mathrm{Cl}(K)) = r_l(\mathrm{Cl}^{split}(K)) + r_l(\mathrm{Cl}_l(K)/\mathrm{Cl}_l^{split}(K)).$$

The first inequality is then obvious. For the second inequality, consider an exact sequence:

$$0 \to \mathbb{Z}_l^2 \to B_l' \to \mathrm{Cl}_l(K)/\mathrm{Cl}_l^{split}(K) \to 0,$$

which shows us $r_l(\mathrm{Cl}_l(K)/\mathrm{Cl}_l^{split}(K)) \leq 2$ and hence we are done. $\square$

Because of the previous lemma it is enough to show that we can construct a sequence K_i with $r_2(\mathrm{Cl}(K_i)) \to \infty$. This easily follows from the following statement which goes back to Gauss' genus theory; see [19]:

Lemma 6.11. *The two rank of the class group $\mathrm{Cl}(K)$ of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$ is $\omega(d) - 1$, where $\omega(d)$, denotes the number of different prime divisors of d.*

Finally summing up all together we obtain:

Corollary 6.12. *Let p_n denote the n-th prime number. Led $d_n = \prod_{i=1}^n p_i$. Among elements of the sequence $K_n = \mathbb{Q}(\sqrt{-d_n})$ of imaginary quadratic fields there are infinitely many fields with pairwise non-isomorphic abelianized absolute Galois groups $\mathcal{G}_{K_n}^{ab}$.*