Global fields and their L-functions

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Title: Global field and their L-functions
Issue Date: 2021-03-02

Chapter 2

Some Remarks With Regard to the Arithmetical Equivalence and Fields Sharing Same L-functions

2.1 Introduction

Having stated basic features of Artin L-functions we will show how to use them to elaborate studies of arithmetically equivalent number fields. There are three related topics we are going to consider in this chapter:

1. The first topic concerns the discussion about the property 4 of Artin L-functions which we called *multiplicative independence over* $\mathbb{Q}$ and analogues of Theorem 1.23 where the field $\mathbb{Q}$ is replaced by an arbitrary number field. Among others this problem was extensively studied by Nagata whose result we will describe in Theorem 2.1 and generalise later to the function field case, see Theorems 3.1 and 3.2 in the next chapter of the thesis.

2. The second part is related to the reconstruction of the isomorphism class of a given number field K by Artin L-functions of different representations attached to Galois extensions of K. The main result of that topic is Theorem 2.4 of Bart de Smit which states that for every number field K there exists an L-function which occurs only for that field. This result will also be generalised later in Theorem 3.3 to the function field side.

3. In the final section of this chapter we consider some extension of Theorem 2.4 as well as its applications towards a proof of the Uchida's part of the Neukirch-Uchida Theorem. The main results are Theorem 2.8 and Corollary 2.9.

2.2 Non-arithmetically equivalent extensions of Number Fields

2.2.1 Nagata's approach

As before let K and K' be two number fields. Let N denote their common Galois closure over $\mathbb{Q}$ and let $G = \mathrm{Gal}(N : \mathbb{Q})$, $H = \mathrm{Gal}(N : K)$, $H' = \mathrm{Gal}(N : K')$. Recall that in notations of Theorem 1.23 by the induction property we have:

$$\zeta_K(s) = L_{\mathbb{Q}}(\mathrm{Ind}_H^G(1_H), s),$$

and therefore by the multiplicative independence of Artin L-functions over $\mathbb{Q}$ we obtain:

$$\zeta_K(s) = \zeta_{K'}(s) \text{ if and only if } \mathrm{Ind}_H^G(1_H) \simeq \mathrm{Ind}_{H'}^G(1_{H'}).$$

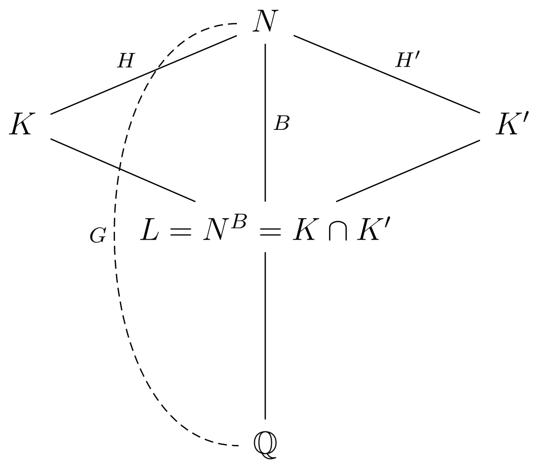
Phrasing this in a different way we regard the trivial representation 1 of the group G and restrict it to subgroups H and H'. Considering two L-functions $L_K(1|_H, s)$ and $L_{K'}(1|_{H'}, s)$ we have: K and K' are arithmetically equivalent if and only if these two L-functions match. There are at least two important questions a reader could ask here. The first one is: what if we pick another irreducible representation ρ of G and consider its restrictions to H and H' and compare the corresponding L-functions? And the second question is: what if we replace the base field $\mathbb{Q}$ with another number field such that multiplicative independence does not hold? how can we detect arithmetical equivalence over that field?

Surprisingly, by using elementary properties of representations of finite groups and properties of Artin L-functions we have discussed above, it is possible to show that the answer to both problems stated above is given by the following result due to K. Nagata, who published [32] in 1986:

Theorem 2.1 (Nagata). *Let K and K' be two finite extensions of a fixed number field L. Let N denote their common Galois closure over L and let $G = \mathrm{Gal}(N : L)$, $H = \mathrm{Gal}(N : K)$, $H' = \mathrm{Gal}(N : K')$. Then K and K' are arithmetically equivalent over L if and only if for every irreducible representation ρ of G we have: $L_K(\rho|_H, s) = L_{K'}(\rho|_{H'}, s)$.*

Proof. See Theorem 3.1 from the next chapter. □

Let us consider a particular instance which explains this lemma. Namely, we focus our attention on Example 1.15 from the introduction. There we picked a Gassmann triple (G, H, H') with G isomorphic to $\mathrm{Gl}_2(\mathbb{F}_p)$. It is easy to see that actually H and H' are subgroups of the proper Borel subgroup $B = \left\{ \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \in G \right\}$ of G and hence one can consider a triple of finite groups (B, H, H') and ask whether this triple is Gassmann or not. By the evaluation of permutation characters one has $\mathrm{Ind}_H^B(1_H) \not\simeq \mathrm{Ind}_{H'}^B(1_{H'})$, which means that (B, H, H') is not a Gassmann triple. Keeping notations of Theorem 1.23 we obtain equality of zeta-functions $\zeta_K(s) = \zeta_{K'}(s)$ and the following Galois correspondence diagram:



(2.1)

In order to see Lemma 2.1 in action we pick $p = 3$, in that case G has order $(p^2-1)(p^2-p) = 48$, B has order $(p-1)^2p = 12$ and index 4 in G and both (H, H') have index $p-1 = 2$ in B. It follows that H and H' are normal subgroups and that there is a non-trivial abelian character χ of B which factors as a non-trivial character through the quotient B/H', i.e. a character with $\ker(\chi) = H$. Then $\chi_{|H} = 1_{|_H}$ and therefore $L_K(\chi_{|H}) = \zeta_K(s)$, meanwhile $L_{K'}(\chi_{|_{H'}})$ is an L-function of a non-trivial abelian character of H' and therefore it has no poles as $s \to 1$, which implies that $L_K(\chi_{|H}) \neq L_{K'}(\chi_{|_{H'}})$.

Magma scripts

As before we add a Magma script to verify the examples we discussed above. We split the script into two parts. The first part is a group-theoretical verification:

```
p := 3; k := GF(p);
G := GL(2,k);
TheBorelGroup := Subgroups(G: OrderEqual := 12)[1]‘subgroup;
TheBorelSubgroups := Subgroups(TheBorelGroup: IndexEqual := 2);
for H in TheBorelSubgroups do
  "Permutation Character:", PermutationCharacter(TheBorelGroup, H‘subgroup);
end for;
```

This script produces the following output, which shows that indeed the corresponding permutation representations are not isomorphic:

```
Permutation Character: ( 2, 0, 2, 0, 2, 0 )
Permutation Character: ( 2, 0, 0, 2, 2, 0 )
Permutation Character: ( 2, 2, 0, 0, 2, 2 )
```

In the second part we construct explicitly number fields K, K', L which fit to the diagram (2.1). We are doing this by using torsion points on elliptic curves, similar to the method introduced in [9]. Recall the following main steps of the algorithm:

1. Pick a general elliptic curve E over $\mathbb{Q}$. We denote by $g(x, y)$ the polynomial y^2-x^3-ax-b which defines E;

2. Find a polynomial $f(x)$ with roots corresponding to the x-coordinates of the 3-torsion points $E[3]$ of E, i.e. a 3-division polynomial of E;
3. Evaluate resultant $z(x)$ of f and g with respect to x;
4. Finally, compute the Galois closure N of the number field defined by $z(x)$.

According to Serre's open image theorem for a general elliptic curve E we have:

$$\mathrm{Gal}(N, \mathbb{Q}) \simeq \mathrm{Gl}_2(\mathbb{F}_p).$$

Here is an implementation of this algorithm:

```
// Steps 1, 2 and 3
a := 1; b := 1;
E := EllipticCurve([a, b]);
K := Rationals();
R<x,y> := PolynomialRing(K,2);
g<x,y> := y^2-x^3-a*x-b;
f := DivisionPolynomial(E, 3);
ResultantOfFandG := Resultant(Evaluate(f, x), g, x);

// Mapping the resultant of f and g to the polynomial ring of one variable
S<z> := PolynomialRing(Rationals());
HomRtoS := hom<R -> S | 0, z>;
h := HomRtoS(ResultantOfFandG);

// The final step: producing explicit equations
FF := NumberField(h);
G, r, N := GaloisGroup(FF) ;
TheBorelGroup := Subgroups(G: IndexEqual := 4)[1]‘subgroup;
TheBorelSubgroups := Subgroups(G: IndexEqual := 8);
B<x> := GaloisSubgroup(N, TheBorelGroup);
B;
for H in TheBorelSubgroups do
  GaloisSubgroup(N, H‘subgroup);
end for;
```

The output of this script is:

```
x^4 - 11648*x^3 + 43792584*x^2 + 350900032*x - 160837688676272
x^8 + 351459648*x^6 + 25734142535892480*x^4 + 4959894048812650728l6128*x^2 +
    662246092057630641285070l205504
((x1 - x5) * ((x2 + (x3 + x4)) - (x6 + (x7 + x8))))
x^8 + 5832*x^6 + 10983114*x^4 - 10052399428083
x1
x^8 + 17496*x^6 - 62710038*x^4 + 6198727824*x^2 - 10052399428083
(x2 + (x3 + x4))
```

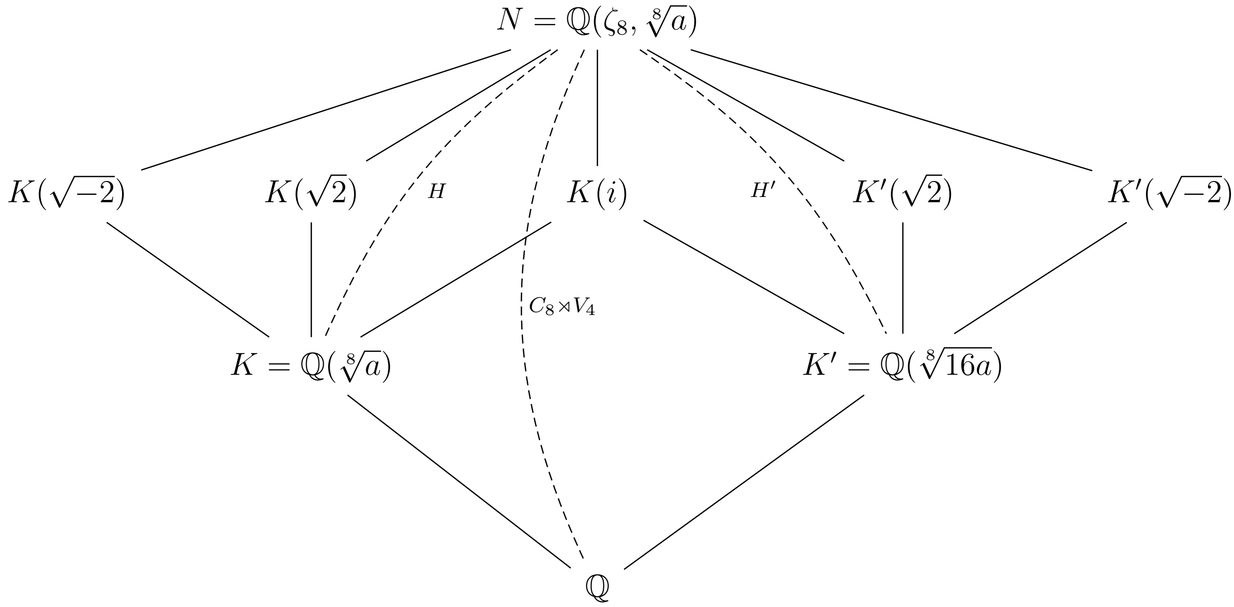
This shows that the number field $L = N^B$ can be defined for instance by the polynomial:

$$x^4 - 11648x^3 + 43792584x^2 + 350900032x - 160837688676272.$$

2.2.2 Yet another example

Slightly generalising Nagata's approach mentioned earlier we could ask: *does there exist a representation χ of H such that for every representation χ' of H' we have $L_K(\chi, s) \neq L_{K'}(\chi', s)$?* Of course if χ is the restriction of a representation ρ of G then the above discussion shows it is not possible to find such a χ, but what if we pick χ which is not the restriction of any representation ρ of G or there are examples where it is not possible to pick such a character? It turns out that if we look thoroughly at fields from example 1.28 then we find out that the answer to the later question is negative: there is an example of a pair (K, K') of non-isomorphic number fields with the following remarkable property. Denoting by N the common normal closure of K and K' there exists a bijection ϕ between the set X of characters of $\mathrm{Gal}(N : K)$ and the set X' of characters of $\mathrm{Gal}(N : K')$ such that for every $\chi \in X$ we have $L_K(\chi, s) = L_{K'}(\phi(\chi), s)$.

Let us consider the example 1.28 more carefully. The normal closure N of $K = \mathbb{Q}(\sqrt[8]{a})$ is $K(\zeta_8)$, where ζ_8 denotes a primitive eighth root of unity. Note that $K(i) = K'(i)$, where $K' = \mathbb{Q}(\sqrt[8]{16a})$. We have $H \simeq H' \simeq V_4$ and the following Galois correspondence diagram:



Lemma 2.2. *There exists a bijection ϕ between characters of H to those of H' such that for every character χ of H holds $L_K(\chi, s) = L_{K'}(\phi(\chi), s)$.*

Proof. This claim is easy to establish from the following observation. First we observe that $\zeta_{K(\sqrt{2})}(s) = \zeta_{K'(\sqrt{2})}(s)$ and $\zeta_{K(\sqrt{-2})}(s) = \zeta_{K'(\sqrt{-2})}(s)$. Indeed, $\zeta_K(s) = \zeta_{K'}(s)$ and $\sqrt{2}$ belongs neither to K nor K', which implies that $K(\sqrt{2})$, $K'(\sqrt{2})$ are arithmetically equivalent and also $K(\sqrt{-2})$, $K'(\sqrt{-2})$. The same argument shows $K(\sqrt{-2})$, $K'(\sqrt{-2})$ are arithmetically equivalent. The group H has three non-trivial characters χ_1, χ_2 and $\chi_1\chi_2$ and up to numeration we have:

$$L_K(\chi_1, s) = \frac{\zeta_{K(\sqrt{2})}(s)}{\zeta_K(s)}, \quad L_K(\chi_2, s) = \frac{\zeta_{K(\sqrt{-2})}(s)}{\zeta_K(s)}, \quad L_K(\chi_1\chi_2, s) = \frac{\zeta_{K(i)}(s)}{\zeta_K(s)}.$$

Replacing K by K' and χ_i by χ_i' we establish the desired bijection. □

Remark 2.3. *Using the above argument for a given pair of arithmetically equivalent number fields K, K' one can construct more examples of pairs of quadratic characters $\chi : \mathcal{G}_K \to \mathbb{C}$, $\chi' : \mathcal{G}_{K'} \to \mathbb{C}$ such that $L_K(\chi, s) = L_{K'}(\chi', s)$. Namely, for a rational prime number p such that $\sqrt{p} \notin K$, K' the number fields $M = K(\sqrt{p})$, $M' = K'(\sqrt{p})$ are also arithmetically equivalent and therefore: $L_K(\chi, s) = \frac{\zeta_M(s)}{\zeta_K(s)} = \frac{\zeta_{M'}(s)}{\zeta_{K'}(s)} = L_{K'}(\chi', s)$, where χ(resp. χ') is the unique non-trivial character of* $\mathrm{Gal}(M : K)$ *(resp.* $\mathrm{Gal}(M' : K')$*).*

2.3 Identifying Number Fields with Artin L-functions

Now it is reasonable to ask the following: could we somehow detect the isomorphism class of a number field K by using Artin L-functions of Galois representations associated to the Galois groups of normal extensions of K? The answer is yes and it is given by Theorem 2.4:

Theorem 2.4. *For each number field K there exists an abelian extension N_K of degree three and a character χ of* $\mathrm{Gal}(N_K : K)$ *such that $L_K(\chi, s)$ occurs only for the isomorphism class of the field K, i.e. if for any other number field K' and any abelian extension $N_{K'}$ of K' there exists a character χ' of* $\mathrm{Gal}(N_{K'} : K')$ *such that $L_K(\chi, s) = L_{K'}(\chi', s)$ then $K \simeq K'$.*

We begin with providing a sketch for the proof of a slightly different version of Theorem 2.4. After that we explain how the statement of Theorem 2.4 follows from what we have discussed.

2.3.1 The First Version of Theorem

We first discuss a proof of another version of Theorem 2.4:

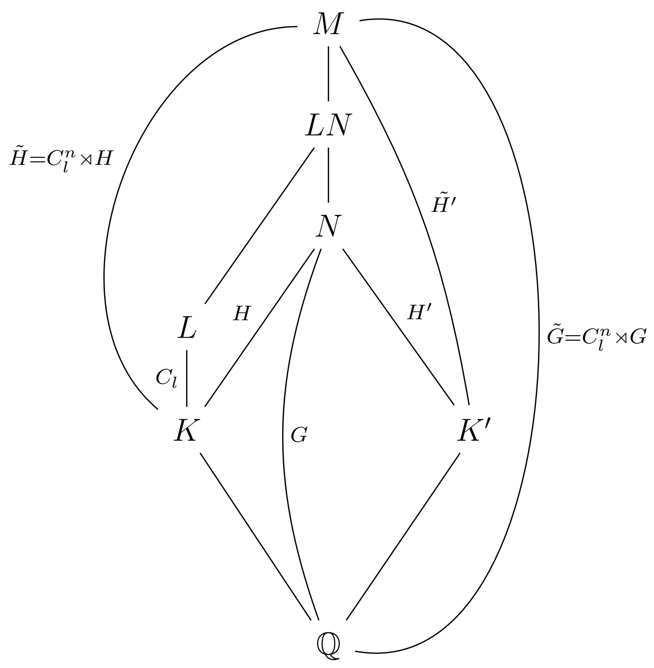
Theorem 2.5. *For every pair of non-isomorphic number fields K, K' with $\zeta_K(s) = \zeta_{K'}(s)$ we may attach a Galois extension M over $\mathbb{Q}$ with the Galois group $\tilde{G}$ which contains both K and K' with $K = M^{\tilde{H}}$, $K' = M^{\tilde{H}'}$ for some subgroups $\tilde{H}$, $\tilde{H}'$ of G and an abelian character χ of $\tilde{H}$ such that for any abelian character χ' of $\tilde{H}'$ we have $L_K(\chi, s) \neq L_{K'}(\chi', s)$. In other words $L_K(\chi, s)$ as an analytic function occurs only for K, but not for K'.*

This goal is achieved in two steps. First we need the following group-theoretical result. Let G be a finite group, H a subgroup of index n, and $C_l = \mu_l \subset \mathbb{C}^\times$ be a cyclic group of order l, where *l is an odd prime*. Consider a G-set G/H of left cosets. We fix some representatives $X_1, \ldots X_n$ of G/H such that X_1 is a coset corresponding to the group H. Let us regard semi-direct products $\tilde{G} = C_l^n \rtimes G$ and $\tilde{H} = C_l^n \rtimes H$, where G acts on the components of C_l^n by permuting them as the cosets $\{X_1, \ldots, X_n\}$. Let χ be the homomorphism from $\tilde{H}$ to the group C_l defined on the element $\tilde{h} = (\zeta_1, \ldots, \zeta_n, h) \in \tilde{H} = C_l^n \rtimes H$ as $\chi(\tilde{h}) = \zeta_1$ i.e. χ is the *projection to the first coordinate*. This is indeed a homomorphism because every $h \in H$ fixes the first coset of G/H. In this setting the following holds:

Theorem 2.6 (Bart de Smit)**.** *For any subgroup $\tilde{H}' \subset \tilde{G}$ and any character $\chi' : \tilde{H}' \to \mathbb{C}^*$ if $\mathrm{Ind}_{\tilde{H}'}^{\tilde{G}}(\chi') \simeq \mathrm{Ind}_{\tilde{H}}^{\tilde{G}}(\chi)$ then $\tilde{H}'$ and $\tilde{H}$ are conjugate in $\tilde{G}$.*

Proof. See section 3.4 from the next chapter. □

Next suppose K is a number field such that $\zeta_K(s)$ does not determine K i.e. there exists a number field K' such that $\zeta_K(s) = \zeta_{K'}(s)$, but $K \not\simeq K'$. Then as before this means the normal closure N of K contains K' and there exists a non-trivial Gassmann triple (G, H, H') with $G = \mathrm{Gal}(N/\mathbb{Q})$, $H = \mathrm{Gal}(N/K)$, $H' = \mathrm{Gal}(N/L)$. In this setting we construct a Galois extension M of $\mathbb{Q}$ containing K and K' such that the Galois group $\mathrm{Gal}(M : \mathbb{Q})$ is $\tilde{G}$ and $K = M^{\tilde{H}}$, $K' = M^{\tilde{H}'}$ for $\tilde{G}$, $\tilde{H}$, $\tilde{H}'$ as in Theorem 2.6. This is possible due to Proposition 9.1 from [13]. See the diagram below:



Now consider the abelian character χ of $\tilde{H}$ as in the statement of Theorem 2.6. Suppose χ' is any abelian character of $\tilde{H}' = \mathrm{Gal}(M : K')$. We have:

$$L_K(\chi, s) = L_{K'}(\chi', s) \Rightarrow K \simeq K'.$$

Indeed, by the induction property we have:

$$L_K(\chi, s) = L_{\mathbb{Q}}(\mathrm{Ind}_{\tilde{H}}^{\tilde{G}}(\chi), s).$$

Therefore:

$$L_K(\chi, s) = L_K(\chi', s) \Leftrightarrow L_{\mathbb{Q}}(\mathrm{Ind}_{\tilde{H}}^{\tilde{G}}(\chi), s) = L_{\mathbb{Q}}(\mathrm{Ind}_{\tilde{H}'}^{\tilde{G}}(\chi'), s) \Leftrightarrow \mathrm{Ind}_{\tilde{H}}^{\tilde{G}}(\chi) \simeq \mathrm{Ind}_{\tilde{H}'}^{\tilde{G}}(\chi'),$$

and by Theorem 2.6 we have that $\tilde{H}$ and $\tilde{H}'$ are conjugate and hence K is isomorphic to K'.

2.3.2 Deducing Theorem 2.4 from Theorem 2.5

Now our goal is to deduce Theorem 2.4 from Theorem 2.5. First we note that denoting by ψ the restriction of χ to the Galois group $\mathrm{Gal}(L : K) = C_l$ we obtain a non-trivial one-dimensional character of C_l. By the inflation property we have $L_K(\chi, s) = L_K(\psi, s)$, where the latter L-function is an abelian L-function of the abelian extension L over K.

In the same setting as before suppose that there exist an abelian extension $N_{K'}$ of K' and a character ψ' of $\mathrm{Gal}(N_{K'} : K')$ such that $L_K(\psi, s) = L_{K'}(\psi', s)$. We would like to show that there exists an abelian character χ' of $\tilde{H}' = \mathrm{Gal}(M : K')$ such that $L_{K'}(\chi', s) = L_K(\psi', s)$ and therefore we can apply Theorem 2.5. In other words we would like to show that the character ψ' can be treated as an abelian character χ' of $\tilde{H}'$ in the setting of Theorem 2.5.

We fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and denote the absolute Galois group of fields K, K', $\mathbb{Q}$ by $\mathcal{G}_K$, $\mathcal{G}_{K'}$ and $\mathcal{G}_{\mathbb{Q}}$ respectively. We consider ψ as a character of $\mathcal{G}_K$ via the projection $\mathcal{G}_K \to \tilde{H}$. In a similar way ψ' is a character of $\mathcal{G}_{K'}$. By the induction property we have:

$$L_K(\psi, s) = L_{\mathbb{Q}}(\mathrm{Ind}_{\mathcal{G}_K}^{\mathcal{G}_{\mathbb{Q}}} \psi, s),$$

and

$$L_{K'}(\psi', s) = L_{\mathbb{Q}}(\mathrm{Ind}_{\mathcal{G}_{K'}}^{\mathcal{G}_{\mathbb{Q}}} \psi', s).$$

Since by our assumptions $L_K(\psi, s) = L_{K'}(\psi', s)$ we have:

$$\mathrm{Ind}_{\mathcal{G}_K}^{\mathcal{G}_{\mathbb{Q}}} \psi \simeq \mathrm{Ind}_{\mathcal{G}_{K'}}^{\mathcal{G}_{\mathbb{Q}}} \psi'.$$

But then kernels of these representations of $\mathcal{G}$ must coincide. Since M is the fixed field of the action of $\ker(\mathrm{Ind}_{\mathcal{G}_K}^{\mathcal{G}_{\mathbb{Q}}} \psi)$ on $\overline{\mathbb{Q}}$ we have that M is also the fixed field of the action of $\ker(\mathrm{Ind}_{\mathcal{G}_{K'}}^{\mathcal{G}_{\mathbb{Q}}} \psi')$ on $\overline{\mathbb{Q}}$. This means that the extension $N_{K'}$ can be embedded into the field M and the character ψ' is an abelian character of $\tilde{H}'$.

2.4 Neukirch-Uchida Theorem

Recall from the first chapter that the famous Neukirch-Uchida theorem states that:

Theorem 2.7. *For given number fields K, K' the existence of a topological isomorphism of profinite groups $\mathcal{G}_K \simeq \mathcal{G}_{K'}$ implies the existence of an isomorphism of fields $K \simeq K'$ themselves.*

The story behind this result is the following. In 1969 Neukirch [34] gave a proof for the case of normal extensions of $\mathbb{Q}$. He proved this by recovering Dedekind zeta-function $\zeta_K(s)$ of K from $\mathcal{G}_K$ in group-theoretical terms and then applying Theorem 1.23 to show that in this case $\zeta_K(s)$ determines the isomorphism class of K. A few years later in 1976 Uchida [56] extended Neukirch's results to the case of arbitrary number fields. Uchida's approach was then also used by himself and others to generalise the Theorem to the case of all global fields. For a modern exposition, see Chapter XII in [37]. Without any doubt Uchida's proof is beautiful and important, but it contains some difficult technical details which make this proof a bit less clear especially for those who are relatively new to the topic. The goal of the present section is to provide an alternative, in some sense more elementary approach to the proof of Uchida's part. The new proof also has another advantage, since it stays closer to Neukirch's original idea. This new approach is based on the following idea.

Given a number field K we associate to it a set Λ_K of Dedekind zeta-functions of finite abelian extensions of K:

$$\Lambda_K = \{\zeta_L(s) \mid L \text{ is a finite abelian extension of } K\}.$$

Our main goal is to prove the following Theorem:

Theorem 2.8. *For every number field K the set Λ_K determines the isomorphism class of K. This means that if for any other number field K' the two sets Λ_K and $\Lambda_{K'}$ coincide, then $K \simeq K'$.*

The following observation shows that Theorem 2.8 allows us to achieve our goal and produce an alternative way to Uchida's part:

Corollary 2.9. *In the above setting suppose that $\mathcal{G}_K \simeq \mathcal{G}_{K'}$. Then $\Lambda_K = \Lambda_{K'}$ and therefore $K \simeq K'$.*

Proof. Indeed, given an isomorphism class of $\mathcal{G}_K$ we consider all closed subgroups of finite index $H \subset \mathcal{G}_K$ such that the quotient $\mathcal{G}_K/H$ is a finite abelian group. By pro-finite Galois theory we have one-to-one correspondence between such H and finite abelian extensions L of K within a fixed algebraic closure $\overline{K}$ given by $H \to (\overline{K})^H$. Now by using Neukirch's Theorem (see chapter 4 in [34]) we reconstruct $\zeta_L(s)$ in a group theoretical manner from H and therefore reconstruct Λ_K from $\mathcal{G}_K$. □

From now on we concentrate our attention on the proof of 2.8.

On the Proof of Theorem 2.8

To deduce Theorem 2.8 we extend Theorem 2.4 by replacing the L-function of the abelian character χ by the Dedekind ζ-function of the abelian extension N_K of K:

Theorem 2.10. *For each number field K there exists an abelian extension N_K of degree three such that the pair $\zeta_{N_K}(s)$, $\zeta_K(s)$ occurs only for the isomorphism class of the field K, i.e. if for any other number field K' and any abelian extension $N_{K'}$ of K' we have $\zeta_K(s) = \zeta_{K'}(s)$ and $\zeta_{N_K}(s) = \zeta_{{N'}_{K'}}(s)$ then $K \simeq K'$.*

Remark 2.11. *Note that the degree of a number field K is determined by $\zeta_K(s)$. Therefore, $\zeta_K(s)$ can be recovered from $\Lambda(K)$ as unique element whose corresponding field has minimal degree.*

The above remark shows that Theorem 2.8 and Theorem 2.10 are equivalent and we can focus on proving the last statement.

2.4.1 The proof

First we fix $l = 3$ and prove the following auxiliary statement:

Lemma 2.12. *In the setting of Theorem 2.6 the induced representation* $\mathrm{Ind}_{\tilde{H}}^{\tilde{G}}(\chi)$ *is an irreducible representation of* $\tilde{G}$.

Proof. In order to verify irreducibility of $\mathrm{Ind}_{\tilde{H}}^{\tilde{G}}(\chi)$ we regard the standard scalar product and show that $(\mathrm{Ind}_{\tilde{H}}^{\tilde{G}}(\chi), \mathrm{Ind}_{\tilde{H}}^{\tilde{G}}(\chi))_{\tilde{G}} = 1$. Applying Frobenius reciprocity:

$$(\mathrm{Ind}_{\tilde{H}}^{\tilde{G}}(\chi), \mathrm{Ind}_{\tilde{H}}^{\tilde{G}}(\chi))_{\tilde{G}} = (\chi, \mathrm{Ind}_{\tilde{H}}^{\tilde{G}}(\chi)|_{\tilde{H}})_{\tilde{H}} = \frac{1}{|\tilde{H}|} \sum_{\tilde{h} \in \tilde{H}} \bar{\chi}(\tilde{h}) \cdot \mathrm{Tr}(\mathrm{Ind}_{\tilde{H}}^{\tilde{G}}(\chi)|_{\tilde{H}}(\tilde{h})). \tag{2.2}$$

Let $\tilde{h} = (\zeta_1, \ldots, \zeta_n, h) \in \tilde{H}$. Then by definition of χ we have $\bar{\chi}(\tilde{h}) = \bar{\zeta}_1$. Now consider the matrix $\mathrm{Ind}_{\tilde{H}}^{\tilde{G}}(\chi)|_{\tilde{H}}(\tilde{h})$. We fix the following representatives for cosets of $\tilde{G}/\tilde{H}$ as $\tilde{X}_i = (1, \ldots, 1, X_i) \in \tilde{G}$, where X_i are the representatives of cosets of G/H we picked before. By definition of the induced representation and because h fixes first conjugacy class of G/H we have that in the top left corner of that matrix ζ_1 is located. Now we fix an integer $1 < i \leq n$ and consider the diagonal element $a_i(\tilde{h})$ in the (i,i)-th position. Consider the permutation of cosets $\tilde{X}_1, \ldots \tilde{X}_n$ by $\tilde{h}$ and denote by j an index such that $\tilde{h}\tilde{X}_i = \tilde{X}_j$. If $i \neq j$ then $a_i(\tilde{h}) = 0$ and therefore such i adds no contribution to the expression (2.2). Otherwise, by definition of the induced representation $\mathrm{Ind}_{\tilde{H}}^{\tilde{G}}(\chi)|_{\tilde{H}}$ we have $a_i(\tilde{h}) = \chi(\tilde{X}_i^{-1}\tilde{h}\tilde{X}_i) = \zeta_{k_i}$ for some index $k_i \in \{2, \ldots, n\}$. In other words, k_i is an index such that $X_i^{-1}X_{k_i} \in H$. For fixed $\tilde{h}$ and i there are elements $\tilde{h}_1$, $\tilde{h}_2$ such that $(\tilde{h}, \tilde{h}_1, \tilde{h}_2)$ pairwise coincide in all coordinates except the k_i-th one. Because $1 + \zeta_{k_i} + \bar{\zeta}_{k_i} = 0$ we have that sum of $a_i(\tilde{h})$ for those $\tilde{h}$, $\tilde{h}_1$, $\tilde{h}_2$ is zero and because they coincide on first coordinate we have $\chi(\tilde{h}) = \chi(\tilde{h}_j)$ for j in $\{1, 2\}$. Therefore for fixed $i > 1$ we have:

$$\sum_{\tilde{h} \in \tilde{H}} \bar{\chi}(\tilde{h}) a_i(\tilde{h}) = 0.$$

Now we consider the expression (2.2):

$$\frac{1}{|\tilde{H}|} \sum_{\tilde{h} \in \tilde{H}} \bar{\chi}(\tilde{h}) \cdot \mathrm{Tr}(\mathrm{Ind}_{\tilde{H}}^{\tilde{G}}(\chi)|_{\tilde{H}}(\tilde{h})) = \frac{1}{|\tilde{H}|} \sum_{\tilde{h} \in \tilde{H}} \bar{\chi}(\tilde{h}) \cdot (\chi(\tilde{h}) + \sum_{i>1}^{i \leq n} a_i(\tilde{h})) =$$

$$= \frac{1}{|\tilde{H}|} \sum_{\tilde{h} \in \tilde{H}} \bar{\chi}(\tilde{h})\chi(\tilde{h}) + \frac{1}{|\tilde{H}|} \sum_{\tilde{h} \in \tilde{H}} (\bar{\chi}(\tilde{h}) \cdot (\sum_{i>1}^{i \leq n} a_i(\tilde{h}))) =$$

$$= \frac{1}{|\tilde{H}|} \sum_{\tilde{h} \in \tilde{H}} 1 + \frac{1}{|\tilde{H}|} \sum_{i>1} (\sum_{\tilde{h} \in \tilde{H}} \bar{\chi}(\tilde{h}) a_i(\tilde{h})) = 1 + 0.$$

□

By using this lemma we can prove the main group theoretical result of this note:

Theorem 2.13. *In the above setting let* $U_{\tilde{H},\chi} = \ker(\chi) = \{h \in \tilde{H} | \chi(h) = 1\}$ *and let* $U_{\tilde{H}',\chi'} = \ker(\chi')$. *Suppose that* $\mathrm{Ind}_{\tilde{H}}^{\tilde{G}}(1) \simeq \mathrm{Ind}_{\tilde{H}'}^{\tilde{G}}(1)$ *and* $\mathrm{Ind}_{U_{\tilde{H},\chi}}^{\tilde{G}}(1) \simeq \mathrm{Ind}_{U_{\tilde{H}',\chi'}}^{\tilde{G}}(1)$. *Then either* $\mathrm{Ind}_{\tilde{H}}^{\tilde{G}}(\chi) \simeq \mathrm{Ind}_{\tilde{H}'}^{\tilde{G}}(\chi')$ *or* $\mathrm{Ind}_{\tilde{H}}^{\tilde{G}}(\chi) \simeq \mathrm{Ind}_{\tilde{H}'}^{\tilde{G}}(\bar{\chi}')$.

Proof. Since $l = 3$ we have that C_l has only three characters $1, \chi, \bar{\chi}$ and therefore:

$$\mathrm{Ind}_{U_{\tilde{H},\chi}}^{\tilde{G}}(1) \simeq \mathrm{Ind}_{\tilde{H}}^{\tilde{G}}(\chi) \oplus \mathrm{Ind}_{\tilde{H}}^{\tilde{G}}(\bar{\chi}) \oplus \mathrm{Ind}_{\tilde{H}}^{\tilde{G}}(1).$$

Hence, from the assumption of the Theorem it follows that:

$$\mathrm{Ind}_{\tilde{H}}^{\tilde{G}}(\chi) \oplus \mathrm{Ind}_{\tilde{H}}^{\tilde{G}}(\bar{\chi}) \simeq \mathrm{Ind}_{\tilde{H}'}^{\tilde{G}}(\chi') \oplus \mathrm{Ind}_{\tilde{H}'}^{\tilde{G}}(\bar{\chi}').$$

In Lemma 2.12 we showed that $\mathrm{Ind}_{\tilde{H}}^{\tilde{G}}(\chi)$, $\mathrm{Ind}_{\tilde{H}}^{\tilde{G}}(\bar{\chi})$ are *irreducible representations* of $\tilde{G}$. But if a direct sum of two irreducible representations of a finite group is isomorphic to a direct sum of two other *non-zero representations* then those representations are pairwise isomorphic up to a permutation. It follows that either $\mathrm{Ind}_{\tilde{H}}^{\tilde{G}}(\chi) \simeq \mathrm{Ind}_{\tilde{H}'}^{\tilde{G}}(\chi')$ or $\mathrm{Ind}_{\tilde{H}}^{\tilde{G}}(\chi) \simeq \mathrm{Ind}_{\tilde{H}'}^{\tilde{G}}(\bar{\chi}')$. □

Finally, we can provide:

Proof of Theorem 2.10. Suppose K is a number field such that $\zeta_K(s)$ does not determine K i.e. there exists a number field K' such that $\zeta_K(s) = \zeta_{K'}(s)$, but $K \not\simeq K'$. Then as before we construct a Galois extension M of $\mathbb{Q}$ as in Theorem 2.4, see figure 2.3.1. Let L' be *any abelian extension* of K' such that $\zeta_{L'} = \zeta_L$. Then L and L' share the same normal closure over $\mathbb{Q}$ and therefore L' is a subfield of M. According to remark 2.11 we also have that the degree of L' over K' is three. Observe that in notations of Theorem 2.13 from the previous section one has: $\mathrm{Gal}(M : L) = \ker(\chi) = U_{\tilde{H},\chi}$ for a non-trivial character χ of $\mathrm{Gal}(L : K)$ and $\mathrm{Gal}(M : L') = \ker(\chi')$ for a non-trivial character χ' of $\mathrm{Gal}(L' : K')$. By the induction property of Artin L-functions we have: $\zeta_L(s) = L_{\mathbb{Q}}(\mathrm{Ind}_{U_{\tilde{H},\chi}}^{\tilde{G}}(1), s)$.

Finally, because of multiplicative independence of L-functions over $\mathbb{Q}$ we have:

$$L_{\mathbb{Q}}(\mathrm{Ind}_{U_{\tilde{H},\chi}}^{\tilde{G}}(1), s) = L_{\mathbb{Q}}(\mathrm{Ind}_{U_{\tilde{H}',\chi'}}^{\tilde{G}}(1), s) \Leftrightarrow \mathrm{Ind}_{U_{\tilde{H},\chi}}^{\tilde{G}}(1) \simeq \mathrm{Ind}_{U_{\tilde{H}',\chi'}}^{\tilde{G}}(1).$$

This means that from the assumptions of Theorem 2.10 we deduced conditions of Theorem 2.13. Therefore because of Theorem 2.6 we have that $\tilde{H}$ and $\tilde{H}'$ are conjugate and hence K is isomorphic to K'. □