Global fields and their L-functions

Solomatin, P.

Author: Solomatin, P.

Title: Global field and their L-functions

Issue Date: 2021-03-02

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Proefschrift
ter verkrijging van
de graad van doctor aan de Universiteit Leiden
op gezag van rector magnificus prof.dr.ir. H.Bijl,
volgens besluit van het college voor promoties
te verdedigen op 2 maart 2021
klokke 16:15 uur

door

Pavel Solomatin
geboren te Moskou, Russische Federatie, in 1991

Promotor: Prof. dr. Bart de Smit
Promotor: Prof. dr. Karim Belabas (Université de Bordeaux)

Samenstelling van de promotiecommissie:
Dr. Elisa Lorenzo García (Université de Rennes 1)
Prof.dr. Frank van der Duijn Schouten
Prof.dr. Marc Hindry (Université Paris VII)
Prof.dr. Michael A. Tsfasman (Laboratoire de Mathématiques de Versailles)
Prof.dr. Ronald van Luijk

This work was funded by Erasmus Mundus ALGANT-DOC and it was carried out at Leiden University and University of Bordeaux

THÈSE EN COTUTELLE PRÉSENTÉE
POUR OBTENIR LE GRADE DE
DOCTEUR
DE L'UNIVERSITÉ DE BORDEAUX
ET DE L'UNIVERSITÉ DE LEYDE

ÉCOLE DOCTORALE DE MATHÉMATIQUES ET INFORMATIQUE
INSTITUT DES MATHÉMATIQUES DE L'UNIVERSITÉ DE LEYDE

SPÉCIALITÉ Mathématiques Pures

Par Pavel Solomatin

Corps globaux et leurs fonctions L

Sous la direction de Bart de Smit et Karim Belabas
Soutenue le 2 mars 2021

Membres du jury:

Marc Hindry	Professeur, Université de Paris	Examinateur
Michael A. Tsfasman	Directeur de recherche, Université de Versailles Saint-Quentin-en-Yvelines	Examinateur
Bart de Smit	Professeur, Universiteit Leiden	Directeur
Karim Belabas	Professeur, Université de Bordeaux	Directeur

Summary of the Thesis

In the first chapter of the present thesis we provide a brief step by step introduction to the topic as well as a survey of the main results of the thesis. During the process we focus primarily on the discussion about *arithmetically equivalent number fields*, also known as *isospectral number fields*. This discussion leads to numerous related notions which are central in this dissertation. Among them the following concepts will play a crucial role: *Artin L-functions, absolute Galois groups, class field theory, representation theory of finite and pro-finite groups.* The principal result behind the theory is the famous Theorem 1.23 which goes back to Gassmann. Together with its corollaries this result provides a powerful framework which illustrates nicely how beautiful the interaction between the notions mentioned above can be. We also recall connections of the topic with the so-called *Grothendieck's Anabelian Geometry.* Among other subjects this theory studies properties of the absolute Galois group $\mathcal{G}_K = \mathrm{Gal}(\overline{K} : K)$ of a number field K as well as the structure of the maximal abelian quotient $\mathcal{G}_K^{ab} = \mathrm{Gal}(K^{ab} : K)$ of $\mathcal{G}_K$. We state our main results in section 1.7. Note that while this part of the dissertation served as an introduction and contains no original results, other chapters represent the original work of the author and have corresponding references to preprint versions available on the Arxiv website: arxiv.org.

In chapter two we extend methods of the framework mentioned above and provide some interesting applications of the theory. In particular, we formulate a bit less-known, but still remarkable Theorem 2.4 due to Professor Bart de Smit. Roughly speaking, this Theorem states that the isomorphism class of a number field K is uniquely determined by the collection of Artin L-functions of abelian characters of the absolute Galois group $\mathcal{G}_K$ of K; see section 2.3. In the section 2.4 of this chapter we also generalise Theorem 2.4 in a way which allows us to produce an alternative approach towards a proof of the famous Neukirch-Uchida theorem for the case of non-normal extensions of number fields. This part of the dissertation occurred in [49].

Then in chapters three and four we provide two different approaches in a direction of a function field analogue of Theorem 2.4. The difference between the two treatments is the following: chapter three regards function fields from an *algebraic point of view*, i.e., function fields as finite extensions of the field $\mathbb{F}_q(X)$. In contrast, in chapter four we consider a more geometric setting such as field of functions on a smooth projective curve defined over a finite field $\mathbb{F}_q$. Despite the fact that the two notions are extremely related, the results we proved seem to be opposite. Chapter three has a large intersection with the pre-print [48], while chapter four is based on [47].

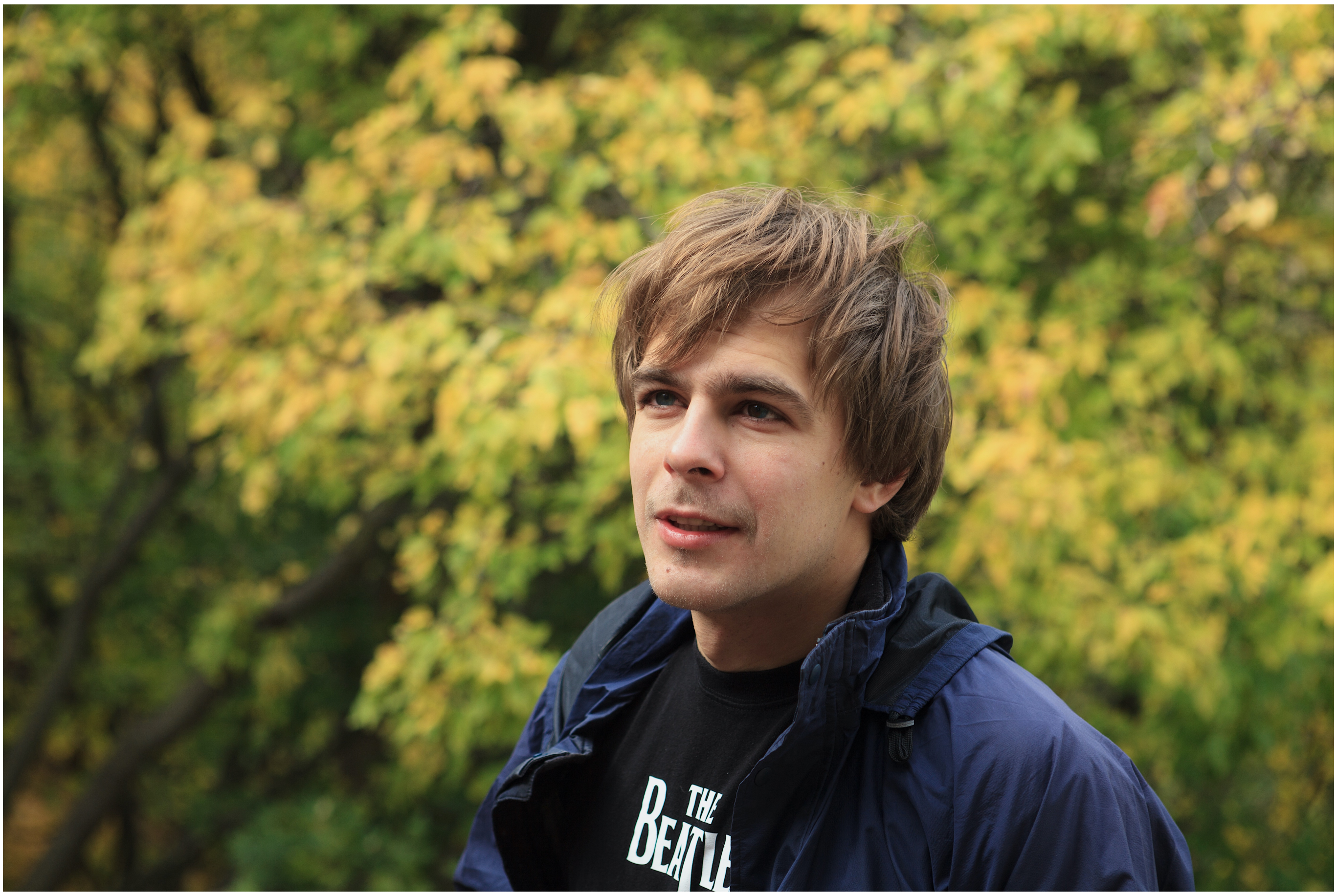
Finally, in chapters five and six we shift our focus towards the description of the isomorphism class of the abelianized absolute Galois group $\mathcal{G}_K^{ab}$ associated to a global function field and an imaginary quadratic number field respectively. In the case of global function fields we obtained a complete description and classified all possible isomorphism classes of $\mathcal{G}_K^{ab}$ in terms of more

elementary invariants attached to K. For the imaginary quadratic field case we improved results of [1]. In particular we proved that there are infinitely many isomorphism types of pro-finite abelian groups which occur as $\mathcal{G}_K^{ab}$ for some imaginary quadratic field K. These parts of the thesis correspond to preprints [11] and [12].

For the sake of coherence, along the way towards our main results we occasionally will discuss some additional questions, lemmas and remarks. At the first sight those might seem to be a little aside from the topic, but actually together with the core content they form essential basis needed for understanding the whole picture. We will also provide many concrete examples as well as scripts written in the language of the computational algebra system called Magma. These scripts can be used by anybody who is curious about constructing more sophisticated instances and checking statements of some of the theorems.

Dedication

*Dedicated to the memory of my Friend, Advisor and Teacher,
Professor Alexey Ivanovich Zykin(13 June 1984 — 22 April 2017).*



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