Patterns on spatially structured domains
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Chapter 6

Parameter-dependent exponential dichotomies for nonlocal differential operators

6.1 Introduction and main result

In this short, final chapter, we extend parts of the theory from Chapter 5 to include MFDEs such as (5.2.1) that depend smoothly on a parameter μ. For each individual μ one can construct the corresponding exponential splitting using our previous results, but this construction contains some noncanonical choices that do not necessarily preserve the smoothness in μ. Often in applications, this smoothness is necessary in order to obtain uniform estimates and close bifurcation arguments.

For example, exponential dichotomies play a major role in the construction and stability analysis [108, 109] of travelling pulse solutions to the FitzHugh-Nagumo LDE (5.1.1). In particular, Hupkes and Sandstede consider a family of linearisations of the Nagumo MFDE of the form

$$cu'(\sigma) \quad = \quad u(\sigma+1) + u(\sigma-1) - 2u(\sigma) + g_u\big(\Theta(\vartheta, c, \rho)(\sigma), a\big)u(\sigma). \qquad (6.1.1)$$

Here, the relevant parameters are the wavespeed c, which should be close to the wavespeed of the travelling front solution (5.1.5), the parameter ρ from the corresponding FitzHugh-Nagumo system, which should be close to 0, and a phase shift ϑ. Using exponential dichotomies for (6.1.1), the authors construct quasi-front and quasi-back solutions to (5.1.1).

Since we work in more or less the same setting as in Chapter 5 and use several key results from that chapter, we will reuse the notation and assumptions introduced there. In particular, we consider the parameter-dependent system

$$\begin{array}{rcl}\dot{x}(t) & = & \sum_{j=-\infty}^{\infty} A_j(t;\mu)x(t+r_j) + \int_{\mathbb{R}} \mathcal{K}(\xi;t;\mu)x(t+\xi)d\xi \\ & := & L(t,\mu)x_t.\end{array} \tag{6.1.2}$$

Here the parameter μ takes values in an open set $U \subset \mathbb{R}^p$, for some integer $p \geq 1$ and the notation x_t was introduced in (5.2.24). The corresponding linear operators $\Lambda(\mu) : W^{1,\infty}(\mathbb{R}; C^M) \to L^\infty(\mathbb{R}; \mathbb{C}^M)$ are given by

$$(\Lambda(\mu)x)(t) \quad = \quad \dot{x}(t) - \sum_{j=-\infty}^{\infty} A_j(t;\mu)x(t+r_j) - \int_{\mathbb{R}} \mathcal{K}(\xi;t;\mu)x(t+\xi)d\xi. \tag{6.1.3}$$

We assume that the system (6.1.2) depends C^k-smoothly on μ in the following sense.

Assumption (HC). The linear operators $\Lambda(\mu)$ corresponding to the system (6.1.2) depends C^k-smoothly on the parameter $\mu \in U$ for some integer $k \geq 0$. In addition, Assumption (HKer) holds for some $\mu_0 \in U$, while Assumptions (HA), (HK) and (HH) hold uniformly for $\mu \in U$. That is, the constant $\tilde{\eta}$ and the upper bounds for the quantities in (5.2.7) and (5.2.8) can be chosen independently of $\mu \in U$. Finally, the limiting operators $\Lambda_{\pm\infty}(\mu)$ depend C^k-smoothly on $\mu \in U$.

Our main result below shows that the exponential splittings which were obtained in §5.5 can be constructed in such a way that the smoothness in the parameter μ is preserved. The concession we have to make is that the space $R(\tau;\mu)$ will be no longer invariant in the sense of Theorem 5.2.8. We view the results in this chapter as another step in the ongoing effort to close the gap between MFDEs with finite-range and with infinite-range interactions. In particular, we expect our results to play an important part in the stability analysis of the FitzHugh-Nagumo LDE with infinite-range interactions (5.1.16), which, at present, is an open problem if $h > 0$ is sufficiently far away from 0.

Theorem 6.1.1 (cf. [104, Thm. 5.1])**.** *Assume that (HC) is satisfied. Then there exists an open neighbourhood $\mu_0 \in U' \subset U$ in such a way that for any $\mu \in U'$ and any $\tau \geq 0$ there exist subspaces $Q(\tau,\mu), R(\tau,\mu) \subset X$ that satisfy the following properties.*

(i) We have the direct sum decomposition

$$X \quad = \quad Q(\tau;\mu) \oplus R(\tau;\mu) \tag{6.1.4}$$

(ii) Each $\phi \in Q(\tau;\mu)$ can be extended to a solution $E_{\tau,\mu}\phi$ of (6.1.2) on the interval $[\tau,\infty)$, while each $\psi \in R(\tau;\mu)$ can be extended to a solution $E_{\tau,\mu}\psi$ of (6.1.2) on the interval $(-\infty,-r_0] \cup [0,\tau]$. [1]

[1] Here the constant $r_0 > 0$ is defined Proposition 5.5.3.

(iii) *The maps $\mu \mapsto \Pi_{Q(\tau;\mu)}$ and $\mu \mapsto \Pi_{R(\tau;\mu)}$ are C^k-smooth and all derivatives can be bounded uniformly for $\tau \geq 0$.*

(iv) *There exist constants $K > 0$ and $\alpha > 0$ in such a way that we have the pointwise exponential estimates for each $\phi \in X$ and each integer $0 \leq \ell \leq k$*

$$\begin{array}{lcll} |D_\mu^\ell E_{\tau,\mu}\Pi_{Q(\tau;\mu)}\phi|(t) & \leq & Ke^{-\alpha|t-\tau|}\|\phi\|_\infty, & \text{for every } t \geq \tau, \\ |D_\mu^\ell E_{\tau,\mu}\Pi_{R(\tau;\mu)}\phi|(t) & \leq & Ke^{-\alpha|t-\tau|}\|\phi\|_\infty, & \text{for every } t \leq \tau, \\ |\Lambda(\mu)E_{\tau,\mu}\Pi_{R(\tau;\mu)}\phi|(t) & \leq & Ke^{-\alpha|t-\tau|}\|\phi\|_\infty, & \text{for every } t \leq \tau. \end{array} \tag{6.1.5}$$

Our results are primarily based on the approach from [104, §3,5], where Hupkes and Verduyn Lunel construct exponential splittings for parameter-dependent MFDEs with finite-range interactions. The main difficulty here is that in [104] these splittings are obtained by solving a linear equation on a space of functions, defined on the interval $D_\tau^\oplus$, with an exponential weight. However, several operators that are involved in this linear equation, such as the inclusion of the space $Q(\tau)$ into such an exponentially weighted space, lose their boundedness if $r_{\min} = -\infty$. As a workaround, we reconsider the problem on a space with a one-sided exponential weight. However, this change complicates several of the key technical computations.

6.2 One-sided exponential weights

We start by expanding the Fredholm theory from [68] for the system (5.2.1) to spaces with a one-sided exponential weight. For any $\eta \in \mathbb{R}$ and $f \in L^1_{\text{loc}}(\mathbb{R};\mathbb{C}^M)$ we introduce the function

$$[e_\eta^+ f](x) \quad = \quad e^{\eta(x^+)}f(x), \tag{6.2.1}$$

where

$$x^+ \quad = \quad \begin{cases} x, & x \geq 0, \\ 0, & x < 0. \end{cases} \tag{6.2.2}$$

This allows us to define the spaces

$$\begin{array}{lcl} L^\infty_{\eta,+}(\mathbb{R};\mathbb{C}^M) & = & \{f \in L^1_{\text{loc}}(\mathbb{R};\mathbb{C}^M) \,|\, e^+_{-\eta}f \in L^\infty(\mathbb{R};\mathbb{C}^M)\}, \\ W^{1,\infty}_{\eta,+}(\mathbb{R};\mathbb{C}^M) & = & \{f \in L^1_{\text{loc}}(\mathbb{R};\mathbb{C}^M) \,|\, e^+_{-\eta}f \in W^{1,\infty}(\mathbb{R};\mathbb{C}^M)\}, \end{array} \tag{6.2.3}$$

with the corresponding norms

$$\begin{array}{lcl} \|f\|_{L^\infty_{\eta,+}(\mathbb{R};\mathbb{C}^M)} & := & \|e^+_{-\eta}f\|_{L^\infty(\mathbb{R};\mathbb{C}^M)}, \\ \|f\|_{W^{1,\infty}_{\eta,+}(\mathbb{R};\mathbb{C}^M)} & := & \|e^+_{-\eta}f\|_{W^{1,\infty}(\mathbb{R};\mathbb{C}^M)}. \end{array} \tag{6.2.4}$$

For sufficiently small $|\eta|$ we can consider the shifted operator $\tilde{\Lambda}_{\eta,+} : W^{1,\infty}(\mathbb{R};\mathbb{C}^M) \to L^\infty(\mathbb{R};\mathbb{C}^M)$ that acts as

$$\tilde{\Lambda}_{\eta,+}x \quad = \quad e^+_\eta \Lambda e^+_{-\eta}x. \tag{6.2.5}$$

Lemma 6.2.1. *Assume that (HA), (HK) and (HH) are satisfied. Pick any $\eta \in \mathbb{R}$ with $|\eta| < \frac{\tilde{\eta}}{4}$. Writing $\tilde{\Delta}^{\pm}_{\eta,+}$ for the characteristic equations defined in (5.2.10) for the operator (6.2.5), we have the identities*

$$\tilde{\Delta}^{+}_{\eta,+}(z) \quad = \quad \Delta^{+}(z-\eta), \qquad \tilde{\Delta}^{-}_{\eta,+}(z) \quad = \quad \Delta^{-}(z). \tag{6.2.6}$$

In addition, the adjoint operator $(\tilde{\Lambda}_{\eta,+})^*$ is given by

$$(\tilde{\Lambda}_{\eta,+})^* \quad = \quad \widetilde{\Lambda^*}_{-\eta,+}. \tag{6.2.7}$$

Proof. For $j \in \mathbb{Z}$ we see that

$$e^{\eta(t^+)}e^{-\eta(t+r_j)^+} \quad = \quad e^{-\eta r_j} \tag{6.2.8}$$

for t sufficiently positive, while

$$e^{\eta(t^+)}e^{-\eta(t+r_j)^+} \quad = \quad 1 \tag{6.2.9}$$

for t sufficiently negative. Similarly for $x \in W^{1,\infty}(\mathbb{R};\mathbb{C}^M)$ we can compute

$$\big(e^{-\eta(t^+)}x(t)\big)' \quad = \quad -\eta e^{-\eta(t^+)}x(t) + e^{-\eta(t^+)}x'(t) \tag{6.2.10}$$

for t sufficiently positive, while

$$\big(e^{-\eta(t^+)}x(t)\big)' \quad = \quad x'(t) \tag{6.2.11}$$

for t sufficiently negative. Finally for $x \in W^{1,\infty}(\mathbb{R};\mathbb{C}^M)$ we see that

$$\begin{array}{lcl} e^{\eta(t^+)}\int_{\mathbb{R}} \mathcal{K}(\xi;t)e^{-\eta(\xi+t)^+}x(\xi+t)d\xi & = & e^{\eta t}\int_{-\infty}^{-t} \mathcal{K}(\xi;t)x(\xi+t)d\xi \\ & & \quad + \int_{-t}^{\infty} \mathcal{K}(\xi;t)e^{-\eta\xi}x(\xi+t)d\xi \end{array} \tag{6.2.12}$$

for t positive, while

$$\begin{array}{lcl} e^{\eta(t^+)}\int_{\mathbb{R}} \mathcal{K}(\xi;t)e^{-\eta(\xi+t)^+}x(\xi+t)d\xi & = & \int_{-\infty}^{-t} \mathcal{K}(\xi;t)x(\xi+t)d\xi \\ & & \quad + e^{-\eta t}\int_{-t}^{\infty} \mathcal{K}(\xi;t)e^{-\eta\xi}x(\xi+t)d\xi \end{array} \tag{6.2.13}$$

for t negative. These computations directly imply the identities (6.2.6).

In addition, a short computation shows that

$$\begin{array}{lcl} \langle y, \tilde{\Lambda}_{\eta,+}x\rangle_{L^2(\mathbb{R};\mathbb{C}^M)} & = & \int y(t)^\dagger e^{\eta(t^+)}(\Lambda e^+_{-\eta}x)(t)dt \\ & = & \int (e^+_{\eta}y)(t)^\dagger(\Lambda e^+_{-\eta}x)(t)dt \\ & = & \int (\Lambda^* e^+_{\eta}y)(t)^\dagger e^{-\eta(t^+)}x(t)dt \\ & = & \int (e^+_{-\eta}\Lambda^* e^+_{\eta}y)(t)^\dagger x(t)dt \\ & = & \langle \widetilde{\Lambda^*}_{-\eta,+}y, x\rangle_{L^2(\mathbb{R};\mathbb{C}^M)}, \end{array} \tag{6.2.14}$$

which implies (6.2.7), as desired. ■

Lemma 6.2.1 allows us to define the Fredholm operators $\Lambda_{(\eta)} : W^{1,\infty}_{\eta,+}(\mathbb{R};\mathbb{C}^M) \to L^{\infty}_{\eta,+}(\mathbb{R};\mathbb{C}^M)$ that act as

$$\Lambda_{(\eta),+} \quad = \quad e^{+}_{\eta} \circ \tilde{\Lambda}_{-\eta,+} \circ e^{+}_{-\eta}. \tag{6.2.15}$$

Our main result here shows that the natural adjoint $\Lambda^{*}_{(-\eta),+} : W^{1,\infty}_{-\eta,+}(\mathbb{R};\mathbb{C}^M) \to L^{\infty}_{-\eta,+}(\mathbb{R};\mathbb{C}^M)$ is given by

$$\Lambda^{*}_{(-\eta),+} \quad = \quad e^{+}_{-\eta} \circ \widetilde{\Lambda^{*}}_{\eta,+} \circ e^{+}_{\eta}. \tag{6.2.16}$$

Note that for $x \in W^{1,\infty}_{\eta,+}(\mathbb{R};\mathbb{C}^M) \cap W^{1,\infty}(\mathbb{R};\mathbb{C}^M)$ and $y \in W^{1,\infty}_{-\eta,+}(\mathbb{R};\mathbb{C}^M) \cap W^{1,\infty}(\mathbb{R};\mathbb{C}^M)$ we simply have

$$\Lambda x \quad = \quad \Lambda_{(\eta),+} x, \qquad \Lambda^{*} y \quad = \quad \Lambda^{*}_{(-\eta),+} y. \tag{6.2.17}$$

The main reasons we constructed the operators $\Lambda_{(\eta),+}$ in this fashion are that it is not a-priori clear that Λ maps $W^{1,\infty}_{\eta,+}(\mathbb{R};\mathbb{C}^M)$ into $L^{\infty}_{\eta,+}(\mathbb{R};\mathbb{C}^M)$ and whether these operators remain Fredholm operators. We note that $\Lambda_{(0),+} = \Lambda$, since we have the identities $W^{1,\infty}_{0,+}(\mathbb{R};\mathbb{C}^M) = W^{1,\infty}(\mathbb{R};\mathbb{C}^M)$ and $L^{\infty}_{0,+}(\mathbb{R};\mathbb{C}^M) = L^{\infty}(\mathbb{R};\mathbb{C}^M)$. The following result is the equivalent of Proposition 5.2.1 for the operator $\Lambda_{(\eta),+}$.

Proposition 6.2.2 (cf. [104, Prop. 3.2])**.** *Assume that (HA), (HK) and (HH) are satisfied. Pick any $\eta \in \mathbb{R}$ with $|\eta| < \frac{\tilde{\eta}}{4}$ for which the characteristic equation $\det \Delta^{+}(z) = 0$ has no roots with $\operatorname{Re} z = \eta$. Then both the operators $\Lambda_{(\eta),+} : W^{1,\infty}_{\eta,+}(\mathbb{R};\mathbb{C}^M) \to L^{\infty}_{\eta,+}(\mathbb{R};\mathbb{C}^M)$ and $\Lambda^{*}_{(-\eta),+} : W^{1,\infty}_{-\eta,+}(\mathbb{R};\mathbb{C}^M) \to L^{\infty}_{-\eta,+}(\mathbb{R};\mathbb{C}^M)$ are Fredholm operators. Moreover, the ranges admit the characterisation*

$$\begin{array}{rcl} \mathcal{R}\big(\Lambda_{(\eta),+}\big) & = & \big\{h \in L^{\infty}(\mathbb{R}) \mid \int_{-\infty}^{\infty} y(t)^{*} h(t) dt = 0 \textit{ for every } y \in \ker(\Lambda^{*}_{(-\eta),+})\big\}, \\ \mathcal{R}\big(\Lambda^{*}_{(-\eta),+}\big) & = & \big\{h \in L^{\infty}(\mathbb{R}) \mid \int_{-\infty}^{\infty} x(t)^{*} h(t) dt = 0 \textit{ for every } x \in \ker(\Lambda_{(\eta),+})\big\}. \end{array} \tag{6.2.18}$$

The Fredholm indices can be computed by

$$\operatorname{ind}(\Lambda_{(\eta),+}) \quad = \quad -\operatorname{ind}(\Lambda^{*}_{(-\eta),+}) \quad = \quad \dim\ker(\Lambda_{(\eta),+}) - \dim\ker(\Lambda_{(-\eta),+}). \tag{6.2.19}$$

Finally, there exist constants $K > 0$ and $0 < \alpha \leq \tilde{\eta}$ so that

$$|e^{+}_{-\eta} x(t)| \quad \leq \quad K e^{-\alpha|t|} \|e^{+}_{-\eta} x\|_{\infty} \tag{6.2.20}$$

holds for any $x \in \ker(\Lambda_{(\eta),+})$ and any $t \in \mathbb{R}$, while the bound

$$|e^{+}_{\eta} x(t)| \quad \leq \quad K e^{-\alpha|t|} \|e^{+}_{\eta} x\|_{\infty} \tag{6.2.21}$$

holds for any $x \in \ker(\Lambda^{*}_{(-\eta),+})$ and any $t \in \mathbb{R}$.

Proof. These results follow from Proposition 5.2.1 and Lemma 6.2.1, together with the identities

$$\begin{array}{lllll} \ker\big(\Lambda_{(\eta),+}\big) & = & e^{+}_{\eta}\ker\big(\tilde{\Lambda}_{-\eta,+}\big), & & \\ \ker\big(\Lambda^{*}_{(-\eta),+}\big) & = & e^{+}_{-\eta}\ker\big(\widetilde{\Lambda^{*}}_{\eta,+}\big) & = & e^{+}_{-\eta}\ker\big((\tilde{\Lambda}_{-\eta,+})^{*}\big), \\ \mathrm{Range}\big(\Lambda_{(\eta),+}\big) & = & e^{+}_{\eta}\mathrm{Range}\big(\tilde{\Lambda}_{-\eta,+}\big), & & \\ \mathrm{Range}\big(\Lambda^{*}_{(-\eta),+}\big) & = & e^{+}_{-\eta}\mathrm{Range}\big(\widetilde{\Lambda^{*}}_{\eta,+}\big) & = & e^{+}_{-\eta}\mathrm{Range}\big((\tilde{\Lambda}_{-\eta,+})^{*}\big). \end{array} \tag{6.2.22}$$

∎

We now shift our attention to the parameter-dependent system (6.1.2). The following result shows that we can find a quasi-inverse for this system that depends smoothly on μ.

Proposition 6.2.3 (cf. [104, Prop. 3.3])**.** *Assume that (HC) is satisfied. Pick any $\eta \in \mathbb{R}$ with $|\eta| < \frac{\tilde{\eta}}{4}$ for which the characteristic equation $\det \Delta^{+}(z) = 0$ for $\mu = \mu_0$ has no roots with $\mathrm{Re}\, z = \eta$. Write $\mathcal{R} = \mathrm{Range}\big(\Lambda_{(\eta),+}(\mu_0)\big)$ and pick a complement $\mathcal{R}^{\perp}$ for $\mathcal{R}$ in $L^{\infty}_{\eta,+}(\mathbb{R};\mathbb{C}^M)$. Then there exists an open neighbourhood $\mu_0 \in U' \subset U$, together with a C^k-smooth function*

$$\mathcal{C}_{(\eta),+} : U' \;\to\; \mathcal{L}\big(L^{\infty}_{\eta,+}(\mathbb{R};\mathbb{C}^M), \mathcal{R}^{\perp}\big) \tag{6.2.23}$$

and a C^k-smooth quasi-inverse

$$\Lambda^{\mathrm{qinv}}_{(\eta),+} : U' \;\to\; \mathcal{L}\big(L^{\infty}_{\eta,+}(\mathbb{R};\mathbb{C}^M), W^{1,\infty}_{\eta,+}(\mathbb{R};\mathbb{C}^M)\big) \tag{6.2.24}$$

that satisfy the following properties.

(i) For any $\mu \in U'$ we have the upper bound

$$\dim\Big(\ker\big(\Lambda_{(\eta),+}(\mu)\big)\Big) \;\le\; \dim\Big(\ker\big(\Lambda_{(\eta),+}(\mu_0)\big)\Big). \tag{6.2.25}$$

(ii) For any $\mu \in U'$ and any $f \in L^{\infty}(\mathbb{R};\mathbb{C}^M)$ we have the identity

$$\Lambda_{(\eta),+}(\mu)\Lambda^{\mathrm{qinv}}_{(\eta),+}(\mu) f \;=\; f + \mathcal{C}_{(\eta),+}(\mu) f. \tag{6.2.26}$$

Moreover, the restriction of the map $\mathcal{C}_{(\eta),+}(\mu_0)$ to $\mathcal{R}$ vanishes identically.

Proof. Upon choosing

$$\begin{array}{lll} \Lambda^{\mathrm{qinv}}_{(\eta),+}(\mu) f & = & \big[\pi_{\mathcal{R}}\Lambda_{(\eta),+}(\mu)\big]^{-1}\pi_{\mathcal{R}} f, \\ \mathcal{C}_{(\eta),+}(\mu) f & = & -\pi_{\mathcal{R}^{\perp}} f + \pi_{\mathcal{R}^{\perp}}\Lambda_{(\eta),+}(\mu)\Lambda^{\mathrm{qinv}}_{(\eta),+}(\mu) f, \end{array} \tag{6.2.27}$$

we can directly follow the proof of [104, Prop. 3.3] to arrive at the desired result. ∎

In a similar fashion, we introduce the function

$$[e_\eta^- f](x) \quad = \quad e^{\eta(x^-)} f(x), \tag{6.2.28}$$

where

$$x^- \quad = \quad \begin{cases} |x|, & x \le 0, \\ 0, & x > 0, \end{cases} \tag{6.2.29}$$

together with the spaces

$$\begin{array}{rcl} L^\infty_{\eta,-}(\mathbb{R};\mathbb{C}^M) & = & \{f \in L^1_{\mathrm{loc}}(\mathbb{R};\mathbb{C}^M) \,|\, e^-_{-\eta} f \in L^\infty(\mathbb{R};\mathbb{C}^M)\}, \\ W^{1,\infty}_{\eta,-}(\mathbb{R};\mathbb{C}^M) & = & \{f \in L^1_{\mathrm{loc}}(\mathbb{R};\mathbb{C}^M) \,|\, e^-_{-\eta} f \in W^{1,\infty}(\mathbb{R};\mathbb{C}^M)\}, \end{array} \tag{6.2.30}$$

with the corresponding norms

$$\begin{array}{rcl} \|f\|_{L^\infty_{\eta,+}(\mathbb{R};\mathbb{C}^M)} & := & \|e^+_{-\eta} f\|_{L^\infty(\mathbb{R};\mathbb{C}^M)}, \\ \|f\|_{W^{1,\infty}_{\eta,+}(\mathbb{R};\mathbb{C}^M)} & := & \|e^+_{-\eta} f\|_{W^{1,\infty}(\mathbb{R};\mathbb{C}^M)}. \end{array} \tag{6.2.31}$$

For sufficiently small $|\eta|$ we can consider the shifted operator $\tilde{\Lambda}_{\eta,-} : W^{1,\infty}(\mathbb{R};\mathbb{C}^M) \to L^\infty(\mathbb{R};\mathbb{C}^M)$ which acts as

$$\tilde{\Lambda}_{\eta,-} x \quad = \quad e^-_\eta \Lambda e^-_{-\eta} x \tag{6.2.32}$$

and we can define the Fredholm operators $\Lambda_{(\eta),-} : W^{1,\infty}_{\eta,-}(\mathbb{R};\mathbb{C}^M) \to L^\infty_{\eta,-}(\mathbb{R};\mathbb{C}^M)$ by

$$\Lambda_{(\eta),-} \quad = \quad e^-_\eta \circ \tilde{\Lambda}_{-\eta,-} \circ e^-_{-\eta}. \tag{6.2.33}$$

Remark 6.2.4. The equivalent statements in Propositions 6.2.2-6.2.3 can be proven for the operator $\Lambda_{(\eta),-}$ under the assumption that the characteristic equation $\det \Delta^-(z) = 0$ has no roots with $\operatorname{Re} z = -\eta$, instead of the condition on Δ^+.

For notational simplicity, we use the shorthand

$$\Lambda^{\mathrm{qinv}}(\mu) \quad := \quad \Lambda^{\mathrm{qinv}}_{(0),+}(\mu) \quad = \quad \Lambda^{\mathrm{qinv}}_{(0),-}(\mu). \tag{6.2.34}$$

The half-line inverses from Lemma 5.5.6 can also be chosen to depend smoothly on the parameter μ. We recall that the intervals $D^\oplus_\tau$ and $D^\ominus_\tau$ were defined in (5.2.32), while the interval D_X was defined in (5.2.22).

Lemma 6.2.5 (cf. [104, Pg. 13])**.** *Assume that (HC) is satisfied. Recall the open neighbourhood U' of μ_0 from Proposition 6.2.3 and fix $\tau \in \mathbb{R}$. Then there exist bounded linear operators*

$$\begin{array}{rcl} \Lambda^{-1}_{+;\tau}(\mu) : L^\infty\big([\tau,\infty);\mathbb{C}^M\big) & \to & W^{1,\infty}(D^\oplus_\tau;\mathbb{C}^M), \\ \Lambda^{-1}_{-;\tau}(\mu) : L^\infty\big((-\infty,\tau];\mathbb{C}^M\big) & \to & W^{1,\infty}(D^\ominus_\tau;\mathbb{C}^M), \end{array} \tag{6.2.35}$$

defined for $\mu \in U'$, in such a way that the identities

$$\begin{array}{rcl} [\Lambda(\mu)\Lambda^{-1}_{+;\tau}(\mu) f](t) & = & f(t), \;\; t \ge \tau, \\ [\Lambda(\mu)\Lambda^{-1}_{-;\tau}(\mu) g](s) & = & g(s), \;\; s \le \tau \end{array} \tag{6.2.36}$$

hold for $f \in L^\infty\big([\tau,\infty);\mathbb{C}^M\big)$ *and* $g \in L^\infty\big((-\infty,\tau];\mathbb{C}^M\big)$. *The operators* $\Lambda_{\pm;\tau}$ *depend* C^k*-smoothly on the parameter* μ.

In addition, if $\tau > 0$ *is sufficiently large, there exists bounded linear operators*

$$\Lambda^{-1}_{\diamond;\tau}(\mu) : L^\infty\big([0,\tau];\mathbb{C}^M\big) \quad\to\quad W^{1,\infty}\big(D_X+\tau;\mathbb{C}^M\big), \tag{6.2.37}$$

defined for $\mu \in U'$, *in such a way that the identity*

$$[\Lambda(\mu)\Lambda^{-1}_{\diamond;\tau}(\mu)f](t) \quad=\quad f(t),\ \ t\in[0,\tau] \tag{6.2.38}$$

holds for $f \in L^\infty\big([0,\tau];\mathbb{C}^M\big)$. *The operators* $\Lambda_{\diamond;\tau}$ *depend* C^k*-smoothly on the parameter* μ.

Proof. Using the quasi-inverse $\Lambda^{\mathrm{qinv}}(\mu)$ instead of the inverse Λ^{-1}, the proof of Lemma 5.5.6 carries over to the current setting. ∎

6.3 Construction of exponential splittings

In this section, we set out to prove Theorem 6.1.1. For $\tau \geq 0$ and $\mu \in U$ we write $Q(\tau,\mu)$ for the space $Q(\tau)$ from Theorem 5.2.8 at this value of μ. In addition, we write $Q(\tau) := Q(\tau,\mu_0)$. Moreover, we introduce, for notational clarity, the evaluation operator ev_t given by

$$\mathrm{ev}_t\phi \quad=\quad \phi_t. \tag{6.3.1}$$

We will be mainly working in the spaces

$$\begin{aligned} BC^{\oplus}_{\tau,\eta} \quad&=\quad \big\{f \in C_b\big(D^{\oplus}_\tau,\mathbb{C}^M\big) \,|\, e^+_{-\eta}f \in C_b\big(D^{\oplus}_\tau,\mathbb{C}^M\big)\big\},\\ BC^{\ominus}_{\tau,\eta} \quad&=\quad \big\{f \in C_b\big(D^{\ominus}_\tau,\mathbb{C}^M\big) \,|\, e^-_{-\eta}f \in C_b\big(D^{\ominus}_\tau,\mathbb{C}^M\big)\big\} \end{aligned} \tag{6.3.2}$$

for $\tau \geq 0$ and $\eta \in \mathbb{R}$, with the corresponding norms

$$\|f\|_{BC^{\oplus}_{\tau,\eta}} \quad=\quad \|e^+_{-\eta}f\|_\infty, \qquad \|f\|_{BC^{\ominus}_{\tau,\eta}} \quad=\quad \|e^-_{-\eta}f\|_\infty. \tag{6.3.3}$$

This choice of spaces is in essential in our analysis and in major contrast to the finite-range setting in [104]. Indeed, there the authors consider weighted spaces, defined on the interval $D^{\oplus}_\tau$, where the weight decays exponentially in positive direction, while it grows exponentially in the direction of $r_{\min}+\tau$. An essential step in the analysis is that the inclusion of the space $Q(\tau)$ into the exponentially weighted space is a bounded linear operator. However, this is the case if and only if $r_{\min} > -\infty$. By contrast, the inclusion of $Q(\tau)$ into the space $BC^{\oplus}_{\tau,\eta}$ is bounded for $\eta < 0$ sufficiently close to 0.

The key ingredients to establish Theorem 6.1.1 are the following two results that we establish in the sequel. Basically, they state that $Q(\tau,\mu)$ and $R(\tau,\mu)$ can be constructed as a graph over $Q(\tau,\mu_0)$ and $R(\tau,\mu_0)$. For $\psi \in Q(\tau,\mu)$, we write $E_{\tau,\mu}\psi$ for the extension of the function ψ. That is, $E_{\tau,\mu}\psi$ is a solution of (6.1.2) on the interval $[\tau,\infty)$.

Proposition 6.3.1 (cf. [104, Lem. 5.2]). *Assume that (HC) is satisfied. Consider the splitting $X = Q(\tau) \oplus R(\tau)$ for $\tau \geq 0$ for the system (6.1.2) at $\mu = \mu_0$. Then there exists an open neighbourhood $\mu_0 \in U' \subset U$, together with C^k-smooth functions $u^*_{Q(\tau)} : U' \to \mathcal{L}(Q(\tau), X)$, defined for $\tau \geq 0$, that satisfy the following properties.*

(i) For each $\mu \in U'$ we have the identity

$$\Pi_{Q(\tau)} u^*_{Q(\tau)}(\mu) \quad = \quad I \tag{6.3.4}$$

and the limit

$$\lim_{\mu \to \mu_0} [I - \Pi_{Q(\tau)}] u^*_{Q(\tau)}(\mu) \quad = \quad 0, \tag{6.3.5}$$

holds uniformly for $\tau \geq 0$.

(ii) For $\mu \in U'$ the operator norms of the maps $u^*_{Q(\tau)}(\mu)$ are bounded uniformly for $\tau \geq 0$.

(iii) For $\mu \in U'$ we have $Q(\tau; \mu) = \mathrm{Range}\big(u^*_{Q(\tau)}(\mu)\big)$.

(iv) There exist constants $K > 0$ and $\alpha > 0$ in so that the bound

$$\big|D^\ell_\mu E_{\tau,\mu} u^*_{Q(\tau)}(\mu)\phi\big|(t) \quad \leq \quad K e^{-\alpha|t-\tau|} \|\phi\|_\infty \tag{6.3.6}$$

holds for each $\mu \in U'$, each $0 \leq \tau \leq t$, each $\phi \in Q(\tau)$ and each integer $0 \leq \ell \leq k$.

Recall that the space $R(\tau, \mu_0)$ is constructed as a finite-dimensional enlargement of the space $\tilde{P}(\tau, \mu_0)$. However, it is unclear whether this finite-dimensional space can be constructed in such a way that it depends smoothly on the parameter μ. As such, we simply construct the space $R(\tau, \mu)$ in a fashion similar to Proposition 6.3.1 and treat this as its definition. The price we have to pay is that this space is no longer invariant.

Proposition 6.3.2 (cf. [104, Lem. 5.3]). *Assume that (HC) is satisfied. Consider the splitting $X = Q(\tau) \oplus R(\tau)$ for $\tau \geq 0$ for the system (6.1.2) at $\mu = \mu_0$. Then there exists an open neighbourhood $\mu_0 \in U' \subset U$, together with C^k-smooth functions $u^*_{R(\tau)} : U' \to \mathcal{L}(R(\tau), X)$, defined for $\tau \geq 0$, that satisfy the following properties.*

(i) For each $\mu \in U'$ we have the identity

$$\Pi_{R(\tau)} u^*_{R(\tau)}(\mu) \quad = \quad I \tag{6.3.7}$$

and the limit

$$\lim_{\mu \to \mu_0} [I - \Pi_{R(\tau)}] u^*_{R(\tau)}(\mu) \quad = \quad 0, \tag{6.3.8}$$

holds uniformly for $\tau \geq 0$.

(ii) For $\mu \in U'$ we have that the operator norms of the maps $u^*_{R(\tau)}(\mu)$ are bounded uniformly for $\tau \geq 0$.

(iii) Writing $R(\tau; \mu) = \mathrm{Range}\big(u^*_{R(\tau)}(\mu)\big)$, each $\psi \in R(\tau; \mu)$ extends to a solution $E_{\tau,\mu}\psi$ of (6.1.2) on the interval $(-\infty, -r_0] \cup [0, \tau]$. In addition, the space $R(\tau; \mu) \subset X$ is closed.

(iv) There exist constants $K > 0$ and $\alpha > 0$ in such a way that we have the bound

$$|D^{\ell}_{\mu} E_{\tau,\mu} u^*_{R(\tau)}(\mu)\phi|(t) \quad \leq \quad K e^{-\alpha|t-\tau|} \|\phi\|_{\infty} \tag{6.3.9}$$

for each $\mu \in U'$, each $t \leq \tau$, each $\phi \in R(\tau)$ and each integer $0 \leq \ell \leq k$.

(v) We have the uniform bound

$$\big|\Lambda(\mu) E_{\tau,\mu} u^*_{R(\tau)}(\mu)\phi\big|(t) \quad \leq \quad K e^{-\alpha|t-\tau|} \|\phi\|_{\infty} \tag{6.3.10}$$

for each $\mu \in U'$, each $t \in [-r_0, 0]$ and each $\phi \in R(\tau)$.

Proof of Theorem 6.1.1. On account of Propositions 6.3.1 and 6.3.2 we can repeat the arguments used in the proof of [104, Thm. 5.1] to arrive at the desired result. ∎

For any $\tau \geq 0$ and $\eta > 0$, we introduce the map $\mathcal{G}_{\tau;\eta} : U \to \mathcal{L}\big(BC^{\oplus}_{\tau,-\eta}\big)$, defined by

$$\mathcal{G}_{\tau;\eta}(\mu)u \quad = \quad \Lambda^{\mathrm{qinv}}_{(-\eta),+}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u - \iota_{\tau;\eta}\Pi_{Q(\tau)}\mathrm{ev}_0\Lambda^{\mathrm{qinv}}_{(-\eta),+}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u. \tag{6.3.11}$$

Here we introduced the notation

$$[L(\mu)u](t) \quad = \quad L(t,\mu)u_t, \tag{6.3.12}$$

together with the map $\iota_{\tau;\eta}$ which is the inclusion from $Q(\tau)$ into $BC^{\oplus}_{\tau,-\eta}$ for $\tau \geq 0$.

The proof of Proposition 6.3.1 consists of a number of steps. We start by showing that the map $\mathcal{G}_{\tau,\alpha}$ from (6.3.11) is well-defined and bounded for some specified $\alpha > 0$. Then we use this map $\mathcal{G}_{\tau,\alpha}$ to construct the functions $u^*_{Q(\tau)}$. Most of our focus will go to the identity $Q(\tau;\mu) = \mathrm{Range}\big(u^*_{Q(\tau)}(\mu)\big)$, since the other bounds and identities follow relatively quickly from the definition.

Lemma 6.3.3. *Consider the setting of Proposition 6.3.1 and suppose that $r_{\min} = -\infty$. Then there exists a constant $\alpha > 0$ so that the map*

$$\mathcal{G}_{\tau} \quad := \quad \mathcal{G}_{\tau;\alpha} \tag{6.3.13}$$

is a well-defined map $\mathcal{G}_{\tau} : U \to \mathcal{L}\big(BC^{\oplus}_{\tau,-\alpha}\big)$. In addition, there exists an open neighbourhood $\mu_0 \in U' \subset U$, together with a constant $C > 0$, so that for all $\mu \in U'$ we have the uniform bounds

$$\|\mathcal{G}_{\tau}(\mu)\| \quad \leq \quad \tfrac{1}{2}, \qquad \|D^{\ell}_{\mu}\mathcal{G}_{\tau}(\mu)\| \quad \leq \quad C \tag{6.3.14}$$

for all $\tau \geq 0$ and all integers $1 \leq \ell \leq k$.

Proof. We let $K \geq 1$ and $0 < \alpha < \tilde{\eta}$ be the constants from Theorem 5.2.8 applied to the system (6.1.2) at $\mu = \mu_0$. Without loss of generality we can assume that α is so small that the characteristic equation $\det \Delta^{+}(z)$ for $\mu = \mu_0$ has no roots with $\mathrm{Re}\, z = -\alpha$, which allows us to consider the quasi-inverse $\Lambda^{\mathrm{qinv}}_{(-\alpha),+}$ from Proposition 6.2.3. We also can assume without loss of generality that $e^{\pm}_{2\alpha} b \in W^{1,\infty}(\mathbb{R};\mathbb{C}^M)$ for any

$b \in \mathcal{B} \cup \mathcal{B}^*$.

We start by showing that the map $\mathcal{G}_\tau := \mathcal{G}_{\tau;\alpha}$ is well-defined by showing that the inclusion map $\iota_{\tau;\alpha}$ and the evaluation operator ev_0 map $Q(\tau)$ into $BC^{\oplus}_{\tau,-\alpha}$ and $BC^{\oplus}_{\tau,-\alpha}$ into X respectively.

On account of Theorem 5.2.4, the map

$$\iota_\tau \quad := \quad \iota_{\tau;\alpha} \tag{6.3.15}$$

is a well-defined and bounded map $\iota_\tau : Q(\tau) \to BC^{\oplus}_{\tau,-\alpha}$, since we assumed that $r_{\min} = -\infty$. In addition, we have the bound

$$\|\iota_\tau \phi\|_{BC^{\oplus}_{\tau,-\alpha}} \quad \leq \quad K_{\mathrm{dec}} \|\phi\|_\infty \tag{6.3.16}$$

for $\phi \in Q(\tau)$.

Let $\phi \in BC^{\oplus}_{\tau,-\alpha}$ be given. Then we obtain the pointwise estimate

$$|(\mathrm{ev}_0 \phi)(t)| \quad = \quad e^{-\alpha(t^+)} |e^{\alpha(t^+)} \phi(t)| \quad \leq \quad e^{-\alpha t} \|\phi\|_{BC^{\oplus}_{\tau,-\alpha}} \tag{6.3.17}$$

for any $t \in D^+_X$, while

$$|(\mathrm{ev}_0 \phi)(t)| \quad = \quad |e^{\alpha(t^+)} \phi(t)| \quad \leq \quad \|\phi\|_{BC^{\oplus}_{\tau,-\alpha}} \tag{6.3.18}$$

for $t \in D^-_X$.

Hence, the norms of the operators ev_0 and ι_τ are bounded by 1 and K_{dec} respectively. In addition, the projections $\Pi_{Q(\tau)}$ are uniformly bounded in norm on account of Theorem 5.2.8. Since the map $\mu \mapsto L(\mu)$ is C^k-smooth, we see that $\mathcal{G}_\tau$ is smooth as a map from U into $\mathcal{L}\big(BC^{\oplus}_{\tau,-\alpha}\big)$. The uniform bounds on the operators ι_τ, $\Pi_{Q(\tau)}$ and ev_0 now yield the uniform bound (6.3.14) for $\tau \geq 0$, integers $1 \leq \ell \leq k$ and μ sufficiently close to μ_0. ∎

In particular, we can define the bounded linear maps

$$\begin{array}{rcl} v^*_{Q(\tau)}(\mu) : Q(\tau) & \to & BC^{\oplus}_{\tau,-\alpha}, \\ \phi & \mapsto & \big[I - \mathcal{G}_\tau(\mu)\big]^{-1} \iota_\tau \phi, \end{array} \tag{6.3.19}$$

together with

$$u^*_{Q(\tau)}(\mu) \quad = \quad \mathrm{ev}_0 v^*_{Q(\tau)}(\mu). \tag{6.3.20}$$

Lemma 6.3.4. *Consider the setting of Lemma 6.3.3. Then the functions $u^*_{Q(\tau)}(\mu)$ defined in (6.3.20) satisfy items (ii) and (iv) of Proposition 6.1.4.*

Proof. The uniform bound on the operator norm of $u^*_{Q(\tau)}(\mu)$ and the exponential estimate (6.3.6) follow directly from the definition (6.3.20), together with the uniform bounds (6.3.14) and (6.3.16). ∎

Lemma 6.3.5. *Consider the setting of Lemma 6.3.3. Then we have the identity (6.3.4) and the limit (6.3.5) holds uniformly for* $\tau \geq 0$.

Proof. Pick any $\tau \geq 0$ and $u \in BC^{\oplus}_{\tau,-\alpha}$. Then we can compute

$$\begin{array}{lcl} \iota_\tau \Pi_{Q(\tau)} \mathrm{ev}_0 \iota_\tau \Pi_{Q(\tau)} \mathrm{ev}_0 u & = & \iota_\tau \Pi_{Q(\tau)} \Pi_{Q(\tau)} \mathrm{ev}_0 u \\ & = & \iota_\tau \Pi_{Q(\tau)} \mathrm{ev}_0 u. \end{array} \tag{6.3.21}$$

In particular, we see from the definition (6.3.11) that

$$\iota_\tau \Pi_{Q(\tau)} \mathrm{ev}_0 \mathcal{G}_\tau(\mu) \quad = \quad 0. \tag{6.3.22}$$

This implies

$$\Pi_{Q(\tau)} \mathrm{ev}_0 \mathcal{G}_\tau(\mu) \quad = \quad 0, \tag{6.3.23}$$

which yields

$$\pi_{Q(\tau)} u^*_{Q(\tau)}(\mu) \quad = \quad \Pi_{Q(\tau)} \mathrm{ev}_0 \big[I - \mathcal{G}_\tau(\mu) \big]^{-1} \iota_\tau \quad = \quad I, \tag{6.3.24}$$

as desired. The remainder term (6.3.5) can be bounded by considering the identity

$$\big[I - \Pi_{Q(\tau)} \big] u^*_{Q(\tau)}(\mu) \quad = \quad \mathrm{ev}_0 \Big[\big[I - \mathcal{G}_\tau(\mu) \big]^{-1} - I \Big] \iota_\tau, \tag{6.3.25}$$

which approaches 0 as $\mu \to \mu_0$, uniformly for $\tau \geq 0$. ∎

We now set out to show that $\mathrm{Range}\big(u^*_{Q(\tau)}(\mu)\big) = Q(\tau, \mu)$. The "$\subset$"-embedding can be established by a relatively direct calculation. The "$\supset$"-embedding follows from the property (6.3.14) for $\mathcal{G}_\tau$.

Lemma 6.3.6. *Consider the setting of Lemma 6.3.3. Then we have the inclusion* $\mathrm{Range}\big(u^*_{Q(\tau)}(\mu)\big) \subset Q(\tau, \mu)$.

Proof. Similarly to (5.5.26), we pick a basis for $\mathrm{Range}\big(\Lambda_{(-\alpha),+}(\mu_0)\big)^\perp$ that consists of continuous functions for which the support is contained in the interval $[-r_0, 0]$. We recall the C^k-smooth operator

$$\mathcal{C}_{(-\alpha),+} : U' \quad \to \quad \mathcal{L}\Big(L^\infty_{(-\alpha),+}(\mathbb{R}; \mathbb{C}^M), \mathrm{Range}\big(\Lambda_{(-\alpha),+}(\mu_0)\big)^\perp \Big) \tag{6.3.26}$$

from Proposition 6.2.3. Recall that α was chosen small enough to have $e^{\pm}_{2\alpha} b \in W^{1,\infty}(\mathbb{R}; \mathbb{C}^M)$ for any $b \in \mathcal{B} \cup \mathcal{B}^*$. Since $\alpha > 0$, we have $L^\infty_{(-\alpha),+}(\mathbb{R}; \mathbb{C}^M) \subset L^\infty(\mathbb{R}; \mathbb{C}^M)$. As such, we have $\Lambda(\mu)x = \Lambda_{(-\alpha),+}(\mu)x$ for any $x \in W^{1,\infty}_{(-\alpha),+}(\mathbb{R}; \mathbb{C}^M)$ and any $\mu \in U'$ on account of (6.2.17). Pick $\phi \in Q(\tau)$ and write

$$u(t) \quad = \quad [v^*_{Q(\tau)}(\mu)\phi](t - \tau), \tag{6.3.27}$$

so that

$$\mathrm{ev}_\tau u \quad = \quad u^*_{Q(\tau)}(\mu)\phi. \tag{6.3.28}$$

Writing

$$u_\tau(t) \quad = \quad u(t+\tau), \tag{6.3.29}$$

we can compute

$$u(t) \quad = \quad [\iota_\tau \phi](t-\tau) + [\mathcal{G}_\tau(\mu) u_\tau](t-\tau) \tag{6.3.30}$$

for $t \in \mathbb{R}$, so that

$$[\Lambda(\mu)u](t) \quad = \quad \big[\Lambda(\mu)\iota_\tau\phi(\cdot - \tau)\big](t) + \big[\Lambda(\mu)\mathcal{G}_\tau(\mu)u_\tau(\cdot - \tau)\big](t). \tag{6.3.31}$$

For $t \in \mathbb{R}$ we can now compute

$$[\Lambda(\mu)\iota_\tau\phi(\cdot - \tau)](t) \quad = \quad \Big[\big[L(\mu_0) - L(\mu)\big]\iota_\tau\phi(\cdot - \tau)\Big](t) + \big[\Lambda(\mu_0)\iota_\tau\phi(\cdot - \tau)\big](t), \tag{6.3.32}$$

together with

$$\begin{array}{rcl} L & := & \big[\Lambda(\mu)\mathcal{G}_\tau(\mu)u_\tau(\cdot - \tau)\big](t) \\ & = & \Big[\big[L(\mu_0) - L(\mu)\big]\mathcal{G}_\tau(\mu)u_\tau(\cdot - \tau)\Big](t) + \Big[\Lambda(\mu_0)[\mathcal{G}_\tau(\mu)u_\tau](\cdot - \tau)\Big](t) \\ & = & \Big[\big[L(\mu_0) - L(\mu)\big]\big[\mathcal{G}_\tau(\mu)[I - \mathcal{G}_\tau(\mu)]^{-1}\iota_\tau\phi(\cdot - \tau)\big]\Big](t) \\ & & \quad +\Big[\Lambda(\mu_0)\Lambda^{\mathrm{qinv}}_{(-\alpha),+}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u_\tau(\cdot - \tau)\Big](t) \\ & & \quad -\Big[\Lambda(\mu_0)\iota_\tau\Pi_{Q(\tau)}\mathrm{ev}_0\Lambda^{\mathrm{qinv}}_{(-\alpha),+}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u_\tau(\cdot - \tau)\Big](t) \\ & := & L_1 + L_2 + L_3. \end{array} \tag{6.3.33}$$

We can compute

$$\begin{array}{rcl} L_1 & = & \Big[\big[L(\mu_0) - L(\mu)\big]\big[\mathcal{G}_\tau(\mu)[I - \mathcal{G}_\tau(\mu)]^{-1}\iota_\tau\phi(\cdot - \tau)\big]\Big](t) \\ & = & -\Big[\big[L(\mu_0) - L(\mu)\big]\iota_\tau\phi(\cdot - \tau)\Big](t) \\ & & \quad +\Big[\big[L(\mu_0) - L(\mu)\big]\big[[I - \mathcal{G}_\tau(\mu)]^{-1}\iota_\tau\phi(\cdot - \tau)\big]\Big](t) \\ & = & -\Big[\big[L(\mu_0) - L(\mu)\big]\iota_\tau\phi(\cdot - \tau)\Big](t) + \Big[\big[L(\mu_0) - L(\mu)\big]u_\tau(\cdot - \tau)\Big](t). \end{array} \tag{6.3.34}$$

Moreover, an application of Proposition 6.2.3 yields

$$\begin{array}{rcl} L_2 & = & \Big[\Lambda(\mu_0)\Lambda^{\mathrm{qinv}}_{(-\alpha),+}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u_\tau(\cdot - \tau)\Big](t) \\ & = & \Big[\big[L(\mu) - L(\mu_0)\big]u_\tau(\cdot - \tau)\Big](t) + \Big[\mathcal{C}_{(-\alpha),+}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u_\tau(\cdot - \tau)\Big](t). \end{array} \tag{6.3.35}$$

Combining (6.3.31), (6.3.32), (6.3.34) and (6.3.35), we obtain

$$
\begin{array}{rcl}
[\Lambda(\mu)u](t) & = & \Big[\big[L(\mu_0) - L(\mu)\big]\iota_\tau\phi(\cdot - \tau)\Big](t) + \big[\Lambda(\mu_0)\iota_\tau\phi(\cdot - \tau)\big](t) \\
& & -\Big[\big[L(\mu_0) - L(\mu)\big]\iota_\tau\phi(\cdot - \tau)\Big](t) + \Big[\big[L(\mu_0) - L(\mu)\big]u_\tau(\cdot - \tau)\Big](t) \\
& & +\Big[\big[L(\mu) - L(\mu_0)\big]u_\tau(\cdot - \tau)\Big](t) \\
& & +\Big[\mathcal{C}_{(-\alpha),+}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u_\tau(\cdot - \tau)\Big](t) \\
& & -\Big[\Lambda(\mu_0)\iota_\tau\Pi_{Q(\tau)}\mathrm{ev}_0\Lambda^{\mathrm{qinv}}_{(-\alpha),+}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u_\tau(\cdot - \tau)\Big](t) \\
& = & \big[\Lambda(\mu_0)\iota_\tau\phi(\cdot - \tau)\big](t) + \Big[\mathcal{C}_{(-\alpha),+}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u_\tau(\cdot - \tau)\Big](t) \\
& & -\Big[\Lambda(\mu_0)\iota_\tau\Pi_{Q(\tau)}\mathrm{ev}_0\Lambda^{\mathrm{qinv}}_{(-\alpha),+}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u_\tau(\cdot - \tau)\Big](t),
\end{array}
\tag{6.3.36}
$$

for any $t \in \mathbb{R}$. For $t \geq \tau$ we obtain

$$\big[\Lambda(\mu_0)\iota_\tau\phi(\cdot - \tau)\big](t) \quad = \quad 0, \tag{6.3.37}$$

since $\phi \in Q(\tau)$. In addition, we recall that we chose $\mathcal{C}_{(-\alpha),+}(\mu_0)v(s)$ to be identically zero for $s \geq 0$. Finally, for $t \geq \tau$ we obtain

$$\Big[\Lambda(\mu_0)\iota_\tau\Pi_{Q(\tau)}\mathrm{ev}_0\Lambda^{\mathrm{qinv}}_{(-\alpha),+}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u_\tau(\cdot - \tau)\Big](t) \quad = \quad 0 \tag{6.3.38}$$

by definition of $Q(\tau)$. Hence we must have

$$
\begin{array}{rcl}
[\Lambda(\mu)u](t) & = & \big[\Lambda(\mu_0)\iota_\tau\phi(\cdot - \tau)\big](t) + \Big[\mathcal{C}_{(-\alpha),+}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u_\tau(\cdot - \tau)\Big](t) \\
& & -\Big[\Lambda(\mu_0)\iota_\tau\Pi_{Q(\tau)}\mathrm{ev}_0\Lambda^{\mathrm{qinv}}_{(-\alpha),+}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u_\tau(\cdot - \tau)\Big](t) \\
& = & 0
\end{array}
\tag{6.3.39}
$$

for any $t \geq \tau$. In particular, we get $u \in \mathcal{Q}(\tau, \mu)$ and thus $u^*_{Q(\tau)}(\mu)\phi \in Q(\tau, \mu)$, as desired. ∎

Lemma 6.3.7. *Consider the setting of Lemma 6.3.3. Then we have the inclusion* $\mathrm{Range}\big(u^*_{Q(\tau)}(\mu)\big) \supset Q(\tau, \mu)$.

Proof. We pick $q^1_\mu \in \mathcal{Q}(\tau, \mu)$ and write

$$
\begin{array}{rcl}
\phi & = & \Pi_{Q(\tau)}\mathrm{ev}_0 q^1_\mu, \\
q^2_\mu(t) & = & [v^*_{Q(\tau)}(\mu)\phi](t - \tau).
\end{array}
\tag{6.3.40}
$$

By Lemma 6.3.6, we see that $q^2_\mu \in \mathcal{Q}(\tau, \mu)$ and therefore also $q_\mu := q^1_\mu - q^2_\mu \in \mathcal{Q}(\tau, \mu)$. Moreover, we can compute

$$
\begin{array}{rcl}
\Pi_{Q(\tau)}\mathrm{ev}_0 q_\mu & = & \Pi_{Q(\tau)}\mathrm{ev}_0 q^1_\mu - \Pi_{Q(\tau)} u^*_{Q(\tau)}(\mu)\phi \\
& = & \phi - \phi \\
& = & 0
\end{array}
\tag{6.3.41}
$$

using (6.3.4). Upon setting

$$q_{\mu_0} \quad = \quad \Lambda^{\mathrm{qinv}}_{(-\alpha),+}(\mu_0)\big[L(\mu) - L(\mu_0)\big]q_\mu - q_\mu, \tag{6.3.42}$$

we note that

$$\begin{array}{lcl} \Lambda(\mu_0)q_{\mu_0} & = & \big[L(\mu) - L(\mu_0)\big]q_\mu + \mathcal{C}_{(-\alpha),+}(\mu_0)\big[L(\mu) - L(\mu_0)\big]q_\mu - \Lambda(\mu_0)q_\mu \\ & = & -\Lambda(\mu)q_\mu + \mathcal{C}_{(-\alpha),+}(\mu_0)\big[L(\mu) - L(\mu_0)\big]q_\mu, \end{array} \tag{6.3.43}$$

since $L(\mu) - L(\mu_0) - \Lambda(\mu_0) = -\Lambda(\mu)$. In particular, we see that the right-hand side of (6.3.43) is zero on the halfline $[\tau, \infty)$, so we must have $q_{\mu_0} \in \mathcal{Q}(\tau)$ and hence

$$\begin{array}{lcl} \mathcal{G}_\tau(\mu)q_\mu & = & q_\mu + q_{\mu_0} - \iota_\tau \Pi_{Q(\tau)}\mathrm{ev}_0[q_\mu + q_{\mu_0}] \\ & = & q_\mu + q_{\mu_0} - q_{\mu_0} \\ & = & q_{\mu_0}. \end{array} \tag{6.3.44}$$

This yields $q_\mu \in \ker(I - \mathcal{G}_\tau(\mu)) = \{0\}$, which implies $\mathrm{ev}_\tau q^1_\mu = \mathrm{ev}_\tau q^2_\mu \in \mathrm{Range}\big(u^*_{Q(\tau)}(\mu)\big)$ and completes the proof. ∎

Proof of Proposition 6.3.1. In the case where $r_{\min} > -\infty$ we can follow the proof of [104, Lem. 5.2], so we assume that $r_{\min} = -\infty$. In that case, the desired result follows directly from Lemmas 6.3.3-6.3.7. ∎

For the proof of Proposition 6.3.2, we can proceed in the same fashion as in the proof of Proposition 6.3.1, where instead of the spaces $BC^{\oplus}_{\tau,-\alpha}$, we use the space $BC^{\ominus}_{\tau,-\alpha}$. It only remains to show that $\mathrm{Range}\big(u^*_{R(\tau)}(\mu)\big) \subset X$ is closed and to establish (6.3.10).

Lemma 6.3.8. *Consider the setting of Proposition 6.3.2. Then* $\mathrm{Range}\big(u^*_{R(\tau)}(\mu)\big) \subset X$ *is closed.*

Proof. Consider a sequence $\{\phi_j\}_{j \geq 1}$ in $R(\tau)$ and, writing $\psi_j = u^*_{R(\tau)}(\mu)\phi_j$, assume that $\psi_j \to \psi_*$. By (6.3.7) we see that $\Pi_{R(\tau)}\psi_j = \phi_j$ and by the continuity of $\Pi_{R(\tau)}$ this yields $\phi_j \to \Pi_{R(\tau)}\psi_* := \phi_*$. Since the operator $u^*_{R(\tau)}(\mu)$ is bounded, we must have $u^*_{R(\tau)}(\mu)\big[\phi_j - \phi_*\big] \to 0$ and therefore $\psi_* = u^*_{R(\tau)}(\mu)\phi_*$, as desired. ∎

Lemma 6.3.9. *Consider the setting of Proposition 6.3.2. Then the uniform bound (6.3.10) holds for each $\mu \in U'$, each $t \in [-r_0, 0]$ and each $\phi \in R(\tau)$.*

Proof. We fix $\mu \in U'$, $-r_0 \leq t \leq 0$ and $\phi \in R(\tau)$ and write

$$u \quad = \quad E_{\tau,\mu} u^*_{R(\tau)}(\mu)\phi. \tag{6.3.45}$$

From (6.3.36) we can derive that

$$\begin{array}{lcl} [\Lambda(\mu)u](t) & = & \big[\Lambda(\mu_0)\iota_\tau\phi(\cdot - \tau)\big](t) + \Big[\mathcal{C}_{(-\alpha),-}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u\Big](t) \\ & & -\Big[\Lambda(\mu_0)\iota_\tau\Pi_{R(\tau)}\mathrm{ev}_0\Lambda^{\mathrm{qinv}}_{(-\alpha),-}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u\Big](t) \\ & := & L_1 + L_2 + L_3. \end{array} \tag{6.3.46}$$

On account of Proposition 5.5.4, we immediately obtain the bound

$$|L_1| \quad = \quad \big|[\Lambda(\mu_0)\iota_\tau\phi(\cdot-\tau)](t)\big| \quad \leq \quad K_1 e^{-\alpha|t-\tau|}\|\phi\|_\infty \tag{6.3.47}$$

for some $K_1 > 0$. Recall that α was chosen small enough to have $e^{\pm}_{2\alpha} b \in W^{1,\infty}(\mathbb{R};\mathbb{C}^M)$ for any $b \in \mathcal{B} \cup \mathcal{B}^*$. Let $\{d^i\}_{i=1}^{n_d}$ denote a basis for $\ker(\Lambda(\mu_0)^*)$. In particular, we can pick a constant $K_2 > 0$ in such a way that the exponential bound

$$|d^i(\xi)| \quad \leq \quad K_2 e^{-2\alpha|\xi|} \tag{6.3.48}$$

holds for any $\xi \in \mathbb{R}$ and any integer $1 \leq i \leq n_d$. Using the representations from Proposition 6.2.3 and from (5.5.22) we can compute

$$\begin{array}{lcl} L_2 & = & \Big[\mathcal{C}_{(-\alpha),-}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u\Big](t) \\ & = & -\pi_{\mathcal{R}^\perp}\Big[\big[L(\mu) - L(\mu_0)\big]u\Big](t) \\ & & \qquad +\pi_{\mathcal{R}^\perp}\Big[\Lambda_{(-\alpha),-}(\mu_0)\big[\Pi_{\mathcal{R}}\Lambda_{(-\alpha),-}(\mu_0)\big]^{-1}\pi_{\mathcal{R}}\big[L(\mu) - L(\mu_0)\big]u\Big](t) \\ & = & -\pi_{\mathcal{R}^\perp}\Big[\big[L(\mu) - L(\mu_0)\big]u\Big](t) \\ & = & \sum\limits_{i=1}^{n_d}\Big[\int\limits_{-\infty}^{\infty} d^i(\xi)^*\big[L(\mu_0) - L(\mu)\big]u(\xi)d\xi\Big]g^i(t). \end{array} \tag{6.3.49}$$

On account of the exponential decay (6.3.9), we can pick a constant $K_3 > 0$, independent of μ and u, for which the bound

$$\big|\big[L(\mu) - L(\mu_0)\big]u\big|(\xi) \quad \leq \quad K_3 e^{-\alpha(\tau-\xi)}\|\phi\|_\infty \tag{6.3.50}$$

holds for any $\xi \leq \tau$, while the bound

$$\big|\big[L(\mu) - L(\mu_0)\big]u\big|(\xi) \quad \leq \quad K_3\|\phi\|_\infty \tag{6.3.51}$$

holds for any $\xi > \tau$. In particular, we can estimate

$$\begin{array}{lcl} |L_2| & \leq & \sum\limits_{i=1}^{n_d}\Big[\int\limits_{-\infty}^{\tau} K_2 e^{-2\alpha|\xi|}K_3 e^{-\alpha(\tau-\xi)}\|\phi\|_\infty d\xi + \int\limits_{\tau}^{\infty} K_2 e^{-2\alpha|\xi|}K_3\|\phi\|_\infty d\xi\Big]|g^i|(t) \\ & \leq & e^{-\alpha\tau}K_2K_3\|\phi\|_\infty\Big[\int\limits_{-\infty}^{0} e^{3\alpha\xi}d\xi + \int\limits_{0}^{\tau} e^{-\alpha\xi}d\xi + (2\alpha)^{-1}\Big]\|g^i\|_\infty \\ & \leq & e^{-\alpha(\tau-t)}K_2K_3\|\phi\|_\infty\Big[\int\limits_{-\infty}^{0} e^{3\alpha\xi}d\xi + \int\limits_{0}^{\infty} e^{-\alpha\xi}d\xi + (2\alpha)^{-1}\Big]\|g^i\|_\infty e^{\alpha r_0}. \end{array} \tag{6.3.52}$$

Finally, we obtain the bound

$$\begin{array}{lcl} |L_3| & = & \Big|\Big[\Lambda(\mu_0)\iota_\tau\Pi_{R(\tau)}\mathrm{ev}_0\Lambda^{\mathrm{qinv}}_{(-\alpha),-}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u\Big](t)\Big| \\ & \leq & Ke^{-\alpha|t-\tau|}\|\Pi_{R(\tau)}\mathrm{ev}_0\Lambda^{\mathrm{qinv}}_{(-\alpha),-}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u\|_\infty \\ & \leq & K_3 e^{-\alpha|t-\tau|}\|\phi\|_\infty \end{array} \tag{6.3.53}$$

for some constant $K_3 > 0$, using the uniform bounds and the exponential decay in Theorem 5.2.8 and the bound (6.3.9).

∎

Proof of Proposition 6.3.2. The desired result follows from Lemmas 6.3.8 and 6.3.9. ∎