Patterns on spatially structured domains

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Chapter 2

Nonlinear stability of pulse solutions for the discrete FitzHugh-Nagumo equation with infinite-range interactions

Abstract. We establish the existence and nonlinear stability of travelling pulse solutions for the discrete FitzHugh-Nagumo equation with infinite-range interactions close to the near-continuum regime. For the verification of the spectral properties, we need to study a functional differential equation of mixed type (MFDE) with unbounded shifts. We avoid the use of exponential dichotomies and phase spaces, by building on a technique developed by Bates, Chen and Chmaj for the discrete Nagumo equation. This allows us to transfer several crucial Fredholm properties from the PDE setting to our discrete setting.

2.1   Introduction

The FitzHugh-Nagumo partial differential equation (PDE) is given by

$$\begin{aligned} u_t &= u_{xx} + g(u; r_0) - w \\ w_t &= \rho(u - \gamma w), \end{aligned} \tag{2.1.1}$$

where $g(\cdot; r_0)$ is the cubic bistable nonlinearity given by

$$g(u; r_0) = u(1 - u)(u - r_0) \tag{2.1.2}$$

and ρ, γ are positive constants. This PDE is commonly used as a simplification of the Hodgkin-Huxley equations, which describe the propagation of signals through nerve fibres. The spatially homogeneous version of this equation was first stated by FitzHugh in 1961 [74] in order to describe the potential felt at a single point along a nerve axon as a signal travels by. A few years later [76], the diffusion term in (2.1.1) was added to describe the dynamics of the full nerve axon instead of just a single point. As early as 1968 [75], FitzHugh released a computer animation based on numerical simulations of (2.1.1). This video clip clearly shows that (2.1.1) admits isolated pulse solutions resembling the spike signals that were measured by Hodgkin and Huxley in the nerve fibres of giant squids [98].

As a consequence of this rich behaviour and the relative simplicity of its structure, (2.1.1) has served as a prototype for several similar systems. For example, memory devices have been designed using a planar version of (2.1.1), which supports stable stationary, radially symmetric spot patterns [120]. In addition, gas discharges have been described using a three-component FitzHugh-Nagumo system [138, 148], for which it is possible to find stable travelling spots [161].

Mathematically, it turned out to be a major challenge to control the interplay between the excitation and recovery dynamics and rigorously construct the travelling pulses visualized by FitzHugh in [75]. Such pulse solutions have the form

$$(u, w)(x, t) = (\overline{u}_0, \overline{w}_0)(x + c_0 t), \tag{2.1.3}$$

in which c_0 is the wavespeed and the wave profile $(\overline{u}_0, \overline{w}_0)$ satisfies the limits

$$\lim_{|\xi| \to \infty} (\overline{u}_0, \overline{w}_0)(\xi) = 0. \tag{2.1.4}$$

Plugging this Ansatz into (2.1.1) and writing $\xi = x + c_0 t$, we see that the profiles are homoclinic solutions to the travelling wave ordinary differential equation (ODE)

$$\begin{aligned} c_0 \overline{u}_0'(\xi) &= \overline{u}_0''(\xi) + g(\overline{u}_0(\xi); r_0) - \overline{w}_0(\xi) \\ c_0 \overline{w}_0'(\xi) &= \rho\big[\overline{u}_0(\xi) - \gamma \overline{w}_0(\xi)\big]. \end{aligned} \tag{2.1.5}$$

The analysis of this equation in the singular limit $\rho \downarrow 0$ led to the birth of many techniques in geometric singular perturbation theory, see for example [118] for an interesting overview. Indeed, the early works [31, 97, 117, 119] used geometric techniques

such as the Conley index, exchange lemmas and differential forms to construct pulses and analyze their stability. A more analytic approach was later developed in [124], where Lin's method was used in the $r_0 \approx \frac{1}{2}$ regime to connect a branch of so-called slow-pulse solutions to (2.1.5) to a branch of fast-pulse solutions. This equation is still under active investigation, see for example [32, 33], where the birth of oscillating tails for the pulse solutions is described as the unstable root r_0 of the nonlinearity g moves towards the stable root at zero.

Many physical, chemical and biological systems have an inherent discrete structure that strongly influences their dynamical behaviour. In such settings *lattice differential equations* (LDEs), i.e. differential equations where the spatial variable can only take values on a lattice such as $\mathbb{Z}^n$, are the natural replacements for PDEs, see for example [6, 109, 130]. Although, mathematically, it is a relatively young field of interest, LDEs have already appeared frequently in the more applied literature. For example, they have been used to describe phase transitions in Ising models [6], crystal growth in materials [28] and phase mixing in martensitic structures [159].

To illustrate these points, let us return to the nerve axon described above and reconsider the propagation of electrical signals through nerve fibres. It is well known that electrical signals can only travel at adequate speeds if the nerve fibre is insulated by a myelin coating. This coating admits regularly spaced gaps at the so-called nodes of Ranvier [143]. Through a process called saltatory conduction, it turns out that excitations of nerves effectively jump from one node to the next [127]. Exploiting this fact, it is possible [123] to model this jumping process with the discrete FitzHugh-Nagumo LDE

$$
\begin{aligned}
\dot{u}_j &= \tfrac{1}{h^2}(u_{j+1} + u_{j-1} - 2u_j) + g(u_j; r_0) - w_j \\
\dot{w}_j &= \rho[u_j - \gamma w_j].
\end{aligned}
\tag{2.1.6}
$$

The variable u_j now represents the potential at the jth node, while the variable w_j denotes a recovery component. The nonlinearity g describes the ionic interactions. Note that this equation arises directly from the FitzHugh-Nagumo PDE upon taking the nearest-neighbour discretisation of the Laplacian on a grid with spacing $h > 0$.

Inspired by the procedure for partial differential equations, one can substitute a travelling pulse Ansatz

$$
(u_j, w_j)(t) = (\overline{u}_h, \overline{w}_h)(hj + c_h t)
\tag{2.1.7}
$$

into (2.1.6). Instead of an ODE, we obtain the system

$$
\begin{aligned}
c_h \overline{u}_h'(\xi) &= \tfrac{1}{h^2}[\overline{u}_h(\xi + h) + \overline{u}_h(\xi - h) - 2\overline{u}_h(\xi)] + g(\overline{u}_h(\xi); r_0) - \overline{w}_h(\xi) \\
c_h \overline{w}_h'(\xi) &= \rho[\overline{u}_h(\xi) - \gamma \overline{w}_h(\xi)]
\end{aligned}
\tag{2.1.8}
$$

in which $\xi = hj + c_h t$. Such equations are called functional differential equations of mixed type (MFDEs), since they contain both advanced (positive) and retarded (negative) shifts.

In [108, 109], Hupkes and Sandstede studied (2.1.6) and showed that, for small values of ρ and r_0 sufficiently far from $\frac{1}{2}$, there exists a locally unique travelling pulse solution of this system and that it is asymptotically stable with an asymptotic phase shift. No restrictions were required on the discretisation distance h, but the results relied heavily on the existence of exponential dichotomies for MFDEs. As a consequence, the techniques developed in [108, 109] can only be used if the discretisation involves finitely many neighbours. Such discretisation schemes are said to have finite range.

Recently, an active interest has arisen in nonlocal equations that feature infinite-range interactions. For example, Ising models have been used to describe the infinite-range interactions between magnetic spins arranged on a grid [6]. In addition, many physical systems, such as amorphous semiconductors [87] and liquid crystals [44], feature nonstandard diffusion processes, which are generated by fractional Laplacians. Such operators are intrinsically nonlocal and, hence, often require infinite-range discretisation schemes [43].

Our primary interest here, however, comes from so-called neural field models, which aim to describe the dynamics of large networks of neurons. These neurons interact with each other by exchanging signals across long distances through their interconnecting nerve axons [15, 23, 24, 142]. It is of course a major challenge to find effective equations to describe such complex interactions. One model that has been proposed [23, Eq. (3.31)] features a FitzHugh-Nagumo type system with infinite-range interactions.

Motivated by the above, we consider a class of infinite-range FitzHugh-Nagumo LDEs that includes the prototype

$$\begin{aligned}
\dot{u}_j &= \frac{\kappa}{h^2} \sum_{k \in \mathbb{Z}_{>0}} e^{-k^2}[u_{j+k} + u_{j-k} - 2u_j] + g(u_j; r_0) - w_j \\
\dot{w}_j &= \rho[u_j - \gamma w_j],
\end{aligned} \tag{2.1.9}$$

in which $\kappa > 0$ is a normalisation constant. In [69], Faye and Scheel studied equations such as (2.1.9) for discretisations with infinite-range interactions featuring exponential decay in the coupling strength. They circumvented the need to use a state space as in [108], which enabled them to construct pulses to (2.1.9) for arbitrary discretisation distance h. Very recently [70], they developed a center manifold approach that allows bifurcation results to be obtained for neural field equations.

In this paper, we also construct pulse solutions to equations such as (2.1.9), but under weaker assumptions on the decay rate of the couplings. Moreover, we will establish the nonlinear stability of these pulse solutions, provided the coupling strength decays exponentially. However, both results do require the discretisation distance h to be very small.

In particular, we will be working in the near-continuum regime. The pulses we construct can be seen as perturbations of the travelling pulse solution of the FitzHugh-Nagumo PDE. However, we will see that the travelling wave equations are highly singular perturbations of (2.1.5), which poses a significant mathematical challenge. On

the other hand, we do not need to use exponential dichotomies directly in our nonlocal setting as in [109]. Instead, we are able to exploit the detailed knowledge that has been obtained using these techniques for the pulses in the PDE setting.

Our approach to tackle the difficulties arising from this singular perturbation is strongly inspired by the work of Bates, Chen and Chmaj. Indeed, in their excellent paper [6], they study a class of systems that includes the infinite-range discrete Nagumo equation

$$\dot{u}_j \;=\; \tfrac{\kappa}{h^2} \sum_{k \in \mathbb{Z}_{>0}} e^{-k^2} [u_{j+k} + u_{j-k} - 2u_j] + g(u_j; r_0), \qquad (2.1.10)$$

in which $\kappa > 0$ is a normalisation constant. This equation can be seen as a discretisation of the Nagumo PDE

$$u_t \;=\; u_{xx} + g(u; r_0). \qquad (2.1.11)$$

The authors show that, under some natural assumptions, these systems admit travelling front solutions for h small enough.

In the remainder of this introduction we outline their approach and discuss our modifications, which significantly broaden the application range of these methods. We discuss these modifications for the prototype (2.1.9), but naturally they can be applied to a broad class of systems.

Transfer of Fredholm properties: Scalar case.

An important role in [6] is reserved for the operator $\mathcal{L}_{h;\overline{u}_{0:\mathrm{sc}};c_{0:\mathrm{sc}}}$ given by

$$\mathcal{L}_{h;\overline{u}_{0:\mathrm{sc}};c_{0:\mathrm{sc}}} v(\xi) \;=\; c_{0:\mathrm{sc}} v'(\xi) - \tfrac{\kappa}{h^2} \sum_{k \in \mathbb{Z}_{>0}} e^{-k^2} \Big[v(\xi + hk) + v(\xi - hk) - 2v(\xi) \Big]$$
$$-g_u(\overline{u}_{0:\mathrm{sc}}(\xi); r_0) v(\xi), \qquad (2.1.12)$$

where $\overline{u}_{0:\mathrm{sc}}$ is the wave solution of the scalar Nagumo PDE (2.1.11) with wavespeed $c_{0:\mathrm{sc}}$. This operator arises as the linearisation of the scalar Nagumo MFDE

$$c_{0:\mathrm{sc}} u'(\xi) = \tfrac{\kappa}{h^2} \sum_{k \in \mathbb{Z}_{>0}} e^{-k^2} \Big[v(\xi + hk) + v(\xi - hk) - 2v(\xi) \Big] + g_u(\overline{u}_{0:\mathrm{sc}}(\xi); r_0) v(\xi), \qquad (2.1.13)$$

around the wave solution $\overline{u}_{0:\mathrm{sc}}$ of the scalar Nagumo PDE (2.1.11). This operator should be compared to

$$\mathcal{L}_{0;\overline{u}_{0:\mathrm{sc}};c_{0:\mathrm{sc}}} v(\xi) \;=\; c_{0:\mathrm{sc}} v'(\xi) - v''(\xi) - g_u(\overline{u}_{0:\mathrm{sc}}(\xi); r_0) v(\xi), \qquad (2.1.14)$$

the linearisation of the scalar Nagumo PDE around its wave solution.

The key contribution in [6] is that the authors fix a constant $\delta > 0$ and use the invertibility of $\mathcal{L}_{0;\overline{u}_{0:\mathrm{sc}};c_{0:\mathrm{sc}}} + \delta$ to show that also $\mathcal{L}_{h;\overline{u}_{0:\mathrm{sc}};c_{0:\mathrm{sc}}} + \delta$ is invertible. In particular, they consider weakly-converging sequences $\{v_n\}$ and $\{w_n\}$ with $(\mathcal{L}_{h;\overline{u}_{0:\mathrm{sc}};c_{0:\mathrm{sc}}} + \delta)v_n = w_n$ and try to find a uniform (in δ and h) upper bound for the L^2-norm of v_n' in terms

of the L^2-norm of w_n. Such a bound is required to rule out the limitless transfer of energy into oscillatory modes, a key complication when taking weak limits. To obtain this bound, the authors exploit the bistable structure of the nonlinearity g to control the behaviour at $\pm\infty$. This allows the local L^2-norm of v_n on a compact set to be uniformly bounded away from zero. Since the operator $\mathcal{L}_{h;\overline{u}_{0:sc};c_{0:sc}} + \delta$ is not self-adjoint, this procedure must be repeated for the adjoint operator.

Transfer of Fredholm properties: System case.

Plugging the travelling pulse Ansatz

$$(u, w)_j(t) \quad = \quad (\overline{u}_h, \overline{w}_h)(hj + c_h t) \tag{2.1.15}$$

into (2.1.9) and writing $\xi = hj + c_h t$, we see that the profiles are homoclinic solutions to the equation

$$
\begin{aligned}
c_h \overline{u}_h'(\xi) &= \frac{\kappa}{h^2} \sum_{k>0} e^{-k^2} \left[\overline{u}_h(\xi + kh) + \overline{u}_h(\xi - kh) - 2\overline{u}_h(\xi) \right] + g(\overline{u}_h(\xi); r_0) - \overline{w}_h(\xi) \\
c_h \overline{w}_h'(\xi) &= \rho \Big(\overline{u}_h(\xi) - \gamma \overline{w}_h(\xi) \Big).
\end{aligned}
\tag{2.1.16}
$$

We start by considering the linearised operator $\mathcal{K}_{h;\overline{u}_0;c_0}$ of the system (2.1.16) around the pulse solution $(\overline{u}_0, \overline{w}_0)$ of the FitzHugh-Nagumo PDE with wavespeed c_0. This operator is given by

$$
\mathcal{K}_{h;\overline{u}_0;c_0} \left(\begin{array}{c} v \\ w \end{array} \right)(\xi) \quad = \quad \left(\begin{array}{c} \mathcal{L}_{h;\overline{u}_0;c_0} v(\xi) + w(\xi) \\ c_0 w'(\xi) - \rho v(\xi) + \rho\gamma w(\xi) \end{array} \right), \tag{2.1.17}
$$

where $\mathcal{L}_{h;\overline{u}_0;c_0}$ is given by equation (2.1.12), but with $\overline{u}_{0:sc}$ replaced by $\overline{u}_0$ and $c_{0:sc}$ by c_0.

In §2.3 we use a Fredholm alternative as described above to establish the invertibility of $\mathcal{K}_{h;\overline{u}_0;c_0} + \delta$ for fixed $\delta > 0$. However, the transition from a scalar equation to a system is far from trivial. Indeed, when transferring the Fredholm properties there are multiple cross terms that need to be controlled. We are aided here by the relative simplicity of the terms in the equation that involve $\overline{w}$. In particular, three of the four matrix-elements of the linearisation (2.1.17) have constant coefficients. We emphasize that it is not sufficient to merely assume that the limiting state $(0, 0)$ is a stable equilibrium of (2.1.9). In [151], we explore a number of structural conditions that allow these types of arguments to be extended to general multi-component systems.

Construction of pulses.

Using these results for $\mathcal{K}_{h;\overline{u}_0;c_0}$, we develop a fixed point argument to show that, for h small enough, the system (2.1.9) has a locally unique travelling pulse solution $(\overline{U}_h(t))_j = (\overline{u}_h, \overline{w}_h)(hj + c_h t)$ which converges to a travelling pulse solution of the FitzHugh-Nagumo PDE as $h \downarrow 0$. This procedure is more or less straightforward and is very similar to the arguments used in [6, §4] which, in turn, closely follow the lines of a standard proof of the implicit function theorem.

Spectral stability.

The natural next step is to study the linear operator $\mathcal{K}_{h;\overline{u}_h;c_h}$ that arises after linearising the system (2.1.9) around its new-found pulse solution. This operator is given by

$$\mathcal{K}_{h;\overline{u}_h;c_h} \begin{pmatrix} v \\ w \end{pmatrix}(\xi) = \begin{pmatrix} \mathcal{L}_{h;\overline{u}_h;c_h} v(\xi) + w(\xi) \\ c_0 w'(\xi) - \rho v(\xi) + \rho\gamma w(\xi) \end{pmatrix}, \qquad (2.1.18)$$

where $\mathcal{L}_{h;\overline{u}_h}$ is given by equation (2.1.12), but with $\overline{u}_{0:sc}$ replaced by $\overline{u}_h$ and $c_{0:sc}$ by c_h. The procedure above can be repeated to show that for fixed $\delta > 0$, it also holds that $\mathcal{K}_{h;\overline{u}_h;c_h} + \delta$ is invertible for h small enough. However, to understand the spectral stability of the pulse, we need to consider the eigenvalue problem

$$\mathcal{K}_{h;\overline{u}_h;c_h} v + \lambda v = 0 \qquad (2.1.19)$$

for fixed values of h and λ ranging throughout a half-plane. Switching between these two points of view turns out to be a delicate task.

We start in §2.5 by showing that $\mathcal{K}_{h;\overline{u}_h;c_h}$ and its adjoint $\mathcal{K}^*_{h;\overline{u}_h;c_h}$ are Fredholm operators with one-dimensional kernels. This is achieved by explicitly constructing a kernel element for $\mathcal{K}^*_{h;\overline{u}_h;c_h}$ that converges to a kernel element of the adjoint of the operator corresponding to the linearised PDE. An abstract perturbation argument then yields the result.

In particular, we see that $\lambda = 0$ is a simple eigenvalue of $\mathcal{K}_{h;\overline{u}_h;c_h}$. In §2.6 we establish that in a suitable half-plane, the spectrum of this operator consists precisely of the points $\{k2\pi i c_h \frac{1}{h} : k \in \mathbb{Z}\}$, which are all simple eigenvalues. We do this by first showing that the spectrum is invariant under the operation $\lambda \mapsto \lambda + \frac{2\pi i c_h}{h}$, which allows us to restrict ourselves to values of λ with imaginary part in between $-\frac{\pi|c_h|}{h}$ and $\frac{\pi|c_h|}{h}$. Note that the period of the spectrum is dependent on h and grows to infinity as $h \downarrow 0$. This is not too surprising, since the spectrum of the linearisation of the PDE around its pulse solution is not periodic. However, this means that we cannot restrict ourselves to a fixed compact subset of the complex plane for all values of h at the same time. In fact, it takes quite some effort to keep the part of the spectrum with large imaginary part under control.

It turns out to be convenient to partition our 'half-strip' into four parts and to calculate the spectrum in each part using different methods. Values close to 0 are analyzed using the Fredholm properties of $\mathcal{K}_{h;\overline{u}_h;c_h}$ exploiting many of the results from §2.5; values with a large real part are considered using standard norm estimates, but values with a large imaginary part are treated using a Fourier transform. The final set to consider is a compact set that is independent of h and bounded away from the origin. This allows us to apply a modified version of the procedure described above that exploits the absence of spectrum in this region for the FitzHugh-Nagumo PDE.

Let us emphasize that our arguments here for bounded values of the spectral parameter λ strongly use the fact that the PDE pulse is spectrally stable. The main

complication to establish the latter fact is the presence of a secondary eigenvalue that is $O(\rho)$-close to the origin. Intuitively, this eigenvalue arises as a consequence of the interaction between the front and back solution to the Nagumo equation that are both part of the singular pulse that arises in the $\rho \downarrow 0$ limit. In the PDE case, Jones [117] and Yanagida [166] essentially used shooting arguments to construct and analyze an Evans function $\mathcal{E}(\lambda)$ that vanishes precisely at eigenvalues. In particular, they computed the sign of $\mathcal{E}'(0)$ and used counting arguments to show that the secondary eigenvalue discussed above lies to the left of the origin. Currently, a program is underway to build a general framework in this spirit based on the Maslov index [10, 37, 101], which also works in multi-dimensional spatial settings. In [46, 47], this framework was applied to an equal-diffusion version of the FitzHugh-Nagumo PDE.

An alternative approach involving Lin's method and exponential dichotomies was pioneered in [124]. Based upon these ideas, stability results have been obtained for the LDE (2.1.6) [109] and the PDE (2.1.1) [32] in the nonhyperbolic regime $r_0 \sim 0$. The first major advantage of this approach is that explicit bifurcation equations can be formulated that allow asymptotic expansions to be developed for the location of the interaction eigenvalue discussed above. The second major advantage is that it allows us to avoid the use of the Evans function, which cannot easily be defined in discrete settings, because MFDEs are ill-posed as initial value problems [144]. We believe that a direct approach along these lines should also be possible for the infinite range system (2.1.9) as soon as exponential dichotomies are available in this setting.

Nonlinear stability.

The final step in our program is to leverage the spectral stability results to obtain a nonlinear stability result. To do so, we follow [109] and derive a formula that links the pointwise Green's function of our general problem (2.1.9) to resolvents of the operator $\mathcal{K}_{h;\overline{u}_h;c_h}$ in §2.7. Since we have already analyzed the latter operator in detail, we readily obtain a spectral decomposition of this Green's function into an explicit neutral part and a residual that decays exponentially in time and space. Therefore, we obtain detailed estimates on the decay rate of the Green's function for the general problem. These Green's functions allow in §2.8 to use multiple fixed point arguments to, eventually, show the nonlinear stability of the family of travelling pulse solutions $\overline{U}_h$. To be more precise, for each initial condition close to $\overline{U}_h(0)$, we show that the solution with that initial condition converges at an exponential rate to the solution $\overline{U}_h(\cdot + \tilde{\theta})$ for a small (and unique) phase shift $\tilde{\theta}$.

We emphasize that by now there are several techniques available to obtain nonlinear stability results in the relatively simple spectral setting encountered in this paper. If a comparison principle is available, which is not the case for the FitzHugh-Nagumo system, one can follow the classic approach developed by Fife and McLeod [73] to show that travelling waves have a large basin of attraction. Indeed, one can construct explicit sub- and super-solutions that trade-off additive perturbations at $t = 0$ to phase-shifts at $t = \infty$. In fact, one can actually use this type of argument to establish the existence of travelling waves by letting an appropriate initial condition evolve and tracking its

asymptotic behaviour [38, 110]. For systems that can be written as gradient flows, which is also not the case here, the existence and stability of travelling waves can be obtained by using an elegant variational technique that was developed by Gallay and Risler [82].

In the spatially continuous setting, it is possible to freeze a travelling wave by passing to a co-moving frame. In our setting, one can achieve this by simply adding a convective term $-c_0 \partial_x(u, w)$ to the right hand side of (2.1.1). The main advantage is that one can immediately use the semigroup $\exp[t\mathcal{L}_0]$ to describe the evolution of the linearised system in this co-moving frame, which is temporally autonomous. Here $\mathcal{L}_0$ is the standard linear operator associated to the linearisation of (2.1.1) around $(\overline{u}_0, \overline{w}_0)$; see (2.2.10). For each $\vartheta \in \mathbb{R}$ one can subsequently construct the stable manifold of $(\overline{u}_0(\cdot + \vartheta), \overline{w}_0(\cdot + \vartheta))$ by applying a fixed point argument to Duhamel's formula. Upon varying ϑ, these stable manifolds span a tubular neighbourhood of the family $(\overline{u}_0, \overline{w}_0)(\cdot + \mathbb{R})$. This readily leads to the desired stability result; see e.g. [121, §4]. We remark here that these stable manifolds are all related to each other via spatial shifts.

In the spatially discrete setting, the wave can no longer be frozen. In particular, the linearisation of (2.1.6) around the pulse (2.1.7) leads to an equation that is temporally shift-periodic. In [41], the authors attack this problem head-on by developing a shift-periodic version of Floquet theory that leads to a nonlinear stability result in ℓ^∞. However, they delicately exploit the geometric structure of ℓ^∞ and it is not clear how more degenerate spectral pictures can be fitted into the framework. These issues are explained in detail in [109, §2].

In [13], the authors found a way to express the Green's function of the temporally shift-periodic linear discrete equation in terms of resolvents of the linear operator $\mathcal{L}_h$ associated to the pulse (2.1.7). Based on this procedure, it is possible to follow the spirit of the powerful pointwise Green's function techniques pioneered by Zumbrun and Howard [168]. Indeed, in [11], a stability result is obtained in the setting of discrete conservation laws, where one encounters curves of essential spectrum that touch the imaginary axis. Using exponential dichotomies in a setting with extended state-spaces $L^2([-h, h]; \mathbb{R}^2) \times \mathbb{R}^2$, pointwise λ-meromorphic expansions were obtained for the operators $[\mathcal{L}_h - \lambda]^{-1}$. This allowed the techniques from [12] to be transferred from the continuous to the discrete setting. A slightly more streamlined approach was developed in [109], which does not need the extended state-space and avoids the use of a variation-of-constants formula. However, exponential dichotomies are still used at certain key points.

In our paper, we follow the spirit of the latter approach and extend it to the present setting with infinite-range interactions. In particular, we show how the use of exponential dichotomies can be eliminated all together, which is a delicate task. In addition, we need to be very careful in many computations since integrals and sums over shifts as in (2.1.16) can no longer be freely exchanged. We emphasize, here, that our techniques do not depend on the specific LDE that we are analyzing. All that is required is the spectral setting described above and the fact that the shifts appearing in the problem

are all rationally related.

Let us mention that it is also possible to bypass the construction of the stable manifolds altogether and employ a direct phase-tracking approach along the lines of [167]. In particular, one can couple the system with an extra equation for the phase. To close the system, one chooses this extra equation in such a way that the resulting nonlinear terms never encounter the nondecaying part of the relevant semigroup. Such an approach has been used in the current spectral setting to show that travelling waves remain stable under the influence of a small stochastic noise term [92].

2.2 Main results

We consider the following system of equations

$$
\begin{array}{rcl}
\dot{u}_j & = & \frac{1}{h^2} \sum_{k>0} \alpha_k [u_{j+k} + u_{j-k} - 2u_j] + g(u_j) - w_j \\
\dot{w}_j & = & \rho[u_j - \gamma w_j],
\end{array}
\tag{2.2.1}
$$

which we refer to as the (spatially) discrete FitzHugh-Nagumo equation with infinite-range interactions. Often, for example in [108, 109], it is assumed that only finitely many of these coefficients α_k are non-zero. However, we will impose the following much weaker conditions here.

Assumption (Hα1). The coefficients $\{\alpha_k\}_{k\in\mathbb{Z}_{>0}}$ satisfy the bound

$$
\sum_{k>0} |\alpha_k| k^2 \quad < \quad \infty,
\tag{2.2.2}
$$

as well as the identity

$$
\sum_{k>0} \alpha_k k^2 \quad = \quad 1.
\tag{2.2.3}
$$

Finally, the inequality

$$
A(z) := \sum_{k>0} \alpha_k \Big(1 - \cos(kz)\Big) \quad > \quad 0
\tag{2.2.4}
$$

holds for all $z \in (0, 2\pi)$.

We note that (2.2.4) is automatically satisfied if $\alpha_1 > 0$ and $\alpha_k \geq 0$ for all $k \in \mathbb{Z}_{>1}$. The conditions in (Hα1) ensure that for $\phi \in L^\infty(\mathbb{R})$ with $\phi'' \in L^2(\mathbb{R})$, we have the limit

$$
\lim_{h\downarrow 0} \|\frac{1}{h^2} \sum_{k>0} \alpha_k \Big[\phi(\cdot + hk) + \phi(\cdot - hk) - 2\phi(\cdot)\Big] - \phi''\|_{L^2(\mathbb{R})} = 0,
\tag{2.2.5}
$$

see Lemma 2.3.5. In particular, we can see (2.2.1) as the discretisation of the FitzHugh-Nagumo PDE (2.1.1) on a grid with distance h. Additional remarks concerning the assumption (Hα1) can be found in [6, §1].

Throughout this paper, we impose the following standard assumptions on the remaining parameters in (2.2.1). The last condition on γ in (HS) ensures that the origin is the only j-independent equilibrium of (2.2.1).

Assumption (HS). The nonlinearity g is given by $g(u) = u(1-u)(u-r_0)$, where $0 < r_0 < 1$. In addition, we have $0 < \rho < 1$ and $0 < \gamma < 4(1-r_0)^{-2}$.

Without explicitly mentioning it, we will allow all constants in this work to depend on r_0, ρ and γ. Dependence on h will always be mentioned explicitly. We will mainly work on the Sobolev spaces

$$
\begin{aligned}
H^1(\mathbb{R}) &= \{f : \mathbb{R} \to \mathbb{R} | f, f' \in L^2(\mathbb{R})\}, \\
H^2(\mathbb{R}) &= \{f : \mathbb{R} \to \mathbb{R} | f, f', f'' \in L^2(\mathbb{R})\},
\end{aligned}
\tag{2.2.6}
$$

with their standard norms

$$
\begin{aligned}
\|f\|_{H^1(\mathbb{R})} &= \left(\|f\|^2_{L^2(\mathbb{R})} + \|f'\|^2_{L^2(\mathbb{R})} \right)^{\frac{1}{2}}, \\
\|f\|_{H^2(\mathbb{R})} &= \left(\|f\|^2_{L^2(\mathbb{R})} + \|f'\|^2_{L^2(\mathbb{R})} + \|f''\|^2_{L^2(\mathbb{R})} \right)^{\frac{1}{2}}.
\end{aligned}
\tag{2.2.7}
$$

Our goal is to construct pulse solutions of (2.2.1) as small perturbations to the travelling pulse solutions of the FitzHugh-Nagumo PDE. These latter pulses satisfy the system

$$
\begin{aligned}
c_0 \overline{u}_0' &= \overline{u}_0'' + g(\overline{u}_0) - \overline{w}_0 \\
c_0 \overline{w}_0' &= \rho(\overline{u}_0 - \gamma \overline{w}_0)
\end{aligned}
\tag{2.2.8}
$$

with the boundary conditions

$$
\lim_{|\xi| \to \infty} (\overline{u}_0, \overline{w}_0)(\xi) = (0,0).
\tag{2.2.9}
$$

If $(\overline{u}_0, \overline{w}_0)$ is a solution of (2.2.8) with wavespeed c_0, then the linearisation of (2.2.8) around this solution is characterized by the operator $\mathcal{L}_0 : H^2(\mathbb{R}) \times H^1(\mathbb{R}) \to L^2(\mathbb{R}) \times L^2(\mathbb{R})$ that acts as

$$
\mathcal{L}_0 \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} c_0 \frac{d}{d\xi} - \frac{d^2}{d\xi^2} - g_u(\overline{u}_0) & 1 \\ -\rho & c_0 \frac{d}{d\xi} + \gamma \rho \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}.
\tag{2.2.10}
$$

The existence of such pulse solutions for the case when ρ is close to 0 is established in [118, §5.3]. Here, we do not require $\rho > 0$ to be small, but we simply impose the following condition.

Assumption (HP1). There exists a solution $(\overline{u}_0, \overline{w}_0)$ of (2.2.8) that satisfies the conditions (2.2.9) and has wavespeed $c_0 \neq 0$. Furthermore, the operator $\mathcal{L}_0$ is Fredholm with index zero and it has a simple eigenvalue in zero.

Recall that an eigenvalue λ of a Fredholm operator L is said to be *simple* if the kernel of $L - \lambda$ is spanned by one vector v and the equation $(L-\lambda)w = v$ does not have a solution w. Note that if L has a formal adjoint L^*, this is equivalent to the condition that $\langle v, w \rangle \neq 0$ for all nontrivial $w \in \ker(L^* - \overline{\lambda})$.

We note that the conditions on $\mathcal{L}_0$ formulated in (HP1) were established in [117] for small $\rho > 0$. In addition, these conditions imply that $\overline{u}_0'$ and $\overline{w}_0'$ decay exponentially.

We emphasize, however, that there exists a choice of parameters for which the condition (HP1) is not satisfied [34].

In order to find travelling pulse solutions of (2.2.1), we substitute the Ansatz

$$(u, w)_j(t) \quad = \quad (\overline{u}_h, \overline{w}_h)(hj + c_h t), \tag{2.2.11}$$

into (2.2.1) to obtain the system

$$
\begin{aligned}
c_h \overline{u}'_h(\xi) &= \tfrac{1}{h^2} \sum_{k>0} \alpha_k \Big[\overline{u}_h(\xi + hk) + \overline{u}_h(\xi - hk) - 2\overline{u}_h(\xi) \Big] + g\big(\overline{u}_h(\xi)\big) - \overline{w}_h(\xi) \\
c_h \overline{w}'_h(\xi) &= \rho[\overline{u}_h(\xi) - \gamma \overline{w}_h(\xi)],
\end{aligned}
\tag{2.2.12}
$$

in which $\xi = hj + c_h t$. The boundary conditions are given by

$$\lim_{|\xi| \to \infty} (\overline{u}_h, \overline{w}_h)(\xi) \quad = \quad (0, 0). \tag{2.2.13}$$

The existence of such solutions is established in our first main theorem.

Theorem 2.2.1 (see §2.4). *Assume that (HP1), (HS) and (Hα1) are satisfied. There exists a positive constant h_* such that for all $h \in (0, h_*)$, the problem (2.2.12) with boundary conditions (2.2.13) admits at least one solution $(c_h, \overline{u}_h, \overline{w}_h)$, which is locally unique in $\mathbb{R} \times H^1(\mathbb{R}) \times H^1(\mathbb{R})$ up to translation and which has the property that*

$$\lim_{h \downarrow 0} (c_h - c_0, \overline{u}_h - \overline{u}_0, \overline{w}_h - \overline{w}_0) \quad = \quad (0, 0, 0) \qquad in \ \mathbb{R} \times H^1(\mathbb{R}) \times H^1(\mathbb{R}). \tag{2.2.14}$$

Note that this result is very similar to [69, Cor. 2.1]. However, Faye and Scheel impose an extra assumption, similar to (Hα2) below, which we do not need in our proof. This is a direct consequence of the strength of the method from [6] that we described in §2.1.

Building on the existence of the travelling pulse solution, the natural next step is to show that our new-found pulse is asymptotically stable. However, we now do need to impose an extra condition on the coefficients $\{\alpha_k\}_{k>0}$.

Assumption (Hα2). The coefficients $\{\alpha_k\}_{k>0}$ satisfy the bound

$$\sum_{k>0} |\alpha_k| e^{k\nu} \quad < \quad \infty \tag{2.2.15}$$

for some $\nu > 0$.

Note that the prototype equation (2.1.9) indeed satisfies both assumptions (Hα1) and (Hα2). An example of a system which satisfies (Hα1), but not (Hα2) is given by

$$
\begin{aligned}
\dot{u}_j &= \tfrac{\kappa}{h^2} \sum_{k>0} \tfrac{1}{k^4} [u_{j+k} + u_{j-k} - 2u_j] + g(u_j) - w_j \\
\dot{w}_j &= \rho[u_j - \gamma w_j],
\end{aligned}
\tag{2.2.16}
$$

in which $\kappa = \tfrac{6}{\pi^2}$ is the normalisation constant.

Moreover, we need to impose an extra condition on the operator $\mathcal{L}_0$ given by (2.2.10). This spectral stability condition is established in [63, Thm. 2] together with [166, Thm. 3.1] for the case where ρ is close to 0.

Assumption (HP2). There exists a constant $\lambda_* > 0$ such that for each $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geq -\lambda_*$ and $\lambda \neq 0$, the operator

$$\mathcal{L}_0 + \lambda : H^2(\mathbb{R}) \times H^1(\mathbb{R}) \to L^2(\mathbb{R}) \times L^2(\mathbb{R}) \tag{2.2.17}$$

is invertible.

To determine if the pulse solution described in Theorem 2.2.1 is nonlinearly stable, we must first linearise (2.2.12) around this pulse and determine the spectral stability. The linearised operator now takes the form

$$L_h \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} c_h \frac{d}{d\xi} - \Delta_h - g_u(\overline{u}_h) & 1 \\ -\rho & c_h \frac{d}{d\xi} + \gamma\rho \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}. \tag{2.2.18}$$

Here the operator Δ_h is given by

$$\Delta_h \phi(\xi) = \frac{1}{h^2} \sum_{k>0} \alpha_k \Big(\phi(\xi + hk) + \phi(\xi - hk) - 2\phi(\xi) \Big). \tag{2.2.19}$$

As usual, we define the *spectrum*, $\sigma(L)$, of a bounded linear operator $L : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to L^2(\mathbb{R}) \times L^2(\mathbb{R})$, as

$$\sigma(L) = \{\lambda \in \mathbb{C} : L - \lambda \text{ is not invertible}\}. \tag{2.2.20}$$

Our second main theorem describes the spectrum of this operator L_h, or rather of $-L_h$, in a suitable half-plane.

Theorem 2.2.2 (see §2.6). *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. There exist constants $\lambda_3 > 0$ and $h_{**} > 0$ such that for all $h \in (0, h_{**})$, the spectrum of the operator $-L_h$ in the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z \geq -\lambda_3\}$ consists precisely of the points $k 2\pi i c_h \frac{1}{h}$ for $k \in \mathbb{Z}$, which are all simple eigenvalues of L_h.*

We emphasize that λ_3 does not depend on h. The translational invariance of (2.2.12) guarantees that $\lambda = 0$ is an eigenvalue of $-L_h$. In Lemma 2.6.1 we show that the spectrum of the operator L_h is periodic with period $2\pi i c_h \frac{1}{h}$, which means that the eigenvalues $k 2\pi i c_h \frac{1}{h}$ for $k \in \mathbb{Z}$ all have the same properties as the zero eigenvalue.

Our final result concerns the nonlinear stability of our pulse solution, which we represent with the shorthand

$$\left[\overline{U}_h(t) \right]_j = (\overline{u}_h, \overline{w}_h)(hj + c_h t). \tag{2.2.21}$$

The perturbations are measured in the spaces ℓ^p, which are defined by

$$\ell^p = \Big\{ V \in (\mathbb{R}^2)^{\mathbb{Z}} : \|V\|_{\ell^p} := \Big[\sum_{j \in \mathbb{Z}} |V_j|^p \Big]^{\frac{1}{p}} < \infty \Big\} \tag{2.2.22}$$

for $1 \leq p < \infty$ and

$$\ell^\infty = \Big\{ V \in (\mathbb{R}^2)^{\mathbb{Z}} : \|V\|_{\ell^\infty} := \sup_{j \in \mathbb{Z}} |V_j| < \infty \Big\}. \tag{2.2.23}$$

Theorem 2.2.3 (see §2.8). *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. Fix $0 < h \leq h_{**}$ and $1 \leq p \leq \infty$. Then there exist constants $\delta > 0$, $C > 0$ and $\beta > 0$, which may depend on h but not on p, such that for all initial conditions $U^0 \in \ell^p$ with $\|U^0 - \overline{U}_h(0)\|_{\ell^p} < \delta$, there exists an asymptotic phase shift $\tilde{\theta} \in \mathbb{R}$ such that the solution $U = (u, w)$ of (2.2.1) with $U(0) = U^0$ satisfies the bound*

$$\|U(t) - \overline{U}_h(t + \tilde{\theta})\|_{\ell^p} \quad \leq \quad Ce^{-\beta t}\|U^0 - \overline{U}_h(0)\|_{\ell^p} \tag{2.2.24}$$

for all $t > 0$.

2.3 The singular perturbation

The main difficulty in analysing the travelling wave MFDE (2.2.12) is that it is a singular perturbation of the ODE (2.2.8). Indeed, the second derivative in (2.2.8) is replaced by the linear operator $\Delta_h : H^1(\mathbb{R}) \to L^2(\mathbb{R})$ that acts as

$$\Delta_h\phi(\xi) \quad = \quad \frac{1}{h^2} \sum_{k>0} \alpha_k \Big(\phi(\xi + hk) + \phi(\xi - hk) - 2\phi(\xi) \Big). \tag{2.3.1}$$

We will see in Lemma 2.3.5 that for all $\phi \in L^\infty(\mathbb{R})$ with $\phi'' \in L^2(\mathbb{R})$, we have that $\lim_{h\downarrow 0} \|\Delta_h\phi - \phi''\|_{L^2} = 0$. Hence, the bounded operator Δ_h converges pointwise on a dense subset of $H^1(\mathbb{R})$ to an unbounded operator on that same dense subset. In particular, the norm of the operator Δ_h grows to infinity as $h \downarrow 0$.

Since there are no second derivatives involved in (2.2.12), we have to view it as an equation posed on the space $H^1(\mathbb{R}) \times H^1(\mathbb{R})$, while the ODE (2.2.8) is posed on the space $H^2(\mathbb{R}) \times H^1(\mathbb{R})$. From now on we write

$$
\begin{aligned}
\mathbf{H}^1 &:= H^1(\mathbb{R}) \times H^1(\mathbb{R}), \\
\mathbf{L}^2 &:= L^2(\mathbb{R}) \times L^2(\mathbb{R}).
\end{aligned}
\tag{2.3.2}
$$

The main results in this section will be used in several different settings. In order to accommodate this, we introduce the following conditions.

Assumption (hFam). For each $h > 0$ there is a pair $(\tilde{u}_h, \tilde{w}_h) \in \mathbf{H}^1$ and a constant $\tilde{c}_h$ such that $(\tilde{u}_h, \tilde{w}_h) - (\overline{u}_0, \overline{w}_0) \to 0$ in $\mathbf{H}^1$ and $\tilde{c}_h \to c_0$ as $h \downarrow 0$.

In the proof of Theorem 2.2.1 we choose $(\tilde{u}_h, \tilde{w}_h)$ and $\tilde{c}_0$ to be $(\overline{u}_0, \overline{w}_0)$ and c_0 for all values of h. However, in §2.5 we let $(\tilde{u}_h, \tilde{w}_h)$ be the travelling pulse $(\overline{u}_h, \overline{w}_h)$ from Theorem 2.2.1 and we let $\tilde{c}_h$ be its wave speed c_h.

If (hFam) is satisfied, then for $\delta > 0$ and $h > 0$ we define the operators

$$\overline{\mathcal{L}}_{h,\delta}^+ \quad = \quad \begin{pmatrix} \tilde{c}_h \frac{d}{d\xi} - \Delta_h - g_u(\tilde{u}_h) + \delta & 1 \\ -\rho & \tilde{c}_h \frac{d}{d\xi} + \gamma\rho + \delta \end{pmatrix} \tag{2.3.3}$$

and

$$\overline{\mathcal{L}}_{h,\delta}^{-} = \begin{pmatrix} -\tilde{c}_h \frac{d}{d\xi} - \Delta_h - g_u(\tilde{u}_h) + \delta & -\rho \\ 1 & -\tilde{c}_h \frac{d}{d\xi} + \gamma\rho + \delta \end{pmatrix}. \tag{2.3.4}$$

These operators are bounded linear functions from $\mathbf{H}^1$ to $\mathbf{L}^2$. We see that $\overline{\mathcal{L}}_{h,\delta}^{-}$ is the adjoint operator of $\overline{\mathcal{L}}_{h,\delta}^{+}$, in the sense that

$$\langle (\phi, \psi), \overline{\mathcal{L}}_{h,\delta}^{+}(\theta, \chi) \rangle = \langle \overline{\mathcal{L}}_{h,\delta}^{-}(\phi, \psi), (\theta, \chi) \rangle \tag{2.3.5}$$

holds for all $(\phi, \psi), (\theta, \chi) \in \mathbf{H}^1$. Here we have introduced the notation

$$\begin{aligned} \langle (\phi, \psi), (\theta, \chi) \rangle &= \langle \phi, \theta \rangle + \langle \psi, \chi \rangle \\ &= \int_{-\infty}^{\infty} \Big(\phi(x)\theta(x) + \psi(x)\chi(x) \Big) dx \end{aligned} \tag{2.3.6}$$

for $(\phi, \psi), (\theta, \chi) \in \mathbf{L}^2$.

Since, at some point, we want to consider complex-valued functions, we also work in the spaces $H_{\mathbb{C}}^2(\mathbb{R})$, $H_{\mathbb{C}}^1(\mathbb{R})$ and $L_{\mathbb{C}}^2(\mathbb{R})$, which are given by

$$\begin{aligned} H_{\mathbb{C}}^2(\mathbb{R}) &= \{f + gi | f, g \in H^2(\mathbb{R})\}, \\ H_{\mathbb{C}}^1(\mathbb{R}) &= \{f + gi | f, g \in H^1(\mathbb{R})\}, \\ L_{\mathbb{C}}^2(\mathbb{R}) &= \{f + gi | f, g \in L^2(\mathbb{R})\}. \end{aligned} \tag{2.3.7}$$

These spaces are equipped with the inner product

$$\langle \phi, \psi \rangle = \int \Big(f_1(x) + i g_1(x) \Big) \Big(f_2(x) - i g_2(x) \Big) dx \tag{2.3.8}$$

for $\phi = f_1 + i g_1, \psi = f_2 + i g_2$. As before, we write

$$\begin{aligned} \mathbf{H}_{\mathbb{C}}^1 &= H_{\mathbb{C}}^1(\mathbb{R}) \times H_{\mathbb{C}}^1(\mathbb{R}) \\ \mathbf{L}_{\mathbb{C}}^2 &= L_{\mathbb{C}}^2(\mathbb{R}) \times L_{\mathbb{C}}^2(\mathbb{R}). \end{aligned} \tag{2.3.9}$$

Each operator L from $\mathbf{H}^1$ to $\mathbf{L}^2$ can be extended to an operator from $\mathbf{H}_{\mathbb{C}}^1$ to $\mathbf{L}_{\mathbb{C}}^2$ by writing

$$L(f + ig) = Lf + iLg. \tag{2.3.10}$$

It is well-known that this complexification preserves adjoints, invertibility, inverses, injectivity, surjectivity and boundedness, see for example [146]. If $\lambda \in \mathbb{C}$ then the operators $\overline{\mathcal{L}}_{h,\lambda}^{\pm}$ are defined analogously to their real counterparts, but now we view them as operators from $H_{\mathbb{C}}^1(\mathbb{R}) \times H_{\mathbb{C}}^1(\mathbb{R})$ to $L_{\mathbb{C}}^2(\mathbb{R}) \times L_{\mathbb{C}}^2(\mathbb{R})$. Whenever it is clear that we are working in the complex setting we drop the subscript $\mathbb{C}$ from the spaces $\mathbf{H}_{\mathbb{C}}^1$ and $\mathbf{L}_{\mathbb{C}}^2$ and simply write $\mathbf{H}^1$ and $\mathbf{L}^2$.

We also introduce the operators $\mathcal{L}_0^{\pm} : H^2(\mathbb{R}) \times H^1(\mathbb{R}) \to L^2(\mathbb{R}) \times L^2(\mathbb{R})$, that act as

$$\mathcal{L}_0^+ = \begin{pmatrix} c_0 \frac{d}{d\xi} - \frac{d^2}{d\xi^2} - g_u(\overline{u}_0) & 1 \\ -\rho & c_0 \frac{d}{d\xi} + \gamma\rho \end{pmatrix} \qquad (2.3.11)$$

and

$$\mathcal{L}_0^- = \begin{pmatrix} -c_0 \frac{d}{d\xi} - \frac{d^2}{d\xi^2} - g_u(\overline{u}_0) & -\rho \\ 1 & -c_0 \frac{d}{d\xi} + \gamma\rho \end{pmatrix}. \qquad (2.3.12)$$

These operators can be viewed as the formal $h \downarrow 0$ limits of the operators $\overline{\mathcal{L}}_{h,0}^{\pm}$. Upon introducing the notation

$$(\phi_0^+, \psi_0^+) = \frac{(\overline{u}_0', \overline{w}_0')}{\|(\overline{u}_0', \overline{w}_0')\|_{\mathbf{L}^2}}, \qquad (2.3.13)$$

we see that $\mathcal{L}_0^+(\phi_0^+, \psi_0^+) = 0$ by differentiating (2.2.8).

To set the stage, we summarize several basic properties of $\mathcal{L}_0^{\pm}$. The proof of this result follows the standard procedure described in [6, Lem. 3.1] and, as such, will be omitted. The last property references a spectral set M, on which we impose the following condition.

Assumption (hM). The set $M \subset \mathbb{C}$ is compact with $0 \notin M$. In addition, recalling the constant λ_* appearing in (HP2), we have $\operatorname{Re} z \geq -\lambda_*$ for all $z \in M$.

In §2.6 the set M will be fixed as the final region of our spectral analysis, which we will refer to as R_4.

Lemma 2.3.1. *Assume that (HP1), (HS) and (Hα1) are satisfied. Then the following results hold.*

1. *We have that $(\phi_0^+, \psi_0^+) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$ and $\ker(\mathcal{L}_0^+) = \operatorname{span}\{(\phi_0^+, \psi_0^+)\}$.*

2. *There exist $(\phi_0^-, \psi_0^-) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$ with $\|(\phi_0^-, \psi_0^-)\|_{\mathbf{L}^2} = 1$, with $\langle(\overline{u}_0', \overline{w}_0'), (\phi_0^-, \psi_0^-)\rangle > 0$ and $\ker(\mathcal{L}_0^-) = \operatorname{span}\{(\phi_0^-, \psi_0^-)\}$.*

3. *For every $(\theta, \chi) \in \mathbf{L}^2$ the problem $\mathcal{L}_0^{\pm}(\phi, \psi) = (\theta, \chi)$ with $(\phi, \psi) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$ and $\langle(\phi, \psi), (\phi_0^{\pm}, \psi_0^{\pm})\rangle = 0$ has a unique solution (ϕ, ψ) if and only if $\langle(\theta, \chi), (\phi_0^{\mp}, \psi_0^{\mp})\rangle = 0$.*

4. *There exists a positive constant C_1 such that*

$$\|(\phi, \psi)\|_{H^2(\mathbb{R}) \times H^1(\mathbb{R})} \leq C_1 \|\mathcal{L}_0^{\pm}(\phi, \psi)\|_{\mathbf{L}^2} \qquad (2.3.14)$$

for all $(\phi, \psi) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$ with $\langle(\phi, \psi), (\phi_0^{\pm}, \psi_0^{\pm})\rangle = 0$.

5. *There exists a positive constant C_2 and a small constant $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ we have*

$$\|(\mathcal{L}_0^{\pm} + \delta)^{-1}(\theta, \chi)\|_{H^2(\mathbb{R}) \times H^1(\mathbb{R})} \leq C_2 \Big[\|(\theta, \chi)\|_{\mathbf{L}^2} + \tfrac{1}{\delta}|\langle(\theta, \chi), (\phi_0^{\mp}, \psi_0^{\mp})\rangle|\Big] \qquad (2.3.15)$$

for all $(\theta, \chi) \in \mathbf{L}^2$.

6. If (HP2) is also satisfied, then for each $M \subset \mathbb{C}$ that satisfies (hM), there exists a constant $C_3 > 0$ such that the uniform bound

$$\|(\mathcal{L}_0^\pm + \lambda)^{-1}(\theta, \chi)\|_{H^2_\mathbb{C}(\mathbb{R}) \times H^1_\mathbb{C}(\mathbb{R})} \leq C_3 \|(\theta, \chi)\|_{L^2_\mathbb{C}} \qquad (2.3.16)$$

holds for all $(\theta, \chi) \in L^2_\mathbb{C}$ and all $\lambda \in M$.

The main goal of this section is to prove the following two propositions, which transfer parts (5) and (6) of Lemma 2.3.1 to the discrete setting.

Proposition 2.3.2. *Assume that (hFam), (HP1), (HS) and (Hα1) are satisfied. There exists a positive constant C_0' and a positive function $h_0'(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$, depending only on the choice of $(\tilde{u}_h, \tilde{w}_h)$ and $\tilde{c}_h$, such that for every $0 < \delta < \delta_0$ and every $h \in (0, h_0'(\delta))$, the operators $\overline{\mathcal{L}}_{h,\delta}^\pm$ are homeomorphisms from $\mathbf{H}^1$ to $\mathbf{L}^2$ that satisfy the bounds*

$$\|(\overline{\mathcal{L}}_{h,\delta}^\pm)^{-1}(\theta, \chi)\|_{\mathbf{H}^1} \quad \leq \quad C_0' \Big[\|(\theta, \chi)\|_{\mathbf{L}^2} + \tfrac{1}{\delta} |\langle (\theta, \chi), (\phi_0^\mp, \psi_0^\mp) \rangle| \Big] \qquad (2.3.17)$$

for all $(\theta, \chi) \in \mathbf{L}^2$.

Proposition 2.3.3. *Assume that (hFam), (HP1),(HP2), (HS) and (Hα1) are satisfied. Let $M \subset \mathbb{C}$ satisfy (hM). Then there exists a constant $h_M > 0$, depending only on M and the choice of $(\tilde{u}_h, \tilde{w}_h)$ and $\tilde{c}_h$, such that for all $0 < h \leq h_M$ and all $\lambda \in M$ the operator $\overline{\mathcal{L}}_{h,\lambda}^\pm$ is a homeomorphism from $\mathbf{H}^1$ to $\mathbf{L}^2$.*

2.3.1 Strategy

Our techniques here are inspired strongly by the approach developed in [6, §2-4]. Indeed, Proposition 2.3.2 and Proposition 2.3.4 are the equivalents of [6, Thm. 3] and [6, Lem. 3.2] respectively. The difference between our results and those in [6] is that Bates, Chen and Chmaj study the discrete Nagumo equation, which can be seen as the one-dimensional fast component of the FitzHugh-Nagumo equation by setting $\rho = 0$ in (2.2.1). In addition, the results in [6] are restricted to $\lambda \in \mathbb{R}$, while we allow $\lambda \in \mathbb{C}$ in Proposition 2.3.3. These differences play a crucial role in the proof of Lemma 2.3.10 below.

Recall the constant $\delta_0 > 0$ appearing in Lemma 2.3.1. For $0 < \delta < \delta_0$ and $h > 0$ we define the quantities

$$\overline{\Lambda}^\pm(h, \delta) \quad = \quad \inf_{\|(\phi, \psi)\|_{\mathbf{H}^1} = 1} \left[\|\overline{\mathcal{L}}_{h,\delta}^\pm(\phi, \psi)\|_{\mathbf{L}^2} + \tfrac{1}{\delta} \left| \langle \overline{\mathcal{L}}_{h,\delta}^\pm(\phi, \psi), (\phi_0^\mp, \psi_0^\mp) \rangle \right| \right], \qquad (2.3.18)$$

together with

$$\overline{\Lambda}^\pm(\delta) \quad = \quad \liminf_{h \downarrow 0} \overline{\Lambda}^\pm(h, \delta). \qquad (2.3.19)$$

Similarly for $M \subset \mathbb{C}$ that satisfies (hM) and $h > 0$ we define

$$\overline{\Lambda}^\pm(h, M) \quad = \quad \inf_{\|(\phi, \psi)\|_{\mathbf{H}^1} = 1, \ \lambda \in M} \left[\|\overline{\mathcal{L}}_{h,\lambda}^\pm(\phi, \psi)\|_{\mathbf{L}^2} \right], \qquad (2.3.20)$$

together with

$$\overline{\Lambda}^{\pm}(M) \quad = \quad \liminf_{h\downarrow 0} \overline{\Lambda}^{\pm}(h, M). \tag{2.3.21}$$

The key ingredients that we need to establish Propositions 2.3.2 and 2.3.3 are lower bounds on the quantities $\overline{\Lambda}^{\pm}(\delta)$ and $\overline{\Lambda}^{\pm}(M)$. These are provided in the result below, which we consider to be the technical heart of this section.

Proposition 2.3.4. *Assume that (hFam), (HP1), (HS) and (Hα1) are satisfied. There exists a positive constant C_0, depending only on our choice of $(\tilde{u}_h, \tilde{w}_h)$ and $\tilde{c}_h$, such that $\overline{\Lambda}^{\pm}(\delta) > C_0$ for all $0 < \delta < \delta_0$. Similarly if $M \subset \mathbb{C}$ satisfies (hM), then there exists a positive constant C_M, depending only on M and our choice of $(\tilde{u}_h, \tilde{w}_h)$ and $\tilde{c}_h$, such that $\overline{\Lambda}^{\pm}(M) > \mathbb{C}_M$.*

Proof of Proposition 2.3.2. Let $\delta > 0$ be fixed and set $C'_0 = \frac{2}{C_0}$. Since $\overline{\Lambda}^{\pm}(\delta) \geq \frac{2}{C'_0}$, the definition (2.3.19) implies that there exists $h'_0(\delta)$ such that $\overline{\Lambda}(h, \delta) \geq \frac{1}{C'_0}$ for all $h \in (0, h'_0(\delta)]$. Now pick $h \in (0, h'_0(\delta)]$.

First of all, $\overline{\mathcal{L}}^{\pm}_{h,\delta}$ is a bounded operator from $\mathbf{H}^1$ to $\mathbf{L}^2$. Since $\overline{\Lambda}^{\pm}(h, \delta)$ is strictly positive, this implies that $\overline{\mathcal{L}}^{\pm}_{h,\delta}$ is a homeomorphism from $\mathbf{H}^1$ to its image $\overline{\mathcal{L}}^{\pm}_{h,\delta}(\mathbf{H}^1)$. Furthermore, the norm of the inverse $(\overline{\mathcal{L}}^{\pm}_{h,\delta})^{-1}$ from $\overline{\mathcal{L}}^{\pm}_{h,\delta}(\mathbf{H}^1) \subset \mathbf{L}^2$ is bounded by $\frac{1}{\overline{\Lambda}^{\pm}(h,\delta)} \leq C'_0$. Since $\overline{\mathcal{L}}^{\pm}_{h,\delta}$ is bounded, it follows that $\overline{\mathcal{L}}^{\pm}_{h,\delta}(\mathbf{H}^1)$ is closed in $\mathbf{L}^2$.

For the remainder of this proof, we only consider the operators $\overline{\mathcal{L}}^{+}_{h,\delta}$, noting that their counterparts $\overline{\mathcal{L}}^{-}_{h,\delta}$ can be treated in an identical fashion.

Seeking a contradiction, let us assume that $\overline{\mathcal{L}}^{+}_{h,\delta}(\mathbf{H}^1) \neq \mathbf{L}^2$, which implies that there exists a nonzero $(\theta, \chi) \in \mathbf{L}^2$ orthogonal to $\overline{\mathcal{L}}^{+}_{h,\delta}(\mathbf{H}^1)$. For any $\phi \in C^\infty_c(\mathbb{R})$, we hence obtain

$$\begin{aligned}
0 &= \langle \overline{\mathcal{L}}^{+}_{h,\delta}(\phi, 0), (\theta, \chi) \rangle \\
&= \langle \tilde{c}_h \phi' - \Delta_h \phi - g_u(\tilde{u}_h)\phi + \delta\phi, \theta \rangle + \langle -\rho\phi, \chi \rangle \\
&= \tilde{c}_h \langle \phi', \theta \rangle + \langle \phi, -\Delta_h \theta - g_u(\tilde{u}_h)\theta + \delta\theta - \rho\chi \rangle.
\end{aligned} \tag{2.3.22}$$

By definition this implies that θ has a weak derivative and that $\tilde{c}_h \theta' = -\Delta_h \theta - g_u(\tilde{u}_h)\theta + \delta\theta - \rho\chi \in L^2(\mathbb{R})$. In particular, we see that $\theta \in H^1(\mathbb{R})$.

For any $\psi \in C^\infty_c(\mathbb{R})$ a similar computation yields

$$\begin{aligned}
0 &= \langle \overline{\mathcal{L}}^{+}_{h,\delta}(0, \psi), (\theta, \chi) \rangle \\
&= \langle \psi, \theta \rangle + \langle \tilde{c}_h \psi' + (\gamma\rho + \delta)\psi, \chi \rangle \\
&= \tilde{c}_h \langle \psi', \chi \rangle + \langle \psi, \theta + (\gamma\rho + \delta)\chi \rangle.
\end{aligned} \tag{2.3.23}$$

Again, this means that χ has a weak derivative and in fact $\tilde{c}_h \chi' = \theta + (\gamma\rho + \delta)\chi$. In particular, it follows that $\chi \in H^1(\mathbb{R})$.

We, therefore, conclude that

$$
\begin{aligned}
0 &= \langle \overline{\mathcal{L}}_{h,\delta}^{+}(\phi, \psi), (\theta, \chi) \rangle \\
&= \langle (\phi, \psi), (\overline{\mathcal{L}}_{h,\delta}^{-}(\theta, \chi) \rangle
\end{aligned}
\tag{2.3.24}
$$

holds for all $(\phi, \psi) \in \mathbf{H}^1$. Since $\mathbf{H}^1$ is dense in $\mathbf{L}^2$ this implies that $\overline{\mathcal{L}}_{h,\delta}^{-}(\theta, \chi) = 0$. Since we already know that $\overline{\mathcal{L}}_{h,\delta}^{-}$ is injective, this means that $(\theta, \chi) = 0$, which gives a contradiction. Hence, we must have $\overline{\mathcal{L}}_{h,\delta}^{+}(\mathbf{H}^1) = \mathbf{L}^2$, as desired. ∎

Proof of Proposition 2.3.3. The result follows in the same fashion as outlined in the proof of Proposition 2.3.2 above. ∎

2.3.2 Preliminaries

Our goal here is to establish some basic facts concerning the operator Δ_h. In particular, we extend the real-valued results from [6] to complex-valued functions. We emphasize that the inequalities in Lemma 2.3.6 in general do not hold for the imaginary parts of these inner products.

Lemma 2.3.5 ([6, Lem. 2.1]). *Assume that (Hα1) is satisfied. The following three properties hold.*

1. *For all $\phi \in L^\infty(\mathbb{R})$ with $\phi'' \in L^2(\mathbb{R})$ we have $\lim_{h \downarrow 0} \|\Delta_h \phi - \phi''\|_{L^2} = 0$.*

2. *For all $\phi \in H^1(\mathbb{R})$ and $h > 0$ we have $\langle \Delta_h \phi, \phi' \rangle = 0$.*

3. *For all $\phi, \psi \in L^2(\mathbb{R})$ and $h > 0$ we have $\langle \Delta_h \phi, \psi \rangle = \langle \phi, \Delta_h \psi \rangle$ and $\langle \Delta_h \phi, \phi \rangle \leq 0$.*

Lemma 2.3.6. *Assume that (Hα1) is satisfied and pick $f \in H^1_{\mathbb{C}}(\mathbb{R})$. Then the following properties hold.*

1. *For all $h > 0$ we have $\mathrm{Re} \langle -\Delta_h f, f \rangle \geq 0$.*

2. *For all $h > 0$ we have $\mathrm{Re} \langle \Delta_h f, f' \rangle = 0$.*

3. *We have $\mathrm{Re} \langle f, f' \rangle = 0$.*

4. *For all $\lambda \in \mathbb{C}$ we have $\mathrm{Re} \langle \lambda f, f' \rangle = 2 \, (\mathrm{Im} \, \lambda) \langle \mathrm{Re} \, f, \mathrm{Im} \, f' \rangle$.*

Proof. Write $f = \phi + i\psi$ with $\phi, \psi \in H^1(\mathbb{R})$. Lemma 2.3.5 implies that

$$
\begin{aligned}
\mathrm{Re} \langle -\Delta_h f, f \rangle &= \mathrm{Re} \int \Big(-\Delta_h \phi - i\Delta_h \psi \Big)(x) \Big(\phi - i\psi \Big)(x) dx \\
&= \int (-\Delta_h \phi)(x)\phi(x) + (-\Delta_h \psi)(x)\psi(x) dx \\
&= \langle -\Delta_h \phi, \phi \rangle + \langle -\Delta_h \psi, \psi \rangle \\
&\geq 0.
\end{aligned}
\tag{2.3.25}
$$

Similarly we have

$$
\begin{aligned}
\operatorname{Re} \langle \Delta_h f, f' \rangle &= \langle -\Delta_h \phi, \phi' \rangle + \langle -\Delta_h \psi, \psi' \rangle \\
&= 0.
\end{aligned}
\tag{2.3.26}
$$

For $\lambda \in \mathbb{C}$ we may compute

$$
\begin{aligned}
\operatorname{Re} \langle \lambda f, f' \rangle &= \operatorname{Re} \int \Big(\lambda \phi(x) + \lambda i \psi(x) \Big) \Big(\phi'(x) - i \psi'(x) \Big) dx \\
&= (\operatorname{Re} \lambda) \langle \phi, \phi' \rangle + (\operatorname{Im} \lambda) \langle \phi, \psi' \rangle - (\operatorname{Im} \lambda) \langle \psi, \phi' \rangle + (\operatorname{Re} \lambda) \langle \psi, \psi' \rangle \\
&= 0 + 2 \, (\operatorname{Im} \lambda) \langle \phi, \psi' \rangle + 0 \\
&= 2 \, (\operatorname{Im} \lambda) \langle \phi, \psi' \rangle.
\end{aligned}
\tag{2.3.27}
$$

Taking $\lambda = 1$ gives the third property. ∎

2.3.3 Proof of Proposition 2.3.4

We now set out to prove Proposition 2.3.4. In Lemmas 2.3.7 and 2.3.8, we construct weakly converging sequences that realize the infima in (2.3.18)-(2.3.21). In Lemmas 2.3.9-2.3.11, we exploit the structure of our operators (2.3.3) and (2.3.4) to recover bounds on the derivatives of these sequences that are typically lost when taking weak limits. Recall the constant $C_2 > 0$ defined in Lemma 2.3.1, which does not depend on $\delta > 0$.

Lemma 2.3.7. *Assume that (hFam), (HP1), (HS) and (Hα1) are satisfied. Consider the setting of Proposition 2.3.4 and fix $0 < \delta < \delta_0$. Then there exists a sequence $\{(h_j, \phi_j, \psi_j)\}_{j \geq 0}$ in $(0,1) \times \boldsymbol{H}^1$ with the following properties.*

1. *We have $\lim_{j \to \infty} h_j = 0$ and $\|(\phi_j, \psi_j)\|_{\boldsymbol{H}^1} = 1$ for all $j \geq 0$.*

2. *The sequence $(\theta_j, \chi_j) = \overline{\mathcal{L}}^+_{h_j, \delta}(\phi_j, \psi_j)$ satisfies*

$$
\lim_{j \to \infty} \left[\|(\theta_j, \chi_j)\|_{\boldsymbol{L}^2} + \tfrac{1}{\delta} |\langle (\theta_j, \chi_j), (\phi_0^-, \psi_0^-) \rangle| \right] = \overline{\Lambda}^+(\delta).
\tag{2.3.28}
$$

3. *There exist $(\phi, \psi) \in \boldsymbol{H}^1$ and $(\theta, \chi) \in \boldsymbol{L}^2$ such that $(\phi_j, \psi_j) \rightharpoonup (\phi, \psi)$ weakly in $\boldsymbol{H}^1$ and such that $(\theta_j, \chi_j) \rightharpoonup (\theta, \chi)$ weakly in $\boldsymbol{L}^2$ as $j \to \infty$.*

4. *We have $(\phi_j, \psi_j) \to (\phi, \psi)$ in $L^2_{\mathrm{loc}}(\mathbb{R}) \times L^2_{\mathrm{loc}}(\mathbb{R})$ as $j \to \infty$.*

5. *The pair (ϕ, ψ) is a weak solution to $(\overline{\mathcal{L}}^+_0 + \delta)(\phi, \psi) = (\theta, \chi)$.*

6. *We have the bound*

$$
\|(\phi, \psi)\|_{H^2(\mathbb{R}) \times H^1(\mathbb{R})} \leq C_2 \overline{\Lambda}^+(\delta).
\tag{2.3.29}
$$

The same statements hold upon replacing $\overline{\mathcal{L}}^+_{h,\delta}$, $\overline{\Lambda}^+$ and $\overline{\mathcal{L}}^+_0$ by $\overline{\mathcal{L}}^-_{h,\delta}$, $\overline{\Lambda}^-$ and $\overline{\mathcal{L}}^-_0$.

Proof. Let $0 < \delta < \delta_0$ be fixed. By definition of $\overline{\Lambda}^+(\delta)$ there exists a sequence $\{(h_j, \phi_j, \psi_j)\}$ in $(0,1) \times \mathbf{H}^1$ such that (1) and (2) hold. Taking a subsequence if necessary, we may assume that there exist $(\phi, \psi) \in \mathbf{H}^1$ and $(\theta, \chi) \in \mathbf{L}^2$ such that $(\phi_j, \psi_j) \to (\phi, \psi)$ in $L^2_{\text{loc}}(\mathbb{R}) \times L^2_{\text{loc}}(\mathbb{R})$ and weakly in $\mathbf{H}^1$ as $j \to \infty$ and such that $(\theta_j, \chi_j) \rightharpoonup (\theta, \chi)$ weakly in $\mathbf{L}^2$. By the weak lower-semicontinuity of the $\mathbf{L}^2$-norm, we obtain

$$\|(\theta, \chi)\|_{\mathbf{L}^2} + \tfrac{1}{\delta} |\langle (\theta, \chi), (\phi_0^-, \psi_0^-) \rangle| \leq \overline{\Lambda}^+(\delta). \tag{2.3.30}$$

For any pair of test functions $(\zeta_1, \zeta_2) \in C_c^\infty(\mathbb{R}) \times C_c^\infty(\mathbb{R})$ we have

$$\begin{aligned}
\langle (\theta_j, \chi_j), (\zeta_1, \zeta_2) \rangle &= \langle \overline{\mathcal{L}}_{h_j,\delta}^+(\phi_j, \psi_j), (\zeta_1, \zeta_2) \rangle \\
&= \langle (\phi_j, \psi_j), \overline{\mathcal{L}}_{h_j,\delta}^-(\zeta_1, \zeta_2) \rangle.
\end{aligned} \tag{2.3.31}$$

Since $\overline{u}_0$ is a bounded function, the limit $\tilde{u}_h - \overline{u}_0 \to 0$ in H^1 implies that also $\tilde{u}_h \to \overline{u}_0$ in L^∞. In particular, we can choose $h' > 0$ and $N > 0$ in such a way that $|\tilde{u}_h| \leq N$ and $|\overline{u}_0| \leq N$ for all $0 < h \leq h'$. Since g_u is Lipschitz continuous on $[-N, N]$, there is a constant $K > 0$ such that $|g_u(x) - g_u(y)| \leq K|x - y|$ for all $x, y \in [-N, N]$. We obtain

$$\begin{aligned}
\lim_{h\downarrow 0} \|g_u(\tilde{u}_h) - g_u(\overline{u}_0)\|_{L^2}^2 &= \lim_{h\downarrow 0} \int (g_u(\tilde{u}_h) - g_u(\overline{u}_0))^2 dx \\
&\leq \lim_{h\downarrow 0} \int K^2 (\tilde{u}_h - \overline{u}_0)^2 dx \\
&\leq \lim_{h\downarrow 0} K^2 \|\tilde{u}_h - \overline{u}_0\|_{\mathbf{L}^2}^2 \\
&= 0,
\end{aligned} \tag{2.3.32}$$

together with

$$\begin{aligned}
\lim_{h\downarrow 0} \|g_u(\tilde{u}_h)\zeta_1 - g_u(\overline{u}_0)\zeta_1\|_{L^2} &\leq \lim_{h\downarrow 0} \|\zeta_1\|_\infty \|g_u(\tilde{u}_h) - g_u(\overline{u}_0)\|_{L^2} \\
&= 0.
\end{aligned} \tag{2.3.33}$$

Furthermore, we know that $\tilde{c}_h \to c_0$ as $h \downarrow 0$, which gives

$$\begin{aligned}
\lim_{h\downarrow 0} \|\tilde{c}_h \zeta_1' - c_0 \zeta_1'\|_{L^2} &= \lim_{h\downarrow 0} \|\tilde{c}_h \zeta_2' - c_0 \zeta_2'\|_{L^2} \\
&= 0.
\end{aligned} \tag{2.3.34}$$

Finally, Lemma 2.3.5 implies

$$\lim_{h\downarrow 0} \|\Delta_h \zeta_1 - \zeta_1''\|_{L^2} = 0, \tag{2.3.35}$$

which means that

$$\|\overline{\mathcal{L}}_{h_j,\delta}^-(\zeta_1, \zeta_2) - (\overline{\mathcal{L}}_0^- + \delta)(\zeta_1, \zeta_2)\|_{\mathbf{L}^2} \to 0 \tag{2.3.36}$$

as $j \to \infty$. Sending $j \to \infty$ in (2.3.31), this yields

$$\langle (\theta, \chi), (\zeta_1, \zeta_2) \rangle = \langle (\phi, \psi), (\overline{\mathcal{L}}_0^- + \delta)(\zeta_1, \zeta_2) \rangle. \tag{2.3.37}$$

In particular, we see that (ϕ, ψ) is a weak solution to $(\overline{\mathcal{L}}_0^+ + \delta)(\phi, \psi) = (\theta, \chi)$. Since $\phi \in H^1$, $\psi \in L^2$, $\theta \in L^2$ and

$$\phi'' = c_0 \phi' - g_u(\overline{u}_0)\phi + \delta \phi + \psi - \theta, \tag{2.3.38}$$

we get $\phi'' \in L^2$ and, hence, $\phi \in H^2$. Since we already know that $\psi \in H^1$, we may apply Lemma 2.3.1 and (2.3.30) to obtain

$$\begin{aligned}\|(\phi, \psi)\|_{H^2(\mathbb{R}) \times H^1(\mathbb{R})} &\leq C_2[\|(\theta, \chi)\|_{\mathbf{L}^2} + \tfrac{1}{\delta}|\langle(\theta, \chi), (\phi_0^-, \psi_0^-)\rangle|] \\ &\leq C_2 \overline{\Lambda}^+(\delta).\end{aligned} \tag{2.3.39}$$

$\blacksquare$

The next result is the analogue of Lemma 2.3.7 for the setting where we are considering a spectral set $M \subset \mathbb{C}$ that satisfies (hM). The proof is omitted as it is almost identical to that of Lemma 2.3.7. We recall the constant $C_3 > 0$ from Lemma 2.3.1, which only depends on the choice of the set $M \subset \mathbb{C}$.

Lemma 2.3.8. *Assume that (HP1),(HP2), (HS) and (Hα1) are satisfied. Let $M \subset \mathbb{C}$ satisfy (hM). There exists a sequence $\{(\lambda_j, h_j, \phi_j, \psi_j)\}$ in $M \times (0, 1) \times \mathbf{H}^1$ with the following properties.*

1. *We have $\lim\limits_{j \to \infty} h_j = 0$, $\lim\limits_{j \to \infty} \lambda_j = \lambda$ for some $\lambda \in M$ and $\|(\phi_j, \psi_j)\|_{\mathbf{H}^1} = 1$ for all j.*

2. *The pair $(\theta_j, \chi_j) = \overline{\mathcal{L}}_{h_j, \lambda_j}^+(\phi_j, \psi_j)$ satisfies*

   $$\lim\limits_{j \to \infty} \|(\theta_j, \chi_j)\|_{\mathbf{L}^2} = \overline{\Lambda}^+(M). \tag{2.3.40}$$

3. *There exist $(\phi, \psi) \in \mathbf{H}^1$ and $(\theta, \chi) \in \mathbf{L}^2$ such that as $j \to \infty$ $(\phi_j, \psi_j) \to (\phi, \psi)$ in $L^2_{\mathrm{loc}}(\mathbb{R}) \times L^2_{\mathrm{loc}}(\mathbb{R})$ and weakly in $\mathbf{H}^1$ and such that $(\theta_j, \chi_j) \rightharpoonup (\theta, \chi)$ weakly in $\mathbf{L}^2$.*

4. *The pair (ϕ, ψ) is a weak solution to $(\overline{\mathcal{L}}_0^+ + \lambda)(\phi, \psi) = (\theta, \chi)$.*

5. *We have the bound*

   $$\|(\phi, \psi)\|_{H^2(\mathbb{R}) \times H^1(\mathbb{R})} \leq C_3 \overline{\Lambda}^+(M). \tag{2.3.41}$$

The same statements hold upon replacing $\overline{\mathcal{L}}_{h, \lambda_j}^+$, $\overline{\Lambda}^+(M)$ and $\overline{\mathcal{L}}_0^+$ by $\overline{\mathcal{L}}_{h, \lambda_j}^-$, $\overline{\Lambda}^-$ and $\overline{\mathcal{L}}_0^-$.

In our arguments below, we often consider the sequences $\{(h_j, \phi_j, \psi_j)\}$ and $\{(\lambda_j, h_j, \phi_j, \psi_j)\}$ defined in Lemmas 2.3.7 and 2.3.8 in a similar fashion. To streamline our notation, we simply write $\{(\lambda_j, h_j, \phi_j, \psi_j)\}$ for all these sequences, with the understanding that $\lambda_j = \delta$ when referring to Lemma 2.3.7. As argued in the proof of Lemma 2.3.7, it is possible to choose $\overline{h} > 0$ in such a way that

$$\begin{aligned}c_* &:= \inf_{0 < h \leq \overline{h}} |\tilde{c}_h| &> 0, \\ g_* &:= \sup_{0 < h \leq \overline{h}} \|g_u(\tilde{u}_h)\|_\infty &< \infty.\end{aligned} \tag{2.3.42}$$

By taking a subsequence if necessary, we assume from now on that $h_j < \overline{h}$ for all j.

It remains to find a positive lower bound for $\|(\phi, \psi)\|_{\mathbf{L}^2}$. An essential step to accomplish this is to keep the derivatives (ϕ'_j, ψ'_j) under control. This can be achieved by exploiting the results for Δ_h derived in §2.3.2.

Lemma 2.3.9. *Assume that (hFam), (HP1), (HS) and (Hα1) are satisfied. Consider the setting of Proposition 2.3.4 and Lemma 2.3.7 or Lemma 2.3.8. Then there exists a constant $B > 0$, depending only on M and our choice of $(\tilde{u}_h, \tilde{w}_h)$ and $\tilde{c}_h$, such that for all j we have the bound*

$$B\|(\phi_j, \psi_j)\|_{\boldsymbol{L}^2}^2 \quad \geq \quad c_*^2\|(\phi'_j, \psi'_j)\|_{\boldsymbol{L}^2}^2 - 4\|(\theta_j, \chi_j)\|_{\boldsymbol{L}^2}^2. \tag{2.3.43}$$

Proof. We first consider the sequence for $\overline{\Lambda}^+$. Using $\overline{\mathcal{L}}^+_{h_j, \lambda_j}(\phi_j, \psi_j) = (\theta_j, \chi_j)$ and $\mathrm{Re}\,\langle\Delta_{h_j}\phi_j, \phi'_j\rangle = 0 = \mathrm{Re}\,\langle\phi_j, \phi'_j\rangle = \mathrm{Re}\,\langle\psi_j, \psi'_j\rangle$, which follow from Lemma 2.3.6, we obtain

$$
\begin{aligned}
\mathrm{Re}\,\langle(\theta_j, \chi_j), (\phi'_j, \psi'_j)\rangle &= \mathrm{Re}\,\langle\overline{\mathcal{L}}^+_{h_j, \lambda_j}(\phi_j, \psi_j), (\phi'_j, \psi'_j)\rangle \\
&= \mathrm{Re}\,\langle\tilde{c}_{h_j}\phi'_j - \Delta_{h_j}\phi_j - g_u(\tilde{u}_{h_j})\phi_j + \lambda_j\phi_j + \psi_j, \phi'_j\rangle \\
&\quad + \mathrm{Re}\,\langle-\rho\phi_j + \tilde{c}_{h_j}\psi'_j + \gamma\rho\psi_j + \lambda_j\psi_j, \psi'_j\rangle \\
&= \tilde{c}_{h_j}\|\phi'_j\|_{L^2}^2 - \mathrm{Re}\,\langle g_u(\tilde{u}_{h_j})\phi_j, \phi'_j\rangle + \mathrm{Re}\,\langle\psi_j, \phi'_j\rangle \\
&\quad + \mathrm{Re}\,\langle\lambda_j\phi_j, \phi'_j\rangle - \rho\mathrm{Re}\,\langle\phi_j, \psi'_j\rangle \\
&\quad + \tilde{c}_{h_j}\|\psi'_j\|_{L^2}^2 + \mathrm{Re}\,\langle\lambda_j\psi_j, \psi'_j\rangle \\
&= \tilde{c}_{h_j}\|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2}^2 - \mathrm{Re}\,\langle g_u(\tilde{u}_{h_j})\phi_j, \phi'_j\rangle + (1 + \rho)\langle\psi_j, \phi'_j\rangle \\
&\quad + \mathrm{Re}\,\langle\lambda_j(\phi_j, \psi_j), (\phi'_j, \psi'_j)\rangle.
\end{aligned}
\tag{2.3.44}
$$

We write $\lambda_{\max} = \delta_0$ in the setting of Lemma 2.3.7 or $\lambda_{\max} = \max\{|z| : z \in M\}$ in the setting of Lemma 2.3.8. We write

$$G \quad = \quad \lambda_{\max}\|(\phi_j, \psi_j)\|_{\mathbf{L}^2}\|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2} + g_*\|\phi_j\|_{L^2}\|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2}. \tag{2.3.45}$$

Using the Cauchy-Schwarz inequality, we now obtain

$$
\begin{aligned}
G &\geq \lambda_{\max}\|(\phi_j, \psi_j)\|_{\mathbf{L}^2}\|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2} + \|g_u(\tilde{u}_{h_j})\|_{L^\infty}\|\phi_j\|_{L^2}\|\phi'_j\|_{L^2} \\
&\geq \mathrm{sign}(\tilde{c}_{h_j})\Big(-\mathrm{Re}\,\langle\lambda_j(\phi_j, \psi_j), (\phi'_j, \psi'_j)\rangle + \mathrm{Re}\,\langle g_u(\tilde{u}_{h_j})\phi_j, \phi'_j\rangle\Big) \\
&= \mathrm{sign}(\tilde{c}_{h_j})\Big(\tilde{c}_{h_j}\|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2}^2 + (1 + \rho)\mathrm{Re}\,\langle\psi_j, \phi'_j\rangle - \mathrm{Re}\,\langle(\theta_j, \chi_j), (\phi'_j, \psi'_j)\rangle\Big) \\
&\geq |\tilde{c}_{h_j}|\|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2}^2 - (1 + \rho)\|\psi_j\|_{L^2}\|\phi'_j\|_{L^2} - \|(\theta_j, \chi_j)\|_{\mathbf{L}^2}\|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2} \\
&\geq c_*\|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2}^2 - (1 + \rho)\|\psi_j\|_{L^2}\|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2} - \|(\theta_j, \chi_j)\|_{\mathbf{L}^2}\|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2}.
\end{aligned}
\tag{2.3.46}
$$

This implies

$$c_*\|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2} \quad \leq \quad g_*\|\phi_j\|_{L^2} + (1 + \rho)\|\psi_j\|_{L^2} + \|(\theta_j, \chi_j)\|_{\mathbf{L}^2} + \lambda_{\max}\|(\phi_j, \psi_j)\|_{\mathbf{L}^2}. \tag{2.3.47}$$

Squaring this equation and using the standard inequality $2\mu\omega \leq \mu^2 + \omega^2$, this implies that

$$
\begin{aligned}
c_*^2\|(\phi_j', \psi_j')\|_{\mathbf{L}^2}^2 \quad \leq \quad & 4g_*^2\|\phi_j\|_{L^2}^2 + 4(1+\rho)^2\|\psi_j\|_{L^2}^2 \\
& +4\|(\theta_j, \chi_j)\|_{\mathbf{L}^2}^2 + 4\lambda_{\max}^2\|(\phi_j, \psi_j)\|_{\mathbf{L}^2}^2.
\end{aligned}
\tag{2.3.48}
$$

In particular, we see

$$
4\Big(\max\{g_*^2, (1+\rho)^2\} + \lambda_{\max}^2\Big)\|(\phi_j, \psi_j)\|_{\mathbf{L}^2}^2 \quad \geq \quad c_*^2\|(\phi_j', \psi_j')\|_{\mathbf{L}^2}^2 - 4\|(\theta_j, \chi_j)\|_{\mathbf{L}^2}^2.
\tag{2.3.49}
$$

We now look at the sequence for $\overline{\Lambda}^-$. Using $\overline{\mathcal{L}}^-_{h_j, \lambda_j}(\phi_j, \psi_j) = (\theta_j, \chi_j)$ and $\mathrm{Re}\,\langle \Delta_{h_j}\phi_j, \phi_j'\rangle = 0 = \mathrm{Re}\,\langle \phi_j, \phi_j'\rangle = \mathrm{Re}\,\langle \psi_j, \psi_j'\rangle$, which follow from Lemma 2.3.6, we obtain

$$
\begin{aligned}
\mathrm{Re}\,\langle (\theta_j, \chi_j), (\phi_j', \psi_j')\rangle \quad = \quad & \mathrm{Re}\,\langle \overline{\mathcal{L}}^-_{h_j, \lambda_j}(\phi_j, \psi_j), (\phi_j', \psi_j')\rangle \\
= \quad & \mathrm{Re}\,\langle -\tilde{c}_{h_j}\phi_j' - \Delta_{h_j}\phi_j - g_u(\tilde{u}_h)\phi_j + \lambda_j\phi_j - \rho\psi_j, \phi_j'\rangle \\
& +\mathrm{Re}\,\langle \phi_j - \tilde{c}_h\psi_j' + \gamma\rho\psi_j + \lambda_j\psi_j, \psi_j'\rangle \\
= \quad & -\tilde{c}_{h_j}\|\phi_j'\|_{L^2}^2 - \mathrm{Re}\,\langle g_u(\tilde{u}_h)\phi_j, \phi_j'\rangle - \rho\mathrm{Re}\,\langle \psi_j, \phi_j'\rangle \\
& +\mathrm{Re}\,\langle \lambda_j\phi_j, \phi_j'\rangle + \mathrm{Re}\,\langle \phi_j, \psi_j'\rangle \\
& -\tilde{c}_{h_j}\|\psi_j'\|_{L^2}^2 + \mathrm{Re}\,\langle \lambda_j\psi_j, \psi_j'\rangle \\
= \quad & -\tilde{c}_{h_j}\|(\phi_j', \psi_j')\|_{\mathbf{L}^2}^2 - \mathrm{Re}\,\langle g_u(\tilde{u}_h)\phi_j, \phi_j'\rangle + (1+\rho)\langle \psi_j, \phi_j'\rangle \\
& +\mathrm{Re}\,\langle \lambda_j(\phi_j, \psi_j), (\phi_j', \psi_j')\rangle.
\end{aligned}
\tag{2.3.50}
$$

We write

$$
G \quad = \quad \lambda_{\max}\|(\phi_j, \psi_j)\|_{\mathbf{L}^2}\|(\phi_j', \psi_j')\|_{\mathbf{L}^2} + g_*\|\phi_j\|_{L^2}\|(\phi_j', \psi_j')\|_{\mathbf{L}^2}.
\tag{2.3.51}
$$

Using the Cauchy-Schwarz inequality we now obtain

$$
\begin{aligned}
G \quad \geq \quad & \lambda_{\max}\|(\phi_j, \psi_j)\|_{\mathbf{L}^2}\|(\phi_j', \psi_j')\|_{\mathbf{L}^2} + \|g_u(\tilde{u}_{h_j})\|_{L^\infty}\|\phi_j\|_{L^2}\|\phi_j'\|_{L^2} \\
\geq \quad & -\mathrm{sign}(\tilde{c}_{h_j})\Big(-\mathrm{Re}\,\langle \lambda_j(\phi_j, \psi_j), (\phi_j', \psi_j')\rangle + \mathrm{Re}\,\langle g_u(\tilde{u}_{h_j})\phi_j, \phi_j'\rangle\Big) \\
= \quad & -\mathrm{sign}(\tilde{c}_{h_j})\Big(-\tilde{c}_{h_j}\|(\phi_j', \psi_j')\|_{\mathbf{L}^2}^2 - (1+\rho)\mathrm{Re}\,\langle \psi_j, \phi_j'\rangle - \mathrm{Re}\,\langle (\theta_j, \chi_j), (\phi_j', \psi_j')\rangle\Big) \\
\geq \quad & |\tilde{c}_{h_j}|\|(\phi_j', \psi_j')\|_{\mathbf{L}^2}^2 - (1+\rho)\|\psi_j\|_{L^2}\|\phi_j'\|_{L^2} - \|(\theta_j, \chi_j)\|_{\mathbf{L}^2}\|(\phi_j', \psi_j')\|_{\mathbf{L}^2} \\
\geq \quad & c_*\|(\phi_j', \psi_j')\|_{\mathbf{L}^2}^2 - (1+\rho)\|\psi_j\|_{L^2}\|(\phi_j', \psi_j')\|_{\mathbf{L}^2} - \|(\theta_j, \chi_j)\|_{\mathbf{L}^2}\|(\phi_j', \psi_j')\|_{\mathbf{L}^2}.
\end{aligned}
\tag{2.3.52}
$$

This is the same equation that we derived for $\overline{\Lambda}^+$. Hence, we again obtain

$$
B\|(\phi_j, \psi_j)\|_{\mathbf{L}^2}^2 \quad \geq \quad c_*^2\|(\phi_j', \psi_j')\|_{\mathbf{L}^2}^2 - 4\|(\theta_j, \chi_j)\|_{\mathbf{L}^2}^2,
\tag{2.3.53}
$$

where

$$
B \quad = \quad 4\Big(\max\{g_*^2, (1+\rho)^2\} + \lambda_{\max}^2\Big).
\tag{2.3.54}
$$

■

The next step is to show that the L^2-mass of ϕ_j can be concentrated in a compact interval. We heavily exploit the bistable structure of the nonlinearity g to accomplish this. Moreover, we are aided by the fact that the off-diagonal elements are constant, which allows us to keep the cross-terms under control. In fact, one might be tempted to think that it is sufficient to note that the eigenvalues of the matrix $\begin{pmatrix} -g_u(0) & 1 \\ -\rho & \gamma\rho \end{pmatrix}$ all have positive real part, as then one would be able to find a basis in which this matrix is positive definite. However, passing over to another basis destroys the structure of the diffusion terms and, therefore, does not give any insight.

Lemma 2.3.10. *Assume that (hFam), (HP1), (HS) and (Hα1) are satisfied. Consider the setting of Proposition 2.3.4 and Lemma 2.3.7 or Lemma 2.3.8. There exist positive constants a and m, depending only on our choice of $(\tilde{u}_h, \tilde{w}_h)$, such that we have the following inequality for all j*

$$\tfrac{1}{\rho}(a + g_*) \int_{|x| \leq m} |\phi_j(x)|^2 dx \geq \left(\tfrac{1}{2}\min\{a, \tfrac{1}{2}\rho\gamma\} + \lambda_{\min}\right)\|(\phi_j, \psi_j)\|_{\boldsymbol{L}^2}^2$$
$$- \tfrac{1}{2\min\{a, \frac{1}{2}\rho\gamma\}}\|(\theta_j, \chi_j)\|_{\boldsymbol{L}^2}^2 - \beta\|(\theta_j, \chi_j)\|_{\boldsymbol{L}^2}^2.$$
(2.3.55)

Here we write $\lambda_{\min} = 0$ in the setting of Lemma 2.3.7 or $\lambda_{\min} = \min\{\text{Re }\lambda : \lambda \in M\}$ in the setting of Lemma 2.3.8, together with

$$\beta = \tfrac{1-\rho}{\rho}\tfrac{1}{4(\frac{\rho}{1-\rho}\frac{1}{2}\gamma\rho + \gamma\rho + \lambda_{\min})}.$$
(2.3.56)

Proof. Again we first look at the sequence for $\overline{\Lambda}^+$. We know that $\tilde{u}_h - \overline{u}_0 \to 0$ in H^1 as $h \downarrow 0$. Hence, it follows that $\tilde{u}_h - \overline{u}_0 \to 0$ in L^∞ and, therefore, also $g_u(\tilde{u}_h) - g_u(\overline{u}_0) \to 0$ in L^∞ as $h \downarrow 0$. By the bistable nature of our nonlinearity g, we can choose m to be a positive constant such that for all $h \in [0, \overline{h}]$ (by making $\overline{h}$ smaller if necessary)

$$\min_{|x| \geq m}[-g_u(\tilde{u}_h(x))] \geq a := \tfrac{1}{2}r_0 > 0.$$
(2.3.57)

Here r_0 is the constant appearing in the choice of our function g in (HS). Then we obtain, using $\text{Re }\langle \phi_j', \phi_j \rangle = \text{Re }\langle \psi_j', \psi_j \rangle = 0$ and $\text{Re }\langle -\Delta_{h_j}\phi_j, \phi_j \rangle \geq 0$, which we know

from Lemma 2.3.6, that

$$
\begin{aligned}
\operatorname{Re}\langle(\theta_j,\chi_j),(\phi_j,\psi_j)\rangle &= \operatorname{Re}\langle\overline{\mathcal{L}}^+_{h_j,\lambda_j}(\phi_j,\psi_j),(\phi_j,\psi_j)\rangle \\
&\geq \operatorname{Re}\langle -g_u(\tilde{u}_{h_j})\phi_j,\phi_j\rangle + \operatorname{Re}\langle\psi_j,\phi_j\rangle \\
&\quad -\rho\operatorname{Re}\langle\psi_j,\phi_j\rangle + \gamma\rho\|\psi_j\|^2_{L^2} + \lambda_{\min}\|(\phi_j,\psi_j)\|^2 \\
&\geq \min_{|x|\geq m}\{-g_u(\tilde{u}_{h_j}(x))\}\int_{|x|\geq m}|\phi_j(x)|^2 dx \\
&\quad -\|g_u(\tilde{u}_{h_j})\|_{L^\infty}\int_{|x|\leq m}|\phi_j(x)|^2 dx + (1-\rho)\operatorname{Re}\langle\psi_j,\phi_j\rangle \\
&\quad +\gamma\rho\|\psi_j\|^2_{L^2} + \lambda_{\min}\|(\phi_j,\psi_j)\|^2 \\
&\geq a\|\phi_j\|^2_{L^2} - (a+g_*)\int_{|x|\leq m}|\phi_j(x)|^2 dx + (1-\rho)\operatorname{Re}\langle\psi_j,\phi_j\rangle \\
&\quad +\gamma\rho\|\psi_j\|^2_{L^2} + \lambda_{\min}\|(\phi_j,\psi_j)\|^2.
\end{aligned}
\tag{2.3.58}
$$

We assumed that $0 < \rho < 1$ so we see that $\frac{1-\rho}{-\rho} < 0$. We set

$$
\beta_j^+ = \frac{1}{4(\frac{\rho}{1-\rho}\frac{1}{2}\gamma\rho + \gamma\rho + \operatorname{Re}\lambda_j)}.
\tag{2.3.59}
$$

Now we obtain

$$
\begin{aligned}
\operatorname{Re}\langle\chi_j,\psi_j\rangle &\leq \|\chi_j\|_{L^2}\|\psi_j\|_{L^2} \\
&= \frac{1}{\sqrt{2(\frac{\rho}{1-\rho}\frac{1}{2}\gamma\rho+\gamma\rho+\operatorname{Re}\lambda_j)}}\|\chi_j\|_{L^2}\sqrt{2(\tfrac{\rho}{1-\rho}\tfrac{1}{2}\gamma\rho+\gamma\rho+\operatorname{Re}\lambda_j)}\|\psi_j\|_{L^2} \\
&\leq \frac{1}{4(\frac{\rho}{1-\rho}\frac{1}{2}\gamma\rho+\gamma\rho+\operatorname{Re}\lambda_j)}\|\chi_j\|^2_{L^2} + (\tfrac{\rho}{1-\rho}\tfrac{1}{2}\gamma\rho+\gamma\rho+\operatorname{Re}\lambda_j)\|\psi_j\|^2_{L^2} \\
&= \beta_j^+\|\chi_j\|^2_{L^2} + (\tfrac{\rho}{1-\rho}\tfrac{1}{2}\gamma\rho+\gamma\rho+\operatorname{Re}\lambda_j)\|\psi_j\|^2_{L^2}.
\end{aligned}
\tag{2.3.60}
$$

Note that the denominator $4(\frac{\rho}{1-\rho}\frac{1}{2}\gamma\rho + \gamma\rho + \operatorname{Re}\lambda_j)$ is never zero since we can assume that λ_* is small enough to have $\operatorname{Re}\lambda_j \geq -\lambda_* > -\gamma\rho$. Using the identity

$$
\chi_j = -\rho\phi_j + \tilde{c}_{h_j}\psi_j' + \gamma\rho\psi_j + \lambda_j\psi_j
\tag{2.3.61}
$$

and the fact that $\operatorname{Re}\langle\psi_j',\psi_j\rangle = 0$, we also have

$$
\operatorname{Re}\langle\chi_j,\psi_j\rangle = -\rho\operatorname{Re}\langle\phi_j,\psi_j\rangle + (\gamma\rho+\operatorname{Re}\lambda_j)\|\psi_j\|^2_{L^2}.
\tag{2.3.62}
$$

Hence, we must have that

$$
\begin{aligned}
(1-\rho)\operatorname{Re}\langle\phi_j,\psi_j\rangle &= \tfrac{1-\rho}{\rho}\Big(-\operatorname{Re}\langle\chi_j,\psi_j\rangle + (\gamma\rho+\operatorname{Re}\lambda_j)\|\psi_j\|^2_{L^2}\Big) \\
&\geq \tfrac{1-\rho}{\rho}\Big(-\beta_j^+\|\chi_j\|^2_{L^2} - (\tfrac{\rho}{1-\rho}\tfrac{1}{2}\gamma\rho+\gamma\rho+\operatorname{Re}\lambda_j)\|\psi_j\|^2_{L^2} \\
&\quad + (\gamma\rho+\operatorname{Re}\lambda_j)\|\psi_j\|^2_{L^2}\Big) \\
&= -\tfrac{1-\rho}{\rho}\beta_j^+\|\chi_j\|^2_{L^2} - \tfrac{1}{2}\gamma\rho\|\psi_j\|^2_{L^2}.
\end{aligned}
\tag{2.3.63}
$$

Combining this bound with (2.3.58) yields the estimate

$$
\begin{aligned}
\operatorname{Re}\langle(\theta_j,\chi_j),(\phi_j,\psi_j)\rangle \;\geq\; & a\|\phi_j\|_{L^2}^2 - (a+g_*)\int_{|x|\leq m}|\phi_j(x)|^2dx + (1-\rho)\operatorname{Re}\langle\psi_j,\phi_j\rangle \\
& +\gamma\rho\|\psi_j\|_{L^2}^2 + \lambda_{\min}\|(\phi_j,\psi_j)\|^2 \\
\geq\; & a\|\phi_j\|_{L^2}^2 - (a+g_*)\int_{|x|\leq m}|\phi_j(x)|^2dx \\
& +\tfrac{1}{2}\gamma\rho\|\psi_j\|_{L^2}^2 + \lambda_{\min}\|(\phi_j,\psi_j)\|^2 - \tfrac{1-\rho}{\rho}\beta_j^+\|\chi_j\|_{L^2}^2 .
\end{aligned}
\tag{2.3.64}
$$

We now look at the sequence for $\overline{\Lambda}^{-}$. Let m and a be as before. Then we obtain, using $\overline{\mathcal{L}}_{h_j,\lambda_j}(\phi_j,\psi_j)=(\theta_j,\chi_j)$, $\operatorname{Re}\langle\phi_j',\phi_j\rangle = \operatorname{Re}\langle\psi_j',\psi_j\rangle = 0$ and $\operatorname{Re}\langle-\Delta_{h_j}\phi_j,\phi_j\rangle \geq 0$ that

$$
\begin{aligned}
\operatorname{Re}\langle(\theta_j,\chi_j),(\phi_j,\psi_j)\rangle \;=\; & \operatorname{Re}\langle\overline{\mathcal{L}}_{h_j,\delta}(\phi_j,\psi_j),(\phi_j,\psi_j)\rangle \\
\geq\; & \operatorname{Re}\langle-g_u(\tilde{u}_h)\phi_j,\phi_j\rangle + (1-\rho)\operatorname{Re}\langle\psi_j,\phi_j\rangle \\
& +\gamma\rho\|\psi_j\|_{L^2}^2 + \lambda_{\min}\|(\phi_j,\psi_j)\|_{\mathbf{L}^2}^2 .
\end{aligned}
\tag{2.3.65}
$$

We set

$$
\beta_j^{-} \;=\; \frac{1}{4(\frac{1}{1-\rho}\frac{1}{2}\gamma\rho+\gamma\rho+\operatorname{Re}\lambda_j)} .
\tag{2.3.66}
$$

Arguing as in (2.3.60) with different constants, we obtain

$$
\begin{aligned}
\operatorname{Re}\langle\theta_j,\phi_j\rangle \;\geq\; & -\|\theta_j\|_{L^2}\|\phi_j\|_{L^2} \\
\geq\; & -\tfrac{1}{4(a+\operatorname{Re}\lambda_j)}\|\theta_j\|_{L^2}^2 - (a+\operatorname{Re}\lambda_j)\|\phi_j\|_{L^2}^2 \\
=\; & -\beta_j^{-}\|\theta_j\|_{L^2}^2 - (a+\operatorname{Re}\lambda_j)\|\phi_j\|_{L^2}^2 .
\end{aligned}
\tag{2.3.67}
$$

Note that the denominator $4(a+\operatorname{Re}\lambda_j)$ is never zero since we can assume that λ_* is small enough to have $\operatorname{Re}\lambda_j \geq -\lambda_* > -a$. Using the identity

$$
\theta_j \;=\; -\tilde{c}_{h_j}\phi_j' - \Delta_h\phi_j - g_u(\tilde{u}_h)\phi_j + \lambda_j\psi_j - \rho\phi_j
\tag{2.3.68}
$$

and the fact that $\operatorname{Re}\langle\phi_j',\phi_j\rangle = 0$, we also have

$$
\begin{aligned}
\operatorname{Re}\langle\theta_j,\phi_j\rangle \;=\; & \operatorname{Re}\langle-\Delta_h\phi_j,\phi_j\rangle + \operatorname{Re}\langle-g_u(\tilde{u}_h)\phi_j,\phi_j\rangle \\
& +\operatorname{Re}\lambda_j\|\psi_j\|_{L^2}^2 - \rho\operatorname{Re}\langle\phi_j,\psi_j\rangle .
\end{aligned}
\tag{2.3.69}
$$

Hence, we must have that

$$
\begin{aligned}
(1-\rho)\operatorname{Re}\langle\phi_j,\psi_j\rangle \;=\; & \tfrac{1-\rho}{\rho}\Big(-\operatorname{Re}\langle\theta_j,\phi_j\rangle + \operatorname{Re}\langle-\Delta_h\phi_j,\phi_j\rangle \\
& +\operatorname{Re}\langle-g_u(\tilde{u}_h)\phi_j,\phi_j\rangle + \operatorname{Re}\lambda_j\|\psi_j\|_{L^2}^2\Big) \\
\geq\; & \tfrac{1-\rho}{\rho}\Big(-\beta_j^{-}\|\theta_j\|_{L^2}^2 - \big(a+\operatorname{Re}\lambda_j\big)\|\phi_j\|_{L^2}^2 \\
& +\operatorname{Re}\langle-g_u(\tilde{u}_h)\phi_j,\phi_j\rangle + \operatorname{Re}\lambda_j\|\psi_j\|_{L^2}^2\Big) \\
=\; & \tfrac{1-\rho}{\rho}\Big(-\beta_j^{-}\|\theta_j\|_{L^2}^2 - a\|\phi_j\|_{L^2}^2 + \operatorname{Re}\langle-g_u(\tilde{u}_h)\phi_j,\phi_j\rangle\Big) .
\end{aligned}
\tag{2.3.70}
$$

Combining this with the estimate (2.3.65) and noting that $\frac{1-\rho}{\rho} + 1 = \frac{1}{\rho}$ yields

$$
\begin{aligned}
\mathrm{Re}\,\langle(\theta_j, \chi_j), (\phi_j, \psi_j)\rangle \;\geq\;& \tfrac{1}{\rho}\mathrm{Re}\,\langle -g_u(\tilde{u}_h)\phi_j, \phi_j\rangle + \lambda_{\min}\|(\phi_j, \psi_j)\|_{\mathbf{L}^2}^2 \\
& + \gamma\rho\|\psi_j\|_{L^2}^2 - a\tfrac{1-\rho}{\rho}\|\phi_j\|_{L^2}^2 - \tfrac{1-\rho}{\rho}\beta_j^-\|\theta_j\|_{L^2}^2 \\
\geq\;& \tfrac{1}{\rho}\Big(\min_{|x|\geq m}\{-g_u(\tilde{u}_h(x))\}\int_{|x|\geq m}|\phi_j|^2 dx \\
& -\|g_u(\tilde{u}_h)\|_{L^\infty}\int_{|x|\leq m}|\phi_j|^2 dx\Big) + \lambda_{\min}\|(\phi_j, \psi_j)\|_{\mathbf{L}^2}^2 \\
& + \gamma\rho\|\psi_j\|_{L^2}^2 - a\tfrac{1-\rho}{\rho}\|\phi_j\|_{L^2}^2 - \tfrac{1-\rho}{\rho}\beta_j^-\|\theta_j\|_{L^2}^2 \\
\geq\;& a\|\phi_j\|_{L^2}^2 - \tfrac{1}{\rho}(a+g_*)\int_{|x|\leq m}|\phi_j|^2 dx + \gamma\rho\|\psi_j\|_{L^2}^2 \\
& + \lambda_{\min}\|(\phi_j, \psi_j)\|_{\mathbf{L}^2}^2 - \tfrac{1-\rho}{\rho}\beta_j^-\|\theta_j\|_{L^2}^2.
\end{aligned}
\tag{2.3.71}
$$

Upon setting

$$
\beta \;=\; \tfrac{1-\rho}{\rho}\min\Big\{\tfrac{1}{4\left(\frac{\rho}{1-\rho}\frac{1}{2}\gamma\rho+\gamma\rho+\lambda_{\min}\right)}, \tfrac{1}{4(a+\lambda_{\min})}\Big\},
\tag{2.3.72}
$$

we note that $\frac{1-\rho}{\rho}\beta_j^+ \leq \beta$ and $\frac{1-\rho}{\rho}\beta_j^- \leq \beta$ for all j since $\rho < 1$ and since β_j^+ and β_j^- are maximal for $\mathrm{Re}\,\lambda = \lambda_{\min}$. Therefore, in both cases, we obtain

$$
\begin{aligned}
\tfrac{1}{\rho}(a+g_*)\int_{|x|\leq m}|\phi_j(x)|^2 dx \;\geq\;& a\|\phi_j\|_{L^2}^2 + \tfrac{1}{2}\rho\gamma\|\psi_j\|_{L^2}^2 - \mathrm{Re}\,\langle(\theta_j, \chi_j), (\phi_j, \psi_j)\rangle \\
& - \beta\|(\theta_j, \chi_j)\|_{\mathbf{L}^2}^2 + \lambda_{\min}\|(\phi_j, \psi_j)\|_{\mathbf{L}^2}^2 \\
\geq\;& \Big(\min\{a, \tfrac{1}{2}\rho\gamma\} + \lambda_{\min}\Big)\|(\phi_j, \psi_j)\|_{\mathbf{L}^2}^2 \\
& - \tfrac{\|(\theta_j, \chi_j)\|_{\mathbf{L}^2}}{\sqrt{\min\{a, \frac{1}{2}\rho\gamma\}}}\sqrt{\min\{a, \tfrac{1}{2}\rho\gamma\}}\|(\phi_j, \psi_j)\|_{\mathbf{L}^2} \\
& - \beta\|(\theta_j, \chi_j)\|_{\mathbf{L}^2}^2
\end{aligned}
\tag{2.3.73}
$$

and thus, again using the inequality $2\mu\omega \leq \mu^2 + \omega^2$ for $\mu, \omega \in \mathbb{R}$,

$$
\begin{aligned}
\tfrac{1}{\rho}(a+g_*)\int_{|x|\leq m}|\phi_j(x)|^2 dx \;\geq\;& \Big(\min\{a, \tfrac{1}{2}\rho\gamma\} + \lambda_{\min}\Big)\|(\phi_j, \psi_j)\|_{\mathbf{L}^2}^2 \\
& - \tfrac{1}{2}\tfrac{1}{\min\{a, \frac{1}{2}\rho\gamma\}}\|(\theta_j, \chi_j)\|_{\mathbf{L}^2}^2 \\
& - \tfrac{1}{2}\min\{a, \tfrac{1}{2}\rho\gamma\}\|(\phi_j, \psi_j)\|_{\mathbf{L}^2}^2 - \beta\|(\theta_j, \chi_j)\|_{\mathbf{L}^2}^2 \\
=\;& \Big(\tfrac{1}{2}\min\{a, \tfrac{1}{2}\rho\gamma\} + \lambda_{\min}\Big)\|(\phi_j, \psi_j)\|_{\mathbf{L}^2}^2 \\
& - \tfrac{1}{2\min\{a, \frac{1}{2}\rho\gamma\}}\|(\theta_j, \chi_j)\|_{\mathbf{L}^2}^2 - \beta\|(\theta_j, \chi_j)\|_{\mathbf{L}^2}^2,
\end{aligned}
\tag{2.3.74}
$$

as desired. ∎

Lemma 2.3.11. *Assume that (hFam), (HP1), (HS) and (Hα1) are satisfied. Consider the setting of Proposition 2.3.4 and Lemma 2.3.7 or Lemma 2.3.8. There exist positive constants C_4 and C_5, depending only on M and our choice of $(\tilde{u}_h, \tilde{w}_h)$ and $\tilde{c}_h$, such*

that for all j we have

$$\frac{1}{\rho}(a + g_*) \int_{|x| \leq m} |\phi_j^2(x)| dx \geq C_4 - C_5 \|(\theta_j, \chi_j)\|_{L^2}^2. \tag{2.3.75}$$

Proof. Without loss of generality we assume that $\frac{1}{2}\min\{a, \frac{1}{2}\rho\gamma\} + \lambda_{\min} > 0$. Write

$$\mu = \frac{\frac{1}{2}\min\{a, \frac{1}{2}\rho\gamma\} + \lambda_{\min}}{c_*^2 + B}. \tag{2.3.76}$$

Adding μ times equation (2.3.43) to equation (2.3.55) gives

$$
\begin{aligned}
\frac{1}{\rho}(a + g_*) \int_{|x| \leq m} |\phi_j(x)|^2 dx \geq{} & \mu c_*^2 \|(\phi_j', \psi_j')\|_{L^2}^2 - 4\mu \|(\theta_j, \chi_j)\|_{L^2}^2 \\
& + \frac{1}{2}(\min\{a, \frac{1}{2}\rho\gamma\} + \lambda_{\min})\|(\phi_j, \psi_j)\|_{L^2}^2 \\
& - \frac{1}{2(\min\{a, \frac{1}{2}\rho\gamma\} + \lambda_{\min})}\|(\theta_j, \chi_j)\|_{L^2}^2 \\
& - \beta\|(\theta_j, \chi_j)\|_{L^2}^2 - B\mu\|(\phi_j, \psi_j)\|_{L^2}^2.
\end{aligned}
\tag{2.3.77}
$$

We hence obtain

$$
\begin{aligned}
\frac{1}{\rho}(a + g_*) \int_{|x| \leq m} |\phi_j(x)|^2 dx \geq{} & -C_5\|(\theta_j, \chi_j)\|_{L^2}^2 + \mu c_*^2\|(\phi_j', \psi_j')\|_{L^2}^2 \\
& + \frac{1}{2}(\min\{a, \frac{1}{2}\rho\gamma\} + \lambda_{\min})\|(\phi_j, \psi_j)\|_{L^2}^2 \\
& - B\mu\|(\phi_j, \psi_j)\|_{L^2}^2,
\end{aligned}
\tag{2.3.78}
$$

where

$$
\begin{aligned}
C_5 ={} & 4\mu + \frac{1}{2(\min\{a, \frac{1}{2}\rho\gamma\} + \lambda_{\min})} + \beta \\
>{} & 0.
\end{aligned}
\tag{2.3.79}
$$

This allows us to compute

$$
\begin{aligned}
\frac{1}{\rho}(a + g_*) \int_{|x| \leq m} |\phi_j(x)|^2 dx \geq{} & -C_5\|(\theta_j, \chi_j)\|_{L^2}^2 + \mu c_*^2\|(\phi_j', \psi_j')\|_{L^2}^2 \\
& + \frac{1}{2}(\min\{a, \frac{1}{2}\rho\gamma\} + \lambda_{\min})\|(\phi_j, \psi_j)\|_{L^2}^2 \\
& - B\mu\|(\phi_j, \psi_j)\|_{L^2}^2 \\
={} & -C_5\|(\theta_j, \chi_j)\|_{L^2}^2 + \mu c_*^2\|(\phi_j', \psi_j')\|_{L^2} \\
& + (\mu(c_*^2 + B) - B\mu)\|(\phi_j, \psi_j)\|_{L^2}^2 \\
={} & \mu c_*^2\|(\phi_j, \psi_j)\|_{H^1}^2 - C_5\|(\theta_j, \chi_j)\|_{L^2}^2 \\
={} & C_4 - C_5\|(\theta_j, \chi_j)\|_{L^2}^2,
\end{aligned}
\tag{2.3.80}
$$

where $C_4 = \mu c_*^2 > 0$. $\blacksquare$

Proof of Proposition 2.3.4. We first choose $0 < \delta < \delta_0$ and consider the setting of Lemma 2.3.7. Sending $j \to \infty$ in (2.3.75), Lemma 2.3.7 implies

$$
\begin{aligned}
C_4 - C_5\overline{\Lambda}^{\pm}(\delta)^2 \leq{} & C_4 - C_5 \lim_{j \to \infty} \|(\theta_j, \chi_j)\|_{L^2}^2 \\
\leq{} & \frac{1}{\rho}(a + g_*) \int_{|x| \leq m} |\phi|^2 dx \\
\leq{} & \frac{1}{\rho}(a + g_*)\|(\phi, \psi)\|_{H^2(\mathbb{R}) \times H^1(\mathbb{R})}^2 \\
\leq{} & \frac{1}{\rho}(a + g_*)C_2^2\overline{\Lambda}^{+}(\delta)^2.
\end{aligned}
\tag{2.3.81}
$$

Solving this quadratic inequality, we obtain

$$
\begin{aligned}
\overline{\Lambda}^{\pm}(\delta) \;\; &\geq \;\; \sqrt{\frac{C_4}{\frac{1}{\rho}(a+g_*)C_2^2+C_5}} \\
&:= \;\; \mathbb{C}_0.
\end{aligned}
\tag{2.3.82}
$$

The analogous computation in the setting of Lemma 2.3.8 yields

$$
\begin{aligned}
\overline{\Lambda}^{+}(M) \;\; &\geq \;\; \sqrt{\frac{C_4}{\frac{1}{\rho}(a+g_*)C_3^2+C_5}} \\
&:= \;\; C_M.
\end{aligned}
\tag{2.3.83}
$$

∎

2.4 Existence of pulse solutions

In this section, we prove our first main result, Theorem 2.2.1. In particular, we construct solutions to (2.2.12) by writing

$$
(\overline{u}_h, \overline{w}_h) \;\; = \;\; (\overline{u}_0, \overline{w}_0) + (\phi_h, \psi_h)
\tag{2.4.1}
$$

and exploiting the linear results of §2.3. Here $(\overline{u}_0, \overline{w}_0)$ is the pulse solution of the PDE (2.1.1).

The arguments presented in this section are strongly reminiscent of a standard proof of the implicit function theorem. However, the singular nature of the $h \downarrow 0$ limit requires some minor adjustments pertaining to the linearisation that is used. In particular, we fix a small $\delta > 0$ that will be determined later and consider the linear operator

$$
\mathcal{L}_{h,\delta}^{+} : \mathbf{H}^1 \;\; \rightarrow \;\; \mathbf{L}^2,
\tag{2.4.2}
$$

defined by

$$
\mathcal{L}_{h,\delta}^{+} \;\; = \;\; \begin{pmatrix} c_0 \frac{d}{d\xi} - \Delta_h - g_u(\overline{u}_0) + \delta & 1 \\ -\rho & c_0 \frac{d}{d\xi} + \gamma\rho + \delta \end{pmatrix}.
\tag{2.4.3}
$$

This operator arises as the linearisation of (2.2.1) around the pulse solution $(\overline{u}_0, \overline{w}_0)$ of (2.1.1). A short computation shows that our travelling wave triplet $(c_h, \phi_h, \psi_h) \in \mathbb{R} \times \mathbf{H}^1$ must satisfy

$$
\mathcal{L}_{h,\delta}^{+}(\phi_h, \psi) \;\; = \;\; \mathcal{R}(c_h, \phi_h, \psi_h),
\tag{2.4.4}
$$

where

$$
\mathcal{R}(c, \phi, \psi) \;\; = \;\; \Big((c_0 - c)(\overline{u}_0' + \phi') + (\Delta_h - \tfrac{d^2}{d\xi^2})\overline{u}_0 + \delta\phi + \mathcal{N}(\overline{u}_0, \phi), (c_0 - c)(\overline{w}_0' + \psi') \Big).
\tag{2.4.5}
$$

Here we have introduced the nonlinearity

$$
\mathcal{N}(\overline{u}_0, \phi) \;\; = \;\; g(\overline{u}_0 + \phi) - g(\overline{u}_0) - g_u(\overline{u}_0)\phi.
\tag{2.4.6}
$$

Corollary 2.4.1. *Assume that (HP1), (HS) and (Hα1) are satisfied. There exists a positive constant C_0 and a positive function $h_0(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $\delta > 0$ and all $h \in (0, h_0(\delta))$, the operator $\mathcal{L}_{h,\delta}^+$ is a homeomorphism for which we have the bound*

$$\|(\mathcal{L}_{h,\delta}^+)^{-1}(\theta, \chi)\|_{\mathbf{H}^1} \quad \leq \quad C_0\|(\theta, \chi)\|_{\mathbf{L}^2} \tag{2.4.7}$$

for all $(\theta, \chi) \in \mathbf{L}^2$ that satisfy $\langle(\theta, \chi), (\phi_0^-, \psi_0^-)\rangle = 0$.

Proof. This is immediate by choosing $(\tilde{u}_h, \tilde{w}_h) = (\overline{u}_0, \overline{w}_0)$ and $\tilde{c}_h = c_0$ for all h in (hFam) and applying Proposition 2.3.2. ∎

Let η be a small positive constant to be determined later. We define

$$X_\eta \quad = \quad \{(\phi, \psi) \in \mathbf{H}^1 : \|(\phi, \psi)\|_{\mathbf{H}^1} \leq \eta\}. \tag{2.4.8}$$

For every $(\phi, \psi) \in X_\eta$, we define $c_h = c_h(\phi, \psi)$ to be the constant

$$c_h(\phi, \psi) \quad = \quad c_0 + \frac{\langle\Delta_h\overline{u}_0 - \overline{u}_0'', \phi_0^-\rangle + \delta\langle\phi, \phi_0^-\rangle + \langle\mathcal{N}(\overline{u}_0, \phi), \phi_0^-\rangle}{\langle\overline{u}_0', \phi_0^-\rangle + \langle\phi', \phi_0^-\rangle + \langle\overline{w}_0', \psi_0^-\rangle + \langle\psi', \psi_0^-\rangle}. \tag{2.4.9}$$

When this expression is well-defined, this choice ensures that

$$\langle\mathcal{R}(c_h(\phi, \psi), \phi, \psi), (\phi_0^-, \psi_0^-)\rangle \quad = \quad 0. \tag{2.4.10}$$

We define $T : X_\eta \subset \mathbf{H}^1 \to \mathbf{H}^1$ by

$$T(\phi, \psi) \quad = \quad (\mathcal{L}_{h,\delta}^+)^{-1}\mathcal{R}(c_h(\phi, \psi), \phi, \psi). \tag{2.4.11}$$

Our goal is to show T maps X_η into itself and is a contraction, since then the fixed point (ϕ_h, ψ_h) leads to a travelling pulse solution of (2.2.12) via (2.4.1) and (2.4.9).

Exploiting (2.4.10), Corollary 2.4.1 implies that there exists a constant $C_0 > 0$ such that for all $\Psi = (\phi, \psi) \in X_\eta$ we have the bound

$$\|T(\Psi)\|_{\mathbf{H}^1} \quad \leq \quad C_0\|\mathcal{R}(c_h(\Psi), \Psi)\|_{\mathbf{L}^2}, \tag{2.4.12}$$

while for all $\Psi_1 = (\phi_1, \psi_1), \Psi_2 = (\phi_2, \psi_2) \in X_\eta$ we have the bound

$$\|T(\Psi_1) - T(\Psi_2)\|_{\mathbf{H}^1} \quad \leq \quad C_0\|\mathcal{R}(c_h(\Psi_1), \Psi_1) - \mathcal{R}(c_h(\Psi_2), \Psi_2)\|_{\mathbf{L}^2}. \tag{2.4.13}$$

In the remainder of this section we, therefore, set out to estimate the right-hand sides of (2.4.12) and (2.4.13). We start by estimating the nonlinear term $\mathcal{N}(\overline{u}_0, \cdot)$.

Lemma 2.4.2. *Assume that (HP1), (HS) and (Hα1) are satisfied. Then there exists a constant $K > 0$ such that for all $0 < \eta \leq 1$, $(\phi, \psi) \in X_\eta, (\phi_1, \psi_1) \in X_\eta$ and $(\phi_2, \psi_2) \in X_\eta$ we have the pointwise inequalities*

$$\begin{aligned}
|\mathcal{N}(\overline{u}_0, \phi)| \quad &\leq \quad K\eta|\phi|, \\
|\mathcal{N}(\overline{u}_0, \phi_1) - \mathcal{N}(\overline{u}_0, \phi_2)| \quad &\leq \quad \eta K|\phi_1 - \phi_2|.
\end{aligned} \tag{2.4.14}$$

Proof. To estimate the nonlinear term $\mathcal{N}(\overline{u}_0, \phi)$, we first recall the embedding $\|\phi\|_{L^\infty} \leq \|\phi\|_{H^1} \leq \eta \leq 1$ for every $(\phi, \psi) \in X_\eta$. Setting $K = \max\{6, \sup_{|s| \leq \|\overline{u}_0\|_\infty} |g_{uu}(s)|\}$, a Taylor expansion around $\overline{u}_0$ allows us to obtain the pointwise inequalities

$$
\begin{aligned}
|\mathcal{N}(\overline{u}_0, \phi)| &= |-g(\overline{u}_0 + \phi) + g(\overline{u}_0) + g_u(\overline{u}_0)| \\
&= |-g(\overline{u}_0) - \phi g_u(\overline{u}_0) - \tfrac{1}{2}\phi^2 g_{uu}(\xi) + g(\overline{u}_0) + g_u(\overline{u}_0)| \\
&= |-\tfrac{1}{2}\phi^2 g_{uu}(\xi)| \\
&\leq \tfrac{1}{2}K\eta|\phi| \\
&\leq K\eta|\phi|,
\end{aligned}
\tag{2.4.15}
$$

where ξ is between $\overline{u}_0$ and $\overline{u}_0 + \phi$. Note that $g_{uuu} = 6$ is constant. Furthermore, for $(\phi_1, \psi_1) \in X_\eta$ and $(\phi_2, \psi_2) \in X_\eta$, a Taylor expansion around $\overline{u}_0$ yields the pointwise inequalities

$$
\begin{aligned}
|\mathcal{N}(\overline{u}_0, \phi_1) - \mathcal{N}(\overline{u}_0, \phi_2)| &= \Big| -g(\overline{u}_0 + \phi_1) + g(\overline{u}_0) + g_u(\overline{u}_0)\phi_1 \\
&\qquad + g(\overline{u}_0 + \phi_2) - g(\overline{u}_0) - g_u(\overline{u}_0)\phi_2 \Big| \\
&= \Big| -\tfrac{1}{2}g_{uu}(\overline{u}_0)\phi_1^2 + \tfrac{1}{2}g_{uu}(\overline{u}_0)\phi_2^2 - \tfrac{1}{6}g_{uuu}(\xi_1)\phi_1^3 + \tfrac{1}{6}g_{uuu}(\xi_2)\phi_2^3 \Big| \\
&\leq \tfrac{1}{2}|g_{uu}(\overline{u}_0)||\phi_1^2 - \phi_2^2| + \tfrac{1}{6}6|\phi_1^3 - \phi_2^3| \\
&\leq \tfrac{1}{2}|g_{uu}(\overline{u}_0)|\Big[|\phi_1||\phi_1 - \phi_2| + |\phi_2||\phi_1 - \phi_2|\Big] \\
&\qquad + |\phi_1||\phi_1^2 - \phi_2^2| + |\phi_1 - \phi_2||\phi_2^2| \\
&\leq \tfrac{1}{2}\tfrac{1}{2}K\Big[2\eta|\phi_1 - \phi_2|\Big] + \eta\Big[2\eta|\phi_1 - \phi_2|\Big] + \eta^2|\phi_1 - \phi_2| \\
&\leq \eta K|\phi_1 - \phi_2|,
\end{aligned}
\tag{2.4.16}
$$

where ξ_1 is between $\overline{u}_0$ and $\overline{u}_0 + \phi_1$ and ξ_2 is between $\overline{u}_0$ and $\overline{u}_0 + \phi_2$. ∎

Pick $(\phi, \psi) \in X_\eta$. Recall that we chose (ϕ_0^-, ψ_0^-) so that $\langle (\phi_0^-, \psi_0^-), (\overline{u}_0', \overline{w}_0') \rangle > 0$. Let $s > 0$ be defined as

$$
s = 2 \frac{1}{\langle (\phi_0^-, \psi_0^-), (\overline{u}_0', \overline{w}_0') \rangle}.
\tag{2.4.17}
$$

For notational compactness, we write

$$
\sigma(\phi, \psi) = \langle \overline{u}_0', \phi_0^- \rangle + \langle \phi', \phi_0^- \rangle + \langle \overline{w}_0', \psi_0^- \rangle + \langle \psi', \psi_0^- \rangle
\tag{2.4.18}
$$

for $(\phi, \psi) \in X_\eta$. We also write

$$
\eta_0 = \min\{1, s^{-1}\}.
\tag{2.4.19}
$$

Lemma 2.4.3. *Assume that (HP1), (HS) and (Hα1) are satisfied. Fix $0 < \eta \leq \eta_0$. Then for all $\Psi = (\phi, \psi) \in X_\eta, \Psi_1 = (\phi_1, \psi_1) \in X_\eta$ and $\Psi_2 = (\phi_2, \psi_2) \in X_\eta$ we have the bounds*

$$
0 < [\sigma(\Psi)]^{-1} \leq s,
\tag{2.4.20}
$$

together with

$$|\sigma(\Psi_1) - \sigma(\Psi_2)| \quad \leq \quad \|\Psi_1' - \Psi_2'\|_{L^2}. \qquad (2.4.21)$$

Proof. Using Cauchy-Schwartz, we obtain that

$$
\begin{aligned}
\sigma(\phi, \psi) &= \langle \overline{u}_0', \phi_0^- \rangle + \langle \phi', \phi_0^- \rangle + \langle \overline{w}_0', \psi_0^- \rangle + \langle \psi', \psi_0^- \rangle \\
&\geq 2s^{-1} + \langle (\phi', \psi'), (\phi_0^-, \psi_0^-) \rangle \\
&\geq 2s^{-1} - \eta \\
&\geq s^{-1},
\end{aligned}
\qquad (2.4.22)
$$

which yields (2.4.20). In particular we see that

$$\frac{1}{\sigma(\Psi)} \quad \leq \quad s. \qquad (2.4.23)$$

The remaining inequality (2.4.21) follows immediately from Cauchy-Schwarz. ∎

Lemma 2.4.4. *Assume that (HP1), (HS) and (Hα1) are satisfied. Recall the constant K from Lemma 2.4.2 and the constant s from (2.4.17). Then for all $0 < \eta \leq \eta_0$, $\Psi \in X_\eta, \Psi_1 \in X_\eta$ and $\Psi_2 \in X_\eta$ we have the inequality*

$$|c_h(\Psi) - c_0| \quad \leq \quad s\Big(\|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} + \delta\eta + K\eta^2 \Big), \qquad (2.4.24)$$

together with

$$|c_h(\Psi_1) - c_h(\Psi_2)| \quad \leq \quad s\|\Psi_1 - \Psi_2\|_{H^1} \Big(s\|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} + 2(\delta + K\eta) \Big). \qquad (2.4.25)$$

Proof. By (2.4.20) we have that $[\sigma(\Psi)]^{-1} \leq s$ for all $\Psi \in X_\eta$. By definition of $c_h(\Psi)$ and Lemma 2.4.2, we obtain for all $\Psi = (\phi, \psi) \in X_\eta$ that

$$
\begin{aligned}
|c_h(\Psi) - c_0| &= \left| \frac{\langle \Delta_h \overline{u}_0 - \overline{u}_0'', \phi_0^- \rangle + \delta\langle \phi, \phi_0^- \rangle + \langle \mathcal{N}(\overline{u}_0, \phi), \phi_0^- \rangle}{\sigma(\Psi)} \right| \\
&\leq s|\langle \Delta_h \overline{u}_0 - \overline{u}_0'', \phi_0^- \rangle + \delta\langle \phi, \phi_0^- \rangle + \langle \mathcal{N}(\overline{u}_0, \phi), \phi_0^- \rangle| \\
&\leq s\left(\|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} \|\phi_0^-\|_{L^2} + \delta\|\phi\|_{L^2} \|\phi_0^-\|_{L^2} \right) + sK\eta\|\|\phi\|\|_{L^2} \\
&\leq s\left(\|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} \|(\phi_0^-, \psi_0^-)\|_{\mathbf{L}^2} + \delta\|\phi\|_{L^2} \|(\phi_0^-, \psi_0^-)\|_{\mathbf{L}^2} \right) + sK\eta\|\phi\|_{L^2} \\
&= s\Big(\|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} + \delta\|\phi\|_{L^2} + K\eta\|\phi\|_{L^2} \Big) \\
&\leq s\Big(\|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} + (\delta + K\eta)\eta \Big).
\end{aligned}
\qquad (2.4.26)
$$

For notational compactness we write

$$d(\Psi) \quad = \quad \langle \Delta_h \overline{u}_0 - \overline{u}_0'', \phi_0^- \rangle + \delta\langle \phi, \phi_0^- \rangle + \langle \mathcal{N}(\overline{u}_0, \phi), \phi_0^- \rangle. \qquad (2.4.27)$$

Then we obtain with (2.4.20) that for all $\Psi_1 = (\phi_1, \psi_1) \in X_\eta$ and $\Psi_2 = (\phi_2, \psi_2) \in X_\eta$

$$
\begin{aligned}
|c_h(\Psi_1) - c_h(\Psi_2)| &= \left| \frac{d(\Psi_1)}{\sigma(\Psi_1)} - \frac{d(\Psi_2)}{\sigma(\Psi_2)} \right| \\
&= \left| \frac{d(\Psi_1)\sigma(\Psi_2) - d(\Psi_2)\sigma(\Psi_1)}{\sigma(\Psi_1)\sigma(\Psi_2)} \right| \\
&\leq \frac{|d(\Psi_2)||\sigma(\Psi_2) - \sigma(\Psi_1)| + |d(\Psi_1) - d(\Psi_2)||\sigma(\Psi_2)|}{|\sigma(\Psi_1)||\sigma(\Psi_2)|} \\
&\leq s^2 |d(\Psi_2)||\sigma(\Psi_2) - \sigma(\Psi_1)| + s|d(\Psi_1) - d(\Psi_2)|.
\end{aligned}
\tag{2.4.28}
$$

Observe, using Lemma 2.4.2, that

$$
\begin{aligned}
|d(\Psi_2)| &\leq \|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} + \delta \|\phi_2\|_{L^2} + \|\mathcal{N}(\overline{u}_0, \phi_2)\|_{L^2} \\
&\leq \|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} + \delta\eta + K\eta\|\phi_2\|_{L^2} \\
&\leq \|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} + \delta\eta + K\eta^2
\end{aligned}
\tag{2.4.29}
$$

and

$$
\begin{aligned}
|d(\Psi_1) - d(\Psi_2)| &\leq \delta\|\phi_1 - \phi_2\|_{L^2} + \|\mathcal{N}(\overline{u}_0, \phi_1) - \mathcal{N}(\overline{u}_0, \phi_2)\|_{L^2} \\
&\leq \delta\|\phi_1 - \phi_2\|_{L^2} + \eta K\|\phi_1 - \phi_2\|_{L^2} \\
&\leq (\delta + K\eta)\|\phi_1 - \phi_2\|_{L^2}.
\end{aligned}
\tag{2.4.30}
$$

Using Lemma 2.4.3, we hence see that

$$
\begin{aligned}
|c_h(\Psi_1) - c_h(\Psi_2)| &\leq s^2 |d(\Psi_2)||\sigma(\Psi_2) - \sigma(\Psi_1)| + s|d(\Psi_1) - d(\Psi_2)| \\
&\leq s^2 \left(\|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} + \delta\eta + K\eta^2 \right) |\sigma(\Psi_2) - \sigma(\Psi_1)| \\
&\qquad + s(\delta + K\eta)\|\phi_1 - \phi_2\|_{L^2} \\
&\leq s^2 \left(\|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} + \delta\eta + K\eta^2 \right) \|\Psi_1 - \Psi_2\|_{\mathbf{H}^1} \\
&\qquad + s(\delta + K\eta)\|\phi_1 - \phi_2\|_{L^2} \\
&\leq s\|\Psi_1 - \Psi_2\|_{\mathbf{H}^1} \left(s\|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} + (\delta + K\eta)(1 + s\eta) \right) \\
&\leq s\|\Psi_1 - \Psi_2\|_{\mathbf{H}^1} \left(s\|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} + 2(\delta + K\eta) \right).
\end{aligned}
\tag{2.4.31}
$$

∎

Lemma 2.4.5. *Assume that (HP1), (HS) and (Hα1) are satisfied. Recall the constant K from Lemma 2.4.2 and the constant s from (2.4.17). Then for all $0 < \eta \leq \eta_0, \Psi \in X_\eta, \Psi_1 \in X_\eta$ and $\Psi_2 \in X_\eta$ we have the inequality*

$$
\|\mathcal{R}(c_h(\Psi), \Psi)\|_{\mathbf{L}^2} \leq \left[1 + s\left(\|\overline{u}_0'\|_{L^2} + \|\overline{w}_0'\|_{L^2} + \eta \right) \right] \left[\|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} + \delta\eta + K\eta^2 \right],
\tag{2.4.32}
$$

together with

$$\|\mathcal{R}(c_h(\Psi_1), \Psi_1) - \mathcal{R}(c_h(\Psi_2), \Psi_2)\|_{L^2} \leq \|\Psi_1 - \Psi_2\|_{H^1}\left(2 + 2s\eta + s\|\overline{u}_0'\|_{L^2} + s\|\overline{w}_0'\|_{L^2}\right)$$
$$\times\left(s\|\Delta_h\overline{u}_0 - \overline{u}_0''\|_{L^2} + 2(\delta + K\eta)\right). \tag{2.4.33}$$

Proof. For any $\Psi = (\phi, \psi) \in X_\eta$, Lemma 2.4.4, together with the definition of $\mathcal{R}(c_h(\Psi), \Psi)$, allows us to estimate

$$\|\mathcal{R}(c_h(\Psi), \Psi)\|_{\mathbf{L}^2} \leq |c_0 - c_h(\Psi)|\left(\|\overline{u}_0'\|_{L^2} + \|\phi'\|_{L^2}\right) + \|\Delta_h\overline{u}_0 - \overline{u}_0''\|_{L^2} + \delta\eta + K\eta^2$$
$$+ |c_0 - c_h(\Psi)|\left(\|\overline{w}_0'\|_{L^2} + \|\psi'\|_{L^2}\right)$$
$$\leq s\left(\|\Delta_h\overline{u}_0 - \overline{u}_0''\|_{L^2} + \delta\eta + K\eta\right)\left(\|\overline{u}_0'\|_{L^2} + \|\overline{w}_0'\|_{L^2} + \eta\right)$$
$$+ \|\Delta_h\overline{u}_0 - \overline{u}_0''\|_{L^2} + \delta\eta + K\eta^2$$
$$= \left[1 + s\left(\|\overline{u}_0'\|_{L^2} + \|\overline{w}_0'\|_{L^2} + \eta\right)\right]\left[\|\Delta_h\overline{u}_0 - \overline{u}_0''\|_{L^2} + \delta\eta + K\eta^2\right]. \tag{2.4.34}$$

For $\Psi_1 = (\phi_1, \psi_1) \in X_\eta$ and $\Psi_2 = (\phi_2, \psi_2) \in X_\eta$ we write

$$d(\Psi_1, \Psi_2) := \|\mathcal{R}(c_h(\Psi_1), \Psi_1) - \mathcal{R}(c_h(\Psi_2), \Psi_2)\|_{\mathbf{L}^2}. \tag{2.4.35}$$

Substituting (2.4.5), we compute

$$d(\Psi_1, \Psi_2) \leq \left\|\left((c_0 - c_h(\Psi_1))(\phi_1' - \phi_2') + (c_h(\Psi_1) - c_h(\Psi_2))(\overline{u}_0' - \phi_2')\right.\right.$$
$$\left.\left. + \delta(\phi_1 - \phi_2) + (\mathcal{N}(\overline{u}_0, \phi_2) - \mathcal{N}(\overline{u}_0, \phi_1))\right)\right\|_{L^2}$$
$$+ \left\|\left((c_0 - c_h(\Psi_1))(\psi_1' - \psi_2') + (c_h(\Psi_1) - c_h(\Psi_2))(\overline{w}_0' - \psi_2')\right)\right\|_{L^2}$$
$$\leq \left(|c_h(\Psi_1) - c_0| + \delta + K\eta\right)\|\phi_1 - \phi_2\|_{H^1}$$
$$+ \left(\|\overline{u}_0'\|_{L^2} + \eta\right)\left|c_h(\Psi_1) - c_h(\Psi_2)\right| + \left|c_h(\Psi_1) - c_0\right|\|\psi_1 - \psi_2\|_{H^1}$$
$$+ \left(\|\overline{w}_0'\|_{L^2} + \eta\right)\left|c_h(\Psi_1) - c_h(\Psi_2)\right|. \tag{2.4.36}$$

Another application of Lemma 2.4.4 yields the desired bound

$$d(\Psi_1, \Psi_2) \leq \left(s\left(\|\Delta_h\overline{u}_0 - \overline{u}_0''\|_{L^2} + (\delta + K)\eta\right) + \delta + K\eta\right)\|\phi_1 - \phi_2\|_{H^1}$$
$$+ (\|\overline{u}_0'\|_{L^2} + \eta)s\|\Psi_1 - \Psi_2\|_{\mathbf{H}^1}\left(s\|\Delta_h\overline{u}_0 - \overline{u}_0''\|_{L^2} + 2(\delta + K\eta)\right)$$
$$+ s\left(\|\Delta_h\overline{u}_0 - \overline{u}_0''\|_{L^2} + (\delta + K\eta)\eta\right)\|\psi_1 - \psi_2\|_{H^1}$$
$$+ (\|\overline{w}_0'\|_{L^2} + \eta)s^2\|\Psi_1 - \Psi_2\|_{\mathbf{H}^1}\left(\|\Delta_h\overline{u}_0 - \overline{u}_0''\|_{L^2} + 2\hat{\sigma}(\delta + K\eta)\right)$$
$$\leq \|\Psi_1 - \Psi_2\|_{\mathbf{H}^1}\left(2 + 2s\eta + s\|\overline{u}_0'\|_{L^2} + s\|\overline{w}_0'\|_{L^2}\right)$$
$$\times\left(s\|\Delta_h\overline{u}_0 - \overline{u}_0''\|_{L^2} + 2(\delta + K\eta)\right). \tag{2.4.37}$$

∎

With these estimates in hand, we can choose our parameters δ and η to ensure that the map T maps X_η into itself and is a contraction. This allows us to prove our first main theorem.

Proof of Theorem 2.2.1. We let

$$C_6 \;=\; \max\Big\{C_0\Big(1 + s\|\overline{u}'_0\|_{L^2} + s\|\overline{w}'_0\|_{L^2} + s\Big), C_0\Big(4 + s\|\overline{u}'_0\|_{L^2} + s\|\overline{w}'_0\|_{L^2}\Big)\Big\}, \tag{2.4.38}$$

which is independent of δ, h and $\eta \in (0, s^{-1}]$. Using Lemma 2.4.5 together with (2.4.12) and (2.4.13), we see that for all $0 < \eta \le \eta_0$, $\Psi = (\phi, \psi) \in X_\eta$, $\Psi_1 = (\phi_1, \psi_1) \in X_\eta$ and $\Psi_2 = (\phi_2, \psi_2) \in X_\eta$ we have

$$\|T(\Psi)\|_{H^1} \;\le\; C_6\Big(\|\Delta_h \overline{u}_0 - \overline{u}''_0\|_{L^2} + \delta\eta + K\eta^2\Big) \tag{2.4.39}$$

and

$$\|T(\Psi_1) - T(\Psi_2)\|_{\mathbf{H^1}} \;\le\; C_6\Big(s\|\Delta_h \overline{u}_0 - \overline{u}''_0\| + 2(\delta + K\eta)\Big)\|\Psi_1 - \Psi_2\|_{\mathbf{H^1}}. \tag{2.4.40}$$

We fix

$$\begin{aligned}
\delta &= \tfrac{1}{8C_6} \\
\eta &= \min\{\eta_0, \tfrac{1}{8MC_6}\},
\end{aligned} \tag{2.4.41}$$

so that indeed $\eta \le \eta_0$. Using the notation from Corollary 2.4.1, we pick $0 < h_* \le h_0(\delta)$ in such a way that

$$\sup_{h \in (0, h_*)} \|\Delta_h \overline{u}_0 - \overline{u}''_0\|_{L^2} \;\le\; \tfrac{\eta}{8C_6}. \tag{2.4.42}$$

Then we see for $h \in (0, h_*)$ that

$$\begin{aligned}
\|T(\Psi)\|_{\mathbf{H^1}} &\le C_6\Big(\|\Delta_h \overline{u}_0 - \overline{u}''_0\|_{L^2} + \delta\eta + K\eta^2\Big) \\
&\le C_6\Big(\tfrac{\eta}{8C_6} + \tfrac{1}{8C_6}\eta + K\tfrac{1}{8MC_6}\eta\Big) \\
&\le \eta
\end{aligned} \tag{2.4.43}$$

and

$$\begin{aligned}
\|T(\Psi_1) - T(\Psi_2)\|_{\mathbf{H^1}} &\le C_6\Big(s\|\Delta_h \overline{u}_0 - \overline{u}''_0\| + 2(\delta + K\eta)\Big)\|\Psi_1 - \Psi_2\|_{\mathbf{H^1}} \\
&\le C_6\Big(s\tfrac{\eta}{8C_6} + 2(\tfrac{1}{8C_6} + K\tfrac{1}{8MC_6})\Big)\|\Psi_1 - \Psi_2\|_{\mathbf{H^1}} \\
&\le \tfrac{3}{4}\|\Psi_1 - \Psi_2\|_{\mathbf{H^1}}.
\end{aligned} \tag{2.4.44}$$

In particular, T maps X_η into itself and is a contraction. The local uniqueness of the family $(c_h, \overline{u}_h, \overline{w}_h)$ follows directly from the uniqueness of fixed points of contraction mappings. This completes the proof. ∎

2.5 The point and essential spectrum

In this section, we discuss several properties of the operator that arises after linearising the travelling pulse MFDE (2.2.12) around our wave solution $(\overline{u}_h, \overline{w}_h)$. The main goals are to determine the Fredholm properties of this operator. In particular, we show that both the linearised operator and its adjoint have Fredholm index 0 and that they both have a one-dimensional kernel. Moreover, we construct a family of kernel elements of the adjoint operator that converges to (ϕ_0^-, ψ_0^-), the kernel element of the operator $\mathcal{L}_0^-$.

Pick $0 < h < \min\{h_*, \overline{h}\}$, where h_* is given in Theorem 2.2.1 and $\overline{h}$ is characterized by (2.3.42). We recall the operator $L_h : \mathbf{H}^1 \to \mathbf{L}^2$, introduced in §2.2, which acts as

$$L_h \;=\; \begin{pmatrix} c_h \frac{d}{d\xi} - \Delta_h - g_u(\overline{u}_h) & 1 \\ -\rho & c_h \frac{d}{d\xi} + \gamma\rho \end{pmatrix}. \qquad (2.5.1)$$

In addition, we write $L_h^* : \mathbf{H}^1 \to \mathbf{L}^2$ for the formal adjoint of L_h, which is given by

$$L_h^* \;=\; \begin{pmatrix} -c_h \frac{d}{d\xi} - \Delta_h - g_u(\overline{u}_h) & -\rho \\ 1 & -c_h \frac{d}{d\xi} + \gamma\rho \end{pmatrix}. \qquad (2.5.2)$$

We emphasize that L_h and L_h^* correspond to the operators $\overline{\mathcal{L}}_{h,0}^+$ and $\overline{\mathcal{L}}_{h,0}^-$ defined in §2.3 respectively upon writing

$$\begin{aligned} (\tilde{u}_h, \tilde{w}_h) &= (\overline{u}_h, \overline{w}_h), \\ \tilde{c}_h &= c_h \end{aligned} \qquad (2.5.3)$$

for the family featuring in (hFam). Finally, we introduce the notation

$$\begin{aligned} \Phi_h^+ &= (\phi_h^+, \psi_h^+) &= \tfrac{1}{\|(\overline{u}_h', \overline{w}_h')\|_{\mathbf{L}^2}}(\overline{u}_h', \overline{w}_h'), \\ \Phi_0^+ &= (\phi_0^+, \psi_0^+), \\ \Phi_0^- &= (\phi_0^-, \psi_0^-). \end{aligned} \qquad (2.5.4)$$

The results of this section should be seen as a bridge between the singular perturbation theory developed in §2.3 and the spectral analysis preformed in §2.6. Indeed, one might be tempted to think that most of the work required for the spectral analysis of the operator L_h can already be found in Proposition 2.3.2 and Proposition 2.3.3. However, the problem is that we have no control over the δ-dependence of the interval $(0, h_0'(\delta))$, which contains all values of h for which $L_h + \delta = \overline{\mathcal{L}}_{h,\delta}^+$ is invertible. In particular, for fixed $h > 0$ we cannot directly conclude that $\overline{\mathcal{L}}_{h,\delta}^+$ is invertible for all δ in a subset of the positive real axis.

Our main task in this section is, therefore, to remove the δ-dependence and study L_h and L_h^* directly. The main conclusions are summarized in the results below.

Proposition 2.5.1. *Assume that (HP1), (HS), (Hα1) and (Hα2) are satisfied. Then there exists a constant $\tilde{\lambda} > 0$ such that for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -\tilde{\lambda}$ and all $0 < h < \min\{h_*, \overline{h}\}$ the operator $L_h + \lambda$ is Fredholm with index 0.*

Proposition 2.5.2. *Assume that (HP1), (HS), (Hα1) and (Hα2) are satisfied. Then there exists a constant $h_{**} > 0$, together with a family $\Phi_h^- = (\phi_h^-, \psi^-) \in \boldsymbol{H}^1$, defined for $0 < h < h_{**}$, such that the following properties hold.*

1. *For each $0 < h < h_{**}$ we have the identities*

$$
\begin{aligned}
\ker(L_h) &= \text{span}\{\Phi_h^+\} \\
&= \{\Psi \in \boldsymbol{L}^2 : \langle \Psi, \Theta \rangle = 0 \text{ for all } \Theta \in \text{Range}(L_h^*)\}
\end{aligned} \tag{2.5.5}
$$

   and

$$
\begin{aligned}
\ker(L_h^*) &= \text{span}\{\Phi_h^-\} \\
&= \{\Psi \in \boldsymbol{L}^2 : \langle \Psi, \Theta \rangle = 0 \text{ for all } \Theta \in \text{Range}(L_h)\}.
\end{aligned} \tag{2.5.6}
$$

2. *The family Φ_h^- converges to Φ_0^- in $\boldsymbol{H}^1$ as $h \downarrow 0$.*

3. *Upon introducing the spaces*

$$
X_h = \{\Theta \in \boldsymbol{H}^1 : \langle \Phi_h^-, \Theta \rangle = 0\} \tag{2.5.7}
$$

   and

$$
Y_h = \{\Theta \in \boldsymbol{L}^2 : \langle \Phi_h^-, \Theta \rangle = 0\}, \tag{2.5.8}
$$

   the operator $L_h : X_h \to Y_h$ is invertible and there exists a constant $C_{\text{unif}} > 0$ such that for each $0 < h < h_{**}$ we have the uniform bound

$$
\|L_h^{-1}\|_{\mathcal{B}(Y_h, X_h)} \leq C_{\text{unif}}. \tag{2.5.9}
$$

A direct consequence of these results is that the zero eigenvalue of L_h is simple. In addition, these results allow us to construct a quasi-inverse for L_h that we use heavily in §2.6 and §2.7.

Corollary 2.5.3. *Assume that (HP1), (HS), (Hα1) and (Hα2) are satisfied. Then for any $0 < h < h_{**}$ the zero eigenvalue of L_h is simple.*

Proof. We can assume that $\langle \Phi_h^-, \Phi_h^+ \rangle \neq 0$ for all $0 < h < h_{**}$, since by Proposition 2.5.2 $\langle \Phi_h^-, \Phi_h^+ \rangle \to \langle \Phi_0^-, \Phi_0^+ \rangle \neq 0$. Equation (2.5.6) now implies that $\Phi_h^+ \notin \text{Range}(L_h)$, which together with (2.5.5) completes the proof. ∎

Corollary 2.5.4. *Assume that (HP1), (HS), (Hα1) and (Hα2) are satisfied. There exist linear maps*

$$
\begin{aligned}
\gamma_h : \boldsymbol{L}^2 &\to \mathbb{R} \\
L_h^{\text{qinv}} : \boldsymbol{L}^2 &\to \boldsymbol{H}^1,
\end{aligned} \tag{2.5.10}
$$

such that for all $\Theta \in \boldsymbol{L}^2$ and each $0 < h < h_{**}$ the pair

$$
(\gamma, \Psi) = (\gamma_h \Theta, L_h^{\text{qinv}} \Theta) \tag{2.5.11}
$$

is the unique solution to the problem

$$L_h \Psi = \Theta + \gamma \Phi_h^+ \tag{2.5.12}$$

that satisfies the normalisation condition

$$\langle \Phi_h^-, \Psi \rangle = 0. \tag{2.5.13}$$

Proof. Fix $0 < h < h_{**}$ and $\Theta \in \mathbf{L}^2$. Upon defining

$$\gamma_h[\Theta] = -\frac{\langle \Phi_h^-, \Theta \rangle}{\langle \Phi_h^-, \Phi_h^+ \rangle}, \tag{2.5.14}$$

we see that $\Theta + \gamma_h[\Theta]\Phi_h^+ \in Y_h$. In particular, Proposition 2.5.2 implies that the problem

$$L_h \Psi = \Theta + \gamma_h[\Theta]\Phi_h^+ \tag{2.5.15}$$

has a unique solution $\Psi \in X_h$, which we refer to as $L_h^{\mathrm{qinv}}\Theta$. ∎

The results in [68, 130] allow us to read off the Fredholm properties of L_h from the behaviour of this operator in the limits $\xi \to \pm\infty$. In particular, we let $L_{h,\infty}$ be the operator defined by

$$
\begin{aligned}
L_{h,\infty} &= \begin{pmatrix} c_h \frac{d}{d\xi} - \Delta_h - \lim_{\xi \to \infty} g_u(\overline{u}_h(\xi)) & 1 \\ -\rho & c_h \frac{d}{d\xi} + \gamma\rho \end{pmatrix} \\
&= \begin{pmatrix} c_h \frac{d}{d\xi} - \Delta_h - g_u(0) & 1 \\ -\rho & c_h \frac{d}{d\xi} + \gamma\rho \end{pmatrix}.
\end{aligned}
\tag{2.5.16}
$$

This system has constant coefficients. For $\lambda \in \mathbb{C}$ we introduce the notation

$$L_{h,\infty;\lambda} = L_{h,\infty} + \lambda. \tag{2.5.17}$$

We show that for λ in a suitable right half-plane the operator $L_{h,\infty;\lambda}$ is hyperbolic in the sense of [68, 130], i.e. we write

$$
\begin{aligned}
\Delta_{L_{h,\infty;\lambda}}(z) &= \left[L_{h,\infty;\lambda} e^{z\xi} \right](0) \\
&= \begin{pmatrix} c_h z - \frac{1}{h^2} \left[\sum_{k>0} \alpha_k \left(e^{khz} + e^{-khz} - 2 \right) \right] - g_u(0) + \lambda & 1 \\ -\rho & c_h z + \gamma\rho + \lambda \end{pmatrix}
\end{aligned}
\tag{2.5.18}
$$

and show that $\det(\Delta_{L_{h,\infty;\lambda}}(iy)) \neq 0$ for all $y \in \mathbb{R}$. In the terminology of [68, 130], this means that $L_h + \lambda$ is asymptotically hyperbolic. This allows us to compute the Fredholm index of $L_h + \lambda$.

Remark 2.5.5. From this section onward we assume that (Hα2) is satisfied. This is done for technical reasons, allowing us to apply the results from [68]. In particular, this condition implies that the function $\Delta_{L_{h,\infty;\lambda}}(z)$ defined in (2.5.18) is well-defined in a vertical strip $|\mathrm{Re}\,(z)| < \nu$.

Lemma 2.5.6. *Assume that (HP1), (HS), (Hα1) and (Hα2) are satisfied. There exists a constant $\tilde{\lambda} > 0$ such that for all $0 < h < \min\{h_*, \overline{h}\}$ and all $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > -\tilde{\lambda}$ the operator $L_{h,\infty;\lambda}$ is hyperbolic and thus the operator $L_h + \lambda$ is asymptotically hyperbolic.*

Proof. Remembering that $-g_u(0) = r_0 > 0$ and picking $y \in \mathbb{R}$, we compute

$$
\Delta_{L_{h,\infty;\lambda}}(iy) = \begin{pmatrix} c_h iy + \frac{1}{h^2}\left[\sum_{k>0} \alpha_k \left(2 - 2\cos(khy)\right)\right] + r_0 + \lambda & 1 \\ -\rho & c_h iy + \gamma\rho \end{pmatrix}
$$

$$
= \begin{pmatrix} c_h iy + \frac{1}{h^2} A(hy) + r_0 + \lambda & 1 \\ -\rho & c_h iy + \gamma\rho + \lambda \end{pmatrix},
$$

$$(2.5.19)$$

where $A(hy) \geq 0$ is defined in (Hα1). We hence see

$$
\det(\Delta_{L_{h,\infty;\lambda}}(iy)) = \left(c_h iy + \tfrac{1}{h^2} A(hy) + r_0 + \lambda\right)\left(c_h iy + \gamma\rho + \lambda\right) + \rho. \quad (2.5.20)
$$

Let $\tilde{\lambda} = \frac{1}{4}\min\{\gamma\rho, r_0\}$ and assume that $\operatorname{Re} \lambda > -\tilde{\lambda}$. If $y \neq -\frac{\operatorname{Im}\lambda}{c_h}$ then we obtain

$$
\begin{aligned}
\operatorname{Im}\left(\det(\Delta_{L_{h,\infty;\lambda}}(iy))\right) &= (c_h y + \operatorname{Im}\lambda)(\gamma\rho + \operatorname{Re}\lambda) \\
&\qquad + (\tfrac{1}{h^2} A(hy) + r_0 + \operatorname{Re}\lambda)(c_h y + \operatorname{Im}\lambda) \\
&= (c_h y + \operatorname{Im}\lambda)(\gamma\rho + \tfrac{1}{h^2}A(hy) + r_0 + 2\operatorname{Re}\lambda) \\
&\neq 0,
\end{aligned}
$$

$$(2.5.21)$$

since $\gamma\rho + \frac{1}{h^2} A(hy) + r_0 + \operatorname{Re}\lambda > 0$. For $y = -\frac{\operatorname{Im}\lambda}{c_h}$ we obtain

$$
\begin{aligned}
\operatorname{Re}\left(\det(\Delta_{L_{h,\infty;\lambda}}(iy))\right) &= \left(\tfrac{1}{h^2}A(hy) + r_0 + \operatorname{Re}\lambda\right)\left(\gamma\rho + \operatorname{Re}\lambda\right) + \rho \\
&> \rho \\
&> 0.
\end{aligned}
$$

$$(2.5.22)$$

In particular, we see that $\det(\Delta_{L_{h,\infty;\lambda}}(iy)) \neq 0$ for all $y \in \mathbb{R}$, as desired. ∎

Before we consider the Fredholm properties of $L_h + \lambda$, we establish a technical estimate for the function $\Delta_{L_{h,\infty;\lambda}}$, which we need in §2.7 later on.

Lemma 2.5.7. *Assume that (HP1), (HS), (Hα1) and (Hα2) are satisfied. Fix $0 < h < \min\{h_*, \overline{h}\}$ and $S \subset \mathbb{C}$ compact in such a way that $\operatorname{Re} \lambda > -\tilde{\lambda}$ for all $\lambda \in S$. Then there exist constants $\kappa > 0$ and $\Gamma > 0$, possibly depending on h and S, such that for all $z = x + iy \in \mathbb{C}$ with $|x| \leq \kappa$ and all $\lambda \in S$ we have the bound*

$$
|\det(\Delta_{L_{h,\infty;\lambda}}(z))| \geq \tfrac{1}{\Gamma}. \quad (2.5.23)
$$

Proof. Using assumption (Hα2), we can pick $\kappa_1 > 0$ and $\Gamma_1 > 0$ in such a way that the bound

$$
\begin{aligned}
|\tfrac{1}{h^2} A(hz)| &:= \left| \tfrac{1}{h^2}\left[\sum_{k>0} \alpha_k \left(2 - e^{khz} - e^{-khz}\right)\right]\right| \\
&\leq \tfrac{1}{h^2} \sum_{k>0} |\alpha_k| \left(e^{hk|x|} + 3\right) \\
&\leq \Gamma_1
\end{aligned}
$$

$$(2.5.24)$$

holds for all $z = x + iy \in \mathbb{C}$ with $|x| \leq \kappa_1$.

Observe that for $z = x + iy \in \mathbb{C}$ and $\lambda \in S$ we have

$$\text{Re}\left(\det(\Delta_{L_{h,\infty;\lambda}}(z))\right) = \left(c_h x + \tfrac{1}{h^2}\text{Re}\,A(hz) + r_0 + \text{Re}\,\lambda\right)\left(c_h x + \gamma\rho + \text{Re}\,\lambda\right)$$
$$- (c_h y + \text{Im}\,\lambda)^2 - (c_h y + \text{Im}\,\lambda)\tfrac{1}{h^2}(\text{Im}\,A(hz)) + \rho.$$
$$(2.5.25)$$

Since S is compact we can find $Y > 0$ such that for all $z = x + iy \in \mathbb{C}$ with $|y| \geq Y$ and $|x| \leq k_1$ and all $\lambda \in S$ we have

$$\left|\text{Re}\left(\det(\Delta_{L_{h,\infty;\lambda}}(z))\right)\right| \geq \tfrac{1}{2}c_h^2 y^2$$
$$\geq \tfrac{1}{2}c_h^2 Y^2. \qquad (2.5.26)$$

Seeking a contradiction, let us assume that for each $0 < \kappa \leq \kappa_1$ and each $\Gamma > 0$ there exist $\lambda \in S$ and $z = x + iy \in \mathbb{C}$ with $|x| \leq \kappa$ and $|y| \leq Y$ for which

$$|\det(\Delta_{L_{h,\infty;\lambda}}(z))| < \tfrac{1}{\Gamma}. \qquad (2.5.27)$$

Then we can construct a sequence $\{\kappa_n, z_n, \lambda_n\}$ with $0 < \kappa_n \leq \kappa_1$ for each n, $\kappa_n \to 0$, $\lambda_n \in S$ for each n and $z_n = x_n + iy_n \in \mathbb{C}$ with $|x_n| \leq \kappa_n$ and $|y_n| \leq Y$ in such a way that $|\det(\Delta_{L_{h,\infty;\lambda_n}}(z_n))| < \tfrac{1}{n}$ for each n. By taking a subsequence if necessary we see that $\lambda_n \to \lambda$ for some $\lambda \in S$ and $z_n \to iy$ for some $y \in \mathbb{R}$ with $|y| \leq Y$. Since $\det(\Delta_{L_{h,\infty;\lambda}}(z))$ is continuous as a function of λ and z, it follows that

$$\det(\Delta_{L_{h,\infty;\lambda}}(iy)) = \lim_{n\to\infty}\det(\Delta_{L_{h,\infty;\lambda_n}}(z_n))$$
$$= 0, \qquad (2.5.28)$$

which contradicts Lemma 2.5.6. Hence, we can find $\kappa > 0$ and $\Gamma > 0$ as desired. $\blacksquare$

Proof of Proposition 2.5.1. We have already seen in Lemma 2.5.6 that $L_h + \lambda$ is asymptotically hyperbolic in the sense of [68, 130]. Now according to [68, Thm. 1.6], we obtain that $L_h + \lambda$ is a Fredholm operator and that the following identities hold

$$\dim\left(\ker(L_h + \lambda)\right) = \text{codim}\left(\text{Range}(L_h^* + \overline{\lambda})\right),$$
$$\text{codim}\left(\text{Range}(L_h + \lambda)\right) = \dim\left(\ker(L_h^* + \overline{\lambda})\right), \qquad (2.5.29)$$
$$\text{ind}(L_h + \lambda) = -\text{ind}(L_h^* + \overline{\lambda}),$$

where

$$\text{ind}(L_h + \lambda) = \dim\left(\ker(L_h + \lambda)\right) - \text{codim}\left(\text{Range}(L_h + \lambda)\right) \qquad (2.5.30)$$

is the Fredholm index of $L_h + \lambda$.

We follow the proof of [130, Thm. B]. For $0 \leq \vartheta \leq 1$, we let the operator $L^\vartheta(h)$ be defined by

$$L^\vartheta(h) = (1 - \vartheta)(L_h + \lambda) + \vartheta(L_{h,\infty} + \lambda). \qquad (2.5.31)$$

Note that the operator $L^\vartheta(h)$ is asymptotically hyperbolic for each ϑ and thus [68, Thm. 1.6] implies that these operators $L^\vartheta(h)$ are Fredholm. Moreover, the family $L^\vartheta(h)$ varies continuously with ϑ in $\mathcal{B}(\mathbf{H}^1, \mathbf{L}^2)$, which means the Fredholm index is constant. In particular, we see that

$$
\begin{aligned}
\operatorname{ind}(L_h + \lambda) &= \operatorname{ind}(L_{h,\infty} + \lambda) \\
&= 0,
\end{aligned}
\tag{2.5.32}
$$

where the last equality follows from [68, Thm. 1.7].   ∎

We can now concentrate on the kernel of L_h. The goal is to exclude kernel elements other than Φ_h^+. In order to accomplish this, we construct a quasi-inverse for L_h by mimicking the approach of [111, Prop. 3.2]. As a preparation, we obtain the following technical result.

Lemma 2.5.8. *Assume that (HP1), (HS) and (Hα1) are satisfied. Recall the constant δ_0 from Lemma 2.3.1. Let $0 < \lambda < \min\{\frac{1}{2}, \delta_0\}$ be given. Then there exist constants $0 < h_1^* \le \min\{h_*, \overline{h}\}$ and $\kappa > 0$ such that for all $0 < h \le h_1^*$ we have*

$$
\begin{aligned}
\langle \Phi_0^-, (L_h + \lambda)^{-1} \Phi_0^+ \rangle &> \tfrac{1}{2}\lambda^{-1} \langle \Phi_0^-, \Phi_h^+ \rangle \\
&> \tfrac{1}{2}\lambda^{-1}\kappa \\
&> 0.
\end{aligned}
\tag{2.5.33}
$$

Proof. We know from Lemma 2.3.1 that $\langle \Phi_0^-, \Phi_0^+ \rangle > 0$. Since Φ_h^+ converges to Φ_0^+ in $\mathbf{L}^2$, it follows that $\langle \Phi_0^-, \Phi_h^+ \rangle$ converges to $\langle \Phi_0^-, \Phi_0^+ \rangle > 0$. Fix $h_1^* \le \min\{h_*, \overline{h}, h_0'(\lambda)\}$ in such a way that

$$
\|\Phi_0^+ - \Phi_h^+\|_{\mathbf{L}^2} < \tfrac{1}{2}\frac{\langle \Phi_0^-, \Phi_h^+ \rangle}{2C_{\mathrm{unif}}}
\tag{2.5.34}
$$

holds for all $0 \le h \le h_1^*$, where

$$
C_{\mathrm{unif}} = 4C_0'
\tag{2.5.35}
$$

and C_0' is defined in Proposition 2.3.2. The factor 4 in the definition is for technical reasons in a later proof. We assume from now on that $0 < h \le h_1^*$. Using $L_h\Phi_h^+ = 0$ we readily see

$$
(L_h + \lambda)^{-1}\Phi_h^+ = \lambda^{-1}\Phi_h^+.
\tag{2.5.36}
$$

Recall that $\|\Phi_0^-\|_{\mathbf{L}^2} = 1$. Since $1 < \lambda^{-1}$, we may use Proposition 2.3.2 to obtain

$$
\begin{aligned}
\|(L_h + \lambda)^{-1}\Phi_0^+ - \lambda^{-1}\Phi_h^+\|_{\mathbf{L}^2} &= \|(L_h + \lambda)^{-1}[\Phi_0^+ - \Phi_h^+]\|_{\mathbf{L}^2} \\
&\le C_{\mathrm{unif}}\Big[\|\Phi_h^+ - \Phi_0^+\|_{\mathbf{L}^2} + \lambda^{-1}|\langle \Phi_h^+ - \Phi_0^+, \Phi_0^- \rangle|\Big] \\
&< C_{\mathrm{unif}}\lambda^{-1}\|\Phi_0^+ - \Phi_h^+\|_{\mathbf{L}^2}\Big(1 + \|\Phi_0^-\|_{\mathbf{L}^2}\Big) \\
&= 2C_{\mathrm{unif}}\lambda^{-1}\|\Phi_0^+ - \Phi_h^+\|_{\mathbf{L}^2}.
\end{aligned}
\tag{2.5.37}
$$

Remembering $\langle \Phi_0^-, \Phi_h^+ \rangle > 0$ and using Cauchy-Schwarz, we see that

$$
\begin{aligned}
|\langle \tfrac{\Phi_0^-}{\langle \Phi_0^-, \Phi_h^+ \rangle}, (L_h + \lambda)^{-1}\Phi_0^+ \rangle - \lambda^{-1}| \quad &= \quad |\langle \tfrac{\Phi_0^-}{\langle \Phi_0^-, \Phi_h^+ \rangle}, (L_h + \lambda)^{-1}\Phi_0^+ - \lambda^{-1}\Phi_h^+ \rangle| \\
&< \quad \tfrac{\|\Phi_0^-\|_{\mathbf{L}^2}}{\langle \Phi_0^-, \Phi_h^+ \rangle} 2C_{\mathrm{unif}}\lambda^{-1}\|\Phi_0^+ - \Phi_h^+\|_{\mathbf{L}^2} \\
&\leq \quad \tfrac{1}{\langle \Phi_0^-, \Phi_h^+ \rangle} 2C_{\mathrm{unif}}\lambda^{-1}\tfrac{1}{2}\tfrac{\langle \Phi_0^-, \Phi_h^+ \rangle}{2C_{\mathrm{unif}}} \\
&= \quad \tfrac{1}{2}\lambda^{-1}.
\end{aligned}
\tag{2.5.38}
$$

Hence, we must have

$$
\langle \Phi_0^-, (L_h + \lambda)^{-1}\Phi_0^+ \rangle \quad > \quad \tfrac{1}{2}\lambda^{-1}\langle \Phi_0^-, \Phi_h^+ \rangle > 0.
\tag{2.5.39}
$$

∎

Lemma 2.5.9. *Assume that (HP1), (HS) and (Hα1) are satisfied. There exists $0 < h_{**} \leq \min\{h_*, \overline{h}\}$ together with linear maps*

$$
\begin{aligned}
\tilde{\gamma}_h^+ : \boldsymbol{L}^2 \quad &\to \quad \mathbb{R} \\
\tilde{L}_h^{\mathrm{qinv}} : \boldsymbol{L}^2 \quad &\to \quad \boldsymbol{H}^1,
\end{aligned}
\tag{2.5.40}
$$

defined for all $0 < h < h_{**}$, such that for all $\Theta \in \boldsymbol{L}^2$ the pair

$$
(\gamma, \Psi) \quad = \quad (\tilde{\gamma}_h^+ \Theta, \tilde{L}_h^{\mathrm{qinv}}\Theta)
\tag{2.5.41}
$$

is the unique solution to the problem

$$
L_h \Psi \quad = \quad \Theta + \gamma \Phi_0^+
\tag{2.5.42}
$$

that satisfies the normalisation condition

$$
\langle \Phi_0^-, \Psi \rangle \quad = \quad 0.
\tag{2.5.43}
$$

In addition, there exists $C > 0$ such that for all $0 < h < h_{**}$ and all $\Theta \in \boldsymbol{L}^2$ we have the bound

$$
|\tilde{\gamma}_h^+ \Theta| + \|\tilde{L}_h^{\mathrm{qinv}}\Theta\|_{\boldsymbol{H}^1} \quad \leq \quad C\|\Theta\|_{\boldsymbol{L}^2}.
\tag{2.5.44}
$$

Proof. Fix $0 < \lambda < \min\{\tfrac{1}{2}, \delta_0\}$ and let $0 < h \leq \min\{h_*, \overline{h}, h_0'(\lambda)\}$ be given, where $h_0'(\lambda)$ is defined in Proposition 2.3.2. For now, all constants will not depend on our choice of λ. We define the set

$$
Z^1 \quad = \quad \{\Psi \in \mathbf{H}^1 : \langle \Phi_0^-, \Psi \rangle = 0\}.
\tag{2.5.45}
$$

Pick $\Theta \in \mathbf{L}^2$. We look for a solution $(\gamma, \Psi) \in \mathbb{R} \times Z^1$ of the problem

$$
\Psi \quad = \quad (L_h + \lambda)^{-1}[\Theta + \gamma\Phi_0^+ + \lambda\Psi].
\tag{2.5.46}
$$

By Lemma 2.5.8 we have $\langle \Phi_0^-, (L_h + \lambda)^{-1}\Phi_0^+ \rangle \neq 0$. Hence, for given $\Theta \in \mathbf{L}^2, \Psi \in Z^1, h, \lambda$, we may write

$$
\gamma(\Psi, \Theta, h, \lambda) \quad = \quad -\frac{\langle \Phi_0^-, (L_h + \lambda)^{-1}(\Theta + \lambda\Psi) \rangle}{\langle \Phi_0^-, (L_h + \lambda)^{-1}\Phi_0^+ \rangle},
\tag{2.5.47}
$$

which is the unique value for γ for which

$$(L_h + \lambda)^{-1}[\Theta + \gamma \Phi_0^+ + \lambda \Psi] \quad \in \quad Z^1. \tag{2.5.48}$$

Recall the constant C_{unif} from (2.5.35). With Proposition 2.3.2 we obtain

$$
\begin{aligned}
|\langle \Phi_0^-, (L_h + \lambda)^{-1}(\Theta + \lambda \Psi)\rangle| \quad &\leq \quad \|\Phi_0^-\|_{\mathbf{L}^2} C_{\mathrm{unif}}\Big[\|\Theta + \lambda \Psi\|_{\mathbf{L}^2} + \tfrac{1}{\lambda}|\langle \Theta + \lambda \Psi, \Phi_0^-\rangle|\Big] \\
&\leq \quad \|\Phi_0^-\|_{\mathbf{L}^2} C_{\mathrm{unif}}\Big[(1 + \tfrac{1}{\lambda})\|\Theta\|_{\mathbf{L}^2} + \lambda \|\Psi\|_{\mathbf{L}^2}\Big] \\
&\leq \quad C_1\Big[\lambda^{-1}\|\Theta\|_{\mathbf{L}^2} + \lambda \|\Psi\|_{\mathbf{L}^2}\Big]
\end{aligned}
\tag{2.5.49}
$$

for some C_1 that is independent of h, λ. Here we used that $\lambda < 1$ and thus $1 + \frac{1}{\lambda} < \frac{2}{\lambda}$. Exploiting $\lambda < \frac{1}{2}$ and applying Lemma 2.5.8, we see that

$$
\begin{aligned}
|\gamma(\Psi, \Theta, h, \lambda)| \quad &= \quad |\langle \Phi_0^-, (L_h + \lambda)^{-1}(\Theta + \lambda \Psi)\rangle| \tfrac{1}{|\langle \Phi_0^-, (L_h + \lambda)^{-1}\Phi_0^+\rangle|} \\
&\leq \quad C_1\Big[\lambda^{-1}\|\Theta\|_{\mathbf{L}^2} + \lambda \|\Psi\|_{\mathbf{L}^2}\Big]\tfrac{1}{\frac{1}{2}\lambda^{-1}\langle \Phi_0^-, \Phi_h^+\rangle} \\
&\leq \quad C_1\Big[\kappa\|\Theta\|_{\mathbf{L}^2} + \kappa\lambda^2\|\Psi\|_{\mathbf{L}^2}\Big] \\
&\leq \quad C_2\Big[\|\Theta\|_{\mathbf{L}^2} + \lambda^2\|\Psi\|_{\mathbf{L}^2}\Big].
\end{aligned}
\tag{2.5.50}
$$

Here we used that $\langle \Phi_0^-, \Phi_h^+\rangle$ converges to $\langle \Phi_0^-, \Phi_0^+\rangle > 0$, which means that $\langle \Phi_0^-, \Phi_h^+\rangle$ can be bounded away from zero. For $\Psi \in Z^1$ we write

$$t(\Psi) \quad = \quad \Theta + \gamma(\Psi, \Theta, h, \lambda)\Phi_0^+ + \lambda \Psi \tag{2.5.51}$$

and

$$T(\Psi) \quad = \quad (L_h + \lambda)^{-1} t(\Psi). \tag{2.5.52}$$

For $\Psi \in Z^1$ Proposition 2.3.2 implies

$$
\begin{aligned}
\|T(\Psi)\|_{\mathbf{H}^1} \quad &\leq \quad C_{\mathrm{unif}}\Big[\|\Theta + \gamma(\Psi, \Theta, h, \lambda)\Phi_0^+ + \lambda \Psi\|_{\mathbf{L}^2} \\
&\qquad\qquad + \tfrac{1}{\lambda}|\langle \Theta + \gamma(\Psi, \Theta, h, \lambda)\Phi_0^+ + \lambda \Psi, \Phi_0^-\rangle|\Big] \\
&\leq \quad C_3\Big[\tfrac{1}{\lambda}\|\Theta\|_{\mathbf{L}^2} + \lambda \|\Psi\|_{\mathbf{L}^2}\Big] \\
&\leq \quad C_3\Big[\tfrac{1}{\lambda}\|\Theta\|_{\mathbf{L}^2} + \lambda \|\Psi\|_{\mathbf{H}^1}\Big].
\end{aligned}
\tag{2.5.53}
$$

For $\Psi_1, \Psi_2 \in Z^1$, a second application of Proposition 2.3.2 yields

$$
\begin{aligned}
|\gamma(\Psi_1, \Theta, h, \lambda) - \gamma(\Psi_2, \Theta, h, \lambda)| \quad &= \quad \left|\tfrac{\langle \Phi_0^-, (L_h + \lambda)^{-1}(\lambda\Psi_1 - \lambda\Psi_2)\rangle}{\langle \Phi_0^-, (L_h + \lambda)^{-1}\Phi_0^+\rangle}\right| \\
&\leq \quad \tfrac{1}{\langle \Phi_0^-, (L_h + \lambda)^{-1}\Phi_0^+\rangle} C_{\mathrm{unif}}\Big[\lambda\|\Psi_1 - \Psi_2\|_{\mathbf{L}^2} \\
&\qquad\qquad + \tfrac{1}{\lambda}|\langle \lambda\Psi_1 - \lambda\Psi_2, \Phi_0^-\rangle|\Big] \\
&\leq \quad C_4\lambda\Big[\lambda\|\Psi_1 - \Psi_2\|_{\mathbf{L}^2} + 0\Big] \\
&\leq \quad C_4\lambda^2\|\Psi_1 - \Psi_2\|_{\mathbf{H}^1}.
\end{aligned}
\tag{2.5.54}
$$

Applying Proposition 2.3.2 for the final time, we see

$$
\begin{aligned}
\|T(\Psi_1) - T(\Psi_2)\|_{\mathbf{H}^1} \ \leq\ & C_{\mathrm{unif}}\Big[\|t(\Psi_1) - t(\Psi_2)\|_{\mathbf{L}^2} + \tfrac{1}{\lambda}|\langle t(\Psi_1) - t(\Psi_2), \Phi_0^-\rangle|\Big] \\
\leq\ & C_{\mathrm{unif}}\Big[\|t(\Psi_1) - t(\Psi_2)\|_{\mathbf{L}^2} \\
& + \tfrac{1}{\lambda}\langle\big(\gamma(\Psi_1, \Theta, h, \lambda) - \gamma(\Psi_2, \Theta, h, \lambda)\big)\Phi_0^+, \Phi_0^-\rangle \\
& + \tfrac{1}{\lambda}\langle\lambda(\Psi_1 - \Psi_2), \Phi_0^-\rangle\Big] \\
\leq\ & C_{\mathrm{unif}}\Big[\|t(\Psi_1) - t(\Psi_2)\|_{\mathbf{L}^2} + \tfrac{1}{\lambda}\Big(C_4\lambda^2\|\Psi_1 - \Psi_2\|_{\mathbf{H}^1} + 0\Big)\Big] \\
\leq\ & C_{\mathrm{unif}}C_4\lambda^2\|\Psi_1 - \Psi_2\|_{\mathbf{H}^1} + C_{\mathrm{unif}}\lambda\|\Psi_1 - \Psi_2\|_{\mathbf{H}^1} \\
& + C_{\mathrm{unif}}C_4\lambda\|\Psi_1 - \Psi_2\|_{\mathbf{H}^1} \\
\leq\ & C_5\lambda\|\Psi_1 - \Psi_2\|_{\mathbf{H}^1}.
\end{aligned}
$$
$$(2.5.55)$$

In view of these bounds, we pick λ to be small enough to have $C_3\lambda < \frac{1}{2}$ and $C_5\lambda < \frac{1}{2}$. Since this λ is now fixed, we can allow the constants in the final part of the proof to depend on λ. In addition, we write $h_{**} = \min\{h_1^*, h_0'(\lambda)\}$ and pick $0 < h < h_{**}$. Then $T : Z^1 \to Z^1$ is a contraction, so the fixed point theorem implies that there is a unique $\tilde{L}_h^{\mathrm{qinv}}(\Theta) \in Z^1$ for which

$$
\tilde{L}_h^{\mathrm{qinv}}(\Theta) \ =\ (L_h + \lambda)^{-1}\Big[\Theta + \gamma(\tilde{L}_h^{\mathrm{qinv}}(\Theta), \Theta, h, \lambda)\Phi_0^+ + \lambda\tilde{L}_h^{\mathrm{qinv}}(\Theta)\Big]. \qquad (2.5.56)
$$

Furthermore, we have

$$
\begin{aligned}
\tfrac{1}{2}\|\tilde{L}_h^{\mathrm{qinv}}(\Theta)\|_{\mathbf{H}^1} \ &\leq\ (1 - \lambda C_3)\|\tilde{L}_h^{\mathrm{qinv}}(\Theta)\|_{\mathbf{H}^1} \\
&\leq\ C_3\lambda^{-1}\|\Theta\|_{\mathbf{L}^2} \\
&\leq\ C_6\|\Theta\|_{\mathbf{L}^2}.
\end{aligned}
$$
$$(2.5.57)$$

Writing $\tilde{\gamma}_h^+(\Theta) = \gamma(\tilde{L}_h^{\mathrm{qinv}}(\Theta), \Theta, h, \lambda)$, we compute

$$
\begin{aligned}
|\tilde{\gamma}_h^+(\Theta)| \ &\leq\ C_2[\|\Theta\|_{\mathbf{L}^2} + \lambda^2\|\Theta\|_{\mathbf{L}^2}] \\
&\leq\ C_7\|\Theta\|_{\mathbf{L}^2}.
\end{aligned}
$$
$$(2.5.58)$$

Finally we see that (2.5.46) is in fact equivalent to (2.5.42)-(2.5.43), so in fact $\tilde{L}_h^{\mathrm{qinv}}(\Theta)$ and $\tilde{\gamma}_h^+(\Theta)$ do not depend on λ. In addition, the constants h_{**}, C_6 and C_7 above only depend on the one fixed λ and, as such, do not depend on h or Θ. ∎

Lemma 2.5.10. *Assume that (HP1), (HS), (Hα1) and (Hα2) are satisfied. Let $0 < h < h_{**}$ be given. Then we have the inclusion*

$$
\begin{aligned}
\mathrm{span}\{\Phi_h^+\} \ &\subset\ \ker(L_h) \\
&=\ \{\Psi \in \mathbf{L}^2 : \langle\Psi, \Theta\rangle = 0 \text{ for all } \Theta \in \mathrm{Range}(L_h^*)\},
\end{aligned}
$$
$$(2.5.59)$$

where L_h^* is the formal adjoint of L_h.

Proof. By differentiating the differential equation (2.2.12) we see that $L_h \Phi_h^+ = 0$. We know that $(\overline{u}_h, \overline{w}_h) - (\overline{u}_0, \overline{w}_0) \to 0 \in \mathbf{H}^1$. Since $(\overline{u}_0', \overline{w}_0')$ decays exponentially, we get $(\overline{u}_0', \overline{w}_0') \in \mathbf{L}^2$. Hence, we can assume that h_{**} is small enough such that $\Phi_h^+ \in \mathbf{L}^2$ for all $0 < h < h_{**}$. Since $L_h \Phi_h^+ = 0$ we obtain from the differential equation that also $(\Phi_h^+)' \in \mathbf{L}^2$. In particular, we see that $\Phi_h^+ \in \mathbf{H}^1$ and, hence, $\Phi_h^+ \in \ker(L_h)$. ∎

Lemma 2.5.11. *Assume that (HP1), (HS), (Hα1) and (Hα2) are satisfied. Let $0 < h < h_{**}$ be given. Then we have*

$$
\begin{aligned}
\ker(L_h) &= \operatorname{span}\{\Phi_h^+\} \\
&= \{\Psi \in \mathbf{L}^2 : \langle \Psi, \Theta \rangle = 0 \text{ for all } \Theta \in \operatorname{Range}(L_h^*)\},
\end{aligned}
\tag{2.5.60}
$$

where L_h^* is the formal adjoint of L_h.

Proof. We show that $\dim(\ker(L_h)) = 1$. Since $\Phi_h^+ \in \ker(L_h)$, we assume that there exists $\Psi \in \ker(L_h)$ in such a way that Ψ is not a scalar multiple of Φ_h^+.

Suppose first that $\langle \Psi, \Phi_0^- \rangle = 0$. Then Lemma 2.5.9 yields by linearity of $\tilde{L}_h^{\text{qinv}}$ that

$$
\begin{aligned}
\Psi &= \tilde{L}_h^{\text{qinv}}[0] \\
&= 0,
\end{aligned}
\tag{2.5.61}
$$

which gives a contradiction. Hence, we suppose that $\langle \Psi, \Phi_0^- \rangle \neq 0$. In the proof of Lemma 2.5.8 we saw that $\langle \Phi_h^+, \Phi_0^- \rangle \neq 0$. As such, we can pick $a, b \in \mathbb{R} \setminus \{0\}$ in such a way that

$$
\langle a\Phi_h^+ + b\Psi, \Phi_0^- \rangle = 0. \tag{2.5.62}
$$

Again it follows from Lemma 2.5.9 that $a\Phi_h^+ + b\Psi = 0$ which gives a contradiction. Therefore, such a kernel element Ψ does not exist. Since we already know that $\Phi_h^+ \in \ker(L_h)$, we must have $\dim(\ker(L_h)) = 1$, which completes the proof. ∎

The remaining major goal of this section is to find a family of elements $\Phi_h^- \in \ker(L_h^*)$ which satisfies $\Phi_h^- \to \Phi_0^-$ as $h \downarrow 0$. To establish this, we repeat part of the process above for the adjoint operator L_h^*. The key difference is that we must construct the family Φ_h^- by hand. This requires a significant refinement of the argument used above to characterize $\ker(L_h^*)$.

First we need a technical result, similar to Lemma 2.5.8.

Lemma 2.5.12. *Assume that (HP1), (HS) and (Hα1) are satisfied. Fix $0 < \lambda < \frac{1}{2}$ and $0 < h \leq \min\{h_{**}, h_0'(\lambda)\}$, where $h_0'(\lambda)$ is defined in Proposition 2.3.2. Then we have*

$$
\langle \Phi_0^+, (L_h^* + \lambda)^{-1}\Phi_0^- \rangle > \frac{\langle \Phi_0^+, \Phi_0^- \rangle}{2}\lambda^{-1}. \tag{2.5.63}
$$

Proof. Lemma 2.3.1 implies that $\langle \Phi_0^+, \Phi_0^- \rangle > 0$. Remembering that

$$L_h^* - L_0^* = \begin{pmatrix} (c_0 - c_h)\frac{d}{d\xi} - (\Delta_h - \frac{d^2}{d\xi^2}) + (g_u(\overline{u}_0) - g_u(\overline{u}_h)) & 0 \\ 0 & (c_0 - c_h)\frac{d}{d\xi} \end{pmatrix}$$

$$(2.5.64)$$

and that $L_0^* \Phi_0^- = 0$, we obtain

$$
\begin{aligned}
(L_h^* + \lambda)\left[(L_h^* + \lambda)^{-1}\Phi_0^- - (L_0^* + \lambda)^{-1}\Phi_0^-\right] &= \Phi_0^- - \Phi_0^- + (L_h^* - L_0^*)(L_0^* + \lambda)^{-1}\Phi_0^- \\
&= (L_h^* - L_0^*)\lambda^{-1}\Phi_0^-.
\end{aligned}
$$

$$(2.5.65)$$

Recall the constant C_{unif} from (2.5.35). Proposition 2.3.2 yields

$$
\begin{aligned}
\|(L_h^* + \lambda)^{-1}\Phi_0^- - (L_0^* + \lambda)^{-1}\Phi_0^-\|_{\mathbf{L}^2} &\leq C_{\text{unif}}\Big[\|(L_h^* - L_0^*)\lambda^{-1}\Phi_0^-\|_{\mathbf{L}^2} \\
&\qquad + |\langle (L_h^* - L_0^*)\lambda^{-1}\Phi_0^-, \Phi_0^+ \rangle|\Big] \\
&\leq C_{\text{unif}}(1 + \lambda^{-1})\|(L_h^* - L_0^*)\lambda^{-1}\Phi_0^-\|_{\mathbf{L}^2}.
\end{aligned}
$$

$$(2.5.66)$$

Using Lemma 2.3.5 and the fact that c_h converges to c_0 and $g_u(\overline{u}_h)$ to $g_u(\overline{u}_0)$, it follows that

$$C_{\text{unif}}(1 + \lambda^{-1})\|(L_h^* - L_0^*)\lambda^{-1}\Phi_0^-\|_{\mathbf{L}^2} \rightarrow 0 \qquad (2.5.67)$$

as $h \downarrow 0$. Possibly after decreasing $h_0'(\lambda) > 0$, we hence see that

$$
\begin{aligned}
\langle \Phi_0^+, (L_h^* + \lambda)^{-1}\Phi_0^- \rangle &= \langle \Phi_0^+, (L_0^* + \lambda)^{-1}\Phi_0^- \rangle + \langle \Phi_0^+, (L_h^* + \lambda)^{-1}\Phi_0^- - (L_0^* + \lambda)^{-1}\Phi_0^- \rangle \\
&= \lambda^{-1}\langle \Phi_0^+, \Phi_0^- \rangle + \langle \Phi_0^+, (L_h^* + \lambda)^{-1}\Phi_0^- - (L_0^* + \lambda)^{-1}\Phi_0^- \rangle \\
&> \frac{\langle \Phi_0^+, \Phi_0^- \rangle}{2}\lambda^{-1}
\end{aligned}
$$

$$(2.5.68)$$

holds for all $0 < h < \min\{h_{**}, h_0'(\lambda)\}$. $\blacksquare$

Lemma 2.5.13. *Assume that (HP1), (HS), (Hα1) and (Hα2) are satisfied. Fix $0 < h < h_{**}$. There exist linear maps*

$$
\begin{aligned}
\tilde{\gamma}_h^- : \boldsymbol{L}^2 &\rightarrow \mathbb{R}, \\
\tilde{L}_h^{*,\text{qinv}} : \boldsymbol{L}^2 &\rightarrow \boldsymbol{H}^1
\end{aligned}
$$

$$(2.5.69)$$

such that for all $\Theta \in \boldsymbol{L}^2$ the pair

$$(\gamma, \Psi) = (\tilde{\gamma}_h^-\Theta, \tilde{L}_h^{*,\text{qinv}}\Theta) \qquad (2.5.70)$$

is the unique solution to the problem

$$L_h^*\Psi = \Theta + \gamma\Phi_0^- \qquad (2.5.71)$$

that satisfies the normalisation condition

$$\langle \Phi_0^+, \Psi \rangle \quad = \quad 0. \tag{2.5.72}$$

Furthermore, there exists $C^* > 0$, such that for all $0 < h < h_{**}$ and all $\Theta \in \boldsymbol{L}^2$ we have the bound

$$|\tilde{\gamma}_h^- \Theta| + \|\tilde{L}_h^{*,\mathrm{qinv}} \Theta\|_{\boldsymbol{H}^1} \quad \leq \quad C^* \|\Theta\|_{\boldsymbol{L}^2}. \tag{2.5.73}$$

Proof. We define the set

$$Z^1 \quad = \quad \{ \Psi \in \mathbf{H}^1 : \langle \Phi_0^+, \Psi \rangle = 0 \}. \tag{2.5.74}$$

Pick $\Theta \in \mathbf{L}^2$. We look for a solution $(\gamma, \Psi) \in \mathbb{R} \times Z^1$ of the problem

$$\Psi \quad = \quad (L_h^* + \lambda)^{-1} [\Theta + \gamma \Phi_0^- + \lambda \Psi]. \tag{2.5.75}$$

Lemma 2.5.12 implies that $\langle \Phi_0^+, (L_h^* + \lambda)^{-1} \Phi_0^- \rangle \neq 0$. Hence, for given $\Theta \in \mathbf{L}^2, \Psi \in Z^1, h, \lambda$, we may write

$$\gamma(\Psi, \Theta, h, \lambda) \quad = \quad -\frac{\langle \Phi_0^+, (L_h^* + \lambda)^{-1}(\Theta + \lambda \Psi) \rangle}{\langle \Phi_0^+, (L_h^* + \lambda)^{-1} \Phi_0^- \rangle}, \tag{2.5.76}$$

which is the unique value for γ for which

$$(L_h^* + \lambda)^{-1} [\Theta + \gamma \Phi_0^- + \lambda \Psi] \quad \in \quad Z^1. \tag{2.5.77}$$

From now on the proof is identical to that of Lemma 2.5.9, so we omit it. ∎

Lemma 2.5.14. *Assume that (HP1), (HS), (Hα1) and (Hα2) are satisfied. For each $0 < h < h_{**}$ there exists an element $\Phi_h^- \in \ker(L_h^*)$ such that the family Φ_h^- converges to Φ_0^- in $\boldsymbol{H}^1$ as $h \downarrow 0$.*

Proof. We repeat some of the steps of the proof of Lemma 2.5.11, but now for L_h^*.

Fix $0 < h < h_{**}$. Since $\dim(\ker(L_h^*)) = 1$ by Proposition 2.5.1 and Lemma 2.5.11, we can pick $\Phi \in \ker(L_h^*)$ with $\Phi \neq 0$. If we would have $\langle \Phi, \Phi_0^+ \rangle = 0$, then we would obtain

$$\begin{aligned} 0 \quad &= \quad L_h^{*,\mathrm{qinv}}[0] \\ &= \quad \Phi, \end{aligned} \tag{2.5.78}$$

which leads to a contradiction. Hence, we can define the kernel element Φ_h^- of L_h^* as follows: Φ_h^- is the unique kernel element of L_h^* with $\langle \Phi_h^-, \Phi_0^+ \rangle = \langle \Phi_0^-, \Phi_0^+ \rangle$. Since we see that

$$\langle \Phi_0^- - \Phi_h^-, \Phi_0^+ \rangle \quad = \quad 0, \tag{2.5.79}$$

we obtain, upon defining

$$\Theta_h \quad := \quad L_h^* \Phi_0^-, \tag{2.5.80}$$

that

$$\Phi_0^- - \Phi_h^- \quad = \quad L_h^{*,\mathrm{qinv}}[\Theta_h]. \tag{2.5.81}$$

Using Lemma 2.5.13, we can estimate

$$
\begin{aligned}
\|\Phi_0^- - \Phi_h^-\|_{\mathbf{H}^1} &= \|L_h^{*,\mathrm{qinv}}[\Theta_h]\|_{\mathbf{H}^1} \\
&\leq C_-\|\Theta_h\|_{\mathbf{L}^2}.
\end{aligned}
\tag{2.5.82}
$$

From the proof of Lemma 2.5.12 we know that $\Theta_h \to 0$ as $h \downarrow 0$ in $\mathbf{L}^2$. Therefore, we see that $\Phi_h^- \to \Phi_0^-$ as $h \downarrow 0$ in $\mathbf{H}^1$. ∎

In the final part of this section we establish item (3) of Proposition 2.5.2. To this end, we recall the spaces

$$
X_h = \{\Theta \in \mathbf{H}^1 : \langle \Phi_h^-, \Theta \rangle = 0\}
\tag{2.5.83}
$$

and

$$
Y_h = \{\Theta \in \mathbf{L}^2 : \langle \Phi_h^-, \Theta \rangle = 0\},
\tag{2.5.84}
$$

together with the constant C_{unif} from (2.5.35).

Lemma 2.5.15. *Assume that (HP1), (HS), (Hα1) and (Hα2) are satisfied. For each $0 < h < h_{**}$ we have that $L_h : X_h \to Y_h$ is invertible and we have the uniform bound*

$$
\|L_h^{-1}\| \leq C_{\mathrm{unif}}.
\tag{2.5.85}
$$

Proof. Fix $0 < h < h_{**}$. Clearly $L_h : X_h \to Y_h$ is a bounded bijective linear map, so the Banach isomorphism theorem implies that $L_h^{-1} : Y_h \to X_h$ is bounded. Now let $\delta > 0$ be a small constant such that $\delta C_{\mathrm{unif}} < 1$. Without loss of generality we assume that $0 < h_{**} \leq h_0'(\delta)$ and that $\|\Phi_h^- - \Phi_0^-\|_{\mathbf{H}^1} \leq \delta$ for all $0 < h < h_{**}$. This is possible by Lemma 2.5.14.

Pick any $\Psi \in X_h$. Remembering that $\langle \Psi, \Phi_h^- \rangle = 0$ and $\langle L_h\Psi, \Phi_h^- \rangle = 0$, we obtain the estimate

$$
\begin{aligned}
\tfrac{1}{\delta}|\langle (L_h + \delta)\Psi, \Phi_0^- \rangle| &= \tfrac{1}{\delta}|\langle (L_h + \delta)\Psi, \Phi_0^- - \Phi_h^- \rangle| \\
&\leq \tfrac{1}{\delta}\|(L_h + \delta)\Psi\|_{\mathbf{L}^2}\delta \\
&\leq \|L_h\Psi\|_{\mathbf{L}^2} + \delta\|\Psi\|_{\mathbf{H}^1}.
\end{aligned}
\tag{2.5.86}
$$

Applying Proposition 2.3.2, we hence see

$$
\begin{aligned}
\|\Psi\|_{\mathbf{H}^1} &\leq \tfrac{1}{4}C_{\mathrm{unif}}[\|(L_h + \delta)\Psi\|_{\mathbf{L}^2} + \tfrac{1}{\delta}|\langle (L_h + \delta)\Psi, \Phi_0^- \rangle|] \\
&\leq \tfrac{1}{4}C_{\mathrm{unif}}[2\|L_h\Psi\|_{\mathbf{L}^2} + 2\delta\|\Psi\|_{\mathbf{H}^1}] \\
&\leq \tfrac{1}{2}C_{\mathrm{unif}}\|L_h\Psi\|_{\mathbf{L}^2} + \tfrac{1}{2}\|\Psi\|_{\mathbf{H}^1}.
\end{aligned}
\tag{2.5.87}
$$

We, therefore, get the bound

$$
\|\Psi\|_{\mathbf{H}^1} \leq C_{\mathrm{unif}}\|L_h\Psi\|_{\mathbf{L}^2},
\tag{2.5.88}
$$

which yields the desired estimate $\|L_h^{-1}\| \leq C_{\mathrm{unif}}$. ∎

Proof of Proposition 2.5.2. This result follows directly from Lemmas 2.5.11, 2.5.14 and 2.5.15. ∎

2.6   The resolvent set

In this section, we prove Theorem 2.2.2 by explicitly determining the spectrum of the operator $-L_h$ defined in (2.2.18) in a suitable half-plane. Our approach hinges on the periodicity of this spectrum, which we describe in our first result.

Lemma 2.6.1. *Assume that (HP1), (HS), (Hα1) and (Hα2) are satisfied. Fix $0 < h < h_{**}$. Then the spectrum of L_h is invariant under the operation $\lambda \mapsto \lambda + 2\pi i c_h \frac{1}{h}$.*

In particular, we can restrict our attention to values with imaginary part in between $-\frac{\pi c_h}{h}$ and $\frac{\pi c_h}{h}$. We divide our 'half-strip' into four regions and in each region we calculate the spectrum. Values close to 0 (region R_1) will be treated in Proposition 2.6.2; values with a large real part (region R_2) in Proposition 2.6.3 and values with a large imaginary part (region R_3) in Proposition 2.6.6. In Corollary 2.6.7 we discuss the remaining intermediate subset (region R_4), which is compact and independent of h. The regions are illustrated in Figure 2.1 below.
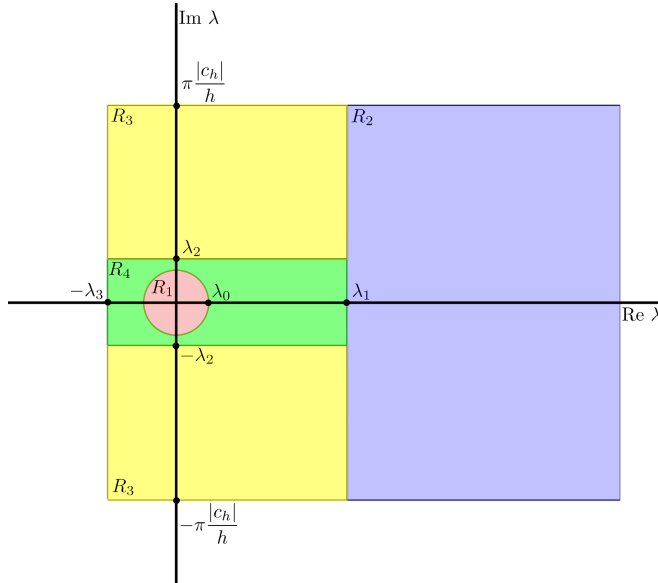

Figure 2.1: *Illustration of the regions R_1, R_2, R_3 and R_4. Note that the regions R_2 and R_3 grow when h decreases, while the regions R_1 and R_4 are independent of h.*

From this section onward we need to assume that (HP2) is satisfied. Indeed, this allows us to lift the invertibility of $L_0 + \lambda$ to $L_h + \lambda$ simultaneously for all λ in appropriate compact sets.

Proof of Lemma 2.6.1. Fix $k \in \mathbb{Z}$ and write $p = 2\pi i k \frac{1}{h}$. We define the exponential shift operator e_ω by

$$[e_\omega V](x) \quad = \quad e^{\omega x} V(x). \tag{2.6.1}$$

For any $\lambda \in \mathbb{C}$, $\Psi = (\phi, \psi) \in \mathbf{H}^1$ and $x \in \mathbb{R}$ we obtain

$$
\begin{aligned}
(e_{-p}\Delta_h e_p)\phi(x) &= e^{-px}\Delta_h(e_p\phi)(x) \\
&= \tfrac{1}{h^2} \sum_{l>0} \alpha_l (e^{plh}\phi(x+lh) + e^{-plh}\phi(x-lh) - 2\phi(x)) \\
&= \tfrac{1}{h^2} \sum_{l>0} \alpha_l (\phi(x+lh) + \phi(x-lh) - 2\phi(x)) \\
&= \Delta_h \phi(x),
\end{aligned}
\tag{2.6.2}
$$

since $plh \in 2\pi i \mathbb{Z}$ for all $l > 0$. In particular, we can compute

$$
\begin{aligned}
[e_{-p}(L_h - \lambda)e_p\Psi](x) &= e^{-px}[(L_h - \lambda)e_p\Psi](x) \\
&= e^{-px}\left(\begin{array}{c} c_h \frac{d}{d\xi}(e^{px}\phi(x)) - \Delta_h(e_p\phi)(x) \\ -\rho e^{px} + c_h \frac{d}{d\xi}(e^{px}\psi(x)) \end{array}\right) \\
&\quad + e^{-px}\left(\begin{array}{c} -g_u(\overline{u}_h)e^{px}\phi(x) + e^{px}\psi(x) - \lambda e^{px}\phi(x) \\ +\gamma\rho e^{px}\psi(x) - \lambda e^{px}\psi(x) \end{array}\right) \\
&= \left(\begin{array}{c} pc_h\phi(x) + c_h\phi'(x) - g_u(\overline{u}_h)\phi(x) + \psi(x) \\ -\rho\phi(x) + pc_h\psi(x) + c_h\psi'(x) + \gamma\rho\psi(x) - \lambda\psi(x) \end{array}\right) \\
&\quad + \left(\begin{array}{c} -\Delta_h\phi(x) - \lambda\phi(x) \\ 0 \end{array}\right) \\
&= (L_h - \lambda + pc_h)\Psi(x).
\end{aligned}
\tag{2.6.3}
$$

Since e_p and e_{-p} are invertible operators on $\mathbf{H}^1$ and $\mathbf{L}^2$ respectively, we know that the spectrum of L_h equals that of $e_{-p}L_h e_p$ and thus that of $L_h + pc_h$. ∎

Region R_1.

Since L_h has a simple eigenvalue at zero, it is relatively straightforward to construct a small neighbourhood around the origin that contains no other part of the spectrum. Exploiting the results from §2.5, it is possible to control the size of this neighbourhood as $h \downarrow 0$.

Proposition 2.6.2. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. There exists a constant $\lambda_0 > 0$ such that for all $0 < h < h_{**}$ the operator $L_h + \lambda : \mathbf{H}^1 \to \mathbf{L}^2$ is invertible for all $\lambda \in \mathbb{C}$ with $0 < |\lambda| < \lambda_0$.*

Proof. Fix $0 < h < h_{**}$ and $\Theta \in \mathbf{L}^2$. We recall the notation $(\gamma_h[\Theta], L_h^{\mathrm{qinv}}\Theta)$ from Corollary 2.5.4 for the unique solution (γ, Ψ) of the equation

$$
L_h\Psi = \Theta + \gamma\Phi_h^+
\tag{2.6.4}
$$

in the space

$$
X_h = \{\Theta \in \mathbf{H}^1 : \langle \Phi_h^-, \Theta \rangle = 0\}.
\tag{2.6.5}
$$

Also recall the space

$$
Y_h = \{\Theta \in \mathbf{L}^2 : \langle \Phi_h^-, \Theta \rangle = 0\}.
\tag{2.6.6}
$$

Now, for $\lambda \in \mathbb{C}$ with $|\lambda|$ small enough, but $\lambda \neq 0$, we want to solve the equation $L_h \Psi = \lambda \Psi + \Theta$. Upon writing

$$\Psi \;\; = \;\; L_h^{\mathrm{qinv}} \Theta + \lambda^{-1} \gamma_h[\Theta] \Phi_h^+ + \tilde{\Psi}, \tag{2.6.7}$$

with $\tilde{\Psi} \in X_h$, we see that

$$\begin{aligned}
(L_h - \lambda)\Psi \;\; &= \;\; (L_h - \lambda) L_h^{\mathrm{qinv}} \Theta + \lambda^{-1}(L_h - \lambda)\gamma_h[\Theta]\Phi_h^+ + (L_h - \lambda)\tilde{\Psi} \\
&= \;\; \Theta + \gamma_h[\Theta]\Phi_h^+ - \lambda L_h^{\mathrm{qinv}}\Theta - \gamma_h[\Theta]\Phi_h^+ + (L_h - \lambda)\tilde{\Psi}.
\end{aligned} \tag{2.6.8}$$

In particular, we must find a solution $\tilde{\Psi} \in X_h$ for the equation

$$L_h \tilde{\Psi} \;\; = \;\; \lambda \tilde{\Psi} + \lambda L_h^{\mathrm{qinv}} \Theta, \tag{2.6.9}$$

which we can rewrite as

$$[I - \lambda L_h^{-1}]\tilde{\Psi} \;\; = \;\; \lambda L_h^{-1} L_h^{\mathrm{qinv}} \Theta. \tag{2.6.10}$$

Note that $L_h^{-1} : X_h \to X_h$ is also a bounded operator since $X_h \subset Y_h$. Since

$$\begin{aligned}
\|L_h^{-1}\Psi\|_{\mathbf{H}^1} \;\; &\leq \;\; C_{\mathrm{unif}}\|\Psi\|_{\mathbf{L}^2} \\
&\leq \;\; C_{\mathrm{unif}}\|\Psi\|_{\mathbf{H}^1},
\end{aligned} \tag{2.6.11}$$

we obtain

$$\|L_h^{-1}\|_{\mathcal{B}(X_h, X_h)} \;\; \leq \;\; C_{\mathrm{unif}}. \tag{2.6.12}$$

We choose λ_0 in such a way that $0 < \lambda_0 C_{\mathrm{unif}} < 1$. Then it is well-known that $I - \lambda L_h^{-1}$ is invertible as an operator on X_h for $0 < |\lambda| < \lambda_0$. Since $\lambda L_h^{-1} L_h^{\mathrm{qinv}} \Theta \in X_h$, we see that (2.6.10) indeed has a unique solution $\tilde{\Psi} \in X_h$. Hence, the equation $(L_h - \lambda)\Psi = \Theta$ always has a unique solution. Proposition 2.5.1 states that $L_h - \lambda$ is Fredholm with index 0, which now implies that $L_h - \lambda$ is indeed invertible. ∎

Region R_2.

We now show that in an appropriate right half-plane, which can be chosen independently of h, the spectrum of $-L_h$ is empty. The proof proceeds via a relatively direct estimate that is strongly inspired by [6, Lem. 3.1].

Proposition 2.6.3. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. There exists a constant $\lambda_1 > 0$ such that for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \geq \lambda_1$ and all $0 < h < h_{**}$ the operator $L_h + \lambda$ is invertible.*

Proof. Write

$$\lambda_1 \;\; = \;\; 1 + g_* + \tfrac{1}{2}(1 - \rho), \tag{2.6.13}$$

where g_* is defined in (2.3.42). Pick any $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geq \lambda_1$ and any $0 < h < h_{**}$. Let $\Psi = (\phi, \psi) \in \mathbf{H}^1$ be arbitrary and set $\Theta = L_h \Psi + \lambda \Psi$. Then we see that

$$
\begin{aligned}
\|\Psi\|_{\mathbf{L}^2} \|\Theta\|_{\mathbf{L}^2} \quad &\geq \quad \operatorname{Re} \langle L_h \Psi + \lambda \Psi, \Psi \rangle \\
&\geq \quad \operatorname{Re} \langle -\Delta_h \phi, \phi \rangle - \|g_u(\overline{u}_h)\|_{L^\infty} \|\phi\|_{L^2}^2 \\
&\qquad -(1-\rho) |\operatorname{Re} \langle \phi, \psi \rangle| + \gamma \rho \|\psi\|_{L^2}^2 + \operatorname{Re} \lambda \|\Psi\|_{\mathbf{L}^2}^2 \\
&\geq \quad -g_* \|\phi\|_{L^2}^2 - (1-\rho) |\operatorname{Re} \langle \phi, \psi \rangle| + \gamma \rho \|\psi\|_{L^2}^2 + \operatorname{Re} \lambda \|\Psi\|_{\mathbf{L}^2}^2 \\
&\geq \quad -g_* \|\phi\|_{L^2}^2 - (1-\rho) \|\phi\|_{L^2} \|\psi\|_{L^2} + \gamma \rho \|\psi\|_{L^2}^2 + \operatorname{Re} \lambda \|\Psi\|_{\mathbf{L}^2}^2 \\
&\geq \quad -(g_* + \tfrac{1}{2}(1-\rho)) \|\Psi\|_{\mathbf{L}^2}^2 + \operatorname{Re} \lambda \|\Psi\|_{\mathbf{L}^2}^2 .
\end{aligned}
\tag{2.6.14}
$$

Hence, we obtain

$$
\left(\operatorname{Re} \lambda - (g_* + \tfrac{1}{2}(1-\rho)) \right) \|\Psi\|_{\mathbf{L}^2} \quad \leq \quad \|\Theta\|_{\mathbf{L}^2}. \tag{2.6.15}
$$

Since $\operatorname{Re} \lambda \geq 1 + g_* + \tfrac{1}{2}(1-\rho)$, we obtain the bound $\|\Psi\|_{\mathbf{L}^2} \leq \|\Theta\|_{\mathbf{L}^2}$.

In particular, if $\Theta = 0$ then we necessarily have $\Psi = 0$, which implies that $L_h + \lambda$ is injective. Since also $\operatorname{ind}(L_h + \lambda) = 0$ by Proposition 2.5.1, this means that $L_h + \lambda$ is invertible. $\blacksquare$

Region R_3.

This region is the most delicate to handle on account of the periodicity of the spectrum. Indeed, one cannot simply take $\operatorname{Im} \lambda \to \pm \infty$ in a fashion that is uniform in h. We pursue a direct approach here, using the Fourier transform to isolate the problematic part of $L_h + \lambda$, which has constant coefficients. The corresponding portion of the resolvent can be estimated in a controlled way by rescaling the imaginary part of λ. We remark that an alternative approach could be to factor out the periodicity in a more operator-theoretic setting, but we do not pursue such an argument here.

Pick $\lambda \in \mathbb{C}$ with $\lambda_0 < |\operatorname{Im} \lambda| \leq \frac{|c_h|}{h} \pi$ and write

$$
\lambda \quad = \quad \lambda_{\mathrm{r}} + i \lambda_{\mathrm{im}} \tag{2.6.16}
$$

with $\lambda_{\mathrm{r}}, \lambda_{\mathrm{im}} \in \mathbb{R}$. Introducing the new variable $\tau = \operatorname{Im} \lambda \xi$, we can write the eigenvalue problem $(L_h + \lambda)(v, w) = 0$ in the form

$$
\begin{aligned}
c_h v_\tau(\tau) \quad &= \quad \tfrac{1}{\lambda_{\mathrm{im}} h^2} \sum_{k>0} \alpha_k \Big[v(\tau + kh\lambda_{\mathrm{im}}) + v(\tau - kh\lambda_{\mathrm{im}}) - 2v(\tau) \Big] \\
&\qquad + \tfrac{1}{\lambda_{\mathrm{im}}} g_u\big(\overline{u}_h(\tau)\big) v(\tau) - iv(\tau) - \tfrac{1}{\lambda_{\mathrm{im}}} \lambda_{\mathrm{r}} v(\tau) - \tfrac{1}{\lambda_{\mathrm{im}}} w(\tau), \tag{2.6.17} \\
c_h w_\tau(\tau) \quad &= \quad \tfrac{1}{\lambda_{\mathrm{im}}} \Big(\rho v(\tau) - \rho \gamma w(\tau) + \lambda w(\tau) \Big).
\end{aligned}
$$

Our computations below show that the leading order terms in the appropriate $|\lambda_{\mathrm{im}}| \to$

∞ limit are encoded by the 'homogeneous operator' $H_{h,\lambda}$ that acts as

$$H_{h,\lambda}v(\tau) \;=\; c_h v_\tau(\tau) + iv(\tau) - \tfrac{1}{\lambda_{\mathrm{im}}h^2}\sum_{k>0}\alpha_k\Big[v(\tau+kh\lambda)+v(\tau-kh\lambda)-2v(\tau)\Big].$$
(2.6.18)

Writing $\mathcal{H}_{h,\lambda}$ for the Fourier symbol associated to $H_{h,\lambda}$, we see that

$$
\begin{aligned}
\mathcal{H}_{h,\lambda}(i\omega) &= c_h i\omega + i - \tfrac{1}{\lambda_{\mathrm{im}}h^2}\sum_{k>0}\alpha_k\Big[\exp(ihk\lambda_{\mathrm{im}}\omega)+\exp(-ihk\lambda_{\mathrm{im}}\omega)-2\Big] \\
&= c_h i\omega + i - \tfrac{2}{\lambda_{\mathrm{im}}h^2}\sum_{k>0}\alpha_k\Big[\cos(hk\lambda_{\mathrm{im}}\omega)-1\Big].
\end{aligned}
$$
(2.6.19)

Lemma 2.6.4. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. There exist small constants $\varepsilon > 0$, $h_* > 0$ and $\omega_0 > 0$ so that for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$, all $0 < h < h_*$ and all $\omega \in \mathbb{R}$, the inequality*

$$|\mathrm{Im}\,\mathcal{H}_{h,\lambda}(i\omega)| \;<\; \varepsilon$$
(2.6.20)

can only be satisfied if the inequalities

$$
\begin{aligned}
|c_h\omega| &\leq \tfrac{3}{2} \\
|\omega| &\geq \omega_0
\end{aligned}
$$
(2.6.21)

both hold.

Proof. Note that

$$|\mathrm{Im}\,\mathcal{H}_{h,\lambda}(i\omega)| \;=\; |c_h\omega + 1|.$$
(2.6.22)

In particular, upon choosing $\varepsilon = \tfrac{1}{4}$, we see that

$$|\mathrm{Im}\,\mathcal{H}_{h,\lambda}(i\omega)| \;<\; \varepsilon$$
(2.6.23)

implies

$$\big||c_h\omega|-1\big| \;\leq\; |c_h\omega+1| \;<\; \varepsilon$$
(2.6.24)

and hence

$$\tfrac{1}{2} \;<\; 1-\varepsilon \;\leq\; |c_h\omega| \;\leq\; 1+\varepsilon \;<\; \tfrac{3}{2}.$$
(2.6.25)

Since $c_h \to c_0 \neq 0$ as $h \downarrow 0$, the desired inequalities (2.6.21) follow. ∎

Lemma 2.6.5. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. Then there exists a constant $C > 0$ such that for all $\omega \in \mathbb{R}$ and $0 < h < h_{**}$ and all $\lambda \in \mathbb{C}$ with $|\lambda| > \lambda_0$ and $|\mathrm{Im}\,\lambda| \leq \tfrac{|c_h|}{h}\pi$, we have the inequality*

$$|\mathcal{H}_{h,\lambda}(i\omega)| \geq \tfrac{1}{C}.$$
(2.6.26)

Proof. We show that $\mathcal{H}_{h,\lambda}(i\omega)$ is bounded away from 0, uniformly in h, λ and ω. To do so, we show that the real part of $\mathcal{H}_{h,\lambda}(i\omega)$ can be bounded away from zero, whenever

the imaginary part is small, i.e. when (2.6.21) holds.

Recall the function $A(y) = \sum_{k>0} \alpha_k[1 - \cos(ky)]$ defined in Assumption (Hα1), which satisfies $A(y) > 0$ for $y \in (0, 2\pi)$. A direct calculation shows that $A'(0) = 0$ and

$$\begin{aligned} A''(0) &= \sum_{k>0} \alpha_k k^2 \\ &= 1. \end{aligned} \qquad (2.6.27)$$

Hence, we can pick $d_0 > 0$ in such a way that

$$\tfrac{1}{y^2} A(y) > d_0 \qquad (2.6.28)$$

holds for all $0 < |y| \le \tfrac{3}{2}\pi$.

Writing $\mu = h\lambda_{\mathrm{im}}\omega$, we see

$$\begin{aligned} \mathrm{Re}\,\mathcal{H}_{h,\lambda}(i\omega) &= \tfrac{2\omega^2 \lambda_{\mathrm{im}}}{\mu^2} \sum_{k>0} \alpha_k \left[1 - \cos(k\mu)\right] \\ &= \tfrac{2\omega^2 \lambda_{\mathrm{im}}}{\mu^2} A(\mu). \end{aligned} \qquad (2.6.29)$$

Now fix ω, h, λ for which $|\mathrm{Im}\,\mathcal{H}_{h,\lambda}(i\omega)| < \varepsilon$. The conditions (2.6.21) now imply that $|\omega| \ge \omega_0$ and $|\mu| \le h\frac{|c_h|}{h}\pi|\omega| \le \tfrac{3}{2}\pi$. Using (2.6.28), we hence see that

$$\begin{aligned} |\mathrm{Re}\,\mathcal{H}_{h,\lambda}(i\omega)| &= |\tfrac{2\omega^2 \lambda_{\mathrm{im}}}{\mu^2} A(\mu)| \\ &\ge 2|\lambda_{\mathrm{im}}|\omega^2 d_0 \\ &\ge 2\lambda_0 \omega_0^2 d_0, \end{aligned} \qquad (2.6.30)$$

which shows that $\mathcal{H}_{h,\lambda}(i\omega)$ can indeed be uniformly bounded away from zero. ∎

Proposition 2.6.6. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. There exist constants $\lambda_2 > 0$ and $\lambda_3 > 0$ such that for all $\lambda \in \mathbb{C}$ with $\lambda_2 \le |\mathrm{Im}\,\lambda| \le \frac{|c_h|}{2h}2\pi$ and $-\lambda_3 \le |\mathrm{Re}\,\lambda| \le \lambda_1$ and all $0 < h < h_{**}$ the operator $L_h + \lambda$ is invertible.*

Proof. Since Proposition 2.5.1 implies that $L_h + \lambda$ is Fredholm with index zero, it suffices to prove that $L_h + \lambda$ is injective.

Let $\lambda_3 = \min\{\tfrac{1}{2}\rho\gamma, \lambda_*, \tilde{\lambda}\}$, where λ_* is defined in (HP2) and $\tilde{\lambda}$ is defined in Proposition 2.5.1. Pick $\lambda \in \mathbb{C}$ with $\lambda_0 \le |\mathrm{Im}\,\lambda| \le \frac{|c_h|}{2h}2\pi$ and $-\lambda_3 \le |\mathrm{Re}\,\lambda| \le \lambda_1$. Write $\lambda = \lambda_{\mathrm{r}} + i\lambda_{\mathrm{im}}$ as before. Suppose $\Psi = (v, w)$ satisfies $(L_h + \lambda)\Psi = 0$.

Write $\hat{v}$ and $\hat{w}$ for the Fourier transforms of v and w respectively. For $f \in L^2$ with Fourier transform $\hat{f}$, the identity

$$H_{h,\lambda}v = f \qquad (2.6.31)$$

implies that

$$\mathcal{H}_{h,\lambda}(i\omega)\hat{v}(i\omega) = \hat{f}(i\omega). \tag{2.6.32}$$

In particular, we obtain

$$\hat{v}(i\omega) = \frac{1}{\mathcal{H}_{h,\lambda}(i\omega)}\hat{f}(i\omega), \tag{2.6.33}$$

which using Lemma 2.6.5 implies that

$$\|v\|_{L^2} \leq C\|f\|_{L^2} \tag{2.6.34}$$

for some constant $C > 0$ that is independent of h, λ and ω.

Since Ψ is an eigenfunction, (2.6.17) hence yields

$$\|v\|_{L^2} \leq C\frac{1}{|\lambda_{\mathrm{im}}|}(g_* + |\lambda_{\mathrm{r}}|)\|v\|_{L^2} + C\frac{1}{|\lambda_{\mathrm{im}}|}\|w\|_{L^2}. \tag{2.6.35}$$

Furthermore, applying a Fourier Transform to the second line of (2.6.17), we find

$$\lambda_{\mathrm{im}}c_h i\omega\hat{w}(i\omega) = \rho\hat{v}(i\omega) - \rho\gamma\hat{w}(i\omega) + \lambda\hat{w}(i\omega). \tag{2.6.36}$$

Our choice $\lambda_3 \leq \frac{1}{2}\rho\gamma$ implies that $-\rho\gamma + \lambda_{\mathrm{r}}$ is bounded away from 0. We may hence write

$$\hat{w}(i\omega) = \frac{1}{\rho\gamma - \lambda_{\mathrm{r}} + i(\omega\lambda_{\mathrm{im}}c_h - \lambda_{\mathrm{im}})}\rho\hat{v}(i\omega), \tag{2.6.37}$$

which yields the bound

$$\|w\|_{L^2} \leq C'\|v\|_{L^2} \tag{2.6.38}$$

for some constant $C' > 0$. Therefore, we obtain that

$$\|v\|_{L^2} \leq C''\frac{1}{|\lambda_{\mathrm{im}}|}\|v\|_{L^2} \tag{2.6.39}$$

for some constant C'', which is independent of λ, h and v. Clearly this is impossible for $v \neq 0$ if

$$|\lambda_{\mathrm{im}}| \geq \lambda_2 := 2C''. \tag{2.6.40}$$

Furthermore, if $v = 0$, then clearly also $w = 0$. Therefore, we have $\Psi = 0$, allowing us to conclude that $L_h + \lambda$ is invertible. ∎

Region R_4.

We conclude our spectral analysis by considering the remaining region R_4. This region is compact and bounded away from the origin, allowing us to directly apply the theory developed in §2.3.

Corollary 2.6.7. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. For all $\lambda \in \mathbb{C}$ with $|\lambda| \geq \lambda_0$, $-\lambda_3 \leq |\mathrm{Re}\,\lambda| \leq \lambda_1$ and $|\mathrm{Im}\,\lambda| \leq \lambda_2$ and all $0 < h < h_{**}$ the operator $L_h + \lambda$ is invertible.*

Proof. The statement follows by applying Proposition 2.3.3 with the choices $(\tilde{u}_h, \tilde{w}_h) = (\overline{u}_h, \overline{w}_h)$, $\tilde{c}_h = c_h$ and $M = R_4$. ∎

Proof of Theorem 2.2.2. The result follows directly from Lemma 2.6.1, Proposition 2.6.2, Proposition 2.6.3, Proposition 2.6.6 and Corollary 2.6.7. ∎

2.7 Green's functions

In order to establish the nonlinear stability of the pulse solution $(\overline{u}_h, \overline{w}_h)$, we need to consider two types of Green's functions. In particular, we first study $G_\lambda(\xi, \xi_0)$, which can roughly be seen as a solution of the equation

$$\Big[(L_h + \lambda)G_\lambda(\cdot, \xi_0)\Big](\xi) = \delta(\xi - \xi_0), \tag{2.7.1}$$

where δ is the Dirac delta-distribution. We then use these functions to build a Green's function $\mathcal{G}$ for the linearisation of the LDE (2.2.1) around the travelling pulse solution.

An important difficulty in comparison to the PDE setting is caused by the discreteness of the spatial variable j. In particular, we cannot use a frame of reference in which the solution $(\overline{u}_h, \overline{w}_h)$ is constant without changing the structure of the equation (2.2.1). The Green's function $\mathcal{G}$ will hence be the solution to a non-autonomous problem that satisfies a shift-periodicity condition. Nevertheless, one can follow the techniques in [13] to express $\mathcal{G}$ in terms of a contour integral involving the functions G_λ.

A significant part of our effort here is concerned with the construction of these latter functions. Indeed, previous approaches in [11, 109] all used exponential dichotomies or variation-of-constants formula's for MFDEs with finite-range interactions. These tools are no longer available for use in the present infinite-range setting. In particular, we construct the functions G_λ in a direct fashion using only Fredholm properties of the operators $L_h + \lambda$. This makes it somewhat involved to recover the desired exponential decay rates and to properly isolate the meromorphic terms of order $O(\lambda^{-1})$.

From now on, we will no longer explicitly use the h-dependence of our system. To simplify our notation, we fix $0 < h < h_{**}$ and write

$$\begin{aligned}
L &:= L_h, \\
L_\infty &:= L_{h;\infty}, \\
\overline{U} &= (\overline{u}, \overline{w}) &= (\overline{u}_h, \overline{w}_h), \\
\Phi^\pm &= (\phi^\pm, \psi^\pm) &:= (\phi_h^\pm, \psi_h^\pm), \\
c &:= c_h.
\end{aligned} \tag{2.7.2}$$

We emphasize that from now on all our constants may (and will) depend on h.

We will loosely follow §2 of [109], borrowing a number of results from [13, 102] at appropriate times. In particular, we start by considering the linearisation of the original LDE (2.2.1) around the travelling pulse solution $\overline{U}(t)$ given by (2.2.21). To this end, we introduce the Hilbert space

$$\mathbb{L}^2 := \{V \in (\mathrm{Mat}_2(\mathbb{R}))^{\mathbb{Z}} : \sum_{j \in \mathbb{Z}} |V(j)|^2 < \infty\}, \tag{2.7.3}$$

in which $\mathrm{Mat}_2(\mathbb{R})$ is the space of 2×2-matrices with real coefficients which we equip with the maximum-norm $|\cdot|$. For any $\mathcal{V} \in \mathbb{L}^2$, we often write $\mathcal{V} = \begin{pmatrix} \mathcal{V}^{(1,1)} & \mathcal{V}^{(1,2)} \\ \mathcal{V}^{(2,1)} & \mathcal{V}^{(2,2)} \end{pmatrix}$,

when we need to refer to the component sequences $\mathcal{V}^{(i,j)} \in \ell^2(\mathbb{Z}; \mathbb{R})$. For any $t \in \mathbb{R}$ we now introduce the linear operator $\mathcal{A}(t) : \mathbb{L}^2 \to \mathbb{L}^2$ that acts as

$$\mathcal{A}(t) \cdot \mathcal{V} \quad = \quad \frac{1}{c} \begin{pmatrix} A^{(1,1)}(t) & A^{(1,2)}(t) \\ A^{(2,1)}(t) & A^{(2,2)}(t) \end{pmatrix} \begin{pmatrix} \mathcal{V}^{(1,1)} & \mathcal{V}^{(1,2)} \\ \mathcal{V}^{(2,1)} & \mathcal{V}^{(2,2)} \end{pmatrix}, \tag{2.7.4}$$

where

$$\begin{aligned}
(A^{(1,1)}(t)v)_j &= \frac{1}{h^2} \sum_{k>0} \alpha_k [v_{j+k} + v_{j-k} - 2v_j] + g_u\Big(\overline{u}(hj + ct)\Big) v_j \\
(A^{(1,2)}(t)w)_j &= -w_j \\
(A^{(2,1)}(t)v)_j &= \rho v_j \\
(A^{(2,2)}(t)v)_j &= -\rho \gamma w_j
\end{aligned} \tag{2.7.5}$$

for $v \in \ell^2(\mathbb{Z}; \mathbb{R})$ and $w \in \ell^2(\mathbb{Z}; \mathbb{R})$. With all this notation in hand, we can write the desired linearisation as the ODE

$$\frac{d}{dt}\mathcal{V}(t) \quad = \quad \mathcal{A}(t) \cdot \mathcal{V}(t) \tag{2.7.6}$$

posed on $\mathbb{L}^2$.

Fix $t_0 \in \mathbb{R}$ and $j_0 \in \mathbb{Z}$. Consider the function

$$\mathbb{R} \ni t \mapsto \mathcal{G}^{j_0}(t, t_0) \quad = \quad \{\mathcal{G}_j^{j_0}(t, t_0)\}_{j \in \mathbb{Z}} \in \mathbb{L}^2 \tag{2.7.7}$$

that is uniquely determined by the initial value problem

$$\begin{cases} \frac{d}{dt}\mathcal{G}^{j_0}(t, t_0) &= \mathcal{A}(t) \cdot \mathcal{G}^{j_0}(t, t_0) \\ \mathcal{G}_j^{j_0}(t_0, t_0) &= \delta_j^{j_0} I. \end{cases} \tag{2.7.8}$$

Here we have introduced

$$\delta_j^{j_0} \quad = \quad \begin{cases} 1 \text{ if } j = j_0 \\ 0 \text{ else,} \end{cases} \tag{2.7.9}$$

where $I \in \mathrm{Mat}_2(\mathbb{R})$ is the identity matrix. We remark that $\mathcal{G}_j^{j_0}(t, t_0)$ is an element of $\mathrm{Mat}_2(\mathbb{R})$ for each $j \in \mathbb{Z}$.

This function $\mathcal{G}$ is called the Green's function for the linearisation around our travelling pulse. Indeed, the general solution of the inhomogeneous equation

$$\begin{cases} \frac{dV}{dt} &= \mathcal{A}(t) \cdot V(t) + F(t) \\ V(0) &= V^0, \end{cases} \tag{2.7.10}$$

where now $V(t) \in \ell^2(\mathbb{Z}; \mathbb{R}^2) \cong \ell^2(\mathbb{Z}; \mathbb{R}^{2 \times 1})$ and $F(t) \in \ell^2(\mathbb{Z}; \mathbb{R}^2) \cong \ell^2(\mathbb{Z}; \mathbb{R}^{2 \times 1})$, is given by

$$V_j(t) \quad = \quad \sum_{j_0 \in \mathbb{Z}} \mathcal{G}_j^{j_0}(t, 0) V_{j_0}^0 + \int_0^t \sum_{j_0 \in \mathbb{Z}} \mathcal{G}_j^{j_0}(t, t_0) F_{j_0}(t_0) \, dt_0. \tag{2.7.11}$$

Introducing the standard convolution operator $*$, this can be written in the abbreviated form

$$V \;=\; \mathcal{G}(t,0) * V^0 + \int_0^t \mathcal{G}(t,t_0) * F(t_0) \; dt_0. \tag{2.7.12}$$

The main result of this section is the following proposition, which shows that we can decompose the Green's function $\mathcal{G}$ into a part that decays exponentially and a neutral part associated with translation along the family of travelling pulses.

Proposition 2.7.1. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. For any pair $t \geq t_0$ and any $j, j_0 \in \mathbb{Z}$, we have the representation*

$$\mathcal{G}_j^{j_0}(t,t_0) \;=\; \mathcal{E}_j^{j_0}(t,t_0) + \tilde{\mathcal{G}}_j^{j_0}(t,t_0), \tag{2.7.13}$$

in which

$$\mathcal{E}_j^{j_0}(t,t_0) \;=\; \frac{h}{\Omega} \left(\begin{array}{cc} \phi^-(hj_0 + ct_0)\phi^+(hj + ct) & \psi^-(hj_0 + ct_0)\phi^+(hj + ct) \\ \phi^-(hj_0 + ct_0)\psi^+(hj + ct) & \psi^-(hj_0 + ct_0)\psi^+(hj + ct) \end{array} \right), \tag{2.7.14}$$

while $\tilde{\mathcal{G}}$ satisfies the found

$$|\tilde{\mathcal{G}}_j^{j_0}(t,t_0)| \;\leq\; Ke^{-\tilde{\delta}(t-t_0)}e^{-\tilde{\delta}|hj+ct-hj_0-ct_0|} \tag{2.7.15}$$

for some $K > 0$ and $\tilde{\delta} > 0$. The constant $\Omega > 0$ is given by

$$\Omega \;=\; \langle \Phi^-, \Phi^+ \rangle. \tag{2.7.16}$$

Furthermore, for any $t \geq t_0$ we have the representation

$$\mathcal{G}_j^{j_0}(t,t_0) \;=\; \sum_{i \in \mathbb{Z}} \left[\mathcal{E}_j^i(t,t_0)\mathcal{E}_i^{j_0}(t_0,t_0) + \tilde{\mathcal{G}}_j^i(t,t_0)(\delta_i^{j_0} I - \mathcal{E}_i^{j_0}(t_0,t_0)) \right], \tag{2.7.17}$$

which can be abbreviated as

$$\mathcal{G}(t,t_0) \;=\; \mathcal{E}(t,t_0) * \mathcal{E}(t_0,t_0) + \tilde{\mathcal{G}}(t,t_0) * \left(I - \mathcal{E}(t_0,t_0) \right). \tag{2.7.18}$$

2.7.1 Construction of the Green's function

In this subsection, we set out to define the functions G_λ in a more rigorous fashion. In addition, we use these Green's functions to formulate a powerful representation formula for $\mathcal{G}$, see Proposition 2.7.4 below, following the approach developed in [13].

A key role in our analysis is reserved for the operator $L_{\infty;\lambda}$ and the function $\Delta_{L_{\infty;\lambda}}$ from Lemma 2.5.6. We will show that $L_{\infty;\lambda}$ has a Green's function $G_{\infty;\lambda}$ which takes values in the space $\text{Mat}_2(\mathbb{R})$ and has some useful properties. To this end, we recall the constant $\tilde{\lambda}$ from Lemma 2.5.6. For each $\lambda \in \mathbb{C}$ with $\text{Re}\,\lambda \geq -\frac{\tilde{\lambda}}{2}$, we may now define $G_{\infty;\lambda} : \mathbb{R} \to \text{Mat}_2(\mathbb{R})$ by writing

$$G_{\infty;\lambda}(\xi) \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\eta\xi}(\Delta_{L_{\infty;\lambda}}(i\eta))^{-1} \; d\eta. \tag{2.7.19}$$

We also introduce the notation

$$G_\infty \;=\; G_{\infty;0}. \tag{2.7.20}$$

Here (Hα2) is essential to ensure that these Green's functions decay exponentially.

Lemma 2.7.2. *Assume that (HP1), (HP2), (HS), (Hα1) and (Hα2) are satisfied. Fix $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq -\frac{\tilde{\lambda}}{2}$. The function $G_{\infty;\lambda}$ is bounded and continuous on $\mathbb{R} \setminus \{0\}$ and C^1-smooth on $\mathbb{R} \setminus h\mathbb{Z}$. Furthermore, $(L_\infty + \lambda)G_{\infty;\lambda}(\cdot - \xi_0)$ is constantly zero except at $\xi = \xi_0 + h\mathbb{Z}$ and satisfies the identity*

$$\int_{-\infty}^{\infty} \left[(L_\infty + \lambda)G_{\infty;\lambda}(\cdot - \xi_0) \right](\xi) f(\xi) \, d\xi \;\; = \;\; f(\xi_0) \tag{2.7.21}$$

for all $\xi \in \mathbb{R}$ and all $f \in \boldsymbol{H}^1$.

Finally for each $\chi > 0$ there exist constants $K_* > 0$ and $\beta_* > 0$, which may depend on χ, such that for each $\lambda \in \mathbb{C}$ with $-\frac{\tilde{\lambda}}{2} \leq \operatorname{Re}\lambda \leq \chi$ and $|\operatorname{Im}\lambda| \leq \frac{\pi|c|}{h}$ we have the bound

$$|G_{\infty;\lambda}(\xi - \xi_0)| \leq K_* e^{-\beta_* |\xi - \xi_0|} \tag{2.7.22}$$

for all $\xi, \xi_0 \in \mathbb{R}$.

Pick $\lambda \in \mathbb{C} \setminus \sigma(-L)$ with $\operatorname{Re}\lambda \geq -\frac{\tilde{\lambda}}{2}$. Observe that

$$L - L_\infty \;\; = \;\; \begin{pmatrix} -g_u(\overline{u}) + r_0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.7.23}$$

We know that $G_{\infty;\lambda}(\cdot - \xi_0) \in L^2(\mathbb{R}, \operatorname{Mat}_2(\mathbb{R}))$ since it decays exponentially. This means that we also have the inclusion

$$[L - L_\infty]G_{\infty;\lambda}(\cdot - \xi_0) \;\; \in \;\; L^2(\mathbb{R}, \operatorname{Mat}_2(\mathbb{C})). \tag{2.7.24}$$

Hence, it is possible to define the function G_λ by writing

$$G_\lambda(\xi, \xi_0) \;\; = \;\; G_{\infty;\lambda}(\xi - \xi_0) - \left[(\lambda + L)^{-1}[L - L_\infty]G_{\infty;\lambda}(\cdot - \xi_0) \right](\xi). \tag{2.7.25}$$

The next result shows that G_λ can be interpreted as the Green's function of $L + \lambda$. It is based on [109, Lem. 2.6].

Lemma 2.7.3. *Assume that (HP1), (HP2), (HS), (Hα1) and (Hα2) are satisfied. For $\lambda \in \mathbb{C} \setminus \sigma(-L)$ with $\operatorname{Re}\lambda \geq -\frac{\tilde{\lambda}}{2}$ we have that $G_\lambda(\cdot, y)$ is continuous on $\mathbb{R} \setminus \{y\}$ and C^1-smooth on $\mathbb{R} \setminus \{y + kh : k \in \mathbb{Z}\}$. Furthermore, it satisfies*

$$\int_{-\infty}^{\infty} \left[(\lambda + L)G_\lambda(\cdot, \xi_0) \right](\xi) f(\xi) \, d\xi \;\; = \;\; f(\xi_0) \tag{2.7.26}$$

for all $\xi \in \mathbb{R}$ and all $f \in \boldsymbol{H}^1$.

The link between our two types of Green's functions is provided by the following key result. It is based on [13, Thm. 4.2], where it was used to study one-sided spatial discretisation schemes for systems with conservation laws.

Proposition 2.7.4. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. Let $\chi > \lambda_{\mathrm{unif}}$ be given, where λ_{unif} is as in Lemma 2.7.7. For all $t \geq t_0$ the Green's function $\mathcal{G}_j^{j_0}(t, t_0)$ of (2.7.8) is given by*

$$\mathcal{G}_j^{j_0}(t, t_0) \;\; = \;\; -\frac{h}{2\pi i} \int_{\chi - \frac{i\pi c}{h}}^{\chi + \frac{i\pi c}{h}} e^{\lambda(t - t_0)} G_\lambda(hj + ct, hj_0 + ct_0) d\lambda \tag{2.7.27}$$

where G_λ is the Green's function of $\lambda + L$ as defined in (2.7.25).

Our first task is to collect several basic facts concerning the operators L_h and L_∞ that will allow us to establish Lemma's 2.7.2 and 2.7.3. In particular, we need to isolate and explicitly compute the part of the Fourier integral (2.7.19) that behave as $|\eta|^{-1}$ and $|\eta|^{-2}$ as $\eta \to \pm\infty$, as these lead to the discontinuities in $G_{\infty;\lambda}$ and its derivative.

Lemma 2.7.5. *Assume that (Hα1) and (Hα2) are satisfied. Consider any bounded function $f : \mathbb{R} \to \mathbb{R}$ which is continuous everywhere except at some $\xi_0 \in \mathbb{R}$. Then $\Delta_h f$ is continuous everywhere except at $\{\xi_0 + hk : k \in \mathbb{Z}\}$. Moreover, if f is differentiable except at ξ_0 and f' is bounded, then $\Delta_h f$ is differentiable everywhere except at $\{\xi_0 + hk : k \in \mathbb{Z}\}$ and $[\Delta_h f]'(\xi) = [\Delta_h f'](\xi)$.*

Proof. For convenience we set $\xi_0 = 0$. Pick $\xi \in \mathbb{R}$ with $\xi \notin \{kh : k \in \mathbb{Z}\}$. Then f is continuous in each point $\xi + kh$ for $k \in \mathbb{Z}$. Choose $\varepsilon > 0$. Since f is bounded and $\sum_{j=1}^{\infty} |\alpha_j| < \infty$, we can pick $K > 0$ in such a way that

$$2\|f\|_\infty \frac{1}{h^2} \sum_{j=K}^{\infty} |\alpha_j| \;<\; \frac{\varepsilon}{2}. \tag{2.7.28}$$

For $j \in \{1, ..., K-1\}$ we can pick $\delta_j > 0$ in such a way that

$$\frac{1}{h^2} |\alpha_j| \Big| f(\xi + y + hj) - f(\xi + hj) \Big| \;<\; \frac{\varepsilon}{2^{K+1}} \tag{2.7.29}$$

for all $y \in \mathbb{R}$ with $|y| < \delta_j$. Let $\delta = \min\{\delta_j : 1 \le j < K\} > 0$. Then for $y \in \mathbb{R}$ with $|y| < \delta$ we obtain

$$
\begin{aligned}
|\Delta_h f(\xi + y) - \Delta_h f(\xi)| \;&\le\; \frac{1}{h^2} \sum_{j=K}^{\infty} |\alpha_j| \Big(|f(\xi + y + jh)| + |f(\xi + jh)| \Big) \\
&\quad + \frac{1}{h^2} \sum_{j=1}^{K-1} |\alpha_j| \Big| f(\xi + y + jh) - f(\xi + jh) \Big| \\
&\le\; \frac{2}{h^2} \sum_{j=K}^{\infty} |\alpha_j| \|f\|_\infty + \sum_{j=1}^{K-1} \frac{\varepsilon}{2^{K+1}} \\
&<\; \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&=\; \varepsilon.
\end{aligned}
\tag{2.7.30}
$$

So $\Delta_h f$ is continuous outside of $\{kh : k \in \mathbb{Z}\}$.

Writing

$$f_n(\xi) \;=\; \frac{1}{h^2} \sum_{j=1}^{n} \alpha_j \Big[f(\xi + hj) + f(\xi - hj) - 2f(\xi) \Big] \tag{2.7.31}$$

for $n \in \mathbb{Z}_{>0}$, we can compute

$$f_n'(\xi) \;=\; \frac{1}{h^2} \sum_{j=1}^{n} \alpha_j \Big[f'(\xi + hj) + f'(\xi - hj) - 2f'(\xi) \Big]. \tag{2.7.32}$$

This allows us to estimate

$$|f_n'(\xi) - (\Delta_h f')(\xi)| \;\le\; \frac{1}{h^2} \sum_{j=n+1}^{\infty} |\alpha_j| 4 \|f'\|_\infty. \tag{2.7.33}$$

In particular, the sequence $\{f_n'\}$ converges uniformly to $\Delta_h f'$ from which it follows that

$$
\begin{aligned}
(\Delta_h f)'(\xi) &= \tfrac{1}{h^2} \sum_{j=1}^\infty \alpha_j \Big[f'(\xi + hj) + f'(\xi - hj) - 2f'(\xi) \Big] \\
&= (\Delta_h f')(\xi).
\end{aligned}
\tag{2.7.34}
$$

∎

Proof of Lemma 2.7.2. Pick $\chi > 0$ and set

$$
R = \{ \lambda \in \mathbb{C} : -\tfrac{\tilde{\lambda}}{2} \le \operatorname{Re} \lambda \le \chi \text{ and } |\operatorname{Im} \lambda| \le \tfrac{\pi |c|}{h} \}.
\tag{2.7.35}
$$

The proof of Lemma 2.5.7 implies that we can choose $\beta_* > 0$ and $K_* > 0$ in such a way that

$$
\| \Delta_{L_\infty;\lambda}(z)^{-1} \| \le \tfrac{K_*}{1 + |\operatorname{Im} z|}
\tag{2.7.36}
$$

for all $\lambda \in R$ and all $z \in \mathbb{C}$ with $|\operatorname{Re} z| \le 2\beta_*$. In particular, it follows that $(y \mapsto \Delta_{L_\infty;\lambda}(iy)^{-1}) \in L^2(\mathbb{R})$. By the Plancherel Theorem it follows that $G_{\infty;\lambda}$ is a well-defined function in $L^2(\mathbb{R})$. In particular, it is bounded. Shifting the integration path in (2.7.19) in the standard fashion described in [103, 130], we obtain the bound

$$
|G_{\infty;\lambda}(\xi - \xi_0)| \le K_* e^{-\beta_* |\xi - \xi_0|}
\tag{2.7.37}
$$

for all $\xi, \xi_0 \in \mathbb{R}$ and $\lambda \in R$.

We loosely follow the approach of [102, §5.1], which considers a similar setting for Green's functions for Banach space-valued operators with finite range interactions. Pick $\lambda \in R$. We rewrite the definition of $\Delta_{L_\infty;\lambda}$ given in (2.5.18) in the more general form

$$
\tfrac{1}{c} \Delta_{L_\infty;\lambda}(z) = z - B_{\infty;\lambda} e^{z \cdot},
\tag{2.7.38}
$$

For $\alpha \in \mathbb{R}$ close to 0 we introduce the expression $\mathcal{R}_{L_\infty;\lambda;\alpha}$ by

$$
\mathcal{R}_{L_\infty;\lambda;\alpha}(z) = c\Delta_{L_\infty;\lambda}(z)^{-1} - \tfrac{1}{z - \alpha} - \tfrac{B_{\infty;\lambda} e^{z \cdot} - \alpha}{(z - \alpha)^2}
\tag{2.7.39}
$$

for $z \in \mathbb{C}$ unequal to α and $|\operatorname{Re} z| \le 2\beta_*$. Since we can compute

$$
\begin{aligned}
c\Delta_{L_\infty;\lambda}(z)^{-1} &= \Big[z - \alpha + \big(\alpha - B_{\infty;\lambda} e^{z \cdot} \big) \Big]^{-1} \\
&= (z - \alpha)^{-1} \Big[1 + (z - \alpha)^{-1} \big(\alpha - B_{\infty;\lambda} e^{z \cdot} \big) \Big]^{-1} \\
&= (z - \alpha)^{-1} \Big[1 - (z - \alpha)^{-1} \big(\alpha - B_{\infty;\lambda} e^{z \cdot} \big) + \mathcal{O}((z - \alpha)^{-2}) \Big],
\end{aligned}
\tag{2.7.40}
$$

we obtain the estimate

$$
|\mathcal{R}_{L_\infty;\lambda;\alpha}(iy)| \le \tfrac{K_*}{1 + |y|^3},
\tag{2.7.41}
$$

for all $y \in \mathbb{R}$, possibly after increasing K_*.

Exploiting the decomposition (2.7.39), we write

$$
G_{\infty;\lambda} = \tfrac{1}{c} \mathcal{M}_\alpha + \tfrac{1}{c} \mathcal{R}_\alpha,
\tag{2.7.42}
$$

where we have introduced

$$
\begin{array}{rcl}
\mathcal{M}_\alpha(\xi) & = & \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\eta\xi} \left(\frac{1}{i\eta-\alpha} - \frac{B_{\infty;\lambda} e^{i\eta} - \alpha}{(i\eta-\alpha)^2} \right) d\eta, \\
\mathcal{R}_\alpha(\xi) & = & \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\eta\xi} \mathcal{R}_{L_{\infty;\lambda};\alpha}(i\eta) \, d\eta
\end{array}
\tag{2.7.43}
$$

for any $\alpha \in \mathbb{R} \setminus \{0\}$ and $\xi \in \mathbb{R}$. Using [102, Lem. 5.8] we can explicitly compute

$$
\mathcal{M}_\alpha(\xi) = -e^{\alpha\xi} H(-\xi) - \Big[B_{\infty;\lambda} - \alpha \Big] \Big(\cdot e^{\alpha\cdot} H(-\cdot) \Big)(\xi),
\tag{2.7.44}
$$

where we have introduced the Heaviside function H as

$$
H(\xi) = \begin{cases} I, & \xi > 0 \\ \frac{1}{2} I, & \xi = 0 \\ 0, & \xi < 0. \end{cases}
\tag{2.7.45}
$$

Since $\xi \mapsto \xi e^{\alpha\xi} H(-\xi)$ is continuous everywhere and differentiable outside of $\xi = 0$, Lemma 2.7.5 implies that $\mathcal{M}_\alpha$ is continuous everywhere outside of $\xi = 0$ and differentiable outside of $\{hk : k \in \mathbb{Z}\}$. Moreover, we have the jump discontinuity

$$
\mathcal{M}_\alpha(0^+) - \mathcal{M}_\alpha(0^-) = I
\tag{2.7.46}
$$

and we can easily compute

$$
\mathcal{M}'_\alpha(\xi) = \alpha \mathcal{M}_\alpha(\xi) - [B_{\infty;\lambda} - \alpha] \Big[e^{\alpha\cdot} H(-\cdot) \Big](\xi),
\tag{2.7.47}
$$

from which it follows that

$$
\begin{array}{rcl}
\frac{1}{c} L_{\infty;\lambda} \mathcal{M}_\alpha(\xi) & = & \mathcal{M}'_\alpha(\xi) - B_{\infty;\lambda} \mathcal{M}_\alpha(\xi) \\
& = & -\alpha e^{\alpha\xi} H(-\xi) - \alpha \Big[B_{\infty;\lambda} - \alpha \Big] \Big(\cdot e^{\alpha\cdot} H(-\cdot) \Big)(\xi) \\
& & -[B_{\infty;\lambda} - \alpha] \Big[e^{\alpha\cdot} H(-\cdot) \Big](\xi) + B_{\infty;\lambda} \Big[e^{\alpha\cdot} H(-\cdot) \Big](\xi) \\
& & +B_{\infty;\lambda} \Big[[B_{\infty;\lambda} - \alpha] \Big(\cdot e^{\alpha\cdot} H(-\cdot) \Big)(*) \Big](\xi) \\
& = & [B_{\infty;\lambda} - \alpha] \Big[[B_{\infty;\lambda} - \alpha] \Big(\cdot e^{\alpha\cdot} H(-\cdot) \Big)(*) \Big](\xi).
\end{array}
\tag{2.7.48}
$$

Since $\mathcal{R}_{L_{\infty;\lambda};\alpha} \in L^1(\mathbb{R})$ we see that $\mathcal{R}_\alpha$ is continuous. Therefore, $G_{\infty;\lambda}$ is continuous outside of $\xi = 0$. Similarly to [102, Eq. (5.79)] we observe that

$$
\frac{1}{c} \Delta_{L_{\infty;\lambda}}(z) \mathcal{R}_{L_{\infty;\lambda};\alpha}(z) = \frac{(B_{\infty;\lambda} e^{z\cdot} - \alpha)^2}{(z-\alpha)^2},
\tag{2.7.49}
$$

which yields

$$
\begin{array}{rcl}
\frac{1}{c} L_{\infty;\lambda} \mathcal{R}_\alpha(\xi) & = & \mathcal{R}'_\alpha(\xi) - B_{\infty;\lambda} \mathcal{R}_\alpha(\xi) \\
& = & \frac{1}{2\pi c} \int_{-\infty}^{\infty} e^{i\xi y} \Delta_{L_{\infty;\lambda}}(iy) \mathcal{R}_{L_{\infty;\lambda};\alpha}(iy) dy \\
& = & \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi y} \frac{(B_{\infty;\lambda} e^{iy\cdot} - \alpha)^2}{(iy-\alpha)^2} dy \\
& = & -[B_{\infty;\lambda} - \alpha] \Big[[B_{\infty;\lambda} - \alpha] \Big(\cdot e^{\alpha\cdot} H(-\cdot) \Big)(*) \Big](\xi),
\end{array}
\tag{2.7.50}
$$

using [102, Lem. 5.8]. In particular, we see that

$$L_{\infty;\lambda}G_{\infty;\lambda}(\xi) \quad = \quad 0 \tag{2.7.51}$$

for all ξ outside of $\{hk : k \in \mathbb{Z}\}$. Lemma 2.7.5 subsequently shows that $G_{\infty;\lambda}$ is C^1-smooth outside of $\{hk : k \in \mathbb{Z}\}$.

Fix $f \in \mathbf{H}^1$. For any $\delta > 0$ we may compute

$$
\begin{aligned}
0 \quad &= \quad \int_\delta^\infty \Big[L_{\infty;\lambda}G_{\infty;\lambda}(\cdot) \Big](\xi)f(\xi)d\xi \\
&= \quad \Big[cG_{\infty;\lambda}f \Big]_\delta^\infty - \int_\delta^\infty cG_{\infty;\lambda}(\xi)f'(\xi) + [cB_{\infty;\lambda}G_{\infty;\lambda}](\xi)f(\xi),
\end{aligned}
\tag{2.7.52}
$$

together with

$$
0 \quad = \quad \Big[cG_{\infty;\lambda}f \Big]_{-\infty}^{-\delta} - \int_{-\infty}^{-\delta} cG_{\infty;\lambda}(\xi)f'(\xi) + [cB_{\infty;\lambda}G_{\infty;\lambda}](\xi)f(\xi).
\tag{2.7.53}
$$

Using (2.7.46) we can hence compute

$$
\begin{aligned}
\int_{-\infty}^\infty \Big[L_{\infty;\lambda}G_{\infty;\lambda}(\cdot) \Big](\xi)f(\xi)d\xi \quad &= \quad \lim_{\delta\downarrow 0} \Big[cG_{\infty;\lambda}f \Big]_\delta^\infty - \Big[cG_{\infty;\lambda}f \Big]_{-\infty}^{-\delta} \\
&= \quad f(0).
\end{aligned}
\tag{2.7.54}
$$

■

Lemma 2.7.6. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. Fix $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq -\frac{\tilde{\lambda}}{2}$. Then there exist constants $K > 0$ and $\beta > 0$ so that for any $g \in \mathbf{L}^2$ and $f \in \mathbf{H}^1$ that satisfy $(L + \lambda)f = g$, the pointwise bound*

$$
|f(\xi)| \quad \leq \quad Ke^{-\alpha|\xi|}\|f\|_\infty + K \int_{-\infty}^\infty e^{-|\eta-\xi|}g(\eta)d\eta
\tag{2.7.55}
$$

holds for all $\xi \in \mathbb{R}$.

Proof. On account of Lemma 2.7.2 we can lift the results from [130, Prop. 5.2-5.3] to our current infinite range setting. The proof of these results are identical, since the estimate [130, Eq. (5.4)] still holds in our setting on account of (Hα2). A more detailed description for this procedure can be found in [20, Lem. 4.1-Lem. 4.3]. ■

Proof of Lemma 2.7.3. Pick $\lambda \in \mathbb{C} \setminus \sigma(-L)$ and compute

$$
\begin{aligned}
(\lambda + L)G_\lambda(\cdot, \xi_0) \quad &= \quad (\lambda + L)G_{\infty;\lambda}(\cdot - \xi_0) - [L - L_\infty]G_{\infty;\lambda}(\cdot - \xi_0) \\
&= \quad (\lambda + L_\infty)G_{\infty;\lambda}(\cdot - \xi_0).
\end{aligned}
\tag{2.7.56}
$$

The last statement follows immediately from this identity.

Write

$$
\hat{G}_{\infty;\lambda}(\cdot - \xi_0) \quad = \quad [L - L_\infty]G_{\infty;\lambda}(\cdot - \xi_0).
\tag{2.7.57}
$$

We have already seen that $\hat{G}_{\infty;\lambda}(\cdot - \xi_0) \in L^2(\mathbb{R}, \mathrm{Mat}_2(\mathbb{C}))$. Hence, it follows that

$$(\lambda + L)^{-1}\hat{G}_{\infty;\lambda}(\cdot - \xi_0) \quad \in \quad H^1(\mathbb{R}, \mathrm{Mat}_2(\mathbb{C})). \tag{2.7.58}$$

In particular, this function is continuous. Together with Lemma 2.7.2 we obtain that $G_\lambda(\cdot, \xi_0)$ is continuous on $\mathbb{R} \setminus \{\xi_0\}$.

Set $H = (\lambda + L)^{-1}\hat{G}_{\infty;\lambda}$ and write $H = \begin{pmatrix} H^{(1,1)} & H^{(1,2)} \\ H^{(2,1)} & H^{(2,2)} \end{pmatrix}$. Using the definition of L we see that

$$c\tfrac{d}{d\xi}H \quad = \quad -\lambda H - \hat{G}_\infty - \tilde{H}, \tag{2.7.59}$$

where

$$\tilde{H} \quad = \quad -\begin{pmatrix} -\Delta_h H^{(1,1)} - g_u(\overline{u})H^{(1,1)} + H^{(2,1)} & \Delta_h H^{(1,2)} - g_u(\overline{u})H^{(1,2)} + H^{(2,2)} \\ -\rho H^{(1,1)} + \gamma\rho H^{(2,1)} & -\rho H^{(1,2)} + \gamma\rho H^{(2,2)} \end{pmatrix}. \tag{2.7.60}$$

Since $\overline{u}' \in H^1$ and, hence, $\overline{u}'$ is continuous, we must have that $\overline{u}$ is continuous. As argued before $\Delta_h H^{(1,1)}$ and $\Delta_h H^{(1,2)}$ are also continuous. Hence, we see that $c\tfrac{d}{d\xi}H$ is continuous on $\mathbb{R} \setminus \{\xi_0\}$ and thus that $\tfrac{d}{d\xi}H$ is continuous on $\mathbb{R} \setminus \{\xi_0\}$. Therefore, we obtain that $G_\lambda(\cdot, \xi_0)$ is C^1-smooth on $\mathbb{R} \setminus \{\xi_0 + kh : k \in \mathbb{Z}\}$. ∎

We now proceed to the verification of the integral representation (2.7.27). As a preparation, we need to show that whenever λ has a sufficiently large real part, the function G_λ is bounded uniformly by a constant. This result is based on [13, Lem. 4.1].

Lemma 2.7.7. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. Then there exist constants K and λ_{unif} so that the Green's function G_λ enjoys the uniform estimate*

$$|G_\lambda(\xi, \xi_0)| \quad \leq \quad K, \tag{2.7.61}$$

for all $\xi, \xi_0 \in \mathbb{R}$, whenever $\mathrm{Re}\,\lambda > \lambda_{\mathrm{unif}}$.

Proof. We write $L = c\tfrac{d}{d\xi} + B$ with

$$B \quad = \quad \begin{pmatrix} -\Delta_h - g_u(\overline{u}) & 1 \\ -\rho & \gamma\rho \end{pmatrix}. \tag{2.7.62}$$

We introduce G_λ^0 as the Green's function of $(\lambda + c\tfrac{d}{d\xi})$ viewed as a map from $\mathbf{H}^1$ to $\mathbf{L}^2$. Luckily, it is well-known that this Green's function admits the estimate

$$|G_\lambda^0(\xi, \xi_0)| \quad \leq \quad \tfrac{1}{|c|}e^{-\mathrm{Re}\,\lambda|\xi-\xi_0|/|c|}. \tag{2.7.63}$$

We can look for the Green's function G_λ as the solution of the fixed point problem

$$G_\lambda(\xi, \xi_0) \quad = \quad G_\lambda^0(\xi, \xi_0) + \int_\mathbb{R} G_\lambda(\xi, z)(BG_\lambda^0)(z, \xi_0)dz. \tag{2.7.64}$$

Since $\lambda + L$ is invertible by Theorem 2.2.2, G_λ must necessarily satisfy the fixed point problem (2.7.64).

For a matrix $A \in \mathrm{Mat}_2(\mathbb{C})$ we write $A = \begin{pmatrix} A^{(1,1)} & A^{(1,2)} \\ A^{(2,1)} & A^{(2,2)} \end{pmatrix}$. We make the decomposition

$$B \quad = \quad B_0 + B_1, \tag{2.7.65}$$

where

$$
\begin{aligned}
B_0 &= \begin{pmatrix} -\Delta_h & 0 \\ 0 & 0 \end{pmatrix}, \\
B_1 &= \begin{pmatrix} -g_u(\overline{u}) & 1 \\ -\rho & \gamma\rho \end{pmatrix}.
\end{aligned}
\tag{2.7.66}
$$

We estimate

$$
\begin{aligned}
|(B_0 G_\lambda^0)(\xi, \xi_0)| &= |\Delta_h G_\lambda^0(\xi, \xi_0)^{(1,1)}| \\
&\leq \sum_{j=1}^{\infty} \left[\frac{1}{h^2} |\alpha_j| \left(|G_\lambda^0(\xi + hj, \xi_0)^{(1,1)}| + |G_\lambda^0(\xi - hj, \xi_0)^{(1,1)}| \right. \right. \\
&\qquad\qquad \left. \left. + 2|G_\lambda^0(\xi, \xi_0)^{(1,1)}| \right) \right] \\
&\leq \frac{1}{|c|} \sum_{j=1}^{\infty} \left[\frac{1}{h^2} |\alpha_j| \left(e^{-\mathrm{Re}\,\lambda |\xi + hj - \xi_0|/|c|} + e^{-\mathrm{Re}\,\lambda |\xi - hj - \xi_0|/|c|} \right. \right. \\
&\qquad\qquad \left. \left. + 2 e^{-\mathrm{Re}\,\lambda |\xi - \xi_0|/|c|} \right) \right]
\end{aligned}
\tag{2.7.67}
$$

and observe that

$$
\begin{aligned}
\int_{\mathbb{R}} |(B_0 G_\lambda^0)(\xi, \xi_0)| \, d\xi &\leq \frac{1}{|c|} \left(\sum_{j=1}^{\infty} 4 \left[\frac{1}{h^2} |\alpha_j| \frac{1}{\mathrm{Re}\,\lambda/|c|} \right] \right) \\
&= \frac{4}{h^2 \mathrm{Re}\,\lambda} \sum_{j=1}^{\infty} |\alpha_j|.
\end{aligned}
\tag{2.7.68}
$$

We now fix $G \in L^\infty(\mathbb{R}^2, \mathrm{Mat}_2(\mathbb{C}))$ and consider the expressions

$$
\begin{aligned}
\mathcal{I}_0 &= \int_{\mathbb{R}} \left[G(\xi, z)(B_0 G_\lambda^0)(z, \xi_0) \right]^{(1,1)} dz, \\
\mathcal{I}_1 &= \int_{\mathbb{R}} \left[G(\xi, z)(B_1 G_\lambda^0)(z, \xi_0) \right]^{(1,1)} dz.
\end{aligned}
\tag{2.7.69}
$$

Using Fubini's theorem for positive functions to switch the integral and the sum, we obtain the estimates

$$
\begin{aligned}
|\mathcal{I}_0| &\leq \|G\|_{L^\infty} \int_{\mathbb{R}} |(B_0 G_\lambda^0)(z, \xi_0)| \, dz \\
&\leq \|G\|_{L^\infty} \frac{4}{h^2 \mathrm{Re}\,\lambda} \sum_{j=1}^{\infty} |\alpha_j|
\end{aligned}
\tag{2.7.70}
$$

and

$$
\begin{aligned}
|\mathcal{I}_1| &\leq \|G\|_{L^\infty} \int_{\mathbb{R}} \Big(|g_u(\overline{u}(z))||G_\lambda^0(z,\xi_0)^{(1,1)}| + \rho|G_\lambda^0(z,\xi_0)^{(1,1)}| \\
&\qquad + (1+\gamma\rho)|G_\lambda^0(z,\xi_0)^{(2,1)}| \Big) dz \\
&\leq \|G\|_{L^\infty} \tfrac{1}{|c|} \int_{\mathbb{R}} \Big(\big(|g_u(\overline{u}(z))| + \rho + 1 + \gamma\rho \big) e^{-\mathrm{Re}\,\lambda|z-\xi_0|/|c|} \Big) dz \qquad (2.7.71) \\
&\leq \|G\|_{L^\infty} \tfrac{1}{|c|} \Big(\|g_u(\overline{u})\|_{L^\infty} + \rho + 1 + \gamma\rho \Big) \left(\tfrac{1}{\mathrm{Re}\,\lambda/|c|} \right) \\
&\leq \|G\|_{L^\infty} \Big(g_* + \rho + 1 + \gamma\rho \Big) \left(\tfrac{1}{\mathrm{Re}\,\lambda} \right).
\end{aligned}
$$

Similar estimates hold for the other components of $\int_{\mathbb{R}} G(\xi,z)(BG_\lambda^0)(z,\xi_0)dz$. Therefore, the mapping $G \mapsto \int_{\mathbb{R}} G(\xi,z)(BG_\lambda^0)(z,\xi_0)dz$ is a contraction in $L^\infty(\mathbb{R}^2, \mathrm{Mat}_2(\mathbb{C}))$ for $\mathrm{Re}\,\lambda > \lambda_{\mathrm{unif}}$ for λ_{unif} large enough, with λ_{unif} possibly dependent of $h \in (0, h_{**})$. Hence, we get a unique bounded solution of (2.7.64), which must be G_λ. The desired bound on G_λ is now immediate. ∎

Proof of Proposition 2.7.4. Fix $j_0 \in \mathbb{Z}$ and $t_0 \in \mathbb{R}$. Since (2.7.8) is merely a linear ODE in the Banach space $\mathbb{L}^2$, it follows from the Cauchy-Lipschitz theorem that (2.7.8) indeed has a unique solution $\mathcal{V} : [t_0, \infty) \to \mathbb{L}^2$. For any $Z \in C_c^\infty(\mathbb{R}; \mathbb{L}^2)$, an integration by parts yields

$$
\begin{aligned}
-Z_{j_0}(t_0) &= \int_{t_0}^\infty \sum_{j \in \mathbb{Z}} \left[\left(\tfrac{d\mathcal{V}_j}{dt}(t) - (\mathcal{A}(t) \cdot \mathcal{V}(t))_j \right) Z_j(t) \right] dt - \sum_{j \in \mathbb{Z}} \mathcal{V}_j(t_0) Z_j(t_0) \\
&= \int_{t_0}^\infty \sum_{j \in \mathbb{Z}} \left[-\tfrac{dZ_j}{dt}(t) \mathcal{V}_j(t) - (\mathcal{A}(t) \cdot \mathcal{V})_j(t) Z_j(t) \right] dt.
\end{aligned}
$$
$$(2.7.72)$$

We want to show that the function $V_j(t) := \mathcal{G}_j^{j_0}(t, t_0)$ defined by (2.7.27) coincides with $\mathcal{V}$ on $[t_0, \infty)$. To accomplish this, we define

$$
I = \int_{t_0}^\infty \sum_{j \in \mathbb{Z}} \left[-\tfrac{dZ_j}{dt}(t) V_j(t) - (\mathcal{A}(t) \cdot V(t))_j Z_j(t) \right] dt \qquad (2.7.73)
$$

and show that V is a weak solution to (2.7.8) in the sense that

$$
I = -Z_{j_0}(t_0) \qquad (2.7.74)
$$

holds for all $Z \in C_c^\infty(\mathbb{R}; \mathbb{L}^2)$. Indeed, the uniqueness of weak solutions then implies that $V = \mathcal{V}$.

Note first that $V(t) = 0$ for $t < t_0$, which can be seen by using (2.7.61) and taking $\chi \to \infty$ in (2.7.27). We write $y = hj_0 + ct_0$, $\chi_- = \chi - \tfrac{i\pi c}{h}$ and $\chi_+ = \chi + \tfrac{i\pi c}{h}$. We see that

$$
I = \int_{-\infty}^\infty \sum_{j \in \mathbb{Z}} \left[-\tfrac{dZ_j}{dt}(t) V_j(t) - (\mathcal{A}(t) \cdot V(t))_j Z_j(t) \right] dt, \qquad (2.7.75)
$$

since $V(t) = 0$ for $t < t_0$. Moreover, we write

$$
\mathbb{G}_j(t) = G_\lambda(hj + ct, y). \qquad (2.7.76)
$$

Using our definition of $V(t)$, we have

$$I \quad = \quad -\tfrac{h}{2\pi i} \int_{\chi_-}^{\chi_+} \sum_{j \in \mathbb{Z}} \left[\int_{-\infty}^{\infty} \mathcal{I}_j(t, \lambda) dt \right] d\lambda, \tag{2.7.77}$$

where

$$\mathcal{I}_j(t, \lambda) \quad = \quad e^{\lambda(t-t_0)} \left[-\mathbb{G}_j(t) \tfrac{dZ_j}{dt}(t) - (\mathcal{A}(t) \cdot \mathbb{G}(t))_j Z_j(t) \right]. \tag{2.7.78}$$

The permutation of the summations and integrations is allowed by Lebesgue's theorem, because Z and $\frac{dZ}{dt}$ are compactly supported and G_λ is uniformly bounded by (2.7.61). Fix $\chi_- \le \lambda \le \chi_+$ and $j \in \mathbb{Z}$. Using the change of variable $x = hj + ct$ we obtain

$$\int_{-\infty}^{\infty} \mathcal{I}_j(t, \lambda) dt \quad = \quad \tfrac{1}{c} \int_{x=-\infty}^{x=\infty} \left[-c\mathbb{G}_j \left(\tfrac{x-hj}{c} \right) \tfrac{d\mathcal{Z}_j}{dx} + \lambda \mathbb{G}_j \left(\tfrac{x-hj}{c} \right) \mathcal{Z}_j(x, \lambda) \right. \\ \left. - \left(\mathcal{A} \left(\tfrac{x-hj}{c} \right) \cdot \mathbb{G} \left(\tfrac{x-hj}{c} \right) \right)_j \mathcal{Z}_j(x, \lambda) \right] dx, \tag{2.7.79}$$

where

$$\mathcal{Z}_j(x, \lambda) \quad = \quad e^{\lambda((x-hj)/c-t_0)} Z_j \left(\tfrac{x-hj}{c} \right). \tag{2.7.80}$$

Exploiting the fact that Z_j and, therefore, $\mathcal{Z}_j$ is compactly supported, (2.7.26) yields

$$\int_{-\infty}^{\infty} \mathcal{I}_j(t, \lambda) dt \quad = \quad \tfrac{1}{c} \int_{-\infty}^{\infty} [(L + \lambda) G_\lambda(x, y) \mathcal{Z}_j(x, \lambda)] dx \\ = \quad \tfrac{1}{c} \mathcal{Z}_j(y). \tag{2.7.81}$$

Now since Z_j is compactly supported, we can exchange sums and integrals in equation (2.7.77). This allows us to compute

$$\begin{aligned} I \quad &= \quad -\tfrac{h}{2\pi i} \tfrac{1}{c} \int_{\chi_-}^{\chi_+} \sum_{j \in \mathbb{Z}} \int_{-\infty}^{\infty} \mathcal{I}_j(t, \lambda) dt d\lambda \\ &= \quad -\tfrac{h}{2\pi i} \tfrac{1}{c} \int_{\chi_-}^{\chi_+} \sum_{j \in \mathbb{Z}} \mathcal{Z}_j(y, \lambda) d\lambda \\ &= \quad -\tfrac{h}{2\pi i c} \int_{\chi_-}^{\chi_+} \sum_{j \in \mathbb{Z}} e^{\lambda \frac{(hj_0 - hj)}{c}} Z_j \left(\tfrac{(hj_0 - hj)}{c} + t_0 \right) d\lambda \\ &= \quad -\tfrac{h}{2\pi i c} \sum_{j \in \mathbb{Z}} \int_{\chi_-}^{\chi_+} e^{\lambda \frac{(hj_0 - hj)}{c}} Z_j \left(\tfrac{(hj_0 - hj)}{c} + t_0 \right) d\lambda \\ &= \quad -\tfrac{h}{2\pi i c} \sum_{j \in \mathbb{Z}} 2 \tfrac{\pi i c}{h} \delta_j^{j_0} Z_j \left(\tfrac{(hj_0 - hj)}{c} + t_0 \right) \\ &= \quad -Z_{j_0}(t_0), \end{aligned} \tag{2.7.82}$$

as desired.   ∎

2.7.2   Meromorphic expansion of G_λ

In this subsection we set out to explicitly isolate the pole at $\lambda = 0$ in the meromorphic expansion of G_λ. In addition, we show that both parts of this decomposition decay

exponentially in a λ-uniform fashion. This will allow us to shift the integration path in (2.7.27) to the left of the imaginary axis. The decomposition (2.7.13) for the Green's function $\mathcal{G}$ together with the exponential decay estimates (2.7.15) can subsequently be read off from the shifted contour integral.

Lemma 2.7.8. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. There exist constants $K_1 > 0, K_2 > 0, \delta > 0$ and $\tilde{\delta} > 0$ such that*

$$
\begin{aligned}
|\Phi^+(\xi)| &\leq K_1 e^{-\delta|\xi|}\|\Phi^+\|_\infty, \\
|\Phi^-(\xi)| &\leq K_2 e^{-\tilde{\delta}|\xi|}\|\Phi^-\|_\infty
\end{aligned}
\tag{2.7.83}
$$

for all $\xi \in \mathbb{R}$.

Proof. We obtain from Lemma 2.7.6 that there are constants $\delta > 0$ and $K_1 > 0$ for which

$$
|\Psi(\xi)| \leq K_1 e^{-\delta|\xi|}\|\Psi\|_\infty + K_1 \int_{-\infty}^\infty e^{-\delta|\xi-\eta|}|\Theta(\eta)|d\eta
\tag{2.7.84}
$$

holds for each $\Psi \in \mathbf{H}^1$, where $\Theta = L\Psi$. Since $L\Phi^+ = 0$ we conclude that

$$
|\Phi^+(\xi)| \leq K_1 e^{-\delta|\xi|}\|\Phi^+\|_\infty
\tag{2.7.85}
$$

for all ξ. Note that the operator L^* is also asymptotically hyperbolic. Hence, there are $\tilde{\delta} > 0$ and $K_2 > 0$ for which

$$
|\Psi(\xi)| \leq K_2 e^{-\tilde{\delta}|\xi|}\|\Psi\|_\infty + K_2 \int_{-\infty}^\infty e^{-\tilde{\delta}|\xi-\eta|}|\Theta(\eta)|d\eta
\tag{2.7.86}
$$

holds for each $\Psi \in \mathbf{H}^1$, where $\Theta = L^*\Psi$. Since $L^*\Phi^- = 0$ we obtain that

$$
|\Phi^-(\xi)| \leq K_2 e^{-\tilde{\delta}|\xi|}\|\Phi^-\|_\infty
\tag{2.7.87}
$$

for all ξ. ∎

Lemma 2.7.9. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. Then there exist constants $K_3 > 0$ and $\delta > 0$ such that*

$$
|(\Phi^\pm)'(\xi)| \leq K_3 e^{-\delta|\xi|}
\tag{2.7.88}
$$

for all $\xi \in \mathbb{R}$.

Proof. Lemma 2.7.8 implies that

$$
\begin{aligned}
|\Delta_h \phi^+(\xi)| &\leq \tfrac{1}{h^2} K_1 \sum_{k>0} |\alpha_k|(e^{-\delta|\xi+hk|} + e^{-\delta|\xi-hk|} + 2e^{-\delta|\xi|}) \\
&\leq K_1 e^{-\delta|\xi|}(\tfrac{1}{h^2} \sum_{k>0} |\alpha_k|(2e^{\delta hk} + 2)),
\end{aligned}
\tag{2.7.89}
$$

where the last sum converges by (Hα2), possibly after decreasing $\delta > 0$. Using the fact that

$$
(\Phi^+)' = \tfrac{1}{c} \begin{pmatrix} \Delta_h \phi^+ + g_u(\overline{u})\phi^+ - \psi^+ \\ \rho\phi^+ - \rho\gamma\psi^+ \end{pmatrix}
\tag{2.7.90}
$$

we hence see that there exists a constant $K_3 > 0$ such that

$$|(\Phi^+)'(\xi)| \;\; \leq \;\; K_3 e^{-\delta|\xi|}. \tag{2.7.91}$$

The proof for the bound on $(\Phi^-)'$ is identical. ∎

We recall the spaces

$$\begin{aligned}
X &:= X_h = \{\Theta \in \mathbf{H}^1 : \langle \Phi^-, \Theta \rangle = 0\} \\
Y &:= Y_h = \{\Theta \in \mathbf{L}^2 : \langle \Phi^-, \Theta \rangle = 0\},
\end{aligned} \tag{2.7.92}$$

together with the operators L^{-1} in the spaces $\mathcal{B}(X, X)$ and in $\mathcal{B}(Y, X)$ that were defined in Proposition 2.5.2. We also recall the notation $L^{\mathrm{qinv}}\Theta$ that was introduced in Corollary 2.5.4 for the unique solution Ψ of the equation

$$L\Psi \;\; = \;\; \Theta - \frac{\langle \Phi^-, \Theta \rangle}{\langle \Phi^-, \Phi^+ \rangle} \Phi^+ \tag{2.7.93}$$

in the space X, which is given explicitly by

$$L^{\mathrm{qinv}}\Theta \;\; = \;\; L^{-1}\Big[\Theta - \frac{\langle \Phi^-, \Theta \rangle}{\langle \Phi^-, \Phi^+ \rangle} \Phi^+\Big]. \tag{2.7.94}$$

We now exploit these operators to decompose the Green's function of $\lambda + L$ into a meromorphic and an analytic part. This result is based on [109, Lem. 2.7].

Lemma 2.7.10. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. There exists a constant $0 < \overline{\lambda} \leq \lambda_0$ such that for all $0 < |\lambda| < \overline{\lambda}$ we have the representation*

$$G_\lambda(\xi, \xi_0) \;\; = \;\; E_\lambda(\xi, \xi_0) + \tilde{G}_\lambda(\xi, \xi_0) \tag{2.7.95}$$

Here the meromorphic (in λ) term can be written as

$$E_\lambda(\xi, \xi_0) \;\; = \;\; -\frac{1}{\lambda\Omega} \begin{pmatrix} \phi^-(\xi_0)\phi^+(\xi) & \psi^-(\xi_0)\phi^+(\xi) \\ \phi^-(\xi_0)\psi^+(\xi) & \psi^-(\xi_0)\psi^+(\xi) \end{pmatrix} \tag{2.7.96}$$

and the analytic (in λ) term $\tilde{G}_\lambda$ is given by

$$\begin{aligned}
\tilde{G}_\lambda(\xi, \xi_0) \;\; = \;\; & G_{\infty;\lambda}(\xi - \xi_0) - \Big[[I + \lambda L^{-1}]^{-1} L^{\mathrm{qinv}}(L - L_\infty)G_{\infty;\lambda}(\cdot - \xi_0)\Big](\xi) \\
& -\frac{1}{\Omega}\langle \Phi^-, G_{\infty;\lambda}(\cdot - \xi_0)\rangle \Phi^+(\xi).
\end{aligned} \tag{2.7.97}$$

Here we recall the notation

$$\Omega \;\; = \;\; \langle \Phi^-, \Phi^+ \rangle. \tag{2.7.98}$$

Proof. Pick $\lambda \in \mathbb{C}$ with $0 < |\lambda| < \lambda_0$. By the proof of Proposition 2.6.2 we see that

$$(L + \lambda)^{-1}\Theta \;\; = \;\; \lambda^{-1}\frac{\langle \Phi^-, \Theta \rangle}{\Omega}\Phi^+ + L^{\mathrm{qinv}}\Theta - [I + \lambda L^{-1}]^{-1}\lambda L^{-1}L^{\mathrm{qinv}}\Theta \tag{2.7.99}$$

for $\Theta \in \mathbf{L}^2$. We now compute

$$\begin{aligned}
\langle \Phi^-, (L - L_\infty)G_{\infty;\lambda}(\cdot - \xi_0)\rangle \;\; &= \;\; \langle \Phi^-, -L_\infty G_{\infty;\lambda}(\cdot - \xi_0)\rangle \\
&= \;\; -\Phi^-(\xi_0) + \lambda\langle \Phi^-, G_{\infty;\lambda}(\cdot - \xi_0)\rangle.
\end{aligned} \tag{2.7.100}$$

In particular, writing

$$\hat{L} = L - L_\infty, \tag{2.7.101}$$

we obtain

$$
\begin{aligned}
(L + \lambda)^{-1}\hat{L}G_{\infty;\lambda}(\cdot - \xi_0) &= \tfrac{1}{\lambda\Omega} \begin{pmatrix} \phi^-(\xi_0)\phi^+ & \psi^-(\xi_0)\phi^+ \\ \phi^-(\xi_0)\psi^+ & \psi^-(\xi_0)\psi^+ \end{pmatrix} \\
&\quad + \tfrac{\langle \Phi^-, G_{\infty;\lambda}(\cdot-\xi_0)\rangle}{\Omega}\Phi^+ + L^{\mathrm{qinv}}\hat{L}G_{\infty;\lambda}(\cdot - \xi_0) \\
&\quad - [I + \lambda L^{-1}]^{-1}\lambda L^{-1}L^{\mathrm{qinv}}\hat{L}G_{\infty;\lambda}(\cdot - \xi_0).
\end{aligned}
\tag{2.7.102}
$$

We may hence write

$$G_\lambda(\xi, \xi_0) = E_\lambda(\xi, \xi_0) + \tilde{G}_\lambda(\xi, \xi_0) \tag{2.7.103}$$

with

$$E_\lambda(\xi, \xi_0) = -\tfrac{1}{\lambda\Omega} \begin{pmatrix} \phi^-(\xi_0)\phi^+(\xi) & \psi^-(\xi_0)\phi^+(\xi) \\ \phi^-(\xi_0)\psi^+(\xi) & \psi^-(\xi_0)\psi^+(\xi) \end{pmatrix} \tag{2.7.104}$$

and

$$
\begin{aligned}
\tilde{G}_\lambda(\cdot, \xi_0) &= G_{\infty;\lambda}(\cdot - \xi_0) - L^{\mathrm{qinv}}\hat{L}G_{\infty;\lambda}(\cdot - \xi_0) \\
&\quad + [I + \lambda L^{-1}]^{-1}\lambda L^{-1}L^{\mathrm{qinv}}\hat{L}G_{\infty;\lambda}(\cdot - \xi_0) \\
&\quad - \tfrac{1}{\Omega}\langle \Phi^-, G_{\infty;\lambda}(\cdot - \xi_0)\rangle\Phi^+ \\
&= G_{\infty;\lambda}(\cdot - \xi_0) - [I + \lambda L^{-1}]^{-1}L^{\mathrm{qinv}}\hat{L}G_{\infty;\lambda}(\cdot - \xi_0) \\
&\quad - \tfrac{1}{\Omega}\langle \Phi^-, G_{\infty;\lambda}(\cdot - \xi_0)\rangle\Phi^+.
\end{aligned}
\tag{2.7.105}
$$

Clearly E_λ is meromorphic in λ, while $\tilde{G}_\lambda$ is analytic in λ in the region $|\lambda| < \lambda_0$. ∎

We fix $\chi > \lambda_{\mathrm{unif}}$, where λ_{unif} was defined in Lemma 2.7.3, and set

$$R = \{\lambda \in \mathbb{C} : -\tfrac{\tilde{\lambda}}{2} \leq \operatorname{Re}\lambda \leq \chi \text{ and } |\operatorname{Im}\lambda| \leq \tfrac{\pi|c|}{h}\}. \tag{2.7.106}$$

We now set out to obtain an estimate on the function $\tilde{G}_\lambda$ from Lemma 2.7.10 by exploiting the asymptotic hyperbolicity of L. We treat each of the terms in (2.7.97) separately in the results below.

Lemma 2.7.11. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. There exist constants $K_4 > 0$ and $\tilde{\chi} > 0$ such that for all $\lambda \in R$*

$$\left| \langle \Phi^-, (L - L_\infty)G_{\infty;\lambda}(\cdot - \xi_0)\rangle \right| \leq K_4 e^{-\tilde{\chi}|\xi_0|}. \tag{2.7.107}$$

Proof. We reuse the notation $\hat{L} = L - L_\infty$ from the previous proof. Lemma 2.7.2 implies that we can pick constants $\beta_* > 0$ and $K_* > 0$ in such a way that

$$|G_{\infty;\lambda}(\xi - \xi_0)| \leq K_* e^{-\beta_*|\xi - \xi_0|} \tag{2.7.108}$$

for all values of ξ, ξ_0. Recall the constants $K_2, \tilde{\delta}$ from Lemma 2.7.8 and set $K_3 = K_2 \|\Phi^-\|_\infty$. Then we obtain

$$
\begin{aligned}
|\langle \Phi^-, \hat{L} G_{\infty;\lambda}(\cdot - \xi_0) \rangle| &\leq \int_{-\infty}^\infty K_3 e^{-\tilde{\delta}|\xi|} g_* K_* e^{-\beta_*|\xi - \xi_0|} \, d\xi \\
&= K_3 g_* K_* \left(\tfrac{1}{\tilde{\delta} + \beta_*} (e^{-\tilde{\delta}|\xi_0|} + e^{-\beta_*|\xi_0|}) + \tfrac{1}{\beta_* - \tilde{\delta}} (e^{-\tilde{\delta}|\xi_0|} - e^{-\beta_*|\xi_0|}) \right) \\
&\leq K_3 g_* K_* \left(\tfrac{1}{\tilde{\delta} + \beta_*} 2 e^{-\min\{\tilde{\delta}, \beta_*\}|\xi_0|} + \tfrac{1}{|\beta_* - \tilde{\delta}|} 2 e^{-\min\{\tilde{\delta}, \beta_*\}|\xi_0|} \right) \\
&= K_4 e^{-\tilde{\chi}|\xi_0|}
\end{aligned}
$$

(2.7.109)

for some $K_4 > 0$ and $\tilde{\chi} > 0$. ∎

Lemma 2.7.12. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. There exist constants $K_{10} > 0$ and $\tilde{\gamma} > 0$ such that for all $\lambda \in R$*

$$
\begin{aligned}
\left| \left[L^{\mathrm{qinv}}(L - L_\infty) G_{\infty;\lambda}(\cdot - \xi_0) \right](\xi) \right| &\leq K_{10} e^{-\tilde{\gamma}|\xi|} e^{-\tilde{\gamma}|\xi_0|} \\
&\leq K_{10} e^{-\tilde{\gamma}|\xi - \xi_0|}.
\end{aligned}
$$

(2.7.110)

Proof. We reuse the notation $\hat{L} = L - L_\infty$ from the previous proof. Recall the constants K_1, δ from Lemma 2.7.8. Writing

$$
H_{\xi_0}(\xi) = \left[L^{\mathrm{qinv}} \hat{L} G_{\infty;\lambda}(\cdot - \xi_0) \right](\xi), \tag{2.7.111}
$$

we may use Lemma 2.7.6 to estimate

$$
|H_{\xi_0}(\xi)| \leq K_1 e^{-\delta|\xi|} \|H_{\xi_0}\|_\infty + K_1 \int_{-\infty}^\infty e^{-\delta|\xi - \eta|} |L H_{\xi_0}(\eta)| d\eta. \tag{2.7.112}
$$

Recalling (2.7.92)-(2.7.94), we obtain

$$
\begin{aligned}
\|H_{\xi_0}\|_\infty &\leq \|H_{\xi_0}\|_{\mathbf{H}^1} \\
&\leq C_{\mathrm{unif}} \| \hat{L} G_{\infty;\lambda}(\cdot - \xi_0) - \tfrac{\langle \Phi^-, \hat{L} G_{\infty;\lambda}(\cdot - \xi_0) \rangle}{\Omega} \Phi^+ \|_{\mathbf{L}^2} \\
&\leq C_{\mathrm{unif}} \left(1 + \tfrac{\|\Phi^-\|_{\mathbf{L}^2}}{\Omega} \|\Phi^+\|_{\mathbf{L}^2} \right) \| \hat{L} G_{\infty;\lambda}(\cdot - \xi_0) \|_{\mathbf{L}^2} \\
&\leq K_5 \| \hat{L} G_{\infty;\lambda}(\cdot - \xi_0) \|_{\mathbf{L}^2}
\end{aligned}
$$

(2.7.113)

for some constant $K_5 > 0$.

Using Lemma 2.7.8 we see that there exists a constant $K_6 > 0$ for which

$$
\begin{aligned}
|\overline{u}(\xi)| &= |\int_\infty^\xi \overline{u}'(\xi') \, d\xi'| \\
&\leq \int_\xi^\infty K_1 \|(\overline{u}', \overline{w}')\|_\infty e^{-\delta|\xi'|} \, d\xi' \\
&= K_6 e^{-\delta|\xi|}
\end{aligned}
$$

(2.7.114)

holds for all $\xi \in \mathbb{R}$. Recall that

$$
\hat{L} = \begin{pmatrix} -g_u(\overline{u}) + r_0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.7.115}
$$

Observe that $-g_u(0) + r_0 = 0$. Then we obtain that

$$| -g_u(\overline{u}(\xi)) + r_0| \leq K_7 e^{-\delta|\xi|} \tag{2.7.116}$$

for all $\xi \in \mathbb{R}$ and for some constant $K_7 > 0$. Lemma 2.7.2 implies that

$$|G_{\infty;\lambda}(\xi - \xi_0)| \leq K_* e^{-\beta_*|\xi - \xi_0|} \tag{2.7.117}$$

for all $\xi \in \mathbb{R}$. Therefore, we must have

$$
\begin{aligned}
\|\hat{L}G_{\infty;\lambda}(\cdot - \xi_0)\|_{\mathbf{L}^2}^2 &\leq \int_{\mathbb{R}} K_7^2 K_*^2 e^{-2\delta|\xi|} e^{-2\beta_*|\xi - \xi_0|} \, d\xi \\
&\leq K_8 e^{-2\tilde{\gamma}|\xi_0|}
\end{aligned}
\tag{2.7.118}
$$

for some constants $K_8 > 0$, $\tilde{\gamma} > 0$ with $\tilde{\gamma} \leq \beta_*$, $\tilde{\gamma} \leq \frac{1}{2}\delta$ and $\tilde{\gamma} \leq \frac{1}{2}\tilde{\chi}$. In particular, we obtain the estimate

$$\|H_{\xi_0}\|_\infty \leq K_5 \sqrt{K_8} e^{-\tilde{\gamma}|\xi_0|}. \tag{2.7.119}$$

In a similar fashion, using Lemma 2.7.11, we see that

$$
\begin{aligned}
|LH_{\xi_0}(\xi)| &\leq \left| \left[\hat{L}G_{\infty;\lambda}(\cdot - \xi_0)\right](\xi) - \frac{\langle \Phi^-, \hat{L}G_{\infty;\lambda}(\cdot - \xi_0)\rangle}{\Omega} \Phi^+(\xi) \right| \\
&\leq K_7 K_* e^{-\delta|\xi|} e^{-\beta_*|\xi - \xi_0|} + \frac{1}{\Omega} K_4 e^{-\tilde{\chi}|\xi_0|} K_1 e^{-\delta|\xi|} \\
&\leq K_9 \left[e^{-2\tilde{\gamma}|\xi|} e^{-\tilde{\gamma}|\xi - \xi_0|} + e^{-\tilde{\gamma}|\xi_0|} e^{-\tilde{\gamma}|\xi|} \right]
\end{aligned}
\tag{2.7.120}
$$

for all $\xi \in \mathbb{R}$ and some constant $K_9 > 0$. Combining (2.7.112) with (2.7.113) and (2.7.118), we hence obtain

$$
\begin{aligned}
|H_{\xi_0}(\xi)| &\leq K_1 e^{-\delta|\xi|} \|H_{\xi_0}\|_\infty + K_1 \int_{-\infty}^\infty e^{-\delta|\xi - \eta|} |LH_{\xi_0}(\eta)| d\eta \\
&\leq K_1 e^{-\delta|\xi|} K_5 \sqrt{K_8} e^{-\tilde{\gamma}|\xi_0|} \\
&\quad + K_1 \int_{-\infty}^\infty e^{-\delta|\xi - \eta|} K_9 \left[e^{-2\tilde{\gamma}|\eta|} e^{-\tilde{\gamma}|\eta - \xi_0|} + e^{-\tilde{\gamma}|\xi_0|} e^{-\tilde{\gamma}|\eta|} \right] d\eta \\
&\leq K_1 e^{-\delta|\xi|} K_5 \sqrt{K_8} e^{-\tilde{\gamma}|\xi_0|} + K_1 \int_{-\infty}^\infty e^{-\delta|\xi - \eta|} 2K_9 e^{-\tilde{\gamma}|\eta|} e^{-\tilde{\gamma}|\xi_0|} d\eta \\
&\leq K_{10} e^{-\tilde{\gamma}|\xi|} e^{-\tilde{\gamma}|\xi_0|} \\
&\leq K_{10} e^{-\tilde{\gamma}|\xi - \xi_0|}
\end{aligned}
\tag{2.7.121}
$$

for some constant $K_{10} > 0$. ∎

Remark 2.7.13. In the proof of Lemma 2.7.12, in particular in (2.7.116), we explicitly used that $\overline{U}$ is a pulse solution, instead of a traveling front solution. If one would want to transfer these results to a more general system where the waves have different limits at $\xi = \pm\infty$, then Lemma 2.7.12 would only hold for $\xi_0 \geq 0$. However, the definition (2.7.25) remains valid upon using the reference system at $\xi = -\infty$ instead of $\xi = +\infty$. This new formulation allows the desired estimates for $\xi_0 \leq 0$ to be recovered.

Lemma 2.7.14. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. There exist constants $K_{13} > 0$ and $\omega > 0$ such that the function $\tilde{G}_\lambda$ from Lemma 2.7.10 satisfies the bound*

$$|\tilde{G}_\lambda(\xi, \xi_0)| \quad \leq \quad K_{13} e^{-\omega|\xi - \xi_0|} \tag{2.7.122}$$

for all ξ, ξ_0 and all $0 < |\lambda| < \overline{\lambda}$.

Proof. As before, we write

$$H_{\xi_0}(\xi) \quad = \quad L^{\mathrm{qinv}} \hat{L} G_{\infty;\lambda}(\cdot - \xi_0)(\xi). \tag{2.7.123}$$

Using Lemma 2.7.6 Lemma 2.7.12 and (2.7.119) and recalling (2.7.92)-(2.7.94), we obtain the estimate

$$
\begin{aligned}
|L^{-1} H_{\xi_0}(\xi)| \quad &\leq \quad K_1 e^{-\alpha|\xi|} \|L^{-1} H_{\xi_0}\|_\infty + K_1 \int_{-\infty}^{\infty} e^{-\alpha|\xi - \eta|} |H_{\xi_0}(\eta)| d\eta \\
&\leq \quad K_1 e^{-\alpha|\xi|} C_{\mathrm{unif}} \|H_{\xi_0}\|_{\mathbf{L}^2} + K_1 \int_{-\infty}^{\infty} e^{-\alpha|\xi - \eta|} K_{10} e^{-\tilde{\gamma}|\eta - \xi_0|} d\eta \\
&\leq \quad K_1 e^{-\alpha|\xi|} C_{\mathrm{unif}} K_5 \sqrt{K_8} e^{-\tilde{\gamma}|\xi_0|} + K_1 \int_{-\infty}^{\infty} e^{-\alpha|\xi - \eta|} K_{10} e^{-\tilde{\gamma}|\eta - \xi_0|} d\eta \\
&\leq \quad K_{10} K_{11} e^{-\tilde{\gamma}|\xi - \xi_0|}
\end{aligned}
\tag{2.7.124}
$$

for some constants $K_{11} > 0$ and $2\tilde{\gamma} \leq \alpha$. Using Proposition 2.5.2 and (2.7.119) we obtain that

$$\|(L^{-1})^n H_{\xi_0}\|_{\mathbf{H}^1} \quad \leq \quad K_5 \sqrt{K_8} (C_{\mathrm{unif}})^n e^{-\tilde{\gamma}|\xi_0|} \tag{2.7.125}$$

for all $n \in \mathbb{Z}_{>0}$. Continuing in this fashion, we see that

$$|(L^{-1})^n H_{\xi_0}(\xi)| \quad \leq \quad K_{10} K_{11}^n e^{-\tilde{\gamma}|\xi - \xi_0|} \tag{2.7.126}$$

for all $n \in \mathbb{Z}_{>0}$. If we set

$$\overline{\lambda} \quad = \quad \min\{\tfrac{\tilde{\lambda}}{2}, \lambda_0, \chi, \tfrac{1}{C_{\mathrm{unif}} K_5 \sqrt{K_8}}, \tfrac{1}{K_{11}}\}, \tag{2.7.127}$$

then for each $n \in \mathbb{Z}_{>0}$ and each $0 < |\lambda| < \overline{\lambda}$ we have

$$\|(-\lambda)^n (L^{-1})^n H_{\xi_0}\|_{\mathbf{H}^1} \quad \leq \quad \tfrac{1}{2}. \tag{2.7.128}$$

In particular, it follows that

$$\sum_{n=0}^{N} (-\lambda)^n (L^{-1})^n H_{\xi_0} \quad \to \quad [I + \lambda L^{-1}]^{-1} H_{\xi_0} \tag{2.7.129}$$

in $\mathbf{H}^1$ as $N \to \infty$. Since $\mathbf{H}^1$-convergence implies point-wise convergence, we conclude that

$$
\begin{aligned}
\left| [I + \lambda L^{-1}]^{-1} H_{\xi_0}(\xi) \right| \quad &= \quad \left| \sum_{n=0}^{\infty} (-\lambda)^n (L^{-1})^n H_{\xi_0}(\xi) \right| \\
&\leq \quad \sum_{n=0}^{\infty} \overline{\lambda}^n K_{11} K_{12}^n e^{-\tilde{\gamma}|\xi - \xi_0|} \\
&\leq \quad \tfrac{K_{11}}{1 - \overline{\lambda} K_{11}} e^{-\tilde{\gamma}|\xi - \xi_0|} \\
&:= \quad K_{12} e^{-\tilde{\gamma}|\xi - \xi_0|}
\end{aligned}
\tag{2.7.130}
$$

for all $\xi \in \mathbb{R}$ and for some constant $K_{12} > 0$.

Combining this estimate with Lemma 2.7.8 and Lemma 2.7.12 yields the desired bound

$$
\begin{aligned}
|\tilde{G}_\lambda(\xi, \xi_0)| &= \Big| G_{\infty;\lambda}(\xi - \xi_0) - \Big[[I + \lambda L^{-1}]^{-1} L^{\mathrm{qinv}} \hat{L} G_{\infty;\lambda}(\cdot - \xi_0) \Big](\xi) \\
&\qquad - \tfrac{1}{\Omega} \langle \Phi^-, G_\infty(\cdot - \xi_0) \rangle \Phi^+(\xi) \Big| \\
&\leq K_* e^{-\beta_* |\xi - \xi_0|} + K_{12} e^{-\tilde{\gamma} |\xi - \xi_0|} + K_4 \tfrac{1}{\Omega} e^{-\tilde{\chi}|\xi_0|} K_1 e^{-\delta|\xi|} \|\Phi^+\|_\infty \\
&\leq K_{13} e^{-\omega|\xi - \xi_0|}
\end{aligned}
\tag{2.7.131}
$$

for some constants $K_{13} > 0$ and $\omega > 0$. ∎

We write
$$
S = \{ -\overline{\lambda} + i\omega : \omega \in [-\tfrac{\pi|c|}{h}, \tfrac{\pi|c|}{h}] \},
\tag{2.7.132}
$$
where $\overline{\lambda}$ is defined in the proof of Lemma 2.7.14.

Lemma 2.7.15. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. Then there exist constants $K > 0$ and $\hat{\beta} > 0$ such that for all $\lambda \in S$ we have the bound*
$$
|G_\lambda(\xi, \xi_0)| \leq K e^{-\hat{\beta}|\xi - \xi_0|}
\tag{2.7.133}
$$
for all ξ, ξ_0.

Proof. Fix $\lambda_0 \in S$. For $\lambda \in S$ sufficiently close to λ_0 we have
$$
\begin{aligned}
\big[L + \lambda\big]^{-1} &= \big[L + \lambda_0 + \lambda - \lambda_0\big]^{-1} \\
&= \Big[\big(L + \lambda_0\big)\big(I + (L + \lambda_0)^{-1}(\lambda - \lambda_0)\big) \Big]^{-1} \\
&= \big[I + (L + \lambda_0)^{-1}(\lambda - \lambda_0)\big]^{-1} \big[L + \lambda_0\big]^{-1}.
\end{aligned}
\tag{2.7.134}
$$
In particular, upon writing
$$
H_{\xi_0}(\xi) = \Big[[L + \lambda_0]^{-1} \hat{L} G_{\lambda;\infty}(\cdot - \xi_0) \Big](\xi),
\tag{2.7.135}
$$
we see that
$$
G_\lambda(\xi, \xi_0) - G_{\infty;\lambda}(\xi - \xi_0) = \Big[[I + (L + \lambda_0)^{-1}(\lambda - \lambda_0)]^{-1} H_{\xi_0} \Big](\xi).
\tag{2.7.136}
$$
Using Lemma 2.7.6 we can pick constants $k_{\lambda_0} > 0$ and $\alpha_{\lambda_0} > 0$ in such a way that
$$
|H_{\xi_0}(\xi)| \leq k_{\lambda_0} e^{-\alpha_{\lambda_0}|\xi|} \|H_{\xi_0}\|_\infty + k_{\lambda_0} \int_{-\infty}^{\infty} e^{-\alpha_{\lambda_0}|\xi - \eta|} |(L + \lambda_0) H_{\xi_0}(\eta)| d\eta.
\tag{2.7.137}
$$
Recall the constant C_S appearing in Proposition 2.3.3. This allows us to estimate
$$
\begin{aligned}
\|H_{\xi_0}\|_\infty &\leq \|H_{\xi_0}\|_{\mathbf{H}^1} \\
&\leq C_S \|\hat{L} G_{\lambda_0}(\xi, \xi_0)\|_{\mathbf{L}^2} \\
&\leq C_S \sqrt{K_8} e^{-\tilde{\gamma}|\xi_0|}.
\end{aligned}
\tag{2.7.138}
$$

This yields the bound

$$
\begin{aligned}
|H_{\xi_0}(\xi)| &\leq k_{\lambda_0} e^{-\alpha_{\lambda_0}|\xi|} C_S \sqrt{K_8} e^{-\tilde{\gamma}|\xi_0|} + k_{\lambda_0} \int_{-\infty}^{\infty} e^{-\alpha_{\lambda_0}|\xi-\eta|} |\hat{L}G_{\lambda;\infty}(\eta,\xi_0)| d\eta \\
&\leq k_{\lambda_0} e^{-\alpha_{\lambda_0}|\xi|} C_S \sqrt{K_8} e^{-\tilde{\gamma}|\xi_0|} + k_{\lambda_0} \int_{-\infty}^{\infty} e^{-\alpha_{\lambda_0}|\xi-\eta|} K_7 K_* e^{-\delta|\eta|} e^{-2\beta_*|\eta-\xi_0|} d\eta \\
&\leq k_{\lambda_0;2} e^{-\alpha_{\lambda_0;2}|\xi-\xi_0|}
\end{aligned}
$$
(2.7.139)

for some constants $k_{\lambda_0;2}, \alpha_{\lambda_0;2}$, which may depend on λ_0, but not on λ. Arguing as in (2.7.124), we obtain

$$
\begin{aligned}
\left|[L+\lambda_0]^{-1} H_{\xi_0}(\xi)\right| &\leq k_{\lambda_0} e^{-\alpha_{\lambda_0}|\xi|} \|[L+\lambda_0]^{-1} H_{\xi_0}\|_\infty \\
&\quad + k_{\lambda_0} \int_{-\infty}^{\infty} e^{-\alpha_{\lambda_0}|\xi-\eta|} |H_{\xi_0}(\eta)| d\eta \\
&\leq k_{\lambda_0;2} k_{\lambda_0;3} e^{-\alpha_{\lambda_0;2}|\xi-\xi_0|}
\end{aligned}
$$
(2.7.140)

for some constant $k_{\lambda_0;3} > 0$, which may depend on λ_0, but not on λ. Following the same steps as the proof of Lemma 2.7.14 and setting

$$
\varepsilon_{\lambda_0} = \min\{\tfrac{1}{k_{\lambda_0} C_S \sqrt{K_8}}, \tfrac{1}{k_{\lambda_0;3}}\},
$$
(2.7.141)

we conclude that

$$
\begin{aligned}
|G_\lambda(\xi,\xi_0) - G_{\infty;\lambda}(\xi-\xi_0)| &= \left|\left[[I + [L+\lambda_0]^{-1}(\lambda-\lambda_0)]^{-1} H_{\xi_0}\right](\xi)\right| \\
&\leq k_{\lambda_0;4} e^{-\alpha_{\lambda_0;2}|\xi-\xi_0|}
\end{aligned}
$$
(2.7.142)

holds for each $\lambda \in S$ with $|\lambda - \lambda_0| < \varepsilon_{\lambda_0}$, for some constant $k_{\lambda_0;4} > 0$, which may depend on λ_0. In particular, we obtain that

$$
\begin{aligned}
|G_\lambda(\xi,\xi_0)| &\leq k_{\lambda_0;4} e^{-\alpha_{\lambda_0;2}|\xi-\xi_0|} + K_* e^{-\beta_*|\xi-\xi_0|} \\
&\leq k_{\lambda_0;5} e^{-\alpha_{\lambda_0;2}|\xi-\xi_0|}
\end{aligned}
$$
(2.7.143)

holds for each $\lambda \in S$ with $|\lambda - \lambda_0| < \varepsilon_{\lambda_0}$, for some constant $k_{\lambda_0;5} > 0$, which may depend on λ_0.

Since S is compact we can find $\lambda_1, ..., \lambda_n \in S$ in such a way that

$$
S \subset \bigcup_{i=1}^{n} \{\lambda \in \mathbb{C} : |\lambda - \lambda_i| < \varepsilon_{\lambda_i}\}.
$$
(2.7.144)

Setting

$$
\begin{aligned}
K &= \max\{k_{\lambda_i;5} : i \in \{1,...n\}\}, \\
\tilde{\beta} &= \min\{\alpha_{\lambda_i;2} : i \in \{1,...,n\}\},
\end{aligned}
$$
(2.7.145)

we conclude that

$$
|G_\lambda(\xi,\xi_0)| \leq K e^{-\tilde{\beta}|\xi-\xi_0|}
$$
(2.7.146)

holds for all $\lambda \in S$ and all $\xi, \xi_0 \in \mathbb{R}$. ∎

2.7.3 Decomposition into stable and center modes

In this final subsection we establish Proposition 2.7.1. In particular, the decomposition (2.7.13) and the exponential bounds (2.7.15) for the Green's function $\mathcal{G}$ can be found by using the splitting of G_λ obtained in §2.7.2. This is performed in Lemma 2.7.16, which is based on [109, Cor. 2.8].

We subsequently carefully study the terms appearing in (2.7.13) and show that they can be interpreted as a spectral decomposition that splits the flow associated to the linear system (2.7.6) into two invariant subspaces. The stable component decays exponentially in a uniform fashion, while the center component can be described explicitly.

Lemma 2.7.16. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. For any pair $t \geq t_0$ and any $j, j_0 \in \mathbb{Z}$, we have the representation*

$$\mathcal{G}_j^{j_0}(t, t_0) \quad = \quad \mathcal{E}_j^{j_0}(t, t_0) + \tilde{\mathcal{G}}_j^{j_0}(t, t_0) \tag{2.7.147}$$

in which

$$\mathcal{E}_j^{j_0}(t, t_0) \quad = \quad \frac{h}{\Omega} \begin{pmatrix} \phi^-(hj_0 + ct_0)\phi^+(hj + ct) & \psi^-(hj_0 + ct_0)\phi^+(hj + ct) \\ \phi^-(hj_0 + ct_0)\psi^+(hj + ct) & \psi^-(hj_0 + ct_0)\psi^+(hj + ct) \end{pmatrix}, \tag{2.7.148}$$

while $\tilde{\mathcal{G}}$ satisfies the bound

$$|\tilde{\mathcal{G}}_j^{j_0}(t, t_0)| \quad \leq \quad K e^{-\tilde{\beta}(t - t_0)} e^{-\tilde{\beta}|hj + ct - hj_0 - ct_0|} \tag{2.7.149}$$

for some $K > 0$ and $\tilde{\beta} > 0$.

Proof. Recall the representation of $\mathcal{G}_j^{j_0}$ from Proposition 2.7.4. Note that $G_\lambda(\xi, \xi_0)$ is meromorphic for λ in the strip $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq -\lambda_3, |\operatorname{Im}\lambda| \leq \frac{c\pi}{h}\}$ with a simple pole at $\lambda = 0$ by Lemma 2.7.10, Lemma 2.7.3 and Theorem 2.2.2. Lemma 2.7.10 also implies that the residue of $G_\lambda(\xi, \xi_0)$ in $\lambda = 0$ is given by

$$\operatorname{Res}(G_\lambda(\xi, \xi_0), 0) \quad = \quad -\frac{1}{\Omega} \begin{pmatrix} \phi^-(\xi_0)\phi^+(\xi) & \psi^-(\xi_0)\phi^+(\xi) \\ \phi^-(\xi_0)\psi^+(\xi) & \psi^-(\xi_0)\psi^+(\xi) \end{pmatrix}. \tag{2.7.150}$$

We write

$$H(\cdot, \xi_0) \quad = \quad e^{2\pi i \frac{1}{h} k \xi_0}(L + \lambda + 2\pi i k \frac{c}{h})e_{-2\pi i \frac{1}{h} k} G_\lambda(\cdot, \xi_0). \tag{2.7.151}$$

In a similar fashion as in the proof of Lemma 2.6.1 we see that for $k \in \mathbb{Z}$ we have

$$(L + \lambda + 2\pi i k \frac{c}{h})e_{-2\pi i \frac{1}{h} k} \quad = \quad e_{-2\pi i \frac{1}{h} k}(L + \lambda). \tag{2.7.152}$$

Therefore, it follows that

$$\begin{aligned} H(\cdot, \xi_0) \quad &= \quad e^{2\pi i \frac{1}{h} k \xi_0}(L + \lambda + 2\pi i k \frac{c}{h})e_{-2\pi i \frac{1}{h} k} G_\lambda(\cdot, \xi_0) \\ &= \quad e^{2\pi i \frac{1}{h} k \xi_0} e_{-2\pi i \frac{1}{h} k}(L + \lambda) G_\lambda(\cdot, \xi_0). \end{aligned} \tag{2.7.153}$$

For any $f \in \mathbf{H}^1$ we may hence compute

$$
\begin{aligned}
\int H(\xi, \xi_0) f(\xi_0) \, d\xi_0 &= \int e^{2\pi i \frac{1}{h} k \xi_0} e^{-2\pi i \frac{1}{h} k \xi} (L + \lambda) G_\lambda(\cdot, \xi_0)(\xi) f(\xi_0) \, d\xi_0 \\
&= e^{-2\pi i \frac{1}{h} k \xi} [e^{2\pi i \frac{1}{h} k \xi} f(\xi)] \\
&= f(\xi).
\end{aligned}
\tag{2.7.154}
$$

Therefore, by the invertibility of $L + \lambda + 2\pi i k \frac{c}{h}$, we must have

$$
G_{\lambda + 2\pi i k \frac{c}{h}}(\xi, \xi_0) = e^{2\pi i k \frac{1}{h}(\xi_0 - \xi)} G_\lambda(\xi, \xi_0).
\tag{2.7.155}
$$

Now recall the constants χ, χ_+, χ_- from (the proof of) Proposition 2.7.4 and define

$$
\begin{aligned}
\overline{\lambda}^- &= -\frac{\overline{\lambda}}{2} - i \frac{\pi c}{h} \\
\overline{\lambda}^+ &= -\frac{\overline{\lambda}}{2} + i \frac{\pi c}{h}.
\end{aligned}
\tag{2.7.156}
$$

Writing $x = hj + ct, y = hj_0 + ct_0$, we see that

$$
\begin{aligned}
\int_{\overline{\lambda}^-}^{\chi_-} e^{\lambda(t - t_0)} G_\lambda(x, y) \, d\lambda &= \int_{\overline{\lambda}^+}^{\chi_+} e^{(\lambda + 2\pi i \frac{c}{h})(t - t_0)} e^{-2\pi i \frac{1}{h}(y - x)} G_\lambda(x, y) \, d\lambda \\
&= \int_{\overline{\lambda}^+}^{\chi_+} e^{\lambda(t - t_0)} G_\lambda(x, y) \, d\lambda.
\end{aligned}
\tag{2.7.157}
$$

Hence, if we integrate the function $e^{\lambda(t - t_0)} G_\lambda(hj + ct, hj_0 + ct_0)$ along the rectangle with edges $-\frac{\overline{\lambda}}{2} - i \frac{\pi c}{h}, -\frac{\overline{\lambda}}{2} + i \frac{\pi c}{h}, \chi - i \frac{\pi c}{h}$ and $\chi + i \frac{\pi c}{h}$, then the integrals from $\chi - i \frac{\pi c}{h}$ to $-\frac{\overline{\lambda}}{2} - i \frac{\pi c}{h}$ and from $-\frac{\overline{\lambda}}{2} + i \frac{\pi c}{h}$ to $\chi + i \frac{\pi c}{h}$ cancel each other out. In particular, again writing $x = hj + ct, y = hj_0 + ct_0$, the residue theorem implies

$$
\begin{aligned}
\mathcal{G}_j^{j_0}(t, t_0) &= \frac{-h}{2\pi i} \int_{\chi - i \frac{\pi c}{h}}^{\chi + i \frac{\pi c}{h}} e^{\lambda(t - t_0)} G_\lambda(x, y) \, d\lambda \\
&= \frac{h}{2\pi i} \int_{-\frac{\overline{\lambda}}{2} - i \frac{\pi c}{h}}^{-\frac{\overline{\lambda}}{2} + i \frac{\pi c}{h}} e^{\lambda(t - t_0)} G_\lambda(x, y) \, d\lambda + \frac{h}{\Omega} \begin{pmatrix} \phi^-(y)\phi^+(x) & \psi^-(y)\phi^+(x) \\ \phi^-(y)\psi^+(x) & \psi^-(y)\psi^+(x) \end{pmatrix}.
\end{aligned}
\tag{2.7.158}
$$

Using Lemma 2.7.15 we also get the estimate

$$
\left| \frac{h}{2\pi i} \int_{-\frac{\overline{\lambda}}{2} - i \frac{\pi c}{h}}^{-\frac{\overline{\lambda}}{2} + i \frac{\pi c}{h}} e^{\lambda(t - t_0)} G_\lambda(x, y) \, d\lambda \right| \leq \frac{h}{2\pi} \frac{2c\pi}{h} e^{-\overline{\lambda}(t - t_0)} K e^{-\tilde{\beta}|x - y|},
\tag{2.7.159}
$$

which yields the desired bound (2.7.149). ∎

For any $t \in \mathbb{R}$, we introduce the suggestive notation

$$
\Pi^c(t) = \mathcal{E}(t, t)
\tag{2.7.160}
$$

together with

$$\Pi^s(t) \;\; = \;\; I - \Pi^c(t). \tag{2.7.161}$$

Recalling the notation introduced in (2.7.12), we set out to show that $\Pi^c(t) * \Pi^c(t) = \Pi^c(t)$ and $\Pi^s(t) * \Pi^s(t) = \Pi^s(t)$. Later on, we will view these operators as projections that correspond to the center and stable parts of the flow induced by $\mathcal{G}$ respectively.

To establish the identity $\Pi^c(t) * \Pi^c(t) = \Pi^c(t)$, it suffices to show that

$$\begin{aligned}
\begin{pmatrix} \phi^-(x_{j_0})\phi^+(x_j) & \psi^-(x_{j_0})\phi^+(x_j) \\ \phi^-(x_{j_0})\psi^+(x_j) & \psi^-(x_{j_0})\psi^+(x_j) \end{pmatrix} &\;=\; \frac{h}{\Omega} \sum_{i\in\mathbb{Z}} \begin{pmatrix} \phi^-(x_i)\phi^+(x_j) & \psi^-(x_i)\phi^+(x_j) \\ \phi^-(x_i)\psi^+(x_j) & \psi^-(x_i)\psi^+(x_j) \end{pmatrix} \\
&\quad\times \begin{pmatrix} \phi^-(x_{j_0})\phi^+(x_i) & \psi^-(x_{j_0})\phi^+(x_i) \\ \phi^-(x_{j_0})\psi^+(x_i) & \psi^-(x_{j_0})\psi^+(x_i) \end{pmatrix},
\end{aligned} \tag{2.7.162}$$

in which $x_i = hi + ct$ for $i \in \mathbb{Z}$. We now write our linear operator in the form

$$L\Psi(\xi) \;\;=\;\; c\frac{d}{d\xi}\Psi(\xi) + \sum_{j=-\infty}^{\infty} A_j(\xi)\Psi(\xi + jh), \tag{2.7.163}$$

where

$$A_j(\xi) \;\;=\;\; \begin{cases} \begin{pmatrix} \frac{1}{h^2}\alpha_{|j|} & 0 \\ 0 & 0 \end{pmatrix} & \text{if } j \neq 0 \\[2em] \begin{pmatrix} -2\frac{1}{h^2}\sum\limits_{k>0}\alpha_k + g_u(\overline{u}(\xi)) & 1 \\ -\rho & \rho\gamma \end{pmatrix} & \text{if } j = 0. \end{cases} \tag{2.7.164}$$

Before we continue, we first prove a small lemma that will help us to relate discrete inner products with their continuous counterparts.

Lemma 2.7.17. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. For all $\xi \in \mathbb{R}$ we have the identity*

$$c\begin{pmatrix} \phi^-(\xi)\phi^+(\xi) \\ \psi^-(\xi)\psi^+(\xi) \end{pmatrix} \;\;=\;\; \sum_{j=-\infty}^{\infty} \int_0^{hj} B(\xi+\theta-hj)A_j(\xi+\theta-hj)\Phi^+(\xi+\theta)d\theta, \tag{2.7.165}$$

where

$$B(\xi) \;\;=\;\; \begin{pmatrix} \phi^-(\xi) & 0 \\ 0 & \psi^-(\xi) \end{pmatrix}. \tag{2.7.166}$$

Proof. Our strategy is to differentiate both sides of (2.7.165) and to show their derivatives are equal. Starting with the first component, we pick $N \in \mathbb{Z}_{>0} \cup \{\infty\}$ and write

$$D(N) \;\; := \;\; \frac{d}{d\xi} \sum_{j=-N}^{N} \int_0^{hj} \phi^-(\xi+\theta-hj)\Big[A_j(\xi+\theta-hj)\Phi^+(\xi+\theta)\Big]^{(1)} d\theta. \tag{2.7.167}$$

For finite N, we may compute

$$
\begin{aligned}
D(N) &= \tfrac{d}{d\xi} \sum_{j=-N}^{N} \int_{\xi-hj}^{\xi} \phi^-(\theta) \Big[A_j(\theta) \Phi^+(\theta + hj) \Big]^{(1)} d\theta \\
&= \sum_{j=-N}^{N} \phi^-(\xi) \Big[A_j(\xi) \Phi^+(\xi + hj) \Big]^{(1)} \\
&\quad - \sum_{j=-N}^{N} \phi^-(\xi - hj) \Big[A_j(\xi - hj) \Phi^+(\xi) \Big]^{(1)}.
\end{aligned}
\tag{2.7.168}
$$

Now for $j > 0$ we have $|A_j(\xi)\Phi^+(\xi + hj)| \le \tfrac{1}{h^2}|\alpha_j|$, so the partial sums converge uniformly. Hence, it follows that

$$
\begin{aligned}
D(\infty) &= \sum_{j=-\infty}^{\infty} \phi^-(\xi) \Big[A_j(\xi) \Phi^+(\xi + hj) \Big]^{(1)} \\
&\quad - \sum_{j=-\infty}^{\infty} \phi^-(\xi - hj) \Big[A_j(\xi - hj) \Phi^+(\xi) \Big]^{(1)} \\
&= \phi^-(\xi)c(\phi^+)'(\xi) + c(\phi^-)'(\xi)\phi^+(\xi) \\
&= c(\phi^-\phi^+)'(\xi),
\end{aligned}
\tag{2.7.169}
$$

since $\Phi^+ \in \ker(L)$ and $\Phi^- \in \ker(L^*)$.

We now set out to show that both sides of (2.7.165) converge to zero as $\xi \to \infty$. Pick $\varepsilon > 0$ and let $N \in \mathbb{Z}_{>0}$ be large enough to ensure that

$$
\sum_{j \ge N} \tfrac{1}{h^2} j|\alpha_j| \;\le\; \tfrac{\varepsilon}{4(1+\|\phi^-\|_\infty)\|\Phi^+\|_\infty}.
\tag{2.7.170}
$$

In addition, let Ξ be large enough to have

$$
|\phi^-(\xi)| \;\le\; \frac{\varepsilon}{4(1+\sum_{j=-N}^{N} |j\alpha_{|j|}|)\|\Phi^+\|_\infty}
\tag{2.7.171}
$$

for all $\xi \ge \Xi - N$. This Ξ exists since $\phi^- \in H^1$. For such ξ we may estimate

$$
\Big| \sum_{j=-\infty}^{\infty} \int_0^{hj} \phi^-(\xi + \theta - hj) \Big[A_j(\xi + \theta - hj)\Phi^+(\xi + \theta) \Big]^{(1)} d\theta \Big| \;<\; \varepsilon,
\tag{2.7.172}
$$

which allows us to compute

$$
\begin{aligned}
\lim_{\xi \to \infty} \sum_{j=-\infty}^{\infty} \int_0^{hj} \phi^-(\xi + \theta - hj) \Big[A_j(\xi + \theta - hj)\Phi^+(\xi + \theta) \Big]^{(1)} d\theta &= 0 \\
&= \lim_{\xi \to \infty} c\phi^-(\xi)\phi^+(\xi).
\end{aligned}
\tag{2.7.173}
$$

With that we have proved our claim. Furthermore, we can repeat the arguments above to obtain

$$
c\psi^-(\xi)\psi^+(\xi) \;=\; \sum_{j=-\infty}^{\infty} \int_0^{hj} \psi^-(\xi + \theta - hj) \Big[A_j(\xi + \theta - hj)\Phi^+(\xi + \theta) \Big]^{(2)} d\theta.
\tag{2.7.174}
$$

∎

We are now ready to show that $\Pi^c(t) * \Pi^c(t) = \Pi^c(t)$ and $\Pi^s(t) * \Pi^s(t) = \Pi^s(t)$. This result is based on the first part of [109, Lem. 2.9].

Lemma 2.7.18. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. Then $\Pi^c(t) * \Pi^c(t) = \Pi^c(t)$ and $\Pi^s(t) * \Pi^s(t) = \Pi^s(t)$ for all $t \in \mathbb{R}$.*

Proof. For $k \in \mathbb{Z}$ we write $x_k = hk + ct$. In addition, for notational convenience we write $B_j(\theta) = \left[A_j(\theta)\Phi^+(\theta+hj) \right]^{(1)}$ for $j \in \mathbb{Z}$ and $\theta \in \mathbb{R}$. Using the results from Lemma 2.7.17 we may compute

$$
\begin{aligned}
c \sum_{k=-\infty}^{\infty} \phi^-(x_k)\phi^+(x_k) &= \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \int_0^{hj} \phi^-(x_k + \theta - hj)B_j(x_k + \theta - hj)d\theta \\
&= \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \int_{x_{k-j}}^{x_k} \phi^-(\theta)B_j(\theta)d\theta \\
&= \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} j\phi^-(\theta)B_j(\theta)d\theta,
\end{aligned}
$$

$$(2.7.175)$$

where we were allowed to interchange the two infinite sums because

$$
\begin{aligned}
\left| \sum_{k=-N}^{N} \int_{x_{k-j}}^{x_k} \phi^-(\theta)B_j(\theta)d\theta \right| &\leq \left| \int_{-\infty}^{\infty} j\phi^-(\theta)B_j(\theta)d\theta \right| \\
&\leq \|\phi^-\|_1 \frac{1}{h^2} |j\alpha_{|j|}| \|\phi^+\|_\infty
\end{aligned}
$$

$$(2.7.176)$$

holds for all $N \in \mathbb{Z}_{>0}$ and $j \in \mathbb{Z}$. This expression is summable over j, allowing us to apply Lebesgue's theorem. On the other hand, we have

$$
\begin{aligned}
c \int_{-\infty}^{\infty} \phi^-(\xi)\phi^+(\xi)\, d\xi &= \int_{-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \int_0^{hj} \phi^-(\xi + \theta - hj)B_j(\xi + \theta - hj)d\theta\, d\xi \\
&= \sum_{j=-\infty}^{\infty} \int_0^{hj} \int_{-\infty}^{\infty} \phi^-(\xi + \theta - hj)B_j(\xi + \theta - hj)\, d\xi d\theta \\
&= \sum_{j=-\infty}^{\infty} \int_0^{hj} \int_{-\infty}^{\infty} \phi^-(\xi - hj)B_j(\xi - hj)d\xi d\theta \\
&= \sum_{j=-\infty}^{\infty} hj \int_{-\infty}^{\infty} \phi^-(\xi - hj)B_j(\xi - hj)d\xi.
\end{aligned}
$$

$$(2.7.177)$$

Interchanging the integral with the sum was allowed since ϕ^- and ϕ^+ decay exponentially, say $|\phi^-(x)| \leq \kappa e^{-\delta|x|}$ and $|\phi^+(x)| \leq \kappa e^{-\delta|x|}$. In particular, for each $N \in \mathbb{Z}_{>0}$ and each $\xi \in \mathbb{R}$ we have

$$
\left| \sum_{j=-N}^{N} \int_0^{hj} \phi^-(\xi + \theta - hj)B_j(\xi + \theta - hj)d\theta \right| \leq \sum_{j=-\infty}^{\infty} h\kappa^2 e^{-\delta|\xi|} |j\alpha_{|j|}| \|\Phi^+\|_\infty,
$$

$$(2.7.178)$$

which is integrable in ξ. Furthermore, the interchanging of the two integrals was allowed, since by the exponential decay of ϕ^- we also see that for each $j \in \mathbb{Z}, \xi \in \mathbb{R}$ and

$\theta \in (0, hj)$ we have

$$|\phi^-(\xi + \theta - hj)B_j(\xi + \theta - hj)| \leq \kappa e^{-\delta|\xi+\theta-hj|}|\alpha_{|j|}|\|\Phi^+\|_\infty. \qquad (2.7.179)$$

This is an integrable function for $(\xi, \theta) \in \mathbb{R} \times (0, hj)$, allowing us to apply Fubini's theorem.

In particular, we see that

$$\int_{-\infty}^{\infty} \phi^-(\xi)\phi^+(\xi) \, d\xi = h \sum_{k=-\infty}^{\infty} \phi^-(x_k)\phi^+(x_k). \qquad (2.7.180)$$

In the same way we obtain

$$\int_{-\infty}^{\infty} \psi^-(\xi)\psi^+(\xi) \, d\xi = h \sum_{k=-\infty}^{\infty} \psi^-(x_k)\psi^+(x_k). \qquad (2.7.181)$$

By writing out the sums it now follows that indeed (2.7.162) holds. ∎

Proof of Proposition 2.7.1. The calculations above imply that $\mathcal{E}(t, t_0) = \mathcal{E}(t, t_0) * \Pi^c(t_0)$, which means that we must also have $\mathcal{E}(t, t_0) * \Pi^s(t_0) = 0$.

Observe that for any $t_0 \in \mathbb{R}$, the function $V_j(t) := \begin{pmatrix} \phi^+(hj + ct) \\ \psi^+(hj + ct) \end{pmatrix}$ is the unique solution to (2.7.6) with $V_j(t_0) = \begin{pmatrix} \phi^+(hj + ct_0) \\ \psi^+(hj + ct_0) \end{pmatrix}$. Hence, by the definition of the Green's function $\mathcal{G}(t, t_0)$ we see that

$$V(t) = \mathcal{G}(t, t_0) * V(t_0) \qquad (2.7.182)$$

for all $t \in \mathbb{R}$. Furthermore, we recall that

$$\mathcal{E}_j^{j_0}(t_0, t_0) = V_j(t_0)\Phi^-(hj_0 + ct_0). \qquad (2.7.183)$$

For $j, j_0 \in \mathbb{Z}$ we may hence compute

$$\begin{aligned}
\left[\mathcal{G}(t, t_0) * \Pi^c(t_0)\right]_j^{j_0} &= \sum_{i \in \mathbb{Z}} \mathcal{G}_j^i(t, t_0) * \mathcal{E}_i^{j_0}(t, t_0) \\
&= \frac{h}{\Omega} \sum_{i \in \mathbb{Z}} \mathcal{G}_j^i(t, t_0)_i * V_i(t_0)\Phi^-(hj_0 + ct_0) \\
&= \frac{h}{\Omega} V_j(t)\Phi^-(hj_0 + ct_0) \\
&= \mathcal{E}_j^{j_0}(t, t_0).
\end{aligned} \qquad (2.7.184)$$

In particular, we obtain $\mathcal{G}(t, t_0) * \Pi^c(t_0) = \mathcal{E}(t, t_0)$ and thus

$$\begin{aligned}
\tilde{\mathcal{G}}(t, t_0) * \Pi^c(t_0) &= \mathcal{G}(t, t_0) * \Pi^c(t_0) - \mathcal{E}(t, t_0) * \Pi^c(t_0) \\
&= \mathcal{E}(t, t_0) - \mathcal{E}(t, t_0) \\
&= 0.
\end{aligned} \qquad (2.7.185)$$

Therefore, we must have

$$
\begin{aligned}
\mathcal{G}(t, t_0) &= \mathcal{E}(t, t_0) + \tilde{\mathcal{G}}(t, t_0) \\
&= \mathcal{E}(t, t_0) * \left(\Pi^c(t_0) + \Pi^s(t_0) \right) + \tilde{\mathcal{G}}(t, t_0) * \left(\Pi^c(t_0) + \Pi^s(t_0) \right) \qquad (2.7.186) \\
&= \mathcal{E}(t, t_0) * \Pi^c(t_0) + \tilde{\mathcal{G}}(t, t_0) * \Pi^s(t_0),
\end{aligned}
$$

which completes the proof. ∎

2.8 Nonlinear stability

In this section, we will finally prove Theorem 2.2.3, along the lines of the approach described in [109]. The main contribution here is that we give a detailed description of the manner in which one can account for the shift-periodicity of the underlying problem when constructing the stable manifolds for the family $\overline{U}(\cdot + \vartheta)$.

Recall from §2.2 that the space ℓ^p is defined by

$$
\ell^p = \{ V \in (\mathbb{R}^2)^{\mathbb{Z}} : \|V\|_{\ell^p} := \sum_{j \in \mathbb{Z}} |V_j|^p < \infty \} \qquad (2.8.1)
$$

for $1 \le p < \infty$ and

$$
\ell^\infty = \{ V \in (\mathbb{R}^2)^{\mathbb{Z}} : \|V\|_{\ell^\infty} := \sup_{j \in \mathbb{Z}} |V_j| < \infty \}. \qquad (2.8.2)
$$

In addition, we recall the notation $(\overline{U})_j(t) = \left(\overline{u}(hj + ct), \overline{w}(hj + ct) \right)$ and we let $\tilde{\beta} > 0$ be the constant appearing in Proposition 2.7.1.

Exploiting Lemma 2.7.8 we see that

$$
\|\mathcal{E}_j^{j_0}(t, t_0)\| \le C_1 e^{-\tilde{\beta}|hj + ct - hj_0 - ct_0|} \qquad (2.8.3)
$$

for some constant $C_1 > 0$. Lemma 2.7.18 hence allows us to define $\Pi^c(t) \in \mathcal{B}(\ell^p; \ell^p)$ and $\Pi^s(t) \in \mathcal{B}(\ell^p, \ell^p)$ by writing

$$
\begin{aligned}
\Pi^c(t)V &= \mathcal{E}(t, t) * V, \\
\Pi^s(t)V &= \left[I - \Pi^c(t) \right] V.
\end{aligned} \qquad (2.8.4)
$$

The proof of our nonlinear stability result proceeds in two main steps. In particular, we first construct the stable manifolds of the solutions $(\overline{u}, \overline{w})(\cdot - \tilde{\theta})$ for each $\tilde{\theta} \in \mathbb{R}$. This result is based on the first half of the proof of [109, Prop. 2.1].

Proposition 2.8.1. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. There exists a constant $\eta > 0$, independent of p, such that for each $\tilde{\theta} \in \mathbb{R}$ and each $W_s \in \text{Range}(\Pi^s(\tilde{\theta}))$ with $\|W_s\|_{\ell^p} < \eta$ there is a unique function $\mathcal{U}_*^{\tilde{\theta}}(W_s) : [0, \infty) \to \ell^p$ such that*

1. $U(t) = \overline{U}(t + \tilde{\theta}) + \mathcal{U}_*^{\tilde{\theta}}(W_s)(t)$ *is a solution of (2.2.1) for all* $t \geq 0$,

2. $\mathcal{U}_*^{\tilde{\theta}}(W_s)(t)$ *decays exponentially to* 0 *as* $t \to \infty$,

3. $\Pi^s(\tilde{\theta})\mathcal{U}_*^{\tilde{\theta}}(W_s)(0) = W_s$.

In addition, there exist constants $C_6 > 0$ *and* $C_{13} > 0$, *independent of* p, *such that the estimate*

$$\|\Pi^c(\tilde{\theta})\mathcal{U}_*^{\tilde{\theta}}(W_s)(0)\|_{\ell^p} \leq C_6 \|W_s\|_{\ell^p}^2 \tag{2.8.5}$$

holds for all $\tilde{\theta} \in \mathbb{R}$ *and each* $W_s \in \mathrm{Range}(\Pi^s(\tilde{\theta}))$ *with* $\|W_s\|_{\ell^p} < \eta$, *while the estimate*

$$\|\Pi^c(\tilde{\theta}_1)\mathcal{U}_*^{\tilde{\theta}_1}(W_s^1)(0) - \Pi^c(\tilde{\theta}_2)\mathcal{U}_*^{\tilde{\theta}_2}(W_s^2)(0)\|_{\ell^p} \leq C_{13}\Big[\|W_s^1\|_{\ell^p} + \|W_s^2\|_{\ell^p}\Big]$$
$$\times \Big[\|W_s^1 - W_s^2\|_{\ell^p} + |\theta_1 - \theta_2|\Big] \tag{2.8.6}$$

holds for all $W_s^1 \in \mathrm{Range}(\Pi^s(\tilde{\theta}_1))$, *all* $W_s^2 \in \mathrm{Range}(\Pi^s(\tilde{\theta}_2))$ *and all* $\tilde{\theta}_1 \in \mathbb{R}$ *and* $\tilde{\theta}_2 \in \mathbb{R}$ *with* $\|W_s^1\|_{\ell^p} < \eta$, $\|W_s^2\|_{\ell^p} < \eta$ *and* $|\tilde{\theta}_2 - \tilde{\theta}_1| < \eta$.

It then suffices to show that the space around the family of travelling pulse solutions can be completely covered by these stable manifolds. We remark that $\tilde{\theta}$ in the result below will correspond with the asymptotic phase shift.

Proposition 2.8.2. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. Then there exists a constant* $\delta > 0$, *which does not depend on* p, *such that for all initial conditions* $U^0 \in \ell^p$ *with* $\|U^0 - \overline{U}(0)\|_{\ell^p} < \delta$ *there exists* $\tilde{\theta} \in \mathbb{R}$ *and* $W_s \in \mathrm{Range}(\Pi^s(\tilde{\theta}))$ *such that*

$$U^0 = \overline{U}(\tilde{\theta}) + \mathcal{U}_*^{\tilde{\theta}}(W_s). \tag{2.8.7}$$

We write the LDE (2.2.1) as

$$\frac{d}{dt}V(t) = \mathcal{F}\Big(V(t)\Big), \tag{2.8.8}$$

where

$$\mathcal{F}\Big(V(t)\Big)_j = \frac{1}{c}\left(\begin{array}{c} \frac{1}{h^2}\sum_{k>0}\alpha_k\Big[V_{j+k}^{(1)}(t) + V_{j-k}^{(1)}(t) - 2V_j^{(1)}(t)\Big] + g(V_j^{(1)}(t)) - V_j^{(2)}(t) \\ \rho[V_j^{(1)}(t) - \gamma V_j^{(2)}(t)] \end{array} \right). \tag{2.8.9}$$

Then we see that $\mathcal{A}(t) = D\mathcal{F}\Big(\overline{U}(t)\Big)$, where $\mathcal{A}(t)$ is defined in (2.7.6). We now write

$$\mathcal{N}_\theta\Big(t, V(t)\Big) = \mathcal{F}\Big(V(t) + \overline{U}(t + \theta)\Big) - \mathcal{F}(\overline{U}(t + \theta)) - D\mathcal{F}\Big(\overline{U}(t + \theta)\Big)V(t) \tag{2.8.10}$$

and set out to solve the differential equation

$$\frac{d}{dt}V(t) = D\mathcal{F}\Big(\overline{U}(t + \theta)\Big)V(t) + \mathcal{N}_\theta\Big(t, V(t)\Big). \tag{2.8.11}$$

Indeed, if V satisfies (2.8.11), then we see that

$$\frac{d}{dt}\Big(\overline{U}(t+\theta)+V(t)\Big) = \mathcal{F}\Big(\overline{U}(t+\theta)\Big)+D\mathcal{F}\Big(\overline{U}(t+\theta)\Big)V(t)+\mathcal{N}_\theta\Big(t,V(t)\Big)$$
$$= \mathcal{F}\Big(\overline{U}(t+\theta)+V(t)\Big),$$

(2.8.12)

which means that $\overline{U}(\cdot+\theta)+V$ is indeed a solution of (2.2.1).

Our goal is to construct decaying solutions to (2.8.11) for multiple values of θ using a single Green's function. To this end, we write

$$\mathcal{M}^{\tilde\theta}\Big(\theta,t,V\Big) = \mathcal{N}_\theta\Big(t,V(t)\Big)+D\mathcal{F}\Big(\overline{U}(t+\theta)\Big)V(t)-D\mathcal{F}\Big(\overline{U}(t+\tilde\theta)\Big)V(t).$$

(2.8.13)

This allow us to rewrite (2.8.11) as

$$\frac{d}{dt}V(t) = D\mathcal{F}\Big(\overline{U}(t+\tilde\theta)\Big)V(t)+\mathcal{M}^{\tilde\theta}\Big(\theta,t,V\Big).$$ (2.8.14)

Lemma 2.8.3. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. Then $\mathcal{M}^{\tilde\theta}(\theta,t,\cdot)$ maps ℓ^p into itself and there exists a constant $C_2 > 0$, independent of p and $\tilde\theta$, so that we have the estimate*

$$\|\mathcal{M}^{\tilde\theta}(\theta,t,V)\|_{\ell^p} \leq C_2\|V\|_{\ell^p}^2+C_2|\tilde\theta-\theta|\|V\|_{\ell^p},$$ (2.8.15)

for $V \in \ell^p$ with $\|V\|_{\ell^p} \leq 1$ and $\theta \in \mathbb{R}$ with $|\theta-\tilde\theta| \leq 1$, together with

$$\|\mathcal{M}^{\tilde\theta}(\theta_1,t,V_1)-\mathcal{M}^{\tilde\theta}(\theta_2,t,V_2)\|_{\ell^p} \leq C_2\|V_1-V_2\|_{\ell^p}\Big[\|V_1\|_{\ell^p}+\|V_2\|_{\ell^p}$$
$$+|\tilde\theta-\theta_1|+|\theta_2-\tilde\theta|\Big]$$
$$+C_2|\tilde\theta_1-\tilde\theta_2|\Big[\|V_1\|_{\ell^p}+\|V_2\|_{\ell^p}\Big]$$

(2.8.16)

for $V_1 \in \ell^p$, $V_2 \in \ell^p$, $\theta_1 \in \mathbb{R}$ and $\theta_2 \in \mathbb{R}$ with $\|V_1\|_{\ell^p} \leq 1, \|V_2\|_{\ell^p} \leq 1, |\theta_1-\tilde\theta| \leq 1$ and $|\theta_2-\tilde\theta| \leq 1$.

Proof. A Taylor expansion around $\overline{U}^{(1)}(t+\tilde\theta)$ yields the pointwise identity

$$\mathcal{M}^{\tilde\theta}(\theta,t,V)^{(1)} = \frac{1}{c}\Big(g\big(V^{(1)}+\overline{U}^{(1)}(t+\theta)\big)-g_u\big(\overline{U}^{(1)}(t+\tilde\theta)\big)V^{(1)}-g\big(\overline{U}^{(1)}(t+\theta)\big)\Big)$$
$$= \frac{1}{c}\Big(\frac{1}{2}g_{uu}(\xi_1)(V^{(1)})^2+\big[g_u\big(\overline{U}^{(1)}(t+\theta)\big)-g_u\big(\overline{U}^{(1)}(t+\tilde\theta)\big)\big]V^{(1)}\Big)$$
$$= \frac{1}{c}\Big(\frac{1}{2}g_{uu}(\xi_1)(V^{(1)})^2+\frac{1}{2}g_{uu}(\xi_2)\big[\overline{U}^{(1)}(t+\theta)-\overline{U}^{(1)}(t+\tilde\theta)\big]V^{(1)}\Big)$$
$$= \frac{1}{c}\Big(\frac{1}{2}g_{uu}(\xi_1)(V^{(1)})^2+\frac{1}{2}g_{uu}(\xi_2)\big[\frac{d}{dt}\overline{U}^{(1)}(\xi_3)\big]|\theta-\tilde\theta|V^{(1)}\Big),$$
$$\mathcal{M}^{\tilde\theta}(\theta,t,V)^{(2)} = 0,$$

(2.8.17)

where ξ_1 is between $\overline{U}^{(1)}(t+\tilde{\theta})$ and $\overline{U}^{(1)}(t+\tilde{\theta})+V$, ξ_2 is between $\overline{U}^{(1)}(t+\tilde{\theta})$ and $\overline{U}^{(1)}(t)$ and ξ_3 is between $t+\theta$ and $t+\tilde{\theta}$. For a bounded function f, we have the pointwise bound

$$
\begin{aligned}
|g_{uu}(f)| &= |6f + 2r_0 + 2| \\
&\leq 6\|f\|_\infty + 2r_0 + 2.
\end{aligned}
\tag{2.8.18}
$$

Therefore we get the pointwise bound

$$
\begin{aligned}
|\mathcal{M}^{\tilde{\theta}}(\theta,t,V)| &\leq \left| \frac{1}{c}\frac{1}{2} \left(\begin{array}{c} \left(6\|\overline{u}\|_\infty + 2r_0 + 2\right)\|\overline{u}'\|_\infty|\theta - \tilde{\theta}|V^{(1)} \\ 0 \end{array} \right) \right| \\
&\quad + \left| \frac{1}{c}\frac{1}{2} \left(\begin{array}{c} \left(6\|V^{(1)}\|_{\ell^\infty} + 6\|\overline{u}\|_\infty + 2r_0 + 2\right)(V^{(1)})^2 \\ 0 \end{array} \right) \right| \\
&\leq \frac{1}{|c|}\frac{1}{2}\left(6\|\overline{u}\|_\infty + 2r_0 + 2\right)\|\overline{u}'\|_\infty|\theta - \tilde{\theta}||V| \\
&\quad + \frac{1}{|c|}\frac{1}{2}(6\|V\|_{\ell^\infty} + 6\|\overline{u}\|_\infty + 2r_0 + 2)|(V^{(1)})|^2.
\end{aligned}
\tag{2.8.19}
$$

Furthermore, for $1 \leq p < \infty$ we see

$$
\begin{aligned}
\left\| \left(\begin{array}{c} |(V^{(1)})|^2 \\ 0 \end{array} \right) \right\|_{\ell^p} &= \left(\sum_{j\in\mathbb{Z}} |V_j^{(1)}|^{2p} \right)^{\frac{1}{p}} \\
&\leq \left(\left(\sum_{j\in\mathbb{Z}} |V_j^{(1)}|^p\right) \left(\sup_{j\in\mathbb{Z}} |V_j^{(1)}|^p\right) \right)^{\frac{1}{p}} \\
&\leq \|V\|_{\ell^p}^2,
\end{aligned}
\tag{2.8.20}
$$

which clearly also holds for $p = \infty$ upon skipping the intermediate two steps. We hence obtain the bound

$$
\begin{aligned}
\|\mathcal{M}^{\tilde{\theta}}(\theta,t,V)\|_{\ell^p} &\leq \frac{1}{|c|}\frac{1}{2}\left(6\|\overline{u}\|_\infty + 2r_0 + 2\right)\|\overline{u}'\|_\infty|\theta - \tilde{\theta}|\|V\|_{\ell^p} \\
&\quad + \frac{1}{|c|}\frac{1}{2}(6\|\overline{u}\|_\infty + 2r_0 + 8)\|V\|_{\ell^p}^2 \\
&\leq C_2\|V\|_{\ell^p}^2 + C_2|\theta - \tilde{\theta}|\|V\|_{\ell^p},
\end{aligned}
\tag{2.8.21}
$$

for some constant $C_2 > 0$, which is independent of p and $\tilde{\theta}$.

We now write

$$
\begin{aligned}
d\mathcal{M} &= \mathcal{M}^{\tilde{\theta}}\left(\theta_1,t,V_1\right) - \mathcal{M}^{\tilde{\theta}}\left(\theta_2,t,V_2\right) \\
&= \frac{1}{c} \left(\begin{array}{c} g\left(V_1^{(1)} + \overline{U}^{(1)}(t+\theta_1)\right) - g_u\left(\overline{U}^{(1)}(t+\tilde{\theta})\right)V_1^{(1)} - g\left(\overline{U}^{(1)}(t+\theta_1)\right) \\ 0 \end{array} \right) \\
&\quad + \frac{1}{c} \left(\begin{array}{c} -g\left(V_2^{(1)} + \overline{U}^{(1)}(t+\theta_2)\right) + g_u\left(\overline{U}^{(1)}(t+\tilde{\theta})\right)V_2^{(1)} + g\left(\overline{U}^{(1)}(t+\theta_2)\right) \\ 0 \end{array} \right).
\end{aligned}
\tag{2.8.22}
$$

Using Taylor expansions around $\overline{U}^{(1)}(t+\theta_1)$ and $\overline{U}^{(1)}(t+\theta_2)$, we obtain the pointwise identities

$$
g\left(V_1^{(1)}+\overline{U}^{(1)}(t+\theta_1)\right)-g\left(\overline{U}^{(1)}(t+\theta_1)\right) = \tfrac{1}{6}g_{uuu}(\xi_1)(V_1^{(1)})^3
$$
$$
+\tfrac{1}{2}g_{uu}(\overline{U}^{(1)}(t+\theta_1))(V_1^{(1)})^2
$$
$$
+g_u\left(\overline{U}^{(1)}(t+\theta_1)\right)V_1^{(1)},
$$
$$
g\left(V_2^{(1)}+\overline{U}^{(1)}(t+\theta_2)\right)-g\left(\overline{U}^{(1)}(t+\theta_2)\right) = \tfrac{1}{6}g_{uuu}(\xi_2)(V_2^{(1)})^3
$$
$$
+\tfrac{1}{2}g_{uu}(\overline{U}^{(1)}(t+\theta_2))(V_2^{(1)})^2
$$
$$
+g_u\left(\overline{U}^{(1)}(t+\theta_2)\right)V_2^{(1)},
$$
$$
(2.8.23)
$$

where ξ_1 is in between $\overline{U}^{(1)}(t+\theta_1)$ and $V_1^{(1)}+\overline{U}^{(1)}(t+\theta_1)$ and ξ_2 is in between $\overline{U}^{(1)}(t+\theta_2)$ and $\overline{U}^{(1)}(t+\theta_2)+V_2$. This allows us to collect all terms of the same order together and write

$$
d\mathcal{M} = dM_1 + dM_2 + dM_3, \qquad\qquad (2.8.24)
$$

where

$$
dM_1 = \tfrac{1}{c}\begin{pmatrix} \tfrac{1}{6}g_{uuu}(\xi_1)(V_1^{(1)})^3 - \tfrac{1}{6}g_{uuu}(\xi_2)(V_2^{(1)})^3 \\ 0 \end{pmatrix},
$$

$$
dM_2 = \tfrac{1}{c}\begin{pmatrix} \tfrac{1}{2}g_{uu}\left(\overline{U}^{(1)}(t+\theta_1)\right)(V_1^{(1)})^2 - \tfrac{1}{2}g_{uu}\left(\overline{U}^{(1)}(t+\theta_2)\right)(V_2^{(1)})^2 \\ 0 \end{pmatrix},
$$

$$
dM_3 = \tfrac{1}{c}\begin{pmatrix} \left[g_u\left(\overline{U}^{(1)}(t+\theta_1)\right)-g_u\left(\overline{U}^{(1)}(t+\tilde{\theta})\right)\right]V_1^{(1)} \\ 0 \end{pmatrix}
$$
$$
+\tfrac{1}{c}\begin{pmatrix} -\left[g_u\left(\overline{U}^{(1)}(t+\theta_2)\right)-g_u\left(\overline{U}^{(1)}(t+\tilde{\theta})\right)\right]V_2^{(1)} \\ 0 \end{pmatrix}.
$$
$$
(2.8.25)
$$

Note that $g_{uuu} = 6$ is constant. A Taylor expansion around $\overline{U}^{(1)}(t+\theta_1)$ yields the pointwise identity

$$
dM_2^{(1)} = \tfrac{1}{c}\left(\tfrac{1}{2}\left(g_{uu}(\overline{U}^{(1)}(t+\theta_1))-g_{uu}(\overline{U}^{(1)}(t+\theta_2))\right)(V_1^{(1)})^2\right)
$$
$$
+\tfrac{1}{c}\left(-\tfrac{1}{2}g_{uu}(\overline{U}^{(1)}(t+\theta_2))\left((V_2^{(1)})^2-(V_1^{(1)})^2\right)\right)
$$
$$
= \tfrac{1}{c}\left(3\left(\overline{U}^{(1)}(t+\theta_1)-\overline{U}^{(1)}(t+\theta_2)\right)(V_1^{(1)})^2\right)
$$
$$
(2.8.26)
$$
$$
+\tfrac{1}{c}\left(-\tfrac{1}{2}g_{uu}(\overline{U}^{(1)}(t+\theta_2))\left((V_2^{(1)})^2-(V_1^{(1)})^2\right)\right)
$$
$$
= \tfrac{1}{c}\left(3(\overline{U}^{(1)})'(\xi_3)(\theta_1-\theta_2)(V_1^{(1)})^2\right)
$$
$$
+\tfrac{1}{c}\left(-\tfrac{1}{2}g_{uu}(\overline{U}^{(1)}(t+\theta_2))\left((V_2^{(1)})^2-(V_1^{(1)})^2\right)\right),
$$

where ξ_3 is in between $t + \theta_1$ and $t + \theta_2$. Using Taylor expansions around $\overline{U}^{(1)}(t + \theta_2)$ and $\overline{U}^{(1)}(t + \tilde{\theta})$, we obtain the pointwise identity

$$
\begin{aligned}
dM_3^{(1)} &= \frac{1}{c}\Big[g_u\big(\overline{U}^{(1)}(t + \theta_1)\big) - g_u\big(\overline{U}^{(1)}(t + \tilde{\theta})\big)\Big]V_1^{(1)} \\
&\quad - \frac{1}{c}\Big[g_u\big(\overline{U}^{(1)}(t + \theta_2)\big) - g_u\big(\overline{U}^{(1)}(t + \tilde{\theta})\big)\Big]V_2^{(1)} \\
&= \frac{1}{c}\Big(\big[g_{uu}(\xi_4)(\theta_1 - \theta_2)\big]V_1^{(1)} - \big[g_{uu}(\xi_5)(\theta_2 - \tilde{\theta})\big]V_2^{(1)}\Big),
\end{aligned}
\tag{2.8.27}
$$

where ξ_4 is in between $\overline{U}^{(1)}(t + \theta_1)$ and $\overline{U}^{(1)}(t + \theta_2)$ and ξ_5 is in between $\overline{U}^{(1)}(t)$ and $\overline{U}^{(1)}(t + \theta_2)$. We estimate

$$
\begin{aligned}
\|d\mathcal{M}_1\|_{\ell^p} &\leq \frac{1}{|c|}\|V_1^3 - V_2^3\|_{\ell^p} \\
&\leq \frac{1}{|c|}\Big[\|V_1\|_{\ell^\infty}\|V_1^2 - V_2^2\|_{\ell^p} + \|V_1 - V_2\|_{\ell^\infty}\|V_2^2\|_{\ell^p}\Big] \\
&\leq \frac{1}{|c|}\Big[\|V_1\|_{\ell^p}\big[\|V_1\|_{\ell^p} + \|V_2\|_{\ell^p}\big]\|V_1 - V_2\|_{\ell^p} + \|V_1 - V_2\|_{\ell^p}\|V_2\|_{\ell^p}^2\Big],
\end{aligned}
\tag{2.8.28}
$$

together with

$$
\begin{aligned}
\|d\mathcal{M}_2\|_{\ell^p} &\leq \frac{1}{|c|}\Big[3\|(\overline{U}^{(1)})'(\xi_3)(\theta_1 - \theta_2)\|_\infty\|V_1\|_{\ell^p}^2 \\
&\quad + \big(6\|\overline{u}\|_\infty + 2r_0 + 2\big)\Big[\|V_1\|_{\ell^\infty}\|V_1 - V_2\|_{\ell^p} + \|V_1 - V_2\|_{\ell^\infty}\|V_2\|_{\ell^p}\Big]\Big] \\
&\leq \frac{1}{|c|}\Big[3\|\overline{u}'\|_\infty|\theta_1 - \theta_2|\|V_1\|_{\ell^p}^2 \\
&\quad + \big(6\|\overline{u}\|_\infty + 2r_0 + 2\big)\Big[\|V_1\|_{\ell^p} + \|V_2\|_{\ell^p}\Big]\|V_1 - V_2\|_{\ell^p}\Big]
\end{aligned}
\tag{2.8.29}
$$

and

$$
\begin{aligned}
\|d\mathcal{M}_3\|_{\ell^p} &\leq \frac{1}{|c|}\Big[\|g_{uu}(\xi_4)(\theta_1 - \theta_2)\|_\infty\|V_1\|_{\ell^p} + \|g_{uu}(\xi_5)(\theta_2 - \tilde{\theta})\|_\infty\|V_1 - V_2\|_{\ell^p}\Big] \\
&\leq \frac{1}{|c|}\Big[\big(6\|\overline{u}\|_\infty + 2r_0 + 2\big)|\theta_1 - \theta_2|\|V_1\|_{\ell^p} \\
&\quad + \big(6\|\overline{u}\|_\infty + 2r_0 + 2\big)|\theta_2 - \tilde{\theta}|\|V_1 - V_2\|_{\ell^p}\Big].
\end{aligned}
\tag{2.8.30}
$$

Combining these estimates yields

$$
\begin{aligned}
\|d\mathcal{M}\|_{\ell^p} &\leq C_2\Big[\|V_1\|_{\ell^p} + \|V_2\|_{\ell^p} + |\tilde{\theta} - \theta_1| + |\theta_2 - \tilde{\theta}|\Big]\|V_1 - V_2\|_{\ell^p} \\
&\quad + C_2|\tilde{\theta}_1 - \tilde{\theta}_2|\Big[\|V_1\|_{\ell^p} + \|V_2\|_{\ell^p}\Big].
\end{aligned}
\tag{2.8.31}
$$

$\blacksquare$

Lemma 2.8.4. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. Then there exists a constant $C_3 > 0$, independent of p, such that for $V \in \ell^p$ we have the bound*

$$
\|\tilde{\mathcal{G}}(t, t_0)V\|_{\ell^p} \leq C_3 e^{-\tilde{\beta}(t - t_0)}\|V\|_{\ell^p},
\tag{2.8.32}
$$

for all $t, t_0 \in \mathbb{R}$.

Proof. We let $f(t, t_0)_j = e^{-\tilde{\beta}|ct - ct_0 + hj|}$ and write $\tilde{V}_j = |V_j|$ for $V \in \ell^p$. Using Lemma 2.7.16 and Young's inequality we obtain

$$
\begin{aligned}
\|\tilde{\mathcal{G}}(t, t_0)V\|_{\ell^p} &\leq 8C e^{-\tilde{\beta}(t-t_0)} \|f(t, t_0) * \tilde{V}\|_{\ell^p(\mathbb{Z}, \mathbb{R})} \\
&\leq 8C e^{-\tilde{\beta}(t-t_0)} \|f(t, t_0)\|_{\ell^1(\mathbb{Z}, \mathbb{R})} \|\tilde{V}\|_{\ell^p(\mathbb{Z}, \mathbb{R})} \\
&= 8C e^{-\tilde{\beta}(t-t_0)} \|f(t, t_0)\|_{\ell^1(\mathbb{Z}, \mathbb{R})} \|V\|_{\ell^p} \\
&\leq C_3 e^{-\tilde{\beta}(t-t_0)} \|V\|_{\ell^p},
\end{aligned}
\tag{2.8.33}
$$

where

$$
C_3 = 8C \sup_{x \in [0]} \sum_{j \in \mathbb{Z}} e^{-\tilde{\beta}|hj + x|} < \infty.
\tag{2.8.34}
$$

Note that C_3 is independent of p. ∎

From the defining system (2.7.8), it is clear that for each $\tilde{\theta} \in \mathbb{R}$ the Green's function of the linearisation of (2.2.1) around the wave $\overline{U}(\cdot + \tilde{\theta})$ is given by $\mathcal{G}(t + \tilde{\theta}, t_0 + \tilde{\theta})$.

Fix $W_s \in \text{Range}\left(\Pi^s(\tilde{\theta})\right)$ and consider the fixed point problem

$$
\begin{aligned}
V(t) &= \tilde{\mathcal{G}}(t + \tilde{\theta}, \tilde{\theta})W_s + \int_0^t \tilde{\mathcal{G}}(t + \tilde{\theta}, t_0 + \tilde{\theta})\Pi^s(t_0 + \tilde{\theta})\mathcal{M}^{\tilde{\theta}}\left(\theta, t_0, V(t_0)\right) dt_0 \\
&\quad + \int_\infty^t \mathcal{E}(t + \tilde{\theta}, t_0 + \tilde{\theta})\Pi^c(t_0 + \tilde{\theta})\mathcal{M}^{\tilde{\theta}}\left(\theta, t_0, V(t_0)\right) dt_0.
\end{aligned}
\tag{2.8.35}
$$

We aim to construct decaying solutions to (2.8.14) by solving this fixed point problem in the space

$$
BC_{-\tilde{\beta}/2}([0, \infty), \ell^p) := \{V \in C([0, \infty), \ell^p) : \|V\|_{-\tilde{\beta}/2} < \infty\},
\tag{2.8.36}
$$

where

$$
\|V\|_{-\tilde{\beta}/2} = \sup_{\xi \in [0, \infty)} e^{\frac{\tilde{\beta}}{2}\xi} \|V(\xi)\|_{\ell^p}
\tag{2.8.37}
$$

Here the integrals are taken component-wise, but we see that for each $1 \leq p \leq \infty$ this component-wise integral corresponds to the Bochner integral.

Lemma 2.8.5. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. If the function* $V \in BC_{-\tilde{\beta}/2}([0, \infty), \ell^p)$ *satisfies the fixed point problem (2.8.35), then* V *satisfies (2.8.14) and, hence,* $V(t) + \overline{U}(t + \theta)$ *is a solution of (2.2.1).*

Proof. If $V(t)$ satisfies this fixed point problem then we see that

$$
\begin{aligned}
\tfrac{d}{dt}\Big(\tilde{\mathcal{G}}(t+\tilde{\theta},\tilde{\theta})W_s\Big) &= DF\Big(\overline{U}(t+\tilde{\theta})\Big)\tilde{\mathcal{G}}(t+\tilde{\theta},\tilde{\theta})W_s \\
&\quad + DF\Big(\overline{U}(t+\tilde{\theta})\Big)\mathcal{E}(t+\tilde{\theta},\tilde{\theta})\Pi^c(\tilde{\theta})W_s \\
&\quad - \tfrac{d}{dt}\Big(\mathcal{E}(t+\tilde{\theta},\tilde{\theta})\Pi^c(\tilde{\theta})W_s\Big) \\
&= DF\Big(\overline{U}(t+\tilde{\theta})\Big)\tilde{\mathcal{G}}(t+\tilde{\theta},0)W_s + 0 - 0 \\
&= DF\Big(\overline{U}(t+\tilde{\theta})\Big)\tilde{\mathcal{G}}(t+\tilde{\theta},\tilde{\theta})W_s.
\end{aligned}
\tag{2.8.38}
$$

Writing

$$
\mathcal{D}(t) = \tfrac{d}{dt}V(t) - \tfrac{d}{dt}\Big(\tilde{\mathcal{G}}(t+\tilde{\theta},\tilde{\theta})W_s\Big),
\tag{2.8.39}
$$

we can compute

$$
\begin{aligned}
\mathcal{D}(t) &= \int_0^t \tfrac{d}{dt}\Big[\tilde{\mathcal{G}}(t+\tilde{\theta},t_0+\tilde{\theta})\Pi^s(t_0+\tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta,t_0,V(t_0)\Big)\Big]dt_0 \\
&\quad + \int_\infty^t \tfrac{d}{dt}\Big[\mathcal{E}(t+\tilde{\theta},t_0+\tilde{\theta})\Pi^c(t_0+\tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta,t_0,V(t_0)\Big)\Big]dt_0 \\
&\quad + \tilde{\mathcal{G}}(t+\tilde{\theta},t+\tilde{\theta})\Pi^s(t+\tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta,t,V(t)\Big) \\
&\quad + \mathcal{E}(t+\tilde{\theta},t+\tilde{\theta})\Pi^c(t+\tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta,t,V(t)\Big) \\
&= \int_0^t \tfrac{d}{dt}\Big[\mathcal{G}(t+\tilde{\theta},t_0+\tilde{\theta})\Pi^s(t_0+\tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta,t_0,V(t_0)\Big)\Big]dt_0 \\
&\quad + \int_\infty^t \tfrac{d}{dt}\Big[\mathcal{G}(t+\tilde{\theta},t_0+\tilde{\theta})\Pi^c(t_0+\tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta,t_0,V(t_0)\Big)\Big]dt_0 \\
&\quad + \mathcal{G}(t+\tilde{\theta},t+\tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta,t,V(t)\Big).
\end{aligned}
\tag{2.8.40}
$$

Exploiting $\mathcal{G}(t+\tilde{\theta},t+\tilde{\theta}) = I$, this yields

$$
\begin{aligned}
\mathcal{D}(t) &= \int_0^t DF\Big(\overline{U}(t+\tilde{\theta})\Big)\mathcal{G}(t+\tilde{\theta},t_0+\tilde{\theta})\Pi^s(t_0+\tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta,t_0,V(t_0)\Big)dt_0 \\
&\quad + \int_\infty^t DF\Big(\overline{U}(t+\tilde{\theta})\Big)\mathcal{G}(t+\tilde{\theta},t_0+\tilde{\theta})\Pi^c(t_0+\tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta,t_0,V(t_0)\Big)dt_0 \\
&\quad + \mathcal{M}^{\tilde{\theta}}\Big(\theta,t,V(t)\Big) \\
&= DF\Big(\overline{U}(t+\tilde{\theta})\Big)\int_0^t \tilde{\mathcal{G}}(t+\tilde{\theta},t_0+\tilde{\theta})\Pi^s(t_0+\tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta,t_0,V(t_0)\Big)dt_0 \\
&\quad + DF\Big(\overline{U}(t+\tilde{\theta})\Big)\int_\infty^t \mathcal{E}(t+\tilde{\theta},t_0+\tilde{\theta})\Pi^c(t_0+\tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta,t_0,V(t_0)\Big)dt_0 \\
&\quad + \mathcal{M}^{\tilde{\theta}}\Big(\theta,t,V(t)\Big)
\end{aligned}
\tag{2.8.41}
$$

and thus

$$
\begin{aligned}
\tfrac{d}{dt}V(t) &= DF\Big(\overline{U}(t+\tilde{\theta})\Big)V(t) + \mathcal{M}^{\tilde{\theta}}\Big(\theta,t,V(t)\Big) \\
&= DF\Big(\overline{U}(t+\theta)\Big)V(t) + \mathcal{N}_\theta\Big(t,V(t)\Big).
\end{aligned}
\tag{2.8.42}
$$

■

Lemma 2.8.6. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. There exists $\eta > 0$, independent of p and $\tilde{\theta}$, so that for all $W_s \in \mathrm{Range}\Big(\Pi^s(\tilde{\theta})\Big)$ that have $\|W_s\|_{\ell^p} \leq \eta$ and all $|\theta - \tilde{\theta}| \leq \eta$, the fixed point problem (2.8.35) has a unique solution $\mathcal{W}^{\tilde{\theta}}_{*;\theta}(W_s)$ in the space $BC_{-\tilde{\beta}/2}([0, \infty), \ell^p)$.*

Proof. We first rewrite (2.8.35) as

$$V = T(W_s, V), \tag{2.8.43}$$

where

$$
\begin{aligned}
T(W_s, V) &= \tilde{\mathcal{G}}(t + \tilde{\theta}, \tilde{\theta})W_s + \int_0^t \tilde{\mathcal{G}}(t + \tilde{\theta}, t_0 + \tilde{\theta})\Pi^s(t_0 + \tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\tilde{\theta}, t_0, V(t_0)\Big)dt_0 \\
&\quad + \int_\infty^t \mathcal{E}(t + \tilde{\theta}, t_0 + \tilde{\theta})\Pi^c(t_0 + \tilde{\theta})\mathcal{M}^\theta\Big(\theta, t_0, V(t_0)\Big)dt_0.
\end{aligned}
\tag{2.8.44}
$$

Pick $V \in BC_{-\tilde{\beta}/2}([0, \infty), \ell^p)$ with $\|V\|_{-\tilde{\beta}/2} \leq 1$. Writing

$$\mathcal{I} = \int_0^t \tilde{\mathcal{G}}(t + \tilde{\theta}, t_0 + \tilde{\theta})\Pi^s(t_0 + \tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta, t_0, V(t_0)\Big)dt_0, \tag{2.8.45}$$

Lemma 2.8.3 and Lemma 2.8.4 imply

$$
\begin{aligned}
\|\mathcal{I}\|_{\ell^p} &\leq \int_0^t \|\tilde{\mathcal{G}}(t + \tilde{\theta}, t_0 + \tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta, t_0, V(t_0)\Big)\|_{\ell^p}\, dt_0 \\
&\leq \int_0^t C_3 e^{-\tilde{\beta}(t - t_0)}\|\mathcal{M}^{\tilde{\theta}}\Big(\theta, t_0, V(t_0)\Big)\|_{\ell^p}\, dt_0 \\
&\leq \int_0^t C_3 e^{-\tilde{\beta}(t - t_0)}C_2\|V(t_0)\|_{\ell^p}\Big[\|V(t_0)\|_{\ell^p} + |\theta - \tilde{\theta}|\Big]dt_0 \\
&\leq \int_0^t C_3 e^{-\tilde{\beta}(t - t_0)}C_2\|V\|_{-\tilde{\beta}/2}e^{-\tilde{\beta}t_0/2}\Big[e^{-\tilde{\beta}t_0/2}\|V\|_{-\tilde{\beta}/2} + |\theta - \tilde{\theta}|\Big]dt_0 \\
&\leq C_3 C_2\|V\|_{-\tilde{\beta}/2}\Big[te^{-\tilde{\beta}t}\|V\|_{-\tilde{\beta}/2} + \tfrac{2}{\tilde{\beta}}e^{-\tilde{\beta}t/2}|\theta - \tilde{\theta}|\Big].
\end{aligned}
\tag{2.8.46}
$$

Observe that if we multiply this final function with $e^{\tilde{\beta}t/2}$ we still have a bounded function. Since this holds for all p we see that

$$\|\mathcal{I}\|_{-\tilde{\beta}/2} \leq C_4\|V\|_{-\tilde{\beta}/2}\Big[\|V\|_{-\tilde{\beta}/2} + |\theta - \tilde{\theta}|\Big] \tag{2.8.47}$$

for some constant C_4, which is independent of p.

We write

$$\mathcal{J}(t) = \int_t^\infty \mathcal{E}(t + \tilde{\theta}, t_0 + \tilde{\theta})\Pi^c(t_0 + \tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta, t_0, V(t_0)\Big)dt_0. \tag{2.8.48}$$

Mimicking the computation above and using the explicit expression (2.8.3), we see that

$$
\begin{aligned}
\|\mathcal{J}(t)\|_{\ell^p} &\leq \int_\infty^t C_1 C_2\|V\|_{-\tilde{\beta}/2}e^{-\tilde{\beta}t_0/2}\Big[e^{-\tilde{\beta}t_0/2}\|V\|_{-\tilde{\beta}/2} + |\theta - \tilde{\theta}|\Big]dt_0 \\
&= C_1 C_2\|V\|_{-\tilde{\beta}/2}\Big[\tfrac{1}{\tilde{\beta}}e^{-\tilde{\beta}t}\|V\|_{-\tilde{\beta}/2} + \tfrac{2}{\tilde{\beta}}e^{-\tilde{\beta}t/2}|\theta - \tilde{\theta}|\Big].
\end{aligned}
\tag{2.8.49}
$$

Observe that if we multiply this final function with $e^{\frac{\tilde{\beta}}{2}t}$ we still have a bounded function. Since this holds for all p we see that

$$\|\mathcal{J}\|_{-\tilde{\beta}/2} \quad \leq \quad C_5\|V\|_{-\tilde{\beta}/2}\Big[\|V\|_{-\tilde{\beta}/2} + |\theta - \tilde{\theta}|\Big] \tag{2.8.50}$$

for some constant C_5, which is independent of p.

Finally, Lemma 2.8.4 yields the bound

$$\|\tilde{\mathcal{G}}(t+\tilde{\theta}, \tilde{\theta})W_s\|_{\ell^p} \quad \leq \quad C_3 e^{-\tilde{\beta}t}\|W_s\|_{\ell^p}, \tag{2.8.51}$$

which means

$$\|\tilde{\mathcal{G}}(t+\tilde{\theta}, \tilde{\theta})W_s\|_{-\tilde{\beta}/2} \quad \leq \quad C_3\|W_s\|_{\ell^p}. \tag{2.8.52}$$

This yields the bound

$$\|T(W_s, V)\|_{-\tilde{\beta}/2} \quad \leq \quad C_3\|W_s\|_{\ell^p} + (C_4 + C_5)\|V\|_{-\tilde{\beta}/2}\Big[\|V\|_{-\tilde{\beta}/2} + |\theta - \tilde{\theta}|\Big]. \tag{2.8.53}$$

Let $V_1 \in BC_{-\tilde{\beta}/2}([0, \infty), \ell^p)$ and $V_2 \in BC_{-\tilde{\beta}/2}([0, \infty), \ell^p)$ with $\|V_1\|_{-\tilde{\beta}/2} \leq 1$ and $\|V_2\|_{-\tilde{\beta}/2} \leq 1$. Again, we write

$$d\mathcal{M} \quad = \quad \mathcal{M}^{\tilde{\theta}}\Big(\theta, t, V_1(t)\Big) - \mathcal{M}^{\tilde{\theta}}\Big(\theta, t, V_2(t)\Big). \tag{2.8.54}$$

Using Lemma 2.8.3 it follows that

$$\|d\mathcal{M}\|_{\ell^p} \quad \leq \quad C_2\Big[\|V_1\|_{\ell^p} + \|V_2\|_{\ell^p} + |\theta - \tilde{\theta}|\Big]\|V_1 - V_2\|_{\ell^p}. \tag{2.8.55}$$

Mimicking the above computations, this yields

$$\|T(W_s, V_1) - T(W_s, V_2)\|_{-\tilde{\beta}/2} \quad \leq \quad (C_4 + C_5)\|V_1 - V_2\|_{-\tilde{\beta}/2}\Big[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2} + |\theta - \tilde{\theta}|\Big]. \tag{2.8.56}$$

We now fix

$$\delta \quad = \quad \min\{1, \tfrac{1}{4(C_4+C_5)}\} \tag{2.8.57}$$

and

$$\eta \quad = \quad \min\{\tfrac{1}{4(C_4+C_5)}, \tfrac{\delta}{4C_3}\}. \tag{2.8.58}$$

For each $V \in BC_{-\tilde{\beta}/2}([0, \infty), \ell^p)$, $V_1 \in BC_{-\tilde{\beta}/2}([0, \infty), \ell^p)$ and $V_2 \in BC_{-\tilde{\beta}/2}([0, \infty), \ell^p)$ with $\|V\|_{-\tilde{\beta}/2} \leq \delta$, $\|V_1\|_{-\tilde{\beta}/2} \leq \delta$ and $\|V_2\|_{-\tilde{\beta}/2} \leq \delta$, each $\theta \in \mathbb{R}$ with $|\theta - \tilde{\theta}| < \eta$ and each $W_s \in \ell^p$ with $\|W_s\|_{\ell^p} < \eta$, we now obtain

$$\begin{aligned} \|T(W_s, V)\|_{-\tilde{\beta}/2} \quad &\leq \quad \tfrac{\delta}{4} + \delta\Big[\tfrac{1}{4} + \tfrac{1}{4}\Big] \\ &\leq \quad \delta \end{aligned} \tag{2.8.59}$$

and

$$\|T(W_s, V_1) - T(W_s, V_2)\|_{-\tilde{\beta}/2} \quad \leq \quad \Big[\tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4}\Big]\|V_1 - V_2\|_{-\tilde{\beta}/2}. \tag{2.8.60}$$

Hence we see that the equation (2.8.35) has a unique solution $\mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)$. ∎

Lemma 2.8.7. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. For each $W_s \in \mathrm{Range}\left(\Pi^s(\tilde{\theta})\right)$ with $\|W_s\|_{\ell^p} \leq \eta$ and each $|\theta - \tilde{\theta}| \leq \eta$ we have $\Pi^s(\tilde{\theta})\mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)(0) = W_s$ and $\|\Pi^c(\tilde{\theta})\mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)(0)\|_{\ell^p} \leq C_6\|W_s\|_{\ell^p}^2$ for some constant $C_6 > 0$, which is independent of p and $\tilde{\theta}$.*

Proof. It is clear that

$$
\begin{aligned}
\Pi^s(\tilde{\theta})\tilde{\mathcal{G}}(\tilde{\theta}, t_0 + \tilde{\theta}) &= \tilde{\mathcal{G}}(\tilde{\theta}, \tilde{\theta} + t_0), \\
\Pi^c(\tilde{\theta})\mathcal{E}(\tilde{\theta}, \tilde{\theta} + t_0) &= \mathcal{E}(\tilde{\theta}, t_0 + \tilde{\theta}).
\end{aligned}
\tag{2.8.61}
$$

This allows us to compute

$$
\begin{aligned}
\Pi^s(\tilde{\theta})\mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)(0) &= \Pi^s(\tilde{\theta})\tilde{\mathcal{G}}(\tilde{\theta}, \tilde{\theta})W_s \\
&\quad + \Pi^s(\tilde{\theta}) \int_0^0 \tilde{\mathcal{G}}(\tilde{\theta}, t_0 + \tilde{\theta})\Pi^s(t_0 + \tilde{\theta})\mathcal{M}^{\tilde{\theta}}\left(\theta, t_0, \mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)(t_0)\right)dt_0 \\
&\quad + \Pi^s(\tilde{\theta}) \int_\infty^0 \mathcal{E}(\tilde{\theta}, t_0 + \tilde{\theta})\Pi^c(t_0 + \tilde{\theta})\mathcal{M}^{\tilde{\theta}}\left(\theta, t_0, \mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)(t_0)\right)dt_0 \\
&= \Pi^s(\tilde{\theta})W_s \\
&\quad + \int_\infty^0 \Pi^s(\tilde{\theta})\mathcal{E}(\tilde{\theta}, t_0 + \tilde{\theta})\Pi^c(t_0 + \tilde{\theta})\mathcal{M}^{\tilde{\theta}}\left(\theta, t_0, \mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)(t_0)\right)dt_0 \\
&= W_s + 0 \\
&= W_s,
\end{aligned}
\tag{2.8.62}
$$

together with

$$
\begin{aligned}
\Pi^c(\tilde{\theta})\mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)(0) &= \Pi^c(\tilde{\theta})\tilde{\mathcal{G}}(\tilde{\theta}, \tilde{\theta})W_s \\
&\quad + \Pi^c(\tilde{\theta}) \int_0^0 \tilde{\mathcal{G}}(\tilde{\theta}, t_0 + \tilde{\theta})\Pi^s(t_0 + \tilde{\theta})\mathcal{M}^{\tilde{\theta}}\left(\theta, t_0, \mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)(t_0)\right)dt_0 \\
&\quad + \Pi^c(\tilde{\theta}) \int_\infty^0 \mathcal{E}(\tilde{\theta}, t_0 + \tilde{\theta})\Pi^c(t_0 + \tilde{\theta})\mathcal{M}^{\tilde{\theta}}\left(\theta, t_0, \mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)(t_0)\right)dt_0 \\
&= \int_\infty^0 \mathcal{E}(\tilde{\theta}, t_0 + \tilde{\theta})\Pi^c(t_0 + \tilde{\theta})\mathcal{M}^{\tilde{\theta}}\left(\theta, t_0, \mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)(t_0)\right)dt_0.
\end{aligned}
\tag{2.8.63}
$$

We assume without loss of generality that η is small enough to ensure

$$
(C_4 + C_5)\|\mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)\|_{-\tilde{\beta}/2}\left[\|\mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)\|_{-\tilde{\beta}/2} + |\theta - \tilde{\theta}|\right] \leq \tfrac{1}{2}\|\mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)\|_{-\tilde{\beta}/2}.
\tag{2.8.64}
$$

Using (2.8.53), we obtain

$$\|\int_\infty^0 \mathcal{E}(\tilde\theta, t_0 + \tilde\theta)\Pi^c(t_0 + \tilde\theta)\mathcal{M}^{\tilde\theta}\Big(\theta, t_0, \mathcal{W}^{\tilde\theta}_{*;\theta}(W_s)(t_0)\Big)dt_0\|_{\ell^p} \begin{array}{l} \leq \quad C_5\Big[\||\mathcal{W}^{\tilde\theta}_{*;\theta}(W_s)\||_{-\tilde\beta/2} \\ \qquad\qquad + |\theta - \tilde\theta|\Big] \\ \leq \quad 4C_5 C_3^2 \|W_s\|_{\ell^p}^2 \\ := \quad C_6 \|W_s\|_{\ell^p}^2. \end{array}$$

(2.8.65)

This yields the desired estimate

$$\|\Pi^c(\tilde\theta)\mathcal{W}^{\tilde\theta}_{*;\theta}(W_s)\|_{\ell^p} \quad \leq \quad C_6 \|W_s\|_{\ell^p}^2. \tag{2.8.66}$$

∎

Exploiting Lemma 2.7.8 and Lemma 2.7.9, we pick $C_7 > 0$ in such a way that

$$|\Phi^\pm(\xi)| + |(\Phi^\pm)'(\xi)| \quad \leq \quad C_7 e^{-\tilde\beta|\xi|} \tag{2.8.67}$$

holds for all $\xi \in \mathbb{R}$, decreasing $\tilde\beta$ if necessary.

Lemma 2.8.8. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. Then there exists a constant $C_9 > 0$, independent of p such that for each $\theta \in \mathbb{R}$ we have the bound*

$$\|\Phi^+(h \cdot +\theta)\|_{\ell^p} \quad \leq \quad C_9. \tag{2.8.68}$$

In addition, for each $\theta \in \mathbb{R}$ and each sequence $\{\xi(j)\}$ with $\|\xi_j\|_\infty \leq 1$, we have the bound

$$\|(\Phi^-)'(h \cdot +\xi(\cdot))\|_{\ell^1} \quad \leq \quad C_9. \tag{2.8.69}$$

Proof. Note that for each $k \in \mathbb{Z}$ we have

$$\begin{array}{rl} \|\Phi^+(h \cdot +\theta_1)\|_{\ell^p}^p &= \quad \sum_{j\in\mathbb{Z}} |\Phi^+(hj + \theta_1)|^p \\ &= \quad \sum_{j\in\mathbb{Z}} |\Phi^+\Big(hj + hk + (\theta_1 - hk)\Big)|^p \\ &= \quad \sum_{j\in\mathbb{Z}} |\Phi^+\Big(hj + (\theta_1 - hk)\Big)|^p \\ &= \quad \|\Phi^+\Big(h \cdot +(\theta_1 - hk)\Big)\|_{\ell^p}^p. \end{array}$$

(2.8.70)

Hence we assume without loss of generality that $|\theta_1| \leq 1$. We see with (2.8.67) that there is a constant $C_8 > 0$ such that

$$\begin{array}{rl} \|\Phi^+(h \cdot +c\theta_1)\|_{\ell^p} &\leq \quad C_8 \|e^{-\tilde\beta|h\cdot+\theta_1|}\||_{\ell^p(\mathbb{Z},\mathbb{R})} \\ &\leq \quad C_8 \|e^{-\tilde\beta|h\cdot|}e^{\tilde\beta|\theta_1|}\|_{\ell^p(\mathbb{Z},\mathbb{R})} \\ &\leq \quad C_8 e^{\tilde\beta|\theta_1|} \|e^{-\tilde\beta p|h\cdot|}\|_{\ell^1(\mathbb{Z},\mathbb{R})}^{\frac{1}{p}} \\ &\leq \quad 2C_8 \max\{1, \|e^{-\tilde\beta|h\cdot|}\|_{\ell^1(\mathbb{Z},\mathbb{R})}\} \\ &:= \quad C_9, \end{array}$$

(2.8.71)

if we assume that $\tilde{\beta}$ is small enough such that $e^{\tilde{\beta}|\theta_1|} \leq e^{\tilde{\beta}} \leq 2$. A similar calculation yields

$$
\begin{aligned}
\|(\Phi^-)'(h \cdot + \xi(\cdot))\|_{\ell^1} &\leq C_8 \|e^{-\tilde{\beta}|h \cdot + \xi(\cdot)|}\|_{\ell^1(\mathbb{Z},\mathbb{R})} \\
&\leq C_8 \|e^{-\tilde{\beta}|h \cdot|} e^{\tilde{\beta}}\|_{\ell^1(\mathbb{Z},\mathbb{R})} \\
&\leq C_8 e^{\tilde{\beta}} \|e^{-\tilde{\beta}|h \cdot|}\|_{\ell^1(\mathbb{Z},\mathbb{R})} \\
&\leq 2C_8 \max\{1, \|e^{-\tilde{\beta}|h \cdot|}\|_{\ell^1(\mathbb{Z},\mathbb{R})}\} \\
&= C_9.
\end{aligned} \tag{2.8.72}
$$

∎

We have

$$
(\Pi^c(\theta)V)_j = \frac{1}{\Omega} \sum_{j_0 \in \mathbb{Z}} \langle \Phi^-(hj_0 + c\theta), V_{j_0} \rangle \Phi^+(hj + c\tilde{\theta}). \tag{2.8.73}
$$

For notational compactness we write

$$
\Pi^c(\theta)V = \overline{\lambda}^c(\theta)(V)\Phi^+(h \cdot + c\theta). \tag{2.8.74}
$$

Lemma 2.8.9. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. For $V \in \ell^p$ and $\theta_1, \theta_2 \in \mathbb{R}$ with $|\theta_2 - \theta_1| \leq 1$ we have the bounds*

$$
\|\Pi^c(\theta_1)\Pi^s(\theta_2)V\|_{\ell^p} \leq C_{10}|\theta_1 - \theta_2|\|V\|_{\ell^p} \tag{2.8.75}
$$

and

$$
\|\Pi^s(\theta_1)\Pi^c(\theta_2)V\|_{\ell^p} \leq C_{10}|\theta_1 - \theta_2|\|V\|_{\ell^p} \tag{2.8.76}
$$

for some constant $C_{10} > 0$ which does not depend on p.

Proof. Writing

$$
\mathcal{P} = \overline{\lambda}^c(\theta_1)\Big(\Pi^s(\theta_2)V\Big), \tag{2.8.77}
$$

we obtain

$$
\begin{aligned}
\|\mathcal{P}\|_{\ell^p} &= \Big|\frac{1}{\Omega} \sum_{j_0 \in \mathbb{Z}} \langle \Phi^-(hj_0 + c\theta_1), (\Pi^s(\theta_2)V)_{j_0} \rangle\Big| \\
&= \Omega^{-1} \Big| \sum_{j_0 \in \mathbb{Z}} \langle \Phi^-(hj_0 + c\theta_1) - \Phi^-(hj_0 + c\theta_2), (\Pi^s(\theta_2)V)_{j_0} \rangle \Big| \\
&\leq \Omega^{-1} \sum_{j_0 \in \mathbb{Z}} |\Phi^-(hj_0 + c\theta_1) - \Phi^-(hj_0 + c\theta_2)| |(\Pi^s(\theta_2)V)_{j_0}| \\
&\leq \Omega^{-1} \|\Phi^-(h \cdot + c\theta_1) - \Phi^-(h \cdot + c\theta_2)\|_{\ell^1} \|\Pi^s(\theta_2)V\|_{\ell^\infty} \\
&\leq \Omega^{-1} |\theta_1 - \theta_2| \|c(\Phi^-)'(h \cdot + \xi(\cdot))\|_{\ell^1} \|\Pi^s(\theta_2)V\|_{\ell^p},
\end{aligned} \tag{2.8.78}
$$

where each $\xi(j)$ is in between $c\theta_1$ and $c\theta_2$. Thus we obtain with Lemma 2.8.4 and Lemma 2.8.8

$$
\begin{aligned}
\|\Pi^c(\theta_1)\Pi^s(\theta_2)V\|_{\ell^p} &\leq \tfrac{1}{\Omega}C_9|c||\theta_1 - \theta_2|\|(\Pi^s(\theta_2)V)\|_{\ell^p}C_9 \\
&\leq \tfrac{1}{\Omega}C_9|c||\theta_1 - \theta_2|C_3\|V\|_{\ell^p}C_9 \\
&\leq \tfrac{1}{2}C_{10}|\theta_1 - \theta_2|\|V\|_{\ell^p}
\end{aligned} \tag{2.8.79}
$$

for some constant $C_{10} > 0$ which is independent of p.

Furthermore we can compute

$$
\begin{aligned}
\Pi^s(\theta_1)\Pi^c(\theta_2)V &= \left[I - \Pi^c(\theta_1)\right]\left[I - \Pi^s(\theta_2)\right]V \\
&= V - \Pi^c(\theta_1)V - \Pi^s(\theta_2)V + \Pi^c(\theta_1)\Pi^s(\theta_2)V \qquad (2.8.80) \\
&= -\Pi^c(\theta_1)V + \Pi^c(\theta_2)V + \Pi^c(\theta_1)\Pi^s(\theta_2)V.
\end{aligned}
$$

This allows us to estimate

$$
\begin{aligned}
\|-\Pi^c(\theta_1)V + \Pi^c(\theta_2)V\|_{\ell^p} &\leq |\overline{\lambda}^c(\theta_1)(V) - \overline{\lambda}^c(\theta_2)(V)| \|\Phi^+(\theta_1)\|_{\ell^p} \\
&\quad + |\lambda(\theta_2)| \|\Phi^+(\theta_1) - \Phi^+(\theta_2)\|_{\ell^p} \\
&\leq C_9|\theta_1 - \theta_2|C_9\|V\|_{\ell^p} \\
&\quad + C_9|\theta_2| |\theta_1 - \theta_2| \|(\Phi^-)'(h \cdot + \eta(\cdot))\|_{\ell^1}\|V\|_{\ell^p} \\
&\leq C_9|\theta_1 - \theta_2|C_9\|V\|_{\ell^p} + C_9|\theta_2||\theta_1 - \theta_2|C_9\|V\|_{\ell^p} \\
&\leq \tfrac{1}{2}C_{10}|\theta_1 - \theta_2|\|V\|_{\ell^p},
\end{aligned}
$$
$$(2.8.81)$$

where each $\eta(j)$ is between $c\theta_1$ and $c\theta_2$. We thus obtain

$$
\|\Pi^s(\theta_1)\Pi^c(\theta_2)V\|_{\ell^p} \quad \leq \quad C_{10}|\theta_1 - \theta_2|\|V\|_{\ell^p}. \qquad (2.8.82)
$$

∎

Proof of Proposition 2.8.1. We set

$$
\mathcal{U}_*^{\tilde{\theta}}(W_s) \quad = \quad \mathcal{W}_{*;\tilde{\theta}}^{\tilde{\theta}}(W_s) \qquad (2.8.83)
$$

for all $\tilde{\theta} \in \mathbb{R}$.

Fix $\tilde{\theta} \in \mathbb{R}$ and pick $\theta \in \mathbb{R}$ with $|\theta - \tilde{\theta}| \leq \eta$ and pick $W_s \in \mathrm{Range}(\Pi^s(\theta))$ with $\|W_s\|_{\ell^p} \leq \eta$. By uniqueness of the solution to (2.8.35) it follows that

$$
\mathcal{U}_*^{\theta}(W_s) \quad = \quad \mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s^0) \qquad (2.8.84)
$$

for some $W_s^0 \in \mathrm{Range}\left(\Pi^s(\tilde{\theta})\right)$. Since $\Pi^s(\tilde{\theta})\mathcal{W}_{*;\theta}^{\tilde{\theta}}(\cdot)(0)$ is the identity map on $\mathrm{Range}\left(\Pi^s(\tilde{\theta})\right)$, it follows that

$$
W_s^0 \quad = \quad \Pi^s(\tilde{\theta})\mathcal{U}_*^{\theta}(W_s)(0). \qquad (2.8.85)
$$

We now see

$$
\begin{aligned}
W_s^0 - W_s &= \Pi^s(\tilde{\theta})\mathcal{U}_*^{\theta}(W_s)(0) - W_s \\
&= \Pi^s(\tilde{\theta})\left[\mathcal{U}_*^{\theta}(W_s)(0) - W_s\right] + \Pi^s(\tilde{\theta})W_s - W_s \\
&= \Pi^s(\tilde{\theta})\Pi^c(\theta)\mathcal{U}_*^{\theta}(W_s)(0) + W_s - \Pi^c(\tilde{\theta})W_s - W_s \\
&= \Pi^s(\tilde{\theta})\Pi^c(\theta)\mathcal{U}_*^{\theta}(W_s)(0) - \Pi^c(\tilde{\theta})\Pi^s(\theta)W_s.
\end{aligned}
$$
$$(2.8.86)$$

Lemma 2.8.9 hence implies

$$
\begin{aligned}
\|W_s^0 - W_s\|_{\ell^p} &\leq C_{10}|\tilde{\theta} - \theta|\|\mathcal{U}_*^\theta(W_s)(0)\|_{\ell^p} + C_{10}|\tilde{\theta} - \theta|\|W_s\|_{\ell^p} \\
&\leq C_{10}|\tilde{\theta} - \theta|\Big[\|W_s\|_{\ell^p} + C_6\|W_s\|_{\ell^p}^2\Big] + C_{10}|\tilde{\theta} - \theta|\|W_s\|_{\ell^p} \quad (2.8.87) \\
&\leq C_{11}|\tilde{\theta} - \theta|\|W_s\|_{\ell^p}.
\end{aligned}
$$

Now fix $W_s^{\tilde{\theta}} \in \text{Range}\Big(\Pi^s(\tilde{\theta})\Big)$ with $\|W_s^{\tilde{\theta}}\|_{\ell^p} \leq \eta$. Then we can compute

$$
\begin{aligned}
\Pi^c(\theta)\mathcal{U}_*^\theta(W_s)(0) - \Pi^c(\tilde{\theta})\mathcal{U}_*^{\tilde{\theta}}(W_s^{\tilde{\theta}})(0) &= \mathcal{U}_*^\theta(W_s)(0) - W_s - \Pi^c(\tilde{\theta})\mathcal{W}_{*;\tilde{\theta}}^{\tilde{\theta}}(W_s^{\tilde{\theta}}) \\
&= \mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s^0)(0) - W_s - \mathcal{W}_{*;\tilde{\theta}}^{\tilde{\theta}}(W_s^{\tilde{\theta}})(0) + W_s^{\tilde{\theta}} \\
&= W_s^0 - W_s + \mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s^0)(0) \\
&\quad -W_s^0 - \mathcal{W}_{*;\tilde{\theta}}^{\tilde{\theta}}(W_s^{\tilde{\theta}})(0) + W_s^{\tilde{\theta}}.
\end{aligned}
$$
$$(2.8.88)$$

Writing

$$
\begin{aligned}
V_1 &= \mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s^0), \\
V_2 &= \mathcal{W}_{*;\tilde{\theta}}^{\tilde{\theta}}(W_s^{\tilde{\theta}}),
\end{aligned}
\quad (2.8.89)
$$

we can mimic the steps in (2.8.56) to obtain the estimate

$$
\begin{aligned}
\|V_1(0) - W_s^0 - V_2(0) + W_s^{\tilde{\theta}}\|_{\ell^p} &\leq C_{12}\Big[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2} + |\tilde{\theta} - \theta|\Big]\|V_1 - V_2\|_{-\tilde{\beta}/2} \\
&\quad + C_{12}|\tilde{\theta} - \theta|\Big[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2}\Big]
\end{aligned}
$$
$$(2.8.90)$$

for some constant $C_{12} > 0$, which is independent of p and $\tilde{\theta}$. Without loss of generality we can assume that η is small enough to ensure

$$
C_{12}\Big[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2} + |\tilde{\theta} - \theta|\Big] \leq \tfrac{1}{2}. \quad (2.8.91)
$$

An estimate similar to (2.8.56) therefore yields

$$
\begin{aligned}
\|V_1 - V_2\|_{-\tilde{\beta}/2} &\leq C_3\|W_s^0 - W_s^{\tilde{\theta}}\|_{\ell^p} \\
&\quad + C_{12}\Big[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2} + |\tilde{\theta} - \theta|\Big]\|V_1 - V_2\|_{-\tilde{\beta}/2} \\
&\quad + C_{12}|\tilde{\theta} - \theta|\Big[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2}\Big] \\
&\leq C_3\|W_s - W_s^{\tilde{\theta}}\|_{\ell^p} + C_3\|W_s - W_s^0\|_{\ell^p} + \tfrac{1}{2}\|V_1 - V_2\|_{-\tilde{\beta}/2} \\
&\quad + C_{12}|\tilde{\theta} - \theta|\Big[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2}\Big] \\
&\leq C_3\|W_s - W_s^{\tilde{\theta}}\|_{\ell^p} + C_3 C_{11}|\tilde{\theta} - \theta|\|W_s\|_{\ell^p} + \tfrac{1}{2}\|V_1 - V_2\|_{-\tilde{\beta}/2} \\
&\quad + C_{12}|\tilde{\theta} - \theta|\Big[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2}\Big],
\end{aligned}
$$
$$(2.8.92)$$

and thus

$$\begin{aligned}
\|V_1 - V_2\|_{-\tilde{\beta}/2} \leq{} & 2C_3\|W_s - W_s^{\tilde{\theta}}\|_{\ell^p} + 2C_3 C_{11}|\tilde{\theta} - \theta|\|W_s\|_{\ell^p} \\
& + 2C_{12}|\tilde{\theta} - \theta|\Big[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2}\Big].
\end{aligned} \tag{2.8.93}$$

Exploiting (2.8.65), this yields

$$\begin{aligned}
dP \; :={} & \|\Pi^c(\theta)\mathcal{U}_*^\theta(W_s)(0) - \Pi^c(\tilde{\theta})\mathcal{U}_*^{\tilde{\theta}}(W_s^{\tilde{\theta}})(0)\|_{\ell^p} \\
\leq{} & C_{11}|\tilde{\theta} - \theta|\|W_s\|_{\ell^p} + C_{12}\Big[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2} + |\tilde{\theta} - \theta|\Big]\|V_1 - V_2\|_{-\tilde{\beta}/2} \\
& + C_{12}|\tilde{\theta} - \theta|\Big[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2}\Big] \\
\leq{} & C_{13}\Big[\|W_s\|_{\ell^p} + \|W_s^{\tilde{\theta}}\|_{\ell^p} + |\tilde{\theta} - \theta|\Big]\|W_s - W_s^{\tilde{\theta}}\|_{\ell^p} \\
& + C_{13}|\tilde{\theta} - \theta|\Big[\|W_s\|_{\ell^p} + \|W_s^{\tilde{\theta}}\|_{\ell^p}\Big]
\end{aligned} \tag{2.8.94}$$

for some constant C_{13}, which is independent of p and $\tilde{\theta}$. ∎

We now expand upon the ideas developed in the second half of [109, Prop. 2.1] to foliate the state space surrounding the travelling pulses $\overline{U}(\cdot + \tilde{\theta})$ by the stable manifolds constructed above. We proceed by showing that these stable manifolds depend continuously on $\tilde{\theta}$. This allows us to set up an appropriate fixed point problem to establish Proposition 2.8.2.

We write

$$\overline{U}(\tilde{\theta}) \;=\; \overline{U}(0) - \tilde{\theta}\overline{U}'(0) + \mathcal{N}_1^{\tilde{\theta}}. \tag{2.8.95}$$

Lemma 2.8.10. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. Then we have the bounds*

$$\|\mathcal{N}_1^{\tilde{\theta}}\|_{\ell^p} \;\leq\; C_{14}\tilde{\theta}^2 \tag{2.8.96}$$

and

$$\|\mathcal{N}_1^{\tilde{\theta}_2} - \mathcal{N}_1^{\tilde{\theta}_1}\|_{\ell^p} \;\leq\; C_{15}(|\tilde{\theta}_1| + |\tilde{\theta}_2|)|\tilde{\theta}_1 - \tilde{\theta}_2|, \tag{2.8.97}$$

for $\tilde{\theta}, \tilde{\theta}_1, \tilde{\theta}_2 \in [-\eta, \eta]$ and for some constants $C_{14} > 0$ and $C_{15} > 0$, which do not depend on p.

Proof. Using Lemma 2.7.9 we see that there exists a sequence $\{\xi_j\}$ with $|\xi_j| \leq |\tilde{\theta}|$ such that

$$\begin{aligned}
\|\mathcal{N}_1^{\tilde{\theta}}\|_{\ell^p} &= \tfrac{1}{2}\|\{\tilde{\theta}^2\overline{U}''(\xi_j)\}\|_{\ell^p} \\
&\leq \tfrac{1}{2}C_7\tilde{\theta}^2\|\{e^{-\tilde{\beta}|hj + c\xi_j|}\}\|_p \\
&\leq \tfrac{1}{2}C_7\tilde{\theta}^2 e^{\tilde{\beta}|c\tilde{\theta}|}\|\{e^{-\tilde{\beta}|hj|}\}\|_p \\
&\leq C_7\tilde{\theta}^2\|\{e^{-\tilde{\beta}|hj|}\}\|_p \\
&:= C_{14}\tilde{\theta}^2,
\end{aligned} \tag{2.8.98}$$

where C_{14} does not depend on p as before. We can hence write

$$\overline{U}(\tilde{\theta}) \quad = \quad \overline{U}(0) - \tilde{\theta}\overline{U}'(0) + \mathcal{N}_1^{\tilde{\theta}} \tag{2.8.99}$$

with

$$\|\mathcal{N}_1^{\tilde{\theta}}\|_{\ell^p} \quad \leq \quad C_{14}\tilde{\theta}^2. \tag{2.8.100}$$

Furthermore, using Lemma 2.7.9, we see that we can find sequences $\{\xi_j\}$ and $\{\eta_j\}$ with ξ_j between $hj + c\tilde{\theta}_1$ and $hj + c\tilde{\theta}_2$ and η_j between hj and $hj + c\tilde{\theta}_1$ so that

$$
\begin{aligned}
\|\mathcal{N}_1^{\tilde{\theta}_2} - \mathcal{N}_1^{\tilde{\theta}_1}\|_{\ell^p} \quad &= \quad \|\overline{U}(\tilde{\theta}_1) + \tilde{\theta}_1\overline{U}'(0) - \overline{U}(\tilde{\theta}_2) - \tilde{\theta}_2\overline{U}'(0)\|_{\ell^p} \\
&\leq \quad |\tilde{\theta}_1 - \tilde{\theta}_2|^2\|\{\overline{U}''(\xi_j)\}\|_{\ell^p} + |\tilde{\theta}_1 - \tilde{\theta}_2|\|\overline{U}'(0) - \overline{U}'(\tilde{\theta}_1)\|_{\ell^p} \\
&\leq \quad |\tilde{\theta}_1 - \tilde{\theta}_2|^2\|\{\overline{U}''(\xi_j)\}\|_{\ell^p} + |\tilde{\theta}_1 - \tilde{\theta}_2||\tilde{\theta}_1|\|\{\overline{U}''(\eta_j)\}\|_{\ell^p} \\
&\leq \quad C_{15}(|\tilde{\theta}_1| + |\tilde{\theta}_2|)|\tilde{\theta}_1 - \tilde{\theta}_2|,
\end{aligned}
\tag{2.8.101}
$$

similarly to the calculations from Lemma 2.8.3. ∎

We write

$$\mathcal{N}_2^{\tilde{\theta}}(W) \quad = \quad \mathcal{U}_*^{\tilde{\theta}}(\Pi^s(\tilde{\theta})W)(0) - \Pi^s(\tilde{\theta})W \tag{2.8.102}$$

for $W \in \text{Range}(\Pi^s(0))$ with $\|\Pi^s(\tilde{\theta})W\|_{\ell^p} < \eta$. We note that Lemma 2.8.4 implies that this inequality holds if $\|W\|_{\ell^p} < \frac{\eta}{C_3}$.

Lemma 2.8.11. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. Recall the constants C_6 and C_{13} appearing in Proposition 2.8.1 and Lemma 2.8.4. Then for any $\tilde{\theta} \in [-\eta, \eta]$ and $W \in \text{Range}\left(\Pi^s(0)\right)$ with $\|W\|_{\ell^p} < \frac{\eta}{C_3}$ we have the bound*

$$\|\mathcal{N}_2^{\tilde{\theta}}(W)\|_{\ell^p} \quad \leq \quad C_6\|V\|_{\ell^p}^2. \tag{2.8.103}$$

In addition, for any $\tilde{\theta}_1, \tilde{\theta}_2 \in [-\eta, \eta]$ and $W_1, W_2 \in \text{Range}\left(\Pi^s(0)\right)$ with $\|W_1\|_{\ell^p} < \frac{\eta}{C_3}$ and $\|W_2\|_{\ell^p} < \frac{\eta}{C_3}$, we have

$$
\begin{aligned}
\|\mathcal{N}_2^{\tilde{\theta}_2}(W_2) - \mathcal{N}_2^{\tilde{\theta}_1}(W_1)\|_{\ell^p} \quad &\leq \quad C_{13}\Big[\|W_1\|_{\ell^p} + \|W_2\|_{\ell^p} + |\theta_1 - \theta_2|\Big]\|W_1 - W_2\|_{\ell^p} \\
&\quad + C_{13}|\theta_1 - \theta_2|\Big[\|W_1\|_{\ell^p} + \|W_2\|_{\ell^p}\Big].
\end{aligned}
\tag{2.8.104}
$$

Proof. Note that

$$\Pi^s(\tilde{\theta})\mathcal{N}_2^{\tilde{\theta}}(W) = 0, \tag{2.8.105}$$

so that

$$
\begin{aligned}
\mathcal{N}_2^{\tilde{\theta}}(W) \quad &= \quad \Pi^c(\tilde{\theta})U_*^{\tilde{\theta}}(W)(0) - \Pi^c(\tilde{\theta})\Pi^s(\tilde{\theta})W \\
&= \quad \Pi^c(\tilde{\theta})U_*^{\tilde{\theta}}(W)(0).
\end{aligned}
\tag{2.8.106}
$$

Therefore, both bounds follow from Proposition 2.8.1. ∎

Let $\delta > 0$ be a small constant, which we will determine later. Pick U^0 in such a way that $\|U^0 - \overline{U}(0)\|_{\ell^p} < \delta$. We write $U^0 = \overline{U}(0) + V_0$.

Our goal is to find a small $W \in \text{Range}\big(\Pi^s(0)\big)$ and a small $\tilde{\theta}$ in such a way that

$$V^0 + \overline{U}(0) \quad = \quad \overline{U}(\tilde{\theta}) + U_*^{\tilde{\theta}}(\Pi^s(\tilde{\theta})W)(0). \tag{2.8.107}$$

Using our notation from above we see that

$$
\begin{aligned}
\overline{U}(\tilde{\theta}) + U_*^{\tilde{\theta}}(\Pi^s(\tilde{\theta})W)(0) \quad &= \quad \overline{U}(\tilde{\theta}) + \Pi^s(\tilde{\theta})W + \mathcal{N}_2^{\tilde{\theta}}(W) \\
&= \quad \overline{U}(0) + \tilde{\theta}\overline{U}'(0) + \mathcal{N}_1^{\tilde{\theta}} + \Pi^s(\tilde{\theta})W + \mathcal{N}_2^{\tilde{\theta}}(W) \\
&= \quad \overline{U}(0) + \tilde{\theta}\overline{U}'(0) + \mathcal{N}_1^{\tilde{\theta}} + W - \Pi^c(\tilde{\theta})W + \mathcal{N}_2^{\tilde{\theta}}(W),
\end{aligned}
\tag{2.8.108}
$$

which means that (2.8.107) can be written as

$$V^0 \quad = \quad \tilde{\theta}\overline{U}'(0) + \mathcal{N}_1^{\tilde{\theta}} + W - \Pi^c(\tilde{\theta})W + \mathcal{N}_2^{\tilde{\theta}}(W) \tag{2.8.109}$$

We write $\overline{\lambda}^c : \text{Range}(\Pi^c(0)) \to \mathbb{R}$ for the map $\mu\overline{U}' \mapsto \mu$. This allow us to rephrase (2.8.109) as the fixed point problem

$$
\begin{cases}
\Pi^s(0)V^0 \quad &= \quad \Pi^s(0)\mathcal{N}_1^{\tilde{\theta}} + W - \Pi^s(0)\Pi^c(\tilde{\theta})(W) + \Pi^s(0)\Big(\mathcal{N}_2^{\tilde{\theta}}(W)\Big) \\
\overline{\lambda}^c\Big[\Pi^c(0)V^0\Big] \quad &= \quad \tilde{\theta} + \overline{\lambda}^c\Big[\Pi^c(0)\mathcal{N}_1^{\tilde{\theta}}\Big] - \overline{\lambda}^c\Big[\Pi^c(0)\Pi^c(\tilde{\theta})(W)\Big] + \overline{\lambda}^c\Big[\Pi^c(0)\Big(\mathcal{N}_2^{\tilde{\theta}}(W)\Big)\Big].
\end{cases}
\tag{2.8.110}
$$

We show that equation (2.8.110) has a solution in the space

$$X_{\kappa, \varepsilon_\theta} \quad := \quad \{V \in \text{Range}(\Pi^s(0)) : \|V\|_{\ell^p} \le \kappa\} \times [-\varepsilon_\theta, \varepsilon_\theta] \tag{2.8.111}$$

for some $\kappa, \varepsilon_\theta$ which we will determine later. Without loss of generality we assume that $\kappa, \varepsilon_\theta$ are small enough such that all previous inequalities hold.

Lemma 2.8.12. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. There are small constants $\delta > 0$ $\kappa > 0$ and $\varepsilon_\theta > 0$, independent of p, such that for each $V^0 \in \ell^p$ with $\|V^0\|_{\ell^p} < \delta$ the fixed point problem (2.8.110) has a unique solution $(W, \tilde{\theta}) \in X_{\kappa, \varepsilon_\theta}$. Moreover there is a constant $C_{19} > 0$ such that we have the bound*

$$\|W\|_{\ell^p} \quad \le \quad C_{19}\|V^0\|_{\ell^p}. \tag{2.8.112}$$

Proof. We show that the map

$$
\begin{aligned}
T : (W, \theta) \quad \mapsto \quad &\begin{pmatrix} \Pi^s(0)V^0 - \Pi^s(0)\mathcal{N}_1^{\tilde{\theta}} + \Pi^s(0)\Pi^c(\tilde{\theta})(W) - \Pi^s(0)(\mathcal{N}_2^{\tilde{\theta}}(W)) \\ \overline{\lambda}^c\Big[\Pi^c(0)V^0\Big] - \overline{\lambda}^c\Big[\Pi^c(0)\mathcal{N}_1^{\tilde{\theta}}\Big] \end{pmatrix} \\
&+ \begin{pmatrix} 0 \\ \overline{\lambda}^c\Big[\Pi^c(0)\Pi^c(\tilde{\theta})(W)\Big] - \overline{\lambda}^c\Big[\Pi^c(0)(\mathcal{N}_2^{\tilde{\theta}}(W)\Big] \end{pmatrix}
\end{aligned}
\tag{2.8.113}
$$

maps $X_{\kappa,\varepsilon_\theta}$ into $X_{\kappa,\varepsilon_\theta}$ and is a contraction. Recall the constant C_1 from (2.8.3). Note that

$$
\begin{aligned}
\Pi^c(0)\Pi^c(\tilde\theta)(W) &= \Pi^c(0)(W) - \Pi^c(0)\Pi^s(\tilde\theta)(W) \\
&= -\Pi^c(0)\Pi^s(\tilde\theta)(W)
\end{aligned}
\tag{2.8.114}
$$

We see, using Lemma 2.8.9, Lemma 2.8.10 and Lemma 2.8.11, and setting

$$
C_{16} = C_1 \sum_{j\in\mathbb{Z}} e^{-\tilde\beta|hj|},
\tag{2.8.115}
$$

that

$$
\begin{aligned}
\|T(W,\tilde\theta)^{(1)}\|_{\ell^p} &\leq (1+C_{16})\|V^0 - \mathcal{N}_1^{\tilde\theta} - \mathcal{N}_2^{\tilde\theta}(W)\|_{\ell^p} + C_{10}|\tilde\theta|\|W\|_{\ell^p} \\
&\leq (1+C_{16})\Big(\|V^0\|_{\ell^p} + C_{14}\tilde\theta^2 + C_6\|W\|_{\ell^p}^2\Big) + C_{10}|\tilde\theta|\|W\|_{\ell^p} \\
&\leq C_{17}\Big[\|V^0\|_{\ell^p} + \tilde\theta^2 + \|W\|_{\ell^p}^2 + |\tilde\theta|\|W\|_{\ell^p}\Big]
\end{aligned}
\tag{2.8.116}
$$

and

$$
\begin{aligned}
|T(W,\tilde\theta)^{(2)}| &\leq \frac{C_{16}(\|V^0\|_{\ell^p} + C_{14}\tilde\theta^2 + C_6\|W\|_{\ell^p}^2) + (1+C_{16})C_{10}|\tilde\theta|\|W\|_{\ell^p}}{\Omega} \\
&\leq C_{17}\Big[\|V^0\|_{\ell^p} + \tilde\theta^2 + \|W\|_{\ell^p}^2 + |\tilde\theta|\|W\|_{\ell^p}\Big]
\end{aligned}
\tag{2.8.117}
$$

for some constant $C_{17} > 0$, which is independent of p. Note that

$$
\begin{aligned}
\Pi^c(\tilde\theta_2)(W_2) - \Pi^c(\tilde\theta_1)(W_1) &= \Pi^c(\tilde\theta_2)W_2 - \Pi^c(\tilde\theta_1)W_2 + \Pi^c(\tilde\theta_1)(W_1 - W_2) \\
&= \Pi^c(\tilde\theta_2)W_2 - \Pi^c(\tilde\theta_1)W_2 \\
&\quad + \Pi^c(\tilde\theta_1)(W_1 - W_2) - \Pi^c(0)(W_1 - W_2).
\end{aligned}
\tag{2.8.118}
$$

Using Lemma 2.8.9, (2.8.81), Lemma 2.8.10 and Lemma 2.8.11 we obtain

$$
\begin{aligned}
\|T(W_1,\tilde\theta_1)^{(1)} - T(W_2,\tilde\theta_2)^{(1)}\|_{\ell^p} &\leq (1+C_{16})\Big(\|\mathcal{N}_1^{\tilde\theta_2} - \mathcal{N}_1^{\tilde\theta_1}\|_{\ell^p} \\
&\quad + \|\Pi^c(\tilde\theta_2)(W_2) - \Pi^c(\tilde\theta_1)(W_1)\|_{\ell^p} \\
&\quad + \|\mathcal{N}_2^{\tilde\theta_2}(W_2) - \mathcal{N}_2^{\tilde\theta_1}(W_1)\|_{\ell^p}\Big) \\
&\leq (1+C_{16})\Big(C_{15}(|\tilde\theta_1| + |\tilde\theta_2|)|\tilde\theta_2 - \tilde\theta_1| \\
&\quad + C_{10}|\tilde\theta_1 - \tilde\theta_2|\|W_2\| + C_{10}|\tilde\theta_1|\|W_1 - W_2\| \\
&\quad + C_{13}\Big[\|W_1\|_{\ell^p} + \|W_2\|_{\ell^p} + |\tilde\theta_1 - \tilde\theta_2|\Big]\|W_1 - W_2\|_{\ell^p} \\
&\quad + C_{13}|\tilde\theta_2 - \tilde\theta_1|\Big[\|W_1\|_{\ell^p} + \|W_2\|_{\ell^p}\Big]\Big) \\
&\leq C_{18}\Big[|\tilde\theta_1| + |\tilde\theta_2| + \|W_1\|_{\ell^p} + \|W_2\|_{\ell^p}\Big] \\
&\quad \times \Big[|\tilde\theta_1 - \tilde\theta_2| + \|W_1 - W_2\|_{\ell^p}\Big],
\end{aligned}
\tag{2.8.119}
$$

together with

$$|T(W_1, \tilde{\theta}_1)^{(2)} - T(W_2, \tilde{\theta}_2)^{(2)}| \leq \frac{C_{16}\|\mathcal{N}_1^{\tilde{\theta}_2} - \mathcal{N}_1^{\tilde{\theta}_1} + \Pi^c(\tilde{\theta}_2)(W_2) - \Pi^c(\tilde{\theta}_1)(W_1) + \mathcal{N}_2^{\tilde{\theta}_2}(W_2) - \mathcal{N}_2^{\tilde{\theta}_1}(W_1)\|_{\ell^p}}{\Omega}$$

$$\leq C_{18}\Big[|\tilde{\theta}_1| + |\tilde{\theta}_2| + \|W_1\|_{\ell^p} + \|W_2\|_{\ell^p}\Big]$$
$$\times \Big[|\tilde{\theta}_1 - \tilde{\theta}_2| + \|W_1 - W_2\|_{\ell^p}\Big]$$

(2.8.120)

for some constant $C_{18} > 0$, which is independent of p. First we let $0 < \kappa_{\max} < 1$ and $0 < \theta_{\max} < 1$ be constants such that all inequalities above hold for all $|\kappa| \leq \kappa_{\max}$ and all $|\theta| \leq \theta_{\max}$. In particular, we demand that $\kappa_{\max} < \eta$, $\kappa_{\max} < \frac{\eta}{C_3}$ and $\theta_{\max} < \eta$. Finally we write

$$\delta = \kappa = \varepsilon_\theta = \frac{1}{20}\min\{\kappa_{\max}, \theta_{\max}, \frac{1}{C_{17}}, \frac{1}{C_{18}}\} > 0. \qquad (2.8.121)$$

With these choices we obtain the estimate

$$\|T(W, \tilde{\theta})^{(1)}\|_{\ell^p} \leq C_{17}\Big[\|V^0\|_{\ell^p} + \tilde{\theta}^2 + \|W\|_{\ell^p}^2 + |\tilde{\theta}|\|W\|_{\ell^p}\Big]$$
$$\leq \frac{1}{20}\kappa + \frac{1}{20}\kappa + \frac{1}{20}\kappa + \frac{1}{20}\kappa \qquad (2.8.122)$$
$$\leq \frac{1}{2}\kappa.$$

Furthermore we see that

$$|T(W, \tilde{\theta})^{(2)}| \leq \frac{1}{2}\varepsilon_\theta. \qquad (2.8.123)$$

Hence we see that the map T indeed maps $X_{\kappa, \varepsilon_\theta}$ into $X_{\kappa, \varepsilon_\theta}$.

In addition, (2.8.119) implies

$$\|T(W_1, \tilde{\theta}_1)^{(1)} - T(W_2, \tilde{\theta}_2)^{(1)}\|_{\ell^p} \leq C_{18}\Big[|\tilde{\theta}_1| + |\tilde{\theta}_2| + \|W_1\|_{\ell^p} + \|W_2\|_{\ell^p}\Big]$$
$$\times \Big[|\tilde{\theta}_1 - \tilde{\theta}_2| + \|W_1 - W_2\|_{\ell^p}\Big]$$
$$\leq \frac{4}{20}|\tilde{\theta}_1 - \tilde{\theta}_2| + \frac{4}{20}\|W_1 - W_2\|_{\ell^p},$$

(2.8.124)

while (2.8.120) yields

$$|T(W_1, \tilde{\theta}_1)^{(2)} - T(W_2, \tilde{\theta}_2)^{(2)}| \leq \frac{4}{20}|\tilde{\theta}_1 - \tilde{\theta}_2| + \frac{4}{20}\|W_1 - W_2\|_{\ell^p}. \qquad (2.8.125)$$

Therefore the map T is a contraction and thus the fixed point problem (2.8.110) has a unique solution $(W, \tilde{\theta})$. Moreover we see that

$$\|(W, \tilde{\theta})\|_{\ell^p \times \mathbb{R}} \leq \|T(W, \tilde{\theta}) - T(0, 0)\|_{\ell^p \times \mathbb{R}} + \|T(0, 0)\|_{\ell^p \times \mathbb{R}}$$
$$\leq \frac{1}{2}\|(W, \tilde{\theta})\|_{\ell^p \times \mathbb{R}} + 2C_{17}\|V^0\|_{\ell^p}, \qquad (2.8.126)$$

which yields

$$\|(W, \tilde{\theta})\|_{\ell^p \times \mathbb{R}} \leq 4C_{17}\|V^0\|_{\ell^p}$$
$$= C_{19}\|V^0\|_{\ell^p} \qquad (2.8.127)$$

as desired. ∎

Proof of Proposition 2.8.2. If $(W, \tilde{\theta})$ satisfies (2.8.110) then we see from (2.8.108) that

$$
\begin{aligned}
U^0 &= \overline{U}(0) + V^0 \\
&= \overline{U}(\tilde{\theta}) + \mathcal{U}_*^{\tilde{\theta}}(W)(0),
\end{aligned}
\tag{2.8.128}
$$

as desired. ∎

Proof of Theorem 2.2.3. Let U be the solution of (2.2.1) with an initial condition $U(0) = U^0$ for which $\|U^0 - \overline{U}(0)\|_{\ell^p} < \delta$. By Proposition 2.8.2 and by uniqueness of the solution we see that

$$
U = \overline{U}(\tilde{\theta}) + \mathcal{U}_*^{\tilde{\theta}}(W)
\tag{2.8.129}
$$

for some small $\tilde{\theta} \in \mathbb{R}$ and $W \in \ell^p$ with

$$
\|W\|_{\ell^p} \leq C_{19}\|U^0 - \overline{U}(0)\|_{\ell^p}.
\tag{2.8.130}
$$

Hence we obtain

$$
\begin{aligned}
\|U(t) - \overline{U}(t + \tilde{\theta})\|_{\ell^p} &\leq e^{-\frac{\tilde{\beta}}{2}t} \|\mathcal{U}_*^{\tilde{\theta}}\left(\Pi^c(\tilde{\theta})(W)\right)\|_{-\tilde{\beta}/2} \\
&\leq 2C_3 e^{-\frac{\tilde{\beta}}{2}t} \|\Pi^c(\tilde{\theta})(W)\|_{\ell^p} \\
&\leq 2C_3 e^{-\frac{\tilde{\beta}}{2}t} C_{16} \|W\|_{\ell^p} \\
&\leq 2C_3 C_{16} C_{19} \|U^0 - \overline{U}(0)\|_{\ell^p} e^{-\frac{\tilde{\beta}}{2}t},
\end{aligned}
\tag{2.8.131}
$$

as desired. ∎