Patterns on spatially structured domains

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Chapter 1

Introduction

Many systems in nature have an underlying spatially discrete structure, which greatly influences their dynamical behaviour. Often, this broken translational and rotational symmetry gives rise to interesting and complex behaviour, which is not present for spatially homogeneous systems. For several systems, this spatially discrete structure is directly visible. For example, one can think of the movement of domain walls [53] or dislocations [35] through crystals. However, the spatially discrete structure can also be more hidden. In particular, let us consider the propagation of electrical signals through nerve fibres. It is well-known that these signals can only move at appropriate speeds if the nerve fibres are insulated by a meylin coating. This coating admits regularly spaced gaps at the so-called nodes of Ranvier [143], see Figure 1.1. The signal moves fast through these coated regions, but loses strength rapidly. In the nodes, the signal moves much slower, while it recovers strength. In particular, the signal appears to hop from one node to the next. This phenomenon is known as saltatory conduction [127].

In many of these processes in nature, the propagation of fixed structures through space and time plays a crucial role. As is the case for spatially continuous systems, travelling waves form the basic building blocks for the complex behaviour and patterns spatially discrete systems can exhibit. Travelling wave solutions have a fixed shape, called the *wave profile*, and travel through time and space with a fixed wavespeed. The propagation of electrical signals through nerve fibers is a key example of the significance of the study of travelling waves in spatially discrete systems.

In §1.1, we further highlight a few of these discrete systems and discuss the mathematical models that are used to describe their behaviour. In §1.2, we focus entirely on the FitzHugh-Nagumo system, which is used to model the signal propagation through nerve fibres and is the main equation under consideration in this thesis. Finally, we elaborate on the most important mathematical techniques that are used in the analysis of our systems in §1.3.

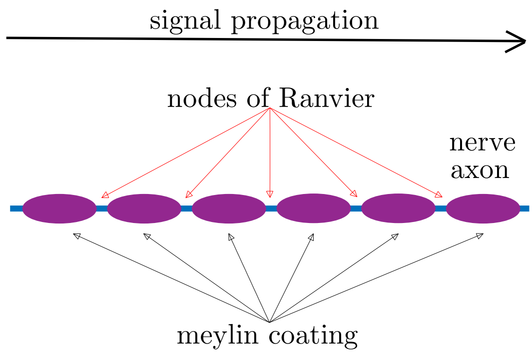


Figure 1.1: *Simplified representation of the meylin coating and nodes of Ranvier in a nerve axon.*

1.1 Scalar LDEs and MFDEs

For systems such as those discussed above, it is essential to incorporate the spatially discrete structure into the models that aim to describe their behaviour. For this purpose, *lattice differential equations* (LDEs) form a natural class of systems to model systems of this type. Indeed, let us consider an infinite chain of particles, indexed by the one-dimensional lattice $j \in \mathbb{Z}$. For the jth particle, we are interested how a specific quantity u_j, for example displacement or electrical potential, evolves in time. Let us assume, for now, that the rate of change of the quantity u_j is only influenced directly by itself and its nearest neighbours u_{j+1} and u_{j-1}. That is, the evolution of the variables u_j for $j \in \mathbb{Z}$ is given by the system of equations

$$\dot{u}_j(t) \quad = \quad f\big(u_{j+1}(t), u_{j-1}(t), u_j(t)\big) \tag{1.1.1}$$

for some function f. We note that the system (1.1.1) is, in fact, a collection of infinitely many, coupled ordinary differential equations (ODEs).

For systems such as (1.1.1), we are mainly interested in travelling wave solutions. Typically, this means that we aim to find a solution $\{u_j\}_{j\in\mathbb{Z}}$ to the system (1.1.1) that takes the form

$$u_j(t) \quad = \quad \overline{u}_0(j + \overline{c}_0 t), \tag{1.1.2}$$

where $\overline{u}_0$ is the wave profile and $\overline{c}_0$ is the wavespeed. Usually, an assumed shape of the solution, called an *Ansatz*, such as (1.1.2) is accompanied by boundary conditions of the form

$$\lim_{\xi\to-\infty} \overline{u}_0(\xi) \quad = \quad u^-, \qquad \lim_{\xi\to\infty} \overline{u}_0(\xi) \quad = \quad u^+. \tag{1.1.3}$$

If $u^- = u^+$ in (1.1.3), we often refer to the travelling wave as a *travelling pulse*, while otherwise it is known as a *travelling front*. In order to establish whether the system (1.1.1) admits a travelling wave solution, we need to substitute the Ansatz (1.1.2) into the LDE (1.1.1) and solve the resulting system. In particular, this procedure yields a so-called *functional differential equation of mixed type* (MFDE), which is given by

$$\overline{c}_0\overline{u}_0'(\xi) \quad = \quad f\big(\overline{u}_0(\xi+1), \overline{u}_0(\xi-1), \overline{u}_0(\xi)\big) \tag{1.1.4}$$

in which $\xi = j + \overline{c}_0 t$. The 'mixed type' in MFDE refers to the fact that it contains both advanced (forward) and retarded (backward) shifts. The MFDE (1.1.4) is called the *travelling wave equation* for the LDE (1.1.1).

LDEs form a relatively young field of interest for mathematicians. In the applied literature, however, LDEs have appeared significantly more frequently. For systems with an inherent discrete structure, LDEs can be seen as the natural replacement for partial differential equations (PDEs). LDEs can both arise as a discretisation of a PDE or as a system that has no direct spatially continuous equivalent. LDEs have been shown to display unexpected and complex dynamical behaviour. We will illustrate this behaviour with a few prominent examples.

1.1.1 The FPUT lattice

The Fermi-Pasta-Ulam-Tsingou (FPUT) lattice is an infinite chain of particles, which are coupled by identical springs to their neighbours. This system is a generalization of a system with finitely many particles, which was studied numerically in [49, 72]. The corresponding FPUT LDE aims to capture the dynamical behaviour of position of these particles. When the particles are identical, the lattice is called a monoatomic lattice. In this case, we can derive from Newton's second law that the FPUT LDE is given by

$$\ddot{u}_j \quad = \quad F(u_{j+1} - u_j) - F(u_j - u_{j-1}), \tag{1.1.5}$$

where the function F represents the spring force. The existence of solitary travelling wave solutions for the system (1.1.5), i.e. travelling wave solutions of which the wave profile decays exponentially, has been shown in [77–81].

When the particles are not identical, these solitary travelling wave solutions no longer capture the behaviour of the particles. In particular, let us consider the diatomic lattice, i.e. when the mass of the particles alternates between the two values 1 and $m \neq 1$, see Figure 1.2. The diatomic FPUT LDE has been studied in various parameter regimes, such as the small mass $m \ll 1$ regime [100], the equal mass $m \approx 1$ regime [66] and the long wave regime [67]. Travelling wave solutions for these systems are usually constructed as *perturbations* of travelling wave solutions for a monoatomic lattice. That it, the travelling wave solution is constructed as the sum of the monoatomic wave and another part, which is small in terms of the relevant parameter regime. For the small mass and long wave regimes, the solitary travelling waves are singularly perturbed into a travelling wave profile which asymptotes into a periodic solution with a very small amplitude. The amplitude of these "ripples" is small beyond all orders in the relevant parameter. This category of travelling waves is often referred to as nanopterons, see [21] for an interesting overview. For the near-equal mass regime, the travelling wave profile also asymptotes to a periodic solution, but the amplitude of this periodic solution is only algebraically small. Such a travelling wave profile is called a micropteron.

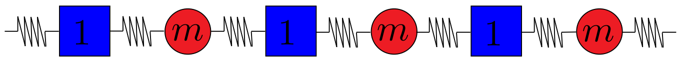


Figure 1.2: *Illustration of the diatomic FPUT lattice with alternating particles with masses* 1 *and* m*.*

1.1.2 The Nagumo equation

The Nagumo or Allen-Cahn PDE is given by

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + g(u(x,t);r). \tag{1.1.6}$$

Here the bistable nonlinearity g is, typically, given by the cubic polynomial $g(u;r) = u(1-u)(u-r)$ with $0 < r < 1$. The Nagumo PDE has been commonly used as a model where two biological species or material states compete in a spatial domain [3]. Due to its relative simplicity, the PDE (1.1.6) has served as a prototype to understand basic concepts in the theory of reaction-diffusion systems. This system is known to admit travelling front solutions of the form

$$u(x,t) = \Phi(x+ct), \qquad \lim_{\xi \to -\infty} \Phi(\xi) = 0, \qquad \lim_{\xi \to \infty} \Phi(\xi) = 1, \tag{1.1.7}$$

which can be constructed explicitly. This travelling front solution satisfies the travelling wave ODE

$$c\Phi'(\xi) = \Phi''(\xi) + g(\Phi(\xi);r). \tag{1.1.8}$$

In addition, there is a one-to-one correspondence between the wavespeed c and the parameter r. Due to the symmetry of the system, travelling waves are *pinned* for $r = \frac{1}{2}$, i.e. the wavespeed c is 0, while the waves move for $r \neq \frac{1}{2}$. It is well-known that these travelling wave solutions are stable under perturbations that do not need to be small [73].

The natural way to discretize the Nagumo PDE (1.1.6) is to consider the LDE

$$\dot{u}_j(t) = d\big[u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)\big] + g(u_j(t);r), \tag{1.1.9}$$

which we will refer to as the Nagumo LDE. There are many similarities and differences between the PDE (1.1.6) and the LDE (1.1.9). Although the LDE (1.1.9) is no longer explicitly solvable, it is well-known that it admits travelling wave solutions, which must, hence, satisfy the travelling wave MFDE

$$cu'(\xi) = d\big[u(\xi+1) + u(\xi-1) - 2u(\xi)\big] + g(u(\xi);r). \tag{1.1.10}$$

In addition, for a given value of $d > 0$ the wavespeed c is uniquely determined by the parameter r [39, 131]. Usually, the *comparison principle* is used to prove these types of results. The comparison principle states, informally, that a subsolution of an elliptic or parabolic equation stays below a supersolution. The comparison principle can be applied to both the PDE (1.1.6) and the LDE (1.1.9).

However, the relation between the wavespeed c and the parameter r is no longer one-to-one. In particular, when $d > 0$ is sufficiently small there is a nontrivial interval $r \in [r_-, r_+]$ for which the LDE (1.1.9) admits travelling wave solutions with wavespeed $c = 0$. This phenomenon is known as *propagation failure* and has been shown to be a common feature of discrete systems [99]. However, we do emphasize that for $c \neq 0$ the $r(c)$ relation remains single-valued.

There are many possible extensions and generalizations to the Nagumo LDE (1.1.9). Here, we will discuss a few results to showcase the rich behaviour of the Nagumo LDE. A more comprehensive overview can be found in [105].

Bichromatic waves. In contrast to the PDE (1.1.6), the LDE (1.1.9) has infinitely many equilibria. Let us consider equilibria of the form

$$u_j = \begin{cases} \overline{u}_e & \text{if } j \text{ is even,} \\ \overline{u}_o & \text{if } j \text{ is odd.} \end{cases} \tag{1.1.11}$$

Such a 2-periodic equilibrium must satisfy the system of equations

$$\begin{aligned} 0 &= 2d(\overline{u}_e - \overline{u}_o) + g(\overline{u}_o; r), \\ 0 &= 2d(\overline{u}_o - \overline{u}_e) + g(\overline{u}_e; r). \end{aligned} \tag{1.1.12}$$

For $d = 0$, the system (1.1.12) decouples and immediately yields the solutions $\overline{u}_e, \overline{u}_o \in \{0, r, 1\}$. In particular, the system (1.1.12) has 9 distinct solutions for $d = 0$. As such, using the implicit function theorem, we can continue these 9 solutions for sufficiently small $d > 0$ until these continuations start to intertwine. We say that a pair $(\overline{u}_e, \overline{u}_o)$ which satisfies (1.1.12) is of type $w \in \{\mathfrak{0}, \mathfrak{r}, \mathfrak{1}\}^2$ if it lies on the branch of the equilibrium w of (1.1.12) for $d = 0$. We are mainly interested in equilibria of type $w \in \{\mathfrak{0}, \mathfrak{1}\}^2$, since these equilibria are stable. In particular, let us write $u_{\mathfrak{01}}(r, d)$ for the solution of (1.1.12) of type $\mathfrak{01}$. Since the equilibrium $u_{\mathfrak{01}}(r, d)$ is stable, a so-called monotonic iteration scheme [39] can be used to show that the LDE (1.1.9) admits so-called *bichromotic waves*. That is, solutions of the form

$$u_j(t) = \begin{cases} \Phi_e(j + c_{\mathfrak{01}}(r, d)t), & \text{if } j \text{ is even,} \\ \Phi_o(j + c_{\mathfrak{01}}(r, d)t), & \text{if } j \text{ is odd} \end{cases} \tag{1.1.13}$$

with boundary conditions

$$\lim_{\xi \to -\infty} (\Phi_e, \Phi_0)(\xi) = (0, 0), \qquad \lim_{\xi \to -\infty} (\Phi_e, \Phi_0)(\xi) = u_{\mathfrak{01}}(r, d) \tag{1.1.14}$$

as long as $d > 0$ remains small enough. See also Figure 1.3. Let us write

$$d_{\mathtt{o1}}(r) \quad = \quad \sup\{d > 0 : \text{there exists an equilibrium of (1.1.12) of type } \mathtt{o1}\}, \tag{1.1.15}$$

so that bichromatic waves exist for $0 < d < d_{\mathtt{o1}}(r)$. A more interesting, and delicate, question is whether these bichromatic waves are pinned or if they are moving. In particular, let us write

$$d_{\mathtt{o1}}(r)^* \quad = \quad \sup\{d > 0 : c_{\mathtt{o1}}(r, d) = 0\}. \tag{1.1.16}$$

One of the main results of [106] is that, if $r \in (0, 1)$ is sufficiently far away from 0, we have the strict inequality

$$d_{\mathtt{o1}}(r)^* \quad < \quad d_{\mathtt{o1}}(r). \tag{1.1.17}$$

That is, the bichromatic wave is not pinned for values of d in the nontrivial interval $\big(d_{\mathtt{o1}}(r)^*, d_{\mathtt{o1}}(r)\big)$. Related results can be found in [107, 159, 160, 162].

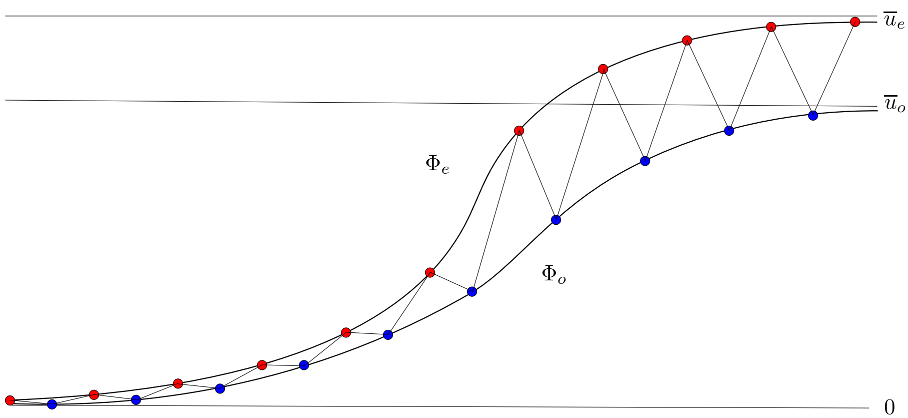


Figure 1.3: *This bichromatic wave with waveprofiles Φ_e and Φ_o connects the homogeneous state* 0 *to the heterogeneous state* $(\overline{u}_e, \overline{u}_0)$.

Infinite-range interactions. In [6], Bates, Chen and Chmaj considered a version of the Nagumo LDE (1.1.9) which features infinite-range interactions. This system is given by

$$u_j'(t) \quad = \quad d \sum_{k=1}^{\infty} \alpha_k \big[u_{j+k}(t) + u_{j-k}(t) - 2u_j(t)\big] + g(u_j(t); r). \tag{1.1.18}$$

Writing $d = \frac{1}{h^2}$, the system (1.1.18) can be seen as an infinite-range discretisation of the Nagumo PDE (1.1.6) on a grid with spacing $h > 0$. By using the Ansatz

$$u_j(t) \quad = \quad \overline{u}_h(hj + \overline{c}_h t), \tag{1.1.19}$$

the corresponding travelling wave equation is an MFDE which features infinite-range interactions and is given by

$$\overline{c}_h \overline{u}_h'(\xi) \quad = \quad \tfrac{1}{h^2} \sum_{k=1}^{\infty} \alpha_k \big[\overline{u}_h(\xi + kh) + \overline{u}_h(\xi - kh) - 2\overline{u}_h(\xi) \big] + g(\overline{u}_h(\xi); r). \tag{1.1.20}$$

In order to make sure the discretised Laplacian still behaves like a Laplacian, the authors impose the following limits on the grow of the coefficients $\{\alpha_k\}_{k \geq 1}$

$$\sum_{k=1}^{\infty} |\alpha_k| k^2 \quad < \quad \infty, \qquad \sum_{k=1}^{\infty} \alpha_k k^2 \quad = \quad 1, \tag{1.1.21}$$

together with the spectral bounds

$$\sum_{k=1}^{\infty} \alpha_k \cos(kz) \quad \geq \quad 0, \ \text{ for } z \in [0, 2\pi]. \tag{1.1.22}$$

In particular, upon defining the operator

$$(\Delta_h \phi)(\xi) \quad = \quad \tfrac{1}{h^2} \sum_{k=1}^{\infty} \alpha_k \big[\phi(\xi + kh) + \phi(\xi - kh) - 2\phi(\xi) \big], \tag{1.1.23}$$

we have the limit

$$\lim_{h \downarrow 0} \| \Delta_h \phi - \phi'' \|_{L^2(\mathbb{R};\mathbb{R})} \quad = \quad 0 \tag{1.1.24}$$

for sufficiently smooth and bounded functions ϕ as long as the conditions (1.1.21)-(1.1.22) hold. Note that not all coefficients $\{\alpha_k\}_{k \geq 1}$ need to be positive. In particular, the comparison principle is not necessarily available for the system (1.1.18).

Due to the limit (1.1.24), Bates, Chen and Chmaj aimed to find travelling wave solutions to the LDE (1.1.18) in the *near-continuum regime* $h \ll 1$. In particular, the authors constructed travelling waves for (1.1.18) as perturbations of the travelling waves for the PDE (1.1.6). However, the transition from the local second derivative operator to the nonlocal infinite-range difference operator is highly singular. To resolve this issue, the authors pioneered a method to lift certain properties of the continuous system to the spatially discrete system. We will refer to this method as the *spectral convergence method*. We will return to this method later in much more detail, as it plays an essential role in this thesis.

The fully discrete Nagumo equation Even though the spatial coordinate is discretised for LDEs, the temporal coordinate remains continuous. In [111], Hupkes and Van Vleck considered temporal discretisations of the Nagumo LDE (1.1.9), or, equivalently, spatial-temporal discretisations of the Nagumo PDE (1.1.6) in order to understand the impact of discretisation schemes on the solutions that these schemes aim to approximate. For the backward-Euler discretisation scheme, the corresponding evolution takes the form

$$\tfrac{1}{\Delta t}[U_j(n\Delta t) - U_j((n-1)\Delta t)] = d[U_{j+1} + U_{j-1} - 2U_j](n\Delta t) + g(U_j(n\Delta t); r), \qquad (1.1.25)$$

where we have $n \in \mathbb{Z}$ and $\Delta t > 0$ is called the *time-step*. Note that the system (1.1.25) is no longer a differential equation. The backward-Euler discretisation scheme is used because of several useful stability properties. This discretisation scheme is, in fact, the first of six so-called *backwards differentiation formula* (BDF) discretisation methods.

A travelling wave Ansatz for the system (1.1.25) with wavespeed c takes the form

$$U_j(n\Delta_t) = \Phi(j + nc\Delta t). \qquad (1.1.26)$$

Therefore, the corresponding travelling wave equation is given by

$$\tfrac{1}{\Delta t}\big[\Phi(\xi) - \Phi(\xi - c\Delta t)\big] = d\big[\Phi(\xi+1) + \Phi(\xi-1) - 2\Phi(\xi)\big] + g(\Phi(\xi); r). \qquad (1.1.27)$$

Something interesting should be noted: if $c\Delta t$ is a rational number, then the domain of the equation (1.1.27) can be restricted to a rational subset of the real line. This restriction turns out to be a key ingredient to construct travelling wave solutions to the system (1.1.25). Indeed, Hupkes and Van Vleck showed that, if $M := (c\Delta t)^{-1}$ is rational and sufficiently large, the system (1.1.25) admits travelling wave solutions. They employed the restriction on the domain of (1.1.27) to establish an interpolation scheme to link the system (1.1.27) to finitely many copies of the Nagumo MFDE (1.1.10). Then, the authors used the previously mentioned spectral convergence method to lift the Fredholm properties of this spatially discrete system to the fully discrete system.

There is an interesting nonuniqueness in the system (1.1.25). Indeed, the travelling wave profile is constructed as a perturbation of the restriction of the original, continuous wave profile Φ to the discrete domain. In particular, this means that for any irrational phase shift ϑ, the profile that is obtained by perturbing off $\Phi(\cdot + \vartheta)$ could potentially yield a different travelling wave solution to the system (1.1.25) with the same parameter values, see Figure 1.4. However, this phase shift might change the wavespeed. This nonuniqueness is not present for the Nagumo PDE (1.1.6) or LDE (1.1.9).

In addition, Hupkes and Van Vleck showed that for the backward-Euler discretisation scheme the previously mentioned $r(c)$ relation is multivalued, even for $c \neq 0$. This is in major contrast to the spatially discrete setting. In this part of the analysis, the authors relied heavily of the inclusion of the system (1.1.27) into an MFDE which admits a comparison principle. This is not possible for the other five BDF discretisation schemes. To alleviate this, Hupkes and Van Vleck also provided numerical evidence that the $r(c)$ relation is multivalued for at least the second BDF discretisation scheme.

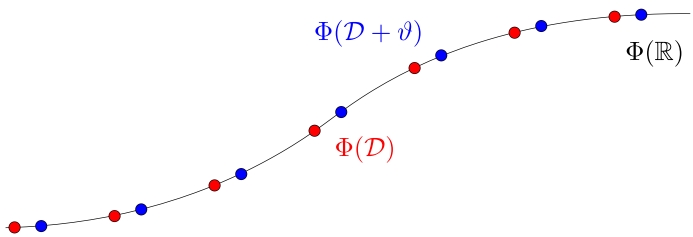


Figure 1.4: *The travelling wave profiles Φ and $\Phi(\cdot+\vartheta)$, defined on the domain $\mathcal{D}$, potentially yield two different travelling wave solutions to the system (1.1.25) for the same parameter values, but possible different wavespeed.*

1.2 The FitzHugh-Nagumo system

Let us return to the propagation of electrical signals through nerve fibres. Naturally, it is a challenge to find effective equations describing this behaviour. Initially, models describing this behaviour did not take the discrete structure into account directly. Based on experiments on giant squids, the first model was formulated in the 1950s and consists of a system of four nonlinear equations, called the Hodgkin-Huxley equations [98]. However, due to the high complexity of this system, an analytical approach to understand the dynamical behaviour of this system turned out to be a major challenge. Instead, in 1961, FitzHugh formulated a spatially homogeneous system to describe the potential felt by a single point on the nerve axon as the signal travels by [74]. A few years later, FitzHugh [76] and Nagumo [137] added a diffusion term to this system to describe the dynamics on the full line. Indeed, they formulated what is now known as the FitzHugh-Nagumo partial differential equation (PDE). This PDE is given by

$$\begin{array}{rcl} \frac{\partial u}{\partial t}(x,t) & = & \frac{\partial^2 u}{\partial x^2}(x,t) + g(u(x,t);r) - w(x,t), \\ \frac{\partial w}{\partial t}(x,t) & = & \rho\big[u(x,t) - \gamma w(x,t)\big]. \end{array} \tag{1.2.1}$$

In this system, the variable $u(x,t)$ describes the potential felt on the space point x at the time t, while $w(x,t)$ describes a recovery component. The Nagumo PDE (1.1.6) can be seen as a simplified version of the FitzHugh-Nagumo PDE (1.2.1). The bistable nonlinearity g is, as before, given by the cubic polynomial $g(u;r) = u(1-u)(u-r)$. In addition, $\rho > 0$ and $\gamma > 0$ are positive constants. As early as 1968 [75], FitzHugh released a computer simulation which clearly shows that the system (1.2.1) admits travelling pulse solutions, which resemble the spike signals found experimentally in the nerve axon of the giant squid by Hodgkin and Huxley. As such, the FitzHugh-Nagumo PDE is commonly used as a simplification of the Hodgkin-Huxley equations.

Mathematically, the FitzHugh-Nagumo PDE turned out to be a very interesting equation due to the combination of the relative simplicity of its structure with the rich behaviour of its dynamics. Indeed, the mathematical construction and analysis of the travelling pulse solutions as observed by FitzHugh turned out to be a major challenge that is still on-going. In particular, let us set out to find a solution (u, w) to the system (1.2.1) that takes the form

$$(u, w)(x, t) \quad = \quad (\overline{u}_0, \overline{w}_0)(x + \overline{c}_0 t), \tag{1.2.2}$$

where $\overline{u}_0$ and $\overline{w}_0$ are the wave profiles and $\overline{c}_0$ is the wavespeed. The wave profiles $\overline{u}_0$ and $\overline{w}_0$ must satisfy the limits

$$\lim_{\xi \to \pm\infty} (\overline{u}_0, \overline{w}_0)(\xi) \quad = \quad (0, 0) \tag{1.2.3}$$

to turn it into a pulse instead of merely a wave. We substitute the Ansatz (1.2.2) into the PDE (1.2.1) to obtain the ODE

$$\begin{aligned} \overline{c}_0 \overline{u}_0'(\xi) &= \overline{u}_0''(\xi) + g(\overline{u}_0(\xi); r) - \overline{w}_0(\xi), \\ \overline{c}_0 \overline{w}_0'(\xi) &= \rho\big[\overline{u}_0(\xi) - \gamma \overline{w}_0(\xi)\big], \end{aligned} \tag{1.2.4}$$

where $\xi = x + \overline{c}_0 t$. Travelling pulse solutions to the PDE (1.2.1) are homoclinic solutions to the ODE (1.2.4).

Typically, the system (1.2.4) has been studied in the $\rho \ll 1$ regime. Then $\rho \downarrow 0$ limit is singular, as substituting $\rho = 0$ in (1.2.4), effectively, yields a scalar equation, instead of a system of equations. Moreover, if we, instead, first rescale the variable ξ in (1.2.4) by ρ and then take the limit $\rho \downarrow 0$, we obtain a different limiting system. The first limiting system is called the fast limiting system, while the second is called the slow limiting system. As such, the system (1.2.4) is a so-called *fast-slow system*. The analysis of the system (1.2.4) in both $\rho \downarrow 0$ limits has led to the discovery of many new techniques in the field of singular perturbation theory. We refer to [118] for an interesting overview of these techniques. A recent overview of the existence and stability of pulse solutions for the PDE (1.2.1) can be found in [34]. Finally, we want to mention that, recently, several results have been developed [92–94] for the existence and nonlinear stability for pulse solutions of FitzHugh-Nagumo systems with added random noise.

However, all previously mentioned results feature the FitzHugh-Nagumo PDE (1.2.1). Since this equation is spatially homogeneous, it does not directly take the discrete properties of the nerve axon it is aiming to simulate, into account. As such, it has been proposed [123] to, instead, model the signal propagation through nerve fibres using a FitzHugh-Nagumo LDE, which is given by the system

$$\begin{aligned} u_j'(t) &= \tfrac{1}{h^2}\big[u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)\big] + g(u_j(t); r) - w_j(t), \\ w_j'(t) &= \rho\big[u_j(t) - \gamma w_j(t)\big]. \end{aligned} \tag{1.2.5}$$

The variables u_j and w_j now represent the potential felt and the recovery component at the jth node respectively. We note that the LDE (1.2.5) can be obtained directly from the PDE (1.2.1) by using a nearest neighbour discretisation of the Laplacian on a grid with spatial distance $h > 0$. A travelling pulse solution to the LDE (1.2.5) now takes the form

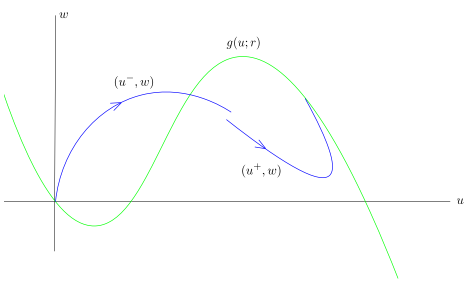
$$(u_j, w_j)(t) \quad = \quad (\overline{u}_h, \overline{w}_h)(hj + \overline{c}_h t) \tag{1.2.6}$$

for some wave profiles $\overline{u}_h$ and $\overline{w}_h$ and wavespeed $\overline{c}_h$. Substituting the Ansatz (1.2.6) into the LDE (1.2.5) yields the MFDE

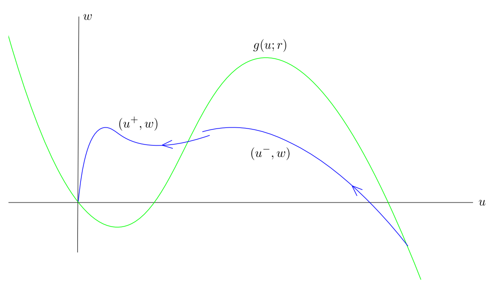
$$\begin{aligned} \overline{c}_h \overline{u}_h'(\xi) &= \tfrac{1}{h^2}\big[\overline{u}_h(\xi+h)+\overline{u}_h(\xi-h)-2\overline{u}_h(\xi)\big]+g(\overline{u}_h(\xi);r)-\overline{w}_h(\xi), \\ \overline{c}_h \overline{w}_h'(\xi) &= \rho\big[\overline{u}_h(\xi)-\gamma\overline{w}_h(\xi)\big], \end{aligned} \tag{1.2.7}$$

in which $\xi = hj + \overline{c}_0 t$. We emphasize that, in contrast to the Nagumo system, the FitzHugh-Nagumo PDE (1.2.1) and LDE (1.2.5) do not admit a comparison principle.

In [108, 109], Hupkes and Sandstede constructed travelling pulse solutions to the system (1.2.5) and showed that these pulses are nonlinearly stable. They assumed that they were in the parameter regime where the travelling front solution $\overline{u}$ for the corresponding Nagumo LDE (1.1.9) has nonzero wavespeed. The main idea in [108, 109] is to use what is known as Lin's method to combine the travelling front $\overline{u}$ and a reflection of this front $\overline{u}$ to obtain so-called quasi-front and quasi-back solutions, see Figure 1.5. These quasi-front and quasi-back solutions have gaps in predetermined finite dimensional spaces, which can be closed by choosing the wavespeed. The existence of these finite dimensional spaces hinges on the existence of exponential dichotomies for the linearization of the MFDE (1.1.10). Exponential dichotomies play an essential role in this thesis and will be discussed in more detail later, see §1.3.3.



(a)



(b)

Figure 1.5: *Quasi-front (a) and quasi-back (b) solutions of the FitzHugh-Nagumo LDE. In both cases, the function u^- is defined on the interval $(-\infty, 1]$, while the function u^+ is defined on the interval $[-1, \infty)$. The difference $u^+ - u^-$, which is defined on the overlapping interval $[-1, 1]$, should be an element of a predetermined finite dimensional space. The construction of such a space is provided by the existence of exponential dichotomies for linear MFDEs.*

Infinite-range interactions In this thesis, we consider several extensions and generalizations of the FitzHugh-Nagumo LDE (1.2.5). Our first model arises in the study of neural field models. Neural field models aim to describe the interactions and dynamics in large networks of neurons. These neurons interact with each other over large distances through the nerve fibres that connect them [15, 23, 24, 142]. Due to the high complexity of these systems, it is a major challenge to find effective equations to describe this dynamical behaviour. In [23, Eq. (3.31)], a model has been proposed that features a FitzHugh-Nagumo type system with infinite-range interactions, which takes the form

$$\begin{aligned} \dot{u}_j(t) &= \tfrac{1}{h^2}\sum_{k=1}^{\infty}\alpha_k\big[u_{j+k}(t)+u_{j-k}(t)-2u_j(t)\big]+g(u_j(t);r)-w_j(t), \\ \dot{w}_j(t) &= \rho\big[u_j(t)-\gamma w_j(t)\big]. \end{aligned} \tag{1.2.8}$$

Here the coefficients $\{\alpha_k\}_{k\geq 1}$ should, at the very least, satisfy the conditions (1.1.21)-(1.1.22) to ensure Laplace-like behaviour. The system (1.2.8) was first studied by Faye and Scheel in [69]. They constructed travelling pulse solutions to the system (1.2.8) under the assumption that the coefficients $\{\alpha_k\}_{k\geq 1}$ decay exponentially. Since, at the time of writing, exponential dichotomies for systems such as (1.2.8) were not available, Faye and Scheel were forced to use a different approach than the one employed by Hupkes and Sandstede for the finite-range version (1.2.5). Indeed, Faye and Scheel used a functional analytic approach to circumvent the use of a state space. However, they did not establish the stability of the pulse solutions they found. In Chapter 2, we expand the previously mentioned spectral convergence method to establish the existence and nonlinear stability of travelling pulse solutions to the system (1.2.8) in the near-continuum regime $h \ll 1$. The stability of pulse solutions outside the near-continuum regime remains an open problem. However, we expect that our results on the existence of exponential dichotomies for MFDEs with infinite-range interactions in Chapters 5-6 are a sufficient theoretical foundation to, eventually, solve this open problem.

Spatial periodicity Recent experiments in optical nanoscopy [50, 51, 165] clearly show that certain proteins in the cytoskeleton of nerve fibres are organised periodically. In particular, this periodicity manifests itself at the nodes of Ranvier. As such, it is natural to consider a spatially periodic version of the FitzHugh-Nagumo LDE (1.2.5). This spatially periodic LDE takes the form

$$\begin{aligned} \dot{u}_j(t) &= d_j\big[u_{j+1}(t)+u_{j-1}(t)-2u_j(t)\big]+g(u_j(t);r_j)-w_j(t), \\ \dot{w}_j(t) &= \rho_j\big[u_j(t)-\gamma_j w_j(t)\big], \end{aligned} \tag{1.2.9}$$

where the 2-periodic coefficients $(d_j, r_j, \rho_j, \gamma_j)$ satisfy

$$(0,\infty)\times(0,1)\times(0,1)\times(0,\infty)\ni(d_j,r_j,\rho_j,\gamma_j) = \begin{cases} (\varepsilon^{-2}, r_o, \rho_o, \gamma_o) & \text{for odd } j, \\ (1, r_e, \rho_e, \gamma_e) & \text{for even } j, \end{cases} \tag{1.2.10}$$

where $0 < \varepsilon \ll 1$. In particular, we have a scale separation between the diffusion coefficients 1 and ε^{-2}. The system (1.2.9) does not have a clear continuum limit. Nonetheless, we obtain the existence and nonlinear stability of travelling pulse solutions to the system (1.2.9) in the $\varepsilon \ll 1$ regime by further developing the spectral convergence method in Chapter 3.

Temporal discretisations Finally in Chapter 4, inspired by the work of Hupkes and Van Vleck [111] which was discussed previously, we consider temporal discretisations of the LDE (1.2.5), using the six BDF discretisation schemes. For the backward-Euler discretisation scheme, the corresponding evolution is given by

$$\begin{aligned} \tfrac{1}{\Delta t}[U_j(n\Delta t) - U_j((n-1)\Delta t)] &= d\big[U_{j+1} + U_{j-1} - 2U_j\big](n\Delta t) + g(U_j(n\Delta t); r) \\ &\quad -W_j(n\Delta t), \\ \tfrac{1}{\Delta t}[W_j(n\Delta t) - W_j((n-1)\Delta t)] &= \rho[U_j(n\Delta t) - \gamma W_j(n\Delta t)], \end{aligned} \tag{1.2.11}$$

for $n \in \mathbb{Z}$ and time-step $\Delta t > 0$. We establish the existence of travelling pulse solutions to the system (1.2.11) by carefully combining the different extensions to the spectral convergence method from [111] and Chapters 2-3. The nonuniqueness of this travelling wave solution, which was previously discussed for the Nagumo system (1.1.25), is present here as well due to the possibility of an irrational phase shift. In addition, we are interested in the $r(c)$ relation for the system (1.2.11). However, the analytical approach employed by Hupkes and Van Vleck for the Nagumo system (1.1.25) relied heavily on the comparison principle, which is not available for FitzHugh-Nagumo systems. Instead, we use numerical simulations to show that the $r(c)$ relation is multivalued for the system (1.2.11), even for $c \neq 0$. This is in major contrast to the FitzHugh-Nagumo PDE (1.2.1) and LDE (1.2.5).

1.3 Techniques

The main techniques to analyze our main systems (1.2.8), (1.2.9) and (1.2.11) fall into two main categories: those that feature the spectral convergence method and those that feature exponential dichotomies. Both of these techniques rely heavily on the Fredholm theory for linear MFDEs. In the remaining part of this chapter, we will discuss these techniques in more detail and explain how they can be applied to our main systems.

1.3.1 Linear Fredholm theory

In the construction and analysis of travelling waves, it is usually essential to understand the underlying linear system. Often, it is useful to consider the Fredholm properties of the corresponding linear operators. If X and Y are normed vector spaces, then we say that a linear operator $T : X \to Y$ is a *Fredholm operator* if the following properties are satisfied.

(i) The kernel satisfies $\dim\big(\ker(T)\big) < \infty$.

(ii) The range satisfies $\mathrm{codim}\big(\mathrm{Range}(T)\big) < \infty$.

(iii) The range $\mathrm{Range}(T)$ is closed.

When T is a Fredholm operator, the Fredholm index of T is given by

$$\mathrm{Ind}(T) = \dim\big(\ker(T)\big) - \mathrm{codim}\big(\mathrm{Range}(T)\big). \tag{1.3.1}$$

Let us now consider the linear MFDE given by

$$cu'(\xi) = d\big[u(\xi+1) + u(\xi-1) - 2u(\xi)\big] + g_u(\overline{u}(\xi);r)u(\xi), \tag{1.3.2}$$

which arises as the linearization of the Nagumo travelling wave MFDE (1.1.10) around a travelling wave solution $\overline{u}$. For clarity, we set $d = c = 1$. We rewrite this MFDE in the more suggestive form

$$u'(\xi) = u(\xi-1) + \big[g_u(\overline{u}(\xi);r) - 2\big]u(\xi+0) + u(\xi+1). \tag{1.3.3}$$

The scalar functions $1, g_u(\overline{u}(\xi);r)-2$ and 1 are called the *coefficients* of the systems and the real numbers $-1, 0$ and 1 are called the *shifts*. The linear operator corresponding to the system (1.3.3) is given by

$$(\Lambda u)(\xi) = u'(\xi) - u(\xi-1) - \big[g_u(\overline{u}(\xi);r) - 2\big]u(\xi+0) - u(\xi+1). \tag{1.3.4}$$

It is not immediately clear on which space the operator Λ from (1.3.4) is posed and how to determine the Fredholm properties of this operator. It turns out to be a natural choice to consider the Sobolev spaces

$$W^{1,p}(\mathbb{R};\mathbb{C}) = \{u \in L^p(\mathbb{R};\mathbb{C}) : u' \in L^p(\mathbb{R};\mathbb{C})\} \tag{1.3.5}$$

for $1 \le p \le \infty$, equipped with the Sobolev norm

$$\|u\|^p_{W^{1,p}(\mathbb{R};\mathbb{C})} = \|u\|^p_{L^p(\mathbb{R};\mathbb{C})} + \|u'\|^p_{L^p(\mathbb{R};\mathbb{C})}. \tag{1.3.6}$$

In this definition, we use u' to denote the weak derivative of a function u. For the space $W^{1,2}(\mathbb{R};\mathbb{C})$ we often use the shorthand $H^1(\mathbb{R};\mathbb{C})$. Using this definition, we view Λ from (1.3.4) as an operator

$$\Lambda : W^{1,p}(\mathbb{R};\mathbb{C}) \to L^p(\mathbb{R};\mathbb{C}). \tag{1.3.7}$$

The works by Rustichini [144, 145] and Mallet-Paret [130] contain the main Fredholm theory for this operator Λ. We recall that the travelling front $\overline{u}$ satisfies the limits

$$\lim_{\xi\to-\infty} \overline{u}(\xi) = 0, \qquad \lim_{\xi\to\infty} \overline{u}(\xi) = 1 \tag{1.3.8}$$

and that $g_u(0;r) = -r$ and $g_u(1;r) = r-1$. Hence, it is natural—and it will also turn out to be useful—to consider the systems

$$\begin{aligned} u'(\xi) &= u(\xi-1) + \lim_{\xi'\to-\infty}\big[g_u(\overline{u}(\xi');r) - 2\big]u(\xi+0) + u(\xi+1) \\ &= u(\xi-1) + \big[-r-2\big]u(\xi+0) + u(\xi+1) \end{aligned} \tag{1.3.9}$$

and

$$\begin{aligned} u'(\xi) &= u(\xi-1) + \lim_{\xi'\to\infty}\big[g_u(\overline{u}(\xi');r) - 2\big]u(\xi+0) + u(\xi+1) \\ &= u(\xi-1) + \big[r-1-2\big]u(\xi+0) + u(\xi+1). \end{aligned} \tag{1.3.10}$$

We refer to the systems (1.3.9) and (1.3.10) as the *limiting systems* of the MFDE (1.3.3) at $-\infty$ and ∞ respectively. We note that the systems (1.3.9) and (1.3.10) are autonomous, since their coefficients do not depend on ξ. Finding a solution to (1.3.9) or (1.3.10) of the form $e^{z\xi}w$ is equivalent to finding a root of the so-called *characteristic function*

$$\begin{aligned} \Delta_-(z) &= z - e^{-z} - \big[-r-2\big]e^{0\cdot z} - e^{z}, \\ \Delta_+(z) &= z - e^{-z} - \big[(r-1)-2\big]e^{0\cdot z} - e^{z}, \end{aligned} \tag{1.3.11}$$

that is, finding $z \in \mathbb{C}$ for which $\Delta_-(z) = 0$ and $\Delta_+(z) = 0$ respectively. Such a scalar z is referred to as a *spatial eigenvalue*. We say that the autonomous system (1.3.9) or (1.3.10) is *hyperbolic* if it has no spatial eigenvalues on the imaginary axis, i.e. $\Delta_-(iy) \neq 0$ respectively $\Delta_+(iy) \neq 0$ for all $y \in \mathbb{R}$. A short computation shows that this is, indeed, the case for the systems (1.3.9) and (1.3.10). Systems with this property are called *asymptotically hyperbolic*. We write Λ_- and Λ_+ for the linear operators corresponding to the systems (1.3.9) and (1.3.10) respectively, which are given by

$$\begin{aligned} (\Lambda_- u)(\xi) &= u'(\xi) - u(\xi-1) - \big[-r-2\big]u(\xi+0) - u(\xi+1), \\ (\Lambda_+ u)(\xi) &= u'(\xi) - u(\xi-1) - \big[(r-1)-2\big]u(\xi+0) - u(\xi+1). \end{aligned} \tag{1.3.12}$$

It turns out that, since the systems (1.3.9) and (1.3.10) are autonomous and hyperbolic, the operators Λ_- and Λ_+ are invertible as operators from $W^{1,p}(\mathbb{R};\mathbb{C})$ to $L^p(\mathbb{R};\mathbb{C})$, independently of $1 \leq p \leq \infty$. In fact, the inverse operators are given explicitly by the Green's function, in the sense that

$$(\Lambda_\pm^{-1} u)(\xi) = \int_{-\infty}^{\infty} G_\pm(\xi-\eta)u(\eta)d\eta, \tag{1.3.13}$$

where the Green's functions $G_\pm$ are given by

$$G_\pm(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\eta\xi}\big(\Delta_\pm(i\eta)\big)^{-1}d\eta. \tag{1.3.14}$$

A non-autonomous system, however, is not necessarily invertible. For example, the derivative $\overline{u}'$ is a kernel element of the system (1.3.3), which can be seen by differentiating the system (1.1.10). The results in [130] show that linear operators corresponding to asymptotically hyperbolic systems are automatically Fredholm operators as operators from $W^{1,p}(\mathbb{R};\mathbb{C})$ to $L^p(\mathbb{R};\mathbb{C})$. In addition, the Fredholm index of such an operator Λ is independent of $1 \leq p \leq \infty$ and the range of Λ can be made explicit by writing

$$\mathrm{Range}(\Lambda) = \big\{u \in L^p(\mathbb{R};\mathbb{C}) : \int_{-\infty}^{\infty} \overline{u(\xi)}v(\xi) = 0 \text{ for all } v \in \ker(\Lambda^*)\big\}. \tag{1.3.15}$$

Here we have introduced the *adjoint operator* Λ^*. The operator $\Lambda^* : W^{1,p}(\mathbb{R};\mathbb{C}) \to L^p(\mathbb{R};\mathbb{C})$ is given by

$$(\Lambda^* u)(\xi) = -u'(\xi) - u(\xi-(-1)) - \big[g_u(\overline{u}(\xi);r) - 2\big]u(\xi-0) - u(\xi-1) \tag{1.3.16}$$

and is called the adjoint operator because it satisfies the identity

$$\langle \Lambda u, v\rangle_{L^2(\mathbb{R};\mathbb{C})} = \langle u, \Lambda^* v\rangle_{L^2(\mathbb{R};\mathbb{C})} \tag{1.3.17}$$

for any pair $u, v \in H^1(\mathbb{R};\mathbb{C})$.

If there exists a homotopy between the systems at $-\infty$ and ∞ and none of the eigenvalues crosses the imaginary axis during this homotopy, then the spectral flow theorem [130, Thm. C] allows us to conclude that the Fredholm index of the corresponding linear operator is, in fact, 0. Such a homotopy is trivially available if the systems at $-\infty$ and ∞ coincide, for example when linearizing around a travelling pulse solution. However, even in such a setting it remains a nontrivial challenge to determine the dimension of the kernel of the linear operator, which is necessary, for example, when determining the spectrum of the linear operator. In that case, other techniques, such as appropriate limits or comparison principles are needed to understand these linear operators in full detail.

For the system (1.3.3), this homotopy can be made explicit. For $0 \le \rho \le 1$, we can consider the linear operator

$$\Lambda_\rho = \rho\Lambda(-\infty) + (1-\rho)\Lambda(\infty). \tag{1.3.18}$$

The corresponding characteristic function is given by

$$\Delta_\rho(z) = z - e^{-z} - [\rho(-r) + (1-\rho)(r-1) - 2]e^{0\cdot z} - e^{z}, \tag{1.3.19}$$

which can easily be seen to have no roots on the imaginary axis. In particular, the system corresponding to the operator Λ_ρ is hyperbolic for each $0 \le \rho \le 1$, which means that the map $\rho \mapsto \Lambda_\rho$ is a homotopy between the systems at $-\infty$ and ∞. In particular, the operator Λ from (1.3.4) is a Fredholm operator with Fredholm index 0. We already observed that the derivative $\overline{u}'$ is a kernel element of Λ. By the definition of the Fredholm index, the codimension of Range(Λ) in $L^p(\mathbb{R};\mathbb{C})$ must be at least one. In addition, the identity (1.3.15) yields that the dimension of $\ker(\Lambda^*)$ must also be at least one. In particular, we have established several strong results on the operator Λ and its adjoint Λ^* using relatively simple computations.

We remark that the results in [131] show that, in this case, the kernels $\ker(\Lambda)$ and $\ker(\Lambda^*)$ are, in fact, precisely one-dimensional on account of the comparison principle.

The Fredholm theory as described above has been extended by Faye and Scheel [68] to include MFDEs which feature infinite range interactions, such as the linearization of the system (1.1.20). However, their restrictions on the coefficients were more severe than those featured in (1.1.21)-(1.1.22). In particular, Faye and Scheel required the coefficients to decay exponentially.

1.3.2 The spectral convergence method

As was stated previously, the spectral convergence method was pioneered by Bates, Chen and Chmaj in [6] in order to construct travelling wave solutions to the Nagumo LDE (1.1.18) with infinite-range interactions. One of the main advantages of this method is that it circumvents the use of a comparison principle or exponential dichotomies. As a consequence, it can be applied to a broader class of coefficients than many other techniques.

To illustrate the spectral convergence method, we focus on its original application to the Nagumo LDE (1.1.18) by following [6]. We fix, for now, a small constant $h > 0$. The main goal of the spectral convergence method is to transfer the known Fredholm properties of the linearization of the continuous system to an appropiate linearization of the discrete system. We need to be a bit careful at this point. Since the end goal is to construct a travelling wave solution to the LDE (1.1.18), we cannot consider a system such as (1.3.2), since it is impossible to linearize around a solution that has not been found yet. Instead, we let u_0 be a travelling front solution of the PDE (1.1.6) with wavespeed c_0 and consider the linearizations of both the ODE (1.1.8) and the MFDE (1.1.20) around the wave u_0. These linearizations yield the linear operators

$$(\mathcal{L}_0 u)(\xi) \quad = \quad c_0 u'(\xi) - u''(\xi) - g_u(u_0(\xi); r)u(\xi) \tag{1.3.20}$$

for the ODE (1.1.8) and

$$(\mathcal{L}_h u)(\xi) \quad = \quad c_0 u'(\xi) - \Delta_h u(\xi) - g_u(u_0(\xi); r)u(\xi) \tag{1.3.21}$$

for the MFDE (1.1.20). Here we recall that the operator Δ_h is given by (1.1.23). First, we need to specify on which spaces the operators $\mathcal{L}_0$ and $\mathcal{L}_h$ are posed, which immediately brings us to the first major complication (and, therefore, strength of the spectral convergence method). The operator $\mathcal{L}_h$ can, and should, clearly be viewed as an operator

$$\mathcal{L}_h : H^1(\mathbb{R};\mathbb{R}) \quad \to \quad L^2(\mathbb{R};\mathbb{R}). \tag{1.3.22}$$

However, since the operator $\mathcal{L}_0$ features a second derivative, it cannot be a well-defined operator on $H^1(\mathbb{R};\mathbb{R})$. Instead, we view it as an operator

$$\mathcal{L}_0 : H^2(\mathbb{R};\mathbb{R}) \quad \to \quad L^2(\mathbb{R};\mathbb{R}), \tag{1.3.23}$$

where we have introduced the space

$$H^2(\mathbb{R};\mathbb{R}) \quad = \quad \{u \in H^1(\mathbb{R};\mathbb{R}) : u'' \in L^2(\mathbb{R};\mathbb{R})\} \tag{1.3.24}$$

with corresponding norm

$$\|u\|^2_{H^2(\mathbb{R};\mathbb{R})} \quad = \quad \|u\|^2_{H^1(\mathbb{R};\mathbb{R})} + \|u''\|^2_{L^2(\mathbb{R};\mathbb{R})}. \tag{1.3.25}$$

In particular, the operators $\mathcal{L}_0$ and $\mathcal{L}_h$ act on different spaces, which makes lifting the Fredholm properties of $\mathcal{L}_0$ to $\mathcal{L}_h$ a delicate effort.

It is well-known that for each $\delta \geq 0$ the operator $\mathcal{L}_0 + \delta$ is a Fredholm operator with Fredholm index 0. In addition, this operator is invertible for $\delta > 0$, while it is has a one-dimensional kernel, spanned by the derivative u_0', for $\delta = 0$. The standard Fredholm theory for ODEs implies that the adjoint operator $\mathcal{L}_0^*$ also has a one-dimensional kernel, spanned by some function ϕ_0^-, i.e. we have

$$\ker(\mathcal{L}_0) \quad = \quad \mathrm{span}\{u_0'\}, \qquad \ker(\mathcal{L}_0^*) \quad = \quad \mathrm{span}\{\phi_0^-\}. \tag{1.3.26}$$

Using standard arguments [6, Lem. 3.1], one can show that there exists a constant $C > 0$ in such a way that the bound

$$\|(\mathcal{L}_0 + \delta)^{-1}\psi\|_{H^2(\mathbb{R};\mathbb{R})} \leq C\big[\|\psi\|_{L^2(\mathbb{R};\mathbb{R})} + \tfrac{1}{\delta}|\langle\psi, \phi_0^-\rangle_{L^2(\mathbb{R};\mathbb{R})}|\big] \quad (1.3.27)$$

holds for all $\delta > 0$ and all $\psi \in L^2(\mathbb{R};\mathbb{R})$. The spectral convergence method aims to show that for all $\delta > 0$ there exists a positive constant $h_0(\delta) > 0$ such that for all $h \in (0, h_0(\delta))$ the operator $\mathcal{L}_h + \delta$ is invertible and that the bound

$$\|(\mathcal{L}_h + \delta)^{-1}\psi\|_{H^1(\mathbb{R};\mathbb{R})} \leq \tilde{C}\big[\|\psi\|_{L^2(\mathbb{R};\mathbb{R})} + \tfrac{1}{\delta}|\langle\psi, \phi_0^-\rangle_{L^2(\mathbb{R};\mathbb{R})}|\big] \quad (1.3.28)$$

holds for all $\psi \in L^2(\mathbb{R};\mathbb{R})$. Here, the constant $\tilde{C}$ should be taken independently of $\delta > 0$ and $0 < h < h_0(\delta)$. Employing the bound (1.3.28), a more or less standard argument, that resembles the proof of the implicit function theorem, can be used to construct the travelling wave solutions to the system (1.1.18).

In order to establish the bound (1.3.28), Bates, Chen and Chmaj consider the quantities

$$\Lambda(h,\delta) = \inf_{\phi \in H^1(\mathbb{R};\mathbb{R}), \|\phi\|_{H^1(\mathbb{R};\mathbb{R})}=1} \big[\|(\mathcal{L}_h + \delta)\phi\|_{L^2(\mathbb{R};\mathbb{R})} + \tfrac{1}{\delta}|\langle(\mathcal{L}_h + \delta)\phi, \phi_0^-\rangle_{L^2(\mathbb{R};\mathbb{R})}|\big] \quad (1.3.29)$$

for $h > 0$ and $\delta > 0$, together with

$$\Lambda(\delta) = \liminf_{h \downarrow 0} \Lambda(h,\delta). \quad (1.3.30)$$

The key ingredient is to construct a lower bound on the quantity $\Lambda(\delta)$, which is uniform in $\delta > 0$. If such a lower bound is found, the invertibility of the operator $\mathcal{L}_h + \delta$ and the bound (1.3.28) can be established relatively easily.

We now fix $\delta > 0$ and consider sequences

$$\{\phi_j\}_{j\geq 1} \subset H^1(\mathbb{R};\mathbb{R}), \qquad \|\phi_j\|_{H^1(\mathbb{R};\mathbb{R})} = 1, \qquad h_j \downarrow 0 \quad (1.3.31)$$

which minimize the quantity $\Lambda(\delta)$. That is, we have the limit

$$\lim_{j\to\infty} \|(\mathcal{L}_{h_j} + \delta)\phi_j\|_{L^2(\mathbb{R};\mathbb{R})} + \tfrac{1}{\delta}|\langle(\mathcal{L}_{h_j} + \delta)\phi_j, \phi_0^-\rangle_{L^2(\mathbb{R};\mathbb{R})}| = \Lambda(\delta). \quad (1.3.32)$$

The existence of these minimizing sequences follows directly from the definition of the quantity $\Lambda(\delta)$. For convenience, we write

$$\psi_j = (\mathcal{L}_{h_j} + \delta)\phi_j \quad (1.3.33)$$

for $j \geq 1$. In order to properly take the $h \downarrow 0$ limit, we consider the weak limits ϕ and ψ of the sequences $\{\phi_j\}_{j\geq 1}$ and $\{\psi_j\}_{j\geq 1}$. The first computational effort is to show that the function ϕ is an element of the space $H^2(\mathbb{R};\mathbb{R})$ and that it is a weak solution of the equation $(\mathcal{L}_0 + \delta)\phi = \psi$ [6, Lem. 3.2]. This computation relies heavily on the limit (1.1.24). As a result, we obtain the lower bound

$$\|\phi\|_{H^2(\mathbb{R}:\mathbb{R})} \leq K\Lambda(\delta) \quad (1.3.34)$$

for some constant $K > 0$.

It remains to find a positive lower bound for the norm $\|\phi\|_{H^2(\mathbb{R}:\mathbb{R})}$. Indeed, a common danger when taking weak limits is that the sequence converges to 0 even though the sequence itself is bounded away in norm from 0. Using the Laplace-like properties of the operator Δ_h, we obtain the estimates

$$\langle \Delta_h v, v' \rangle_{L^2(\mathbb{R};\mathbb{R})} = 0, \qquad \langle \Delta_h v, v \rangle_{L^2(\mathbb{R};\mathbb{R})} \leq 0 \tag{1.3.35}$$

for any function $v \in H^1(\mathbb{R};\mathbb{R})$. Employing the bounds (1.3.35) and remembering that $(\mathcal{L}_{h_j} + \delta)\phi_j = \psi_j$, we can estimate the inner products $\langle \psi_j, \phi_j' \rangle_{L^2(\mathbb{R};\mathbb{R})}$ using the Cauchy-Schwarz inequality to obtain a uniform estimate of the form

$$A_1 \|\phi_j\|^2_{L^2(\mathbb{R};\mathbb{R})} \geq A_2 \|\phi_j'\|^2_{L^2(\mathbb{R};\mathbb{R})} - A_3 \|\psi_j\|^2_{L^2(\mathbb{R};\mathbb{R})}; \tag{1.3.36}$$

see [6, Eq. (3.9)].

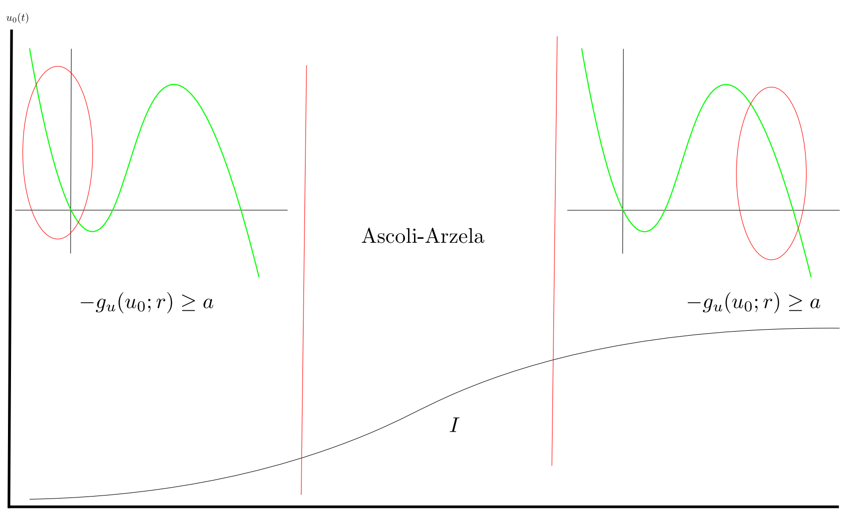


Figure 1.6: *In the spectral convergence method, we pick a compact interval I in such a way that the sign of $-g_u(u_0(x); r)$ for $x \in \mathbb{R} \setminus I$ is fixed. This is allowed because of the bistable nature of the nonlinearity g. Inside I, we employ the Ascoli-Arzela Theorem, while outside I we can use the fixed sign to aid in our estimates.*

At this point in the computation, we employ the bistable nature of the nonlinearity g. Remembering that the front u_0 connects 0 and 1, we can pick a sufficiently large, but bounded, interval I to have $-g_u(u_0(x); r) \geq a$ for x outside I for some fixed constant

$a > 0$; see Figure 1.6. This allows us to estimate the inner product

$$\begin{aligned} \langle -g_u(u_0)\phi_j, \phi_j\rangle_{L^2(\mathbb{R};\mathbb{R})} &= \langle -g_u(u_0)\phi_j, \phi_j\rangle_{L^2(\mathbb{R}\setminus I;\mathbb{R})} + \langle -g_u(u_0)\phi_j, \phi_j\rangle_{L^2(I;\mathbb{R})} \\ &\geq a\|\phi_j\|^2_{L^2(\mathbb{R}\setminus I;\mathbb{R})} - \|g_u(u_0)\|_\infty \|\phi_j\|^2_{L^2(I;\mathbb{R})} \\ &= a\|\phi_j\|^2_{L^2(\mathbb{R};\mathbb{R})} - \big(a + \|g_u(u_0)\|_\infty\big)\|\phi_j\|^2_{L^2(I;\mathbb{R})}. \end{aligned} \tag{1.3.37}$$

Inside I, we can employ the Ascoli-Arzela Theorem to have the limit $\phi_j \to \phi$ in $L^2(I;\mathbb{R})$. As such, on account of (1.3.37) we can estimate the inner products $\langle \psi_j, \phi_j\rangle_{L^2(\mathbb{R};\mathbb{R})}$ to obtain a uniform estimate of the form

$$B_1\|\phi_j\|^2_{L^2(I;\mathbb{R})} \geq B_2\|\phi_j\|^2_{L^2(\mathbb{R};\mathbb{R})} - B_3\|\psi_j\|^2_{L^2(\mathbb{R};\mathbb{R})}; \tag{1.3.38}$$

see [6, Eq. (3.10)]. By properly scaling the inequalities (1.3.36) and (1.3.38) and adding them, we obtain a uniform estimate of the form

$$C_1\|\phi_j\|^2_{L^2(I;\mathbb{R})} \geq C_2\|\phi_j\|^2_{L^2(\mathbb{R};\mathbb{R})} + C_2\|\phi_j'\|^2_{L^2(\mathbb{R};\mathbb{R})} - C_3\|\psi_j\|^2_{L^2(\mathbb{R};\mathbb{R})}. \tag{1.3.39}$$

Remebering that

$$\begin{aligned} \|\phi_j\|^2_{L^2(\mathbb{R};\mathbb{R})} + \|\phi_j'\|^2_{L^2(\mathbb{R};\mathbb{R})} &= \|\phi_j\|^2_{H^1(\mathbb{R};\mathbb{R})} \\ &= 1 \end{aligned} \tag{1.3.40}$$

the inequality (1.3.39) reduces to

$$C_1\|\phi_j\|^2_{L^2(I;\mathbb{R})} \geq C_2 - C_3\|\psi_j\|^2_{L^2(\mathbb{R};\mathbb{R})}. \tag{1.3.41}$$

Because of the strong convergence $\phi_j \to \phi$ in $L^2(I;\mathbb{R})$, the bound (1.3.34) and the limit (1.3.32) we can take the limit $j \to \infty$ in (1.3.41) to obtain

$$\begin{aligned} C_2 - C_3\Lambda(\delta)^2 &\leq C_1\|\phi\|^2_{L^2(I;\mathbb{R})} \\ &\leq C_1\|\phi\|^2_{H^2(\mathbb{R};\mathbb{R})} \\ &\leq C_1 K\Lambda(\delta)^2. \end{aligned} \tag{1.3.42}$$

In particular, Bates, Chen and Chmaj obtain

$$\Lambda(\delta) \geq \sqrt{\tfrac{C_2}{C_1K + C_3}}, \tag{1.3.43}$$

which is a positive constant, as desired.

FitzHugh-Nagumo LDE with infinite-range interactions Our first challenge is to generalize the spectral convergence method to the system (1.2.8). We construct travelling pulse solutions to this system as perturbations of the travelling pulse solutions for the FitzHugh-Nagumo PDE (1.2.1) in the $h \ll 1$ regime. In particular, we only need to assume the conditions (1.1.21)-(1.1.22) instead of exponential decay on the coefficients $\{\alpha_k\}_{k\geq 1}$. The travelling wave equation for (1.2.8) is given by

$$\begin{array}{lcl} \overline{c}_h \overline{u}_h'(\xi) & = & \frac{1}{h^2}\sum_{k=1}^{\infty} \alpha_k\big[\overline{u}_h(\xi + kh) + \overline{u}_h(\xi - kh) - 2\overline{u}_h(\xi)\big] + g(\overline{u}_h(\xi); r) - \overline{w}_h(\xi) \\ \overline{c}_h \overline{w}_h'(\xi) & = & \rho\big[\overline{u}_h(\xi) - \gamma \overline{w}_h(\xi)\big]. \end{array} \tag{1.3.44}$$

However, the generalization of the spectral convergence method from scalar to system equations is far from trivial. In particular, when estimating the equivalents of the inner products $\langle \psi_j, \phi_j\rangle_{L^2(\mathbb{R};\mathbb{R})}$ and $\langle \psi_j, \phi_j'\rangle_{L^2(\mathbb{R};\mathbb{R})}$ as described above, there are various cross-terms we need to keep under control. Luckily, we are aided by the relative simplicity of the second component of (1.3.44). In particular, the off-diagonal elements of the linearization of the MFDE (1.3.44) are constant multiples of each other, which allows us to combine their contributions and absorb them in the diagonal terms. We also generalize the spectral convergence method to yield uniform bounds for values of δ in compact subsets of the complex plane $\mathbb{C} \setminus \{0\}$.

We write $(\overline{u}_h, \overline{w}_h)$ for the new-found travelling pulse solution to (1.2.8) with wavespeed c_h. The next step is to establish the spectral stability of this travelling pulse solution. As such, we linearize the MFDE (1.3.44) around this travelling pulse solution. The corresponding linear operator is given by

$$L_h \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} c_h\frac{d}{d\xi} - \Delta_h - g_u(\overline{u}_h) & 1 \\ -\rho & c_h\frac{d}{d\xi} + \gamma\rho \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}, \tag{1.3.45}$$

where we recall the operator Δ_h from (1.1.23).

We first note that the spectrum of the operator L_h from (1.3.45) is periodic with period $2\pi i\frac{c_h}{h}$. This period grows to infinity as $h \downarrow 0$, which makes sense since the spectrum of the operator $\mathcal{L}_0$ from (1.3.23) is not periodic. That the spectrum is periodic, can be seen as follows. For $p \in \mathbb{C}$, we consider the multiplication operator

$$[e_p\phi](\xi) = e^{p\xi}\phi(\xi). \tag{1.3.46}$$

We fix $p = 2\pi i\frac{1}{h}$. For $k \in \mathbb{Z}$, we observe that $phk \in 2\pi i\mathbb{Z}$, so that we can compute

$$\begin{array}{lcl} \big[e_{-p}[e_p\phi(\cdot + kh)]\big](\xi) & = & e^{-p\xi}[e_p\phi](\xi + hk) \\ & = & e^{-p\xi}e^{p\xi + pkh}\phi(\xi + hk) \\ & = & \phi(\xi + hk). \end{array} \tag{1.3.47}$$

In particular, we see that

$$e_{-p}\Delta_h e_p = \Delta_h. \tag{1.3.48}$$

Since we also have

$$\begin{array}{lcl} \big[e_{-p}c_h\frac{d}{d\xi}[e_p\phi]\big](\xi) & = & c_h e^{-p\xi}\frac{d}{d\xi}(e^{p\xi}\phi(\xi)) \\ & = & c_h\phi(\xi) + pc_h\phi(\xi), \end{array} \tag{1.3.49}$$

we can conclude that

$$e_{-p}L_h e_p \quad = \quad L_h + pc_h. \tag{1.3.50}$$

Since the operators e_{-p} and e_p are invertible, this means that L_h and $L_h + pc_h$ have the same spectrum, which yields the desired periodicity.

The spectral convergence method does not immediately resolve the spectral stability question. Indeed, for each individual value of $\lambda \neq 0$ we can conclude the invertibility of $L_h + \lambda$ for h sufficiently small. However, what we mean by 'sufficiently small' depends heavily on the choice of λ and can only be made uniform for λ in compact subsets of $\mathbb{C} \setminus \{0\}$. Since the period of the spectrum grows to infinity as $h \downarrow 0$, we can only apply the spectral convergence method if we exclude spectrum in a region close to 0, spectrum with a large real part and spectrum with a large imaginary part. We will discuss these issues in more detail in §2.5-2.6.

Spatially periodic FitzHugh-Nagumo LDE The next extension to the spectral convergence method is to construct travelling pulse solutions to the system (1.2.9) in the $\varepsilon \ll 1$ regime. The spatial periodicity of this system also returns in the travelling wave Ansatz, which takes the form

$$(u,w)_j(t) \quad = \quad \begin{cases} (\overline{u}_o, \overline{w}_o)(j+ct) & \text{when } j \text{ is odd,} \\ (\overline{u}_e, \overline{w}_e)(j+ct) & \text{when } j \text{ is even.} \end{cases} \tag{1.3.51}$$

Using the Ansatz (1.3.51), we arrive at the travelling wave MFDE

$$\begin{array}{rcl} c\overline{u}_o'(\xi) & = & \frac{1}{\varepsilon^2}\big(\overline{u}_e(\xi+1) + \overline{u}_e(\xi-1) - 2\overline{u}_o(\xi)\big) + g(\overline{u}_o(\xi); r_o) - \overline{w}_o(\xi), \\ c\overline{w}_o'(\xi) & = & \rho_o[\overline{u}_o(\xi) - \gamma_o \overline{w}_o(\xi)], \\ c\overline{u}_e'(\xi) & = & \big(\overline{u}_o(\xi+1) + \overline{u}_o(\xi-1) - 2\overline{u}_e(\xi)\big) + g(\overline{u}_e(\xi); r_e) - \overline{w}_e(\xi), \\ c\overline{w}_e'(\xi) & = & \rho_e[\overline{u}_e(\xi) - \gamma_e \overline{w}_e(\xi)]. \end{array} \tag{1.3.52}$$

Since we consider the $\varepsilon \ll 1$ regime, we first need to understand the system (1.3.52) for $\varepsilon = 0$. Multiplying the first line of (1.3.52) with ε^2 and taking the limit $\varepsilon \downarrow 0$ yields

$$0 \quad = \quad \overline{u}_e(\xi+1) + \overline{u}_e(\xi-1) - 2\overline{u}_o(\xi). \tag{1.3.53}$$

In particular, we can express $\overline{u}_o$ in terms of $\overline{u}_e$. This means that the third and fourth line of (1.3.52) become

$$\begin{array}{rcl} c\overline{u}_e'(\xi) & = & \frac{1}{2}\big(\overline{u}_e(\xi+2) + \overline{u}_e(\xi-2) - 2\overline{u}_e(\xi)\big) + g(\overline{u}_e(\xi); r_e) - \overline{w}_e(\xi), \\ c\overline{w}_e'(\xi) & = & \rho_e[\overline{u}_e(\xi) - \gamma_e \overline{w}_e(\xi)], \end{array} \tag{1.3.54}$$

which we recognise as a scaled version of the regular FitzHugh-Nagumo LDE (1.2.5). We emphasize that the system (1.3.54) does not contain any odd wave functions, which means that the system decouples at $\varepsilon = 0$. In particular, we know [108, 109] that the system (1.3.54) admits stable travelling pulse solutions $(\overline{u}_{e;0}, \overline{w}_{e;0})$ with wavespeed $\overline{c}_0$. Recalling (1.3.53), we set

$$\overline{u}_{o;0}(\xi) \quad = \quad \tfrac{1}{2}\big[\overline{u}_{e;0}(\xi+1) + \overline{u}_{e;0}(\xi-1)\big]. \tag{1.3.55}$$

Finally, we let $\overline{w}_{o;0}$ be the solution of the linear, inhomogeneous system

$$\overline{c}_0 \overline{w}_o'(\xi) \quad = \quad \rho_o[\overline{u}_{o;0}(\xi) - \gamma_o \overline{w}_o(\xi)]. \tag{1.3.56}$$

As such, the multiplet $\overline{U}_0 = (\overline{u}_{o;0}, \overline{w}_{o;0}, \overline{u}_{e;0}, \overline{w}_{e;0})$ can be seen as the solution of (1.3.52) at $\varepsilon = 0$. Note that the identity (1.3.53) essentially turns the four-component system (1.3.52) into a three-component system at $\varepsilon = 0$. We construct travelling pulse solutions to the LDE (1.2.9) by perturbing them off the function $\overline{U}_0$ by applying the spectral convergence method. However, there are a few major differences with the previous applications of the spectral convergence method. Previously, this method was used to lift Fredholm properties from a continuous to a spatially discrete system, while here we use it to lift Fredholm properties from a three-component to a four-component spatially discrete system. In addition, the different scalings of ε for the diffusion coefficients prevent us from making a direct analogue of the inequalities (1.3.35). Instead, we have to use different scalings in ε for each component to compensate for this imbalance. These different scalings in ε complicate, in turn, the fixed point arguments used to control the nonlinear terms in the construction of the travelling pulse solutions. This complication forces us to take an extra spatial derivative of the system (1.3.52).

For the spectral stability of the travelling pulse solutions to the LDE (1.2.9), we note that the spectrum is periodic with period $2\pi i c_\varepsilon$, similarly to the system (1.2.8). Luckily, this period does not blow up in the $\varepsilon \downarrow 0$ limit here. As such, we only need to exclude spectrum near 0 and with a large real part before we can apply the spectral convergence method.

In this analysis, we do not restrict ourselves to the LDE (1.2.9). Instead, we consider general spatially 2-periodic reaction-diffusion systems with $n+k$ components. Here $n \geq 1$ is the number of components with a nonzero diffusion coefficient, while $k \geq 0$ is the number of components without diffusion (so $n = k = 1$ for the FitzHugh-Nagumo LDE (1.2.9)). In particular, our results also cover the spatially periodic version of the Nagumo LDE (1.1.9) without the use of a comparison principle. However, we need conditions on the end-states that are slightly stronger than the usual temporal stability. Indeed, we need certain submatrices of the corresponding Jacobians to be positive definite, instead of simply spectrally stable. All in all, we have a broad class of systems to which the spectral convergence method can be applied.

Spatially-temporally discrete FitzHugh-Nagumo system Our final application of the spectral convergence method is to the system (1.2.11). In fact, to make the analysis as general as possible, we allow for infinite-range spatial interactions and temporal discretisations of the general $n+k$-component reaction-diffusion LDEs discussed previously.

In this case, we use the spectral convergence method to lift the Fredholm properties of the spatially discrete to the fully discrete system. As discussed previously, we need to assume that $M := (c\Delta t)^{-1}$ is rational to establish an appropriate interpolation scheme. However, Hupkes and Van Vleck relied heavily on the comparison principle to understand this interpolated spatially discrete system. As such, we need to prove several results related to this spatially discrete system from scratch, using the general Fredholm theory for these systems. In addition, the complications we faced previously for infinite-range spatial interactions and nonscalar equations in the spectral convergence method needed to be dealt with here as well.

1.3.3 Exponential dichotomies

The second major technique we develop and employ is the splittings given by exponential dichotomies. There are many ways to look at exponential dichotomies. We take the following general point of view: we say that a linear differential equations is *exponentially dichotomous* if the space of initial conditions, called the *state space*, can be split into a stable and an unstable part. Continuations of stable initial states need to decay exponentially in forward time, while those of unstable states need to decay exponentially in backward time. Let us, for example, consider a linear, autonomous ODE, given by

$$\frac{du}{d\sigma}(\sigma) = Au(\sigma) \tag{1.3.57}$$

where $u(\sigma) \in \mathbb{C}^M$ for $\sigma \in \mathbb{R}$. If the $M \times M$ matrix A is hyperbolic, i.e. has no spectrum on the imaginary axis, then the state space $\mathbb{C}^M$ can be split as

$$\mathbb{C}^M = E_0^s \oplus E_0^u, \tag{1.3.58}$$

where E_0^s is the generalized stable eigenspace and E_0^u is the generalized unstable eigenspace of A. The flow of the ODE (1.3.57) is given by $\Phi(\sigma, \tau) = \exp[A(\sigma - \tau)]$. We note that the spaces E_0^s and E_0^u remain invariant under the flow Φ. Moreover, Φ decays exponentially for $\sigma > \tau$ on E_0^s and for $\sigma < \tau$ on E_0^u. In particular, hyperbolic, autonomous, linear ODEs admit exponential dichotomies.

For non-autonomous, linear ODEs, we need the splitting (1.3.58) to depend on the base time $\tau \in \mathbb{R}$. In particular, we say that the linear ODE

$$\frac{du}{d\sigma}(\sigma) = A(\sigma)u(\sigma) \tag{1.3.59}$$

admits exponential dichotomies on an interval $I \in \{\mathbb{R}, \mathbb{R}^- \mathbb{R}^+\}$ if the following properties are satisfied.

- There exist projection operators $\{P(\tau)\}_{\tau \in I}$ on $\mathbb{C}^M$ which commute with the evolution $\Phi(\sigma, \tau)$.

- The restricted evolutions $\Phi(\sigma, \tau)P(\tau)$ and $\Phi(\sigma, \tau)(I - P(\tau))$ decay exponentially for $\sigma > \tau$ and for $\sigma < \tau$ respectively.

In particular, we have the exponential splitting of the state space $\mathbb{C}^M$ into the range of $P(\tau)$ and the kernel of $P(\tau)$. From this definition, we see that if a system admits exponential dichotomies on an interval $I \in \{\mathbb{R}, \mathbb{R}^- \mathbb{R}^+\}$ then each solution on I can be decomposed in two parts that decay exponentially in forward and backward time respectively.

Exponential dichotomies are closely related to the Fredholm properties of the corresponding linear operators. In particular, we consider the operator

$$\begin{array}{rcl} \Lambda : H^1(\mathbb{R};\mathbb{C}^M) & \to & L^2(\mathbb{R};\mathbb{C}^M), \\ (\Lambda u)(\sigma) & = & \frac{du}{d\sigma}(\sigma) - A(\sigma)u(\sigma). \end{array} \tag{1.3.60}$$

Then it is well-known that the operator Λ is a Fredholm operator if and only if the system (1.3.59) admits exponential dichotomies on $\mathbb{R}^-$ and $\mathbb{R}^+$. In addition, Λ is invertible if and only if (1.3.59) admits exponential dichotomies on $\mathbb{R}$. We refer to the review by Sandstede [147] for more details.

A very powerful and useful result is the so-called roughness theorem, see [45, Chapter 4]. Informally, this result states that exponential dichotomies are preserved when a small perturbation is added to the system. For example, let A be a hyperbolic $M \times M$ matrix and let $B(\sigma)$ be a bounded collection of $M \times M$ matrices which depend continuously on σ. Then the roughness theorem yields that the system

$$\frac{du}{d\sigma}(\sigma) \quad = \quad Au(\sigma) + \delta B(\sigma)u(\sigma) \tag{1.3.61}$$

admits exponential dichotomies on $\mathbb{R}$ when $\delta > 0$ is sufficiently small. Hence, we can conclude that the corresponding linear operator Λ from (1.3.60) is invertible! This means that the inhomogeneous ODE

$$\frac{du}{d\sigma}(\sigma) \quad = \quad Au(\sigma) + \delta B(\sigma)u(\sigma) + f(\sigma) \tag{1.3.62}$$

has a unique solution $u \in H^1(\mathbb{R};\mathbb{C})$ for any function $f \in L^2(\mathbb{R};\mathbb{C})$. We emphasize that these powerful results can be derived with hardly any assumptions on the matrices $B(\sigma)$.

For linear MFDEs such as (1.3.3), a few major complications turn up. First, the space $\mathbb{C}$ is no longer sufficient as a state space. Indeed, for determining $u'(0)$ in (1.3.3) we need to specify the behaviour of u on the entire interval $[-1, 1]$. As such, one usually takes $C_b([-1, 1]; \mathbb{C})$ as a state space. The second major complication is that MFDEs are typically ill-posed. That is, given an initial segment there may not be an extension of that segment that solves the MFDE, or such an extension may not be unique. As such, there is no equivalent of the evolution operator Φ that we defined for ODEs.

These complications were solved simultaneously and independently by Mallet-Paret and Verduyn Lunel [133] and by Härterich, Scheel and Sandstede [96]. We will focus on the former approach. Mallet-Paret and Verduyn Lunel showed that for linear, asymptotically hyperbolic MFDEs such as (1.3.3) we have the splitting

$$C_b([-1, 1]; \mathbb{C}) \quad = \quad P(\tau) + Q(\tau) + \Gamma(\tau). \tag{1.3.63}$$

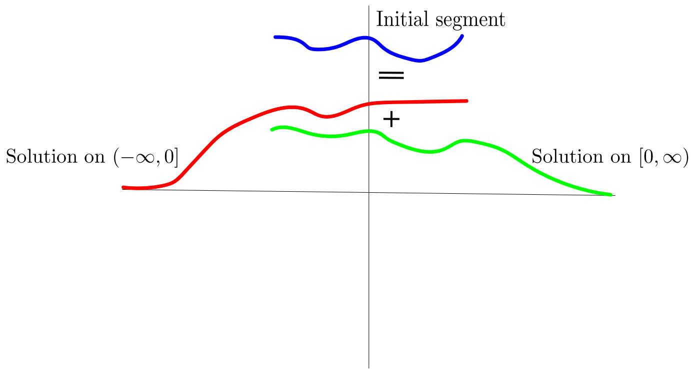


Figure 1.7: *Visual representation of the splitting* $C_b([-1,1];\mathbb{C}) = P(0) \oplus Q(0)$ *for linear MFDEs.*

Here $P(\tau)$ are the initial segments, centered around $\tau \in \mathbb{R}$, which can be extended to solutions of (1.3.3) on $(-\infty, \tau]$, while $Q(\tau)$ are the initial segments that can be extended to solutions of (1.3.3) on $[\tau, \infty)$; see Figure 1.7. One of the main results of [133] is that the extensions of the initial segments in $P(\tau)$ and $Q(\tau)$ decay exponentially as $\xi \to -\infty$ and as $\xi \to \infty$ respectively. Moreover, writing $B(\tau) = P(\tau) \cap Q(\tau)$ for those segments that can be extended to full solutions of (1.3.3), we can divide the space $B(\tau)$ out of $P(\tau)$ to turn (1.3.63) into a direct sum. Finally, the space $\Gamma(\tau)$ is finite dimensional and can be made explicit using the so-called Hale inner product [91]. For the linearized Nagumo MFDE (1.3.3), this Hale inner product takes the form

$$\langle \psi, \phi \rangle_\tau \quad = \quad \psi(0)\phi(0) + \int_{-1}^{0} \psi(s+1)\phi(s)ds - \int_{0}^{1} \psi(s-1)\phi(s)ds \tag{1.3.64}$$

for $\phi, \psi \in C_b([-1,1];\mathbb{C})$. Indeed, the space $\Gamma(\tau)$ from (1.3.63) can be classified by the identity

$$P(\tau) + Q(\tau) \quad = \quad \{\phi \in C_b([-1,1];\mathbb{C}) : \langle b(\tau + \cdot), \phi \rangle_\tau = 0 \text{ for all } b \in \ker(\Lambda^*)\}, \tag{1.3.65}$$

where we recall the operator Λ^* from (1.3.16).

There are two potential concerns that can arise to impact the usefulness of the identity (1.3.65). First, there may be nonzero kernel elements $b \in \ker(\Lambda^*)$ that vanish on the relevant interval $[\tau - 1, \tau + 1]$. In that case, we have $\langle b(\tau + \cdot), \phi \rangle_\tau = 0$ for any function ϕ. Second, the Hale inner product may be degenerate, in the sense that there exists a nonzero function ψ for which $\langle \psi, \phi \rangle_\tau = 0$ for any function ϕ. If both situations do not occur, then the dimension of the space $\Gamma(\tau)$ can easily be determined to be the dimension of the kernel $\ker(\Lambda^*)$. However, if either one of these situations occurs, we can no longer compute this dimension. Luckily, Mallet-Paret and Verduyn

Lunel showed that both situations cannot occur if the coefficients corresponding to the largest and smallest shifts are *atomic*, i.e. invertible on appropriate time-intervals. This is clearly the case for the system (1.3.3).

Exponential dichotomies for MFDEs with infinite-range interactions We extended the results by Mallet-Paret and Verduyn Lunel to include linear MFDEs such as the linearization of the system (1.1.20), which is given by

$$\bar{c}_h u'(\xi) \quad = \quad \sum_{k=1}^{\infty} \alpha_k \big[u(\xi + kh) + u(\xi - kh) - 2u(\xi)\big] + g_u(\overline{u}_h(\xi); r)u(\xi). \qquad (1.3.66)$$

To ensure that the Fredholm theory developed by Faye and Scheel in [69] can be used, we assume that the coefficients $\{\alpha_k\}_{k\geq 1}$ decay exponentially. For the system (1.3.66) the appropriate state space is the space $C_b(\mathbb{R};\mathbb{C})$. Since the system (1.3.66) is asymptotically hyperbolic, we have the splitting

$$C_b(\mathbb{R};\mathbb{C}) \quad = \quad P(\tau) + Q(\tau) + \Gamma(\tau). \qquad (1.3.67)$$

As before, the space $P(\tau)$ contains those initial segments, centered around $\tau \in \mathbb{R}$, which can be extended to solutions of (1.3.66) on $(-\infty, \tau]$, while $Q(\tau)$ are those segments that can be extended to solutions of (1.3.66) on $[\tau, \infty)$. However, extending is not really the appropriate word, since the initial segments are already defined on the entire line. In addition, the segments in $P(\tau)$ decay exponentially as $\xi \to -\infty$ and those in $Q(\tau)$ decay exponentially as $\xi \to \infty$. Moreover, dividing out the solution space $B(\tau) = P(\tau) \cap Q(\tau)$ from $P(\tau)$ turns (1.3.67) into a direct sum. Finally, we regain the identity

$$P(\tau) + Q(\tau) \quad = \quad \{\phi \in C_b(\mathbb{R};\mathbb{C}) : \langle b(\tau + \cdot), \phi\rangle_\tau = 0 \text{ for all } b \in \ker(\Lambda^*)\}, \qquad (1.3.68)$$

where $\langle \cdot, \cdot \rangle_\tau$ is the Hale inner product.

However, the degeneracy issues that were discussed previously are much harder to solve for systems such as (1.3.66). Indeed, the atomicity condition Mallet-Paret and Verduyn Lunel used to exclude these degeneracies explicitly references the largest and the smallest shift. We formulate several new conditions on the coefficients which can, separately, be used to rule out degeneracies. In particular, for the system (1.3.66) one of these conditions entails that the coefficients $\{\alpha_k\}_{k\geq 1}$ should be *cyclic* with respect to the backward shift operator on $\ell^2(\mathbb{N};\mathbb{R})$. That is, the set of sequences $\{\alpha_k\}_{k\geq N}$ for $N \geq 1$ should span a dense subspace of $\ell^2(\mathbb{N};\mathbb{R})$. This condition is, for example, satisfied if the coefficients decay like a Gaussian. However, this condition is not satisfied if $\alpha_k = \exp(-k)$ for each $k \geq 1$, since in that case we have $\{\alpha_k\}_{k\geq N}$ is a scalar multiple of $\{\alpha_k\}_{k\geq 1}$ for each $N \geq 1$.

If the coefficients $\{\alpha_k\}_{k\geq 1}$ of the the system (1.3.66) are positive, we can merely show that the Hale inner product is nondegenerate for kernel elements of the adjoint operator. That is, we can show that if $\langle b(\tau + \cdot), \phi\rangle_\tau = 0$ for all $b \in \ker(\Lambda^*)$, we must have $\phi = 0$. In particular, if $\alpha_k = \exp(-k)$ for $k \geq 1$, we explicitly construct a nonzero function ψ which has $\langle \psi, \phi \rangle_\tau = 0$ for all functions ϕ. Luckily, the nondegeneracy for kernel elements is sufficient to compute the dimension of the space $\Gamma(\tau)$.