Patterns on spatially structured domains

Schouten-Straatman, W.M.

Patterns on Spatially Structured Domains

Proefschrift

ter verkrijging van
de graad van Doctor aan de Universiteit Leiden,
op gezag van Rector Magnificus prof. dr. ir. H. Bijl,
volgens besluit van het College voor Promoties
te verdedigen op dinsdag 2 maart 2021
klokke 11.15 uur

door

Willem Migchel Schouten-Straatman

geboren te Baarn
in 1992

Promotor:
Prof. dr. Arjen Doelman

Copromotor:
Dr. Hermen Jan Hupkes

Promotiecommissie:

Prof. dr. Peter Bates	Michigan State University
Dr. Grégory Faye	Université Toulouse III – Paul Sabatier
Prof. dr. Erik Van Vleck	University of Kansas
Prof. dr. Frank van der Duijn Schouten	
Prof. dr. Roeland Merks	

Print: Haveka — www.haveka.nl

Front Cover:
Whitehoune / stock.adobe.com

This work was supported by the Netherlands Organisation for Scientific Research (NWO), grant 639.032.612.

Contents

1 Introduction **1**

1.1 Scalar LDEs and MFDEs . . . 2

1.1.1 The FPUT lattice . . . 3

1.1.2 The Nagumo equation . . . 4

1.2 The FitzHugh-Nagumo system . . . 9

1.3 Techniques . . . 13

1.3.1 Linear Fredholm theory . . . 13

1.3.2 The spectral convergence method . . . 16

1.3.3 Exponential dichotomies . . . 24

2 Nonlinear stability of pulse solutions for the discrete FitzHugh-Nagumo equation with infinite-range interactions **29**

2.1 Introduction . . . 30

2.2 Main results . . . 38

2.3 The singular perturbation . . . 42

2.3.1 Strategy . . . 45

2.3.2 Preliminaries . . . 47

2.3.3 Proof of Proposition 2.3.4 . . . 48

2.4 Existence of pulse solutions . . . 58

2.5 The point and essential spectrum . . . 65

2.6 The resolvent set . . . 78

2.7 Green's functions . . . 85

2.7.1 Construction of the Green's function . . . 87

2.7.2 Meromorphic expansion of G_λ . . . 96

2.7.3 Decomposition into stable and center modes . . . 105

2.8 Nonlinear stability . . . 111

3 Travelling waves for spatially discrete systems of FitzHugh-Nagumo type with periodic coefficients **133**

3.1 Introduction . . . 134

3.2 Main results . . . 140

3.3 The limiting system . . . 145

3.3.1 Properties of $\overline{L}_o$ . . . 147

3.4 Transfer of Fredholm properties . . . 149

3.4.1 Strategy . . . 153
3.4.2 Proof of Proposition 3.4.7 . . . 156
3.5 Existence of travelling waves . . . 164
3.6 Stability of travelling waves . . . 171
3.6.1 The operator $\overline{\mathcal{L}}_{\varepsilon}$ . . . 173
3.6.2 Spectral stability . . . 175

4 Travelling wave solutions for fully discrete FitzHugh-Nagumo type equations with infinite-range interactions 179
4.1 Introduction . . . 180
4.2 Main result . . . 186
4.2.1 The spatially discrete system . . . 187
4.2.2 Spatially discrete travelling waves . . . 188
4.2.3 The fully discrete system . . . 190
4.2.4 Nonuniqueness and numerical examples . . . 193
4.3 Setup . . . 195
4.4 The limiting system . . . 199
4.5 Linear theory for $\Delta t \to 0$ . . . 205
4.5.1 Strategy . . . 205
4.5.2 Spectral convergence . . . 208
4.5.3 Exponential decay . . . 213
4.6 Proof of main result . . . 218
4.6.1 Existence of solutions . . . 219
4.6.2 Local uniqueness of solutions . . . 227
4.A Auxiliary results . . . 234

5 Exponential dichotomies for nonlocal differential operators with infinite-range interactions 239
5.1 Introduction . . . 240
5.2 Main results . . . 247
5.2.1 State spaces . . . 250
5.2.2 Exponential dichotomies on $\mathbb{R}$ . . . 252
5.2.3 Exponential dichotomies on half-lines . . . 255
5.3 The existence of exponential dichotomies . . . 256
5.3.1 Preliminaries . . . 258
5.3.2 Exponential decay . . . 259
5.3.3 The restriction operators π^+ and π^- . . . 265
5.3.4 Fundamental properties of the Hale inner product . . . 267
5.3.5 Exponential splitting of the state space X . . . 271
5.4 Fredholm properties of the projections $\Pi_{\widehat{P}}$ and $\Pi_{\widehat{Q}}$ . . . 274
5.5 Exponential dichotomies on half-lines . . . 277
5.5.1 Strategy . . . 278
5.5.2 Construction of $\mathcal{Y}(\tau)$ . . . 280
5.5.3 Exponential decay . . . 282
5.5.4 Projection operators . . . 286
5.6 Degeneracies and their avoidance . . . 289

5.6.1 Structural conditions 290
5.6.2 Examples 295
5.6.3 (Co)-dimension counting 300
5.6.4 Cyclic coefficients 302
5.6.5 Nondegeneracy of the Hale inner product 304
5.6.6 The nontriviality condition (HKer) 309

6 Parameter-dependent exponential dichotomies for nonlocal differential operators 311

6.1 Introduction and main result 311
6.2 One-sided exponential weights 313
6.3 Construction of exponential splittings 318

Bibliography 329

Samenvatting 343

Dankwoord 349

Curriculum Vitae 351

Chapter 1

Introduction

Many systems in nature have an underlying spatially discrete structure, which greatly influences their dynamical behaviour. Often, this broken translational and rotational symmetry gives rise to interesting and complex behaviour, which is not present for spatially homogeneous systems. For several systems, this spatially discrete structure is directly visible. For example, one can think of the movement of domain walls [53] or dislocations [35] through crystals. However, the spatially discrete structure can also be more hidden. In particular, let us consider the propagation of electrical signals through nerve fibres. It is well-known that these signals can only move at appropriate speeds if the nerve fibres are insulated by a meylin coating. This coating admits regularly spaced gaps at the so-called nodes of Ranvier [143], see Figure 1.1. The signal moves fast through these coated regions, but loses strength rapidly. In the nodes, the signal moves much slower, while it recovers strength. In particular, the signal appears to hop from one node to the next. This phenomenon is known as saltatory conduction [127].

In many of these processes in nature, the propagation of fixed structures through space and time plays a crucial role. As is the case for spatially continuous systems, travelling waves form the basic building blocks for the complex behaviour and patterns spatially discrete systems can exhibit. Travelling wave solutions have a fixed shape, called the *wave profile*, and travel through time and space with a fixed wavespeed. The propagation of electrical signals through nerve fibers is a key example of the significance of the study of travelling waves in spatially discrete systems.

In §1.1, we further highlight a few of these discrete systems and discuss the mathematical models that are used to describe their behaviour. In §1.2, we focus entirely on the FitzHugh-Nagumo system, which is used to model the signal propagation through nerve fibres and is the main equation under consideration in this thesis. Finally, we elaborate on the most important mathematical techniques that are used in the analysis of our systems in §1.3.

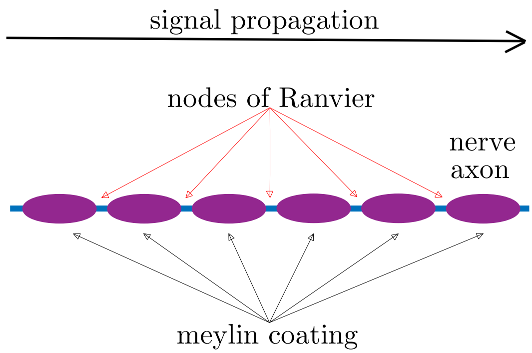


Figure 1.1: *Simplified representation of the meylin coating and nodes of Ranvier in a nerve axon.*

1.1 Scalar LDEs and MFDEs

For systems such as those discussed above, it is essential to incorporate the spatially discrete structure into the models that aim to describe their behaviour. For this purpose, *lattice differential equations* (LDEs) form a natural class of systems to model systems of this type. Indeed, let us consider an infinite chain of particles, indexed by the one-dimensional lattice $j \in \mathbb{Z}$. For the jth particle, we are interested how a specific quantity u_j, for example displacement or electrical potential, evolves in time. Let us assume, for now, that the rate of change of the quantity u_j is only influenced directly by itself and its nearest neighbours u_{j+1} and u_{j-1}. That is, the evolution of the variables u_j for $j \in \mathbb{Z}$ is given by the system of equations

$$\dot{u}_j(t) \quad = \quad f\big(u_{j+1}(t), u_{j-1}(t), u_j(t)\big) \tag{1.1.1}$$

for some function f. We note that the system (1.1.1) is, in fact, a collection of infinitely many, coupled ordinary differential equations (ODEs).

For systems such as (1.1.1), we are mainly interested in travelling wave solutions. Typically, this means that we aim to find a solution $\{u_j\}_{j \in \mathbb{Z}}$ to the system (1.1.1) that takes the form

$$u_j(t) \quad = \quad \overline{u}_0(j + \overline{c}_0 t), \tag{1.1.2}$$

where $\overline{u}_0$ is the wave profile and $\overline{c}_0$ is the wavespeed. Usually, an assumed shape of the solution, called an *Ansatz*, such as (1.1.2) is accompanied by boundary conditions of the form

$$\lim_{\xi \to -\infty} \overline{u}_0(\xi) \quad = \quad u^-, \qquad \lim_{\xi \to \infty} \overline{u}_0(\xi) \quad = \quad u^+. \tag{1.1.3}$$

If $u^- = u^+$ in (1.1.3), we often refer to the travelling wave as a *travelling pulse*, while otherwise it is known as a *travelling front*. In order to establish whether the system (1.1.1) admits a travelling wave solution, we need to substitute the Ansatz (1.1.2) into the LDE (1.1.1) and solve the resulting system. In particular, this procedure yields a so-called *functional differential equation of mixed type* (MFDE), which is given by

$$\overline{c}_0\overline{u}_0'(\xi) = f\big(\overline{u}_0(\xi+1), \overline{u}_0(\xi-1), \overline{u}_0(\xi)\big) \tag{1.1.4}$$

in which $\xi = j + \overline{c}_0 t$. The 'mixed type' in MFDE refers to the fact that it contains both advanced (forward) and retarded (backward) shifts. The MFDE (1.1.4) is called the *travelling wave equation* for the LDE (1.1.1).

LDEs form a relatively young field of interest for mathematicians. In the applied literature, however, LDEs have appeared significantly more frequently. For systems with an inherent discrete structure, LDEs can be seen as the natural replacement for partial differential equations (PDEs). LDEs can both arise as a discretisation of a PDE or as a system that has no direct spatially continuous equivalent. LDEs have been shown to display unexpected and complex dynamical behaviour. We will illustrate this behaviour with a few prominent examples.

1.1.1 The FPUT lattice

The Fermi-Pasta-Ulam-Tsingou (FPUT) lattice is an infinite chain of particles, which are coupled by identical springs to their neighbours. This system is a generalization of a system with finitely many particles, which was studied numerically in [49, 72]. The corresponding FPUT LDE aims to capture the dynamical behaviour of position of these particles. When the particles are identical, the lattice is called a monoatomic lattice. In this case, we can derive from Newton's second law that the FPUT LDE is given by

$$\ddot{u}_j = F(u_{j+1} - u_j) - F(u_j - u_{j-1}), \tag{1.1.5}$$

where the function F represents the spring force. The existence of solitary travelling wave solutions for the system (1.1.5), i.e. travelling wave solutions of which the wave profile decays exponentially, has been shown in [77–81].

When the particles are not identical, these solitary travelling wave solutions no longer capture the behaviour of the particles. In particular, let us consider the diatomic lattice, i.e. when the mass of the particles alternates between the two values 1 and $m \neq 1$, see Figure 1.2. The diatomic FPUT LDE has been studied in various parameter regimes, such as the small mass $m \ll 1$ regime [100], the equal mass $m \approx 1$ regime [66] and the long wave regime [67]. Travelling wave solutions for these systems are usually constructed as *perturbations* of travelling wave solutions for a monoatomic lattice. That it, the travelling wave solution is constructed as the sum of the monoatomic wave and another part, which is small in terms of the relevant parameter regime. For the small mass and long wave regimes, the solitary travelling waves are singularly perturbed into a travelling wave profile which asymptotes into a periodic solution with a very small amplitude. The amplitude of these "ripples" is small beyond all orders in the relevant parameter. This category of travelling waves is often referred to as nanopterons, see [21] for an interesting overview. For the near-equal mass regime, the travelling wave profile also asymptotes to a periodic solution, but the amplitude of this periodic solution is only algebraically small. Such a travelling wave profile is called a micropteron.

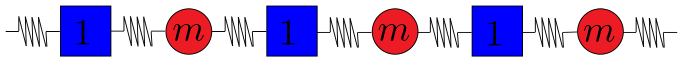


Figure 1.2: *Illustration of the diatomic FPUT lattice with alternating particles with masses* 1 *and* m.

1.1.2 The Nagumo equation

The Nagumo or Allen-Cahn PDE is given by

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + g(u(x,t);r). \tag{1.1.6}$$

Here the bistable nonlinearity g is, typically, given by the cubic polynomial $g(u;r) = u(1-u)(u-r)$ with $0 < r < 1$. The Nagumo PDE has been commonly used as a model where two biological species or material states compete in a spatial domain [3]. Due to its relative simplicity, the PDE (1.1.6) has served as a prototype to understand basic concepts in the theory of reaction-diffusion systems. This system is known to admit travelling front solutions of the form

$$u(x,t) = \Phi(x+ct), \qquad \lim_{\xi\to-\infty} \Phi(\xi) = 0, \qquad \lim_{\xi\to\infty} \Phi(\xi) = 1, \tag{1.1.7}$$

which can be constructed explicitly. This travelling front solution satisfies the travelling wave ODE

$$c\Phi'(\xi) = \Phi''(\xi) + g(\Phi(\xi);r). \tag{1.1.8}$$

In addition, there is a one-to-one correspondence between the wavespeed c and the parameter r. Due to the symmetry of the system, travelling waves are *pinned* for $r = \frac{1}{2}$, i.e. the wavespeed c is 0, while the waves move for $r \neq \frac{1}{2}$. It is well-known that these travelling wave solutions are stable under perturbations that do not need to be small [73].

The natural way to discretize the Nagumo PDE (1.1.6) is to consider the LDE

$$\dot{u}_j(t) = d\big[u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)\big] + g(u_j(t);r), \tag{1.1.9}$$

which we will refer to as the Nagumo LDE. There are many similarities and differences between the PDE (1.1.6) and the LDE (1.1.9). Although the LDE (1.1.9) is no longer explicitly solvable, it is well-known that it admits travelling wave solutions, which must, hence, satisfy the travelling wave MFDE

$$cu'(\xi) = d\big[u(\xi+1) + u(\xi-1) - 2u(\xi)\big] + g(u(\xi);r). \tag{1.1.10}$$

In addition, for a given value of $d > 0$ the wavespeed c is uniquely determined by the parameter r [39, 131]. Usually, the *comparison principle* is used to prove these types of results. The comparison principle states, informally, that a subsolution of an elliptic or parabolic equation stays below a supersolution. The comparison principle can be applied to both the PDE (1.1.6) and the LDE (1.1.9).

However, the relation between the wavespeed c and the parameter r is no longer one-to-one. In particular, when $d > 0$ is sufficiently small there is a nontrivial interval $r \in [r_-, r_+]$ for which the LDE (1.1.9) admits travelling wave solutions with wavespeed $c = 0$. This phenomenon is known as *propagation failure* and has been shown to be a common feature of discrete systems [99]. However, we do emphasize that for $c \neq 0$ the $r(c)$ relation remains single-valued.

There are many possible extensions and generalizations to the Nagumo LDE (1.1.9). Here, we will discuss a few results to showcase the rich behaviour of the Nagumo LDE. A more comprehensive overview can be found in [105].

Bichromatic waves. In contrast to the PDE (1.1.6), the LDE (1.1.9) has infinitely many equilibria. Let us consider equilibria of the form

$$u_j = \begin{cases} \overline{u}_e & \text{if } j \text{ is even,} \\ \overline{u}_o & \text{if } j \text{ is odd.} \end{cases} \tag{1.1.11}$$

Such a 2-periodic equilibrium must satisfy the system of equations

$$\begin{aligned} 0 &= 2d(\overline{u}_e - \overline{u}_o) + g(\overline{u}_o; r), \\ 0 &= 2d(\overline{u}_o - \overline{u}_e) + g(\overline{u}_e; r). \end{aligned} \tag{1.1.12}$$

For $d = 0$, the system (1.1.12) decouples and immediately yields the solutions $\overline{u}_e, \overline{u}_o \in \{0, r, 1\}$. In particular, the system (1.1.12) has 9 distinct solutions for $d = 0$. As such, using the implicit function theorem, we can continue these 9 solutions for sufficiently small $d > 0$ until these continuations start to intertwine. We say that a pair $(\overline{u}_e, \overline{u}_o)$ which satisfies (1.1.12) is of type $w \in \{\mathfrak{o}, \mathfrak{r}, \mathfrak{1}\}^2$ if it lies on the branch of the equilibrium w of (1.1.12) for $d = 0$. We are mainly interested in equilibria of type $w \in \{\mathfrak{o}, \mathfrak{1}\}^2$, since these equilibria are stable. In particular, let us write $u_{\mathfrak{o1}}(r, d)$ for the solution of (1.1.12) of type $\mathfrak{o1}$. Since the equilibrium $u_{\mathfrak{o1}}(r, d)$ is stable, a so-called monotonic iteration scheme [39] can be used to show that the LDE (1.1.9) admits so-called *bichromotic waves*. That is, solutions of the form

$$u_j(t) = \begin{cases} \Phi_e(j + c_{\mathfrak{o1}}(r, d)t), & \text{if } j \text{ is even,} \\ \Phi_o(j + c_{\mathfrak{o1}}(r, d)t), & \text{if } j \text{ is odd} \end{cases} \tag{1.1.13}$$

with boundary conditions

$$\lim_{\xi \to -\infty} (\Phi_e, \Phi_0)(\xi) = (0, 0), \qquad \lim_{\xi \to -\infty} (\Phi_e, \Phi_0)(\xi) = u_{\mathfrak{o1}}(r, d) \tag{1.1.14}$$

as long as $d > 0$ remains small enough. See also Figure 1.3. Let us write

$$d_{\mathfrak{o1}}(r) \quad = \quad \sup\{d > 0 : \text{there exists an equilibrium of (1.1.12) of type } \mathfrak{o1}\}, \tag{1.1.15}$$

so that bichromatic waves exist for $0 < d < d_{\mathfrak{o1}}(r)$. A more interesting, and delicate, question is whether these bichromatic waves are pinned or if they are moving. In particular, let us write

$$d_{\mathfrak{o1}}(r)^* \quad = \quad \sup\{d > 0 : c_{\mathfrak{o1}}(r, d) = 0\}. \tag{1.1.16}$$

One of the main results of [106] is that, if $r \in (0, 1)$ is sufficiently far away from 0, we have the strict inequality

$$d_{\mathfrak{o1}}(r)^* \quad < \quad d_{\mathfrak{o1}}(r). \tag{1.1.17}$$

That is, the bichromatic wave is not pinned for values of d in the nontrivial interval $\big(d_{\mathfrak{o1}}(r)^*, d_{\mathfrak{o1}}(r)\big)$. Related results can be found in [107, 159, 160, 162].

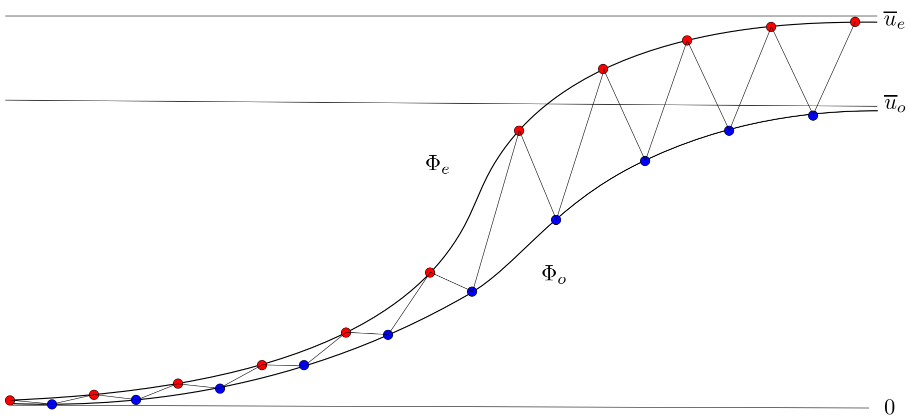


Figure 1.3: *This bichromatic wave with waveprofiles Φ_e and Φ_o connects the homogeneous state 0 to the heterogeneous state $(\overline{u}_e, \overline{u}_0)$.*

Infinite-range interactions. In [6], Bates, Chen and Chmaj considered a version of the Nagumo LDE (1.1.9) which features infinite-range interactions. This system is given by

$$u_j'(t) \quad = \quad d \sum_{k=1}^{\infty} \alpha_k \big[u_{j+k}(t) + u_{j-k}(t) - 2u_j(t)\big] + g(u_j(t); r). \tag{1.1.18}$$

Writing $d = \frac{1}{h^2}$, the system (1.1.18) can be seen as an infinite-range discretisation of the Nagumo PDE (1.1.6) on a grid with spacing $h > 0$. By using the Ansatz

$$u_j(t) \quad = \quad \overline{u}_h(hj + \overline{c}_h t), \tag{1.1.19}$$

the corresponding travelling wave equation is an MFDE which features infinite-range interactions and is given by

$$\overline{c}_h \overline{u}_h'(\xi) \quad = \quad \frac{1}{h^2} \sum_{k=1}^{\infty} \alpha_k \big[\overline{u}_h(\xi + kh) + \overline{u}_h(\xi - kh) - 2\overline{u}_h(\xi)\big] + g(\overline{u}_h(\xi); r). \tag{1.1.20}$$

In order to make sure the discretised Laplacian still behaves like a Laplacian, the authors impose the following limits on the grow of the coefficients $\{\alpha_k\}_{k\geq 1}$

$$\sum_{k=1}^{\infty} |\alpha_k| k^2 \quad < \quad \infty, \qquad \sum_{k=1}^{\infty} \alpha_k k^2 \quad = \quad 1, \tag{1.1.21}$$

together with the spectral bounds

$$\sum_{k=1}^{\infty} \alpha_k \cos(kz) \quad \geq \quad 0, \text{ for } z \in [0, 2\pi]. \tag{1.1.22}$$

In particular, upon defining the operator

$$(\Delta_h \phi)(\xi) \quad = \quad \frac{1}{h^2} \sum_{k=1}^{\infty} \alpha_k \big[\phi(\xi + kh) + \phi(\xi - kh) - 2\phi(\xi)\big], \tag{1.1.23}$$

we have the limit

$$\lim_{h\downarrow 0} \|\Delta_h \phi - \phi''\|_{L^2(\mathbb{R};\mathbb{R})} \quad = \quad 0 \tag{1.1.24}$$

for sufficiently smooth and bounded functions ϕ as long as the conditions (1.1.21)-(1.1.22) hold. Note that not all coefficients $\{\alpha_k\}_{k\geq 1}$ need to be positive. In particular, the comparison principle is not necessarily available for the system (1.1.18).

Due to the limit (1.1.24), Bates, Chen and Chmaj aimed to find travelling wave solutions to the LDE (1.1.18) in the *near-continuum regime* $h \ll 1$. In particular, the authors constructed travelling waves for (1.1.18) as perturbations of the travelling waves for the PDE (1.1.6). However, the transition from the local second derivative operator to the nonlocal infinite-range difference operator is highly singular. To resolve this issue, the authors pioneered a method to lift certain properties of the continuous system to the spatially discrete system. We will refer to this method as the *spectral convergence method*. We will return to this method later in much more detail, as it plays an essential role in this thesis.

The fully discrete Nagumo equation Even though the spatial coordinate is discretised for LDEs, the temporal coordinate remains continuous. In [111], Hupkes and Van Vleck considered temporal discretisations of the Nagumo LDE (1.1.9), or, equivalently, spatial-temporal discretisations of the Nagumo PDE (1.1.6) in order to understand the impact of discretisation schemes on the solutions that these schemes aim to approximate. For the backward-Euler discretisation scheme, the corresponding evolution takes the form

$$\tfrac{1}{\Delta t}[U_j(n\Delta t) - U_j((n-1)\Delta t)] \quad = \quad d[U_{j+1} + U_{j-1} - 2U_j](n\Delta t) + g(U_j(n\Delta t); r), \tag{1.1.25}$$

where we have $n \in \mathbb{Z}$ and $\Delta t > 0$ is called the *time-step*. Note that the system (1.1.25) is no longer a differential equation. The backward-Euler discretisation scheme is used because of several useful stability properties. This discretisation scheme is, in fact, the first of six so-called *backwards differentiation formula* (BDF) discretisation methods.

A travelling wave Ansatz for the system (1.1.25) with wavespeed c takes the form

$$U_j(n\Delta_t) \quad = \quad \Phi(j + nc\Delta t). \tag{1.1.26}$$

Therefore, the corresponding travelling wave equation is given by

$$\tfrac{1}{\Delta t}\big[\Phi(\xi) - \Phi(\xi - c\Delta t)\big] \quad = \quad d\big[\Phi(\xi+1) + \Phi(\xi-1) - 2\Phi(\xi)\big] + g(\Phi(\xi); r). \tag{1.1.27}$$

Something interesting should be noted: if $c\Delta t$ is a rational number, then the domain of the equation (1.1.27) can be restricted to a rational subset of the real line. This restriction turns out to be a key ingredient to construct travelling wave solutions to the system (1.1.25). Indeed, Hupkes and Van Vleck showed that, if $M := (c\Delta t)^{-1}$ is rational and sufficiently large, the system (1.1.25) admits travelling wave solutions. They employed the restriction on the domain of (1.1.27) to establish an interpolation scheme to link the system (1.1.27) to finitely many copies of the Nagumo MFDE (1.1.10). Then, the authors used the previously mentioned spectral convergence method to lift the Fredholm properties of this spatially discrete system to the fully discrete system.

There is an interesting nonuniqueness in the system (1.1.25). Indeed, the travelling wave profile is constructed as a perturbation of the restriction of the original, continuous wave profile Φ to the discrete domain. In particular, this means that for any irrational phase shift ϑ, the profile that is obtained by perturbing off $\Phi(\cdot + \vartheta)$ could potentially yield a different travelling wave solution to the system (1.1.25) with the same parameter values, see Figure 1.4. However, this phase shift might change the wavespeed. This nonuniqueness is not present for the Nagumo PDE (1.1.6) or LDE (1.1.9).

In addition, Hupkes and Van Vleck showed that for the backward-Euler discretisation scheme the previously mentioned $r(c)$ relation is multivalued, even for $c \neq 0$. This is in major contrast to the spatially discrete setting. In this part of the analysis, the authors relied heavily of the inclusion of the system (1.1.27) into an MFDE which admits a comparison principle. This is not possible for the other five BDF discretisation schemes. To alleviate this, Hupkes and Van Vleck also provided numerical evidence that the $r(c)$ relation is multivalued for at least the second BDF discretisation scheme.

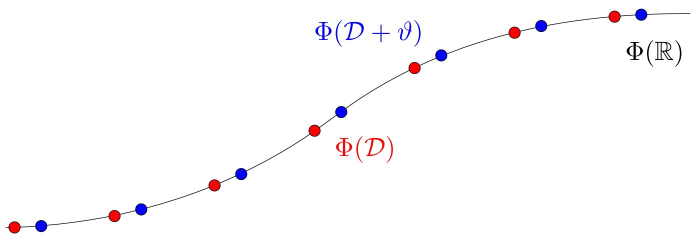


Figure 1.4: *The travelling wave profiles Φ and $\Phi(\cdot+\vartheta)$, defined on the domain $\mathcal{D}$, potentially yield two different travelling wave solutions to the system (1.1.25) for the same parameter values, but possible different wavespeed.*

1.2 The FitzHugh-Nagumo system

Let us return to the propagation of electrical signals through nerve fibres. Naturally, it is a challenge to find effective equations describing this behaviour. Initially, models describing this behaviour did not take the discrete structure into account directly. Based on experiments on giant squids, the first model was formulated in the 1950s and consists of a system of four nonlinear equations, called the Hodgkin-Huxley equations [98]. However, due to the high complexity of this system, an analytical approach to understand the dynamical behaviour of this system turned out to be a major challenge. Instead, in 1961, FitzHugh formulated a spatially homogeneous system to describe the potential felt by a single point on the nerve axon as the signal travels by [74]. A few years later, FitzHugh [76] and Nagumo [137] added a diffusion term to this system to describe the dynamics on the full line. Indeed, they formulated what is now known as the FitzHugh-Nagumo partial differential equation (PDE). This PDE is given by

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) &= \frac{\partial^2 u}{\partial x^2}(x,t) + g(u(x,t);r) - w(x,t), \\ \frac{\partial w}{\partial t}(x,t) &= \rho\big[u(x,t) - \gamma w(x,t)\big]. \end{aligned} \tag{1.2.1}$$

In this system, the variable $u(x,t)$ describes the potential felt on the space point x at the time t, while $w(x,t)$ describes a recovery component. The Nagumo PDE (1.1.6) can be seen as a simplified version of the FitzHugh-Nagumo PDE (1.2.1). The bistable nonlinearity g is, as before, given by the cubic polynomial $g(u;r) = u(1-u)(u-r)$. In addition, $\rho > 0$ and $\gamma > 0$ are positive constants. As early as 1968 [75], FitzHugh released a computer simulation which clearly shows that the system (1.2.1) admits travelling pulse solutions, which resemble the spike signals found experimentally in the nerve axon of the giant squid by Hodgkin and Huxley. As such, the FitzHugh-Nagumo PDE is commonly used as a simplification of the Hodgkin-Huxley equations.

Mathematically, the FitzHugh-Nagumo PDE turned out to be a very interesting equation due to the combination of the relative simplicity of its structure with the rich behaviour of its dynamics. Indeed, the mathematical construction and analysis of the travelling pulse solutions as observed by FitzHugh turned out to be a major challenge that is still on-going. In particular, let us set out to find a solution (u, w) to the system (1.2.1) that takes the form

$$(u, w)(x, t) \quad = \quad (\overline{u}_0, \overline{w}_0)(x + \overline{c}_0 t), \tag{1.2.2}$$

where $\overline{u}_0$ and $\overline{w}_0$ are the wave profiles and $\overline{c}_0$ is the wavespeed. The wave profiles $\overline{u}_0$ and $\overline{w}_0$ must satisfy the limits

$$\lim_{\xi \to \pm\infty} (\overline{u}_0, \overline{w}_0)(\xi) \quad = \quad (0, 0) \tag{1.2.3}$$

to turn it into a pulse instead of merely a wave. We substitute the Ansatz (1.2.2) into the PDE (1.2.1) to obtain the ODE

$$\begin{aligned} \overline{c}_0 \overline{u}_0'(\xi) &= \overline{u}_0''(\xi) + g(\overline{u}_0(\xi); r) - \overline{w}_0(\xi), \\ \overline{c}_0 \overline{w}_0'(\xi) &= \rho\big[\overline{u}_0(\xi) - \gamma \overline{w}_0(\xi)\big], \end{aligned} \tag{1.2.4}$$

where $\xi = x + \overline{c}_0 t$. Travelling pulse solutions to the PDE (1.2.1) are homoclinic solutions to the ODE (1.2.4).

Typically, the system (1.2.4) has been studied in the $\rho \ll 1$ regime. Then $\rho \downarrow 0$ limit is singular, as substituting $\rho = 0$ in (1.2.4), effectively, yields a scalar equation, instead of a system of equations. Moreover, if we, instead, first rescale the variable ξ in (1.2.4) by ρ and then take the limit $\rho \downarrow 0$, we obtain a different limiting system. The first limiting system is called the fast limiting system, while the second is called the slow limiting system. As such, the system (1.2.4) is a so-called *fast-slow system*. The analysis of the system (1.2.4) in both $\rho \downarrow 0$ limits has led to the discovery of many new techniques in the field of singular perturbation theory. We refer to [118] for an interesting overview of these techniques. A recent overview of the existence and stability of pulse solutions for the PDE (1.2.1) can be found in [34]. Finally, we want to mention that, recently, several results have been developed [92–94] for the existence and nonlinear stability for pulse solutions of FitzHugh-Nagumo systems with added random noise.

However, all previously mentioned results feature the FitzHugh-Nagumo PDE (1.2.1). Since this equation is spatially homogeneous, it does not directly take the discrete properties of the nerve axon it is aiming to simulate, into account. As such, it has been proposed [123] to, instead, model the signal propagation through nerve fibres using a FitzHugh-Nagumo LDE, which is given by the system

$$\begin{aligned} u_j'(t) &= \tfrac{1}{h^2}\big[u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)\big] + g(u_j(t); r) - w_j(t), \\ w_j'(t) &= \rho\big[u_j(t) - \gamma w_j(t)\big]. \end{aligned} \tag{1.2.5}$$

The variables u_j and w_j now represent the potential felt and the recovery component at the jth node respectively. We note that the LDE (1.2.5) can be obtained directly from the PDE (1.2.1) by using a nearest neighbour discretisation of the Laplacian on a grid with spatial distance $h > 0$. A travelling pulse solution to the LDE (1.2.5) now takes the form

$$(u_j, w_j)(t) \quad = \quad (\overline{u}_h, \overline{w}_h)(hj + \overline{c}_h t) \tag{1.2.6}$$

for some wave profiles $\overline{u}_h$ and $\overline{w}_h$ and wavespeed $\overline{c}_h$. Substituting the Ansatz (1.2.6) into the LDE (1.2.5) yields the MFDE

$$\begin{aligned} \overline{c}_h \overline{u}_h'(\xi) &= \tfrac{1}{h^2}\big[\overline{u}_h(\xi+h) + \overline{u}_h(\xi-h) - 2\overline{u}_h(\xi)\big] + g(\overline{u}_h(\xi);r) - \overline{w}_h(\xi), \\ \overline{c}_h \overline{w}_h'(\xi) &= \rho\big[\overline{u}_h(\xi) - \gamma\overline{w}_h(\xi)\big], \end{aligned} \tag{1.2.7}$$

in which $\xi = hj + \overline{c}_0 t$. We emphasize that, in contrast to the Nagumo system, the FitzHugh-Nagumo PDE (1.2.1) and LDE (1.2.5) do not admit a comparison principle.

In [108, 109], Hupkes and Sandstede constructed travelling pulse solutions to the system (1.2.5) and showed that these pulses are nonlinearly stable. They assumed that they were in the parameter regime where the travelling front solution $\overline{u}$ for the corresponding Nagumo LDE (1.1.9) has nonzero wavespeed. The main idea in [108, 109] is to use what is known as Lin's method to combine the travelling front $\overline{u}$ and a reflection of this front $\overline{u}$ to obtain so-called quasi-front and quasi-back solutions, see Figure 1.5. These quasi-front and quasi-back solutions have gaps in predetermined finite dimensional spaces, which can be closed by choosing the wavespeed. The existence of these finite dimensional spaces hinges on the existence of exponential dichotomies for the linearization of the MFDE (1.1.10). Exponential dichotomies play an essential role in this thesis and will be discussed in more detail later, see §1.3.3.

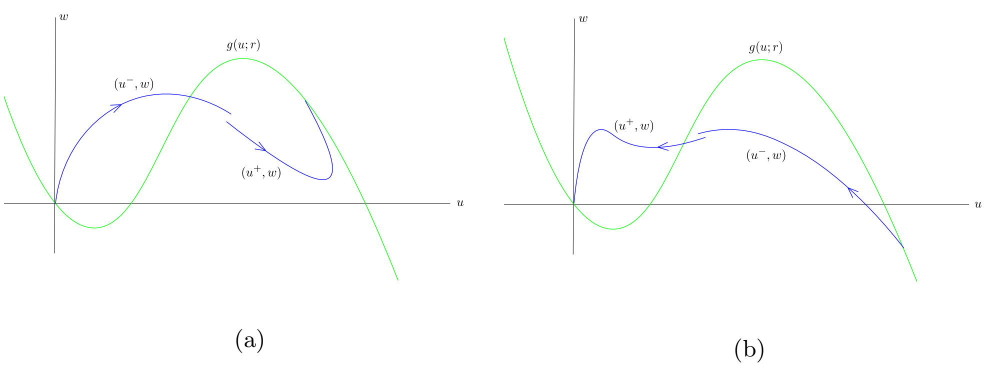


Figure 1.5: *Quasi-front (a) and quasi-back (b) solutions of the FitzHugh-Nagumo LDE. In both cases, the function u^- is defined on the interval $(-\infty, 1]$, while the function u^+ is defined on the interval $[-1, \infty)$. The difference $u^+ - u^-$, which is defined on the overlapping interval $[-1, 1]$, should be an element of a predetermined finite dimensional space. The construction of such a space is provided by the existence of exponential dichotomies for linear MFDEs.*

Infinite-range interactions In this thesis, we consider several extensions and generalizations of the FitzHugh-Nagumo LDE (1.2.5). Our first model arises in the study of neural field models. Neural field models aim to describe the interactions and dynamics in large networks of neurons. These neurons interact with each other over large distances through the nerve fibres that connect them [15, 23, 24, 142]. Due to the high complexity of these systems, it is a major challenge to find effective equations to describe this dynamical behaviour. In [23, Eq. (3.31)], a model has been proposed that features a FitzHugh-Nagumo type system with infinite-range interactions, which takes the form

$$\begin{array}{lcl}\dot{u}_j(t) & = & \frac{1}{h^2}\sum\limits_{k=1}^{\infty}\alpha_k\big[u_{j+k}(t)+u_{j-k}(t)-2u_j(t)\big]+g(u_j(t);r)-w_j(t),\\ \dot{w}_j(t) & = & \rho\big[u_j(t)-\gamma w_j(t)\big].\end{array} \tag{1.2.8}$$

Here the coefficients $\{\alpha_k\}_{k\geq 1}$ should, at the very least, satisfy the conditions (1.1.21)-(1.1.22) to ensure Laplace-like behaviour. The system (1.2.8) was first studied by Faye and Scheel in [69]. They constructed travelling pulse solutions to the system (1.2.8) under the assumption that the coefficients $\{\alpha_k\}_{k\geq 1}$ decay exponentially. Since, at the time of writing, exponential dichotomies for systems such as (1.2.8) were not available, Faye and Scheel were forced to use a different approach than the one employed by Hupkes and Sandstede for the finite-range version (1.2.5). Indeed, Faye and Scheel used a functional analytic approach to circumvent the use of a state space. However, they did not establish the stability of the pulse solutions they found. In Chapter 2, we expand the previously mentioned spectral convergence method to establish the existence and nonlinear stability of travelling pulse solutions to the system (1.2.8) in the near-continuum regime $h \ll 1$. The stability of pulse solutions outside the near-continuum regime remains an open problem. However, we expect that our results on the existence of exponential dichotomies for MFDEs with infinite-range interactions in Chapters 5-6 are a sufficient theoretical foundation to, eventually, solve this open problem.

Spatial periodicity Recent experiments in optical nanoscopy [50, 51, 165] clearly show that certain proteins in the cytoskeleton of nerve fibres are organised periodically. In particular, this periodicity manifests itself at the nodes of Ranvier. As such, it is natural to consider a spatially periodic version of the FitzHugh-Nagumo LDE (1.2.5). This spatially periodic LDE takes the form

$$\begin{array}{lcl}\dot{u}_j(t) & = & d_j\big[u_{j+1}(t)+u_{j-1}(t)-2u_j(t)\big]+g(u_j(t);r_j)-w_j(t),\\ \dot{w}_j(t) & = & \rho_j\big[u_j(t)-\gamma_j w_j(t)\big],\end{array} \tag{1.2.9}$$

where the 2-periodic coefficients $(d_j, r_j, \rho_j, \gamma_j)$ satisfy

$$(0,\infty)\times(0,1)\times(0,1)\times(0,\infty)\ni(d_j,r_j,\rho_j,\gamma_j) \quad = \quad \begin{cases}(\varepsilon^{-2},r_o,\rho_o,\gamma_o) & \text{for odd } j,\\ (1,r_e,\rho_e,\gamma_e) & \text{for even } j,\end{cases} \tag{1.2.10}$$

where $0<\varepsilon\ll 1$. In particular, we have a scale separation between the diffusion coefficients 1 and ε^{-2}. The system (1.2.9) does not have a clear continuum limit. Nonetheless, we obtain the existence and nonlinear stability of travelling pulse solutions to the system (1.2.9) in the $\varepsilon \ll 1$ regime by further developing the spectral convergence method in Chapter 3.

Temporal discretisations Finally in Chapter 4, inspired by the work of Hupkes and Van Vleck [111] which was discussed previously, we consider temporal discretisations of the LDE (1.2.5), using the six BDF discretisation schemes. For the backward-Euler discretisation scheme, the corresponding evolution is given by

$$\begin{aligned}
\tfrac{1}{\Delta t}[U_j(n\Delta t) - U_j((n-1)\Delta t)] &= d\big[U_{j+1} + U_{j-1} - 2U_j\big](n\Delta t) + g(U_j(n\Delta t); r) \\
&\quad - W_j(n\Delta t), \\
\tfrac{1}{\Delta t}[W_j(n\Delta t) - W_j((n-1)\Delta t)] &= \rho[U_j(n\Delta t) - \gamma W_j(n\Delta t)],
\end{aligned} \tag{1.2.11}$$

for $n \in \mathbb{Z}$ and time-step $\Delta t > 0$. We establish the existence of travelling pulse solutions to the system (1.2.11) by carefully combining the different extensions to the spectral convergence method from [111] and Chapters 2-3. The nonuniqueness of this travelling wave solution, which was previously discussed for the Nagumo system (1.1.25), is present here as well due to the possibility of an irrational phase shift. In addition, we are interested in the $r(c)$ relation for the system (1.2.11). However, the analytical approach employed by Hupkes and Van Vleck for the Nagumo system (1.1.25) relied heavily on the comparison principle, which is not available for FitzHugh-Nagumo systems. Instead, we use numerical simulations to show that the $r(c)$ relation is multivalued for the system (1.2.11), even for $c \neq 0$. This is in major contrast to the FitzHugh-Nagumo PDE (1.2.1) and LDE (1.2.5).

1.3 Techniques

The main techniques to analyze our main systems (1.2.8), (1.2.9) and (1.2.11) fall into two main categories: those that feature the spectral convergence method and those that feature exponential dichotomies. Both of these techniques rely heavily on the Fredholm theory for linear MFDEs. In the remaining part of this chapter, we will discuss these techniques in more detail and explain how they can be applied to our main systems.

1.3.1 Linear Fredholm theory

In the construction and analysis of travelling waves, it is usually essential to understand the underlying linear system. Often, it is useful to consider the Fredholm properties of the corresponding linear operators. If X and Y are normed vector spaces, then we say that a linear operator $T : X \to Y$ is a *Fredholm operator* if the following properties are satisfied.

(i) The kernel satisfies $\dim\big(\ker(T)\big) < \infty$.

(ii) The range satisfies $\operatorname{codim}\big(\operatorname{Range}(T)\big) < \infty$.

(iii) The range $\operatorname{Range}(T)$ is closed.

When T is a Fredholm operator, the Fredholm index of T is given by

$$\operatorname{Ind}(T) = \dim\big(\ker(T)\big) - \operatorname{codim}\big(\operatorname{Range}(T)\big). \tag{1.3.1}$$

Let us now consider the linear MFDE given by

$$cu'(\xi) = d\big[u(\xi+1)+u(\xi-1)-2u(\xi)\big]+g_u(\overline{u}(\xi);r)u(\xi), \tag{1.3.2}$$

which arises as the linearization of the Nagumo travelling wave MFDE (1.1.10) around a travelling wave solution $\overline{u}$. For clarity, we set $d=c=1$. We rewrite this MFDE in the more suggestive form

$$u'(\xi) = u(\xi-1)+\big[g_u(\overline{u}(\xi);r)-2\big]u(\xi+0)+u(\xi+1). \tag{1.3.3}$$

The scalar functions $1, g_u(\overline{u}(\xi);r)-2$ and 1 are called the *coefficients* of the systems and the real numbers $-1, 0$ and 1 are called the *shifts*. The linear operator corresponding to the system (1.3.3) is given by

$$(\Lambda u)(\xi) = u'(\xi)-u(\xi-1)-\big[g_u(\overline{u}(\xi);r)-2\big]u(\xi+0)-u(\xi+1). \tag{1.3.4}$$

It is not immediately clear on which space the operator Λ from (1.3.4) is posed and how to determine the Fredholm properties of this operator. It turns out to be a natural choice to consider the Sobolev spaces

$$W^{1,p}(\mathbb{R};\mathbb{C}) = \{u\in L^p(\mathbb{R};\mathbb{C}) : u'\in L^p(\mathbb{R};\mathbb{C})\} \tag{1.3.5}$$

for $1\le p\le\infty$, equipped with the Sobolev norm

$$\|u\|^p_{W^{1,p}(\mathbb{R};\mathbb{C})} = \|u\|^p_{L^p(\mathbb{R};\mathbb{C})}+\|u'\|^p_{L^p(\mathbb{R};\mathbb{C})}. \tag{1.3.6}$$

In this definition, we use u' to denote the weak derivative of a function u. For the space $W^{1,2}(\mathbb{R};\mathbb{C})$ we often use the shorthand $H^1(\mathbb{R};\mathbb{C})$. Using this definition, we view Λ from (1.3.4) as an operator

$$\Lambda : W^{1,p}(\mathbb{R};\mathbb{C}) \to L^p(\mathbb{R};\mathbb{C}). \tag{1.3.7}$$

The works by Rustichini [144, 145] and Mallet-Paret [130] contain the main Fredholm theory for this operator Λ. We recall that the travelling front $\overline{u}$ satisfies the limits

$$\lim_{\xi\to-\infty}\overline{u}(\xi) = 0, \qquad \lim_{\xi\to\infty}\overline{u}(\xi) = 1 \tag{1.3.8}$$

and that $g_u(0;r)=-r$ and $g_u(1;r)=r-1$. Hence, it is natural—and it will also turn out to be useful—to consider the systems

$$\begin{aligned} u'(\xi) &= u(\xi-1)+\lim_{\xi'\to-\infty}\big[g_u(\overline{u}(\xi');r)-2\big]u(\xi+0)+u(\xi+1) \\ &= u(\xi-1)+\big[-r-2\big]u(\xi+0)+u(\xi+1) \end{aligned} \tag{1.3.9}$$

and

$$\begin{aligned} u'(\xi) &= u(\xi-1)+\lim_{\xi'\to\infty}\big[g_u(\overline{u}(\xi');r)-2\big]u(\xi+0)+u(\xi+1) \\ &= u(\xi-1)+\big[r-1-2\big]u(\xi+0)+u(\xi+1). \end{aligned} \tag{1.3.10}$$

We refer to the systems (1.3.9) and (1.3.10) as the *limiting systems* of the MFDE (1.3.3) at $-\infty$ and ∞ respectively. We note that the systems (1.3.9) and (1.3.10) are autonomous, since their coefficients do not depend on ξ. Finding a solution to (1.3.9) or (1.3.10) of the form $e^{z\xi}w$ is equivalent to finding a root of the so-called *characteristic function*

$$\begin{array}{rcl}\Delta_-(z) & = & z - e^{-z} - \big[-r-2\big]e^{0\cdot z} - e^{z},\\ \Delta_+(z) & = & z - e^{-z} - \big[(r-1)-2\big]e^{0\cdot z} - e^{z},\end{array} \tag{1.3.11}$$

that is, finding $z \in \mathbb{C}$ for which $\Delta_-(z) = 0$ and $\Delta_+(z) = 0$ respectively. Such a scalar z is referred to as a *spatial eigenvalue*. We say that the autonomous system (1.3.9) or (1.3.10) is *hyperbolic* if it has no spatial eigenvalues on the imaginary axis, i.e. $\Delta_-(iy) \neq 0$ respectively $\Delta_+(iy) \neq 0$ for all $y \in \mathbb{R}$. A short computation shows that this is, indeed, the case for the systems (1.3.9) and (1.3.10). Systems with this property are called *asymptotically hyperbolic*. We write Λ_- and Λ_+ for the linear operators corresponding to the systems (1.3.9) and (1.3.10) respectively, which are given by

$$\begin{array}{rcl}(\Lambda_- u)(\xi) & = & u'(\xi) - u(\xi-1) - \big[-r-2\big]u(\xi+0) - u(\xi+1),\\ (\Lambda_+ u)(\xi) & = & u'(\xi) - u(\xi-1) - \big[(r-1)-2\big]u(\xi+0) - u(\xi+1).\end{array} \tag{1.3.12}$$

It turns out that, since the systems (1.3.9) and (1.3.10) are autonomous and hyperbolic, the operators Λ_- and Λ_+ are invertible as operators from $W^{1,p}(\mathbb{R};\mathbb{C})$ to $L^p(\mathbb{R};\mathbb{C})$, independently of $1 \leq p \leq \infty$. In fact, the inverse operators are given explicitly by the Green's function, in the sense that

$$(\Lambda_\pm^{-1}u)(\xi) \quad = \quad \int_{-\infty}^{\infty} G_\pm(\xi-\eta)u(\eta)d\eta, \tag{1.3.13}$$

where the Green's functions $G_\pm$ are given by

$$G_\pm(\xi) \quad = \quad \tfrac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\eta\xi}\big(\Delta_\pm(i\eta)\big)^{-1}d\eta. \tag{1.3.14}$$

A non-autonomous system, however, is not necessarily invertible. For example, the derivative $\overline{u}'$ is a kernel element of the system (1.3.3), which can be seen by differentiating the system (1.1.10). The results in [130] show that linear operators corresponding to asymptotically hyperbolic systems are automatically Fredholm operators as operators from $W^{1,p}(\mathbb{R};\mathbb{C})$ to $L^p(\mathbb{R};\mathbb{C})$. In addition, the Fredholm index of such an operator Λ is independent of $1 \leq p \leq \infty$ and the range of Λ can be made explicit by writing

$$\mathrm{Range}(\Lambda) \quad = \quad \big\{u \in L^p(\mathbb{R};\mathbb{C}) : \int_{-\infty}^{\infty}\overline{u(\xi)}v(\xi) = 0 \text{ for all } v \in \ker(\Lambda^*)\big\}. \tag{1.3.15}$$

Here we have introduced the *adjoint operator* Λ^*. The operator $\Lambda^* : W^{1,p}(\mathbb{R};\mathbb{C}) \to L^p(\mathbb{R};\mathbb{C})$ is given by

$$(\Lambda^* u)(\xi) \quad = \quad -u'(\xi) - u(\xi-(-1)) - \big[g_u(\overline{u}(\xi);r) - 2\big]u(\xi-0) - u(\xi-1) \tag{1.3.16}$$

and is called the adjoint operator because it satisfies the identity

$$\langle \Lambda u, v\rangle_{L^2(\mathbb{R};\mathbb{C})} \quad = \quad \langle u, \Lambda^* v\rangle_{L^2(\mathbb{R};\mathbb{C})} \tag{1.3.17}$$

for any pair $u, v \in H^1(\mathbb{R};\mathbb{C})$.

If there exists a homotopy between the systems at $-\infty$ and ∞ and none of the eigenvalues crosses the imaginary axis during this homotopy, then the spectral flow theorem [130, Thm. C] allows us to conclude that the Fredholm index of the corresponding linear operator is, in fact, 0. Such a homotopy is trivially available if the systems at $-\infty$ and ∞ coincide, for example when linearizing around a travelling pulse solution. However, even in such a setting it remains a nontrivial challenge to determine the dimension of the kernel of the linear operator, which is necessary, for example, when determining the spectrum of the linear operator. In that case, other techniques, such as appropriate limits or comparison principles are needed to understand these linear operators in full detail.

For the system (1.3.3), this homotopy can be made explicit. For $0 \le \rho \le 1$, we can consider the linear operator

$$\Lambda_\rho \quad = \quad \rho\Lambda(-\infty) + (1-\rho)\Lambda(\infty). \tag{1.3.18}$$

The corresponding characteristic function is given by

$$\Delta_\rho(z) \quad = \quad z - e^{-z} - [\rho(-r) + (1-\rho)(r-1) - 2]e^{0\cdot z} - e^{z}, \tag{1.3.19}$$

which can easily be seen to have no roots on the imaginary axis. In particular, the system corresponding to the operator Λ_ρ is hyperbolic for each $0 \le \rho \le 1$, which means that the map $\rho \mapsto \Lambda_\rho$ is a homotopy between the systems at $-\infty$ and ∞. In particular, the operator Λ from (1.3.4) is a Fredholm operator with Fredholm index 0. We already observed that the derivative $\overline{u}'$ is a kernel element of Λ. By the definition of the Fredholm index, the codimension of Range(Λ) in $L^p(\mathbb{R};\mathbb{C})$ must be at least one. In addition, the identity (1.3.15) yields that the dimension of $\ker(\Lambda^*)$ must also be at least one. In particular, we have established several strong results on the operator Λ and its adjoint Λ^* using relatively simple computations.

We remark that the results in [131] show that, in this case, the kernels $\ker(\Lambda)$ and $\ker(\Lambda^*)$ are, in fact, precisely one-dimensional on account of the comparison principle.

The Fredholm theory as described above has been extended by Faye and Scheel [68] to include MFDEs which feature infinite range interactions, such as the linearization of the system (1.1.20). However, their restrictions on the coefficients were more severe than those featured in (1.1.21)-(1.1.22). In particular, Faye and Scheel required the coefficients to decay exponentially.

1.3.2 The spectral convergence method

As was stated previously, the spectral convergence method was pioneered by Bates, Chen and Chmaj in [6] in order to construct travelling wave solutions to the Nagumo LDE (1.1.18) with infinite-range interactions. One of the main advantages of this method is that it circumvents the use of a comparison principle or exponential dichotomies. As a consequence, it can be applied to a broader class of coefficients than many other techniques.

To illustrate the spectral convergence method, we focus on its original application to the Nagumo LDE (1.1.18) by following [6]. We fix, for now, a small constant $h > 0$. The main goal of the spectral convergence method is to transfer the known Fredholm properties of the linearization of the continuous system to an appropiate linearization of the discrete system. We need to be a bit careful at this point. Since the end goal is to construct a travelling wave solution to the LDE (1.1.18), we cannot consider a system such as (1.3.2), since it is impossible to linearize around a solution that has not been found yet. Instead, we let u_0 be a travelling front solution of the PDE (1.1.6) with wavespeed c_0 and consider the linearizations of both the ODE (1.1.8) and the MFDE (1.1.20) around the wave u_0. These linearizations yield the linear operators

$$(\mathcal{L}_0 u)(\xi) \quad = \quad c_0 u'(\xi) - u''(\xi) - g_u(u_0(\xi); r)u(\xi) \tag{1.3.20}$$

for the ODE (1.1.8) and

$$(\mathcal{L}_h u)(\xi) \quad = \quad c_0 u'(\xi) - \Delta_h u(\xi) - g_u(u_0(\xi); r)u(\xi) \tag{1.3.21}$$

for the MFDE (1.1.20). Here we recall that the operator Δ_h is given by (1.1.23). First, we need to specify on which spaces the operators $\mathcal{L}_0$ and $\mathcal{L}_h$ are posed, which immediately brings us to the first major complication (and, therefore, strength of the spectral convergence method). The operator $\mathcal{L}_h$ can, and should, clearly be viewed as an operator

$$\mathcal{L}_h : H^1(\mathbb{R}; \mathbb{R}) \quad \to \quad L^2(\mathbb{R}; \mathbb{R}). \tag{1.3.22}$$

However, since the operator $\mathcal{L}_0$ features a second derivative, it cannot be a well-defined operator on $H^1(\mathbb{R}; \mathbb{R})$. Instead, we view it as an operator

$$\mathcal{L}_0 : H^2(\mathbb{R}; \mathbb{R}) \quad \to \quad L^2(\mathbb{R}; \mathbb{R}), \tag{1.3.23}$$

where we have introduced the space

$$H^2(\mathbb{R}; \mathbb{R}) \quad = \quad \{u \in H^1(\mathbb{R}; \mathbb{R}) : u'' \in L^2(\mathbb{R}; \mathbb{R})\} \tag{1.3.24}$$

with corresponding norm

$$\|u\|^2_{H^2(\mathbb{R};\mathbb{R})} \quad = \quad \|u\|^2_{H^1(\mathbb{R};\mathbb{R})} + \|u''\|^2_{L^2(\mathbb{R};\mathbb{R})}. \tag{1.3.25}$$

In particular, the operators $\mathcal{L}_0$ and $\mathcal{L}_h$ act on different spaces, which makes lifting the Fredholm properties of $\mathcal{L}_0$ to $\mathcal{L}_h$ a delicate effort.

It is well-known that for each $\delta \geq 0$ the operator $\mathcal{L}_0 + \delta$ is a Fredholm operator with Fredholm index 0. In addition, this operator is invertible for $\delta > 0$, while it is has a one-dimensional kernel, spanned by the derivative u_0', for $\delta = 0$. The standard Fredholm theory for ODEs implies that the adjoint operator $\mathcal{L}_0^*$ also has a one-dimensional kernel, spanned by some function ϕ_0^-, i.e. we have

$$\ker(\mathcal{L}_0) \quad = \quad \mathrm{span}\{u_0'\}, \qquad \ker(\mathcal{L}_0^*) \quad = \quad \mathrm{span}\{\phi_0^-\}. \tag{1.3.26}$$

Using standard arguments [6, Lem. 3.1], one can show that there exists a constant $C > 0$ in such a way that the bound

$$\|(\mathcal{L}_0 + \delta)^{-1}\psi\|_{H^2(\mathbb{R};\mathbb{R})} \quad \leq \quad C\big[\|\psi\|_{L^2(\mathbb{R};\mathbb{R})} + \tfrac{1}{\delta}|\langle\psi, \phi_0^-\rangle_{L^2(\mathbb{R};\mathbb{R})}|\big] \tag{1.3.27}$$

holds for all $\delta > 0$ and all $\psi \in L^2(\mathbb{R};\mathbb{R})$. The spectral convergence method aims to show that for all $\delta > 0$ there exists a positive constant $h_0(\delta) > 0$ such that for all $h \in (0, h_0(\delta))$ the operator $\mathcal{L}_h + \delta$ is invertible and that the bound

$$\|(\mathcal{L}_h + \delta)^{-1}\psi\|_{H^1(\mathbb{R};\mathbb{R})} \quad \leq \quad \tilde{C}\big[\|\psi\|_{L^2(\mathbb{R};\mathbb{R})} + \tfrac{1}{\delta}|\langle\psi, \phi_0^-\rangle_{L^2(\mathbb{R};\mathbb{R})}|\big] \tag{1.3.28}$$

holds for all $\psi \in L^2(\mathbb{R};\mathbb{R})$. Here, the constant $\tilde{C}$ should be taken independently of $\delta > 0$ and $0 < h < h_0(\delta)$. Employing the bound (1.3.28), a more or less standard argument, that resembles the proof of the implicit function theorem, can be used to construct the travelling wave solutions to the system (1.1.18).

In order to establish the bound (1.3.28), Bates, Chen and Chmaj consider the quantities

$$\Lambda(h,\delta) \quad = \quad \inf_{\phi \in H^1(\mathbb{R};\mathbb{R}), \|\phi\|_{H^1(\mathbb{R};\mathbb{R})}=1} \big[\|(\mathcal{L}_h + \delta)\phi\|_{L^2(\mathbb{R};\mathbb{R})} + \tfrac{1}{\delta}|\langle(\mathcal{L}_h + \delta)\phi, \phi_0^-\rangle_{L^2(\mathbb{R};\mathbb{R})}|\big] \tag{1.3.29}$$

for $h > 0$ and $\delta > 0$, together with

$$\Lambda(\delta) \quad = \quad \liminf_{h \downarrow 0} \Lambda(h,\delta). \tag{1.3.30}$$

The key ingredient is to construct a lower bound on the quantity $\Lambda(\delta)$, which is uniform in $\delta > 0$. If such a lower bound is found, the invertibility of the operator $\mathcal{L}_h + \delta$ and the bound (1.3.28) can be established relatively easily.

We now fix $\delta > 0$ and consider sequences

$$\{\phi_j\}_{j \geq 1} \subset H^1(\mathbb{R};\mathbb{R}), \qquad \|\phi_j\|_{H^1(\mathbb{R};\mathbb{R})} = 1, \qquad h_j \downarrow 0 \tag{1.3.31}$$

which minimize the quantity $\Lambda(\delta)$. That is, we have the limit

$$\lim_{j \to \infty} \|(\mathcal{L}_{h_j} + \delta)\phi_j\|_{L^2(\mathbb{R};\mathbb{R})} + \tfrac{1}{\delta}|\langle(\mathcal{L}_{h_j} + \delta)\phi_j, \phi_0^-\rangle_{L^2(\mathbb{R};\mathbb{R})}| \quad = \quad \Lambda(\delta). \tag{1.3.32}$$

The existence of these minimizing sequences follows directly from the definition of the quantity $\Lambda(\delta)$. For convenience, we write

$$\psi_j \quad = \quad (\mathcal{L}_{h_j} + \delta)\phi_j \tag{1.3.33}$$

for $j \geq 1$. In order to properly take the $h \downarrow 0$ limit, we consider the weak limits ϕ and ψ of the sequences $\{\phi_j\}_{j \geq 1}$ and $\{\psi_j\}_{j \geq 1}$. The first computational effort is to show that the function ϕ is an element of the space $H^2(\mathbb{R};\mathbb{R})$ and that it is a weak solution of the equation $(\mathcal{L}_0 + \delta)\phi = \psi$ [6, Lem. 3.2]. This computation relies heavily on the limit (1.1.24). As a result, we obtain the lower bound

$$\|\phi\|_{H^2(\mathbb{R}:\mathbb{R})} \quad \leq \quad K\Lambda(\delta) \tag{1.3.34}$$

for some constant $K > 0$.

It remains to find a positive lower bound for the norm $\|\phi\|_{H^2(\mathbb{R};\mathbb{R})}$. Indeed, a common danger when taking weak limits is that the sequence converges to 0 even though the sequence itself is bounded away in norm from 0. Using the Laplace-like properties of the operator Δ_h, we obtain the estimates

$$\langle \Delta_h v, v' \rangle_{L^2(\mathbb{R};\mathbb{R})} = 0, \qquad \langle \Delta_h v, v \rangle_{L^2(\mathbb{R};\mathbb{R})} \leq 0 \tag{1.3.35}$$

for any function $v \in H^1(\mathbb{R};\mathbb{R})$. Employing the bounds (1.3.35) and remembering that $(\mathcal{L}_{h_j} + \delta)\phi_j = \psi_j$, we can estimate the inner products $\langle \psi_j, \phi_j' \rangle_{L^2(\mathbb{R};\mathbb{R})}$ using the Cauchy-Schwarz inequality to obtain a uniform estimate of the form

$$A_1 \|\phi_j\|^2_{L^2(\mathbb{R};\mathbb{R})} \geq A_2 \|\phi_j'\|^2_{L^2(\mathbb{R};\mathbb{R})} - A_3 \|\psi_j\|^2_{L^2(\mathbb{R};\mathbb{R})}; \tag{1.3.36}$$

see [6, Eq. (3.9)].

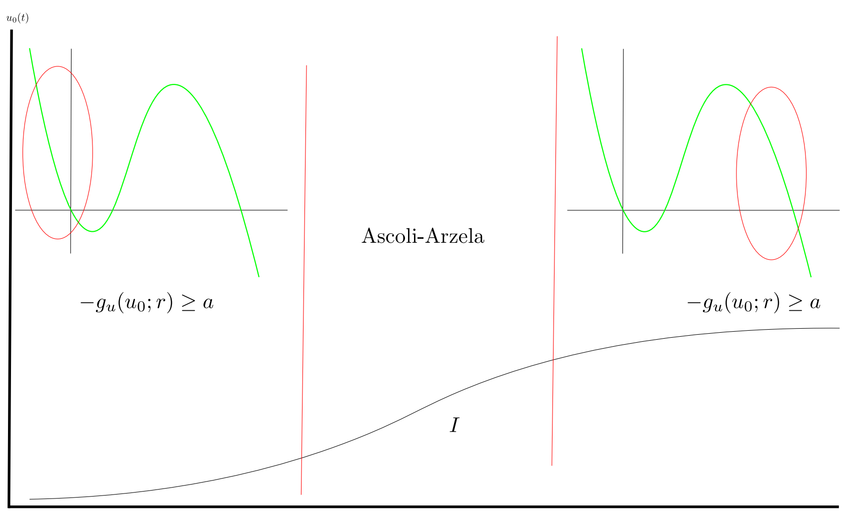


Figure 1.6: *In the spectral convergence method, we pick a compact interval I in such a way that the sign of $-g_u(u_0(x); r)$ for $x \in \mathbb{R} \setminus I$ is fixed. This is allowed because of the bistable nature of the nonlinearity g. Inside I, we employ the Ascoli-Arzela Theorem, while outside I we can use the fixed sign to aid in our estimates.*

At this point in the computation, we employ the bistable nature of the nonlinearity g. Remembering that the front u_0 connects 0 and 1, we can pick a sufficiently large, but bounded, interval I to have $-g_u(u_0(x); r) \geq a$ for x outside I for some fixed constant

$a > 0$; see Figure 1.6. This allows us to estimate the inner product

$$\begin{aligned} \langle -g_u(u_0)\phi_j, \phi_j \rangle_{L^2(\mathbb{R};\mathbb{R})} &= \langle -g_u(u_0)\phi_j, \phi_j \rangle_{L^2(\mathbb{R}\setminus I;\mathbb{R})} + \langle -g_u(u_0)\phi_j, \phi_j \rangle_{L^2(I;\mathbb{R})} \\ &\geq a\|\phi_j\|^2_{L^2(\mathbb{R}\setminus I;\mathbb{R})} - \|g_u(u_0)\|_\infty \|\phi_j\|^2_{L^2(I;\mathbb{R})} \\ &= a\|\phi_j\|^2_{L^2(\mathbb{R};\mathbb{R})} - \big(a + \|g_u(u_0)\|_\infty\big)\|\phi_j\|^2_{L^2(I;\mathbb{R})}. \end{aligned} \tag{1.3.37}$$

Inside I, we can employ the Ascoli-Arzela Theorem to have the limit $\phi_j \to \phi$ in $L^2(I;\mathbb{R})$. As such, on account of (1.3.37) we can estimate the inner products $\langle \psi_j, \phi_j \rangle_{L^2(\mathbb{R};\mathbb{R})}$ to obtain a uniform estimate of the form

$$B_1\|\phi_j\|^2_{L^2(I;\mathbb{R})} \geq B_2\|\phi_j\|^2_{L^2(\mathbb{R};\mathbb{R})} - B_3\|\psi_j\|^2_{L^2(\mathbb{R};\mathbb{R})}; \tag{1.3.38}$$

see [6, Eq. (3.10)]. By properly scaling the inequalities (1.3.36) and (1.3.38) and adding them, we obtain a uniform estimate of the form

$$C_1\|\phi_j\|^2_{L^2(I;\mathbb{R})} \geq C_2\|\phi_j\|^2_{L^2(\mathbb{R};\mathbb{R})} + C_2\|\phi_j'\|^2_{L^2(\mathbb{R};\mathbb{R})} - C_3\|\psi_j\|^2_{L^2(\mathbb{R};\mathbb{R})}. \tag{1.3.39}$$

Remebering that

$$\begin{aligned} \|\phi_j\|^2_{L^2(\mathbb{R};\mathbb{R})} + \|\phi_j'\|^2_{L^2(\mathbb{R};\mathbb{R})} &= \|\phi_j\|^2_{H^1(\mathbb{R};\mathbb{R})} \\ &= 1 \end{aligned} \tag{1.3.40}$$

the inequality (1.3.39) reduces to

$$C_1\|\phi_j\|^2_{L^2(I;\mathbb{R})} \geq C_2 - C_3\|\psi_j\|^2_{L^2(\mathbb{R};\mathbb{R})}. \tag{1.3.41}$$

Because of the strong convergence $\phi_j \to \phi$ in $L^2(I;\mathbb{R})$, the bound (1.3.34) and the limit (1.3.32) we can take the limit $j \to \infty$ in (1.3.41) to obtain

$$\begin{aligned} C_2 - C_3\Lambda(\delta)^2 &\leq C_1\|\phi\|^2_{L^2(I;\mathbb{R})} \\ &\leq C_1\|\phi\|^2_{H^2(\mathbb{R};\mathbb{R})} \\ &\leq C_1 K \Lambda(\delta)^2. \end{aligned} \tag{1.3.42}$$

In particular, Bates, Chen and Chmaj obtain

$$\Lambda(\delta) \geq \sqrt{\tfrac{C_2}{C_1 K + C_3}}, \tag{1.3.43}$$

which is a positive constant, as desired.

FitzHugh-Nagumo LDE with infinite-range interactions Our first challenge is to generalize the spectral convergence method to the system (1.2.8). We construct travelling pulse solutions to this system as perturbations of the travelling pulse solutions for the FitzHugh-Nagumo PDE (1.2.1) in the $h \ll 1$ regime. In particular, we only need to assume the conditions (1.1.21)-(1.1.22) instead of exponential decay on the coefficients $\{\alpha_k\}_{k\geq 1}$. The travelling wave equation for (1.2.8) is given by

$$\begin{array}{lcl} \bar{c}_h \overline{u}_h'(\xi) & = & \frac{1}{h^2}\sum_{k=1}^{\infty} \alpha_k \big[\overline{u}_h(\xi+kh) + \overline{u}_h(\xi-kh) - 2\overline{u}_h(\xi)\big] + g(\overline{u}_h(\xi);r) - \overline{w}_h(\xi) \\ \bar{c}_h \overline{w}_h'(\xi) & = & \rho\big[\overline{u}_h(\xi) - \gamma\overline{w}_h(\xi)\big]. \end{array} \tag{1.3.44}$$

However, the generalization of the spectral convergence method from scalar to system equations is far from trivial. In particular, when estimating the equivalents of the inner products $\langle \psi_j, \phi_j\rangle_{L^2(\mathbb{R};\mathbb{R})}$ and $\langle \psi_j, \phi_j'\rangle_{L^2(\mathbb{R};\mathbb{R})}$ as described above, there are various cross-terms we need to keep under control. Luckily, we are aided by the relative simplicity of the second component of (1.3.44). In particular, the off-diagonal elements of the linearization of the MFDE (1.3.44) are constant multiples of each other, which allows us to combine their contributions and absorb them in the diagonal terms. We also generalize the spectral convergence method to yield uniform bounds for values of δ in compact subsets of the complex plane $\mathbb{C}\setminus\{0\}$.

We write $(\overline{u}_h, \overline{w}_h)$ for the new-found travelling pulse solution to (1.2.8) with wavespeed c_h. The next step is to establish the spectral stability of this travelling pulse solution. As such, we linearize the MFDE (1.3.44) around this travelling pulse solution. The corresponding linear operator is given by

$$L_h\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} c_h\frac{d}{d\xi} - \Delta_h - g_u(\overline{u}_h) & 1 \\ -\rho & c_h\frac{d}{d\xi} + \gamma\rho \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix}, \tag{1.3.45}$$

where we recall the operator Δ_h from (1.1.23).

We first note that the spectrum of the operator L_h from (1.3.45) is periodic with period $2\pi i\frac{c_h}{h}$. This period grows to infinity as $h \downarrow 0$, which makes sense since the spectrum of the operator $\mathcal{L}_0$ from (1.3.23) is not periodic. That the spectrum is periodic, can be seen as follows. For $p \in \mathbb{C}$, we consider the multiplication operator

$$[e_p\phi](\xi) = e^{p\xi}\phi(\xi). \tag{1.3.46}$$

We fix $p = 2\pi i\frac{1}{h}$. For $k \in \mathbb{Z}$, we observe that $phk \in 2\pi i\mathbb{Z}$, so that we can compute

$$\begin{array}{lcl} \big[e_{-p}[e_p\phi(\cdot + kh)]\big](\xi) & = & e^{-p\xi}[e_p\phi](\xi+hk) \\ & = & e^{-p\xi}e^{p\xi+pkh}\phi(\xi+hk) \\ & = & \phi(\xi+hk). \end{array} \tag{1.3.47}$$

In particular, we see that

$$e_{-p}\Delta_h e_p = \Delta_h. \tag{1.3.48}$$

Since we also have

$$\begin{array}{lcl} \big[e_{-p}c_h\frac{d}{d\xi}[e_p\phi]\big](\xi) & = & c_h e^{-p\xi}\frac{d}{d\xi}(e^{p\xi}\phi(\xi)) \\ & = & c_h\phi(\xi) + pc_h\phi(\xi), \end{array} \tag{1.3.49}$$

we can conclude that

$$e_{-p} L_h e_p \quad = \quad L_h + p c_h. \tag{1.3.50}$$

Since the operators e_{-p} and e_p are invertible, this means that L_h and $L_h + pc_h$ have the same spectrum, which yields the desired periodicity.

The spectral convergence method does not immediately resolve the spectral stability question. Indeed, for each individual value of $\lambda \neq 0$ we can conclude the invertibility of $L_h + \lambda$ for h sufficiently small. However, what we mean by 'sufficiently small' depends heavily on the choice of λ and can only be made uniform for λ in compact subsets of $\mathbb{C} \setminus \{0\}$. Since the period of the spectrum grows to infinity as $h \downarrow 0$, we can only apply the spectral convergence method if we exclude spectrum in a region close to 0, spectrum with a large real part and spectrum with a large imaginary part. We will discuss these issues in more detail in §2.5-2.6.

Spatially periodic FitzHugh-Nagumo LDE The next extension to the spectral convergence method is to construct travelling pulse solutions to the system (1.2.9) in the $\varepsilon \ll 1$ regime. The spatial periodicity of this system also returns in the travelling wave Ansatz, which takes the form

$$(u, w)_j(t) \quad = \quad \begin{cases} (\overline{u}_o, \overline{w}_o)(j + ct) & \text{when } j \text{ is odd}, \\ (\overline{u}_e, \overline{w}_e)(j + ct) & \text{when } j \text{ is even}. \end{cases} \tag{1.3.51}$$

Using the Ansatz (1.3.51), we arrive at the travelling wave MFDE

$$\begin{aligned} c\overline{u}_o'(\xi) &= \tfrac{1}{\varepsilon^2}\big(\overline{u}_e(\xi+1) + \overline{u}_e(\xi-1) - 2\overline{u}_o(\xi)\big) + g(\overline{u}_o(\xi); r_o) - \overline{w}_o(\xi), \\ c\overline{w}_o'(\xi) &= \rho_o[\overline{u}_o(\xi) - \gamma_o \overline{w}_o(\xi)], \\ c\overline{u}_e'(\xi) &= \big(\overline{u}_o(\xi+1) + \overline{u}_o(\xi-1) - 2\overline{u}_e(\xi)\big) + g(\overline{u}_e(\xi); r_e) - \overline{w}_e(\xi), \\ c\overline{w}_e'(\xi) &= \rho_e[\overline{u}_e(\xi) - \gamma_e \overline{w}_e(\xi)]. \end{aligned} \tag{1.3.52}$$

Since we consider the $\varepsilon \ll 1$ regime, we first need to understand the system (1.3.52) for $\varepsilon = 0$. Multiplying the first line of (1.3.52) with ε^2 and taking the limit $\varepsilon \downarrow 0$ yields

$$0 \quad = \quad \overline{u}_e(\xi+1) + \overline{u}_e(\xi-1) - 2\overline{u}_o(\xi). \tag{1.3.53}$$

In particular, we can express $\overline{u}_o$ in terms of $\overline{u}_e$. This means that the third and fourth line of (1.3.52) become

$$\begin{aligned} c\overline{u}_e'(\xi) &= \tfrac{1}{2}\big(\overline{u}_e(\xi+2) + \overline{u}_e(\xi-2) - 2\overline{u}_e(\xi)\big) + g(\overline{u}_e(\xi); r_e) - \overline{w}_e(\xi), \\ c\overline{w}_e'(\xi) &= \rho_e[\overline{u}_e(\xi) - \gamma_e \overline{w}_e(\xi)], \end{aligned} \tag{1.3.54}$$

which we recognise as a scaled version of the regular FitzHugh-Nagumo LDE (1.2.5). We emphasize that the system (1.3.54) does not contain any odd wave functions, which means that the system decouples at $\varepsilon = 0$. In particular, we know [108, 109] that the system (1.3.54) admits stable travelling pulse solutions $(\overline{u}_{e;0}, \overline{w}_{e;0})$ with wavespeed $\overline{c}_0$. Recalling (1.3.53), we set

$$\overline{u}_{o;0}(\xi) = \tfrac{1}{2}\big[\overline{u}_{e;0}(\xi+1) + \overline{u}_{e;0}(\xi-1)\big]. \qquad (1.3.55)$$

Finally, we let $\overline{w}_{o;0}$ be the solution of the linear, inhomogeneous system

$$\overline{c}_0 \overline{w}_o'(\xi) = \rho_o[\overline{u}_{o;0}(\xi) - \gamma_o \overline{w}_o(\xi)]. \qquad (1.3.56)$$

As such, the multiplet $\overline{U}_0 = (\overline{u}_{o;0}, \overline{w}_{o;0}, \overline{u}_{e;0}, \overline{w}_{e;0})$ can be seen as the solution of (1.3.52) at $\varepsilon = 0$. Note that the identity (1.3.53) essentially turns the four-component system (1.3.52) into a three-component system at $\varepsilon = 0$. We construct travelling pulse solutions to the LDE (1.2.9) by perturbing them off the function $\overline{U}_0$ by applying the spectral convergence method. However, there are a few major differences with the previous applications of the spectral convergence method. Previously, this method was used to lift Fredholm properties from a continuous to a spatially discrete system, while here we use it to lift Fredholm properties from a three-component to a four-component spatially discrete system. In addition, the different scalings of ε for the diffusion coefficients prevent us from making a direct analogue of the inequalities (1.3.35). Instead, we have to use different scalings in ε for each component to compensate for this imbalance. These different scalings in ε complicate, in turn, the fixed point arguments used to control the nonlinear terms in the construction of the travelling pulse solutions. This complication forces us to take an extra spatial derivative of the system (1.3.52).

For the spectral stability of the travelling pulse solutions to the LDE (1.2.9), we note that the spectrum is periodic with period $2\pi i c_\varepsilon$, similarly to the system (1.2.8). Luckily, this period does not blow up in the $\varepsilon \downarrow 0$ limit here. As such, we only need to exclude spectrum near 0 and with a large real part before we can apply the spectral convergence method.

In this analysis, we do not restrict ourselves to the LDE (1.2.9). Instead, we consider general spatially 2-periodic reaction-diffusion systems with $n + k$ components. Here $n \geq 1$ is the number of components with a nonzero diffusion coefficient, while $k \geq 0$ is the number of components without diffusion (so $n = k = 1$ for the FitzHugh-Nagumo LDE (1.2.9)). In particular, our results also cover the spatially periodic version of the Nagumo LDE (1.1.9) without the use of a comparison principle. However, we need conditions on the end-states that are slightly stronger than the usual temporal stability. Indeed, we need certain submatrices of the corresponding Jacobians to be positive definite, instead of simply spectrally stable. All in all, we have a broad class of systems to which the spectral convergence method can be applied.

Spatially-temporally discrete FitzHugh-Nagumo system Our final application of the spectral convergence method is to the system (1.2.11). In fact, to make the analysis as general as possible, we allow for infinite-range spatial interactions and temporal discretisations of the general $n + k$-component reaction-diffusion LDEs discussed previously.

In this case, we use the spectral convergence method to lift the Fredholm properties of the spatially discrete to the fully discrete system. As discussed previously, we need to assume that $M := (c\Delta t)^{-1}$ is rational to establish an appropriate interpolation scheme. However, Hupkes and Van Vleck relied heavily on the comparison principle to understand this interpolated spatially discrete system. As such, we need to prove several results related to this spatially discrete system from scratch, using the general Fredholm theory for these systems. In addition, the complications we faced previously for infinite-range spatial interactions and nonscalar equations in the spectral convergence method needed to be dealt with here as well.

1.3.3 Exponential dichotomies

The second major technique we develop and employ is the splittings given by exponential dichotomies. There are many ways to look at exponential dichotomies. We take the following general point of view: we say that a linear differential equations is *exponentially dichotomous* if the space of initial conditions, called the *state space*, can be split into a stable and an unstable part. Continuations of stable initial states need to decay exponentially in forward time, while those of unstable states need to decay exponentially in backward time. Let us, for example, consider a linear, autonomous ODE, given by

$$\frac{du}{d\sigma}(\sigma) = Au(\sigma) \tag{1.3.57}$$

where $u(\sigma) \in \mathbb{C}^M$ for $\sigma \in \mathbb{R}$. If the $M \times M$ matrix A is hyperbolic, i.e. has no spectrum on the imaginary axis, then the state space $\mathbb{C}^M$ can be split as

$$\mathbb{C}^M = E_0^s \oplus E_0^u, \tag{1.3.58}$$

where E_0^s is the generalized stable eigenspace and E_0^u is the generalized unstable eigenspace of A. The flow of the ODE (1.3.57) is given by $\Phi(\sigma, \tau) = \exp[A(\sigma - \tau)]$. We note that the spaces E_0^s and E_0^u remain invariant under the flow Φ. Moreover, Φ decays exponentially for $\sigma > \tau$ on E_0^s and for $\sigma < \tau$ on E_0^u. In particular, hyperbolic, autonomous, linear ODEs admit exponential dichotomies.

For non-autonomous, linear ODEs, we need the splitting (1.3.58) to depend on the base time $\tau \in \mathbb{R}$. In particular, we say that the linear ODE

$$\frac{du}{d\sigma}(\sigma) = A(\sigma)u(\sigma) \tag{1.3.59}$$

admits exponential dichotomies on an interval $I \in \{\mathbb{R}, \mathbb{R}^- \mathbb{R}^+\}$ if the following properties are satisfied.

- There exist projection operators $\{P(\tau)\}_{\tau \in I}$ on $\mathbb{C}^M$ which commute with the evolution $\Phi(\sigma, \tau)$.
- The restricted evolutions $\Phi(\sigma, \tau)P(\tau)$ and $\Phi(\sigma, \tau)(I - P(\tau))$ decay exponentially for $\sigma > \tau$ and for $\sigma < \tau$ respectively.

In particular, we have the exponential splitting of the state space $\mathbb{C}^M$ into the range of $P(\tau)$ and the kernel of $P(\tau)$. From this definition, we see that if a system admits exponential dichotomies on an interval $I \in \{\mathbb{R}, \mathbb{R}^- \mathbb{R}^+\}$ then each solution on I can be decomposed in two parts that decay exponentially in forward and backward time respectively.

Exponential dichotomies are closely related to the Fredholm properties of the corresponding linear operators. In particular, we consider the operator

$$\begin{array}{rcl} \Lambda : H^1(\mathbb{R}; \mathbb{C}^M) & \to & L^2(\mathbb{R}; \mathbb{C}^M), \\ (\Lambda u)(\sigma) & = & \frac{du}{d\sigma}(\sigma) - A(\sigma)u(\sigma). \end{array} \tag{1.3.60}$$

Then it is well-known that the operator Λ is a Fredholm operator if and only if the system (1.3.59) admits exponential dichotomies on $\mathbb{R}^-$ and $\mathbb{R}^+$. In addition, Λ is invertible if and only if (1.3.59) admits exponential dichotomies on $\mathbb{R}$. We refer to the review by Sandstede [147] for more details.

A very powerful and useful result is the so-called roughness theorem, see [45, Chapter 4]. Informally, this result states that exponential dichotomies are preserved when a small perturbation is added to the system. For example, let A be a hyperbolic $M \times M$ matrix and let $B(\sigma)$ be a bounded collection of $M \times M$ matrices which depend continuously on σ. Then the roughness theorem yields that the system

$$\frac{du}{d\sigma}(\sigma) \quad = \quad Au(\sigma) + \delta B(\sigma)u(\sigma) \tag{1.3.61}$$

admits exponential dichotomies on $\mathbb{R}$ when $\delta > 0$ is sufficiently small. Hence, we can conclude that the corresponding linear operator Λ from (1.3.60) is invertible! This means that the inhomogeneous ODE

$$\frac{du}{d\sigma}(\sigma) \quad = \quad Au(\sigma) + \delta B(\sigma)u(\sigma) + f(\sigma) \tag{1.3.62}$$

has a unique solution $u \in H^1(\mathbb{R}; \mathbb{C})$ for any function $f \in L^2(\mathbb{R}; \mathbb{C})$. We emphasize that these powerful results can be derived with hardly any assumptions on the matrices $B(\sigma)$.

For linear MFDEs such as (1.3.3), a few major complications turn up. First, the space $\mathbb{C}$ is no longer sufficient as a state space. Indeed, for determining $u'(0)$ in (1.3.3) we need to specify the behaviour of u on the entire interval $[-1, 1]$. As such, one usually takes $C_b([-1, 1]; \mathbb{C})$ as a state space. The second major complication is that MFDEs are typically ill-posed. That is, given an initial segment there may not be an extension of that segment that solves the MFDE, or such an extension may not be unique. As such, there is no equivalent of the evolution operator Φ that we defined for ODEs.

These complications were solved simultaneously and independently by Mallet-Paret and Verduyn Lunel [133] and by Härterich, Scheel and Sandstede [96]. We will focus on the former approach. Mallet-Paret and Verduyn Lunel showed that for linear, asymptotically hyperbolic MFDEs such as (1.3.3) we have the splitting

$$C_b([-1, 1]; \mathbb{C}) \quad = \quad P(\tau) + Q(\tau) + \Gamma(\tau). \tag{1.3.63}$$

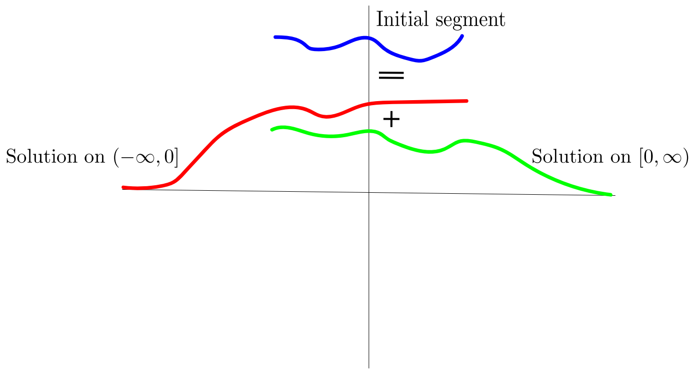


Figure 1.7: *Visual representation of the splitting* $C_b([-1,1];\mathbb{C}) = P(0) \oplus Q(0)$ *for linear MFDEs.*

Here $P(\tau)$ are the initial segments, centered around $\tau \in \mathbb{R}$, which can be extended to solutions of (1.3.3) on $(-\infty, \tau]$, while $Q(\tau)$ are the initial segments that can be extended to solutions of (1.3.3) on $[\tau, \infty)$; see Figure 1.7. One of the main results of [133] is that the extensions of the initial segments in $P(\tau)$ and $Q(\tau)$ decay exponentially as $\xi \to -\infty$ and as $\xi \to \infty$ respectively. Moreover, writing $B(\tau) = P(\tau) \cap Q(\tau)$ for those segments that can be extended to full solutions of (1.3.3), we can divide the space $B(\tau)$ out of $P(\tau)$ to turn (1.3.63) into a direct sum. Finally, the space $\Gamma(\tau)$ is finite dimensional and can be made explicit using the so-called Hale inner product [91]. For the linearized Nagumo MFDE (1.3.3), this Hale inner product takes the form

$$\langle \psi, \phi \rangle_\tau \quad = \quad \psi(0)\phi(0) + \int_{-1}^{0} \psi(s+1)\phi(s)ds - \int_{0}^{1} \psi(s-1)\phi(s)ds \tag{1.3.64}$$

for $\phi, \psi \in C_b([-1,1];\mathbb{C})$. Indeed, the space $\Gamma(\tau)$ from (1.3.63) can be classified by the identity

$$P(\tau) + Q(\tau) \quad = \quad \{\phi \in C_b([-1,1];\mathbb{C}) : \langle b(\tau + \cdot), \phi \rangle_\tau = 0 \text{ for all } b \in \ker(\Lambda^*)\}, \tag{1.3.65}$$

where we recall the operator Λ^* from (1.3.16).

There are two potential concerns that can arise to impact the usefulness of the identity (1.3.65). First, there may be nonzero kernel elements $b \in \ker(\Lambda^*)$ that vanish on the relevant interval $[\tau - 1, \tau + 1]$. In that case, we have $\langle b(\tau + \cdot), \phi \rangle_\tau = 0$ for any function ϕ. Second, the Hale inner product may be degenerate, in the sense that there exists a nonzero function ψ for which $\langle \psi, \phi \rangle_\tau = 0$ for any function ϕ. If both situations do not occur, then the dimension of the space $\Gamma(\tau)$ can easily be determined to be the dimension of the kernel $\ker(\Lambda^*)$. However, if either one of these situations occurs, we can no longer compute this dimension. Luckily, Mallet-Paret and Verduyn

Lunel showed that both situations cannot occur if the coefficients corresponding to the largest and smallest shifts are *atomic*, i.e. invertible on appropriate time-intervals. This is clearly the case for the system (1.3.3).

Exponential dichotomies for MFDEs with infinite-range interactions We extended the results by Mallet-Paret and Verduyn Lunel to include linear MFDEs such as the linearization of the system (1.1.20), which is given by

$$\overline{c}_h u'(\xi) \quad = \quad \sum_{k=1}^{\infty} \alpha_k \big[u(\xi + kh) + u(\xi - kh) - 2u(\xi)\big] + g_u(\overline{u}_h(\xi); r)u(\xi). \qquad (1.3.66)$$

To ensure that the Fredholm theory developed by Faye and Scheel in [69] can be used, we assume that the coefficients $\{\alpha_k\}_{k\geq 1}$ decay exponentially. For the system (1.3.66) the appropriate state space is the space $C_b(\mathbb{R};\mathbb{C})$. Since the system (1.3.66) is asymptotically hyperbolic, we have the splitting

$$C_b(\mathbb{R};\mathbb{C}) \quad = \quad P(\tau) + Q(\tau) + \Gamma(\tau). \qquad (1.3.67)$$

As before, the space $P(\tau)$ contains those initial segments, centered around $\tau \in \mathbb{R}$, which can be extended to solutions of (1.3.66) on $(-\infty, \tau]$, while $Q(\tau)$ are those segments that can be extended to solutions of (1.3.66) on $[\tau, \infty)$. However, extending is not really the appropriate word, since the initial segments are already defined on the entire line. In addition, the segments in $P(\tau)$ decay exponentially as $\xi \to -\infty$ and those in $Q(\tau)$ decay exponentially as $\xi \to \infty$. Moreover, dividing out the solution space $B(\tau) = P(\tau) \cap Q(\tau)$ from $P(\tau)$ turns (1.3.67) into a direct sum. Finally, we regain the identity

$$P(\tau) + Q(\tau) \quad = \quad \{\phi \in C_b(\mathbb{R};\mathbb{C}) : \langle b(\tau + \cdot), \phi\rangle_\tau = 0 \text{ for all } b \in \ker(\Lambda^*)\}, \qquad (1.3.68)$$

where $\langle\cdot,\cdot\rangle_\tau$ is the Hale inner product.

However, the degeneracy issues that were discussed previously are much harder to solve for systems such as (1.3.66). Indeed, the atomicity condition Mallet-Paret and Verduyn Lunel used to exclude these degeneracies explicitly references the largest and the smallest shift. We formulate several new conditions on the coefficients which can, separately, be used to rule out degeneracies. In particular, for the system (1.3.66) one of these conditions entails that the coefficients $\{\alpha_k\}_{k\geq 1}$ should be *cyclic* with respect to the backward shift operator on $\ell^2(\mathbb{N};\mathbb{R})$. That is, the set of sequences $\{\alpha_k\}_{k\geq N}$ for $N \geq 1$ should span a dense subspace of $\ell^2(\mathbb{N};\mathbb{R})$. This condition is, for example, satisfied if the coefficients decay like a Gaussian. However, this condition is not satisfied if $\alpha_k = \exp(-k)$ for each $k \geq 1$, since in that case we have $\{\alpha_k\}_{k\geq N}$ is a scalar multiple of $\{\alpha_k\}_{k\geq 1}$ for each $N \geq 1$.

If the coefficients $\{\alpha_k\}_{k\geq 1}$ of the the system (1.3.66) are positive, we can merely show that the Hale inner product is nondegenerate for kernel elements of the adjoint operator. That is, we can show that if $\langle b(\tau + \cdot), \phi\rangle_\tau = 0$ for all $b \in \ker(\Lambda^*)$, we must have $\phi = 0$. In particular, if $\alpha_k = \exp(-k)$ for $k \geq 1$, we explicitly construct a nonzero function ψ which has $\langle \psi, \phi \rangle_\tau = 0$ for all functions ϕ. Luckily, the nondegeneracy for kernel elements is sufficient to compute the dimension of the space $\Gamma(\tau)$.

Chapter 2

Nonlinear stability of pulse solutions for the discrete FitzHugh-Nagumo equation with infinite-range interactions

Sections 2.1-2.3 and 2.5-2.7 have been published in Discrete & Continuous Dynamical Systems-A 39(9) (2019) 5017–5083 as W.M. Schouten-Straatman and H.J. Hupkes "Nonlinear Stability of Pulse Solutions for the Discrete FitzHugh-Nagumo equation with Infinite-Range Interactions" [150].

Abstract. We establish the existence and nonlinear stability of travelling pulse solutions for the discrete FitzHugh-Nagumo equation with infinite-range interactions close to the near-continuum regime. For the verification of the spectral properties, we need to study a functional differential equation of mixed type (MFDE) with unbounded shifts. We avoid the use of exponential dichotomies and phase spaces, by building on a technique developed by Bates, Chen and Chmaj for the discrete Nagumo equation. This allows us to transfer several crucial Fredholm properties from the PDE setting to our discrete setting.

Key words: Lattice differential equations, FitzHugh-Nagumo system, infinite-range interactions, nonlinear stability, nonstandard implicit function theorem.

2.1 Introduction

The FitzHugh-Nagumo partial differential equation (PDE) is given by

$$\begin{aligned} u_t &= u_{xx} + g(u; r_0) - w \\ w_t &= \rho(u - \gamma w), \end{aligned} \qquad (2.1.1)$$

where $g(\cdot; r_0)$ is the cubic bistable nonlinearity given by

$$g(u; r_0) = u(1 - u)(u - r_0) \qquad (2.1.2)$$

and ρ, γ are positive constants. This PDE is commonly used as a simplification of the Hodgkin-Huxley equations, which describe the propagation of signals through nerve fibres. The spatially homogeneous version of this equation was first stated by FitzHugh in 1961 [74] in order to describe the potential felt at a single point along a nerve axon as a signal travels by. A few years later [76], the diffusion term in (2.1.1) was added to describe the dynamics of the full nerve axon instead of just a single point. As early as 1968 [75], FitzHugh released a computer animation based on numerical simulations of (2.1.1). This video clip clearly shows that (2.1.1) admits isolated pulse solutions resembling the spike signals that were measured by Hodgkin and Huxley in the nerve fibres of giant squids [98].

As a consequence of this rich behaviour and the relative simplicity of its structure, (2.1.1) has served as a prototype for several similar systems. For example, memory devices have been designed using a planar version of (2.1.1), which supports stable stationary, radially symmetric spot patterns [120]. In addition, gas discharges have been described using a three-component FitzHugh-Nagumo system [138, 148], for which it is possible to find stable travelling spots [161].

Mathematically, it turned out to be a major challenge to control the interplay between the excitation and recovery dynamics and rigorously construct the travelling pulses visualized by FitzHugh in [75]. Such pulse solutions have the form

$$(u, w)(x, t) = (\overline{u}_0, \overline{w}_0)(x + c_0 t), \qquad (2.1.3)$$

in which c_0 is the wavespeed and the wave profile $(\overline{u}_0, \overline{w}_0)$ satisfies the limits

$$\lim_{|\xi| \to \infty} (\overline{u}_0, \overline{w}_0)(\xi) = 0. \qquad (2.1.4)$$

Plugging this Ansatz into (2.1.1) and writing $\xi = x + c_0 t$, we see that the profiles are homoclinic solutions to the travelling wave ordinary differential equation (ODE)

$$\begin{aligned} c_0 \overline{u}_0'(\xi) &= \overline{u}_0''(\xi) + g(\overline{u}_0(\xi); r_0) - \overline{w}_0(\xi) \\ c_0 \overline{w}_0'(\xi) &= \rho\big[\overline{u}_0(\xi) - \gamma \overline{w}_0(\xi)\big]. \end{aligned} \qquad (2.1.5)$$

The analysis of this equation in the singular limit $\rho \downarrow 0$ led to the birth of many techniques in geometric singular perturbation theory, see for example [118] for an interesting overview. Indeed, the early works [31, 97, 117, 119] used geometric techniques such as the Conley index, exchange lemmas and differential forms to construct pulses and analyze their stability. A more analytic approach was later developed in [124], where Lin's method was used in the $r_0 \approx \frac{1}{2}$ regime to connect a branch of so-called slow-pulse solutions to (2.1.5) to a branch of fast-pulse solutions. This equation is still under active investigation, see for example [32, 33], where the birth of oscillating tails for the pulse solutions is described as the unstable root r_0 of the nonlinearity g moves towards the stable root at zero.

Many physical, chemical and biological systems have an inherent discrete structure that strongly influences their dynamical behaviour. In such settings *lattice differential equations* (LDEs), i.e. differential equations where the spatial variable can only take values on a lattice such as $\mathbb{Z}^n$, are the natural replacements for PDEs, see for example [6, 109, 130]. Although, mathematically, it is a relatively young field of interest, LDEs have already appeared frequently in the more applied literature. For example, they have been used to describe phase transitions in Ising models [6], crystal growth in materials [28] and phase mixing in martensitic structures [159].

To illustrate these points, let us return to the nerve axon described above and reconsider the propagation of electrical signals through nerve fibres. It is well known that electrical signals can only travel at adequate speeds if the nerve fibre is insulated by a myelin coating. This coating admits regularly spaced gaps at the so-called nodes of Ranvier [143]. Through a process called saltatory conduction, it turns out that excitations of nerves effectively jump from one node to the next [127]. Exploiting this fact, it is possible [123] to model this jumping process with the discrete FitzHugh-Nagumo LDE

$$\begin{array}{rcl} \dot{u}_j &=& \frac{1}{h^2}(u_{j+1} + u_{j-1} - 2u_j) + g(u_j; r_0) - w_j \\ \dot{w}_j &=& \rho[u_j - \gamma w_j]. \end{array} \tag{2.1.6}$$

The variable u_j now represents the potential at the j^{th} node, while the variable w_j denotes a recovery component. The nonlinearity g describes the ionic interactions. Note that this equation arises directly from the FitzHugh-Nagumo PDE upon taking the nearest-neighbour discretisation of the Laplacian on a grid with spacing $h > 0$.

Inspired by the procedure for partial differential equations, one can substitute a travelling pulse Ansatz

$$(u_j, w_j)(t) \;=\; (\overline{u}_h, \overline{w}_h)(hj + c_h t) \tag{2.1.7}$$

into (2.1.6). Instead of an ODE, we obtain the system

$$\begin{array}{rcl} c_h \overline{u}_h'(\xi) &=& \frac{1}{h^2}[\overline{u}_h(\xi + h) + \overline{u}_h(\xi - h) - 2\overline{u}_h(\xi)] + g(\overline{u}_h(\xi); r_0) - \overline{w}_h(\xi) \\ c_h \overline{w}_h'(\xi) &=& \rho[\overline{u}_h(\xi) - \gamma \overline{w}_h(\xi)] \end{array} \tag{2.1.8}$$

in which $\xi = hj + c_h t$. Such equations are called functional differential equations of mixed type (MFDEs), since they contain both advanced (positive) and retarded (negative) shifts.

In [108, 109], Hupkes and Sandstede studied (2.1.6) and showed that, for small values of ρ and r_0 sufficiently far from $\frac{1}{2}$, there exists a locally unique travelling pulse solution of this system and that it is asymptotically stable with an asymptotic phase shift. No restrictions were required on the discretisation distance h, but the results relied heavily on the existence of exponential dichotomies for MFDEs. As a consequence, the techniques developed in [108, 109] can only be used if the discretisation involves finitely many neighbours. Such discretisation schemes are said to have finite range.

Recently, an active interest has arisen in nonlocal equations that feature infinite-range interactions. For example, Ising models have been used to describe the infinite-range interactions between magnetic spins arranged on a grid [6]. In addition, many physical systems, such as amorphous semiconductors [87] and liquid crystals [44], feature nonstandard diffusion processes, which are generated by fractional Laplacians. Such operators are intrinsically nonlocal and, hence, often require infinite-range discretisation schemes [43].

Our primary interest here, however, comes from so-called neural field models, which aim to describe the dynamics of large networks of neurons. These neurons interact with each other by exchanging signals across long distances through their interconnecting nerve axons [15, 23, 24, 142]. It is of course a major challenge to find effective equations to describe such complex interactions. One model that has been proposed [23, Eq. (3.31)] features a FitzHugh-Nagumo type system with infinite-range interactions.

Motivated by the above, we consider a class of infinite-range FitzHugh-Nagumo LDEs that includes the prototype

$$\begin{array}{rcl} \dot{u}_j & = & \frac{\kappa}{h^2} \sum_{k \in \mathbb{Z}_{>0}} e^{-k^2} [u_{j+k} + u_{j-k} - 2u_j] + g(u_j; r_0) - w_j \\ \dot{w}_j & = & \rho[u_j - \gamma w_j], \end{array} \tag{2.1.9}$$

in which $\kappa > 0$ is a normalisation constant. In [69], Faye and Scheel studied equations such as (2.1.9) for discretisations with infinite-range interactions featuring exponential decay in the coupling strength. They circumvented the need to use a state space as in [108], which enabled them to construct pulses to (2.1.9) for arbitrary discretisation distance h. Very recently [70], they developed a center manifold approach that allows bifurcation results to be obtained for neural field equations.

In this paper, we also construct pulse solutions to equations such as (2.1.9), but under weaker assumptions on the decay rate of the couplings. Moreover, we will establish the nonlinear stability of these pulse solutions, provided the coupling strength decays exponentially. However, both results do require the discretisation distance h to be very small.

In particular, we will be working in the near-continuum regime. The pulses we construct can be seen as perturbations of the travelling pulse solution of the FitzHugh-Nagumo PDE. However, we will see that the travelling wave equations are highly singular perturbations of (2.1.5), which poses a significant mathematical challenge. On the other hand, we do not need to use exponential dichotomies directly in our nonlocal setting as in [109]. Instead, we are able to exploit the detailed knowledge that has been obtained using these techniques for the pulses in the PDE setting.

Our approach to tackle the difficulties arising from this singular perturbation is strongly inspired by the work of Bates, Chen and Chmaj. Indeed, in their excellent paper [6], they study a class of systems that includes the infinite-range discrete Nagumo equation

$$\dot{u}_j = \frac{\kappa}{h^2} \sum_{k \in \mathbb{Z}_{>0}} e^{-k^2} [u_{j+k} + u_{j-k} - 2u_j] + g(u_j; r_0), \tag{2.1.10}$$

in which $\kappa > 0$ is a normalisation constant. This equation can be seen as a discretisation of the Nagumo PDE

$$u_t = u_{xx} + g(u; r_0). \tag{2.1.11}$$

The authors show that, under some natural assumptions, these systems admit travelling front solutions for h small enough.

In the remainder of this introduction we outline their approach and discuss our modifications, which significantly broaden the application range of these methods. We discuss these modifications for the prototype (2.1.9), but naturally they can be applied to a broad class of systems.

Transfer of Fredholm properties: Scalar case.

An important role in [6] is reserved for the operator $\mathcal{L}_{h;\overline{u}_{0:\mathrm{sc}};c_{0:\mathrm{sc}}}$ given by

$$\begin{aligned}\mathcal{L}_{h;\overline{u}_{0:\mathrm{sc}};c_{0:\mathrm{sc}}} v(\xi) = {} & c_{0:\mathrm{sc}} v'(\xi) - \frac{\kappa}{h^2} \sum_{k \in \mathbb{Z}_{>0}} e^{-k^2} \Big[v(\xi + hk) + v(\xi - hk) - 2v(\xi) \Big] \\ & - g_u(\overline{u}_{0:\mathrm{sc}}(\xi); r_0) v(\xi),\end{aligned} \tag{2.1.12}$$

where $\overline{u}_{0:\mathrm{sc}}$ is the wave solution of the scalar Nagumo PDE (2.1.11) with wavespeed $c_{0:\mathrm{sc}}$. This operator arises as the linearisation of the scalar Nagumo MFDE

$$c_{0:\mathrm{sc}} u'(\xi) = \frac{\kappa}{h^2} \sum_{k \in \mathbb{Z}_{>0}} e^{-k^2} \Big[v(\xi + hk) + v(\xi - hk) - 2v(\xi) \Big] + g_u(\overline{u}_{0:\mathrm{sc}}(\xi); r_0) v(\xi), \tag{2.1.13}$$

around the wave solution $\overline{u}_{0:\mathrm{sc}}$ of the scalar Nagumo PDE (2.1.11). This operator should be compared to

$$\mathcal{L}_{0;\overline{u}_{0:\mathrm{sc}};c_{0:\mathrm{sc}}} v(\xi) = c_{0:\mathrm{sc}} v'(\xi) - v''(\xi) - g_u(\overline{u}_{0:\mathrm{sc}}(\xi); r_0) v(\xi), \tag{2.1.14}$$

the linearisation of the scalar Nagumo PDE around its wave solution.

The key contribution in [6] is that the authors fix a constant $\delta > 0$ and use the invertibility of $\mathcal{L}_{0;\overline{u}_{0:\mathrm{sc}};c_{0:\mathrm{sc}}} + \delta$ to show that also $\mathcal{L}_{h;\overline{u}_{0:\mathrm{sc}};c_{0:\mathrm{sc}}} + \delta$ is invertible. In particular, they consider weakly-converging sequences $\{v_n\}$ and $\{w_n\}$ with $(\mathcal{L}_{h;\overline{u}_{0:\mathrm{sc}};c_{0:\mathrm{sc}}} + \delta) v_n = w_n$ and try to find a uniform (in δ and h) upper bound for the L^2-norm of v_n' in terms of the L^2-norm of w_n. Such a bound is required to rule out the limitless transfer of energy into oscillatory modes, a key complication when taking weak limits. To obtain this bound, the authors exploit the bistable structure of the nonlinearity g to control the behaviour at $\pm\infty$. This allows the local L^2-norm of v_n on a compact set to be uniformly bounded away from zero. Since the operator $\mathcal{L}_{h;\overline{u}_{0:\mathrm{sc}};c_{0:\mathrm{sc}}} + \delta$ is not self-adjoint, this procedure must be repeated for the adjoint operator.

Transfer of Fredholm properties: System case.

Plugging the travelling pulse Ansatz

$$(u,w)_j(t) \quad = \quad (\overline{u}_h, \overline{w}_h)(hj + c_h t) \tag{2.1.15}$$

into (2.1.9) and writing $\xi = hj + c_h t$, we see that the profiles are homoclinic solutions to the equation

$$\begin{aligned} c_h \overline{u}_h'(\xi) &= \tfrac{\kappa}{h^2} \sum_{k>0} e^{-k^2} \Big[\overline{u}_h(\xi + kh) + \overline{u}_h(\xi - kh) - 2\overline{u}_h(\xi) \Big] + g(\overline{u}_h(\xi); r_0) - \overline{w}_h(\xi) \\ c_h \overline{w}_h'(\xi) &= \rho \Big(\overline{u}_h(\xi) - \gamma \overline{w}_h(\xi) \Big). \end{aligned} \tag{2.1.16}$$

We start by considering the linearised operator $\mathcal{K}_{h;\overline{u}_0;c_0}$ of the system (2.1.16) around the pulse solution $(\overline{u}_0, \overline{w}_0)$ of the FitzHugh-Nagumo PDE with wavespeed c_0. This operator is given by

$$\mathcal{K}_{h;\overline{u}_0;c_0} \begin{pmatrix} v \\ w \end{pmatrix} (\xi) \quad = \quad \begin{pmatrix} \mathcal{L}_{h;\overline{u}_0;c_0} v(\xi) + w(\xi) \\ c_0 w'(\xi) - \rho v(\xi) + \rho\gamma w(\xi) \end{pmatrix}, \tag{2.1.17}$$

where $\mathcal{L}_{h;\overline{u}_0;c_0}$ is given by equation (2.1.12), but with $\overline{u}_{0:\mathrm{sc}}$ replaced by $\overline{u}_0$ and $c_{0:\mathrm{sc}}$ by c_0.

In §2.3 we use a Fredholm alternative as described above to establish the invertibility of $\mathcal{K}_{h;\overline{u}_0;c_0} + \delta$ for fixed $\delta > 0$. However, the transition from a scalar equation to a system is far from trivial. Indeed, when transferring the Fredholm properties there are multiple cross terms that need to be controlled. We are aided here by the relative simplicity of the terms in the equation that involve $\overline{w}$. In particular, three of the four matrix-elements of the linearisation (2.1.17) have constant coefficients. We emphasize that it is not sufficient to merely assume that the limiting state $(0,0)$ is a stable equilibrium of (2.1.9). In [151], we explore a number of structural conditions that allow these types of arguments to be extended to general multi-component systems.

Construction of pulses.

Using these results for $\mathcal{K}_{h;\overline{u}_0;c_0}$, we develop a fixed point argument to show that, for h small enough, the system (2.1.9) has a locally unique travelling pulse solution $(\overline{U}_h(t))_j = (\overline{u}_h, \overline{w}_h)(hj + c_h t)$ which converges to a travelling pulse solution of the FitzHugh-Nagumo PDE as $h \downarrow 0$. This procedure is more or less straightforward and is very similar to the arguments used in [6, §4] which, in turn, closely follow the lines of a standard proof of the implicit function theorem.

Spectral stability.

The natural next step is to study the linear operator $\mathcal{K}_{h;\overline{u}_h;c_h}$ that arises after linearising the system (2.1.9) around its new-found pulse solution. This operator is given by

$$\mathcal{K}_{h;\overline{u}_h;c_h}\begin{pmatrix} v \\ w \end{pmatrix}(\xi) = \begin{pmatrix} \mathcal{L}_{h;\overline{u}_h;c_h}v(\xi) + w(\xi) \\ c_0 w'(\xi) - \rho v(\xi) + \rho\gamma w(\xi) \end{pmatrix}, \qquad (2.1.18)$$

where $\mathcal{L}_{h;\overline{u}_h}$ is given by equation (2.1.12), but with $\overline{u}_{0:sc}$ replaced by $\overline{u}_h$ and $c_{0:\mathrm{sc}}$ by c_h. The procedure above can be repeated to show that for fixed $\delta > 0$, it also holds that $\mathcal{K}_{h;\overline{u}_h;c_h} + \delta$ is invertible for h small enough. However, to understand the spectral stability of the pulse, we need to consider the eigenvalue problem

$$\mathcal{K}_{h;\overline{u}_h;c_h}v + \lambda v = 0 \qquad (2.1.19)$$

for fixed values of h and λ ranging throughout a half-plane. Switching between these two points of view turns out to be a delicate task.

We start in §2.5 by showing that $\mathcal{K}_{h;\overline{u}_h;c_h}$ and its adjoint $\mathcal{K}^*_{h;\overline{u}_h;c_h}$ are Fredholm operators with one-dimensional kernels. This is achieved by explicitly constructing a kernel element for $\mathcal{K}^*_{h;\overline{u}_h;c_h}$ that converges to a kernel element of the adjoint of the operator corresponding to the linearised PDE. An abstract perturbation argument then yields the result.

In particular, we see that $\lambda = 0$ is a simple eigenvalue of $\mathcal{K}_{h;\overline{u}_h;c_h}$. In §2.6 we establish that in a suitable half-plane, the spectrum of this operator consists precisely of the points $\{k2\pi i c_h \frac{1}{h} : k \in \mathbb{Z}\}$, which are all simple eigenvalues. We do this by first showing that the spectrum is invariant under the operation $\lambda \mapsto \lambda + \frac{2\pi i c_h}{h}$, which allows us to restrict ourselves to values of λ with imaginary part in between $-\frac{\pi|c_h|}{h}$ and $\frac{\pi|c_h|}{h}$. Note that the period of the spectrum is dependent on h and grows to infinity as $h \downarrow 0$. This is not too surprising, since the spectrum of the linearisation of the PDE around its pulse solution is not periodic. However, this means that we cannot restrict ourselves to a fixed compact subset of the complex plane for all values of h at the same time. In fact, it takes quite some effort to keep the part of the spectrum with large imaginary part under control.

It turns out to be convenient to partition our 'half-strip' into four parts and to calculate the spectrum in each part using different methods. Values close to 0 are analyzed using the Fredholm properties of $\mathcal{K}_{h;\overline{u}_h;c_h}$ exploiting many of the results from §2.5; values with a large real part are considered using standard norm estimates, but values with a large imaginary part are treated using a Fourier transform. The final set to consider is a compact set that is independent of h and bounded away from the origin. This allows us to apply a modified version of the procedure described above that exploits the absence of spectrum in this region for the FitzHugh-Nagumo PDE.

Let us emphasize that our arguments here for bounded values of the spectral parameter λ strongly use the fact that the PDE pulse is spectrally stable. The main complication to establish the latter fact is the presence of a secondary eigenvalue that is $O(\rho)$-close to the origin. Intuitively, this eigenvalue arises as a consequence of the interaction between the front and back solution to the Nagumo equation that are both part of the singular pulse that arises in the $\rho \downarrow 0$ limit. In the PDE case, Jones [117] and Yanagida [166] essentially used shooting arguments to construct and analyze an Evans function $\mathcal{E}(\lambda)$ that vanishes precisely at eigenvalues. In particular, they computed the sign of $\mathcal{E}'(0)$ and used counting arguments to show that the secondary eigenvalue discussed above lies to the left of the origin. Currently, a program is underway to build a general framework in this spirit based on the Maslov index [10, 37, 101], which also works in multi-dimensional spatial settings. In [46, 47], this framework was applied to an equal-diffusion version of the FitzHugh-Nagumo PDE.

An alternative approach involving Lin's method and exponential dichotomies was pioneered in [124]. Based upon these ideas, stability results have been obtained for the LDE (2.1.6) [109] and the PDE (2.1.1) [32] in the nonhyperbolic regime $r_0 \sim 0$. The first major advantage of this approach is that explicit bifurcation equations can be formulated that allow asymptotic expansions to be developed for the location of the interaction eigenvalue discussed above. The second major advantage is that it allows us to avoid the use of the Evans function, which cannot easily be defined in discrete settings, because MFDEs are ill-posed as initial value problems [144]. We believe that a direct approach along these lines should also be possible for the infinite range system (2.1.9) as soon as exponential dichotomies are available in this setting.

Nonlinear stability.

The final step in our program is to leverage the spectral stability results to obtain a nonlinear stability result. To do so, we follow [109] and derive a formula that links the pointwise Green's function of our general problem (2.1.9) to resolvents of the operator $\mathcal{K}_{h;\overline{u}_h;c_h}$ in §2.7. Since we have already analyzed the latter operator in detail, we readily obtain a spectral decomposition of this Green's function into an explicit neutral part and a residual that decays exponentially in time and space. Therefore, we obtain detailed estimates on the decay rate of the Green's function for the general problem. These Green's functions allow in §2.8 to use multiple fixed point arguments to, eventually, show the nonlinear stability of the family of travelling pulse solutions $\overline{U}_h$. To be more precise, for each initial condition close to $\overline{U}_h(0)$, we show that the solution with that initial condition converges at an exponential rate to the solution $\overline{U}_h(\cdot + \tilde{\theta})$ for a small (and unique) phase shift $\tilde{\theta}$.

We emphasize that by now there are several techniques available to obtain nonlinear stability results in the relatively simple spectral setting encountered in this paper. If a comparison principle is available, which is not the case for the FitzHugh-Nagumo system, one can follow the classic approach developed by Fife and McLeod [73] to show that travelling waves have a large basin of attraction. Indeed, one can construct explicit sub- and super-solutions that trade-off additive perturbations at $t = 0$ to phase-shifts at $t = \infty$. In fact, one can actually use this type of argument to establish the existence of travelling waves by letting an appropriate initial condition evolve and tracking its asymptotic behaviour [38, 110]. For systems that can be written as gradient flows, which is also not the case here, the existence and stability of travelling waves can be obtained by using an elegant variational technique that was developed by Gallay and Risler [82].

In the spatially continuous setting, it is possible to freeze a travelling wave by passing to a co-moving frame. In our setting, one can achieve this by simply adding a convective term $-c_0\partial_x(u, w)$ to the right hand side of (2.1.1). The main advantage is that one can immediately use the semigroup $\exp[t\mathcal{L}_0]$ to describe the evolution of the linearised system in this co-moving frame, which is temporally autonomous. Here $\mathcal{L}_0$ is the standard linear operator associated to the linearisation of (2.1.1) around $(\overline{u}_0, \overline{w}_0)$; see (2.2.10). For each $\vartheta \in \mathbb{R}$ one can subsequently construct the stable manifold of $\big(\overline{u}_0(\cdot+\vartheta), \overline{w}_0(\cdot+\vartheta)\big)$ by applying a fixed point argument to Duhamel's formula. Upon varying ϑ, these stable manifolds span a tubular neighbourhood of the family $(\overline{u}_0, \overline{w}_0)(\cdot+\mathbb{R})$. This readily leads to the desired stability result; see e.g. [121, §4]. We remark here that these stable manifolds are all related to each other via spatial shifts.

In the spatially discrete setting, the wave can no longer be frozen. In particular, the linearisation of (2.1.6) around the pulse (2.1.7) leads to an equation that is temporally shift-periodic. In [41], the authors attack this problem head-on by developing a shift-periodic version of Floquet theory that leads to a nonlinear stability result in ℓ^∞. However, they delicately exploit the geometric structure of ℓ^∞ and it is not clear how more degenerate spectral pictures can be fitted into the framework. These issues are explained in detail in [109, §2].

In [13], the authors found a way to express the Green's function of the temporally shift-periodic linear discrete equation in terms of resolvents of the linear operator $\mathcal{L}_h$ associated to the pulse (2.1.7). Based on this procedure, it is possible to follow the spirit of the powerful pointwise Green's function techniques pioneered by Zumbrun and Howard [168]. Indeed, in [11], a stability result is obtained in the setting of discrete conservation laws, where one encounters curves of essential spectrum that touch the imaginary axis. Using exponential dichotomies in a setting with extended state-spaces $L^2([-h, h]; \mathbb{R}^2) \times \mathbb{R}^2$, pointwise λ-meromorphic expansions were obtained for the operators $[\mathcal{L}_h - \lambda]^{-1}$. This allowed the techniques from [12] to be transferred from the continuous to the discrete setting. A slightly more streamlined approach was developed in [109], which does not need the extended state-space and avoids the use of a variation-of-constants formula. However, exponential dichotomies are still used at certain key points.

In our paper, we follow the spirit of the latter approach and extend it to the present setting with infinite-range interactions. In particular, we show how the use of exponential dichotomies can be eliminated all together, which is a delicate task. In addition, we need to be very careful in many computations since integrals and sums over shifts as in (2.1.16) can no longer be freely exchanged. We emphasize, here, that our techniques do not depend on the specific LDE that we are analyzing. All that is required is the spectral setting described above and the fact that the shifts appearing in the problem are all rationally related.

Let us mention that it is also possible to bypass the construction of the stable manifolds altogether and employ a direct phase-tracking approach along the lines of [167]. In particular, one can couple the system with an extra equation for the phase. To close the system, one chooses this extra equation in such a way that the resulting nonlinear terms never encounter the nondecaying part of the relevant semigroup. Such an approach has been used in the current spectral setting to show that travelling waves remain stable under the influence of a small stochastic noise term [92].

2.2 Main results

We consider the following system of equations

$$\begin{array}{lcl} \dot{u}_j & = & \frac{1}{h^2} \sum_{k>0} \alpha_k [u_{j+k} + u_{j-k} - 2u_j] + g(u_j) - w_j \\ \dot{w}_j & = & \rho[u_j - \gamma w_j], \end{array} \tag{2.2.1}$$

which we refer to as the (spatially) discrete FitzHugh-Nagumo equation with infinite-range interactions. Often, for example in [108, 109], it is assumed that only finitely many of these coefficients α_k are non-zero. However, we will impose the following much weaker conditions here.

Assumption (Hα1). The coefficients $\{\alpha_k\}_{k \in \mathbb{Z}_{>0}}$ satisfy the bound

$$\sum_{k>0} |\alpha_k| k^2 \quad < \quad \infty, \tag{2.2.2}$$

as well as the identity

$$\sum_{k>0} \alpha_k k^2 \quad = \quad 1. \tag{2.2.3}$$

Finally, the inequality

$$A(z) := \sum_{k>0} \alpha_k \Big(1 - \cos(kz)\Big) \quad > \quad 0 \tag{2.2.4}$$

holds for all $z \in (0, 2\pi)$.

We note that (2.2.4) is automatically satisfied if $\alpha_1 > 0$ and $\alpha_k \geq 0$ for all $k \in \mathbb{Z}_{>1}$. The conditions in (Hα1) ensure that for $\phi \in L^\infty(\mathbb{R})$ with $\phi'' \in L^2(\mathbb{R})$, we have the limit

$$\lim_{h \downarrow 0} \| \frac{1}{h^2} \sum_{k>0} \alpha_k \Big[\phi(\cdot + hk) + \phi(\cdot - hk) - 2\phi(\cdot) \Big] - \phi'' \|_{L^2(\mathbb{R})} = 0, \tag{2.2.5}$$

see Lemma 2.3.5. In particular, we can see (2.2.1) as the discretisation of the FitzHugh-Nagumo PDE (2.1.1) on a grid with distance h. Additional remarks concerning the assumption (Hα1) can be found in [6, §1].

Throughout this paper, we impose the following standard assumptions on the remaining parameters in (2.2.1). The last condition on γ in (HS) ensures that the origin is the only j-independent equilibrium of (2.2.1).

Assumption (HS). The nonlinearity g is given by $g(u) = u(1-u)(u-r_0)$, where $0 < r_0 < 1$. In addition, we have $0 < \rho < 1$ and $0 < \gamma < 4(1-r_0)^{-2}$.

Without explicitly mentioning it, we will allow all constants in this work to depend on r_0, ρ and γ. Dependence on h will always be mentioned explicitly. We will mainly work on the Sobolev spaces

$$\begin{aligned} H^1(\mathbb{R}) &= \{f : \mathbb{R} \to \mathbb{R} | f, f' \in L^2(\mathbb{R})\}, \\ H^2(\mathbb{R}) &= \{f : \mathbb{R} \to \mathbb{R} | f, f', f'' \in L^2(\mathbb{R})\}, \end{aligned} \tag{2.2.6}$$

with their standard norms

$$\begin{aligned} \|f\|_{H^1(\mathbb{R})} &= \left(\|f\|^2_{L^2(\mathbb{R})} + \|f'\|^2_{L^2(\mathbb{R})}\right)^{\frac{1}{2}}, \\ \|f\|_{H^2(\mathbb{R})} &= \left(\|f\|^2_{L^2(\mathbb{R})} + \|f'\|^2_{L^2(\mathbb{R})} + \|f''\|^2_{L^2(\mathbb{R})}\right)^{\frac{1}{2}}. \end{aligned} \tag{2.2.7}$$

Our goal is to construct pulse solutions of (2.2.1) as small perturbations to the travelling pulse solutions of the FitzHugh-Nagumo PDE. These latter pulses satisfy the system

$$\begin{aligned} c_0\overline{u}_0' &= \overline{u}_0'' + g(\overline{u}_0) - \overline{w}_0 \\ c_0\overline{w}_0' &= \rho(\overline{u}_0 - \gamma\overline{w}_0) \end{aligned} \tag{2.2.8}$$

with the boundary conditions

$$\lim_{|\xi|\to\infty} (\overline{u}_0, \overline{w}_0)(\xi) = (0, 0). \tag{2.2.9}$$

If $(\overline{u}_0, \overline{w}_0)$ is a solution of (2.2.8) with wavespeed c_0, then the linearisation of (2.2.8) around this solution is characterized by the operator $\mathcal{L}_0 : H^2(\mathbb{R}) \times H^1(\mathbb{R}) \to L^2(\mathbb{R}) \times L^2(\mathbb{R})$ that acts as

$$\mathcal{L}_0 \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} c_0\frac{d}{d\xi} - \frac{d^2}{d\xi^2} - g_u(\overline{u}_0) & 1 \\ -\rho & c_0\frac{d}{d\xi} + \gamma\rho \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}. \tag{2.2.10}$$

The existence of such pulse solutions for the case when ρ is close to 0 is established in [118, §5.3]. Here, we do not require $\rho > 0$ to be small, but we simply impose the following condition.

Assumption (HP1). There exists a solution $(\overline{u}_0, \overline{w}_0)$ of (2.2.8) that satisfies the conditions (2.2.9) and has wavespeed $c_0 \neq 0$. Furthermore, the operator $\mathcal{L}_0$ is Fredholm with index zero and it has a simple eigenvalue in zero.

Recall that an eigenvalue λ of a Fredholm operator L is said to be *simple* if the kernel of $L - \lambda$ is spanned by one vector v and the equation $(L - \lambda)w = v$ does not have a solution w. Note that if L has a formal adjoint L^*, this is equivalent to the condition that $\langle v, w\rangle \neq 0$ for all nontrivial $w \in \ker(L^* - \overline{\lambda})$.

We note that the conditions on $\mathcal{L}_0$ formulated in (HP1) were established in [117] for small $\rho > 0$. In addition, these conditions imply that $\overline{u}_0'$ and $\overline{w}_0'$ decay exponentially.

We emphasize, however, that there exists a choice of parameters for which the condition (HP1) is not satisfied [34].

In order to find travelling pulse solutions of (2.2.1), we substitute the Ansatz

$$(u,w)_j(t) \quad = \quad (\overline{u}_h, \overline{w}_h)(hj + c_h t), \tag{2.2.11}$$

into (2.2.1) to obtain the system

$$\begin{array}{rcl} c_h \overline{u}_h'(\xi) & = & \frac{1}{h^2} \sum_{k>0} \alpha_k \Big[\overline{u}_h(\xi + hk) + \overline{u}_h(\xi - hk) - 2\overline{u}_h(\xi) \Big] + g\big(\overline{u}_h(\xi)\big) - \overline{w}_h(\xi) \\ c_h \overline{w}_h'(\xi) & = & \rho[\overline{u}_h(\xi) - \gamma \overline{w}_h(\xi)], \end{array} \tag{2.2.12}$$

in which $\xi = hj + c_h t$. The boundary conditions are given by

$$\lim_{|\xi| \to \infty} (\overline{u}_h, \overline{w}_h)(\xi) \quad = \quad (0,0). \tag{2.2.13}$$

The existence of such solutions is established in our first main theorem.

Theorem 2.2.1 (see §2.4)**.** *Assume that (HP1), (HS) and (Hα1) are satisfied. There exists a positive constant h_* such that for all $h \in (0, h_*)$, the problem (2.2.12) with boundary conditions (2.2.13) admits at least one solution $(c_h, \overline{u}_h, \overline{w}_h)$, which is locally unique in $\mathbb{R} \times H^1(\mathbb{R}) \times H^1(\mathbb{R})$ up to translation and which has the property that*

$$\lim_{h \downarrow 0} \; (c_h - c_0, \overline{u}_h - \overline{u}_0, \overline{w}_h - \overline{w}_0) \quad = \quad (0,0,0) \qquad \textit{in } \mathbb{R} \times H^1(\mathbb{R}) \times H^1(\mathbb{R}). \tag{2.2.14}$$

Note that this result is very similar to [69, Cor. 2.1]. However, Faye and Scheel impose an extra assumption, similar to (Hα2) below, which we do not need in our proof. This is a direct consequence of the strength of the method from [6] that we described in §2.1.

Building on the existence of the travelling pulse solution, the natural next step is to show that our new-found pulse is asymptotically stable. However, we now do need to impose an extra condition on the coefficients $\{\alpha_k\}_{k>0}$.

Assumption (Hα2). The coefficients $\{\alpha_k\}_{k>0}$ satisfy the bound

$$\sum_{k>0} |\alpha_k| e^{k\nu} \quad < \quad \infty \tag{2.2.15}$$

for some $\nu > 0$.

Note that the prototype equation (2.1.9) indeed satisfies both assumptions (Hα1) and (Hα2). An example of a system which satisfies (Hα1), but not (Hα2) is given by

$$\begin{array}{rcl} \dot{u}_j & = & \frac{\kappa}{h^2} \sum_{k>0} \frac{1}{k^4} [u_{j+k} + u_{j-k} - 2u_j] + g(u_j) - w_j \\ \dot{w}_j & = & \rho[u_j - \gamma w_j], \end{array} \tag{2.2.16}$$

in which $\kappa = \frac{6}{\pi^2}$ is the normalisation constant.

Moreover, we need to impose an extra condition on the operator $\mathcal{L}_0$ given by (2.2.10). This spectral stability condition is established in [63, Thm. 2] together with [166, Thm. 3.1] for the case where ρ is close to 0.

Assumption (HP2). There exists a constant $\lambda_* > 0$ such that for each $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq -\lambda_*$ and $\lambda \neq 0$, the operator

$$\mathcal{L}_0 + \lambda : H^2(\mathbb{R}) \times H^1(\mathbb{R}) \to L^2(\mathbb{R}) \times L^2(\mathbb{R}) \tag{2.2.17}$$

is invertible.

To determine if the pulse solution described in Theorem 2.2.1 is nonlinearly stable, we must first linearise (2.2.12) around this pulse and determine the spectral stability. The linearised operator now takes the form

$$L_h \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} c_h \frac{d}{d\xi} - \Delta_h - g_u(\overline{u}_h) & 1 \\ -\rho & c_h \frac{d}{d\xi} + \gamma\rho \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}. \tag{2.2.18}$$

Here the operator Δ_h is given by

$$\Delta_h \phi(\xi) = \frac{1}{h^2} \sum_{k>0} \alpha_k \Big(\phi(\xi + hk) + \phi(\xi - hk) - 2\phi(\xi) \Big). \tag{2.2.19}$$

As usual, we define the *spectrum*, $\sigma(L)$, of a bounded linear operator $L : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to L^2(\mathbb{R}) \times L^2(\mathbb{R})$, as

$$\sigma(L) = \{\lambda \in \mathbb{C} : L - \lambda \text{ is not invertible}\}. \tag{2.2.20}$$

Our second main theorem describes the spectrum of this operator L_h, or rather of $-L_h$, in a suitable half-plane.

Theorem 2.2.2 (see §2.6)**.** *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. There exist constants $\lambda_3 > 0$ and $h_{**} > 0$ such that for all $h \in (0, h_{**})$, the spectrum of the operator $-L_h$ in the half-plane $\{z \in \mathbb{C} : \operatorname{Re}\ z \geq -\lambda_3\}$ consists precisely of the points $k2\pi i c_h \frac{1}{h}$ for $k \in \mathbb{Z}$, which are all simple eigenvalues of L_h.*

We emphasize that λ_3 does not depend on h. The translational invariance of (2.2.12) guarantees that $\lambda = 0$ is an eigenvalue of $-L_h$. In Lemma 2.6.1 we show that the spectrum of the operator L_h is periodic with period $2\pi i c_h \frac{1}{h}$, which means that the eigenvalues $k2\pi i c_h \frac{1}{h}$ for $k \in \mathbb{Z}$ all have the same properties as the zero eigenvalue.

Our final result concerns the nonlinear stability of our pulse solution, which we represent with the shorthand

$$\Big[\overline{U}_h(t)\Big]_j = (\overline{u}_h, \overline{w}_h)(hj + c_h t). \tag{2.2.21}$$

The perturbations are measured in the spaces ℓ^p, which are defined by

$$\ell^p = \{V \in (\mathbb{R}^2)^{\mathbb{Z}} : \|V\|_{\ell^p} := \Big[\sum_{j\in\mathbb{Z}} |V_j|^p \Big]^{\frac{1}{p}} < \infty\} \tag{2.2.22}$$

for $1 \leq p < \infty$ and

$$\ell^\infty = \{V \in (\mathbb{R}^2)^{\mathbb{Z}} : ||V||_{\ell^\infty} := \sup_{j\in\mathbb{Z}} |V_j| < \infty\}. \tag{2.2.23}$$

Theorem 2.2.3 (see §2.8)**.** *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. Fix $0 < h \leq h_{**}$ and $1 \leq p \leq \infty$. Then there exist constants $\delta > 0$, $C > 0$ and $\beta > 0$, which may depend on h but not on p, such that for all initial conditions $U^0 \in \ell^p$ with $\|U^0 - \overline{U}_h(0)\|_{\ell^p} < \delta$, there exists an asymptotic phase shift $\tilde{\theta} \in \mathbb{R}$ such that the solution $U = (u, w)$ of (2.2.1) with $U(0) = U^0$ satisfies the bound*

$$\|U(t) - \overline{U}_h(t + \tilde{\theta})\|_{\ell^p} \quad \leq \quad Ce^{-\beta t}\|U^0 - \overline{U}_h(0)\|_{\ell^p} \tag{2.2.24}$$

for all $t > 0$.

2.3 The singular perturbation

The main difficulty in analysing the travelling wave MFDE (2.2.12) is that it is a singular perturbation of the ODE (2.2.8). Indeed, the second derivative in (2.2.8) is replaced by the linear operator $\Delta_h : H^1(\mathbb{R}) \to L^2(\mathbb{R})$ that acts as

$$\Delta_h \phi(\xi) \quad = \quad \tfrac{1}{h^2} \sum_{k>0} \alpha_k \Big(\phi(\xi + hk) + \phi(\xi - hk) - 2\phi(\xi) \Big). \tag{2.3.1}$$

We will see in Lemma 2.3.5 that for all $\phi \in L^\infty(\mathbb{R})$ with $\phi'' \in L^2(\mathbb{R})$, we have that $\lim_{h\downarrow 0} \|\Delta_h \phi - \phi''\|_{L^2} = 0$. Hence, the bounded operator Δ_h converges pointwise on a dense subset of $H^1(\mathbb{R})$ to an unbounded operator on that same dense subset. In particular, the norm of the operator Δ_h grows to infinity as $h \downarrow 0$.

Since there are no second derivatives involved in (2.2.12), we have to view it as an equation posed on the space $H^1(\mathbb{R}) \times H^1(\mathbb{R})$, while the ODE (2.2.8) is posed on the space $H^2(\mathbb{R}) \times H^1(\mathbb{R})$. From now on we write

$$\begin{aligned} \mathbf{H}^1 \quad &:= \quad H^1(\mathbb{R}) \times H^1(\mathbb{R}), \\ \mathbf{L}^2 \quad &:= \quad L^2(\mathbb{R}) \times L^2(\mathbb{R}). \end{aligned} \tag{2.3.2}$$

The main results in this section will be used in several different settings. In order to accommodate this, we introduce the following conditions.

Assumption (hFam). For each $h > 0$ there is a pair $(\tilde{u}_h, \tilde{w}_h) \in \mathbf{H}^1$ and a constant $\tilde{c}_h$ such that $(\tilde{u}_h, \tilde{w}_h) - (\overline{u}_0, \overline{w}_0) \to 0$ in $\mathbf{H}^1$ and $\tilde{c}_h \to c_0$ as $h \downarrow 0$.

In the proof of Theorem 2.2.1 we choose $(\tilde{u}_h, \tilde{w}_h)$ and $\tilde{c}_0$ to be $(\overline{u}_0, \overline{w}_0)$ and c_0 for all values of h. However, in §2.5 we let $(\tilde{u}_h, \tilde{w}_h)$ be the travelling pulse $(\overline{u}_h, \overline{w}_h)$ from Theorem 2.2.1 and we let $\tilde{c}_h$ be its wave speed c_h.

If (hFam) is satisfied, then for $\delta > 0$ and $h > 0$ we define the operators

$$\overline{\mathcal{L}}^+_{h,\delta} \quad = \quad \begin{pmatrix} \tilde{c}_h \frac{d}{d\xi} - \Delta_h - g_u(\tilde{u}_h) + \delta & 1 \\ -\rho & \tilde{c}_h \frac{d}{d\xi} + \gamma\rho + \delta \end{pmatrix} \tag{2.3.3}$$

and

$$\overline{\mathcal{L}}^{-}_{h,\delta} = \begin{pmatrix} -\tilde{c}_h \frac{d}{d\xi} - \Delta_h - g_u(\tilde{u}_h) + \delta & -\rho \\ 1 & -\tilde{c}_h \frac{d}{d\xi} + \gamma\rho + \delta \end{pmatrix}. \tag{2.3.4}$$

These operators are bounded linear functions from $\mathbf{H}^1$ to $\mathbf{L}^2$. We see that $\overline{\mathcal{L}}^{-}_{h,\delta}$ is the adjoint operator of $\overline{\mathcal{L}}^{+}_{h,\delta}$, in the sense that

$$\langle (\phi,\psi), \overline{\mathcal{L}}^{+}_{h,\delta}(\theta,\chi)\rangle = \langle \overline{\mathcal{L}}^{-}_{h,\delta}(\phi,\psi), (\theta,\chi)\rangle \tag{2.3.5}$$

holds for all $(\phi,\psi),(\theta,\chi) \in \mathbf{H}^1$. Here we have introduced the notation

$$\begin{aligned} \langle (\phi,\psi),(\theta,\chi)\rangle &= \langle \phi,\theta\rangle + \langle \psi,\chi\rangle \\ &= \int_{-\infty}^{\infty} \Big(\phi(x)\theta(x) + \psi(x)\chi(x)\Big)dx \end{aligned} \tag{2.3.6}$$

for $(\phi,\psi),(\theta,\chi) \in \mathbf{L}^2$.

Since, at some point, we want to consider complex-valued functions, we also work in the spaces $H^2_{\mathbb{C}}(\mathbb{R})$, $H^1_{\mathbb{C}}(\mathbb{R})$ and $L^2_{\mathbb{C}}(\mathbb{R})$, which are given by

$$\begin{aligned} H^2_{\mathbb{C}}(\mathbb{R}) &= \{f + gi \mid f,g \in H^2(\mathbb{R})\}, \\ H^1_{\mathbb{C}}(\mathbb{R}) &= \{f + gi \mid f,g \in H^1(\mathbb{R})\}, \\ L^2_{\mathbb{C}}(\mathbb{R}) &= \{f + gi \mid f,g \in L^2(\mathbb{R})\}. \end{aligned} \tag{2.3.7}$$

These spaces are equipped with the inner product

$$\langle \phi,\psi\rangle = \int \Big(f_1(x) + ig_1(x)\Big)\Big(f_2(x) - ig_2(x)\Big)dx \tag{2.3.8}$$

for $\phi = f_1 + ig_1, \psi = f_2 + ig_2$. As before, we write

$$\begin{aligned} \mathbf{H}^1_{\mathbb{C}} &= H^1_{\mathbb{C}}(\mathbb{R}) \times H^1_{\mathbb{C}}(\mathbb{R}) \\ \mathbf{L}^2_{\mathbb{C}} &= L^2_{\mathbb{C}}(\mathbb{R}) \times L^2_{\mathbb{C}}(\mathbb{R}). \end{aligned} \tag{2.3.9}$$

Each operator L from $\mathbf{H}^1$ to $\mathbf{L}^2$ can be extended to an operator from $\mathbf{H}^1_{\mathbb{C}}$ to $\mathbf{L}^2_{\mathbb{C}}$ by writing

$$L(f + ig) = Lf + iLg. \tag{2.3.10}$$

It is well-known that this complexification preserves adjoints, invertibility, inverses, injectivity, surjectivity and boundedness, see for example [146]. If $\lambda \in \mathbb{C}$ then the operators $\overline{\mathcal{L}}^{\pm}_{h,\lambda}$ are defined analogously to their real counterparts, but now we view them as operators from $H^1_{\mathbb{C}}(\mathbb{R}) \times H^1_{\mathbb{C}}(\mathbb{R})$ to $L^2_{\mathbb{C}}(\mathbb{R}) \times L^2_{\mathbb{C}}(\mathbb{R})$. Whenever it is clear that we are working in the complex setting we drop the subscript $\mathbb{C}$ from the spaces $\mathbf{H}^1_{\mathbb{C}}$ and $\mathbf{L}^2_{\mathbb{C}}$ and simply write $\mathbf{H}^1$ and $\mathbf{L}^2$.

We also introduce the operators $\mathcal{L}_0^{\pm} : H^2(\mathbb{R}) \times H^1(\mathbb{R}) \to L^2(\mathbb{R}) \times L^2(\mathbb{R})$, that act as

$$\mathcal{L}_0^{+} = \begin{pmatrix} c_0 \frac{d}{d\xi} - \frac{d^2}{d\xi^2} - g_u(\overline{u}_0) & 1 \\ -\rho & c_0 \frac{d}{d\xi} + \gamma\rho \end{pmatrix} \tag{2.3.11}$$

and

$$\mathcal{L}_0^{-} = \begin{pmatrix} -c_0 \frac{d}{d\xi} - \frac{d^2}{d\xi^2} - g_u(\overline{u}_0) & -\rho \\ 1 & -c_0 \frac{d}{d\xi} + \gamma\rho \end{pmatrix}. \tag{2.3.12}$$

These operators can be viewed as the formal $h \downarrow 0$ limits of the operators $\overline{\mathcal{L}}_{h,0}^{\pm}$. Upon introducing the notation

$$(\phi_0^+, \psi_0^+) = \frac{(\overline{u}_0', \overline{w}_0')}{\|(\overline{u}_0', \overline{w}_0')\|_{\mathbf{L}^2}}, \tag{2.3.13}$$

we see that $\mathcal{L}_0^+(\phi_0^+, \psi_0^+) = 0$ by differentiating (2.2.8).

To set the stage, we summarize several basic properties of $\mathcal{L}_0^{\pm}$. The proof of this result follows the standard procedure described in [6, Lem. 3.1] and, as such, will be omitted. The last property references a spectral set M, on which we impose the following condition.

Assumption (hM). The set $M \subset \mathbb{C}$ is compact with $0 \notin M$. In addition, recalling the constant λ_* appearing in (HP2), we have $\operatorname{Re} z \geq -\lambda_*$ for all $z \in M$.

In §2.6 the set M will be fixed as the final region of our spectral analysis, which we will refer to as R_4.

Lemma 2.3.1. *Assume that (HP1), (HS) and (Hα1) are satisfied. Then the following results hold.*

1. *We have that $(\phi_0^+, \psi_0^+) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$ and $\ker(\mathcal{L}_0^+) = \operatorname{span}\{(\phi_0^+, \psi_0^+)\}$.*
2. *There exist $(\phi_0^-, \psi_0^-) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$ with $\|(\phi_0^-, \psi_0^-)\|_{\mathbf{L}^2} = 1$, with $\langle(\overline{u}_0', \overline{w}_0'), (\phi_0^-, \psi_0^-)\rangle > 0$ and $\ker(\mathcal{L}_0^-) = \operatorname{span}\{(\phi_0^-, \psi_0^-)\}$.*
3. *For every $(\theta, \chi) \in \mathbf{L}^2$ the problem $\mathcal{L}_0^{\pm}(\phi, \psi) = (\theta, \chi)$ with $(\phi, \psi) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$ and $\langle(\phi, \psi), (\phi_0^{\pm}, \psi_0^{\pm})\rangle = 0$ has a unique solution (ϕ, ψ) if and only if $\langle(\theta, \chi), (\phi_0^{\mp}, \psi_0^{\mp})\rangle = 0$.*
4. *There exists a positive constant C_1 such that*
$$\|(\phi, \psi)\|_{H^2(\mathbb{R}) \times H^1(\mathbb{R})} \leq C_1 \|\mathcal{L}_0^{\pm}(\phi, \psi)\|_{\mathbf{L}^2} \tag{2.3.14}$$
for all $(\phi, \psi) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$ with $\langle(\phi, \psi), (\phi_0^{\pm}, \psi_0^{\pm})\rangle = 0$.
5. *There exists a positive constant C_2 and a small constant $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ we have*
$$\|(\mathcal{L}_0^{\pm} + \delta)^{-1}(\theta, \chi)\|_{H^2(\mathbb{R}) \times H^1(\mathbb{R})} \leq C_2 \Big[\|(\theta, \chi)\|_{\mathbf{L}^2} + \tfrac{1}{\delta} |\langle(\theta, \chi), (\phi_0^{\mp}, \psi_0^{\mp})\rangle| \Big] \tag{2.3.15}$$
for all $(\theta, \chi) \in \mathbf{L}^2$.

6. *If (HP2) is also satisfied, then for each $M \subset \mathbb{C}$ that satisfies (hM), there exists a constant $C_3 > 0$ such that the uniform bound*

$$\|(\mathcal{L}_0^{\pm} + \lambda)^{-1}(\theta, \chi)\|_{H^2_{\mathbb{C}}(\mathbb{R}) \times H^1_{\mathbb{C}}(\mathbb{R})} \leq C_3 \|(\theta, \chi)\|_{\boldsymbol{L}^2_{\mathbb{C}}} \tag{2.3.16}$$

holds for all $(\theta, \chi) \in \boldsymbol{L}^2_{\mathbb{C}}$ and all $\lambda \in M$.

The main goal of this section is to prove the following two propositions, which transfer parts (5) and (6) of Lemma 2.3.1 to the discrete setting.

Proposition 2.3.2. *Assume that (hFam), (HP1), (HS) and (Hα1) are satisfied. There exists a positive constant C_0' and a positive function $h_0'(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$, depending only on the choice of $(\tilde{u}_h, \tilde{w}_h)$ and $\tilde{c}_h$, such that for every $0 < \delta < \delta_0$ and every $h \in (0, h_0'(\delta))$, the operators $\overline{\mathcal{L}}^{\pm}_{h,\delta}$ are homeomorphisms from $\boldsymbol{H}^1$ to $\boldsymbol{L}^2$ that satisfy the bounds*

$$\|(\overline{\mathcal{L}}^{\pm}_{h,\delta})^{-1}(\theta, \chi)\|_{\boldsymbol{H}^1} \quad \leq \quad C_0' \Big[\|(\theta, \chi)\|_{\boldsymbol{L}^2} + \tfrac{1}{\delta} |\langle (\theta, \chi), (\phi_0^{\mp}, \psi_0^{\mp}) \rangle| \Big] \tag{2.3.17}$$

for all $(\theta, \chi) \in \boldsymbol{L}^2$.

Proposition 2.3.3. *Assume that (hFam), (HP1),(HP2), (HS) and (Hα1) are satisfied. Let $M \subset \mathbb{C}$ satisfy (hM). Then there exists a constant $h_M > 0$, depending only on M and the choice of $(\tilde{u}_h, \tilde{w}_h)$ and $\tilde{c}_h$, such that for all $0 < h \leq h_M$ and all $\lambda \in M$ the operator $\overline{\mathcal{L}}^{\pm}_{h,\lambda}$ is a homeomorphism from $\boldsymbol{H}^1$ to $\boldsymbol{L}^2$.*

2.3.1 Strategy

Our techniques here are inspired strongly by the approach developed in [6, §2-4]. Indeed, Proposition 2.3.2 and Proposition 2.3.4 are the equivalents of [6, Thm. 3] and [6, Lem. 3.2] respectively. The difference between our results and those in [6] is that Bates, Chen and Chmaj study the discrete Nagumo equation, which can be seen as the one-dimensional fast component of the FitzHugh-Nagumo equation by setting $\rho = 0$ in (2.2.1). In addition, the results in [6] are restricted to $\lambda \in \mathbb{R}$, while we allow $\lambda \in \mathbb{C}$ in Proposition 2.3.3. These differences play a crucial role in the proof of Lemma 2.3.10 below.

Recall the constant $\delta_0 > 0$ appearing in Lemma 2.3.1. For $0 < \delta < \delta_0$ and $h > 0$ we define the quantities

$$\overline{\Lambda}^{\pm}(h, \delta) \quad = \inf_{\|(\phi, \psi)\|_{\mathbf{H}^1} = 1} \Big[\|\overline{\mathcal{L}}^{\pm}_{h,\delta}(\phi, \psi)\|_{\mathbf{L}^2} + \tfrac{1}{\delta} \Big| \langle \overline{\mathcal{L}}^{\pm}_{h,\delta}(\phi, \psi), (\phi_0^{\mp}, \psi_0^{\mp}) \rangle \Big| \Big], \tag{2.3.18}$$

together with

$$\overline{\Lambda}^{\pm}(\delta) \quad = \quad \liminf_{h \downarrow 0} \overline{\Lambda}^{\pm}(h, \delta). \tag{2.3.19}$$

Similarly for $M \subset \mathbb{C}$ that satisfies (hM) and $h > 0$ we define

$$\overline{\Lambda}^{\pm}(h, M) \quad = \inf_{\|(\phi, \psi)\|_{\mathbf{H}^1} = 1, \ \lambda \in M} \Big[\|\overline{\mathcal{L}}^{\pm}_{h,\lambda}(\phi, \psi)\|_{\mathbf{L}^2} \Big], \tag{2.3.20}$$

together with

$$\overline{\Lambda}^{\pm}(M) \quad = \quad \liminf_{h\downarrow 0} \overline{\Lambda}^{\pm}(h, M). \tag{2.3.21}$$

The key ingredients that we need to establish Propositions 2.3.2 and 2.3.3 are lower bounds on the quantities $\overline{\Lambda}^{\pm}(\delta)$ and $\overline{\Lambda}^{\pm}(M)$. These are provided in the result below, which we consider to be the technical heart of this section.

Proposition 2.3.4. *Assume that (hFam), (HP1), (HS) and (Hα1) are satisfied. There exists a positive constant C_0, depending only on our choice of $(\tilde{u}_h, \tilde{w}_h)$ and $\tilde{c}_h$, such that $\overline{\Lambda}^{\pm}(\delta) > C_0$ for all $0 < \delta < \delta_0$. Similarly if $M \subset \mathbb{C}$ satisfies (hM), then there exists a positive constant C_M, depending only on M and our choice of $(\tilde{u}_h, \tilde{w}_h)$ and $\tilde{c}_h$, such that $\overline{\Lambda}^{\pm}(M) > \mathbb{C}_M$.*

Proof of Proposition 2.3.2. Let $\delta > 0$ be fixed and set $C_0' = \frac{2}{C_0}$. Since $\overline{\Lambda}^{\pm}(\delta) \geq \frac{2}{C_0'}$, the definition (2.3.19) implies that there exists $h_0'(\delta)$ such that $\overline{\Lambda}(h, \delta) \geq \frac{1}{C_0'}$ for all $h \in (0, h_0'(\delta)]$. Now pick $h \in (0, h_0'(\delta)]$.
First of all, $\overline{\mathcal{L}}^{\pm}_{h,\delta}$ is a bounded operator from $\mathbf{H}^1$ to $\mathbf{L}^2$. Since $\overline{\Lambda}^{\pm}(h, \delta)$ is strictly positive, this implies that $\overline{\mathcal{L}}^{\pm}_{h,\delta}$ is a homeomorphism from $\mathbf{H}^1$ to its image $\overline{\mathcal{L}}^{\pm}_{h,\delta}(\mathbf{H}^1)$. Furthermore, the norm of the inverse $(\overline{\mathcal{L}}^{\pm}_{h,\delta})^{-1}$ from $\overline{\mathcal{L}}^{\pm}_{h,\delta}(\mathbf{H}^1) \subset \mathbf{L}^2$ is bounded by $\frac{1}{\overline{\Lambda}^{\pm}(h,\delta)} \leq C_0'$. Since $\overline{\mathcal{L}}^{\pm}_{h,\delta}$ is bounded, it follows that $\overline{\mathcal{L}}^{\pm}_{h,\delta}(\mathbf{H}^1)$ is closed in $\mathbf{L}^2$.

For the remainder of this proof, we only consider the operators $\overline{\mathcal{L}}^{+}_{h,\delta}$, noting that their counterparts $\overline{\mathcal{L}}^{-}_{h,\delta}$ can be treated in an identical fashion.

Seeking a contradiction, let us assume that $\overline{\mathcal{L}}^{+}_{h,\delta}(\mathbf{H}^1) \neq \mathbf{L}^2$, which implies that there exists a nonzero $(\theta, \chi) \in \mathbf{L}^2$ orthogonal to $\overline{\mathcal{L}}^{+}_{h,\delta}(\mathbf{H}^1)$. For any $\phi \in C_c^{\infty}(\mathbb{R})$, we hence obtain

$$\begin{aligned} 0 &= \langle \overline{\mathcal{L}}^{+}_{h,\delta}(\phi, 0), (\theta, \chi) \rangle \\ &= \langle \tilde{c}_h \phi' - \Delta_h \phi - g_u(\tilde{u}_h)\phi + \delta\phi, \theta \rangle + \langle -\rho\phi, \chi \rangle \\ &= \tilde{c}_h \langle \phi', \theta \rangle + \langle \phi, -\Delta_h \theta - g_u(\tilde{u}_h)\theta + \delta\theta - \rho\chi \rangle. \end{aligned} \tag{2.3.22}$$

By definition this implies that θ has a weak derivative and that $\tilde{c}_h \theta' = -\Delta_h \theta - g_u(\tilde{u}_h)\theta + \delta\theta - \rho\chi \in L^2(\mathbb{R})$. In particular, we see that $\theta \in H^1(\mathbb{R})$.

For any $\psi \in C_c^{\infty}(\mathbb{R})$ a similar computation yields

$$\begin{aligned} 0 &= \langle \overline{\mathcal{L}}^{+}_{h,\delta}(0, \psi), (\theta, \chi) \rangle \\ &= \langle \psi, \theta \rangle + \langle \tilde{c}_h \psi' + (\gamma\rho + \delta)\psi, \chi \rangle \\ &= \tilde{c}_h \langle \psi', \chi \rangle + \langle \psi, \theta + (\gamma\rho + \delta)\chi \rangle. \end{aligned} \tag{2.3.23}$$

Again, this means that χ has a weak derivative and in fact $\tilde{c}_h \chi' = \theta + (\gamma\rho + \delta)\chi$. In particular, it follows that $\chi \in H^1(\mathbb{R})$.

We, therefore, conclude that

$$\begin{array}{rcl} 0 & = & \langle \overline{\mathcal{L}}^+_{h,\delta}(\phi,\psi),(\theta,\chi)\rangle \\ & = & \langle (\phi,\psi),(\overline{\mathcal{L}}^-_{h,\delta}(\theta,\chi)\rangle \end{array} \tag{2.3.24}$$

holds for all $(\phi,\psi) \in \mathbf{H}^1$. Since $\mathbf{H}^1$ is dense in $\mathbf{L}^2$ this implies that $\overline{\mathcal{L}}^-_{h,\delta}(\theta,\chi) = 0$. Since we already know that $\overline{\mathcal{L}}^-_{h,\delta}$ is injective, this means that $(\theta,\chi) = 0$, which gives a contradiction. Hence, we must have $\overline{\mathcal{L}}^+_{h,\delta}(\mathbf{H}^1) = \mathbf{L}^2$, as desired. ■

Proof of Proposition 2.3.3. The result follows in the same fashion as outlined in the proof of Proposition 2.3.2 above. ■

2.3.2 Preliminaries

Our goal here is to establish some basic facts concerning the operator Δ_h. In particular, we extend the real-valued results from [6] to complex-valued functions. We emphasize that the inequalities in Lemma 2.3.6 in general do not hold for the imaginary parts of these inner products.

Lemma 2.3.5 ([6, Lem. 2.1])**.** *Assume that (Hα1) is satisfied. The following three properties hold.*

1. *For all $\phi \in L^\infty(\mathbb{R})$ with $\phi'' \in L^2(\mathbb{R})$ we have $\lim_{h\downarrow 0} \|\Delta_h\phi - \phi''\|_{L^2} = 0$.*
2. *For all $\phi \in H^1(\mathbb{R})$ and $h > 0$ we have $\langle \Delta_h\phi, \phi'\rangle = 0$.*
3. *For all $\phi,\psi \in L^2(\mathbb{R})$ and $h > 0$ we have $\langle \Delta_h\phi,\psi\rangle = \langle \phi, \Delta_h\psi\rangle$ and $\langle \Delta_h\phi,\phi\rangle \leq 0$.*

Lemma 2.3.6. *Assume that (Hα1) is satisfied and pick $f \in H^1_{\mathbb{C}}(\mathbb{R})$. Then the following properties hold.*

1. *For all $h > 0$ we have* $\mathrm{Re}\,\langle -\Delta_h f, f\rangle \geq 0$.
2. *For all $h > 0$ we have* $\mathrm{Re}\,\langle \Delta_h f, f'\rangle = 0$.
3. *We have* $\mathrm{Re}\,\langle f, f'\rangle = 0$.
4. *For all $\lambda \in \mathbb{C}$ we have* $\mathrm{Re}\,\langle \lambda f, f'\rangle = 2\ (\mathrm{Im}\,\lambda)\langle \mathrm{Re}\,f, \mathrm{Im}\,f'\rangle$.

Proof. Write $f = \phi + i\psi$ with $\phi,\psi \in H^1(\mathbb{R})$. Lemma 2.3.5 implies that

$$\begin{array}{rcl} \mathrm{Re}\,\langle -\Delta_h f, f\rangle & = & \mathrm{Re}\int \Big(-\Delta_h\phi - i\Delta_h\psi\Big)(x)\Big(\phi - i\psi\Big)(x)dx \\ & = & \int(-\Delta_h\phi)(x)\phi(x) + (-\Delta_h\psi)(x)\psi(x)dx \\ & = & \langle -\Delta_h\phi,\phi\rangle + \langle -\Delta_h\psi,\psi\rangle \\ & \geq & 0. \end{array} \tag{2.3.25}$$

Similarly we have

$$\begin{array}{lcl}\mathrm{Re}\,\langle \Delta_h f, f' \rangle & = & \langle -\Delta_h \phi, \phi' \rangle + \langle -\Delta_h \psi, \psi' \rangle \\ & = & 0.\end{array} \tag{2.3.26}$$

For $\lambda \in \mathbb{C}$ we may compute

$$\begin{array}{lcl}\mathrm{Re}\,\langle \lambda f, f' \rangle & = & \mathrm{Re}\int \Big(\lambda\phi(x) + \lambda i \psi(x)\Big)\Big(\phi'(x) - i\psi'(x)\Big)dx \\ & = & (\mathrm{Re}\,\lambda)\langle \phi, \phi' \rangle + (\mathrm{Im}\,\lambda)\langle \phi, \psi' \rangle - (\mathrm{Im}\,\lambda)\langle \psi, \phi' \rangle + (\mathrm{Re}\,\lambda)\langle \psi, \psi' \rangle \\ & = & 0 + 2\ (\mathrm{Im}\,\lambda)\langle \phi, \psi' \rangle + 0 \\ & = & 2\ (\mathrm{Im}\,\lambda)\langle \phi, \psi' \rangle.\end{array} \tag{2.3.27}$$

Taking $\lambda = 1$ gives the third property. ■

2.3.3 Proof of Proposition 2.3.4

We now set out to prove Proposition 2.3.4. In Lemmas 2.3.7 and 2.3.8, we construct weakly converging sequences that realize the infima in (2.3.18)-(2.3.21). In Lemmas 2.3.9-2.3.11, we exploit the structure of our operators (2.3.3) and (2.3.4) to recover bounds on the derivatives of these sequences that are typically lost when taking weak limits. Recall the constant $C_2 > 0$ defined in Lemma 2.3.1, which does not depend on $\delta > 0$.

Lemma 2.3.7. *Assume that (hFam), (HP1), (HS) and (Hα1) are satisfied. Consider the setting of Proposition 2.3.4 and fix $0 < \delta < \delta_0$. Then there exists a sequence $\{(h_j, \phi_j, \psi_j)\}_{j \geq 0}$ in $(0,1) \times \boldsymbol{H}^1$ with the following properties.*

1. *We have $\lim_{j\to\infty} h_j = 0$ and $\|(\phi_j, \psi_j)\|_{\boldsymbol{H}^1} = 1$ for all $j \geq 0$.*
2. *The sequence $(\theta_j, \chi_j) = \overline{\mathcal{L}}^+_{h_j,\delta}(\phi_j, \psi_j)$ satisfies*
$$\lim_{j\to\infty}\Big[\|(\theta_j, \chi_j)\|_{\boldsymbol{L}^2} + \tfrac{1}{\delta}|\langle(\theta_j, \chi_j), (\phi_0^-, \psi_0^-)\rangle|\Big] \;=\; \overline{\Lambda}^+(\delta). \tag{2.3.28}$$
3. *There exist $(\phi, \psi) \in \boldsymbol{H}^1$ and $(\theta, \chi) \in \boldsymbol{L}^2$ such that $(\phi_j, \psi_j) \rightharpoonup (\phi, \psi)$ weakly in $\boldsymbol{H}^1$ and such that $(\theta_j, \chi_j) \rightharpoonup (\theta, \chi)$ weakly in $\boldsymbol{L}^2$ as $j \to \infty$.*
4. *We have $(\phi_j, \psi_j) \to (\phi, \psi)$ in $L^2_{\mathrm{loc}}(\mathbb{R}) \times L^2_{\mathrm{loc}}(\mathbb{R})$ as $j \to \infty$.*
5. *The pair (ϕ, ψ) is a weak solution to $(\overline{\mathcal{L}}^+_0 + \delta)(\phi, \psi) = (\theta, \chi)$.*
6. *We have the bound*
$$\|(\phi, \psi)\|_{H^2(\mathbb{R})\times H^1(\mathbb{R})} \;\leq\; C_2\overline{\Lambda}^+(\delta). \tag{2.3.29}$$

The same statements hold upon replacing $\overline{\mathcal{L}}^+_{h,\delta}$, $\overline{\Lambda}^+$ and $\overline{\mathcal{L}}^+_0$ by $\overline{\mathcal{L}}^-_{h,\delta}$, $\overline{\Lambda}^-$ and $\overline{\mathcal{L}}^-_0$.

Proof. Let $0 < \delta < \delta_0$ be fixed. By definition of $\overline{\Lambda}^+(\delta)$ there exists a sequence $\{(h_j, \phi_j, \psi_j)\}$ in $(0,1) \times \mathbf{H}^1$ such that (1) and (2) hold. Taking a subsequence if necessary, we may assume that there exist $(\phi, \psi) \in \mathbf{H}^1$ and $(\theta, \chi) \in \mathbf{L}^2$ such that $(\phi_j, \psi_j) \to (\phi, \psi)$ in $L^2_{\mathrm{loc}}(\mathbb{R}) \times L^2_{\mathrm{loc}}(\mathbb{R})$ and weakly in $\mathbf{H}^1$ as $j \to \infty$ and such that $(\theta_j, \chi_j) \rightharpoonup (\theta, \chi)$ weakly in $\mathbf{L}^2$. By the weak lower-semicontinuity of the $\mathbf{L}^2$-norm, we obtain

$$\|(\theta, \chi)\|_{\mathbf{L}^2} + \tfrac{1}{\delta}|\langle(\theta, \chi), (\phi_0^-, \psi_0^-)\rangle| \quad \leq \quad \overline{\Lambda}^+(\delta). \tag{2.3.30}$$

For any pair of test functions $(\zeta_1, \zeta_2) \in C_c^\infty(\mathbb{R}) \times C_c^\infty(\mathbb{R})$ we have

$$\begin{aligned} \langle(\theta_j, \chi_j), (\zeta_1, \zeta_2)\rangle &= \langle\overline{\mathcal{L}}^+_{h_j,\delta}(\phi_j, \psi_j), (\zeta_1, \zeta_2)\rangle \\ &= \langle(\phi_j, \psi_j), \overline{\mathcal{L}}^-_{h_j,\delta}(\zeta_1, \zeta_2)\rangle. \end{aligned} \tag{2.3.31}$$

Since $\overline{u}_0$ is a bounded function, the limit $\tilde{u}_h - \overline{u}_0 \to 0$ in H^1 implies that also $\tilde{u}_h \to \overline{u}_0$ in L^∞. In particular, we can choose $h' > 0$ and $N > 0$ in such a way that $|\tilde{u}_h| \leq N$ and $|\overline{u}_0| \leq N$ for all $0 < h \leq h'$. Since g_u is Lipschitz continuous on $[-N, N]$, there is a constant $K > 0$ such that $|g_u(x) - g_u(y)| \leq K|x - y|$ for all $x, y \in [-N, N]$. We obtain

$$\begin{aligned} \lim_{h\downarrow 0} \|g_u(\tilde{u}_h) - g_u(\overline{u}_0)\|^2_{L^2} &= \lim_{h\downarrow 0} \int (g_u(\tilde{u}_h) - g_u(\overline{u}_0))^2 dx \\ &\leq \lim_{h\downarrow 0} \int K^2(\tilde{u}_h - \overline{u}_0)^2 dx \\ &\leq \lim_{h\downarrow 0} K^2\|\tilde{u}_h - \overline{u}_0\|^2_{\mathbf{L}^2} \\ &= 0, \end{aligned} \tag{2.3.32}$$

together with

$$\begin{aligned} \lim_{h\downarrow 0} \|g_u(\tilde{u}_h)\zeta_1 - g_u(\overline{u}_0)\zeta_1\|_{L^2} &\leq \lim_{h\downarrow 0} \|\zeta_1\|_\infty \|g_u(\tilde{u}_h) - g_u(\overline{u}_0)\|_{L^2} \\ &= 0. \end{aligned} \tag{2.3.33}$$

Furthermore, we know that $\tilde{c}_h \to c_0$ as $h \downarrow 0$, which gives

$$\begin{aligned} \lim_{h\downarrow 0} \|\tilde{c}_h\zeta_1' - c_0\zeta_1'\|_{L^2} &= \lim_{h\downarrow 0} \|\tilde{c}_h\zeta_2' - c_0\zeta_2'\|_{L^2} \\ &= 0. \end{aligned} \tag{2.3.34}$$

Finally, Lemma 2.3.5 implies

$$\lim_{h\downarrow 0} \|\Delta_h\zeta_1 - \zeta_1''\|_{L^2} \quad = \quad 0, \tag{2.3.35}$$

which means that

$$\|\overline{\mathcal{L}}^-_{h_j,\delta}(\zeta_1, \zeta_2) - (\overline{\mathcal{L}}_0^- + \delta)(\zeta_1, \zeta_2)\|_{\mathbf{L}^2} \quad \to \quad 0 \tag{2.3.36}$$

as $j \to \infty$. Sending $j \to \infty$ in (2.3.31), this yields

$$\langle(\theta, \chi), (\zeta_1, \zeta_2)\rangle \quad = \quad \langle(\phi, \psi), (\overline{\mathcal{L}}_0^- + \delta)(\zeta_1, \zeta_2)\rangle. \tag{2.3.37}$$

In particular, we see that (ϕ, ψ) is a weak solution to $(\overline{\mathcal{L}}_0^+ + \delta)(\phi, \psi) = (\theta, \chi)$. Since $\phi \in H^1$, $\psi \in L^2$, $\theta \in L^2$ and

$$\phi'' = c_0\phi' - g_u(\overline{u}_0)\phi + \delta\phi + \psi - \theta, \tag{2.3.38}$$

we get $\phi'' \in L^2$ and, hence, $\phi \in H^2$. Since we already know that $\psi \in H^1$, we may apply Lemma 2.3.1 and (2.3.30) to obtain

$$\begin{aligned} \|(\phi, \psi)\|_{H^2(\mathbb{R})\times H^1(\mathbb{R})} &\leq C_2[\|(\theta, \chi)\|_{\mathbf{L}^2} + \tfrac{1}{\delta}|\langle(\theta, \chi), (\phi_0^-, \psi_0^-)\rangle|] \\ &\leq C_2\overline{\Lambda}^+(\delta). \end{aligned} \tag{2.3.39}$$

■

The next result is the analogue of Lemma 2.3.7 for the setting where we are considering a spectral set $M \subset \mathbb{C}$ that satisfies (hM). The proof is omitted as it is almost identical to that of Lemma 2.3.7. We recall the constant $C_3 > 0$ from Lemma 2.3.1, which only depends on the choice of the set $M \subset \mathbb{C}$.

Lemma 2.3.8. *Assume that (HP1),(HP2), (HS) and (Hα1) are satisfied. Let $M \subset \mathbb{C}$ satisfy (hM). There exists a sequence $\{(\lambda_j, h_j, \phi_j, \psi_j)\}$ in $M \times (0,1) \times \boldsymbol{H}^1$ with the following properties.*

1. *We have $\lim_{j\to\infty} h_j = 0$, $\lim_{j\to\infty} \lambda_j = \lambda$ for some $\lambda \in M$ and $\|(\phi_j, \psi_j)\|_{\boldsymbol{H}^1} = 1$ for all j.*
2. *The pair $(\theta_j, \chi_j) = \overline{\mathcal{L}}^+_{h_j,\lambda_j}(\phi_j, \psi_j)$ satisfies*
$$\lim_{j\to\infty} \|(\theta_j, \chi_j)\|_{\boldsymbol{L}^2} = \overline{\Lambda}^+(M). \tag{2.3.40}$$
3. *There exist $(\phi, \psi) \in \boldsymbol{H}^1$ and $(\theta, \chi) \in \boldsymbol{L}^2$ such that as $j \to \infty$ $(\phi_j, \psi_j) \to (\phi, \psi)$ in $L^2_{\mathrm{loc}}(\mathbb{R}) \times L^2_{\mathrm{loc}}(\mathbb{R})$ and weakly in $\boldsymbol{H}^1$ and such that $(\theta_j, \chi_j) \rightharpoonup (\theta, \chi)$ weakly in $\boldsymbol{L}^2$.*
4. *The pair (ϕ, ψ) is a weak solution to $(\overline{\mathcal{L}}_0^+ + \lambda)(\phi, \psi) = (\theta, \chi)$.*
5. *We have the bound*
$$\|(\phi, \psi)\|_{H^2(\mathbb{R})\times H^1(\mathbb{R})} \leq C_3\overline{\Lambda}^+(M). \tag{2.3.41}$$

The same statements hold upon replacing $\overline{\mathcal{L}}^+_{h,\lambda_j}$, $\overline{\Lambda}^+(M)$ and $\overline{\mathcal{L}}_0^+$ by $\overline{\mathcal{L}}^-_{h,\lambda_j}$, $\overline{\Lambda}^-$ and $\overline{\mathcal{L}}_0^-$.

In our arguments below, we often consider the sequences $\{(h_j, \phi_j, \psi_j)\}$ and $\{(\lambda_j, h_j, \phi_j, \psi_j)\}$ defined in Lemmas 2.3.7 and 2.3.8 in a similar fashion. To streamline our notation, we simply write $\{(\lambda_j, h_j, \phi_j, \psi_j)\}$ for all these sequences, with the understanding that $\lambda_j = \delta$ when referring to Lemma 2.3.7. As argued in the proof of Lemma 2.3.7, it is possible to choose $\overline{h} > 0$ in such a way that

$$\begin{aligned} c_* &:= \inf\nolimits_{0<h\leq\overline{h}} |\tilde{c}_h| &> 0, \\ g_* &:= \sup\nolimits_{0<h\leq\overline{h}} \|g_u(\tilde{u}_h)\|_\infty &< \infty. \end{aligned} \tag{2.3.42}$$

By taking a subsequence if necessary, we assume from now on that $h_j < \overline{h}$ for all j.

It remains to find a positive lower bound for $\|(\phi, \psi)\|_{\mathbf{L}^2}$. An essential step to accomplish this is to keep the derivatives (ϕ_j', ψ_j') under control. This can be achieved by exploiting the results for Δ_h derived in §2.3.2.

Lemma 2.3.9. *Assume that (hFam), (HP1), (HS) and (Hα1) are satisfied. Consider the setting of Proposition 2.3.4 and Lemma 2.3.7 or Lemma 2.3.8. Then there exists a constant $B > 0$, depending only on M and our choice of $(\tilde{u}_h, \tilde{w}_h)$ and $\tilde{c}_h$, such that for all j we have the bound*

$$B\|(\phi_j, \psi_j)\|^2_{\mathbf{L}^2} \quad \geq \quad c_*^2\|(\phi_j', \psi_j')\|^2_{\mathbf{L}^2} - 4\|(\theta_j, \chi_j)\|^2_{\mathbf{L}^2}. \tag{2.3.43}$$

Proof. We first consider the sequence for $\overline{\Lambda}^+$. Using $\overline{\mathcal{L}}^+_{h_j,\lambda_j}(\phi_j, \psi_j) = (\theta_j, \chi_j)$ and $\mathrm{Re}\,\langle \Delta_{h_j}\phi_j, \phi_j'\rangle = 0 = \mathrm{Re}\,\langle \phi_j, \phi_j'\rangle = \mathrm{Re}\,\langle \psi_j, \psi_j'\rangle$, which follow from Lemma 2.3.6, we obtain

$$\begin{array}{rcl}
\mathrm{Re}\,\langle(\theta_j, \chi_j), (\phi_j', \psi_j')\rangle & = & \mathrm{Re}\,\langle\overline{\mathcal{L}}^+_{h_j,\lambda_j}(\phi_j, \psi_j), (\phi_j', \psi_j')\rangle \\
& = & \mathrm{Re}\,\langle \tilde{c}_{h_j}\phi_j' - \Delta_{h_j}\phi_j - g_u(\tilde{u}_{h_j})\phi_j + \lambda_j\phi_j + \psi_j, \phi_j'\rangle \\
& & \quad +\mathrm{Re}\,\langle -\rho\phi_j + \tilde{c}_{h_j}\psi_j' + \gamma\rho\psi_j + \lambda_j\psi_j, \psi_j'\rangle \\
& = & \tilde{c}_{h_j}\|\phi_j'\|^2_{L^2} - \mathrm{Re}\,\langle g_u(\tilde{u}_{h_j})\phi_j, \phi_j'\rangle + \mathrm{Re}\,\langle\psi_j, \phi_j'\rangle \\
& & \quad +\mathrm{Re}\,\langle\lambda_j\phi_j, \phi_j'\rangle - \rho\mathrm{Re}\,\langle\phi_j, \psi_j'\rangle \\
& & \quad +\tilde{c}_{h_j}\|\psi_j'\|^2_{L^2} + \mathrm{Re}\,\langle\lambda_j\psi_j, \psi_j'\rangle \\
& = & \tilde{c}_{h_j}\|(\phi_j', \psi_j')\|^2_{\mathbf{L}^2} - \mathrm{Re}\,\langle g_u(\tilde{u}_{h_j})\phi_j, \phi_j'\rangle + (1+\rho)\langle\psi_j, \phi_j'\rangle \\
& & \quad +\mathrm{Re}\,\langle\lambda_j(\phi_j, \psi_j), (\phi_j', \psi_j')\rangle.
\end{array} \tag{2.3.44}$$

We write $\lambda_{\max} = \delta_0$ in the setting of Lemma 2.3.7 or $\lambda_{\max} = \max\{|z| : z \in M\}$ in the setting of Lemma 2.3.8. We write

$$G \quad = \quad \lambda_{\max}\|(\phi_j, \psi_j)\|_{\mathbf{L}^2}\|(\phi_j', \psi_j')\|_{\mathbf{L}^2} + g_*\|\phi_j\|_{L^2}\|(\phi_j', \psi_j')\|_{\mathbf{L}^2}. \tag{2.3.45}$$

Using the Cauchy-Schwarz inequality, we now obtain

$$\begin{array}{rcl}
G & \geq & \lambda_{\max}\|(\phi_j, \psi_j)\|_{\mathbf{L}^2}\|(\phi_j', \psi_j')\|_{\mathbf{L}^2} + \|g_u(\tilde{u}_{h_j})\|_{L^\infty}\|\phi_j\|_{L^2}\|\phi_j'\|_{L^2} \\
& \geq & \mathrm{sign}(\tilde{c}_{h_j})\Big(-\mathrm{Re}\,\langle\lambda_j(\phi_j, \psi_j), (\phi_j', \psi_j')\rangle + \mathrm{Re}\,\langle g_u(\tilde{u}_{h_j})\phi_j, \phi_j'\rangle\Big) \\
& = & \mathrm{sign}(\tilde{c}_{h_j})\Big(\tilde{c}_{h_j}\|(\phi_j', \psi_j')\|^2_{\mathbf{L}^2} + (1+\rho)\mathrm{Re}\,\langle\psi_j, \phi_j'\rangle - \mathrm{Re}\,\langle(\theta_j, \chi_j), (\phi_j', \psi_j')\rangle\Big) \\
& \geq & |\tilde{c}_{h_j}|\|(\phi_j', \psi_j')\|^2_{\mathbf{L}^2} - (1+\rho)\|\psi_j\|_{L^2}\|\phi_j'\|_{L^2} - \|(\theta_j, \chi_j)\|_{\mathbf{L}^2}\|(\phi_j', \psi_j')\|_{\mathbf{L}^2} \\
& \geq & c_*\|(\phi_j', \psi_j')\|^2_{\mathbf{L}^2} - (1+\rho)\|\psi_j\|_{L^2}\|(\phi_j', \psi_j')\|_{\mathbf{L}^2} - \|(\theta_j, \chi_j)\|_{\mathbf{L}^2}\|(\phi_j', \psi_j')\|_{\mathbf{L}^2}.
\end{array} \tag{2.3.46}$$

This implies

$$c_*\|(\phi_j', \psi_j')\|_{\mathbf{L}^2} \quad \leq \quad g_*\|\phi_j\|_{L^2} + (1+\rho)\|\psi_j\|_{L^2} + \|(\theta_j, \chi_j)\|_{\mathbf{L}^2} + \lambda_{\max}\|(\phi_j, \psi_j)\|_{\mathbf{L}^2}. \tag{2.3.47}$$

Squaring this equation and using the standard inequality $2\mu\omega \le \mu^2 + \omega^2$, this implies that

$$\begin{array}{rcl} c_*^2\|(\phi_j', \psi_j')\|^2_{\mathbf{L}^2} & \le & 4g_*^2\|\phi_j\|^2_{L^2} + 4(1+\rho)^2\|\psi_j\|^2_{L^2} \\ & & +4\|(\theta_j, \chi_j)\|^2_{\mathbf{L}^2} + 4\lambda^2_{\max}\|(\phi_j, \psi_j)\|^2_{\mathbf{L}^2}. \end{array} \tag{2.3.48}$$

In particular, we see

$$4\Big(\max\{g_*^2, (1+\rho)^2\} + \lambda^2_{\max}\Big)\|(\phi_j, \psi_j)\|^2_{\mathbf{L}^2} \quad \ge \quad c_*^2\|(\phi_j', \psi_j')\|^2_{\mathbf{L}^2} - 4\|(\theta_j, \chi_j)\|^2_{\mathbf{L}^2}. \tag{2.3.49}$$

We now look at the sequence for $\overline{\Lambda}^-$. Using $\overline{\mathcal{L}}^-_{h_j, \lambda_j}(\phi_j, \psi_j) = (\theta_j, \chi_j)$ and $\mathrm{Re}\,\langle \Delta_{h_j}\phi_j, \phi_j'\rangle = 0 = \mathrm{Re}\,\langle \phi_j, \phi_j'\rangle = \mathrm{Re}\,\langle \psi_j, \psi_j'\rangle$, which follow from Lemma 2.3.6, we obtain

$$\begin{array}{rcl} \mathrm{Re}\,\langle (\theta_j, \chi_j), (\phi_j', \psi_j')\rangle & = & \mathrm{Re}\,\langle \overline{\mathcal{L}}^-_{h_j,\lambda_j}(\phi_j, \psi_j), (\phi_j', \psi_j')\rangle \\ & = & \mathrm{Re}\,\langle -\tilde{c}_{h_j}\phi_j' - \Delta_{h_j}\phi_j - g_u(\tilde{u}_h)\phi_j + \lambda_j\phi_j - \rho\psi_j, \phi_j'\rangle \\ & & \quad +\mathrm{Re}\,\langle \phi_j - \tilde{c}_h\psi_j' + \gamma\rho\psi_j + \lambda_j\psi_j, \psi_j'\rangle \\ & = & -\tilde{c}_{h_j}\|\phi_j'\|^2_{L^2} - \mathrm{Re}\,\langle g_u(\tilde{u}_h)\phi_j, \phi_j'\rangle - \rho\mathrm{Re}\,\langle \psi_j, \phi_j'\rangle \\ & & \quad +\mathrm{Re}\,\langle \lambda_j\phi_j, \phi_j'\rangle + \mathrm{Re}\,\langle \phi_j, \psi_j'\rangle \\ & & \quad -\tilde{c}_{h_j}\|\psi_j'\|^2_{L^2} + \mathrm{Re}\,\langle \lambda_j\psi_j, \psi_j'\rangle \\ & = & -\tilde{c}_{h_j}\|(\phi_j', \psi_j')\|^2_{\mathbf{L}^2} - \mathrm{Re}\,\langle g_u(\tilde{u}_h)\phi_j, \phi_j'\rangle + (1+\rho)\langle \psi_j, \phi_j'\rangle \\ & & \quad +\mathrm{Re}\,\langle \lambda_j(\phi_j, \psi_j), (\phi_j', \psi_j')\rangle. \end{array} \tag{2.3.50}$$

We write

$$G \quad = \quad \lambda_{\max}\|(\phi_j, \psi_j)\|_{\mathbf{L}^2}\|(\phi_j', \psi_j')\|_{\mathbf{L}^2} + g_*\|\phi_j\|_{L^2}\|(\phi_j', \psi_j')\|_{\mathbf{L}^2}. \tag{2.3.51}$$

Using the Cauchy-Schwarz inequality we now obtain

$$\begin{array}{rcl} G & \ge & \lambda_{\max}\|(\phi_j, \psi_j)\|_{\mathbf{L}^2}\|(\phi_j', \psi_j')\|_{\mathbf{L}^2} + \|g_u(\tilde{u}_{h_j})\|_{L^\infty}\|\phi_j\|_{L^2}\|\phi_j'\|_{L^2} \\ & \ge & -\mathrm{sign}(\tilde{c}_{h_j})\Big(- \mathrm{Re}\,\langle \lambda_j(\phi_j, \psi_j), (\phi_j', \psi_j')\rangle + \mathrm{Re}\,\langle g_u(\tilde{u}_{h_j})\phi_j, \phi_j'\rangle\Big) \\ & = & -\mathrm{sign}(\tilde{c}_{h_j})\Big(- \tilde{c}_{h_j}\|(\phi_j', \psi_j')\|^2_{\mathbf{L}^2} - (1+\rho)\mathrm{Re}\,\langle \psi_j, \phi_j'\rangle - \mathrm{Re}\,\langle (\theta_j, \chi_j), (\phi_j', \psi_j')\rangle\Big) \\ & \ge & |\tilde{c}_{h_j}|\|(\phi_j', \psi_j')\|^2_{\mathbf{L}^2} - (1+\rho)\|\psi_j\|_{L^2}\|\phi_j'\|_{L^2} - \|(\theta_j, \chi_j)\|_{\mathbf{L}^2}\|(\phi_j', \psi_j')\|_{\mathbf{L}^2} \\ & \ge & c_*\|(\phi_j', \psi_j')\|^2_{\mathbf{L}^2} - (1+\rho)\|\psi_j\|_{L^2}\|(\phi_j', \psi_j')\|_{\mathbf{L}^2} - \|(\theta_j, \chi_j)\|_{\mathbf{L}^2}\|(\phi_j', \psi_j')\|_{\mathbf{L}^2}. \end{array} \tag{2.3.52}$$

This is the same equation that we derived for $\overline{\Lambda}^+$. Hence, we again obtain

$$B\|(\phi_j, \psi_j)\|^2_{\mathbf{L}^2} \quad \ge \quad c_*^2\|(\phi_j', \psi_j')\|^2_{\mathbf{L}^2} - 4\|(\theta_j, \chi_j)\|^2_{\mathbf{L}^2}, \tag{2.3.53}$$

where

$$B \quad = \quad 4\Big(\max\{g_*^2, (1+\rho)^2\} + \lambda^2_{\max}\Big). \tag{2.3.54}$$

■

The next step is to show that the L^2-mass of ϕ_j can be concentrated in a compact interval. We heavily exploit the bistable structure of the nonlinearity g to accomplish this. Moreover, we are aided by the fact that the off-diagonal elements are constant, which allows us to keep the cross-terms under control. In fact, one might be tempted to think that it is sufficient to note that the eigenvalues of the matrix $\begin{pmatrix} -g_u(0) & 1 \\ -\rho & \gamma\rho \end{pmatrix}$ all have positive real part, as then one would be able to find a basis in which this matrix is positive definite. However, passing over to another basis destroys the structure of the diffusion terms and, therefore, does not give any insight.

Lemma 2.3.10. *Assume that (hFam), (HP1), (HS) and (Hα1) are satisfied. Consider the setting of Proposition 2.3.4 and Lemma 2.3.7 or Lemma 2.3.8. There exist positive constants a and m, depending only on our choice of $(\tilde{u}_h, \tilde{w}_h)$, such that we have the following inequality for all j*

$$\begin{aligned} \tfrac{1}{\rho}(a + g_*) \int\limits_{|x| \le m} |\phi_j(x)|^2 dx \quad \ge \quad & \Big(\tfrac{1}{2}\min\{a, \tfrac{1}{2}\rho\gamma\} + \lambda_{\min}\Big) \|(\phi_j, \psi_j)\|^2_{\boldsymbol{L}^2} \\ & - \tfrac{1}{2\min\{a, \frac{1}{2}\rho\gamma\}} \|(\theta_j, \chi_j)\|^2_{\boldsymbol{L}^2} - \beta \|(\theta_j, \chi_j)\|^2_{\boldsymbol{L}^2}. \end{aligned} \tag{2.3.55}$$

Here we write $\lambda_{\min} = 0$ in the setting of Lemma 2.3.7 or $\lambda_{\min} = \min\{\mathrm{Re}\,\lambda : \lambda \in M\}$ in the setting of Lemma 2.3.8, together with

$$\beta \quad = \quad \tfrac{1-\rho}{\rho} \tfrac{1}{4(\frac{\rho}{1-\rho}\frac{1}{2}\gamma\rho + \gamma\rho + \lambda_{\min})}. \tag{2.3.56}$$

Proof. Again we first look at the sequence for $\overline{\Lambda}^+$. We know that $\tilde{u}_h - \overline{u}_0 \to 0$ in H^1 as $h \downarrow 0$. Hence, it follows that $\tilde{u}_h - \overline{u}_0 \to 0$ in L^∞ and, therefore, also $g_u(\tilde{u}_h) - g_u(\overline{u}_0) \to 0$ in L^∞ as $h \downarrow 0$. By the bistable nature of our nonlinearity g, we can choose m to be a positive constant such that for all $h \in [0, \overline{h}]$ (by making $\overline{h}$ smaller if necessary)

$$\min_{|x| \ge m}[-g_u(\tilde{u}_h(x))] \ge a := \frac{1}{2} r_0 > 0. \tag{2.3.57}$$

Here r_0 is the constant appearing in the choice of our function g in (HS). Then we obtain, using $\mathrm{Re}\,\langle \phi_j', \phi_j \rangle = \mathrm{Re}\,\langle \psi_j', \psi_j \rangle = 0$ and $\mathrm{Re}\,\langle -\Delta_{h_j} \phi_j, \phi_j \rangle \ge 0$, which we know from Lemma 2.3.6, that

$$\begin{array}{rcl} \mathrm{Re}\,\langle(\theta_j,\chi_j),(\phi_j,\psi_j)\rangle & = & \mathrm{Re}\,\langle\overline{\mathcal{L}}^{+}_{h_j,\lambda_j}(\phi_j,\psi_j),(\phi_j,\psi_j)\rangle \\ & \geq & \mathrm{Re}\,\langle -g_u(\tilde{u}_{h_j})\phi_j,\phi_j\rangle + \mathrm{Re}\,\langle\psi_j,\phi_j\rangle \\ & & \quad -\rho\mathrm{Re}\,\langle\psi_j,\phi_j\rangle + \gamma\rho\|\psi_j\|^2_{L^2} + \lambda_{\min}\|(\phi_j,\psi_j)\|^2 \\ & \geq & \min_{|x|\geq m}\{-g_u(\tilde{u}_{h_j}(x))\}\int_{|x|\geq m}|\phi_j(x)|^2dx \\ & & \quad -\|g_u(\tilde{u}_{h_j})\|_{L^\infty}\int\limits_{|x|\leq m}|\phi_j(x)|^2dx + (1-\rho)\mathrm{Re}\,\langle\psi_j,\phi_j\rangle \\ & & \quad +\gamma\rho\|\psi_j\|^2_{L^2} + \lambda_{\min}\|(\phi_j,\psi_j)\|^2 \\ & \geq & a\|\phi_j\|^2_{L^2} - (a+g_*)\int\limits_{|x|\leq m}|\phi_j(x)|^2dx + (1-\rho)\mathrm{Re}\,\langle\psi_j,\phi_j\rangle \\ & & \quad +\gamma\rho\|\psi_j\|^2_{L^2} + \lambda_{\min}\|(\phi_j,\psi_j)\|^2. \end{array} \tag{2.3.58}$$

We assumed that $0<\rho<1$ so we see that $\frac{1-\rho}{-\rho}<0$. We set

$$\beta_j^{+} = \frac{1}{4(\frac{\rho}{1-\rho}\frac{1}{2}\gamma\rho+\gamma\rho+\mathrm{Re}\,\lambda_j)}. \tag{2.3.59}$$

Now we obtain

$$\begin{array}{rcl} \mathrm{Re}\,\langle\chi_j,\psi_j\rangle & \leq & \|\chi_j\|_{L^2}\|\psi_j\|_{L^2} \\ & = & \frac{1}{\sqrt{2(\frac{\rho}{1-\rho}\frac{1}{2}\gamma\rho+\gamma\rho+\mathrm{Re}\,\lambda_j)}}\|\chi_j\|_{L^2}\sqrt{2(\frac{\rho}{1-\rho}\frac{1}{2}\gamma\rho+\gamma\rho+\mathrm{Re}\,\lambda_j)}\|\psi_j\|_{L^2} \\ & \leq & \frac{1}{4(\frac{\rho}{1-\rho}\frac{1}{2}\gamma\rho+\gamma\rho+\mathrm{Re}\,\lambda_j)}\|\chi_j\|^2_{L^2} + (\frac{\rho}{1-\rho}\frac{1}{2}\gamma\rho+\gamma\rho+\mathrm{Re}\,\lambda_j)\|\psi_j\|^2_{L^2} \\ & = & \beta_j^{+}\|\chi_j\|^2_{L^2} + (\frac{\rho}{1-\rho}\frac{1}{2}\gamma\rho+\gamma\rho+\mathrm{Re}\,\lambda_j)\|\psi_j\|^2_{L^2}. \end{array} \tag{2.3.60}$$

Note that the denominator $4(\frac{\rho}{1-\rho}\frac{1}{2}\gamma\rho+\gamma\rho+\mathrm{Re}\,\lambda_j)$ is never zero since we can assume that λ_* is small enough to have $\mathrm{Re}\,\lambda_j\geq-\lambda_*>-\gamma\rho$. Using the identity

$$\chi_j = -\rho\phi_j + \tilde{c}_{h_j}\psi_j' + \gamma\rho\psi_j + \lambda_j\psi_j \tag{2.3.61}$$

and the fact that $\mathrm{Re}\,\langle\psi_j',\psi_j\rangle=0$, we also have

$$\mathrm{Re}\,\langle\chi_j,\psi_j\rangle = -\rho\mathrm{Re}\,\langle\phi_j,\psi_j\rangle + (\gamma\rho+\mathrm{Re}\,\lambda_j)\|\psi_j\|^2_{L^2}. \tag{2.3.62}$$

Hence, we must have that

$$\begin{array}{rcl} (1-\rho)\mathrm{Re}\,\langle\phi_j,\psi_j\rangle & = & \frac{1-\rho}{\rho}\Big(-\mathrm{Re}\,\langle\chi_j,\psi_j\rangle + (\gamma\rho+\mathrm{Re}\,\lambda_j)\|\psi_j\|^2_{L^2}\Big) \\ & \geq & \frac{1-\rho}{\rho}\Big(-\beta_j^{+}\|\chi_j\|^2_{L^2} - (\frac{\rho}{1-\rho}\frac{1}{2}\gamma\rho+\gamma\rho+\mathrm{Re}\,\lambda_j)\|\psi_j\|^2_{L^2} \\ & & \quad +(\gamma\rho+\mathrm{Re}\,\lambda_j)\|\psi_j\|^2_{L^2}\Big) \\ & = & -\frac{1-\rho}{\rho}\beta_j^{+}\|\chi_j\|^2_{L^2} - \frac{1}{2}\gamma\rho\|\psi_j\|^2_{L^2}. \end{array} \tag{2.3.63}$$

Combining this bound with (2.3.58) yields the estimate

$$\begin{aligned}\mathrm{Re}\,\langle(\theta_j,\chi_j),(\phi_j,\psi_j)\rangle &\geq a\|\phi_j\|_{L^2}^2-(a+g_*)\int_{|x|\leq m}|\phi_j(x)|^2dx+(1-\rho)\mathrm{Re}\,\langle\psi_j,\phi_j\rangle\\ &\qquad+\gamma\rho\|\psi_j\|_{L^2}^2+\lambda_{\min}\|(\phi_j,\psi_j)\|^2\\ &\geq a\|\phi_j\|_{L^2}^2-(a+g_*)\int_{|x|\leq m}|\phi_j(x)|^2dx\\ &\qquad+\tfrac{1}{2}\gamma\rho\|\psi_j\|_{L^2}^2+\lambda_{\min}\|(\phi_j,\psi_j)\|^2-\tfrac{1-\rho}{\rho}\beta_j^+\|\chi_j\|_{L^2}^2.\end{aligned} \tag{2.3.64}$$

We now look at the sequence for $\overline{\Lambda}^-$. Let m and a be as before. Then we obtain, using $\overline{\mathcal{L}}^-_{h_j,\lambda_j}(\phi_j,\psi_j)=(\theta_j,\chi_j)$, $\mathrm{Re}\,\langle\phi_j',\phi_j\rangle=\mathrm{Re}\,\langle\psi_j',\psi_j\rangle=0$ and $\mathrm{Re}\,\langle-\Delta_{h_j}\phi_j,\phi_j\rangle\geq 0$ that

$$\begin{aligned}\mathrm{Re}\,\langle(\theta_j,\chi_j),(\phi_j,\psi_j)\rangle &= \mathrm{Re}\,\langle\overline{\mathcal{L}}^-_{h_j,\delta}(\phi_j,\psi_j),(\phi_j,\psi_j)\rangle\\ &\geq \mathrm{Re}\,\langle-g_u(\tilde{u}_h)\phi_j,\phi_j\rangle+(1-\rho)\mathrm{Re}\,\langle\psi_j,\phi_j\rangle\\ &\qquad+\gamma\rho\|\psi_j\|_{L^2}^2+\lambda_{\min}\|(\phi_j,\psi_j)\|_{\mathbf{L}^2}^2.\end{aligned} \tag{2.3.65}$$

We set

$$\beta_j^- = \frac{1}{4(\frac{1}{1-\rho}\frac{1}{2}\gamma\rho+\gamma\rho+\mathrm{Re}\,\lambda_j)}. \tag{2.3.66}$$

Arguing as in (2.3.60) with different constants, we obtain

$$\begin{aligned}\mathrm{Re}\,\langle\theta_j,\phi_j\rangle &\geq -\|\theta_j\|_{L^2}\|\phi_j\|_{L^2}\\ &\geq -\tfrac{1}{4(a+\mathrm{Re}\,\lambda_j)}\|\theta_j\|_{L^2}^2-(a+\mathrm{Re}\lambda_j)\|\phi_j\|_{L^2}^2\\ &= -\beta_j^-\|\theta_j\|_{L^2}^2-(a+\mathrm{Re}\,\lambda_j)\|\phi_j\|_{L^2}^2.\end{aligned} \tag{2.3.67}$$

Note that the denominator $4(a+\mathrm{Re}\,\lambda_j)$ is never zero since we can assume that λ_* is small enough to have $\mathrm{Re}\,\lambda_j\geq-\lambda_*>-a$. Using the identity

$$\theta_j = -\tilde{c}_{h_j}\phi_j'-\Delta_h\phi_j-g_u(\tilde{u}_h)\phi_j+\lambda_j\psi_j-\rho\phi_j \tag{2.3.68}$$

and the fact that $\mathrm{Re}\,\langle\phi_j',\phi_j\rangle=0$, we also have

$$\begin{aligned}\mathrm{Re}\,\langle\theta_j,\phi_j\rangle &= \mathrm{Re}\,\langle-\Delta_h\phi_j,\phi_j\rangle+\mathrm{Re}\,\langle-g_u(\tilde{u}_h)\phi_j,\phi_j\rangle\\ &\qquad+\mathrm{Re}\,\lambda_j\|\psi_j\|_{L^2}^2-\rho\mathrm{Re}\,\langle\phi_j,\psi_j\rangle.\end{aligned} \tag{2.3.69}$$

Hence, we must have that

$$\begin{aligned}(1-\rho)\mathrm{Re}\,\langle\phi_j,\psi_j\rangle &= \tfrac{1-\rho}{\rho}\Big(-\mathrm{Re}\,\langle\theta_j,\phi_j\rangle+\mathrm{Re}\,\langle-\Delta_h\phi_j,\phi_j\rangle\\ &\qquad+\mathrm{Re}\,\langle-g_u(\tilde{u}_h)\phi_j,\phi_j\rangle+\mathrm{Re}\,\lambda_j\|\psi_j\|_{L^2}^2\Big)\\ &\geq \tfrac{1-\rho}{\rho}\Big(-\beta_j^-\|\theta_j\|_{L^2}^2-\big(a+\mathrm{Re}\lambda_j\big)\|\phi_j\|_{L^2}^2\\ &\qquad+\mathrm{Re}\,\langle-g_u(\tilde{u}_h)\phi_j,\phi_j\rangle+\mathrm{Re}\,\lambda_j\|\psi_j\|_{L^2}^2\Big)\\ &= \tfrac{1-\rho}{\rho}\Big(-\beta_j^-\|\theta_j\|_{L^2}^2-a\|\phi_j\|_{L^2}^2+\mathrm{Re}\,\langle-g_u(\tilde{u}_h)\phi_j,\phi_j\rangle\Big).\end{aligned} \tag{2.3.70}$$

Combining this with the estimate (2.3.65) and noting that $\frac{1-\rho}{\rho} + 1 = \frac{1}{\rho}$ yields

$$
\begin{array}{rcl}
\mathrm{Re}\,\langle(\theta_j,\chi_j),(\phi_j,\psi_j)\rangle & \geq & \frac{1}{\rho}\mathrm{Re}\,\langle -g_u(\tilde{u}_h)\phi_j,\phi_j\rangle + \lambda_{\min}\|(\phi_j,\psi_j)\|^2_{\mathbf{L}^2} \\
 & & \quad +\gamma\rho\|\psi_j\|^2_{L^2} - a\frac{1-\rho}{\rho}\|\phi_j\|^2_{L^2} - \frac{1-\rho}{\rho}\beta_j^-\|\theta_j\|^2_{L^2} \\
 & \geq & \frac{1}{\rho}\Big(\min_{|x|\geq m}\{-g_u(\tilde{u}_h(x))\}\int_{|x|\geq m}|\phi_j|^2dx \\
 & & \quad -\|g_u(\tilde{u}_h)\|_{L^\infty}\int\limits_{|x|\leq m}|\phi_j|^2dx\Big) + \lambda_{\min}\|(\phi_j,\psi_j)\|^2_{\mathbf{L}^2} \\
 & & \quad +\gamma\rho\|\psi_j\|^2_{L^2} - a\frac{1-\rho}{\rho}\|\phi_j\|^2_{L^2} - \frac{1-\rho}{\rho}\beta_j^-\|\theta_j\|^2_{L^2} \\
 & \geq & a\|\phi_j\|^2_{L^2} - \frac{1}{\rho}(a+g_*)\int\limits_{|x|\leq m}|\phi_j|^2dx + \gamma\rho\|\psi_j\|^2_{L^2} \\
 & & \quad +\lambda_{\min}\|(\phi_j,\psi_j)\|^2_{\mathbf{L}^2} - \frac{1-\rho}{\rho}\beta_j^-\|\theta_j\|^2_{L^2}.
\end{array}
\tag{2.3.71}
$$

Upon setting

$$
\beta \;=\; \frac{1-\rho}{\rho}\min\Big\{\frac{1}{4(\frac{\rho}{1-\rho}\frac{1}{2}\gamma\rho+\gamma\rho+\lambda_{\min})}, \frac{1}{4(a+\lambda_{\min})}\Big\}, \tag{2.3.72}
$$

we note that $\frac{1-\rho}{\rho}\beta_j^+ \leq \beta$ and $\frac{1-\rho}{\rho}\beta_j^- \leq \beta$ for all j since $\rho < 1$ and since β_j^+ and β_j^- are maximal for $\mathrm{Re}\,\lambda = \lambda_{\min}$. Therefore, in both cases, we obtain

$$
\begin{array}{rcl}
\frac{1}{\rho}(a+g_*)\int\limits_{|x|\leq m}|\phi_j(x)|^2dx & \geq & a\|\phi_j\|^2_{L^2} + \frac{1}{2}\rho\gamma\|\psi_j\|^2_{L^2} - \mathrm{Re}\,\langle(\theta_j,\chi_j),(\phi_j,\psi_j)\rangle \\
 & & \quad -\beta\|(\theta_j,\chi_j)\|^2_{\mathbf{L}^2} + \lambda_{\min}\|(\phi_j,\psi_j)\|^2_{\mathbf{L}^2} \\
 & \geq & \Big(\min\{a,\frac{1}{2}\rho\gamma\} + \lambda_{\min}\Big)\|(\phi_j,\psi_j)\|^2_{\mathbf{L}^2} \\
 & & \quad -\frac{\|(\theta_j,\chi_j)\|_{\mathbf{L}^2}}{\sqrt{\min\{a,\frac{1}{2}\rho\gamma\}}}\sqrt{\min\{a,\frac{1}{2}\rho\gamma\}}\|(\phi_j,\psi_j)\|_{\mathbf{L}^2} \\
 & & \quad -\beta\|(\theta_j,\chi_j)\|_{\mathbf{L}^2}
\end{array}
\tag{2.3.73}
$$

and thus, again using the inequality $2\mu\omega \leq \mu^2 + \omega^2$ for $\mu,\omega\in\mathbb{R}$,

$$
\begin{array}{rcl}
\frac{1}{\rho}(a+g_*)\int\limits_{|x|\leq m}|\phi_j(x)|^2dx & \geq & \Big(\min\{a,\frac{1}{2}\rho\gamma\} + \lambda_{\min}\Big)\|(\phi_j,\psi_j)\|^2_{\mathbf{L}^2} \\
 & & \quad -\frac{1}{2}\frac{1}{\min\{a,\frac{1}{2}\rho\gamma\}}\|(\theta_j,\chi_j)\|^2_{\mathbf{L}^2} \\
 & & \quad -\frac{1}{2}\min\{a,\frac{1}{2}\rho\gamma\}\|(\phi_j,\psi_j)\|^2_{\mathbf{L}^2} - \beta\|(\theta_j,\chi_j)\|^2_{\mathbf{L}^2} \\
 & = & \Big(\frac{1}{2}\min\{a,\frac{1}{2}\rho\gamma\} + \lambda_{\min}\Big)\|(\phi_j,\psi_j)\|^2_{\mathbf{L}^2} \\
 & & \quad -\frac{1}{2\min\{a,\frac{1}{2}\rho\gamma\}}\|(\theta_j,\chi_j)\|^2_{\mathbf{L}^2} - \beta\|(\theta_j,\chi_j)\|^2_{\mathbf{L}^2},
\end{array}
\tag{2.3.74}
$$

as desired. ∎

Lemma 2.3.11. *Assume that (hFam), (HP1), (HS) and (Hα1) are satisfied. Consider the setting of Proposition 2.3.4 and Lemma 2.3.7 or Lemma 2.3.8. There exist positive constants C_4 and C_5, depending only on M and our choice of $(\tilde{u}_h,\tilde{w}_h)$ and $\tilde{c}_h$, such* *that for all j we have*

$$\frac{1}{\rho}(a+g_*)\int_{|x|\le m}|\phi_j^2(x)|dx \;\;\ge\;\; C_4 - C_5\|(\theta_j,\chi_j)\|^2_{\mathbf{L}^2}. \tag{2.3.75}$$

Proof. Without loss of generality we assume that $\frac{1}{2}\min\{a,\frac{1}{2}\rho\gamma\}+\lambda_{\min}>0$. Write

$$\mu \;=\; \frac{\frac{1}{2}\min\{a,\frac{1}{2}\rho\gamma\}+\lambda_{\min}}{c_*^2+B}. \tag{2.3.76}$$

Adding μ times equation (2.3.43) to equation (2.3.55) gives

$$\begin{aligned}\frac{1}{\rho}(a+g_*)\int_{|x|\le m}|\phi_j(x)|^2dx \;\;\ge\;\; & \mu c_*^2\|(\phi_j',\psi_j')\|^2_{\mathbf{L}^2} - 4\mu\|(\theta_j,\chi_j)\|^2_{\mathbf{L}^2}\\ &+\tfrac{1}{2}(\min\{a,\tfrac{1}{2}\rho\gamma\}+\lambda_{\min})\|(\phi_j,\psi_j)\|^2_{\mathbf{L}^2}\\ &-\frac{1}{2(\min\{a,\frac{1}{2}\rho\gamma\}+\lambda_{\min})}\|(\theta_j,\chi_j)\|^2_{\mathbf{L}^2}\\ &-\beta\|(\theta_j,\chi_j)\|^2_{\mathbf{L}^2} - B\mu\|(\phi_j,\psi_j)\|^2_{\mathbf{L}^2}.\end{aligned} \tag{2.3.77}$$

We hence obtain

$$\begin{aligned}\frac{1}{\rho}(a+g_*)\int_{|x|\le m}|\phi_j(x)|^2dx \;\;\ge\;\; & -C_5\|(\theta_j,\chi_j)\|^2_{\mathbf{L}^2} + \mu c_*^2\|(\phi_j',\psi_j')\|^2_{\mathbf{L}^2}\\ &+\tfrac{1}{2}(\min\{a,\tfrac{1}{2}\rho\gamma\}+\lambda_{\min})\|(\phi_j,\psi_j)\|^2_{\mathbf{L}^2}\\ &-B\mu\|(\phi_j,\psi_j)\|^2_{\mathbf{L}^2},\end{aligned} \tag{2.3.78}$$

where

$$\begin{aligned}C_5 \;&=\; 4\mu + \frac{1}{2(\min\{a,\frac{1}{2}\rho\gamma\}+\lambda_{\min})} + \beta\\ &>\; 0.\end{aligned} \tag{2.3.79}$$

This allows us to compute

$$\begin{aligned}\frac{1}{\rho}(a+g_*)\int_{|x|\le m}|\phi_j(x)|^2dx \;\;\ge\;\; & -C_5\|(\theta_j,\chi_j)\|^2_{\mathbf{L}^2} + \mu c_*^2\|(\phi_j',\psi_j')\|^2_{\mathbf{L}^2}\\ &+\tfrac{1}{2}(\min\{a,\tfrac{1}{2}\rho\gamma\}+\lambda_{\min})\|(\phi_j,\psi_j)\|^2_{\mathbf{L}^2}\\ &-B\mu\|(\phi_j,\psi_j)\|^2_{\mathbf{L}^2}\\ =\;\; & -C_5\|(\theta_j,\chi_j)\|^2_{\mathbf{L}^2} + \mu c_*^2\|(\phi_j',\psi_j')\|_{\mathbf{L}^2}\\ &+(\mu(c_*^2+B)-B\mu)\|(\phi_j,\psi_j)\|^2_{\mathbf{L}^2}\\ =\;\; & \mu c_*^2\|(\phi_j,\psi_j)\|^2_{\mathbf{H}^1} - C_5\|(\theta_j,\chi_j)\|^2_{\mathbf{L}^2}\\ =\;\; & C_4 - C_5\|(\theta_j,\chi_j)\|^2_{\mathbf{L}^2},\end{aligned} \tag{2.3.80}$$

where $C_4 = \mu c_*^2 > 0$. ■

Proof of Proposition 2.3.4. We first choose $0<\delta<\delta_0$ and consider the setting of Lemma 2.3.7. Sending $j\to\infty$ in (2.3.75), Lemma 2.3.7 implies

$$\begin{aligned}C_4 - C_5\overline{\Lambda}^{\pm}(\delta)^2 \;\;\le\;\; & C_4 - C_5\lim_{j\to\infty}\|(\theta_j,\chi_j)\|^2_{\mathbf{L}^2}\\ \le\;\; & \frac{1}{\rho}(a+g_*)\int_{|x|\le m}|\phi|^2dx\\ \le\;\; & \frac{1}{\rho}(a+g_*)\|(\phi,\psi)\|^2_{H^2(\mathbb{R})\times H^1(\mathbb{R})}\\ \le\;\; & \frac{1}{\rho}(a+g_*)C_2^2\overline{\Lambda}^{+}(\delta)^2.\end{aligned} \tag{2.3.81}$$

Solving this quadratic inequality, we obtain

$$\begin{array}{rcl} \overline{\Lambda}^{\pm}(\delta) & \geq & \sqrt{\frac{C_4}{\frac{1}{\rho}(a+g_*)C_2^2+C_5}} \\ & := & \mathbb{C}_0. \end{array} \tag{2.3.82}$$

The analogous computation in the setting of Lemma 2.3.8 yields

$$\begin{array}{rcl} \overline{\Lambda}^{+}(M) & \geq & \sqrt{\frac{C_4}{\frac{1}{\rho}(a+g_*)C_3^2+C_5}} \\ & := & C_M. \end{array} \tag{2.3.83}$$

■

2.4 Existence of pulse solutions

In this section, we prove our first main result, Theorem 2.2.1. In particular, we construct solutions to (2.2.12) by writing

$$(\overline{u}_h, \overline{w}_h) \quad = \quad (\overline{u}_0, \overline{w}_0) + (\phi_h, \psi_h) \tag{2.4.1}$$

and exploiting the linear results of §2.3. Here $(\overline{u}_0, \overline{w}_0)$ is the pulse solution of the PDE (2.1.1).

The arguments presented in this section are strongly reminiscent of a standard proof of the implicit function theorem. However, the singular nature of the $h \downarrow 0$ limit requires some minor adjustments pertaining to the linearisation that is used. In particular, we fix a small $\delta > 0$ that will be determined later and consider the linear operator

$$\mathcal{L}^{+}_{h,\delta} : \mathbf{H}^1 \quad \to \quad \mathbf{L}^2, \tag{2.4.2}$$

defined by

$$\mathcal{L}^{+}_{h,\delta} \quad = \quad \begin{pmatrix} c_0 \frac{d}{d\xi} - \Delta_h - g_u(\overline{u}_0) + \delta & 1 \\ -\rho & c_0 \frac{d}{d\xi} + \gamma\rho + \delta \end{pmatrix}. \tag{2.4.3}$$

This operator arises as the linearisation of (2.2.1) around the pulse solution $(\overline{u}_0, \overline{w}_0)$ of (2.1.1). A short computation shows that our travelling wave triplet $(c_h, \phi_h, \psi_h) \in \mathbb{R} \times \mathbf{H}^1$ must satisfy

$$\mathcal{L}^{+}_{h,\delta}(\phi_h, \psi) \quad = \quad \mathcal{R}(c_h, \phi_h, \psi_h), \tag{2.4.4}$$

where

$$\mathcal{R}(c, \phi, \psi) \quad = \quad \Big((c_0 - c)(\overline{u}_0' + \phi') + (\Delta_h - \tfrac{d^2}{d\xi^2})\overline{u}_0 + \delta\phi + \mathcal{N}(\overline{u}_0, \phi), (c_0 - c)(\overline{w}_0' + \psi')\Big). \tag{2.4.5}$$

Here we have introduced the nonlinearity

$$\mathcal{N}(\overline{u}_0, \phi) \quad = \quad g(\overline{u}_0 + \phi) - g(\overline{u}_0) - g_u(\overline{u}_0)\phi. \tag{2.4.6}$$

Corollary 2.4.1. *Assume that (HP1), (HS) and (Hα1) are satisfied. There exists a positive constant C_0 and a positive function $h_0(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $\delta > 0$ and all $h \in (0, h_0(\delta))$, the operator $\mathcal{L}^+_{h,\delta}$ is a homeomorphism for which we have the bound*

$$\|(\mathcal{L}^+_{h,\delta})^{-1}(\theta,\chi)\|_{\boldsymbol{H}^1} \quad \leq \quad C_0\|(\theta,\chi)\|_{\boldsymbol{L}^2} \tag{2.4.7}$$

for all $(\theta,\chi) \in \boldsymbol{L}^2$ that satisfy $\langle(\theta,\chi),(\phi_0^-,\psi_0^-)\rangle = 0$.

Proof. This is immediate by choosing $(\tilde{u}_h, \tilde{w}_h) = (\overline{u}_0, \overline{w}_0)$ and $\tilde{c}_h = c_0$ for all h in (hFam) and applying Proposition 2.3.2. ∎

Let η be a small positive constant to be determined later. We define

$$X_\eta \quad = \quad \{(\phi,\psi) \in \mathbf{H}^1 : \|(\phi,\psi)\|_{\mathbf{H}^1} \leq \eta\}. \tag{2.4.8}$$

For every $(\phi,\psi) \in X_\eta$, we define $c_h = c_h(\phi,\psi)$ to be the constant

$$c_h(\phi,\psi) \quad = \quad c_0 + \frac{\langle \Delta_h \overline{u}_0 - \overline{u}_0'', \phi_0^-\rangle + \delta\langle\phi,\phi_0^-\rangle + \langle\mathcal{N}(\overline{u}_0,\phi),\phi_0^-\rangle}{\langle\overline{u}_0',\phi_0^-\rangle + \langle\phi',\phi_0^-\rangle + \langle\overline{w}_0',\psi_0^-\rangle + \langle\psi',\psi_0^-\rangle}. \tag{2.4.9}$$

When this expression is well-defined, this choice ensures that

$$\langle\mathcal{R}\big(c_h(\phi,\psi),\phi,\psi\big),(\phi_0^-,\psi_0^-)\rangle \quad = \quad 0. \tag{2.4.10}$$

We define $T : X_\eta \subset \mathbf{H}^1 \to \mathbf{H}^1$ by

$$T(\phi,\psi) \quad = \quad (\mathcal{L}^+_{h,\delta})^{-1}\mathcal{R}(c_h(\phi,\psi),\phi,\psi). \tag{2.4.11}$$

Our goal is to show T maps X_η into itself and is a contraction, since then the fixed point (ϕ_h,ψ_h) leads to a travelling pulse solution of (2.2.12) via (2.4.1) and (2.4.9).

Exploiting (2.4.10), Corollary 2.4.1 implies that there exists a constant $C_0 > 0$ such that for all $\Psi = (\phi,\psi) \in X_\eta$ we have the bound

$$\|T(\Psi)\|_{\mathbf{H}^1} \quad \leq \quad C_0\|\mathcal{R}(c_h(\Psi),\Psi)\|_{\mathbf{L}^2}, \tag{2.4.12}$$

while for all $\Psi_1 = (\phi_1,\psi_1), \Psi_2 = (\phi_2,\psi_2) \in X_\eta$ we have the bound

$$\|T(\Psi_1) - T(\Psi_2)\|_{\mathbf{H}^1} \quad \leq \quad C_0\|\mathcal{R}(c_h(\Psi_1),\Psi_1) - \mathcal{R}(c_h(\Psi_2),\Psi_2)\|_{\mathbf{L}^2}. \tag{2.4.13}$$

In the remainder of this section we, therefore, set out to estimate the right-hand sides of (2.4.12) and (2.4.13). We start by estimating the nonlinear term $\mathcal{N}(\overline{u}_0,\cdot)$.

Lemma 2.4.2. *Assume that (HP1), (HS) and (Hα1) are satisfied. Then there exists a constant $K > 0$ such that for all $0 < \eta \leq 1$, $(\phi,\psi) \in X_\eta, (\phi_1,\psi_1) \in X_\eta$ and $(\phi_2,\psi_2) \in X_\eta$ we have the pointwise inequalities*

$$\begin{aligned} |\mathcal{N}(\overline{u}_0,\phi)| &\leq K\eta|\phi|, \\ |\mathcal{N}(\overline{u}_0,\phi_1) - \mathcal{N}(\overline{u}_0,\phi_2)| &\leq \eta K|\phi_1 - \phi_2|. \end{aligned} \tag{2.4.14}$$

Proof. To estimate the nonlinear term $\mathcal{N}(\overline{u}_0, \phi)$, we first recall the embedding $\|\phi\|_{L^\infty} \leq \|\phi\|_{H^1} \leq \eta \leq 1$ for every $(\phi, \psi) \in X_\eta$. Setting $K = \max\{6, \sup_{|s| \leq \|\overline{u}_0\|_\infty} |g_{uu}(s)|\}$, a Taylor expansion around $\overline{u}_0$ allows us to obtain the pointwise inequalities

$$\begin{aligned}
|\mathcal{N}(\overline{u}_0, \phi)| &= |-g(\overline{u}_0 + \phi) + g(\overline{u}_0) + g_u(\overline{u}_0)| \\
&= |-g(\overline{u}_0) - \phi g_u(\overline{u}_0) - \tfrac{1}{2}\phi^2 g_{uu}(\xi) + g(\overline{u}_0) + g_u(\overline{u}_0)| \\
&= |-\tfrac{1}{2}\phi^2 g_{uu}(\xi)| \\
&\leq \tfrac{1}{2} K\eta|\phi| \\
&\leq K\eta|\phi|,
\end{aligned} \tag{2.4.15}$$

where ξ is between $\overline{u}_0$ and $\overline{u}_0 + \phi$. Note that $g_{uuu} = 6$ is constant. Furthermore, for $(\phi_1, \psi_1) \in X_\eta$ and $(\phi_2, \psi_2) \in X_\eta$, a Taylor expansion around $\overline{u}_0$ yields the pointwise inequalities

$$\begin{aligned}
|\mathcal{N}(\overline{u}_0, \phi_1) - \mathcal{N}(\overline{u}_0, \phi_2)| &= \Big| - g(\overline{u}_0 + \phi_1) + g(\overline{u}_0) + g_u(\overline{u}_0)\phi_1 \\
&\qquad + g(\overline{u}_0 + \phi_2) - g(\overline{u}_0) - g_u(\overline{u}_0)\phi_2 \Big| \\
&= \Big| - \tfrac{1}{2} g_{uu}(\overline{u}_0)\phi_1^2 + \tfrac{1}{2} g_{uu}(\overline{u}_0)\phi_2^2 - \tfrac{1}{6} g_{uuu}(\xi_1)\phi_1^3 + \tfrac{1}{6} g_{uuu}(\xi_2)\phi_2^3 \Big| \\
&\leq \tfrac{1}{2}|g_{uu}(\overline{u_0})||\phi_1^2 - \phi_2^2| + \tfrac{1}{6} 6|\phi_1^3 - \phi_2^3| \\
&\leq \tfrac{1}{2}|g_{uu}(\overline{u}_0)|\Big[|\phi_1||\phi_1 - \phi_2| + |\phi_2||\phi_1 - \phi_2|\Big] \\
&\qquad + |\phi_1||\phi_1^2 - \phi_2^2| + |\phi_1 - \phi_2||\phi_2^2| \\
&\leq \tfrac{1}{2}\tfrac{1}{2} K\Big[2\eta|\phi_1 - \phi_2|\Big] + \eta\Big[2\eta|\phi_1 - \phi_2|\Big] + \eta^2|\phi_1 - \phi_2| \\
&\leq \eta K|\phi_1 - \phi_2|,
\end{aligned} \tag{2.4.16}$$

where ξ_1 is between $\overline{u}_0$ and $\overline{u}_0 + \phi_1$ and ξ_2 is between $\overline{u}_0$ and $\overline{u}_0 + \phi_2$. ∎

Pick $(\phi, \psi) \in X_\eta$. Recall that we chose (ϕ_0^-, ψ_0^-) so that $\langle(\phi_0^-, \psi_0^-), (\overline{u}_0', \overline{w}_0')\rangle > 0$. Let $s > 0$ be defined as

$$s = 2\frac{1}{\langle(\phi_0^-, \psi_0^-)\rangle, (\overline{u}_0', \overline{w}_0')\rangle}. \tag{2.4.17}$$

For notational compactness, we write

$$\sigma(\phi, \psi) = \langle \overline{u}_0', \phi_0^- \rangle + \langle \phi', \phi_0^- \rangle + \langle \overline{w}_0', \psi_0^- \rangle + \langle \psi', \psi_0^- \rangle \tag{2.4.18}$$

for $(\phi, \psi) \in X_\eta$. We also write

$$\eta_0 = \min\{1, s^{-1}\}. \tag{2.4.19}$$

Lemma 2.4.3. *Assume that (HP1), (HS) and (Hα1) are satisfied. Fix $0 < \eta \leq \eta_0$. Then for all $\Psi = (\phi, \psi) \in X_\eta, \Psi_1 = (\phi_1, \psi_1) \in X_\eta$ and $\Psi_2 = (\phi_2, \psi_2) \in X_\eta$ we have the bounds*

$$0 < [\sigma(\Psi)]^{-1} \leq s, \tag{2.4.20}$$

together with

$$|\sigma(\Psi_1) - \sigma(\Psi_2)| \quad \leq \quad \|\Psi_1' - \Psi_2'\|_{\boldsymbol{L}^2}. \tag{2.4.21}$$

Proof. Using Cauchy-Schwartz, we obtain that

$$\begin{aligned}
\sigma(\phi, \psi) &= \langle \overline{u}_0', \phi_0^- \rangle + \langle \phi', \phi_0^- \rangle + \langle \overline{w}_0', \psi_0^- \rangle + \langle \psi', \psi_0^- \rangle \\
&\geq 2s^{-1} + \langle (\phi', \psi'), (\phi_0^-, \psi_0^-) \rangle \\
&\geq 2s^{-1} - \eta \\
&\geq s^{-1},
\end{aligned} \tag{2.4.22}$$

which yields (2.4.20). In particular we see that

$$\tfrac{1}{\sigma(\Psi)} \quad \leq \quad s. \tag{2.4.23}$$

The remaining inequality (2.4.21) follows immediately from Cauchy-Schwarz. ∎

Lemma 2.4.4. *Assume that (HP1), (HS) and (Hα1) are satisfied. Recall the constant K from Lemma 2.4.2 and the constant s from (2.4.17). Then for all $0 < \eta \leq \eta_0$, $\Psi \in X_\eta, \Psi_1 \in X_\eta$ and $\Psi_2 \in X_\eta$ we have the inequality*

$$|c_h(\Psi) - c_0| \quad \leq \quad s\Big(\|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} + \delta\eta + K\eta^2 \Big), \tag{2.4.24}$$

together with

$$|c_h(\Psi_1) - c_h(\Psi_2)| \quad \leq \quad s\|\Psi_1 - \Psi_2\|_{\boldsymbol{H}^1}\Big(s\|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} + 2(\delta + K\eta) \Big). \tag{2.4.25}$$

Proof. By (2.4.20) we have that $[\sigma(\Psi)]^{-1} \leq s$ for all $\Psi \in X_\eta$. By definition of $c_h(\Psi)$ and Lemma 2.4.2, we obtain for all $\Psi = (\phi, \psi) \in X_\eta$ that

$$\begin{aligned}
|c_h(\Psi) - c_0| &= \Big| \tfrac{\langle \Delta_h \overline{u}_0 - \overline{u}_0'', \phi_0^- \rangle + \delta \langle \phi, \phi_0^- \rangle + \langle \mathcal{N}(\overline{u}_0, \phi), \phi_0^- \rangle}{\sigma(\Psi)} \Big| \\
&\leq s |\langle \Delta_h \overline{u}_0 - \overline{u}_0'', \phi_0^- \rangle + \delta \langle \phi, \phi_0^- \rangle + \langle \mathcal{N}(\overline{u}_0, \phi), \phi_0^- \rangle| \\
&\leq s \left(\|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} \|\phi_0^-\|_{L^2} + \delta \|\phi\|_{L^2} \|\phi_0^-\|_{L^2} \right) + sK\eta \||\phi|\|_{L^2} \\
&\leq s \left(\|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} \|(\phi_0^-, \psi_0^-)\|_{\mathbf{L}^2} + \delta \|\phi\|_{L^2} \|(\phi_0^-, \psi_0^-)\|_{\mathbf{L}^2} \right) + sK\eta \|\phi\|_{L^2} \\
&= s\Big(\|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} + \delta \|\phi\|_{L^2} + K\eta \|\phi\|_{L^2} \Big) \\
&\leq s\Big(\|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} + (\delta + K\eta)\eta \Big).
\end{aligned} \tag{2.4.26}$$

For notational compactness we write

$$d(\Psi) \quad = \quad \langle \Delta_h \overline{u}_0 - \overline{u}_0'', \phi_0^- \rangle + \delta \langle \phi, \phi_0^- \rangle + \langle \mathcal{N}(\overline{u}_0, \phi), \phi_0^- \rangle. \tag{2.4.27}$$

Then we obtain with (2.4.20) that for all $\Psi_1 = (\phi_1, \psi_1) \in X_\eta$ and $\Psi_2 = (\phi_2, \psi_2) \in X_\eta$

$$\begin{array}{rcl} |c_h(\Psi_1) - c_h(\Psi_2)| & = & \left| \frac{d(\Psi_1)}{\sigma(\Psi_1)} - \frac{d(\Psi_2)}{\sigma(\Psi_2)} \right| \\ & = & \left| \frac{d(\Psi_1)\sigma(\Psi_2) - d(\Psi_2)\sigma(\Psi_1)}{\sigma(\Psi_1)\sigma(\Psi_2)} \right| \\ & \leq & \frac{|d(\Psi_2)||\sigma(\Psi_2) - \sigma(\Psi_1)| + |d(\Psi_1) - d(\Psi_2)||\sigma(\Psi_2)|}{|\sigma(\Psi_1)||\sigma(\Psi_2)|} \\ & \leq & s^2|d(\Psi_2)||\sigma(\Psi_2) - \sigma(\Psi_1)| + s|d(\Psi_1) - d(\Psi_2)|. \end{array} \tag{2.4.28}$$

Observe, using Lemma 2.4.2, that

$$\begin{array}{rcl} |d(\Psi_2)| & \leq & \|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} + \delta\|\phi_2\|_{L^2} + \|\mathcal{N}(\overline{u}_0, \phi_2)\|_{L^2} \\ & \leq & \|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} + \delta\eta + K\eta\|\phi_2\|_{L^2} \\ & \leq & \|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} + \delta\eta + K\eta^2 \end{array} \tag{2.4.29}$$

and

$$\begin{array}{rcl} |d(\Psi_1) - d(\Psi_2)| & \leq & \delta\|\phi_1 - \phi_2\|_{L^2} + \|\mathcal{N}(\overline{u}_0, \phi_1) - \mathcal{N}(\overline{u}_0, \phi_2)\|_{L^2} \\ & \leq & \delta\|\phi_1 - \phi_2\|_{L^2} + \eta K\|\phi_1 - \phi_2\|_{L^2} \\ & \leq & (\delta + K\eta)\|\phi_1 - \phi_2\|_{L^2}. \end{array} \tag{2.4.30}$$

Using Lemma 2.4.3, we hence see that

$$\begin{array}{rcl} |c_h(\Psi_1) - c_h(\Psi_2)| & \leq & s^2|d(\Psi_2)||\sigma(\Psi_2) - \sigma(\Psi_1)| + s|d(\Psi_1) - d(\Psi_2)| \\ & \leq & s^2\Big(\|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} + \delta\eta + K\eta^2\Big)|\sigma(\Psi_2) - \sigma(\Psi_1)| \\ & & \quad +s(\delta + K\eta)\|\phi_1 - \phi_2\|_{L^2} \\ & \leq & s^2\Big(\|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} + \delta\eta + K\eta^2\Big)\|\Psi_1 - \Psi_2\|_{\mathbf{H}^1} \\ & & \quad +s(\delta + K\eta)\|\phi_1 - \phi_2\|_{L^2} \\ & \leq & s\|\Psi_1 - \Psi_2\|_{\mathbf{H}^1}\Big(s\|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} + (\delta + K\eta)(1 + s\eta)\Big) \\ & \leq & s\|\Psi_1 - \Psi_2\|_{\mathbf{H}^1}\Big(s\|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} + 2(\delta + K\eta)\Big). \end{array} \tag{2.4.31}$$

■

Lemma 2.4.5. *Assume that (HP1), (HS) and (Hα1) are satisfied. Recall the constant K from Lemma 2.4.2 and the constant s from (2.4.17). Then for all $0 < \eta \leq \eta_0, \Psi \in X_\eta, \Psi_1 \in X_\eta$ and $\Psi_2 \in X_\eta$ we have the inequality*

$$\|\mathcal{R}(c_h(\Psi), \Psi)\|_{\boldsymbol{L}^2} \leq \Big[1 + s\Big(\|\overline{u}_0'\|_{L^2} + \|\overline{w}_0'\|_{L^2} + \eta\Big)\Big]\Big[\|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} + \delta\eta + K\eta^2\Big], \tag{2.4.32}$$

together with

$$\begin{array}{rcl} \|\mathcal{R}(c_h(\Psi_1),\Psi_1)-\mathcal{R}(c_h(\Psi_2),\Psi_2)\|_{\boldsymbol{L}^2} & \leq & \|\Psi_1-\Psi_2\|_{\boldsymbol{H}^1}\Big(2+2s\eta+s\|\overline{u}_0'\|_{L^2}+s\|\overline{w}_0'\|_{L^2}\Big) \\ & & \times\Big(s\|\Delta_h\overline{u}_0-\overline{u}_0''\|_{L^2}+2(\delta+K\eta)\Big). \end{array} \tag{2.4.33}$$

Proof. For any $\Psi=(\phi,\psi)\in X_\eta$, Lemma 2.4.4, together with the definition of $\mathcal{R}(c_h(\Psi),\Psi)$, allows us to estimate

$$\begin{array}{rcl} \|\mathcal{R}(c_h(\Psi),\Psi)\|_{\mathbf{L}^2} & \leq & |c_0-c_h(\Psi)|\Big(\|\overline{u}_0'\|_{L^2}+\|\phi'\|_{L^2}\Big)+\|\Delta_h\overline{u}_0-\overline{u}_0''\|_{L^2}+\delta\eta+K\eta^2 \\ & & +|c_0-c_h(\Psi)|\Big(\|\overline{w}_0'\|_{L^2}+\|\psi'\|_{L^2}\Big) \\ & \leq & s\Big(\|\Delta_h\overline{u}_0-\overline{u}_0''\|_{L^2}+\delta\eta+K\eta\Big)\Big(\|\overline{u}_0'\|_{L^2}+\|\overline{w}_0'\|_{L^2}+\eta\Big) \\ & & +\|\Delta_h\overline{u}_0-\overline{u}_0''\|_{L^2}+\delta\eta+K\eta^2 \\ & = & \Big[1+s\Big(\|\overline{u}_0'\|_{L^2}+\|\overline{w}_0'\|_{L^2}+\eta\Big)\Big]\Big[\|\Delta_h\overline{u}_0-\overline{u}_0''\|_{L^2}+\delta\eta+K\eta^2\Big]. \end{array} \tag{2.4.34}$$

For $\Psi_1=(\phi_1,\psi_1)\in X_\eta$ and $\Psi_2=(\phi_2,\psi_2)\in X_\eta$ we write

$$d(\Psi_1,\Psi_2) \quad := \quad \|\mathcal{R}(c_h(\Psi_1),\Psi_1)-\mathcal{R}(c_h(\Psi_2),\Psi_2)\|_{\mathbf{L}^2}. \tag{2.4.35}$$

Substituting (2.4.5), we compute

$$\begin{array}{rcl} d(\Psi_1,\Psi_2) & \leq & \Big\|\Big((c_0-c_h(\Psi_1))(\phi_1'-\phi_2')+(c_h(\Psi_1)-c_h(\Psi_2))(\overline{u}_0'-\phi_2') \\ & & +\delta(\phi_1-\phi_2)+(\mathcal{N}(\overline{u}_0,\phi_2)-\mathcal{N}(\overline{u}_0,\phi_1))\Big\|_{L^2} \\ & & +\Big\|(c_0-c_h(\Psi_1))(\psi_1'-\psi_2')+(c_h(\Psi_1)-c_h(\Psi_2))(\overline{w}_0'-\psi_2')\Big)\Big\|_{L^2} \\ & \leq & \Big(|c_h(\Psi_1)-c_0|+\delta+K\eta\Big)\|\phi_1-\phi_2\|_{H^1} \\ & & +\Big(\|\overline{u}_0'\|_{L^2}+\eta\Big)\Big|c_h(\Psi_1)-c_h(\Psi_2)\Big|+\Big|c_h(\Psi_1)-c_0\Big|\|\psi_1-\psi_2\|_{H^1} \\ & & +\Big(\|\overline{w}_0'\|_{L^2}+\eta\Big)\Big|c_h(\Psi_1)-c_h(\Psi_2)\Big|. \end{array} \tag{2.4.36}$$

Another application of Lemma 2.4.4 yields the desired bound

$$\begin{array}{rcl} d(\Psi_1,\Psi_2) & \leq & \Big(s\Big(\|\Delta_h\overline{u}_0-\overline{u}_0''\|_{L^2}+(\delta+K)\eta\Big)+\delta+K\eta\Big)\|\phi_1-\phi_2\|_{H^1} \\ & & +(\|\overline{u}_0'\|_{L^2}+\eta)s||\Psi_1-\Psi_2||_{\mathbf{H}^1}\Big(s\|\Delta_h\overline{u}_0-\overline{u}_0''\|_{L^2}+2(\delta+K\eta)\Big) \\ & & +s\Big(\|\Delta_h\overline{u}_0-\overline{u}_0''\|_{L^2}+(\delta+K\eta)\eta\Big)\|\psi_1-\psi_2\|_{H^1} \\ & & +(\|\overline{w}_0'\|_{L^2}+\eta)s^2\|\Psi_1-\Psi_2\|_{\mathbf{H}^1}\Big(\|\Delta_h\overline{u}_0-\overline{u}_0''\|_{L^2}+2\hat{\sigma}(\delta+K\eta)\Big) \\ & \leq & \|\Psi_1-\Psi_2\|_{\mathbf{H}^1}\Big(2+2s\eta+s\|\overline{u}_0'\|_{L^2}+s\|\overline{w}_0'\|_{L^2}\Big) \\ & & \times\Big(s\|\Delta_h\overline{u}_0-\overline{u}_0''\|_{L^2}+2(\delta+K\eta)\Big). \end{array} \tag{2.4.37}$$

■

With these estimates in hand, we can choose our parameters δ and η to ensure that the map T maps X_η into itself and is a contraction. This allows us to prove our first main theorem.

Proof of Theorem 2.2.1. We let

$$C_6 = \max\Big\{C_0\Big(1 + s\|\overline{u}_0'\|_{L^2} + s\|\overline{w}_0'\|_{L^2} + s\Big), C_0\Big(4 + s\|\overline{u}_0'\|_{L^2} + s\|\overline{w}_0'\|_{L^2}\Big)\Big\}, \tag{2.4.38}$$

which is independent of δ, h and $\eta \in (0, s^{-1}]$. Using Lemma 2.4.5 together with (2.4.12) and (2.4.13), we see that for all $0 < \eta \le \eta_0$, $\Psi = (\phi, \psi) \in X_\eta$, $\Psi_1 = (\phi_1, \psi_1) \in X_\eta$ and $\Psi_2 = (\phi_2, \psi_2) \in X_\eta$ we have

$$\|T(\Psi)\|_{H^1} \le C_6\Big(\|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} + \delta\eta + K\eta^2\Big) \tag{2.4.39}$$

and

$$\|T(\Psi_1) - T(\Psi_2)\|_{\mathbf{H}^1} \le C_6\Big(s\|\Delta_h \overline{u}_0 - \overline{u}_0''\| + 2(\delta + K\eta)\Big)\|\Psi_1 - \Psi_2\|_{\mathbf{H}^1}. \tag{2.4.40}$$

We fix

$$\begin{aligned} \delta &= \tfrac{1}{8C_6} \\ \eta &= \min\{\eta_0, \tfrac{1}{8MC_6}\}, \end{aligned} \tag{2.4.41}$$

so that indeed $\eta \le \eta_0$. Using the notation from Corollary 2.4.1, we pick $0 < h_* \le h_0(\delta)$ in such a way that

$$\sup_{h \in (0, h_*)} \|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} \le \tfrac{\eta}{8C_6}. \tag{2.4.42}$$

Then we see for $h \in (0, h_*)$ that

$$\begin{aligned} \|T(\Psi)\|_{\mathbf{H}^1} &\le C_6\Big(\|\Delta_h \overline{u}_0 - \overline{u}_0''\|_{L^2} + \delta\eta + K\eta^2\Big) \\ &\le C_6\Big(\tfrac{\eta}{8C_6} + \tfrac{1}{8C_6}\eta + K\tfrac{1}{8MC_6}\eta\Big) \\ &\le \eta \end{aligned} \tag{2.4.43}$$

and

$$\begin{aligned} \|T(\Psi_1) - T(\Psi_2)\|_{\mathbf{H}^1} &\le C_6\Big(s\|\Delta_h \overline{u}_0 - \overline{u}_0''\| + 2(\delta + K\eta)\Big)\|\Psi_1 - \Psi_2\|_{\mathbf{H}^1} \\ &\le C_6\Big(s\tfrac{\eta}{8C_6} + 2(\tfrac{1}{8C_6} + K\tfrac{1}{8MC_6})\Big)\|\Psi_1 - \Psi_2\|_{\mathbf{H}^1} \\ &\le \tfrac{3}{4}\|\Psi_1 - \Psi_2\|_{\mathbf{H}^1}. \end{aligned} \tag{2.4.44}$$

In particular, T maps X_η into itself and is a contraction. The local uniqueness of the family $(c_h, \overline{u}_h, \overline{w_h})$ follows directly from the uniqueness of fixed points of contraction mappings. This completes the proof. ■

2.5 The point and essential spectrum

In this section, we discuss several properties of the operator that arises after linearising the travelling pulse MFDE (2.2.12) around our wave solution $(\overline{u}_h, \overline{w}_h)$. The main goals are to determine the Fredholm properties of this operator. In particular, we show that both the linearised operator and its adjoint have Fredholm index 0 and that they both have a one-dimensional kernel. Moreover, we construct a family of kernel elements of the adjoint operator that converges to (ϕ_0^-, ψ_0^-), the kernel element of the operator $\mathcal{L}_0^-$.

Pick $0 < h < \min\{h_*, \overline{h}\}$, where h_* is given in Theorem 2.2.1 and $\overline{h}$ is characterized by (2.3.42). We recall the operator $L_h : \mathbf{H}^1 \to \mathbf{L}^2$, introduced in §2.2, which acts as

$$L_h = \begin{pmatrix} c_h \frac{d}{d\xi} - \Delta_h - g_u(\overline{u}_h) & 1 \\ -\rho & c_h \frac{d}{d\xi} + \gamma\rho \end{pmatrix}. \tag{2.5.1}$$

In addition, we write $L_h^* : \mathbf{H}^1 \to \mathbf{L}^2$ for the formal adjoint of L_h, which is given by

$$L_h^* = \begin{pmatrix} -c_h \frac{d}{d\xi} - \Delta_h - g_u(\overline{u}_h) & -\rho \\ 1 & -c_h \frac{d}{d\xi} + \gamma\rho \end{pmatrix}. \tag{2.5.2}$$

We emphasize that L_h and L_h^* correspond to the operators $\overline{\mathcal{L}}_{h,0}^+$ and $\overline{\mathcal{L}}_{h,0}^-$ defined in §2.3 respectively upon writing

$$\begin{aligned} (\tilde{u}_h, \tilde{w}_h) &= (\overline{u}_h, \overline{w}_h), \\ \tilde{c}_h &= c_h \end{aligned} \tag{2.5.3}$$

for the family featuring in (hFam). Finally, we introduce the notation

$$\begin{aligned} \Phi_h^+ &= (\phi_h^+, \psi_h^+) = \tfrac{1}{\|(\overline{u}_h', \overline{w}_h')\|_{\mathbf{L}^2}}(\overline{u}_h', \overline{w}_h'), \\ \Phi_0^+ &= (\phi_0^+, \psi_0^+), \\ \Phi_0^- &= (\phi_0^-, \psi_0^-). \end{aligned} \tag{2.5.4}$$

The results of this section should be seen as a bridge between the singular perturbation theory developed in §2.3 and the spectral analysis preformed in §2.6. Indeed, one might be tempted to think that most of the work required for the spectral analysis of the operator L_h can already be found in Proposition 2.3.2 and Proposition 2.3.3. However, the problem is that we have no control over the δ-dependence of the interval $(0, h_0'(\delta))$, which contains all values of h for which $L_h + \delta = \overline{\mathcal{L}}_{h,\delta}^+$ is invertible. In particular, for fixed $h > 0$ we cannot directly conclude that $\overline{\mathcal{L}}_{h,\delta}^+$ is invertible for all δ in a subset of the positive real axis.

Our main task in this section is, therefore, to remove the δ-dependence and study L_h and L_h^* directly. The main conclusions are summarized in the results below.

Proposition 2.5.1. *Assume that (HP1), (HS), (Hα1) and (Hα2) are satisfied. Then there exists a constant $\tilde{\lambda} > 0$ such that for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -\tilde{\lambda}$ and all $0 < h < \min\{h_*, \overline{h}\}$ the operator $L_h + \lambda$ is Fredholm with index 0.*

Proposition 2.5.2. *Assume that (HP1), (HS), (Hα1) and (Hα2) are satisfied. Then there exists a constant $h_{**} > 0$, together with a family $\Phi_h^- = (\phi_h^-, \psi^-) \in \boldsymbol{H}^1$, defined for $0 < h < h_{**}$, such that the following properties hold.*

1. *For each $0 < h < h_{**}$ we have the identities*

$$\begin{aligned} \ker(L_h) &= \operatorname{span}\{\Phi_h^+\} \\ &= \{\Psi \in \boldsymbol{L}^2 : \langle \Psi, \Theta \rangle = 0 \textit{ for all } \Theta \in \operatorname{Range}(L_h^*)\} \end{aligned} \tag{2.5.5}$$

and

$$\begin{aligned} \ker(L_h^*) &= \operatorname{span}\{\Phi_h^-\} \\ &= \{\Psi \in \boldsymbol{L}^2 : \langle \Psi, \Theta \rangle = 0 \textit{ for all } \Theta \in \operatorname{Range}(L_h)\}. \end{aligned} \tag{2.5.6}$$

2. *The family Φ_h^- converges to Φ_0^- in $\boldsymbol{H}^1$ as $h \downarrow 0$.*

3. *Upon introducing the spaces*

$$X_h = \{\Theta \in \boldsymbol{H}^1 : \langle \Phi_h^-, \Theta \rangle = 0\} \tag{2.5.7}$$

and

$$Y_h = \{\Theta \in \boldsymbol{L}^2 : \langle \Phi_h^-, \Theta \rangle = 0\}, \tag{2.5.8}$$

the operator $L_h : X_h \to Y_h$ is invertible and there exists a constant $C_{\text{unif}} > 0$ such that for each $0 < h < h_{**}$ we have the uniform bound

$$\|L_h^{-1}\|_{\mathcal{B}(Y_h, X_h)} \leq C_{\text{unif}}. \tag{2.5.9}$$

A direct consequence of these results is that the zero eigenvalue of L_h is simple. In addition, these results allow us to construct a quasi-inverse for L_h that we use heavily in §2.6 and §2.7.

Corollary 2.5.3. *Assume that (HP1), (HS), (Hα1) and (Hα2) are satisfied. Then for any $0 < h < h_{**}$ the zero eigenvalue of L_h is simple.*

Proof. We can assume that $\langle \Phi_h^-, \Phi_h^+ \rangle \neq 0$ for all $0 < h < h_{**}$, since by Proposition 2.5.2 $\langle \Phi_h^-, \Phi_h^+ \rangle \to \langle \Phi_0^-, \Phi_0^+ \rangle \neq 0$. Equation (2.5.6) now implies that $\Phi_h^+ \notin \operatorname{Range}(L_h)$, which together with (2.5.5) completes the proof. ■

Corollary 2.5.4. *Assume that (HP1), (HS), (Hα1) and (Hα2) are satisfied. There exist linear maps*

$$\begin{aligned} \gamma_h : \boldsymbol{L}^2 &\to \mathbb{R} \\ L_h^{\text{qinv}} : \boldsymbol{L}^2 &\to \boldsymbol{H}^1, \end{aligned} \tag{2.5.10}$$

such that for all $\Theta \in \boldsymbol{L}^2$ and each $0 < h < h_{**}$ the pair

$$(\gamma, \Psi) = (\gamma_h \Theta, L_h^{\text{qinv}} \Theta) \tag{2.5.11}$$

is the unique solution to the problem

$$L_h\Psi = \Theta + \gamma\Phi_h^+ \tag{2.5.12}$$

that satisfies the normalisation condition

$$\langle \Phi_h^-, \Psi\rangle = 0. \tag{2.5.13}$$

Proof. Fix $0 < h < h_{**}$ and $\Theta \in \mathbf{L}^2$. Upon defining

$$\gamma_h[\Theta] = -\frac{\langle \Phi_h^-, \Theta\rangle}{\langle \Phi_h^-, \Phi_h^+\rangle}, \tag{2.5.14}$$

we see that $\Theta + \gamma_h[\Theta]\Phi_h^+ \in Y_h$. In particular, Proposition 2.5.2 implies that the problem

$$L_h\Psi = \Theta + \gamma_h[\Theta]\Phi_h^+ \tag{2.5.15}$$

has a unique solution $\Psi \in X_h$, which we refer to as $L_h^{\mathrm{qinv}}\Theta$. ∎

The results in [68, 130] allow us to read off the Fredholm properties of L_h from the behaviour of this operator in the limits $\xi \to \pm\infty$. In particular, we let $L_{h,\infty}$ be the operator defined by

$$\begin{aligned} L_{h,\infty} &= \begin{pmatrix} c_h\frac{d}{d\xi} - \Delta_h - \lim_{\xi\to\infty} g_u(\overline{u}_h(\xi)) & 1 \\ -\rho & c_h\frac{d}{d\xi} + \gamma\rho \end{pmatrix} \\ &= \begin{pmatrix} c_h\frac{d}{d\xi} - \Delta_h - g_u(0) & 1 \\ -\rho & c_h\frac{d}{d\xi} + \gamma\rho \end{pmatrix}. \end{aligned} \tag{2.5.16}$$

This system has constant coefficients. For $\lambda \in \mathbb{C}$ we introduce the notation

$$L_{h,\infty;\lambda} = L_{h,\infty} + \lambda. \tag{2.5.17}$$

We show that for λ in a suitable right half-plane the operator $L_{h,\infty;\lambda}$ is hyperbolic in the sense of [68, 130], i.e. we write

$$\begin{aligned} \Delta_{L_{h,\infty;\lambda}}(z) &= \Big[L_{h,\infty;\lambda}e^{z\xi}\Big](0) \\ &= \begin{pmatrix} c_h z - \frac{1}{h^2}\Big[\sum_{k>0}\alpha_k\Big(e^{khz} + e^{-khz} - 2\Big)\Big] - g_u(0) + \lambda & 1 \\ -\rho & c_h z + \gamma\rho + \lambda \end{pmatrix} \end{aligned} \tag{2.5.18}$$

and show that $\det(\Delta_{L_{h,\infty;\lambda}}(iy)) \neq 0$ for all $y \in \mathbb{R}$. In the terminology of [68, 130], this means that $L_h + \lambda$ is asymptotically hyperbolic. This allows us to compute the Fredholm index of $L_h + \lambda$.

Remark 2.5.5. From this section onward we assume that (Hα2) is satisfied. This is done for technical reasons, allowing us to apply the results from [68]. In particular, this condition implies that the function $\Delta_{L_{h,\infty;\lambda}}(z)$ defined in (2.5.18) is well-defined in a vertical strip $|\mathrm{Re}\ (z)| < \nu$.

Lemma 2.5.6. *Assume that (HP1), (HS), (Hα1) and (Hα2) are satisfied. There exists a constant* $\tilde{\lambda} > 0$ *such that for all* $0 < h < \min\{h_*, \overline{h}\}$ *and all* $\lambda \in \mathbb{C}$ *with* $\mathrm{Re}\,\lambda > -\tilde{\lambda}$ *the operator* $L_{h,\infty;\lambda}$ *is hyperbolic and thus the operator* $L_h + \lambda$ *is asymptotically hyperbolic.*

Proof. Remembering that $-g_u(0) = r_0 > 0$ and picking $y \in \mathbb{R}$, we compute

$$\begin{array}{rcl} \Delta_{L_{h,\infty;\lambda}}(iy) & = & \begin{pmatrix} c_h iy + \frac{1}{h^2}\Big[\sum_{k>0} \alpha_k\Big(2 - 2\cos(khy)\Big)\Big] + r_0 + \lambda & 1 \\ -\rho & c_h iy + \gamma\rho \end{pmatrix} \\ & = & \begin{pmatrix} c_h iy + \frac{1}{h^2}A(hy) + r_0 + \lambda & 1 \\ -\rho & c_h iy + \gamma\rho + \lambda \end{pmatrix}, \end{array} \tag{2.5.19}$$

where $A(hy) \geq 0$ is defined in (Hα1). We hence see

$$\det(\Delta_{L_{h,\infty;\lambda}}(iy)) \;=\; \Big(c_h iy + \tfrac{1}{h^2}A(hy) + r_0 + \lambda\Big)\Big(c_h iy + \gamma\rho + \lambda\Big) + \rho. \tag{2.5.20}$$

Let $\tilde{\lambda} = \frac{1}{4}\min\{\gamma\rho, r_0\}$ and assume that $\mathrm{Re}\,\lambda > -\tilde{\lambda}$. If $y \neq -\frac{\mathrm{Im}\,\lambda}{c_h}$ then we obtain

$$\begin{array}{rcl} \mathrm{Im}\Big(\det(\Delta_{L_{h,\infty;\lambda}}(iy))\Big) & = & (c_h y + \mathrm{Im}\,\lambda)(\gamma\rho + \mathrm{Re}\,\lambda) \\ & & +(\frac{1}{h^2}A(hy) + r_0 + \mathrm{Re}\,\lambda)(c_h y + \mathrm{Im}\,\lambda) \\ & = & (c_h y + \mathrm{Im}\,\lambda)(\gamma\rho + \frac{1}{h^2}A(hy) + r_0 + 2\mathrm{Re}\,\lambda) \\ & \neq & 0, \end{array} \tag{2.5.21}$$

since $\gamma\rho + \frac{1}{h^2}A(hy) + r_0 + \mathrm{Re}\,\lambda > 0$. For $y = -\frac{\mathrm{Im}\,\lambda}{c_h}$ we obtain

$$\begin{array}{rcl} \mathrm{Re}\Big(\det(\Delta_{L_{h,\infty;\lambda}}(iy))\Big) & = & \Big(\frac{1}{h^2}A(hy) + r_0 + \mathrm{Re}\,\lambda\Big)\Big(\gamma\rho + \mathrm{Re}\,\lambda\Big) + \rho \\ & > & \rho \\ & > & 0. \end{array} \tag{2.5.22}$$

In particular, we see that $\det(\Delta_{L_{h,\infty;\lambda}}(iy)) \neq 0$ for all $y \in \mathbb{R}$, as desired. ∎

Before we consider the Fredholm properties of $L_h + \lambda$, we establish a technical estimate for the function $\Delta_{L_{h,\infty;\lambda}}$, which we need in §2.7 later on.

Lemma 2.5.7. *Assume that (HP1), (HS), (Hα1) and (Hα2) are satisfied. Fix* $0 < h < \min\{h_*, \overline{h}\}$ *and* $S \subset \mathbb{C}$ *compact in such a way that* $\mathrm{Re}\,\lambda > -\tilde{\lambda}$ *for all* $\lambda \in S$*. Then there exist constants* $\kappa > 0$ *and* $\Gamma > 0$*, possibly depending on* h *and* S*, such that for all* $z = x + iy \in \mathbb{C}$ *with* $|x| \leq \kappa$ *and all* $\lambda \in S$ *we have the bound*

$$|\det(\Delta_{L_{h,\infty;\lambda}}(z))| \;\geq\; \tfrac{1}{\Gamma}. \tag{2.5.23}$$

Proof. Using assumption (Hα2), we can pick $\kappa_1 > 0$ and $\Gamma_1 > 0$ in such a way that the bound

$$\begin{array}{rcl} |\frac{1}{h^2}A(hz)| & := & \Big|\frac{1}{h^2}\Big[\sum_{k>0}\alpha_k\Big(2 - e^{khz} - e^{-khz}\Big)\Big]\Big| \\ & \leq & \frac{1}{h^2}\sum_{k>0}|\alpha_k|\Big(e^{hk|x|} + 3\Big) \\ & \leq & \Gamma_1 \end{array} \tag{2.5.24}$$

holds for all $z = x + iy \in \mathbb{C}$ with $|x| \le \kappa_1$.

Observe that for $z = x + iy \in \mathbb{C}$ and $\lambda \in S$ we have

$$\begin{array}{rcl} \mathrm{Re}\Big(\det(\Delta_{L_{h,\infty;\lambda}}(z))\Big) & = & \Big(c_h x + \frac{1}{h^2}\mathrm{Re}\,A(hz) + r_0 + \mathrm{Re}\,\lambda\Big)\Big(c_h x + \gamma\rho + \mathrm{Re}\,\lambda\Big) \\ & & -(c_h y + \mathrm{Im}\,\lambda)^2 - (c_h y + \mathrm{Im}\,\lambda)\frac{1}{h^2}(\mathrm{Im}\,A(hz)) + \rho. \end{array} \tag{2.5.25}$$

Since S is compact we can find $Y > 0$ such that for all $z = x + iy \in \mathbb{C}$ with $|y| \ge Y$ and $|x| \le k_1$ and all $\lambda \in S$ we have

$$\begin{array}{rcl} \Big|\mathrm{Re}\Big(\det(\Delta_{L_{h,\infty;\lambda}}(z))\Big)\Big| & \ge & \frac{1}{2}c_h^2 y^2 \\ & \ge & \frac{1}{2}c_h^2 Y^2. \end{array} \tag{2.5.26}$$

Seeking a contradiction, let us assume that for each $0 < \kappa \le \kappa_1$ and each $\Gamma > 0$ there exist $\lambda \in S$ and $z = x + iy \in \mathbb{C}$ with $|x| \le \kappa$ and $|y| \le Y$ for which

$$|\det(\Delta_{L_{h,\infty;\lambda}}(z))| \quad < \quad \tfrac{1}{\Gamma}. \tag{2.5.27}$$

Then we can construct a sequence $\{\kappa_n, z_n, \lambda_n\}$ with $0 < \kappa_n \le \kappa_1$ for each n, $\kappa_n \to 0$, $\lambda_n \in S$ for each n and $z_n = x_n + iy_n \in \mathbb{C}$ with $|x_n| \le \kappa_n$ and $|y_n| \le Y$ in such a way that $|\det(\Delta_{L_{h,\infty;\lambda_n}}(z_n))| < \frac{1}{n}$ for each n. By taking a subsequence if necessary we see that $\lambda_n \to \lambda$ for some $\lambda \in S$ and $z_n \to iy$ for some $y \in \mathbb{R}$ with $|y| \le Y$. Since $\det(\Delta_{L_{h,\infty;\lambda}}(z))$ is continuous as a function of λ and z, it follows that

$$\begin{array}{rcl} \det(\Delta_{L_{h,\infty;\lambda}}(iy)) & = & \lim_{n\to\infty} \det(\Delta_{L_{h,\infty;\lambda_n}}(z_n)) \\ & = & 0, \end{array} \tag{2.5.28}$$

which contradicts Lemma 2.5.6. Hence, we can find $\kappa > 0$ and $\Gamma > 0$ as desired. ■

Proof of Proposition 2.5.1. We have already seen in Lemma 2.5.6 that $L_h + \lambda$ is asymptotically hyperbolic in the sense of [68, 130]. Now according to [68, Thm. 1.6], we obtain that $L_h + \lambda$ is a Fredholm operator and that the following identities hold

$$\begin{array}{rcl} \dim\Big(\ker(L_h + \lambda)\Big) & = & \mathrm{codim}\Big(\mathrm{Range}(L_h^* + \overline{\lambda})\Big), \\ \mathrm{codim}\Big(\mathrm{Range}(L_h + \lambda)\Big) & = & \dim\Big(\ker(L_h^* + \overline{\lambda})\Big), \\ \mathrm{ind}(L_h + \lambda) & = & -\mathrm{ind}(L_h^* + \overline{\lambda}), \end{array} \tag{2.5.29}$$

where

$$\mathrm{ind}(L_h + \lambda) \quad = \quad \dim\Big(\ker(L_h + \lambda)\Big) - \mathrm{codim}\Big(\mathrm{Range}(L_h + \lambda)\Big) \tag{2.5.30}$$

is the Fredholm index of $L_h + \lambda$.

We follow the proof of [130, Thm. B]. For $0 \le \vartheta \le 1$, we let the operator $L^\vartheta(h)$ be defined by

$$L^\vartheta(h) \quad = \quad (1-\vartheta)(L_h + \lambda) + \vartheta(L_{h,\infty} + \lambda). \tag{2.5.31}$$

Note that the operator $L^{\vartheta}(h)$ is asymptotically hyperbolic for each ϑ and thus [68, Thm. 1.6] implies that these operators $L^{\vartheta}(h)$ are Fredholm. Moreover, the family $L^{\vartheta}(h)$ varies continuously with ϑ in $\mathcal{B}(\mathbf{H}^1, \mathbf{L}^2)$, which means the Fredholm index is constant. In particular, we see that

$$\begin{aligned} \mathrm{ind}(L_h + \lambda) &= \mathrm{ind}(L_{h,\infty} + \lambda) \\ &= 0, \end{aligned} \tag{2.5.32}$$

where the last equality follows from [68, Thm. 1.7]. ■

We can now concentrate on the kernel of L_h. The goal is to exclude kernel elements other than Φ_h^+. In order to accomplish this, we construct a quasi-inverse for L_h by mimicking the approach of [111, Prop. 3.2]. As a preparation, we obtain the following technical result.

Lemma 2.5.8. *Assume that (HP1), (HS) and (Hα1) are satisfied. Recall the constant δ_0 from Lemma 2.3.1. Let $0 < \lambda < \min\{\frac{1}{2}, \delta_0\}$ be given. Then there exist constants $0 < h_1^* \leq \min\{h_*, \overline{h}\}$ and $\kappa > 0$ such that for all $0 < h \leq h_1^*$ we have*

$$\begin{aligned} \langle \Phi_0^-, (L_h + \lambda)^{-1} \Phi_0^+ \rangle &> \tfrac{1}{2}\lambda^{-1} \langle \Phi_0^-, \Phi_h^+ \rangle \\ &> \tfrac{1}{2}\lambda^{-1}\kappa \\ &> 0. \end{aligned} \tag{2.5.33}$$

Proof. We know from Lemma 2.3.1 that $\langle \Phi_0^-, \Phi_0^+ \rangle > 0$. Since Φ_h^+ converges to Φ_0^+ in $\mathbf{L}^2$, it follows that $\langle \Phi_0^-, \Phi_h^+ \rangle$ converges to $\langle \Phi_0^-, \Phi_0^+ \rangle > 0$. Fix $h_1^* \leq \min\{h_*, \overline{h}, h_0'(\lambda)\}$ in such a way that

$$\|\Phi_0^+ - \Phi_h^+\|_{\mathbf{L}^2} < \frac{1}{2} \frac{\langle \Phi_0^-, \Phi_h^+ \rangle}{2C_{\mathrm{unif}}} \tag{2.5.34}$$

holds for all $0 \leq h \leq h_1^*$, where

$$C_{\mathrm{unif}} = 4C_0' \tag{2.5.35}$$

and C_0' is defined in Proposition 2.3.2. The factor 4 in the definition is for technical reasons in a later proof. We assume from now on that $0 < h \leq h_1^*$. Using $L_h \Phi_h^+ = 0$ we readily see

$$(L_h + \lambda)^{-1} \Phi_h^+ = \lambda^{-1} \Phi_h^+. \tag{2.5.36}$$

Recall that $\|\Phi_0^-\|_{\mathbf{L}^2} = 1$. Since $1 < \lambda^{-1}$, we may use Proposition 2.3.2 to obtain

$$\begin{aligned} \|(L_h + \lambda)^{-1} \Phi_0^+ - \lambda^{-1} \Phi_h^+\|_{\mathbf{L}^2} &= \|(L_h + \lambda)^{-1} [\Phi_0^+ - \Phi_h^+]\|_{\mathbf{L}^2} \\ &\leq C_{\mathrm{unif}} \Big[\|\Phi_h^+ - \Phi_0^+\|_{\mathbf{L}^2} + \lambda^{-1} |\langle \Phi_h^+ - \Phi_0^+, \Phi_0^- \rangle| \Big] \\ &< C_{\mathrm{unif}} \lambda^{-1} \|\Phi_0^+ - \Phi_h^+\|_{\mathbf{L}^2} \Big(1 + \|\Phi_0^-\|_{\mathbf{L}^2} \Big) \\ &= 2C_{\mathrm{unif}} \lambda^{-1} \|\Phi_0^+ - \Phi_h^+\|_{\mathbf{L}^2}. \end{aligned} \tag{2.5.37}$$

Remembering $\langle \Phi_0^-, \Phi_h^+ \rangle > 0$ and using Cauchy-Schwarz, we see that

$$\begin{array}{lcl} |\langle \frac{\Phi_0^-}{\langle \Phi_0^-, \Phi_h^+ \rangle}, (L_h + \lambda)^{-1}\Phi_0^+ \rangle - \lambda^{-1}| & = & |\langle \frac{\Phi_0^-}{\langle \Phi_0^-, \Phi_h^+ \rangle}, (L_h + \lambda)^{-1}\Phi_0^+ - \lambda^{-1}\Phi_h^+ \rangle| \\ & < & \frac{\|\Phi_0^-\|_{\mathbf{L}^2}}{\langle \Phi_0^-, \Phi_h^+ \rangle} 2C_{\mathrm{unif}}\lambda^{-1}\|\Phi_0^+ - \Phi_h^+\|_{\mathbf{L}^2} \\ & \leq & \frac{1}{\langle \Phi_0^-, \Phi_h^+ \rangle} 2C_{\mathrm{unif}}\lambda^{-1}\frac{1}{2}\frac{\langle \Phi_0^-, \Phi_h^+ \rangle}{2C_{\mathrm{unif}}} \\ & = & \frac{1}{2}\lambda^{-1}. \end{array} \tag{2.5.38}$$

Hence, we must have

$$\langle \Phi_0^-, (L_h + \lambda)^{-1}\Phi_0^+ \rangle \quad > \quad \tfrac{1}{2}\lambda^{-1}\langle \Phi_0^-, \Phi_h^+ \rangle > 0. \tag{2.5.39}$$

■

Lemma 2.5.9. *Assume that (HP1), (HS) and (Hα1) are satisfied. There exists* $0 < h_{**} \leq \min\{h_*, \overline{h}\}$ *together with linear maps*

$$\begin{array}{rcl} \tilde{\gamma}_h^+ : \boldsymbol{L}^2 & \to & \mathbb{R} \\ \tilde{L}_h^{\mathrm{qinv}} : \boldsymbol{L}^2 & \to & \boldsymbol{H}^1, \end{array} \tag{2.5.40}$$

defined for all $0 < h < h_{**}$*, such that for all* $\Theta \in \boldsymbol{L}^2$ *the pair*

$$(\gamma, \Psi) \quad = \quad (\tilde{\gamma}_h^+\Theta, \tilde{L}_h^{\mathrm{qinv}}\Theta) \tag{2.5.41}$$

is the unique solution to the problem

$$L_h\Psi \quad = \quad \Theta + \gamma\Phi_0^+ \tag{2.5.42}$$

that satisfies the normalisation condition

$$\langle \Phi_0^-, \Psi \rangle \quad = \quad 0. \tag{2.5.43}$$

In addition, there exists $C > 0$ *such that for all* $0 < h < h_{**}$ *and all* $\Theta \in \boldsymbol{L}^2$ *we have the bound*

$$|\tilde{\gamma}_h^+\Theta| + \|\tilde{L}_h^{\mathrm{qinv}}\Theta\|_{\boldsymbol{H}^1} \quad \leq \quad C\|\Theta\|_{\boldsymbol{L}^2}. \tag{2.5.44}$$

Proof. Fix $0 < \lambda < \min\{\frac{1}{2}, \delta_0\}$ and let $0 < h \leq \min\{h_*, \overline{h}, h_0'(\lambda)\}$ be given, where $h_0'(\lambda)$ is defined in Proposition 2.3.2. For now, all constants will not depend on our choice of λ. We define the set

$$Z^1 \quad = \quad \{\Psi \in \mathbf{H}^1 : \langle \Phi_0^-, \Psi \rangle = 0\}. \tag{2.5.45}$$

Pick $\Theta \in \mathbf{L}^2$. We look for a solution $(\gamma, \Psi) \in \mathbb{R} \times Z^1$ of the problem

$$\Psi \quad = \quad (L_h + \lambda)^{-1}[\Theta + \gamma\Phi_0^+ + \lambda\Psi]. \tag{2.5.46}$$

By Lemma 2.5.8 we have $\langle \Phi_0^-, (L_h + \lambda)^{-1}\Phi_0^+ \rangle \neq 0$. Hence, for given $\Theta \in \mathbf{L}^2, \Psi \in Z^1, h, \lambda$, we may write

$$\gamma(\Psi, \Theta, h, \lambda) \quad = \quad -\frac{\langle \Phi_0^-, (L_h+\lambda)^{-1}(\Theta + \lambda\Psi) \rangle}{\langle \Phi_0^-, (L_h+\lambda)^{-1}\Phi_0^+ \rangle}, \tag{2.5.47}$$

which is the unique value for γ for which

$$(L_h + \lambda)^{-1}[\Theta + \gamma\Phi_0^+ + \lambda\Psi] \quad \in \quad Z^1. \tag{2.5.48}$$

Recall the constant C_{unif} from (2.5.35). With Proposition 2.3.2 we obtain

$$\begin{aligned} |\langle\Phi_0^-, (L_h+\lambda)^{-1}(\Theta+\lambda\Psi)\rangle| &\leq \|\Phi_0^-\|_{\mathbf{L}^2} C_{\text{unif}}\Big[\|\Theta+\lambda\Psi\|_{\mathbf{L}^2} + \tfrac{1}{\lambda}|\langle\Theta+\lambda\Psi, \Phi_0^-\rangle|\Big] \\ &\leq \|\Phi_0^-\|_{\mathbf{L}^2} C_{\text{unif}}\Big[(1+\tfrac{1}{\lambda})\|\Theta\|_{\mathbf{L}^2} + \lambda\|\Psi\|_{\mathbf{L}^2}\Big] \\ &\leq C_1\Big[\lambda^{-1}\|\Theta\|_{\mathbf{L}^2} + \lambda\|\Psi\|_{\mathbf{L}^2}\Big] \end{aligned} \tag{2.5.49}$$

for some C_1 that is independent of h, λ. Here we used that $\lambda < 1$ and thus $1 + \frac{1}{\lambda} < \frac{2}{\lambda}$. Exploiting $\lambda < \frac{1}{2}$ and applying Lemma 2.5.8, we see that

$$\begin{aligned} |\gamma(\Psi,\Theta,h,\lambda)| &= |\langle\Phi_0^-, (L_h+\lambda)^{-1}(\Theta+\lambda\Psi)\rangle| \tfrac{1}{|\langle\Phi_0^-,(L_h+\lambda)^{-1}\Phi_0^+\rangle|} \\ &\leq C_1\Big[\lambda^{-1}\|\Theta\|_{\mathbf{L}^2} + \lambda\|\Psi\|_{\mathbf{L}^2}\Big]\tfrac{1}{\frac{1}{2}\lambda^{-1}\langle\Phi_0^-,\Phi_h^+\rangle} \\ &\leq C_1\Big[\kappa\|\Theta\|_{\mathbf{L}^2} + \kappa\lambda^2\|\Psi\|_{\mathbf{L}^2}\Big] \\ &\leq C_2\Big[\|\Theta\|_{\mathbf{L}^2} + \lambda^2\|\Psi\|_{\mathbf{L}^2}\Big]. \end{aligned} \tag{2.5.50}$$

Here we used that $\langle\Phi_0^-, \Phi_h^+\rangle$ converges to $\langle\Phi_0^-, \Phi_0^+\rangle > 0$, which means that $\langle\Phi_0^-, \Phi_h^+\rangle$ can be bounded away from zero. For $\Psi \in Z^1$ we write

$$t(\Psi) \quad = \quad \Theta + \gamma(\Psi,\Theta,h,\lambda)\Phi_0^+ + \lambda\Psi \tag{2.5.51}$$

and

$$T(\Psi) \quad = \quad (L_h+\lambda)^{-1}t(\Psi). \tag{2.5.52}$$

For $\Psi \in Z^1$ Proposition 2.3.2 implies

$$\begin{aligned} \|T(\Psi)\|_{\mathbf{H}^1} &\leq C_{\text{unif}}\Big[\|\Theta + \gamma(\Psi,\Theta,h,\lambda)\Phi_0^+ + \lambda\Psi\|_{\mathbf{L}^2} \\ &\qquad + \tfrac{1}{\lambda}|\langle\Theta + \gamma(\Psi,\Theta,h,\lambda)\Phi_0^+ + \lambda\Psi, \Phi_0^-\rangle|\Big] \\ &\leq C_3\Big[\tfrac{1}{\lambda}\|\Theta\|_{\mathbf{L}^2} + \lambda\|\Psi\|_{\mathbf{L}^2}\Big] \\ &\leq C_3\Big[\tfrac{1}{\lambda}\|\Theta\|_{\mathbf{L}^2} + \lambda\|\Psi\|_{\mathbf{H}^1}\Big]. \end{aligned} \tag{2.5.53}$$

For $\Psi_1, \Psi_2 \in Z^1$, a second application of Proposition 2.3.2 yields

$$\begin{aligned} |\gamma(\Psi_1,\Theta,h,\lambda) - \gamma(\Psi_2,\Theta,h,\lambda)| &= \Big|\tfrac{\langle\Phi_0^-,(L_h+\lambda)^{-1}(\lambda\Psi_1-\lambda\Psi_2)\rangle}{\langle\Phi_0^-,(L_h+\lambda)^{-1}\Phi_0^+\rangle}\Big| \\ &\leq \tfrac{1}{\langle\Phi_0^-,(L_h+\lambda)^{-1}\Phi_0^+\rangle} C_{\text{unif}}\Big[\lambda\|\Psi_1-\Psi_2\|_{\mathbf{L}^2} \\ &\qquad + \tfrac{1}{\lambda}|\langle\lambda\Psi_1-\lambda\Psi_2, \Phi_0^-\rangle|\Big] \\ &\leq C_4\lambda\Big[\lambda\|\Psi_1-\Psi_2\|_{\mathbf{L}^2} + 0\Big] \\ &\leq C_4\lambda^2\|\Psi_1-\Psi_2\|_{\mathbf{H}^1}. \end{aligned} \tag{2.5.54}$$

Applying Proposition 2.3.2 for the final time, we see

$$\begin{array}{rcl}
\|T(\Psi_1) - T(\Psi_2)\|_{\mathbf{H}^1} & \leq & C_{\mathrm{unif}}\Big[\|t(\Psi_1) - t(\Psi_2)\|_{\mathbf{L}^2} + \frac{1}{\lambda}|\langle t(\Psi_1) - t(\Psi_2), \Phi_0^- \rangle|\Big] \\
& \leq & C_{\mathrm{unif}}\Big[\|t(\Psi_1) - t(\Psi_2)\|_{\mathbf{L}^2} \\
& & \quad + \frac{1}{\lambda}\langle\big(\gamma(\Psi_1, \Theta, h, \lambda) - \gamma(\Psi_2, \Theta, h, \lambda)\big)\Phi_0^+, \Phi_0^-\rangle \\
& & \quad + \frac{1}{\lambda}\langle\lambda(\Psi_1 - \Psi_2), \Phi_0^-\rangle\Big] \\
& \leq & C_{\mathrm{unif}}\Big[\|t(\Psi_1) - t(\Psi_2)\|_{\mathbf{L}^2} + \frac{1}{\lambda}\Big(C_4\lambda^2\|\Psi_1 - \Psi_2\|_{\mathbf{H}^1} + 0\Big)\Big] \\
& \leq & C_{\mathrm{unif}}C_4\lambda^2\|\Psi_1 - \Psi_2\|_{\mathbf{H}^1} + C_{\mathrm{unif}}\lambda\|\Psi_1 - \Psi_2\|_{\mathbf{H}^1} \\
& & \quad + C_{\mathrm{unif}}C_4\lambda\|\Psi_1 - \Psi_2\|_{\mathbf{H}^1} \\
& \leq & C_5\lambda\|\Psi_1 - \Psi_2\|_{\mathbf{H}^1}.
\end{array} \tag{2.5.55}$$

In view of these bounds, we pick λ to be small enough to have $C_3\lambda < \frac{1}{2}$ and $C_5\lambda < \frac{1}{2}$. Since this λ is now fixed, we can allow the constants in the final part of the proof to depend on λ. In addition, we write $h_{**} = \min\{h_1^*, h_0'(\lambda)\}$ and pick $0 < h < h_{**}$. Then $T : Z^1 \to Z^1$ is a contraction, so the fixed point theorem implies that there is a unique $\tilde{L}_h^{\mathrm{qinv}}(\Theta) \in Z^1$ for which

$$\tilde{L}_h^{\mathrm{qinv}}(\Theta) = (L_h + \lambda)^{-1}\Big[\Theta + \gamma(\tilde{L}_h^{\mathrm{qinv}}(\Theta), \Theta, h, \lambda)\Phi_0^+ + \lambda\tilde{L}_h^{\mathrm{qinv}}(\Theta)\Big]. \tag{2.5.56}$$

Furthermore, we have

$$\begin{array}{rcl}
\frac{1}{2}\|\tilde{L}_h^{\mathrm{qinv}}(\Theta)\|_{\mathbf{H}^1} & \leq & (1 - \lambda C_3)\|\tilde{L}_h^{\mathrm{qinv}}(\Theta)\|_{\mathbf{H}^1} \\
& \leq & C_3\lambda^{-1}\|\Theta\|_{\mathbf{L}^2} \\
& \leq & C_6\|\Theta\|_{\mathbf{L}^2}.
\end{array} \tag{2.5.57}$$

Writing $\tilde{\gamma}_h^+(\Theta) = \gamma(\tilde{L}_h^{\mathrm{qinv}}(\Theta), \Theta, h, \lambda)$, we compute

$$\begin{array}{rcl}
|\tilde{\gamma}_h^+(\Theta)| & \leq & C_2[\|\Theta\|_{\mathbf{L}^2} + \lambda^2\|\Theta\|_{\mathbf{L}^2}] \\
& \leq & C_7\|\Theta\|_{\mathbf{L}^2}.
\end{array} \tag{2.5.58}$$

Finally we see that (2.5.46) is in fact equivalent to (2.5.42)-(2.5.43), so in fact $\tilde{L}_h^{\mathrm{qinv}}(\Theta)$ and $\tilde{\gamma}_h^+(\Theta)$ do not depend on λ. In addition, the constants h_{**}, C_6 and C_7 above only depend on the one fixed λ and, as such, do not depend on h or Θ. ∎

Lemma 2.5.10. *Assume that (HP1), (HS), (Hα1) and (Hα2) are satisfied. Let $0 < h < h_{**}$ be given. Then we have the inclusion*

$$\begin{array}{rcl}
\mathrm{span}\{\Phi_h^+\} & \subset & \ker(L_h) \\
& = & \{\Psi \in \boldsymbol{L}^2 : \langle\Psi, \Theta\rangle = 0 \textit{ for all } \Theta \in \mathrm{Range}(L_h^*)\},
\end{array} \tag{2.5.59}$$

where L_h^* is the formal adjoint of L_h.

Proof. By differentiating the differential equation (2.2.12) we see that $L_h\Phi_h^+ = 0$. We know that $(\overline{u}_h, \overline{w}_h) - (\overline{u}_0, \overline{w}_0) \to 0 \in \mathbf{H}^1$. Since $(\overline{u}_0', \overline{w}_0')$ decays exponentially, we get $(\overline{u}_0', \overline{w}_0') \in \mathbf{L}^2$. Hence, we can assume that h_{**} is small enough such that $\Phi_h^+ \in \mathbf{L}^2$ for all $0 < h < h_{**}$. Since $L_h\Phi_h^+ = 0$ we obtain from the differential equation that also $(\Phi_h^+)' \in \mathbf{L}^2$. In particular, we see that $\Phi_h^+ \in \mathbf{H}^1$ and, hence, $\Phi_h^+ \in \ker(L_h)$. ∎

Lemma 2.5.11. *Assume that (HP1), (HS), (Hα1) and (Hα2) are satisfied. Let $0 < h < h_{**}$ be given. Then we have*

$$\begin{aligned} \ker(L_h) &= \operatorname{span}\{\Phi_h^+\} \\ &= \{\Psi \in \boldsymbol{L}^2 : \langle \Psi, \Theta \rangle = 0 \textit{ for all } \Theta \in \operatorname{Range}(L_h^*)\}, \end{aligned} \tag{2.5.60}$$

where L_h^* is the formal adjoint of L_h.

Proof. We show that $\dim(\ker(L_h)) = 1$. Since $\Phi_h^+ \in \ker(L_h)$, we assume that there exists $\Psi \in \ker(L_h)$ in such a way that Ψ is not a scalar multiple of Φ_h^+.

Suppose first that $\langle \Psi, \Phi_0^- \rangle = 0$. Then Lemma 2.5.9 yieds by linearity of $\tilde{L}_h^{\mathrm{qinv}}$ that

$$\begin{aligned} \Psi &= \tilde{L}_h^{\mathrm{qinv}}[0] \\ &= 0, \end{aligned} \tag{2.5.61}$$

which gives a contradiction. Hence, we suppose that $\langle \Psi, \Phi_0^- \rangle \neq 0$. In the proof of Lemma 2.5.8 we saw that $\langle \Phi_h^+, \Phi_0^- \rangle \neq 0$. As such, we can pick $a, b \in \mathbb{R} \setminus \{0\}$ in such a way that

$$\langle a\Phi_h^+ + b\Psi, \Phi_0^- \rangle = 0. \tag{2.5.62}$$

Again it follows from Lemma 2.5.9 that $a\Phi_h^+ + b\Psi = 0$ which gives a contradiction. Therefore, such a kernel element Ψ does not exist. Since we already know that $\Phi_h^+ \in \ker(L_h)$, we must have $\dim\big(\ker(L_h)\big) = 1$, which completes the proof. ∎

The remaining major goal of this section is to find a family of elements $\Phi_h^- \in \ker(L_h^*)$ which satisfies $\Phi_h^- \to \Phi_0^-$ as $h \downarrow 0$. To establish this, we repeat part of the process above for the adjoint operator L_h^*. The key difference is that we must construct the family Φ_h^- by hand. This requires a significant refinement of the argument used above to characterize $\ker(L_h^*)$.

First we need a technical result, similar to Lemma 2.5.8.

Lemma 2.5.12. *Assume that (HP1), (HS) and (Hα1) are satisfied. Fix $0 < \lambda < \frac{1}{2}$ and $0 < h \leq \min\{h_{**}, h_0'(\lambda)\}$, where $h_0'(\lambda)$ is defined in Proposition 2.3.2. Then we have*

$$\langle \Phi_0^+, (L_h^* + \lambda)^{-1}\Phi_0^- \rangle > \tfrac{\langle \Phi_0^+, \Phi_0^- \rangle}{2}\lambda^{-1}. \tag{2.5.63}$$

Proof. Lemma 2.3.1 implies that $\langle \Phi_0^+, \Phi_0^- \rangle > 0$. Remembering that

$$L_h^* - L_0^* \quad = \quad \begin{pmatrix} (c_0 - c_h)\frac{d}{d\xi} - (\Delta_h - \frac{d^2}{d\xi^2}) + (g_u(\overline{u}_0) - g_u(\overline{u}_h)) & 0 \\ 0 & (c_0 - c_h)\frac{d}{d\xi} \end{pmatrix} \tag{2.5.64}$$

and that $L_0^* \Phi_0^- = 0$, we obtain

$$\begin{array}{rcl} (L_h^* + \lambda)\Big[(L_h^* + \lambda)^{-1}\Phi_0^- - (L_0^* + \lambda)^{-1}\Phi_0^-\Big] & = & \Phi_0^- - \Phi_0^- + (L_h^* - L_0^*)(L_0^* + \lambda)^{-1}\Phi_0^- \\ & = & (L_h^* - L_0^*)\lambda^{-1}\Phi_0^-. \end{array} \tag{2.5.65}$$

Recall the constant C_{unif} from (2.5.35). Proposition 2.3.2 yields

$$\begin{array}{rcl} \|(L_h^* + \lambda)^{-1}\Phi_0^- - (L_0^* + \lambda)^{-1}\Phi_0^-\|_{\mathbf{L}^2} & \leq & C_{\mathrm{unif}}\Big[\|(L_h^* - L_0^*)\lambda^{-1}\Phi_0^-\|_{\mathbf{L}^2} \\ & & +|\langle (L_h^* - L_0^*)\lambda^{-1}\Phi_0^-, \Phi_0^+\rangle|\Big] \\ & \leq & C_{\mathrm{unif}}(1 + \lambda^{-1})\|(L_h^* - L_0^*)\lambda^{-1}\Phi_0^-\|_{\mathbf{L}^2}. \end{array} \tag{2.5.66}$$

Using Lemma 2.3.5 and the fact that c_h converges to c_0 and $g_u(\overline{u}_h)$ to $g_u(\overline{u}_0)$, it follows that

$$C_{\mathrm{unif}}(1 + \lambda^{-1})\|(L_h^* - L_0^*)\lambda^{-1}\Phi_0^-\|_{\mathbf{L}^2} \quad \to \quad 0 \tag{2.5.67}$$

as $h \downarrow 0$. Possibly after decreasing $h_0'(\lambda) > 0$, we hence see that

$$\begin{array}{rcl} \langle \Phi_0^+, (L_h^* + \lambda)^{-1}\Phi_0^- \rangle & = & \langle \Phi_0^+, (L_0^* + \lambda)^{-1}\Phi_0^- \rangle + \langle \Phi_0^+, (L_h^* + \lambda)^{-1}\Phi_0^- - (L_0^* + \lambda)^{-1}\Phi_0^- \rangle \\ & = & \lambda^{-1}\langle \Phi_0^+, \Phi_0^- \rangle + \langle \Phi_0^+, (L_h^* + \lambda)^{-1}\Phi_0^- - (L_0^* + \lambda)^{-1}\Phi_0^- \rangle \\ & > & \frac{\langle \Phi_0^+, \Phi_0^- \rangle}{2}\lambda^{-1} \end{array} \tag{2.5.68}$$

holds for all $0 < h < \min\{h_{**}, h_0'(\lambda)\}$. ∎

Lemma 2.5.13. *Assume that (HP1), (HS), (Hα1) and (Hα2) are satisfied. Fix $0 < h < h_{**}$. There exist linear maps*

$$\begin{array}{rcl} \tilde{\gamma}_h^- : \boldsymbol{L}^2 & \to & \mathbb{R}, \\ \tilde{L}_h^{*,\mathrm{qinv}} : \boldsymbol{L}^2 & \to & \boldsymbol{H}^1 \end{array} \tag{2.5.69}$$

such that for all $\Theta \in \boldsymbol{L}^2$ the pair

$$(\gamma, \Psi) \quad = \quad (\tilde{\gamma}_h^- \Theta, \tilde{L}_h^{*,\mathrm{qinv}}\Theta) \tag{2.5.70}$$

is the unique solution to the problem

$$L_h^* \Psi \quad = \quad \Theta + \gamma \Phi_0^- \tag{2.5.71}$$

that satisfies the normalisation condition

$$\langle \Phi_0^+, \Psi \rangle \quad = \quad 0. \tag{2.5.72}$$

Furthermore, there exists $C^* > 0$, such that for all $0 < h < h_{**}$ and all $\Theta \in \boldsymbol{L}^2$ we have the bound

$$|\tilde{\gamma}_h^- \Theta| + \|\tilde{L}_h^{*,\mathrm{qinv}} \Theta\|_{\boldsymbol{H}^1} \quad \leq \quad C^* \|\Theta\|_{\boldsymbol{L}^2}. \tag{2.5.73}$$

Proof. We define the set

$$Z^1 \quad = \quad \{\Psi \in \mathbf{H}^1 : \langle \Phi_0^+, \Psi \rangle = 0\}. \tag{2.5.74}$$

Pick $\Theta \in \mathbf{L}^2$. We look for a solution $(\gamma, \Psi) \in \mathbb{R} \times Z^1$ of the problem

$$\Psi \quad = \quad (L_h^* + \lambda)^{-1}[\Theta + \gamma \Phi_0^- + \lambda \Psi]. \tag{2.5.75}$$

Lemma 2.5.12 implies that $\langle \Phi_0^+, (L_h^* + \lambda)^{-1} \Phi_0^- \rangle \neq 0$. Hence, for given $\Theta \in \mathbf{L}^2, \Psi \in Z^1, h, \lambda$, we may write

$$\gamma(\Psi, \Theta, h, \lambda) \quad = \quad -\frac{\langle \Phi_0^+, (L_h^* + \lambda)^{-1}(\Theta + \lambda\Psi) \rangle}{\langle \Phi_0^+, (L_h^* + \lambda)^{-1} \Phi_0^- \rangle}, \tag{2.5.76}$$

which is the unique value for γ for which

$$(L_h^* + \lambda)^{-1}[\Theta + \gamma \Phi_0^- + \lambda \Psi] \quad \in \quad Z^1. \tag{2.5.77}$$

From now on the proof is identical to that of Lemma 2.5.9, so we omit it. ■

Lemma 2.5.14. *Assume that (HP1), (HS), (Hα1) and (Hα2) are satisfied. For each $0 < h < h_{**}$ there exists an element $\Phi_h^- \in \ker(L_h^*)$ such that the family Φ_h^- converges to Φ_0^- in $\boldsymbol{H}^1$ as $h \downarrow 0$.*

Proof. We repeat some of the steps of the proof of Lemma 2.5.11, but now for L_h^*.

Fix $0 < h < h_{**}$. Since $\dim(\ker(L_h^*)) = 1$ by Proposition 2.5.1 and Lemma 2.5.11, we can pick $\Phi \in \ker(L_h^*)$ with $\Phi \neq 0$. If we would have $\langle \Phi, \Phi_0^+ \rangle = 0$, then we would obtain

$$\begin{aligned} 0 \quad &= \quad L_h^{*,\mathrm{qinv}}[0] \\ &= \quad \Phi, \end{aligned} \tag{2.5.78}$$

which leads to a contradiction. Hence, we can define the kernel element Φ_h^- of L_h^* as follows: Φ_h^- is the unique kernel element of L_h^* with $\langle \Phi_h^-, \Phi_0^+ \rangle = \langle \Phi_0^-, \Phi_0^+ \rangle$. Since we see that

$$\langle \Phi_0^- - \Phi_h^-, \Phi_0^+ \rangle \quad = \quad 0, \tag{2.5.79}$$

we obtain, upon defining

$$\Theta_h \quad := \quad L_h^* \Phi_0^-, \tag{2.5.80}$$

that

$$\Phi_0^- - \Phi_h^- \quad = \quad L_h^{*,\mathrm{qinv}}[\Theta_h]. \tag{2.5.81}$$

Using Lemma 2.5.13, we can estimate

$$\begin{array}{rcl} \|\Phi_0^- - \Phi_h^-\|_{\mathbf{H}^1} & = & \|L_h^{*,\mathrm{qinv}}[\Theta_h]\|_{\mathbf{H}^1} \\ & \leq & C_- \|\Theta_h\|_{\mathbf{L}^2}. \end{array} \tag{2.5.82}$$

From the proof of Lemma 2.5.12 we know that $\Theta_h \to 0$ as $h \downarrow 0$ in $\mathbf{L}^2$. Therefore, we see that $\Phi_h^- \to \Phi_0^-$ as $h \downarrow 0$ in $\mathbf{H}^1$. ∎

In the final part of this section we establish item (3) of Proposition 2.5.2. To this end, we recall the spaces

$$X_h = \{\Theta \in \mathbf{H}^1 : \langle \Phi_h^-, \Theta \rangle = 0\} \tag{2.5.83}$$

and

$$Y_h = \{\Theta \in \mathbf{L}^2 : \langle \Phi_h^-, \Theta \rangle = 0\}, \tag{2.5.84}$$

together with the constant C_{unif} from (2.5.35).

Lemma 2.5.15. *Assume that (HP1), (HS), (Hα1) and (Hα2) are satisfied. For each $0 < h < h_{**}$ we have that $L_h : X_h \to Y_h$ is invertible and we have the uniform bound*

$$\|L_h^{-1}\| \leq C_{\mathrm{unif}}. \tag{2.5.85}$$

Proof. Fix $0 < h < h_{**}$. Clearly $L_h : X_h \to Y_h$ is a bounded bijective linear map, so the Banach isomorphism theorem implies that $L_h^{-1} : Y_h \to X_h$ is bounded. Now let $\delta > 0$ be a small constant such that $\delta C_{\mathrm{unif}} < 1$. Without loss of generality we assume that $0 < h_{**} \leq h_0'(\delta)$ and that $\|\Phi_h^- - \Phi_0^-\|_{\mathbf{H}^1} \leq \delta$ for all $0 < h < h_{**}$. This is possible by Lemma 2.5.14.

Pick any $\Psi \in X_h$. Remembering that $\langle \Psi, \Phi_h^- \rangle = 0$ and $\langle L_h\Psi, \Phi_h^- \rangle = 0$, we obtain the estimate

$$\begin{array}{rcl} \frac{1}{\delta}|\langle (L_h + \delta)\Psi, \Phi_0^- \rangle| & = & \frac{1}{\delta}|\langle (L_h + \delta)\Psi, \Phi_0^- - \Phi_h^- \rangle| \\ & \leq & \frac{1}{\delta}\|(L_h + \delta)\Psi\|_{\mathbf{L}^2}\delta \\ & \leq & \|L_h\Psi\|_{\mathbf{L}^2} + \delta\|\Psi\|_{\mathbf{H}^1}. \end{array} \tag{2.5.86}$$

Applying Proposition 2.3.2, we hence see

$$\begin{array}{rcl} \|\Psi\|_{\mathbf{H}^1} & \leq & \frac{1}{4}C_{\mathrm{unif}}[\|(L_h + \delta)\Psi\|_{\mathbf{L}^2} + \frac{1}{\delta}|\langle (L_h + \delta)\Psi, \Phi_0^- \rangle|] \\ & \leq & \frac{1}{4}C_{\mathrm{unif}}[2\|L_h\Psi\|_{\mathbf{L}^2} + 2\delta\|\Psi\|_{\mathbf{H}^1}] \\ & \leq & \frac{1}{2}C_{\mathrm{unif}}\|L_h\Psi\|_{\mathbf{L}^2} + \frac{1}{2}\|\Psi\|_{\mathbf{H}^1}. \end{array} \tag{2.5.87}$$

We, therefore, get the bound

$$\|\Psi\|_{\mathbf{H}^1} \leq C_{\mathrm{unif}}\|L_h\Psi\|_{\mathbf{L}^2}, \tag{2.5.88}$$

which yields the desired estimate $\|L_h^{-1}\| \leq C_{\mathrm{unif}}$. ∎

Proof of Proposition 2.5.2. This result follows directly from Lemmas 2.5.11, 2.5.14 and 2.5.15. ∎.

2.6 The resolvent set

In this section, we prove Theorem 2.2.2 by explicitly determining the spectrum of the operator $-L_h$ defined in (2.2.18) in a suitable half-plane. Our approach hinges on the periodicity of this spectrum, which we describe in our first result.

Lemma 2.6.1. *Assume that (HP1), (HS), (Hα1) and (Hα2) are satisfied. Fix $0 < h < h_{**}$. Then the spectrum of L_h is invariant under the operation $\lambda \mapsto \lambda + 2\pi i c_h \frac{1}{h}$.*

In particular, we can restrict our attention to values with imaginary part in between $-\frac{\pi c_h}{h}$ and $\frac{\pi c_h}{h}$. We divide our 'half-strip' into four regions and in each region we calculate the spectrum. Values close to 0 (region R_1) will be treated in Proposition 2.6.2; values with a large real part (region R_2) in Proposition 2.6.3 and values with a large imaginary part (region R_3) in Proposition 2.6.6. In Corollary 2.6.7 we discuss the remaining intermediate subset (region R_4), which is compact and independent of h. The regions are illustrated in Figure 2.1 below.

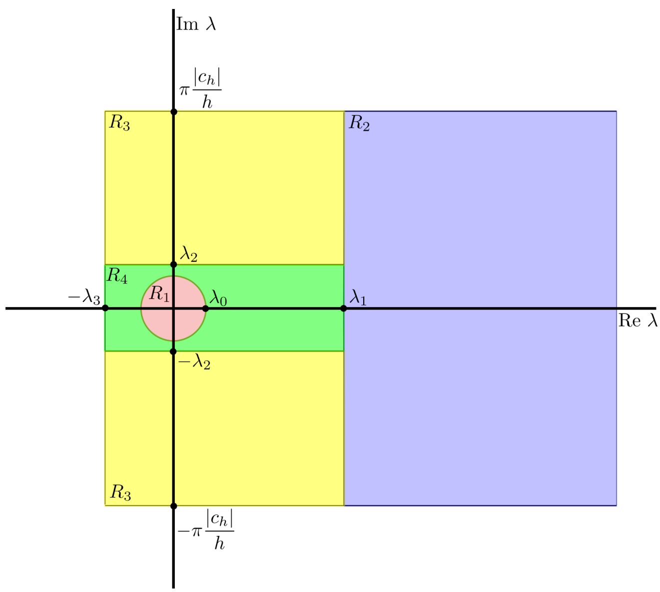


Figure 2.1: *Illustration of the regions R_1, R_2, R_3 and R_4. Note that the regions R_2 and R_3 grow when h decreases, while the regions R_1 and R_4 are independent of h.*

From this section onward we need to assume that (HP2) is satisfied. Indeed, this allows us to lift the invertibility of $L_0+\lambda$ to $L_h+\lambda$ simultaneously for all λ in appropriate compact sets.

Proof of Lemma 2.6.1. Fix $k \in \mathbb{Z}$ and write $p = 2\pi i k\frac{1}{h}$. We define the exponential shift operator e_ω by

$$[e_\omega V](x) \quad = \quad e^{\omega x}V(x). \tag{2.6.1}$$

For any $\lambda \in \mathbb{C}$, $\Psi = (\phi, \psi) \in \mathbf{H}^1$ and $x \in \mathbb{R}$ we obtain

$$\begin{array}{rcl} (e_{-p}\Delta_h e_p)\phi(x) & = & e^{-px}\Delta_h(e_p\phi)(x) \\ & = & \frac{1}{h^2}\sum_{l>0}\alpha_l(e^{plh}\phi(x+lh) + e^{-plh}\phi(x-lh) - 2\phi(x)) \\ & = & \frac{1}{h^2}\sum_{l>0}\alpha_l(\phi(x+lh) + \phi(x-lh) - 2\phi(x)) \\ & = & \Delta_h\phi(x), \end{array} \tag{2.6.2}$$

since $plh \in 2\pi i\mathbb{Z}$ for all $l > 0$. In particular, we can compute

$$\begin{array}{rcl} [e_{-p}(L_h - \lambda)e_p\Psi](x) & = & e^{-px}[(L_h - \lambda)e_p\Psi](x) \\ & = & e^{-px}\begin{pmatrix} c_h\frac{d}{d\xi}(e^{px}\phi(x)) - \Delta_h(e_p\phi)(x) \\ -\rho e^{px} + c_h\frac{d}{d\xi}(e^{px}\psi(x)) \end{pmatrix} \\ & & +e^{-px}\begin{pmatrix} -g_u(\overline{u}_h)e^{px}\phi(x) + e^{px}\psi(x) - \lambda e^{px}\phi(x) \\ +\gamma\rho e^{px}\psi(x) - \lambda e^{px}\psi(x) \end{pmatrix} \\ & = & \begin{pmatrix} pc_h\phi(x) + c_h\phi'(x) - g_u(\overline{u}_h)\phi(x) + \psi(x) \\ -\rho\phi(x) + pc_h\psi(x) + c_h\psi'(x) + \gamma\rho\psi(x) - \lambda\psi(x) \end{pmatrix} \\ & & +\begin{pmatrix} -\Delta_h\phi(x) - \lambda\phi(x) \\ 0 \end{pmatrix} \\ & = & (L_h - \lambda + pc_h)\Psi(x). \end{array} \tag{2.6.3}$$

Since e_p and e_{-p} are invertible operators on $\mathbf{H}^1$ and $\mathbf{L}^2$ respectively, we know that the spectrum of L_h equals that of $e_{-p}L_he_p$ and thus that of $L_h + pc_h$. ∎

Region R_1.

Since L_h has a simple eigenvalue at zero, it is relatively straightforward to construct a small neighbourhood around the origin that contains no other part of the spectrum. Exploiting the results from §2.5, it is possible to control the size of this neighbourhood as $h \downarrow 0$.

Proposition 2.6.2. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. There exists a constant $\lambda_0 > 0$ such that for all $0 < h < h_{**}$ the operator $L_h + \lambda : \boldsymbol{H}^1 \to \boldsymbol{L}^2$ is invertible for all $\lambda \in \mathbb{C}$ with $0 < |\lambda| < \lambda_0$.*

Proof. Fix $0 < h < h_{**}$ and $\Theta \in \mathbf{L}^2$. We recall the notation $(\gamma_h[\Theta], L_h^{\mathrm{qinv}}\Theta)$ from Corollary 2.5.4 for the unique solution (γ, Ψ) of the equation

$$L_h\Psi = \Theta + \gamma\Phi_h^+ \tag{2.6.4}$$

in the space

$$X_h = \{\Theta \in \mathbf{H}^1 : \langle\Phi_h^-, \Theta\rangle = 0\}. \tag{2.6.5}$$

Also recall the space

$$Y_h = \{\Theta \in \mathbf{L}^2 : \langle\Phi_h^-, \Theta\rangle = 0\}. \tag{2.6.6}$$

Now, for $\lambda \in \mathbb{C}$ with $|\lambda|$ small enough, but $\lambda \neq 0$, we want to solve the equation $L_h \Psi = \lambda \Psi + \Theta$. Upon writing

$$\Psi \quad = \quad L_h^{\mathrm{qinv}} \Theta + \lambda^{-1} \gamma_h[\Theta] \Phi_h^+ + \tilde{\Psi}, \tag{2.6.7}$$

with $\tilde{\Psi} \in X_h$, we see that

$$\begin{aligned} (L_h - \lambda)\Psi &= (L_h - \lambda) L_h^{\mathrm{qinv}} \Theta + \lambda^{-1}(L_h - \lambda)\gamma_h[\Theta]\Phi_h^+ + (L_h - \lambda)\tilde{\Psi} \\ &= \Theta + \gamma_h[\Theta]\Phi_h^+ - \lambda L_h^{\mathrm{qinv}} \Theta - \gamma_h[\Theta]\Phi_h^+ + (L_h - \lambda)\tilde{\Psi}. \end{aligned} \tag{2.6.8}$$

In particular, we must find a solution $\tilde{\Psi} \in X_h$ for the equation

$$L_h \tilde{\Psi} \quad = \quad \lambda \tilde{\Psi} + \lambda L_h^{\mathrm{qinv}} \Theta, \tag{2.6.9}$$

which we can rewrite as

$$[I - \lambda L_h^{-1}]\tilde{\Psi} \quad = \quad \lambda L_h^{-1} L_h^{\mathrm{qinv}} \Theta. \tag{2.6.10}$$

Note that $L_h^{-1} : X_h \to X_h$ is also a bounded operator since $X_h \subset Y_h$. Since

$$\begin{aligned} \|L_h^{-1}\Psi\|_{\mathbf{H}^1} &\leq C_{\mathrm{unif}} \|\Psi\|_{\mathbf{L}^2} \\ &\leq C_{\mathrm{unif}} \|\Psi\|_{\mathbf{H}^1}, \end{aligned} \tag{2.6.11}$$

we obtain

$$\|L_h^{-1}\|_{\mathcal{B}(X_h, X_h)} \quad \leq \quad C_{\mathrm{unif}}. \tag{2.6.12}$$

We choose λ_0 in such a way that $0 < \lambda_0 C_{\mathrm{unif}} < 1$. Then it is well-known that $I - \lambda L_h^{-1}$ is invertible as an operator on X_h for $0 < |\lambda| < \lambda_0$. Since $\lambda L_h^{-1} L_h^{\mathrm{qinv}} \Theta \in X_h$, we see that (2.6.10) indeed has a unique solution $\tilde{\Psi} \in X_h$. Hence, the equation $(L_h - \lambda)\Psi = \Theta$ always has a unique solution. Proposition 2.5.1 states that $L_h - \lambda$ is Fredholm with index 0, which now implies that $L_h - \lambda$ is indeed invertible. ∎

Region R_2.

We now show that in an appropriate right half-plane, which can be chosen independently of h, the spectrum of $-L_h$ is empty. The proof proceeds via a relatively direct estimate that is strongly inspired by [6, Lem. 3.1].

Proposition 2.6.3. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. There exists a constant $\lambda_1 > 0$ such that for all $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geq \lambda_1$ and all $0 < h < h_{**}$ the operator $L_h + \lambda$ is invertible.*

Proof. Write

$$\lambda_1 \quad = \quad 1 + g_* + \tfrac{1}{2}(1 - \rho), \tag{2.6.13}$$

where g_* is defined in (2.3.42). Pick any $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \geq \lambda_1$ and any $0 < h < h_{**}$. Let $\Psi = (\phi, \psi) \in \mathbf{H}^1$ be arbitrary and set $\Theta = L_h\Psi + \lambda\Psi$. Then we see that

$$\begin{array}{lcl}
\|\Psi\|_{\mathbf{L}^2}\|\Theta\|_{\mathbf{L}^2} & \geq & \mathrm{Re}\,\langle L_h\Psi + \lambda\Psi, \Psi\rangle \\
& \geq & \mathrm{Re}\,\langle -\Delta_h\phi, \phi\rangle - \|g_u(\overline{u}_h)\|_{L^\infty}\|\phi\|^2_{L^2} \\
& & \qquad -(1-\rho)|\mathrm{Re}\,\langle\phi,\psi\rangle| + \gamma\rho\|\psi\|^2_{L^2} + \mathrm{Re}\,\lambda\|\Psi\|^2_{\mathbf{L}^2} \\
& \geq & -g_*\|\phi\|^2_{L^2} - (1-\rho)|\mathrm{Re}\,\langle\phi,\psi\rangle| + \gamma\rho\|\psi\|^2_{L^2} + \mathrm{Re}\,\lambda\|\Psi\|^2_{\mathbf{L}^2} \\
& \geq & -g_*\|\phi\|^2_{L^2} - (1-\rho)\|\phi\|_{L^2}\|\psi\|_{L^2} + \gamma\rho\|\psi\|^2_{L^2} + \mathrm{Re}\,\lambda\|\Psi\|^2_{\mathbf{L}^2} \\
& \geq & -(g_* + \frac{1}{2}(1-\rho))\|\Psi\|^2_{\mathbf{L}^2} + \mathrm{Re}\,\lambda\|\Psi\|^2_{\mathbf{L}^2}.
\end{array} \tag{2.6.14}$$

Hence, we obtain

$$\Big(\mathrm{Re}\,\lambda - (g_* + \tfrac{1}{2}(1-\rho))\Big)\|\Psi\|_{\mathbf{L}^2} \quad \leq \quad \|\Theta\|_{\mathbf{L}^2}. \tag{2.6.15}$$

Since $\mathrm{Re}\,\lambda \geq 1 + g_* + \frac{1}{2}(1-\rho)$, we obtain the bound $\|\Psi\|_{\mathbf{L}^2} \leq \|\Theta\|_{\mathbf{L}^2}$.

In particular, if $\Theta = 0$ then we necessarily have $\Psi = 0$, which implies that $L_h + \lambda$ is injective. Since also $\mathrm{ind}(L_h + \lambda) = 0$ by Proposition 2.5.1, this means that $L_h + \lambda$ is invertible. ■

Region R_3.

This region is the most delicate to handle on account of the periodicity of the spectrum. Indeed, one cannot simply take $\mathrm{Im}\,\lambda \to \pm\infty$ in a fashion that is uniform in h. We pursue a direct approach here, using the Fourier transform to isolate the problematic part of $L_h + \lambda$, which has constant coefficients. The corresponding portion of the resolvent can be estimated in a controlled way by rescaling the imaginary part of λ. We remark that an alternative approach could be to factor out the periodicity in a more operator-theoretic setting, but we do not pursue such an argument here.

Pick $\lambda \in \mathbb{C}$ with $\lambda_0 < |\mathrm{Im}\,\lambda| \leq \frac{|c_h|}{h}\pi$ and write

$$\lambda \quad = \quad \lambda_{\mathrm{r}} + i\lambda_{\mathrm{im}} \tag{2.6.16}$$

with $\lambda_{\mathrm{r}}, \lambda_{\mathrm{im}} \in \mathbb{R}$. Introducing the new variable $\tau = \mathrm{Im}\,\lambda\xi$, we can write the eigenvalue problem $(L_h + \lambda)(v, w) = 0$ in the form

$$\begin{array}{lcl}
c_h v_\tau(\tau) & = & \frac{1}{\lambda_{\mathrm{im}}h^2}\sum_{k>0}\alpha_k\Big[v(\tau + kh\lambda_{\mathrm{im}}) + v(\tau - kh\lambda_{\mathrm{im}}) - 2v(\tau)\Big] \\
& & \qquad + \frac{1}{\lambda_{\mathrm{im}}}g_u\Big(\overline{u}_h(\tau)\Big)v(\tau) - iv(\tau) - \frac{1}{\lambda_{\mathrm{im}}}\lambda_{\mathrm{r}}v(\tau) - \frac{1}{\lambda_{\mathrm{im}}}w(\tau), \\
c_h w_\tau(\tau) & = & \frac{1}{\lambda_{\mathrm{im}}}\Big(\rho v(\tau) - \rho\gamma w(\tau) + \lambda w(\tau)\Big).
\end{array} \tag{2.6.17}$$

Our computations below show that the leading order terms in the appropriate $|\lambda_{\mathrm{im}}| \to$

∞ limit are encoded by the 'homogeneous operator' $H_{h,\lambda}$ that acts as

$$H_{h,\lambda}v(\tau) = c_h v_\tau(\tau) + iv(\tau) - \frac{1}{\lambda_{\mathrm{im}}h^2}\sum_{k>0}\alpha_k\Big[v(\tau + kh\lambda) + v(\tau - kh\lambda) - 2v(\tau)\Big]. \tag{2.6.18}$$

Writing $\mathcal{H}_{h,\lambda}$ for the Fourier symbol associated to $H_{h,\lambda}$, we see that

$$\begin{aligned} \mathcal{H}_{h,\lambda}(i\omega) &= c_h i\omega + i - \frac{1}{\lambda_{\mathrm{im}}h^2}\sum_{k>0}\alpha_k\Big[\exp(ihk\lambda_{\mathrm{im}}\omega) + \exp(-ihk\lambda_{\mathrm{im}}\omega) - 2\Big] \\ &= c_h i\omega + i - \frac{2}{\lambda_{\mathrm{im}}h^2}\sum_{k>0}\alpha_k\Big[\cos(hk\lambda_{\mathrm{im}}\omega) - 1\Big]. \end{aligned} \tag{2.6.19}$$

Lemma 2.6.4. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. There exist small constants $\varepsilon > 0$, $h_* > 0$ and $\omega_0 > 0$ so that for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, all $0 < h < h_*$ and all $\omega \in \mathbb{R}$, the inequality*

$$|\mathrm{Im}\,\mathcal{H}_{h,\lambda}(i\omega)| < \varepsilon \tag{2.6.20}$$

can only be satisfied if the inequalities

$$\begin{aligned} |c_h\omega| &\le \tfrac{3}{2} \\ |\omega| &\ge \omega_0 \end{aligned} \tag{2.6.21}$$

both hold.

Proof. Note that

$$|\mathrm{Im}\,\mathcal{H}_{h,\lambda}(i\omega)| = |c_h\omega + 1|. \tag{2.6.22}$$

In particular, upon choosing $\varepsilon = \frac{1}{4}$, we see that

$$|\mathrm{Im}\,\mathcal{H}_{h,\lambda}(i\omega)| < \varepsilon \tag{2.6.23}$$

implies

$$\big||c_h\omega| - 1\big| \le |c_h\omega + 1| < \varepsilon \tag{2.6.24}$$

and hence

$$\tfrac{1}{2} < 1 - \varepsilon \le |c_h\omega| \le 1 + \varepsilon < \tfrac{3}{2}. \tag{2.6.25}$$

Since $c_h \to c_0 \neq 0$ as $h \downarrow 0$, the desired inequalities (2.6.21) follow. ∎

Lemma 2.6.5. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. Then there exists a constant $C > 0$ such that for all $\omega \in \mathbb{R}$ and $0 < h < h_{**}$ and all $\lambda \in \mathbb{C}$ with $|\lambda| > \lambda_0$ and $|\mathrm{Im}\,\lambda| \le \frac{|c_h|}{h}\pi$, we have the inequality*

$$|\mathcal{H}_{h,\lambda}(i\omega)| \ge \tfrac{1}{C}. \tag{2.6.26}$$

Proof. We show that $\mathcal{H}_{h,\lambda}(i\omega)$ is bounded away from 0, uniformly in h, λ and ω. To do so, we show that the real part of $\mathcal{H}_{h,\lambda}(i\omega)$ can be bounded away from zero, whenever the imaginary part is small, i.e. when (2.6.21) holds.

Recall the function $A(y) = \sum_{k>0} \alpha_k[1-\cos(ky)]$ defined in Assumption (Hα1), which satisfies $A(y) > 0$ for $y \in (0, 2\pi)$. A direct calculation shows that $A'(0) = 0$ and

$$\begin{array}{rcl} A''(0) & = & \sum_{k>0} \alpha_k k^2 \\ & = & 1. \end{array} \tag{2.6.27}$$

Hence, we can pick $d_0 > 0$ in such a way that

$$\tfrac{1}{y^2} A(y) \quad > \quad d_0 \tag{2.6.28}$$

holds for all $0 < |y| \le \frac{3}{2}\pi$.

Writing $\mu = h\lambda_{\mathrm{im}}\omega$, we see

$$\begin{array}{rcl} \mathrm{Re}\,\mathcal{H}_{h,\lambda}(i\omega) & = & \frac{2\omega^2\lambda_{\mathrm{im}}}{\mu^2} \sum_{k>0} \alpha_k \Big[1-\cos(k\mu)\Big] \\ & = & \frac{2\omega^2\lambda_{\mathrm{im}}}{\mu^2} A(\mu). \end{array} \tag{2.6.29}$$

Now fix ω, h, λ for which $|\mathrm{Im}\,\mathcal{H}_{h,\lambda}(i\omega)| < \varepsilon$. The conditions (2.6.21) now imply that $|\omega| \ge \omega_0$ and $|\mu| \le h\frac{|c_h|}{h}\pi|\omega| \le \frac{3}{2}\pi$. Using (2.6.28), we hence see that

$$\begin{array}{rcl} |\mathrm{Re}\,\mathcal{H}_{h,\lambda}(i\omega)| & = & |\frac{2\omega^2\lambda_{\mathrm{im}}}{\mu^2} A(\mu)| \\ & \ge & 2|\lambda_{\mathrm{im}}|\omega^2 d_0 \\ & \ge & 2\lambda_0\omega_0^2 d_0, \end{array} \tag{2.6.30}$$

which shows that $\mathcal{H}_{h,\lambda}(i\omega)$ can indeed be uniformly bounded away from zero. ■

Proposition 2.6.6. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. There exist constants $\lambda_2 > 0$ and $\lambda_3 > 0$ such that for all $\lambda \in \mathbb{C}$ with $\lambda_2 \le |\mathrm{Im}\,\lambda| \le \frac{|c_h|}{2h}2\pi$ and $-\lambda_3 \le |\mathrm{Re}\,\lambda| \le \lambda_1$ and all $0 < h < h_{**}$ the operator $L_h + \lambda$ is invertible.*

Proof. Since Proposition 2.5.1 implies that $L_h + \lambda$ is Fredholm with index zero, it suffices to prove that $L_h + \lambda$ is injective.

Let $\lambda_3 = \min\{\frac{1}{2}\rho\gamma, \lambda_*, \tilde{\lambda}\}$, where λ_* is defined in (HP2) and $\tilde{\lambda}$ is defined in Proposition 2.5.1. Pick $\lambda \in \mathbb{C}$ with $\lambda_0 \le |\mathrm{Im}\,\lambda| \le \frac{|c_h|}{2h}2\pi$ and $-\lambda_3 \le |\mathrm{Re}\,\lambda| \le \lambda_1$. Write $\lambda = \lambda_{\mathrm{r}} + i\lambda_{\mathrm{im}}$ as before. Suppose $\Psi = (v, w)$ satisfies $(L_h + \lambda)\Psi = 0$.

Write $\hat{v}$ and $\hat{w}$ for the Fourier transforms of v and w respectively. For $f \in L^2$ with Fourier transform $\hat{f}$, the identity

$$H_{h,\lambda} v \quad = \quad f \tag{2.6.31}$$

implies that

$$\mathcal{H}_{h,\lambda}(i\omega)\hat{v}(i\omega) \quad = \quad \hat{f}(i\omega). \tag{2.6.32}$$

In particular, we obtain

$$\hat{v}(i\omega) \quad = \quad \tfrac{1}{\mathcal{H}_{h,\lambda}(i\omega)}\hat{f}(i\omega), \tag{2.6.33}$$

which using Lemma 2.6.5 implies that

$$\|v\|_{L^2} \quad \leq \quad C\|f\|_{L^2} \tag{2.6.34}$$

for some constant $C > 0$ that is independent of h, λ and ω.

Since Ψ is an eigenfunction, (2.6.17) hence yields

$$\|v\|_{L^2} \quad \leq \quad C\tfrac{1}{|\lambda_{\mathrm{im}}|}(g_* + |\lambda_{\mathrm{r}}|)\|v\|_{L^2} + C\tfrac{1}{|\lambda_{\mathrm{im}}|}\|w\|_{L^2}. \tag{2.6.35}$$

Furthermore, applying a Fourier Transform to the second line of (2.6.17), we find

$$\lambda_{\mathrm{im}}c_h i\omega\hat{w}(i\omega) \quad = \quad \rho\hat{v}(i\omega) - \rho\gamma\hat{w}(i\omega) + \lambda\hat{w}(i\omega). \tag{2.6.36}$$

Our choice $\lambda_3 \leq \frac{1}{2}\rho\gamma$ implies that $-\rho\gamma + \lambda_{\mathrm{r}}$ is bounded away from 0. We may hence write

$$\hat{w}(i\omega) \quad = \quad \tfrac{1}{\rho\gamma - \lambda_{\mathrm{r}} + i(\omega\lambda_{\mathrm{im}}c_h - \lambda_{\mathrm{im}})}\rho\hat{v}(i\omega), \tag{2.6.37}$$

which yields the bound

$$\|w\|_{L^2} \quad \leq \quad C'\|v\|_{L^2} \tag{2.6.38}$$

for some constant $C' > 0$. Therefore, we obtain that

$$\|v\|_{L^2} \quad \leq \quad C''\tfrac{1}{|\lambda_{\mathrm{im}}|}\|v\|_{L^2} \tag{2.6.39}$$

for some constant C'', which is independent of λ, h and v. Clearly this is impossible for $v \neq 0$ if

$$|\lambda_{\mathrm{im}}| \quad \geq \quad \lambda_2 \quad := \quad 2C''. \tag{2.6.40}$$

Furthermore, if $v = 0$, then clearly also $w = 0$. Therefore, we have $\Psi = 0$, allowing us to conclude that $L_h + \lambda$ is invertible. ∎

Region R_4.

We conclude our spectral analysis by considering the remaining region R_4. This region is compact and bounded away from the origin, allowing us to directly apply the theory developed in §2.3.

Corollary 2.6.7. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. For all $\lambda \in \mathbb{C}$ with $|\lambda| \geq \lambda_0$, $-\lambda_3 \leq |\mathrm{Re}\,\lambda| \leq \lambda_1$ and $|\mathrm{Im}\,\lambda| \leq \lambda_2$ and all $0 < h < h_{**}$ the operator $L_h + \lambda$ is invertible.*

Proof. The statement follows by applying Proposition 2.3.3 with the choices $(\tilde{u}_h, \tilde{w}_h) = (\overline{u}_h, \overline{w}_h)$, $\tilde{c}_h = c_h$ and $M = R_4$. ∎

Proof of Theorem 2.2.2. The result follows directly from Lemma 2.6.1, Proposition 2.6.2, Proposition 2.6.3, Proposition 2.6.6 and Corollary 2.6.7. ∎

2.7 Green's functions

In order to establish the nonlinear stability of the pulse solution $(\overline{u}_h, \overline{w}_h)$, we need to consider two types of Green's functions. In particular, we first study $G_\lambda(\xi, \xi_0)$, which can roughly be seen as a solution of the equation

$$\Big[(L_h + \lambda)G_\lambda(\cdot, \xi_0)\Big](\xi) = \delta(\xi - \xi_0), \tag{2.7.1}$$

where δ is the Dirac delta-distribution. We then use these functions to build a Green's function $\mathcal{G}$ for the linearisation of the LDE (2.2.1) around the travelling pulse solution.

An important difficulty in comparison to the PDE setting is caused by the discreteness of the spatial variable j. In particular, we cannot use a frame of reference in which the solution $(\overline{u}_h, \overline{w}_h)$ is constant without changing the structure of the equation (2.2.1). The Green's function $\mathcal{G}$ will hence be the solution to a non-autonomous problem that satisfies a shift-periodicity condition. Nevertheless, one can follow the techniques in [13] to express $\mathcal{G}$ in terms of a contour integral involving the functions G_λ.

A significant part of our effort here is concerned with the construction of these latter functions. Indeed, previous approaches in [11, 109] all used exponential dichotomies or variation-of-constants formula's for MFDEs with finite-range interactions. These tools are no longer available for use in the present infinite-range setting. In particular, we construct the functions G_λ in a direct fashion using only Fredholm properties of the operators $L_h + \lambda$. This makes it somewhat involved to recover the desired exponential decay rates and to properly isolate the meromorphic terms of order $O(\lambda^{-1})$.

From now on, we will no longer explicitly use the h-dependence of our system. To simplify our notation, we fix $0 < h < h_{**}$ and write

$$\begin{array}{lllll} L & := & L_h, & & \\ L_\infty & := & L_{h;\infty}, & & \\ \overline{U} & = & (\overline{u}, \overline{w}) & := & (\overline{u}_h, \overline{w}_h), \\ \Phi^\pm & = & (\phi^\pm, \psi^\pm) & := & (\phi_h^\pm, \psi_h^\pm), \\ c & := & c_h. & & \end{array} \tag{2.7.2}$$

We emphasize that from now on all our constants may (and will) depend on h.

We will loosely follow §2 of [109], borrowing a number of results from [13, 102] at appropriate times. In particular, we start by considering the linearisation of the original LDE (2.2.1) around the travelling pulse solution $\overline{U}(t)$ given by (2.2.21). To this end, we introduce the Hilbert space

$$\mathbb{L}^2 \quad := \quad \{V \in (\mathrm{Mat}_2(\mathbb{R}))^{\mathbb{Z}} : \sum_{j\in\mathbb{Z}} |V(j)|^2 < \infty\}, \tag{2.7.3}$$

in which $\mathrm{Mat}_2(\mathbb{R})$ is the space of 2×2-matrices with real coefficients which we equip with the maximum-norm $|\cdot|$. For any $\mathcal{V} \in \mathbb{L}^2$, we often write $\mathcal{V} = \begin{pmatrix} \mathcal{V}^{(1,1)} & \mathcal{V}^{(1,2)} \\ \mathcal{V}^{(2,1)} & \mathcal{V}^{(2,2)} \end{pmatrix}$, when we need to refer to the component sequences $\mathcal{V}^{(i,j)} \in \ell^2(\mathbb{Z};\mathbb{R})$. For any $t \in \mathbb{R}$ we now introduce the linear operator $\mathcal{A}(t) : \mathbb{L}^2 \to \mathbb{L}^2$ that acts as

$$\mathcal{A}(t) \cdot \mathcal{V} = \tfrac{1}{c} \begin{pmatrix} A^{(1,1)}(t) & A^{(1,2)}(t) \\ A^{(2,1)}(t) & A^{(2,2)}(t) \end{pmatrix} \begin{pmatrix} \mathcal{V}^{(1,1)} & \mathcal{V}^{(1,2)} \\ \mathcal{V}^{(2,1)} & \mathcal{V}^{(2,2)} \end{pmatrix}, \tag{2.7.4}$$

where

$$\begin{aligned} (A^{(1,1)}(t)v)_j &= \tfrac{1}{h^2} \sum_{k>0} \alpha_k [v_{j+k} + v_{j-k} - 2v_j] + g_u\Big(\overline{u}(hj+ct)\Big)v_j \\ (A^{(1,2)}(t)w)_j &= -w_j \\ (A^{(2,1)}(t)v)_j &= \rho v_j \\ (A^{(2,2)}(t)v)_j &= -\rho\gamma w_j \end{aligned} \tag{2.7.5}$$

for $v \in \ell^2(\mathbb{Z};\mathbb{R})$ and $w \in \ell^2(\mathbb{Z};\mathbb{R})$. With all this notation in hand, we can write the desired linearisation as the ODE

$$\tfrac{d}{dt}\mathcal{V}(t) = \mathcal{A}(t) \cdot \mathcal{V}(t) \tag{2.7.6}$$

posed on $\mathbb{L}^2$.

Fix $t_0 \in \mathbb{R}$ and $j_0 \in \mathbb{Z}$. Consider the function

$$\mathbb{R} \ni t \mapsto \mathcal{G}^{j_0}(t,t_0) = \{\mathcal{G}^{j_0}_j(t,t_0)\}_{j\in\mathbb{Z}} \in \mathbb{L}^2 \tag{2.7.7}$$

that is uniquely determined by the initial value problem

$$\begin{cases} \frac{d}{dt}\mathcal{G}^{j_0}(t,t_0) & = \mathcal{A}(t) \cdot \mathcal{G}^{j_0}(t,t_0) \\ \mathcal{G}^{j_0}_j(t_0,t_0) & = \delta^{j_0}_j I. \end{cases} \tag{2.7.8}$$

Here we have introduced

$$\delta^{j_0}_j = \begin{cases} 1 \text{ if } j = j_0 \\ 0 \text{ else,} \end{cases} \tag{2.7.9}$$

where $I \in \mathrm{Mat}_2(\mathbb{R})$ is the identity matrix. We remark that $\mathcal{G}^{j_0}_j(t,t_0)$ is an element of $\mathrm{Mat}_2(\mathbb{R})$ for each $j \in \mathbb{Z}$.

This function $\mathcal{G}$ is called the Green's function for the linearisation around our travelling pulse. Indeed, the general solution of the inhomogeneous equation

$$\begin{cases} \frac{dV}{dt} & = \mathcal{A}(t) \cdot V(t) + F(t) \\ V(0) & = V^0, \end{cases} \tag{2.7.10}$$

where now $V(t) \in \ell^2(\mathbb{Z};\mathbb{R}^2) \cong \ell^2(\mathbb{Z};\mathbb{R}^{2\times 1})$ and $F(t) \in \ell^2(\mathbb{Z};\mathbb{R}^2) \cong \ell^2(\mathbb{Z};\mathbb{R}^{2\times 1})$, is given by

$$V_j(t) = \sum_{j_0\in\mathbb{Z}} \mathcal{G}^{j_0}_j(t,0)V^0_{j_0} + \int_0^t \sum_{j_0\in\mathbb{Z}} \mathcal{G}^{j_0}_j(t,t_0)F_{j_0}(t_0)\, dt_0. \tag{2.7.11}$$

Introducing the standard convolution operator $*$, this can be written in the abbreviated form

$$V = \mathcal{G}(t,0) * V^0 + \int_0^t \mathcal{G}(t,t_0) * F(t_0)\, dt_0. \qquad (2.7.12)$$

The main result of this section is the following proposition, which shows that we can decompose the Green's function $\mathcal{G}$ into a part that decays exponentially and a neutral part associated with translation along the family of travelling pulses.

Proposition 2.7.1. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. For any pair $t \geq t_0$ and any $j, j_0 \in \mathbb{Z}$, we have the representation*

$$\mathcal{G}_j^{j_0}(t,t_0) = \mathcal{E}_j^{j_0}(t,t_0) + \tilde{\mathcal{G}}_j^{j_0}(t,t_0), \qquad (2.7.13)$$

in which

$$\mathcal{E}_j^{j_0}(t,t_0) = \frac{h}{\Omega}\begin{pmatrix} \phi^-(hj_0+ct_0)\phi^+(hj+ct) & \psi^-(hj_0+ct_0)\phi^+(hj+ct) \\ \phi^-(hj_0+ct_0)\psi^+(hj+ct) & \psi^-(hj_0+ct_0)\psi^+(hj+ct) \end{pmatrix}, \qquad (2.7.14)$$

while $\tilde{\mathcal{G}}$ satisfies the found

$$|\tilde{\mathcal{G}}_j^{j_0}(t,t_0)| \leq Ke^{-\tilde{\delta}(t-t_0)}e^{-\tilde{\delta}|hj+ct-hj_0-ct_0|} \qquad (2.7.15)$$

for some $K > 0$ and $\tilde{\delta} > 0$. The constant $\Omega > 0$ is given by

$$\Omega = \langle \Phi^-, \Phi^+ \rangle. \qquad (2.7.16)$$

Furthermore, for any $t \geq t_0$ we have the representation

$$\mathcal{G}_j^{j_0}(t,t_0) = \sum_{i\in\mathbb{Z}} \Big[\mathcal{E}_j^i(t,t_0)\mathcal{E}_i^{j_0}(t_0,t_0) + \tilde{\mathcal{G}}_j^i(t,t_0)(\delta_i^{j_0} I - \mathcal{E}_i^{j_0}(t_0,t_0))\Big], \qquad (2.7.17)$$

which can be abbreviated as

$$\mathcal{G}(t,t_0) = \mathcal{E}(t,t_0) * \mathcal{E}(t_0,t_0) + \tilde{\mathcal{G}}(t,t_0) * \Big(I - \mathcal{E}(t_0,t_0)\Big). \qquad (2.7.18)$$

2.7.1 Construction of the Green's function

In this subsection, we set out to define the functions G_λ in a more rigorous fashion. In addition, we use these Green's functions to formulate a powerful representation formula for $\mathcal{G}$, see Proposition 2.7.4 below, following the approach developed in [13].

A key role in our analysis is reserved for the operator $L_{\infty;\lambda}$ and the function $\Delta_{L_{\infty;\lambda}}$ from Lemma 2.5.6. We will show that $L_{\infty;\lambda}$ has a Green's function $G_{\infty;\lambda}$ which takes values in the space $\mathrm{Mat}_2(\mathbb{R})$ and has some useful properties. To this end, we recall the constant $\tilde{\lambda}$ from Lemma 2.5.6. For each $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \geq -\frac{\tilde{\lambda}}{2}$, we may now define $G_{\infty;\lambda} : \mathbb{R} \to \mathrm{Mat}_2(\mathbb{R})$ by writing

$$G_{\infty;\lambda}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\eta\xi}(\Delta_{L_{\infty;\lambda}}(i\eta))^{-1}\, d\eta. \qquad (2.7.19)$$

We also introduce the notation

$$G_\infty = G_{\infty;0}. \qquad (2.7.20)$$

Here (Hα2) is essential to ensure that these Green's functions decay exponentially.

Lemma 2.7.2. *Assume that (HP1), (HP2), (HS), (Hα1) and (Hα2) are satisfied. Fix $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq -\frac{\tilde{\lambda}}{2}$. The function $G_{\infty;\lambda}$ is bounded and continuous on $\mathbb{R} \setminus \{0\}$ and C^1-smooth on $\mathbb{R} \setminus h\mathbb{Z}$. Furthermore, $(L_\infty + \lambda)G_{\infty;\lambda}(\cdot - \xi_0)$ is constantly zero except at $\xi = \xi_0 + h\mathbb{Z}$ and satisfies the identity*

$$\int_{-\infty}^{\infty} \Big[(L_\infty + \lambda)G_{\infty;\lambda}(\cdot - \xi_0)\Big](\xi) f(\xi)\, d\xi \quad = \quad f(\xi_0) \tag{2.7.21}$$

for all $\xi \in \mathbb{R}$ and all $f \in \boldsymbol{H}^1$.

Finally for each $\chi > 0$ there exist constants $K_* > 0$ and $\beta_* > 0$, which may depend on χ, such that for each $\lambda \in \mathbb{C}$ with $-\frac{\tilde{\lambda}}{2} \leq \operatorname{Re}\lambda \leq \chi$ and $|\operatorname{Im}\lambda| \leq \frac{\pi|c|}{h}$ we have the bound

$$|G_{\infty;\lambda}(\xi - \xi_0)| \leq K_* e^{-\beta_*|\xi - \xi_0|} \tag{2.7.22}$$

for all $\xi, \xi_0 \in \mathbb{R}$.

Pick $\lambda \in \mathbb{C} \setminus \sigma(-L)$ with $\operatorname{Re}\lambda \geq -\frac{\tilde{\lambda}}{2}$. Observe that

$$L - L_\infty \quad = \quad \begin{pmatrix} -g_u(\overline{u}) + r_0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.7.23}$$

We know that $G_{\infty;\lambda}(\cdot - \xi_0) \in L^2(\mathbb{R}, \mathrm{Mat}_2(\mathbb{R}))$ since it decays exponentially. This means that we also have the inclusion

$$[L - L_\infty]G_{\infty;\lambda}(\cdot - \xi_0) \quad \in \quad L^2(\mathbb{R}, \mathrm{Mat}_2(\mathbb{C})). \tag{2.7.24}$$

Hence, it is possible to define the function G_λ by writing

$$G_\lambda(\xi, \xi_0) \quad = \quad G_{\infty;\lambda}(\xi - \xi_0) - \Big[(\lambda + L)^{-1}[L - L_\infty]G_{\infty;\lambda}(\cdot - \xi_0)\Big](\xi). \tag{2.7.25}$$

The next result shows that G_λ can be interpreted as the Green's function of $L + \lambda$. It is based on [109, Lem. 2.6].

Lemma 2.7.3. *Assume that (HP1), (HP2), (HS), (Hα1) and (Hα2) are satisfied. For $\lambda \in \mathbb{C} \setminus \sigma(-L)$ with $\operatorname{Re}\lambda \geq -\frac{\tilde{\lambda}}{2}$ we have that $G_\lambda(\cdot, y)$ is continuous on $\mathbb{R} \setminus \{y\}$ and C^1-smooth on $\mathbb{R} \setminus \{y + kh : k \in \mathbb{Z}\}$. Furthermore, it satisfies*

$$\int_{-\infty}^{\infty} \Big[(\lambda + L)G_\lambda(\cdot, \xi_0)\Big](\xi) f(\xi)\, d\xi \quad = \quad f(\xi_0) \tag{2.7.26}$$

for all $\xi \in \mathbb{R}$ and all $f \in \boldsymbol{H}^1$.

The link between our two types of Green's functions is provided by the following key result. It is based on [13, Thm. 4.2], where it was used to study one-sided spatial discretisation schemes for systems with conservation laws.

Proposition 2.7.4. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. Let $\chi > \lambda_{\mathrm{unif}}$ be given, where λ_{unif} is as in Lemma 2.7.7. For all $t \geq t_0$ the Green's function $\mathcal{G}_j^{j_0}(t, t_0)$ of (2.7.8) is given by*

$$\mathcal{G}_j^{j_0}(t, t_0) \quad = \quad -\frac{h}{2\pi i} \int_{\chi - \frac{i\pi c}{h}}^{\chi + \frac{i\pi c}{h}} e^{\lambda(t - t_0)} G_\lambda(hj + ct, hj_0 + ct_0)\, d\lambda \tag{2.7.27}$$

where G_λ is the Green's function of $\lambda + L$ as defined in (2.7.25).

Our first task is to collect several basic facts concerning the operators L_h and L_∞ that will allow us to establish Lemma's 2.7.2 and 2.7.3. In particular, we need to isolate and explicitly compute the part of the Fourier integral (2.7.19) that behave as $|\eta|^{-1}$ and $|\eta|^{-2}$ as $\eta \to \pm\infty$, as these lead to the discontinuities in $G_{\infty;\lambda}$ and its derivative.

Lemma 2.7.5. *Assume that (Hα1) and (Hα2) are satisfied. Consider any bounded function $f : \mathbb{R} \to \mathbb{R}$ which is continuous everywhere except at some $\xi_0 \in \mathbb{R}$. Then $\Delta_h f$ is continuous everywhere except at $\{\xi_0 + hk : k \in \mathbb{Z}\}$. Moreover, if f is differentiable except at ξ_0 and f' is bounded, then $\Delta_h f$ is differentiable everywhere except at $\{\xi_0 + hk : k \in \mathbb{Z}\}$ and $[\Delta_h f]'(\xi) = [\Delta_h f'](\xi)$.*

Proof. For convenience we set $\xi_0 = 0$. Pick $\xi \in \mathbb{R}$ with $\xi \notin \{kh : k \in \mathbb{Z}\}$. Then f is continuous in each point $\xi + kh$ for $k \in \mathbb{Z}$. Choose $\varepsilon > 0$. Since f is bounded and $\sum_{j=1}^{\infty} |\alpha_j| < \infty$, we can pick $K > 0$ in such a way that

$$2\|f\|_\infty \tfrac{1}{h^2} \sum_{j=K}^{\infty} |\alpha_j| \quad < \quad \tfrac{\varepsilon}{2}. \tag{2.7.28}$$

For $j \in \{1, ..., K-1\}$ we can pick $\delta_j > 0$ in such a way that

$$\tfrac{1}{h^2} |\alpha_j| \Big| f(\xi + y + hj) - f(\xi + hj) \Big| \quad < \quad \tfrac{\varepsilon}{2^{K+1}} \tag{2.7.29}$$

for all $y \in \mathbb{R}$ with $|y| < \delta_j$. Let $\delta = \min\{\delta_j : 1 \le j < K\} > 0$. Then for $y \in \mathbb{R}$ with $|y| < \delta$ we obtain

$$\begin{array}{lcl} |\Delta_h f(\xi + y) - \Delta_h f(\xi)| & \le & \tfrac{1}{h^2} \sum_{j=K}^{\infty} |\alpha_j| \Big(|f(\xi + y + jh)| + |f(\xi + jh)| \Big) \\ & & \quad + \tfrac{1}{h^2} \sum_{j=1}^{K-1} |\alpha_j| \Big| f(\xi + y + jh) - f(\xi + jh) \Big| \\ & \le & \tfrac{2}{h^2} \sum_{j=K}^{\infty} |\alpha_j| \|f\|_\infty + \sum_{j=1}^{K-1} \tfrac{\varepsilon}{2^{K+1}} \\ & < & \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} \\ & = & \varepsilon. \end{array} \tag{2.7.30}$$

So $\Delta_h f$ is continuous outside of $\{kh : k \in \mathbb{Z}\}$.

Writing

$$f_n(\xi) \quad = \quad \tfrac{1}{h^2} \sum_{j=1}^{n} \alpha_j \Big[f(\xi + hj) + f(\xi - hj) - 2f(\xi) \Big] \tag{2.7.31}$$

for $n \in \mathbb{Z}_{>0}$, we can compute

$$f_n'(\xi) \quad = \quad \tfrac{1}{h^2} \sum_{j=1}^{n} \alpha_j \Big[f'(\xi + hj) + f'(\xi - hj) - 2f'(\xi) \Big]. \tag{2.7.32}$$

This allows us to estimate

$$|f_n'(\xi) - (\Delta_h f')(\xi)| \quad \le \quad \tfrac{1}{h^2} \sum_{j=n+1}^{\infty} |\alpha_j| 4 \|f'\|_\infty. \tag{2.7.33}$$

In particular, the sequence $\{f_n'\}$ converges uniformly to $\Delta_h f'$ from which it follows that

$$\begin{aligned}(\Delta_h f)'(\xi) &= \tfrac{1}{h^2}\sum_{j=1}^{\infty}\alpha_j\Big[f'(\xi+hj)+f'(\xi-hj)-2f'(\xi)\Big] \\ &= (\Delta_h f')(\xi).\end{aligned} \tag{2.7.34}$$

■

Proof of Lemma 2.7.2. Pick $\chi > 0$ and set

$$R = \{\lambda \in \mathbb{C} : -\tfrac{\tilde{\lambda}}{2} \le \operatorname{Re}\lambda \le \chi \text{ and } |\operatorname{Im}\lambda| \le \tfrac{\pi|c|}{h}\}. \tag{2.7.35}$$

The proof of Lemma 2.5.7 implies that we can choose $\beta_* > 0$ and $K_* > 0$ in such a way that

$$\|\Delta_{L_{\infty;\lambda}}(z)^{-1}\| \le \tfrac{K_*}{1+|\operatorname{Im} z|} \tag{2.7.36}$$

for all $\lambda \in R$ and all $z \in \mathbb{C}$ with $|\operatorname{Re} z| \le 2\beta_*$. In particular, it follows that $(y \mapsto \Delta_{L_{\infty;\lambda}}(iy)^{-1}) \in L^2(\mathbb{R})$. By the Plancherel Theorem it follows that $G_{\infty;\lambda}$ is a well-defined function in $L^2(\mathbb{R})$. In particular, it is bounded. Shifting the integration path in (2.7.19) in the standard fashion described in [103, 130], we obtain the bound

$$|G_{\infty;\lambda}(\xi-\xi_0)| \le K_* e^{-\beta_*|\xi-\xi_0|} \tag{2.7.37}$$

for all $\xi, \xi_0 \in \mathbb{R}$ and $\lambda \in R$.

We loosely follow the approach of [102, §5.1], which considers a similar setting for Green's functions for Banach space-valued operators with finite range interactions. Pick $\lambda \in R$. We rewrite the definition of $\Delta_{L_{\infty;\lambda}}$ given in (2.5.18) in the more general form

$$\tfrac{1}{c}\Delta_{L_{\infty;\lambda}}(z) = z - B_{\infty;\lambda}e^{z\cdot}, \tag{2.7.38}$$

For $\alpha \in \mathbb{R}$ close to 0 we introduce the expression $\mathcal{R}_{L_{\infty;\lambda};\alpha}$ by

$$\mathcal{R}_{L_{\infty;\lambda};\alpha}(z) = c\Delta_{L_{\infty;\lambda}}(z)^{-1} - \tfrac{1}{z-\alpha} - \tfrac{B_{\infty;\lambda}e^{z\cdot}-\alpha}{(z-\alpha)^2} \tag{2.7.39}$$

for $z \in \mathbb{C}$ unequal to α and $|\operatorname{Re} z| \le 2\beta_*$. Since we can compute

$$\begin{aligned} c\Delta_{L_{\infty;\lambda}}(z)^{-1} &= \Big[z-\alpha+\big(\alpha - B_{\infty;\lambda}e^{z\cdot}\big)\Big]^{-1} \\ &= (z-\alpha)^{-1}\Big[1+(z-\alpha)^{-1}\big(\alpha-B_{\infty;\lambda}e^{z\cdot}\big)\Big]^{-1} \\ &= (z-\alpha)^{-1}\Big[1-(z-\alpha)^{-1}\big(\alpha-B_{\infty;\lambda}e^{z\cdot}\big)+\mathcal{O}((z-\alpha)^{-2})\Big],\end{aligned} \tag{2.7.40}$$

we obtain the estimate

$$|\mathcal{R}_{L_{\infty;\lambda};\alpha}(iy)| \le \tfrac{K_*}{1+|y|^3}, \tag{2.7.41}$$

for all $y \in \mathbb{R}$, possibly after increasing K_*.

Exploiting the decomposition (2.7.39), we write

$$G_{\infty;\lambda} = \tfrac{1}{c}\mathcal{M}_\alpha + \tfrac{1}{c}\mathcal{R}_\alpha, \tag{2.7.42}$$

where we have introduced

$$\begin{array}{rcl}\mathcal{M}_\alpha(\xi) & = & \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\eta\xi}\Big(\frac{1}{i\eta-\alpha} - \frac{B_{\infty;\lambda}e^{i\eta\cdot}-\alpha}{(i\eta-\alpha)^2}\Big)\, d\eta, \\ \mathcal{R}_\alpha(\xi) & = & \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\eta\xi}\mathcal{R}_{L_{\infty;\lambda};\alpha}(i\eta)\, d\eta\end{array} \tag{2.7.43}$$

for any $\alpha \in \mathbb{R}\setminus\{0\}$ and $\xi \in \mathbb{R}$. Using [102, Lem. 5.8] we can explicitly compute

$$\mathcal{M}_\alpha(\xi) \;=\; -e^{\alpha\xi}H(-\xi) - \Big[B_{\infty;\lambda} - \alpha\Big]\Big(\cdot\, e^{\alpha\cdot}H(-\cdot)\Big)(\xi), \tag{2.7.44}$$

where we have introduced the Heaviside function H as

$$H(\xi) \;=\; \begin{cases} I, & \xi > 0 \\ \frac{1}{2}I, & \xi = 0 \\ 0, & \xi < 0. \end{cases} \tag{2.7.45}$$

Since $\xi \mapsto \xi e^{\alpha\xi}H(-\xi)$ is continuous everywhere and differentiable outside of $\xi = 0$, Lemma 2.7.5 implies that $\mathcal{M}_\alpha$ is continuous everywhere outside of $\xi = 0$ and differentiable outside of $\{hk : k \in \mathbb{Z}\}$. Moreover, we have the jump discontinuity

$$\mathcal{M}_\alpha(0^+) - \mathcal{M}_\alpha(0^-) \;=\; I \tag{2.7.46}$$

and we can easily compute

$$\mathcal{M}_\alpha'(\xi) \;=\; \alpha\mathcal{M}_\alpha(\xi) - [B_{\infty;\lambda} - \alpha]\Big[e^{\alpha\cdot}H(-\cdot)\Big](\xi), \tag{2.7.47}$$

from which it follows that

$$\begin{array}{rcl}\frac{1}{c}L_{\infty;\lambda}\mathcal{M}_\alpha(\xi) & = & \mathcal{M}_\alpha'(\xi) - B_{\infty;\lambda}\mathcal{M}_\alpha(\xi) \\ & = & -\alpha e^{\alpha\xi}H(-\xi) - \alpha\Big[B_{\infty;\lambda} - \alpha\Big]\Big(\cdot\, e^{\alpha\cdot}H(-\cdot)\Big)(\xi) \\ & & \quad -[B_{\infty;\lambda} - \alpha]\Big[e^{\alpha\cdot}H(-\cdot)\Big](\xi) + B_{\infty;\lambda}\Big[e^{\alpha\cdot}H(-\cdot)\Big](\xi) \\ & & \quad +B_{\infty;\lambda}\Big[[B_{\infty;\lambda} - \alpha]\Big(\cdot\, e^{\alpha\cdot}H(-\cdot)\Big)(*)\Big](\xi) \\ & = & [B_{\infty;\lambda} - \alpha]\Big[[B_{\infty;\lambda} - \alpha]\Big(\cdot\, e^{\alpha\cdot}H(-\cdot)\Big)(*)\Big](\xi).\end{array} \tag{2.7.48}$$

Since $\mathcal{R}_{L_{\infty;\lambda};\alpha} \in L^1(\mathbb{R})$ we see that $\mathcal{R}_\alpha$ is continuous. Therefore, $G_{\infty;\lambda}$ is continuous outside of $\xi = 0$. Similarly to [102, Eq. (5.79)] we observe that

$$\frac{1}{c}\Delta_{L_{\infty;\lambda}}(z)\mathcal{R}_{L_{\infty;\lambda};\alpha}(z) \;=\; \frac{(B_{\infty;\lambda}e^{z\cdot}-\alpha)^2}{(z-\alpha)^2}, \tag{2.7.49}$$

which yields

$$\begin{array}{rcl}\frac{1}{c}L_{\infty;\lambda}\mathcal{R}_\alpha(\xi) & = & \mathcal{R}_\alpha'(\xi) - B_{\infty;\lambda}\mathcal{R}_\alpha(\xi) \\ & = & \frac{1}{2\pi c}\int_{-\infty}^{\infty} e^{i\xi y}\Delta_{L_{\infty;\lambda}}(iy)\mathcal{R}_{L_{\infty;\lambda};\alpha}(iy)dy \\ & = & \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi y}\frac{(B_{\infty;\lambda}e^{iy\cdot}-\alpha)^2}{(iy-\alpha)^2}dy \\ & = & -[B_{\infty;\lambda} - \alpha]\Big[[B_{\infty;\lambda} - \alpha]\Big(\cdot\, e^{\alpha\cdot}H(-\cdot)\Big)(*)\Big](\xi),\end{array} \tag{2.7.50}$$

using [102, Lem. 5.8]. In particular, we see that

$$L_{\infty;\lambda}G_{\infty;\lambda}(\xi) \quad = \quad 0 \tag{2.7.51}$$

for all ξ outside of $\{hk : k \in \mathbb{Z}\}$. Lemma 2.7.5 subsequently shows that $G_{\infty;\lambda}$ is C^1-smooth outside of $\{hk : k \in \mathbb{Z}\}$.

Fix $f \in \mathbf{H}^1$. For any $\delta > 0$ we may compute

$$\begin{array}{lcl} 0 & = & \int_\delta^\infty \Big[L_{\infty;\lambda}G_{\infty;\lambda}(\cdot)\Big](\xi)f(\xi)d\xi \\ & = & \Big[cG_{\infty;\lambda}f\Big]_\delta^\infty - \int_\delta^\infty cG_{\infty;\lambda}(\xi)f'(\xi) + [cB_{\infty;\lambda}G_{\infty;\lambda}](\xi)f(\xi), \end{array} \tag{2.7.52}$$

together with

$$0 \quad = \quad \Big[cG_{\infty;\lambda}f\Big]_{-\infty}^{-\delta} - \int_{-\infty}^{-\delta} cG_{\infty;\lambda}(\xi)f'(\xi) + [cB_{\infty;\lambda}G_{\infty;\lambda}](\xi)f(\xi). \tag{2.7.53}$$

Using (2.7.46) we can hence compute

$$\begin{array}{lcl} \int_{-\infty}^\infty \Big[L_{\infty;\lambda}G_{\infty;\lambda}(\cdot)\Big](\xi)f(\xi)d\xi & = & \lim_{\delta\downarrow 0} \Big[cG_{\infty;\lambda}f\Big]_\delta^\infty - \Big[cG_{\infty;\lambda}f\Big]_{-\infty}^{-\delta} \\ & = & f(0). \end{array} \tag{2.7.54}$$

■

Lemma 2.7.6. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. Fix* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda \geq -\frac{\tilde{\lambda}}{2}$. *Then there exist constants* $K > 0$ *and* $\beta > 0$ *so that for any* $g \in \mathbf{L}^2$ *and* $f \in \mathbf{H}^1$ *that satisfy* $(L + \lambda)f = g$, *the pointwise bound*

$$|f(\xi)| \quad \leq \quad Ke^{-\alpha|\xi|}\|f\|_\infty + K\int_{-\infty}^\infty e^{-|\eta-\xi|}g(\eta)d\eta \tag{2.7.55}$$

holds for all $\xi \in \mathbb{R}$.

Proof. On account of Lemma 2.7.2 we can lift the results from [130, Prop. 5.2-5.3] to our current infinite range setting. The proof of these results are identical, since the estimate [130, Eq. (5.4)] still holds in our setting on account of (Hα2). A more detailed description for this procedure can be found in [20, Lem. 4.1-Lem. 4.3]. ■

Proof of Lemma 2.7.3. Pick $\lambda \in \mathbb{C} \setminus \sigma(-L)$ and compute

$$\begin{array}{lcl} (\lambda + L)G_\lambda(\cdot, \xi_0) & = & (\lambda + L)G_{\infty;\lambda}(\cdot - \xi_0) - [L - L_\infty]G_{\infty;\lambda}(\cdot - \xi_0) \\ & = & (\lambda + L_\infty)G_{\infty;\lambda}(\cdot - \xi_0). \end{array} \tag{2.7.56}$$

The last statement follows immediately from this identity.

Write

$$\hat{G}_{\infty;\lambda}(\cdot - \xi_0) \quad = \quad [L - L_\infty]G_{\infty;\lambda}(\cdot - \xi_0). \tag{2.7.57}$$

We have already seen that $\hat{G}_{\infty;\lambda}(\cdot - \xi_0) \in L^2(\mathbb{R}, \mathrm{Mat}_2(\mathbb{C}))$. Hence, it follows that

$$(\lambda + L)^{-1}\hat{G}_{\infty;\lambda}(\cdot - \xi_0) \quad \in \quad H^1(\mathbb{R}, \mathrm{Mat}_2(\mathbb{C})). \tag{2.7.58}$$

In particular, this function is continuous. Together with Lemma 2.7.2 we obtain that $G_\lambda(\cdot, \xi_0)$ is continuous on $\mathbb{R} \setminus \{\xi_0\}$.

Set $H = (\lambda + L)^{-1}\hat{G}_{\infty;\lambda}$ and write $H = \begin{pmatrix} H^{(1,1)} & H^{(1,2)} \\ H^{(2,1)} & H^{(2,2)} \end{pmatrix}$. Using the definition of L we see that

$$c\tfrac{d}{d\xi}H \quad = \quad -\lambda H - \hat{G}_\infty - \tilde{H}, \tag{2.7.59}$$

where

$$\tilde{H} \quad = \quad -\begin{pmatrix} -\Delta_h H^{(1,1)} - g_u(\overline{u})H^{(1,1)} + H^{(2,1)} & \Delta_h H^{(1,2)} - g_u(\overline{u})H^{(1,2)} + H^{(2,2)} \\ -\rho H^{(1,1)} + \gamma\rho H^{(2,1)} & -\rho H^{(1,2)} + \gamma\rho H^{(2,2)} \end{pmatrix}. \tag{2.7.60}$$

Since $\overline{u}' \in H^1$ and, hence, $\overline{u}'$ is continuous, we must have that $\overline{u}$ is continuous. As argued before $\Delta_h H^{(1,1)}$ and $\Delta_h H^{(1,2)}$ are also continuous. Hence, we see that $c\frac{d}{d\xi}H$ is continuous on $\mathbb{R} \setminus \{\xi_0\}$ and thus that $\frac{d}{d\xi}H$ is continuous on $\mathbb{R} \setminus \{\xi_0\}$. Therefore, we obtain that $G_\lambda(\cdot, \xi_0)$ is C^1-smooth on $\mathbb{R} \setminus \{\xi_0 + kh : k \in \mathbb{Z}\}$. ∎

We now proceed to the verification of the integral representation (2.7.27). As a preparation, we need to show that whenever λ has a sufficiently large real part, the function G_λ is bounded uniformly by a constant. This result is based on [13, Lem. 4.1].

Lemma 2.7.7. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. Then there exist constants K and λ_{unif} so that the Green's function G_λ enjoys the uniform estimate*

$$|G_\lambda(\xi, \xi_0)| \quad \le \quad K, \tag{2.7.61}$$

for all $\xi, \xi_0 \in \mathbb{R}$, whenever $\mathrm{Re}\,\lambda > \lambda_{\mathrm{unif}}$.

Proof. We write $L = c\frac{d}{d\xi} + B$ with

$$B \quad = \quad \begin{pmatrix} -\Delta_h - g_u(\overline{u}) & 1 \\ -\rho & \gamma\rho \end{pmatrix}. \tag{2.7.62}$$

We introduce G^0_λ as the Green's function of $(\lambda + c\frac{d}{d\xi})$ viewed as a map from $\mathbf{H}^1$ to $\mathbf{L}^2$. Luckily, it is well-known that this Green's function admits the estimate

$$|G^0_\lambda(\xi, \xi_0)| \quad \le \quad \tfrac{1}{|c|}e^{-\mathrm{Re}\,\lambda|\xi - \xi_0|/|c|}. \tag{2.7.63}$$

We can look for the Green's function G_λ as the solution of the fixed point problem

$$G_\lambda(\xi, \xi_0) \quad = \quad G^0_\lambda(\xi, \xi_0) + \textstyle\int_{\mathbb{R}} G_\lambda(\xi, z)(BG^0_\lambda)(z, \xi_0)dz. \tag{2.7.64}$$

Since $\lambda + L$ is invertible by Theorem 2.2.2, G_λ must necessarily satisfy the fixed point problem (2.7.64).

For a matrix $A \in \mathrm{Mat}_2(\mathbb{C})$ we write $A = \begin{pmatrix} A^{(1,1)} & A^{(1,2)} \\ A^{(2,1)} & A^{(2,2)} \end{pmatrix}$. We make the decomposition

$$B = B_0 + B_1, \tag{2.7.65}$$

where

$$\begin{aligned} B_0 &= \begin{pmatrix} -\Delta_h & 0 \\ 0 & 0 \end{pmatrix}, \\ B_1 &= \begin{pmatrix} -g_u(\overline{u}) & 1 \\ -\rho & \gamma\rho \end{pmatrix}. \end{aligned} \tag{2.7.66}$$

We estimate

$$\begin{aligned} |(B_0 G^0_\lambda)(\xi,\xi_0)| &= |\Delta_h G^0_\lambda(\xi,\xi_0)^{(1,1)}| \\ &\leq \sum_{j=1}^{\infty} \Big[\tfrac{1}{h^2}|\alpha_j| \Big(|G^0_\lambda(\xi+hj,\xi_0)^{(1,1)}| + |G^0_\lambda(\xi-hj,\xi_0)^{(1,1)}| \\ &\qquad + 2|G^0_\lambda(\xi,\xi_0)^{(1,1)}| \Big)\Big] \\ &\leq \tfrac{1}{|c|}\sum_{j=1}^{\infty} \Big[\tfrac{1}{h^2}|\alpha_j| \Big(e^{-\mathrm{Re}\,\lambda|\xi+hj-\xi_0|/|c|} + e^{-\mathrm{Re}\,\lambda|\xi-hj-\xi_0|/|c|} \\ &\qquad + 2e^{-\mathrm{Re}\,\lambda|\xi-\xi_0|/|c|} \Big)\Big] \end{aligned} \tag{2.7.67}$$

and observe that

$$\begin{aligned} \int_{\mathbb{R}} |(B_0 G^0_\lambda)(\xi,\xi_0)|\, d\xi &\leq \tfrac{1}{|c|}\left(\sum_{j=1}^{\infty} 4\Big[\tfrac{1}{h^2}|\alpha_j|\tfrac{1}{\mathrm{Re}\,\lambda/|c|}\Big]\right) \\ &= \tfrac{4}{h^2 \mathrm{Re}\,\lambda}\sum_{j=1}^{\infty}|\alpha_j|. \end{aligned} \tag{2.7.68}$$

We now fix $G \in L^\infty(\mathbb{R}^2, \mathrm{Mat}_2(\mathbb{C}))$ and consider the expressions

$$\begin{aligned} \mathcal{I}_0 &= \int_{\mathbb{R}} \Big[G(\xi,z)(B_0 G^0_\lambda)(z,\xi_0)\Big]^{(1,1)} dz, \\ \mathcal{I}_1 &= \int_{\mathbb{R}} \Big[G(\xi,z)(B_1 G^0_\lambda)(z,\xi_0)\Big]^{(1,1)} dz. \end{aligned} \tag{2.7.69}$$

Using Fubini's theorem for positive functions to switch the integral and the sum, we obtain the estimates

$$\begin{aligned} |\mathcal{I}_0| &\leq \|G\|_{L^\infty} \int_{\mathbb{R}} |(B_0 G^0_\lambda)(z,\xi_0)|\, dz \\ &\leq \|G\|_{L^\infty} \tfrac{4}{h^2 \mathrm{Re}\,\lambda}\sum_{j=1}^{\infty}|\alpha_j| \end{aligned} \tag{2.7.70}$$

and

$$\begin{array}{rcl}|\mathcal{I}_1| & \leq & \|G\|_{L^\infty} \int_{\mathbb{R}} \Big(|g_u(\overline{u}(z))||G^0_\lambda(z,\xi_0)^{(1,1)}| + \rho |G^0_\lambda(z,\xi_0)^{(1,1)}| \\ & & \quad +(1+\gamma\rho)|G^0_\lambda(z,\xi_0)^{(2,1)}| \Big) dz \\ & \leq & \|G\|_{L^\infty} \frac{1}{|c|} \int_{\mathbb{R}} \Big(\Big(|g_u(\overline{u}(z))| + \rho + 1 + \gamma\rho \Big) e^{-\operatorname{Re}\lambda |z-\xi_0|/|c|} \Big) dz \\ & \leq & \|G\|_{L^\infty} \frac{1}{|c|} \Big(\|g_u(\overline{u})\|_{L^\infty} + \rho + 1 + \gamma\rho \Big) \left(\frac{1}{\operatorname{Re}\lambda/|c|} \right) \\ & \leq & \|G\|_{L^\infty} \Big(g_* + \rho + 1 + \gamma\rho \Big) \left(\frac{1}{\operatorname{Re}\lambda} \right).\end{array} \tag{2.7.71}$$

Similar estimates hold for the other components of $\int_{\mathbb{R}} G(\xi,z)(BG^0_\lambda)(z,\xi_0)dz$. Therefore, the mapping $G \mapsto \int_{\mathbb{R}} G(\xi,z)(BG^0_\lambda)(z,\xi_0)dz$ is a contraction in $L^\infty(\mathbb{R}^2, \mathrm{Mat}_2(\mathbb{C}))$ for $\operatorname{Re}\lambda > \lambda_{\mathrm{unif}}$ for λ_{unif} large enough, with λ_{unif} possibly dependent of $h \in (0, h_{**})$. Hence, we get a unique bounded solution of (2.7.64), which must be G_λ. The desired bound on G_λ is now immediate. ∎

Proof of Proposition 2.7.4. Fix $j_0 \in \mathbb{Z}$ and $t_0 \in \mathbb{R}$. Since (2.7.8) is merely a linear ODE in the Banach space $\mathbb{L}^2$, it follows from the Cauchy-Lipschitz theorem that (2.7.8) indeed has a unique solution $\mathcal{V} : [t_0, \infty) \to \mathbb{L}^2$. For any $Z \in C^\infty_c(\mathbb{R}; \mathbb{L}^2)$, an integration by parts yields

$$\begin{array}{rcl}-Z_{j_0}(t_0) & = & \int_{t_0}^\infty \sum\limits_{j\in\mathbb{Z}} \Big[\Big(\frac{d\mathcal{V}_j}{dt}(t) - (\mathcal{A}(t)\cdot\mathcal{V}(t))_j \Big) Z_j(t) \Big] dt - \sum_{j\in\mathbb{Z}} \mathcal{V}_j(t_0) Z_j(t_0) \\ & = & \int_{t_0}^\infty \sum\limits_{j\in\mathbb{Z}} \Big[-\frac{dZ_j}{dt}(t)\mathcal{V}_j(t) - (\mathcal{A}(t)\cdot\mathcal{V})_j(t) Z_j(t) \Big] dt.\end{array} \tag{2.7.72}$$

We want to show that the function $V_j(t) := \mathcal{G}^{j_0}_j(t,t_0)$ defined by (2.7.27) coincides with $\mathcal{V}$ on $[t_0,\infty)$. To accomplish this, we define

$$I = \int_{t_0}^\infty \sum_{j\in\mathbb{Z}} \Big[-\frac{dZ_j}{dt}(t)V_j(t) - (\mathcal{A}(t)\cdot V(t))_j Z_j(t) \Big] dt \tag{2.7.73}$$

and show that V is a weak solution to (2.7.8) in the sense that

$$I = -Z_{j_0}(t_0) \tag{2.7.74}$$

holds for all $Z \in C^\infty_c(\mathbb{R}; \mathbb{L}^2)$. Indeed, the uniqueness of weak solutions then implies that $V = \mathcal{V}$.

Note first that $V(t) = 0$ for $t < t_0$, which can be seen by using (2.7.61) and taking $\chi \to \infty$ in (2.7.27). We write $y = hj_0 + ct_0$, $\chi_- = \chi - \frac{i\pi c}{h}$ and $\chi_+ = \chi + \frac{i\pi c}{h}$. We see that

$$I = \int_{-\infty}^{\infty} \sum_{j\in\mathbb{Z}} \Big[-\frac{dZ_j}{dt}(t)V_j(t) - (\mathcal{A}(t)\cdot V(t))_j Z_j(t) \Big] dt, \tag{2.7.75}$$

since $V(t) = 0$ for $t < t_0$. Moreover, we write

$$\mathbb{G}_j(t) = G_\lambda(hj + ct, y). \tag{2.7.76}$$

Using our definition of $V(t)$, we have

$$I = -\frac{h}{2\pi i}\int_{\chi_-}^{\chi_+}\sum_{j\in\mathbb{Z}}\Big[\int_{-\infty}^{\infty}\mathcal{I}_j(t,\lambda)dt\Big]d\lambda, \tag{2.7.77}$$

where

$$\mathcal{I}_j(t,\lambda) = e^{\lambda(t-t_0)}\Big[-\mathbb{G}_j(t)\frac{dZ_j}{dt}(t) - (\mathcal{A}(t)\cdot\mathbb{G}(t))_j Z_j(t)\Big]. \tag{2.7.78}$$

The permutation of the summations and integrations is allowed by Lebesgue's theorem, because Z and $\frac{dZ}{dt}$ are compactly supported and G_λ is uniformly bounded by (2.7.61). Fix $\chi_- \le \lambda \le \chi_+$ and $j \in \mathbb{Z}$. Using the change of variable $x = hj + ct$ we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}\mathcal{I}_j(t,\lambda)dt &= \frac{1}{c}\int_{x=-\infty}^{x=\infty}\Big[-c\mathbb{G}_j\Big(\frac{x-hj}{c}\Big)\frac{d\mathcal{Z}_j}{dx} + \lambda\mathbb{G}_j\Big(\frac{x-hj}{c}\Big)\mathcal{Z}_j(x,\lambda)\\ &\quad -\Big(\mathcal{A}\Big(\frac{x-hj}{c}\Big)\cdot\mathbb{G}\Big(\frac{x-hj}{c}\Big)\Big)_j\mathcal{Z}_j(x,\lambda)\Big]dx,\end{aligned} \tag{2.7.79}$$

where

$$\mathcal{Z}_j(x,\lambda) = e^{\lambda((x-hj)/c-t_0)}Z_j\Big(\frac{x-hj}{c}\Big). \tag{2.7.80}$$

Exploiting the fact that Z_j and, therefore, $\mathcal{Z}_j$ is compactly supported, (2.7.26) yields

$$\begin{aligned}\int_{-\infty}^{\infty}\mathcal{I}_j(t,\lambda)dt &= \frac{1}{c}\int_{-\infty}^{\infty}[(L+\lambda)G_\lambda(x,y)\mathcal{Z}_j(x,\lambda)]dx\\ &= \frac{1}{c}\mathcal{Z}_j(y).\end{aligned} \tag{2.7.81}$$

Now since Z_j is compactly supported, we can exchange sums and integrals in equation (2.7.77). This allows us to compute

$$\begin{aligned}I &= -\frac{h}{2\pi i}\frac{1}{c}\int_{\chi_-}^{\chi_+}\sum_{j\in\mathbb{Z}}\int_{-\infty}^{\infty}\mathcal{I}_j(t,\lambda)dtd\lambda\\ &= -\frac{h}{2\pi i}\frac{1}{c}\int_{\chi_-}^{\chi_+}\sum_{j\in\mathbb{Z}}\mathcal{Z}_j(y,\lambda)d\lambda\\ &= -\frac{h}{2\pi ic}\int_{\chi_-}^{\chi_+}\sum_{j\in\mathbb{Z}}e^{\lambda\frac{(hj_0-hj)}{c}}Z_j\Big(\frac{(hj_0-hj)}{c}+t_0\Big)d\lambda\\ &= -\frac{h}{2\pi ic}\sum_{j\in\mathbb{Z}}\int_{\chi_-}^{\chi_+}e^{\lambda\frac{(hj_0-hj)}{c}}Z_j\Big(\frac{(hj_0-hj)}{c}+t_0\Big)d\lambda\\ &= -\frac{h}{2\pi ic}\sum_{j\in\mathbb{Z}}2\frac{\pi ic}{h}\delta_j^{j_0}Z_j\Big(\frac{(hj_0-hj)}{c}+t_0\Big)\\ &= -Z_{j_0}(t_0),\end{aligned} \tag{2.7.82}$$

as desired. ■

2.7.2 Meromorphic expansion of G_λ

In this subsection we set out to explicitly isolate the pole at $\lambda = 0$ in the meromorphic expansion of G_λ. In addition, we show that both parts of this decomposition decay exponentially in a λ-uniform fashion. This will allow us to shift the integration path in (2.7.27) to the left of the imaginary axis. The decomposition (2.7.13) for the Green's function $\mathcal{G}$ together with the exponential decay estimates (2.7.15) can subsequently be read off from the shifted contour integral.

Lemma 2.7.8. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. There exist constants $K_1 > 0, K_2 > 0, \delta > 0$ and $\tilde{\delta} > 0$ such that*

$$\begin{array}{rcl} |\Phi^+(\xi)| & \leq & K_1 e^{-\delta|\xi|}\|\Phi^+\|_\infty, \\ |\Phi^-(\xi)| & \leq & K_2 e^{-\tilde{\delta}|\xi|}\|\Phi^-\|_\infty \end{array} \tag{2.7.83}$$

for all $\xi \in \mathbb{R}$.

Proof. We obtain from Lemma 2.7.6 that there are constants $\delta > 0$ and $K_1 > 0$ for which

$$|\Psi(\xi)| \quad \leq \quad K_1 e^{-\delta|\xi|}\|\Psi\|_\infty + K_1 \int_{-\infty}^{\infty} e^{-\delta|\xi-\eta|}|\Theta(\eta)|d\eta \tag{2.7.84}$$

holds for each $\Psi \in \mathbf{H}^1$, where $\Theta = L\Psi$. Since $L\Phi^+ = 0$ we conclude that

$$|\Phi^+(\xi)| \quad \leq \quad K_1 e^{-\delta|\xi|}\|\Phi^+\|_\infty \tag{2.7.85}$$

for all ξ. Note that the operator L^* is also asymptotically hyperbolic. Hence, there are $\tilde{\delta} > 0$ and $K_2 > 0$ for which

$$|\Psi(\xi)| \quad \leq \quad K_2 e^{-\tilde{\delta}|\xi|}\|\Psi\|_\infty + K_2 \int_{-\infty}^{\infty} e^{-\tilde{\delta}|\xi-\eta|}|\Theta(\eta)|d\eta \tag{2.7.86}$$

holds for each $\Psi \in \mathbf{H}^1$, where $\Theta = L^*\Psi$. Since $L^*\Phi^- = 0$ we obtain that

$$|\Phi^-(\xi)| \quad \leq \quad K_2 e^{-\tilde{\delta}|\xi|}\|\Phi^-\|_\infty \tag{2.7.87}$$

for all ξ. ∎

Lemma 2.7.9. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. Then there exist constants $K_3 > 0$ and $\delta > 0$ such that*

$$|(\Phi^\pm)'(\xi)| \quad \leq \quad K_3 e^{-\delta|\xi|} \tag{2.7.88}$$

for all $\xi \in \mathbb{R}$.

Proof. Lemma 2.7.8 implies that

$$\begin{array}{rcl} |\Delta_h \phi^+(\xi)| & \leq & \frac{1}{h^2} K_1 \sum_{k>0} |\alpha_k|(e^{-\delta|\xi+hk|} + e^{-\delta|\xi-hk|} + 2e^{-\delta|\xi|}) \\ & \leq & K_1 e^{-\delta|\xi|}(\frac{1}{h^2} \sum_{k>0} |\alpha_k|(2e^{\delta hk} + 2)), \end{array} \tag{2.7.89}$$

where the last sum converges by (Hα2), possibly after decreasing $\delta > 0$. Using the fact that

$$(\Phi^+)' \quad = \quad \frac{1}{c}\left(\begin{array}{l} \Delta_h \phi^+ + g_u(\overline{u})\phi^+ - \psi^+ \\ \rho\phi^+ - \rho\gamma\psi^+ \end{array}\right) \tag{2.7.90}$$

we hence see that there exists a constant $K_3 > 0$ such that

$$|(\Phi^+)'(\xi)| \quad \leq \quad K_3 e^{-\delta|\xi|}. \tag{2.7.91}$$

The proof for the bound on $(\Phi^-)'$ is identical. ■

We recall the spaces

$$\begin{array}{rcccl} X & := & X_h & = & \{\Theta \in \mathbf{H}^1 : \langle \Phi^-, \Theta \rangle = 0\} \\ Y & := & Y_h & = & \{\Theta \in \mathbf{L}^2 : \langle \Phi^-, \Theta \rangle = 0\}, \end{array} \tag{2.7.92}$$

together with the operators L^{-1} in the spaces $\mathcal{B}(X, X)$ and in $\mathcal{B}(Y, X)$ that were defined in Proposition 2.5.2. We also recall the notation $L^{\mathrm{qinv}}\Theta$ that was introduced in Corollary 2.5.4 for the unique solution Ψ of the equation

$$L\Psi \quad = \quad \Theta - \tfrac{\langle \Phi^-, \Theta \rangle}{\langle \Phi^-, \Phi^+ \rangle} \Phi^+ \tag{2.7.93}$$

in the space X, which is given explicitly by

$$L^{\mathrm{qinv}}\Theta \quad = \quad L^{-1}\Big[\Theta - \tfrac{\langle \Phi^-, \Theta \rangle}{\langle \Phi^-, \Phi^+ \rangle} \Phi^+\Big]. \tag{2.7.94}$$

We now exploit these operators to decompose the Green's function of $\lambda + L$ into a meromorphic and an analytic part. This result is based on [109, Lem. 2.7].

Lemma 2.7.10. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. There exists a constant $0 < \overline{\lambda} \leq \lambda_0$ such that for all $0 < |\lambda| < \overline{\lambda}$ we have the representation*

$$G_\lambda(\xi, \xi_0) \quad = \quad E_\lambda(\xi, \xi_0) + \tilde{G}_\lambda(\xi, \xi_0) \tag{2.7.95}$$

Here the meromorphic (in λ) term can be written as

$$E_\lambda(\xi, \xi_0) \quad = \quad -\tfrac{1}{\lambda\Omega} \begin{pmatrix} \phi^-(\xi_0)\phi^+(\xi) & \psi^-(\xi_0)\phi^+(\xi) \\ \phi^-(\xi_0)\psi^+(\xi) & \psi^-(\xi_0)\psi^+(\xi) \end{pmatrix} \tag{2.7.96}$$

and the analytic (in λ) term $\tilde{G}_\lambda$ is given by

$$\begin{array}{rcl} \tilde{G}_\lambda(\xi, \xi_0) & = & G_{\infty;\lambda}(\xi - \xi_0) - \Big[[I + \lambda L^{-1}]^{-1} L^{\mathrm{qinv}}(L - L_\infty) G_{\infty;\lambda}(\cdot - \xi_0)\Big](\xi) \\ & & -\tfrac{1}{\Omega}\langle \Phi^-, G_{\infty;\lambda}(\cdot - \xi_0)\rangle \Phi^+(\xi). \end{array} \tag{2.7.97}$$

Here we recall the notation

$$\Omega \quad = \quad \langle \Phi^-, \Phi^+ \rangle. \tag{2.7.98}$$

Proof. Pick $\lambda \in \mathbb{C}$ with $0 < |\lambda| < \lambda_0$. By the proof of Proposition 2.6.2 we see that

$$(L + \lambda)^{-1}\Theta \quad = \quad \lambda^{-1} \tfrac{\langle \Phi^-, \Theta \rangle}{\Omega} \Phi^+ + L^{\mathrm{qinv}}\Theta - [I + \lambda L^{-1}]^{-1} \lambda L^{-1} L^{\mathrm{qinv}}\Theta \tag{2.7.99}$$

for $\Theta \in \mathbf{L}^2$. We now compute

$$\begin{array}{rcl} \langle \Phi^-, (L - L_\infty) G_{\infty;\lambda}(\cdot - \xi_0)\rangle & = & \langle \Phi^-, -L_\infty G_{\infty;\lambda}(\cdot - \xi_0)\rangle \\ & = & -\Phi^-(\xi_0) + \lambda \langle \Phi^-, G_{\infty;\lambda}(\cdot - \xi_0)\rangle. \end{array} \tag{2.7.100}$$

In particular, writing

$$\hat{L} \quad = \quad L - L_\infty, \tag{2.7.101}$$

we obtain

$$\begin{array}{rcl} (L+\lambda)^{-1}\hat{L}G_{\infty;\lambda}(\cdot - \xi_0) & = & \frac{1}{\lambda\Omega}\left(\begin{array}{cc} \phi^-(\xi_0)\phi^+ & \psi^-(\xi_0)\phi^+ \\ \phi^-(\xi_0)\psi^+ & \psi^-(\xi_0)\psi^+ \end{array}\right) \\ & & +\frac{\langle\Phi^-, G_{\infty;\lambda}(\cdot-\xi_0)\rangle}{\Omega}\Phi^+ + L^{\mathrm{qinv}}\hat{L}G_{\infty;\lambda}(\cdot - \xi_0) \\ & & -[I+\lambda L^{-1}]^{-1}\lambda L^{-1}L^{\mathrm{qinv}}\hat{L}G_{\infty;\lambda}(\cdot - \xi_0). \end{array} \tag{2.7.102}$$

We may hence write

$$G_\lambda(\xi,\xi_0) \quad = \quad E_\lambda(\xi,\xi_0) + \tilde{G}_\lambda(\xi,\xi_0) \tag{2.7.103}$$

with

$$E_\lambda(\xi,\xi_0) \quad = \quad -\frac{1}{\lambda\Omega}\left(\begin{array}{cc} \phi^-(\xi_0)\phi^+(\xi) & \psi^-(\xi_0)\phi^+(\xi) \\ \phi^-(\xi_0)\psi^+(\xi) & \psi^-(\xi_0)\psi^+(\xi) \end{array}\right) \tag{2.7.104}$$

and

$$\begin{array}{rcl} \tilde{G}_\lambda(\cdot,\xi_0) & = & G_{\infty;\lambda}(\cdot - \xi_0) - L^{\mathrm{qinv}}\hat{L}G_{\infty;\lambda}(\cdot - \xi_0) \\ & & +[I+\lambda L^{-1}]^{-1}\lambda L^{-1}L^{\mathrm{qinv}}\hat{L}G_{\infty;\lambda}(\cdot - \xi_0) \\ & & -\frac{1}{\Omega}\langle\Phi^-, G_{\infty;\lambda}(\cdot - \xi_0)\rangle\Phi^+ \\ & = & G_{\infty;\lambda}(\cdot - \xi_0) - [I+\lambda L^{-1}]^{-1}L^{\mathrm{qinv}}\hat{L}G_{\infty;\lambda}(\cdot - \xi_0) \\ & & -\frac{1}{\Omega}\langle\Phi^-, G_{\infty;\lambda}(\cdot - \xi_0)\rangle\Phi^+. \end{array} \tag{2.7.105}$$

Clearly E_λ is meromorphic in λ, while $\tilde{G}_\lambda$ is analytic in λ in the region $|\lambda| < \lambda_0$. ∎

We fix $\chi > \lambda_{\mathrm{unif}}$, where λ_{unif} was defined in Lemma 2.7.3, and set

$$R \quad = \quad \{\lambda \in \mathbb{C} : -\frac{\tilde{\lambda}}{2} \le \mathrm{Re}\,\lambda \le \chi \text{ and } |\mathrm{Im}\,\lambda| \le \frac{\pi|c|}{h}\}. \tag{2.7.106}$$

We now set out to obtain an estimate on the function $\tilde{G}_\lambda$ from Lemma 2.7.10 by exploiting the asymptotic hyperbolicity of L. We treat each of the terms in (2.7.97) separately in the results below.

Lemma 2.7.11. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. There exist constants $K_4 > 0$ and $\tilde{\chi} > 0$ such that for all $\lambda \in R$*

$$\left|\langle\Phi^-, (L - L_\infty)G_{\infty;\lambda}(\cdot - \xi_0)\rangle\right| \quad \le \quad K_4 e^{-\tilde{\chi}|\xi_0|}. \tag{2.7.107}$$

Proof. We reuse the notation $\hat{L} = L - L_\infty$ from the previous proof. Lemma 2.7.2 implies that we can pick constants $\beta_* > 0$ and $K_* > 0$ in such a way that

$$|G_{\infty;\lambda}(\xi - \xi_0)| \quad \le \quad K_* e^{-\beta_*|\xi - \xi_0|} \tag{2.7.108}$$

for all values of ξ, ξ_0. Recall the constants $K_2, \tilde{\delta}$ from Lemma 2.7.8 and set $K_3 = K_2 \|\Phi^-\|_\infty$. Then we obtain

$$\begin{array}{rcl} |\langle \Phi^-, \hat{L}G_{\infty;\lambda}(\cdot - \xi_0)\rangle| & \leq & \int_{-\infty}^{\infty} K_3 e^{-\tilde{\delta}|\xi|} g_* K_* e^{-\beta_*|\xi-\xi_0|} \, d\xi \\ & = & K_3 g_* K_* \Big(\frac{1}{\tilde{\delta}+\beta_*}(e^{-\tilde{\delta}|\xi_0|} + e^{-\beta_*|\xi_0|}) + \frac{1}{\beta_*-\tilde{\delta}}(e^{-\tilde{\delta}|\xi_0|} - e^{-\beta_*|\xi_0|}) \Big) \\ & \leq & K_3 g_* K_* \Big(\frac{1}{\tilde{\delta}+\beta_*} 2e^{-\min\{\tilde{\delta},\beta_*\}|\xi_0|} + \frac{1}{|\beta_*-\tilde{\delta}|} 2e^{-\min\{\tilde{\delta},\beta_*\}|\xi_0|} \Big) \\ & = & K_4 e^{-\tilde{\chi}|\xi_0|} \end{array} \tag{2.7.109}$$

for some $K_4 > 0$ and $\tilde{\chi} > 0$. ∎

Lemma 2.7.12. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. There exist constants $K_{10} > 0$ and $\tilde{\gamma} > 0$ such that for all $\lambda \in R$*

$$\begin{array}{rcl} \Big| \Big[L^{\mathrm{qinv}}(L - L_\infty) G_{\infty;\lambda}(\cdot - \xi_0) \Big](\xi) \Big| & \leq & K_{10} e^{-\tilde{\gamma}|\xi|} e^{-\tilde{\gamma}|\xi_0|} \\ & \leq & K_{10} e^{-\tilde{\gamma}|\xi-\xi_0|}. \end{array} \tag{2.7.110}$$

Proof. We reuse the notation $\hat{L} = L - L_\infty$ from the previous proof. Recall the constants K_1, δ from Lemma 2.7.8. Writing

$$H_{\xi_0}(\xi) = \Big[L^{\mathrm{qinv}} \hat{L} G_{\infty;\lambda}(\cdot - \xi_0) \Big](\xi), \tag{2.7.111}$$

we may use Lemma 2.7.6 to estimate

$$|H_{\xi_0}(\xi)| \leq K_1 e^{-\delta|\xi|} \|H_{\xi_0}\|_\infty + K_1 \int_{-\infty}^{\infty} e^{-\delta|\xi-\eta|} |L H_{\xi_0}(\eta)| d\eta. \tag{2.7.112}$$

Recalling (2.7.92)-(2.7.94), we obtain

$$\begin{array}{rcl} \|H_{\xi_0}\|_\infty & \leq & \|H_{\xi_0}\|_{\mathbf{H}^1} \\ & \leq & C_{\mathrm{unif}} || \hat{L} G_{\infty;\lambda}(\cdot - \xi_0) - \frac{\langle \Phi^-, \hat{L} G_{\infty;\lambda}(\cdot - \xi_0)\rangle}{\Omega} \Phi^+ ||_{\mathbf{L}^2} \\ & \leq & C_{\mathrm{unif}} \Big(1 + \frac{\|\Phi^-\|_{\mathbf{L}^2}}{\Omega} \|\Phi^+\|_{\mathbf{L}^2} \Big) \|\hat{L} G_{\infty;\lambda}(\cdot - \xi_0)\|_{\mathbf{L}^2} \\ & \leq & K_5 \|\hat{L} G_{\infty;\lambda}(\cdot - \xi_0)\|_{\mathbf{L}^2} \end{array} \tag{2.7.113}$$

for some constant $K_5 > 0$.

Using Lemma 2.7.8 we see that there exists a constant $K_6 > 0$ for which

$$\begin{array}{rcl} |\overline{u}(\xi)| & = & | \int_{\infty}^{\xi} \overline{u}'(\xi') \, d\xi' | \\ & \leq & \int_{\xi}^{\infty} K_1 \|(\overline{u}', \overline{w}')\|_\infty e^{-\delta|\xi'|} \, d\xi' \\ & = & K_6 e^{-\delta|\xi|} \end{array} \tag{2.7.114}$$

holds for all $\xi \in \mathbb{R}$. Recall that

$$\hat{L} = \left(\begin{array}{cc} -g_u(\overline{u}) + r_0 & 0 \\ 0 & 0 \end{array} \right). \tag{2.7.115}$$

Observe that $-g_u(0) + r_0 = 0$. Then we obtain that

$$|-g_u(\overline{u}(\xi)) + r_0| \quad \leq \quad K_7 e^{-\delta|\xi|} \tag{2.7.116}$$

for all $\xi \in \mathbb{R}$ and for some constant $K_7 > 0$. Lemma 2.7.2 implies that

$$|G_{\infty;\lambda}(\xi - \xi_0)| \quad \leq \quad K_* e^{-\beta_*|\xi-\xi_0|} \tag{2.7.117}$$

for all $\xi \in \mathbb{R}$. Therefore, we must have

$$\begin{aligned} \|\hat{L}G_{\infty;\lambda}(\cdot - \xi_0)\|^2_{\mathbf{L}^2} &\leq \int_{\mathbb{R}} K_7^2 K_*^2 e^{-2\delta|\xi|} e^{-2\beta_*|\xi-\xi_0|} \, d\xi \\ &\leq K_8 e^{-2\tilde{\gamma}|\xi_0|} \end{aligned} \tag{2.7.118}$$

for some constants $K_8 > 0$, $\tilde{\gamma} > 0$ with $\tilde{\gamma} \leq \beta_*$, $\tilde{\gamma} \leq \frac{1}{2}\delta$ and $\tilde{\gamma} \leq \frac{1}{2}\tilde{\chi}$. In particular, we obtain the estimate

$$\|H_{\xi_0}\|_\infty \quad \leq \quad K_5 \sqrt{K_8} e^{-\tilde{\gamma}|\xi_0|}. \tag{2.7.119}$$

In a similar fashion, using Lemma 2.7.11, we see that

$$\begin{aligned} |LH_{\xi_0}(\xi)| &\leq \left| \Big[\hat{L}G_{\infty;\lambda}(\cdot - \xi_0)\Big](\xi) - \tfrac{\langle \Phi^-, \hat{L}G_{\infty;\lambda}(\cdot - \xi_0)\rangle}{\Omega} \Phi^+(\xi) \right| \\ &\leq K_7 K_* e^{-\delta|\xi|} e^{-\beta_*|\xi-\xi_0|} + \tfrac{1}{\Omega} K_4 e^{-\tilde{\chi}|\xi_0|} K_1 e^{-\delta|\xi|} \\ &\leq K_9 \Big[e^{-2\tilde{\gamma}|\xi|} e^{-\tilde{\gamma}|\xi-\xi_0|} + e^{-\tilde{\gamma}|\xi_0|} e^{-\tilde{\gamma}|\xi|} \Big] \end{aligned} \tag{2.7.120}$$

for all $\xi \in \mathbb{R}$ and some constant $K_9 > 0$. Combining (2.7.112) with (2.7.113) and (2.7.118), we hence obtain

$$\begin{aligned} |H_{\xi_0}(\xi)| &\leq K_1 e^{-\delta|\xi|} \|H_{\xi_0}\|_\infty + K_1 \int_{-\infty}^{\infty} e^{-\delta|\xi-\eta|} |LH_{\xi_0}(\eta)| d\eta \\ &\leq K_1 e^{-\delta|\xi|} K_5 \sqrt{K_8} e^{-\tilde{\gamma}|\xi_0|} \\ &\qquad + K_1 \int_{-\infty}^{\infty} e^{-\delta|\xi-\eta|} K_9 \Big[e^{-2\tilde{\gamma}|\eta|} e^{-\tilde{\gamma}|\eta-\xi_0|} + e^{-\tilde{\gamma}|\xi_0|} e^{-\tilde{\gamma}|\eta|} \Big] d\eta \\ &\leq K_1 e^{-\delta|\xi|} K_5 \sqrt{K_8} e^{-\tilde{\gamma}|\xi_0|} + K_1 \int_{-\infty}^{\infty} e^{-\delta|\xi-\eta|} 2K_9 e^{-\tilde{\gamma}|\eta|} e^{-\tilde{\gamma}|\xi_0|} d\eta \\ &\leq K_{10} e^{-\tilde{\gamma}|\xi|} e^{-\tilde{\gamma}|\xi_0|} \\ &\leq K_{10} e^{-\tilde{\gamma}|\xi-\xi_0|} \end{aligned} \tag{2.7.121}$$

for some constant $K_{10} > 0$. ■

Remark 2.7.13. In the proof of Lemma 2.7.12, in particular in (2.7.116), we explicitly used that $\overline{U}$ is a pulse solution, instead of a traveling front solution. If one would want to transfer these results to a more general system where the waves have different limits at $\xi = \pm\infty$, then Lemma 2.7.12 would only hold for $\xi_0 \geq 0$. However, the definition (2.7.25) remains valid upon using the reference system at $\xi = -\infty$ instead of $\xi = +\infty$. This new formulation allows the desired estimates for $\xi_0 \leq 0$ to be recovered.

Lemma 2.7.14. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. There exist constants $K_{13} > 0$ and $\omega > 0$ such that the function $\tilde{G}_\lambda$ from Lemma 2.7.10 satisfies the bound*

$$|\tilde{G}_\lambda(\xi, \xi_0)| \quad \leq \quad K_{13} e^{-\omega|\xi - \xi_0|} \tag{2.7.122}$$

for all ξ, ξ_0 and all $0 < |\lambda| < \overline{\lambda}$.

Proof. As before, we write

$$H_{\xi_0}(\xi) \quad = \quad L^{\mathrm{qinv}} \hat{L} G_{\infty;\lambda}(\cdot - \xi_0)(\xi). \tag{2.7.123}$$

Using Lemma 2.7.6 Lemma 2.7.12 and (2.7.119) and recalling (2.7.92)-(2.7.94), we obtain the estimate

$$\begin{array}{rcl} |L^{-1} H_{\xi_0}(\xi)| & \leq & K_1 e^{-\alpha|\xi|} \|L^{-1} H_{\xi_0}\|_\infty + K_1 \int_{-\infty}^{\infty} e^{-\alpha|\xi - \eta|} |H_{\xi_0}(\eta)| d\eta \\ & \leq & K_1 e^{-\alpha|\xi|} C_{\mathrm{unif}} \|H_{\xi_0}\|_{\mathbf{L}^2} + K_1 \int_{-\infty}^{\infty} e^{-\alpha|\xi-\eta|} K_{10} e^{-\tilde{\gamma}|\eta - \xi_0|} d\eta \\ & \leq & K_1 e^{-\alpha|\xi|} C_{\mathrm{unif}} K_5 \sqrt{K_8} e^{-\tilde{\gamma}|\xi_0|} + K_1 \int_{-\infty}^{\infty} e^{-\alpha|\xi-\eta|} K_{10} e^{-\tilde{\gamma}|\eta-\xi_0|} d\eta \\ & \leq & K_{10} K_{11} e^{-\tilde{\gamma}|\xi - \xi_0|} \end{array} \tag{2.7.124}$$

for some constants $K_{11} > 0$ and $2\tilde{\gamma} \leq \alpha$. Using Proposition 2.5.2 and (2.7.119) we obtain that

$$\|(L^{-1})^n H_{\xi_0}\|_{\mathbf{H}^1} \quad \leq \quad K_5 \sqrt{K_8} (C_{\mathrm{unif}})^n e^{-\tilde{\gamma}|\xi_0|} \tag{2.7.125}$$

for all $n \in \mathbb{Z}_{>0}$. Continuing in this fashion, we see that

$$|(L^{-1})^n H_{\xi_0}(\xi)| \quad \leq \quad K_{10} K_{11}^n e^{-\tilde{\gamma}|\xi - \xi_0|} \tag{2.7.126}$$

for all $n \in \mathbb{Z}_{>0}$. If we set

$$\overline{\lambda} \quad = \quad \min\{\tfrac{\tilde{\lambda}}{2}, \lambda_0, \chi, \tfrac{1}{C_{\mathrm{unif}} K_5 \sqrt{K_8}}, \tfrac{1}{K_{11}}\}, \tag{2.7.127}$$

then for each $n \in \mathbb{Z}_{>0}$ and each $0 < |\lambda| < \overline{\lambda}$ we have

$$\|(-\lambda)^n (L^{-1})^n H_{\xi_0}\|_{\mathbf{H}^1} \quad \leq \quad \tfrac{1}{2}. \tag{2.7.128}$$

In particular, it follows that

$$\sum_{n=0}^{N} (-\lambda)^n (L^{-1})^n H_{\xi_0} \quad \to \quad [I + \lambda L^{-1}]^{-1} H_{\xi_0} \tag{2.7.129}$$

in $\mathbf{H}^1$ as $N \to \infty$. Since $\mathbf{H}^1$-convergence implies point-wise convergence, we conclude that

$$\begin{array}{rcl} \big|[I + \lambda L^{-1}]^{-1} H_{\xi_0}(\xi)\big| & = & \Big| \sum_{n=0}^{\infty} (-\lambda)^n (L^{-1})^n H_{\xi_0}(\xi) \Big| \\ & \leq & \sum_{n=0}^{\infty} \overline{\lambda}^n K_{11} K_{12}^n e^{-\tilde{\gamma}|\xi - \xi_0|} \\ & \leq & \frac{K_{11}}{1 - \overline{\lambda} K_{11}} e^{-\tilde{\gamma}|\xi - \xi_0|} \\ & := & K_{12} e^{-\tilde{\gamma}|\xi - \xi_0|} \end{array} \tag{2.7.130}$$

for all $\xi \in \mathbb{R}$ and for some constant $K_{12} > 0$.

Combining this estimate with Lemma 2.7.8 and Lemma 2.7.12 yields the desired bound

$$\begin{array}{lcl} |\tilde{G}_\lambda(\xi, \xi_0)| & = & \Big|G_{\infty;\lambda}(\xi - \xi_0) - \Big[[I + \lambda L^{-1}]^{-1} L^{\mathrm{qinv}} \hat{L} G_{\infty;\lambda}(\cdot - \xi_0)\Big](\xi) \\ & & \quad - \frac{1}{\Omega}\langle \Phi^-, G_\infty(\cdot - \xi_0)\rangle \Phi^+(\xi)\Big| \\ & \leq & K_* e^{-\beta_*|\xi - \xi_0|} + K_{12} e^{-\tilde{\gamma}|\xi - \xi_0|} + K_4 \frac{1}{\Omega} e^{-\tilde{\chi}|\xi_0|} K_1 e^{-\delta|\xi|} \|\Phi^+\|_\infty \\ & \leq & K_{13} e^{-\omega|\xi - \xi_0|} \end{array} \tag{2.7.131}$$

for some constants $K_{13} > 0$ and $\omega > 0$. ■

We write

$$S \quad = \quad \{-\overline{\lambda} + i\omega : \omega \in [-\tfrac{\pi|c|}{h}, \tfrac{\pi|c|}{h}]\}, \tag{2.7.132}$$

where $\overline{\lambda}$ is defined in the proof of Lemma 2.7.14.

Lemma 2.7.15. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. Then there exist constants $K > 0$ and $\tilde{\beta} > 0$ such that for all $\lambda \in S$ we have the bound*

$$|G_\lambda(\xi, \xi_0)| \quad \leq \quad K e^{-\tilde{\beta}|\xi - \xi_0|} \tag{2.7.133}$$

for all ξ, ξ_0.

Proof. Fix $\lambda_0 \in S$. For $\lambda \in S$ sufficiently close to λ_0 we have

$$\begin{array}{lcl} \Big[L + \lambda\Big]^{-1} & = & \Big[L + \lambda_0 + \lambda - \lambda_0\Big]^{-1} \\ & = & \Big[\Big(L + \lambda_0\Big)\Big(I + (L + \lambda_0)^{-1}(\lambda - \lambda_0)\Big)\Big]^{-1} \\ & = & \Big[I + (L + \lambda_0)^{-1}(\lambda - \lambda_0)\Big]^{-1}\Big[L + \lambda_0\Big]^{-1}. \end{array} \tag{2.7.134}$$

In particular, upon writing

$$H_{\xi_0}(\xi) \quad = \quad \Big[[L + \lambda_0]^{-1} \hat{L} G_{\lambda;\infty}(\cdot - \xi_0)\Big](\xi), \tag{2.7.135}$$

we see that

$$G_\lambda(\xi, \xi_0) - G_{\infty;\lambda}(\xi - \xi_0) \quad = \quad \Big[[I + (L + \lambda_0)^{-1}(\lambda - \lambda_0)]^{-1} H_{\xi_0}\Big](\xi). \tag{2.7.136}$$

Using Lemma 2.7.6 we can pick constants $k_{\lambda_0} > 0$ and $\alpha_{\lambda_0} > 0$ in such a way that

$$|H_{\xi_0}(\xi)| \quad \leq \quad k_{\lambda_0} e^{-\alpha_{\lambda_0}|\xi|} \|H_{\xi_0}\|_\infty + k_{\lambda_0} \int_{-\infty}^{\infty} e^{-\alpha_{\lambda_0}|\xi - \eta|} |(L + \lambda_0) H_{\xi_0}(\eta)| d\eta. \tag{2.7.137}$$

Recall the constant C_S appearing in Proposition 2.3.3. This allows us to estimate

$$\begin{array}{lcl} \|H_{\xi_0}\|_\infty & \leq & \|H_{\xi_0}\|_{\mathbf{H}^1} \\ & \leq & C_S \|\hat{L} G_{\lambda_0}(\xi, \xi_0)\|_{\mathbf{L}^2} \\ & \leq & C_S \sqrt{K_8} e^{-\tilde{\gamma}|\xi_0|}. \end{array} \tag{2.7.138}$$

This yields the bound

$$\begin{array}{rcl} |H_{\xi_0}(\xi)| & \leq & k_{\lambda_0} e^{-\alpha_{\lambda_0}|\xi|} C_S \sqrt{K_8} e^{-\tilde{\gamma}|\xi_0|} + k_{\lambda_0} \int_{-\infty}^{\infty} e^{-\alpha_{\lambda_0}|\xi-\eta|} |\hat{L} G_{\lambda;\infty}(\eta,\xi_0)| d\eta \\ & \leq & k_{\lambda_0} e^{-\alpha_{\lambda_0}|\xi|} C_S \sqrt{K_8} e^{-\tilde{\gamma}|\xi_0|} + k_{\lambda_0} \int_{-\infty}^{\infty} e^{-\alpha_{\lambda_0}|\xi-\eta|} K_7 K_* e^{-\delta|\eta|} e^{-2\beta_*|\eta-\xi_0|} d\eta \\ & \leq & k_{\lambda_0;2} e^{-\alpha_{\lambda_0;2}|\xi-\xi_0|} \end{array} \tag{2.7.139}$$

for some constants $k_{\lambda_0;2}, \alpha_{\lambda_0;2}$, which may depend on λ_0, but not on λ. Arguing as in (2.7.124), we obtain

$$\begin{array}{rcl} \left|[L+\lambda_0]^{-1} H_{\xi_0}(\xi)\right| & \leq & k_{\lambda_0} e^{-\alpha_{\lambda_0}|\xi|} \|[L+\lambda_0]^{-1} H_{\xi_0}\|_\infty \\ & & + k_{\lambda_0} \int_{-\infty}^{\infty} e^{-\alpha_{\lambda_0}|\xi-\eta|} |H_{\xi_0}(\eta)| d\eta \\ & \leq & k_{\lambda_0;2} k_{\lambda_0;3} e^{-\alpha_{\lambda_0;2}|\xi-\xi_0|} \end{array} \tag{2.7.140}$$

for some constant $k_{\lambda_0;3} > 0$, which may depend on λ_0, but not on λ. Following the same steps as the proof of Lemma 2.7.14 and setting

$$\varepsilon_{\lambda_0} = \min\{\tfrac{1}{k_{\lambda_0} C_S \sqrt{K_8}}, \tfrac{1}{k_{\lambda_0;3}}\}, \tag{2.7.141}$$

we conclude that

$$\begin{array}{rcl} |G_\lambda(\xi,\xi_0) - G_{\infty;\lambda}(\xi-\xi_0)| & = & \left|\Big[[I + [L+\lambda_0]^{-1}(\lambda-\lambda_0)]^{-1} H_{\xi_0}\Big](\xi)\right| \\ & \leq & k_{\lambda_0;4} e^{-\alpha_{\lambda_0;2}|\xi-\xi_0|} \end{array} \tag{2.7.142}$$

holds for each $\lambda \in S$ with $|\lambda - \lambda_0| < \varepsilon_{\lambda_0}$, for some constant $k_{\lambda_0;4} > 0$, which may depend on λ_0. In particular, we obtain that

$$\begin{array}{rcl} |G_\lambda(\xi,\xi_0)| & \leq & k_{\lambda_0;4} e^{-\alpha_{\lambda_0;2}|\xi-\xi_0|} + K_* e^{-\beta_*|\xi-\xi_0|} \\ & \leq & k_{\lambda_0;5} e^{-\alpha_{\lambda_0;2}|\xi-\xi_0|} \end{array} \tag{2.7.143}$$

holds for each $\lambda \in S$ with $|\lambda - \lambda_0| < \varepsilon_{\lambda_0}$, for some constant $k_{\lambda_0;5} > 0$, which may depend on λ_0.

Since S is compact we can find $\lambda_1, ..., \lambda_n \in S$ in such a way that

$$S \subset \bigcup_{i=1}^{n} \{\lambda \in \mathbb{C} : |\lambda - \lambda_i| < \varepsilon_{\lambda_i}\}. \tag{2.7.144}$$

Setting

$$\begin{array}{rcl} K & = & \max\{k_{\lambda_i;5} : i \in \{1, ...n\}\}, \\ \tilde{\beta} & = & \min\{\alpha_{\lambda_i;2} : i \in \{1, ..., n\}\}, \end{array} \tag{2.7.145}$$

we conclude that

$$|G_\lambda(\xi,\xi_0)| \leq K e^{-\tilde{\beta}|\xi-\xi_0|} \tag{2.7.146}$$

holds for all $\lambda \in S$ and all $\xi, \xi_0 \in \mathbb{R}$. ∎

2.7.3 Decomposition into stable and center modes

In this final subsection we establish Proposition 2.7.1. In particular, the decomposition (2.7.13) and the exponential bounds (2.7.15) for the Green's function $\mathcal{G}$ can be found by using the splitting of G_λ obtained in §2.7.2. This is performed in Lemma 2.7.16, which is based on [109, Cor. 2.8].

We subsequently carefully study the terms appearing in (2.7.13) and show that they can be interpreted as a spectral decomposition that splits the flow associated to the linear system (2.7.6) into two invariant subspaces. The stable component decays exponentially in a uniform fashion, while the center component can be described explicitly.

Lemma 2.7.16. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. For any pair $t \geq t_0$ and any $j, j_0 \in \mathbb{Z}$, we have the representation*

$$\mathcal{G}_j^{j_0}(t,t_0) = \mathcal{E}_j^{j_0}(t,t_0) + \tilde{\mathcal{G}}_j^{j_0}(t,t_0) \tag{2.7.147}$$

in which

$$\mathcal{E}_j^{j_0}(t,t_0) = \tfrac{h}{\Omega}\begin{pmatrix} \phi^-(hj_0+ct_0)\phi^+(hj+ct) & \psi^-(hj_0+ct_0)\phi^+(hj+ct) \\ \phi^-(hj_0+ct_0)\psi^+(hj+ct) & \psi^-(hj_0+ct_0)\psi^+(hj+ct) \end{pmatrix}, \tag{2.7.148}$$

while $\tilde{\mathcal{G}}$ satisfies the bound

$$|\tilde{\mathcal{G}}_j^{j_0}(t,t_0)| \leq Ke^{-\tilde{\beta}(t-t_0)}e^{-\tilde{\beta}|hj+ct-hj_0-ct_0|} \tag{2.7.149}$$

for some $K > 0$ and $\tilde{\beta} > 0$.

Proof. Recall the representation of $\mathcal{G}_j^{j_0}$ from Proposition 2.7.4. Note that $G_\lambda(\xi,\xi_0)$ is meromorphic for λ in the strip $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq -\lambda_3, |\operatorname{Im}\lambda| \leq \frac{c\pi}{h}\}$ with a simple pole at $\lambda = 0$ by Lemma 2.7.10, Lemma 2.7.3 and Theorem 2.2.2. Lemma 2.7.10 also implies that the residue of $G_\lambda(\xi,\xi_0)$ in $\lambda = 0$ is given by

$$\operatorname{Res}(G_\lambda(\xi,\xi_0),0) = -\tfrac{1}{\Omega}\begin{pmatrix} \phi^-(\xi_0)\phi^+(\xi) & \psi^-(\xi_0)\phi^+(\xi) \\ \phi^-(\xi_0)\psi^+(\xi) & \psi^-(\xi_0)\psi^+(\xi) \end{pmatrix}. \tag{2.7.150}$$

We write

$$H(\cdot,\xi_0) = e^{2\pi i\frac{1}{h}k\xi_0}(L+\lambda+2\pi ik\tfrac{c}{h})e_{-2\pi i\frac{1}{h}k}G_\lambda(\cdot,\xi_0). \tag{2.7.151}$$

In a similar fashion as in the proof of Lemma 2.6.1 we see that for $k \in \mathbb{Z}$ we have

$$(L+\lambda+2\pi ik\tfrac{c}{h})e_{-2\pi i\frac{1}{h}k} = e_{-2\pi i\frac{1}{h}k}(L+\lambda). \tag{2.7.152}$$

Therefore, it follows that

$$\begin{aligned} H(\cdot,\xi_0) &= e^{2\pi i\frac{1}{h}k\xi_0}(L+\lambda+2\pi ik\tfrac{c}{h})e_{-2\pi i\frac{1}{h}k}G_\lambda(\cdot,\xi_0) \\ &= e^{2\pi i\frac{1}{h}k\xi_0}e_{-2\pi i\frac{1}{h}k}(L+\lambda)G_\lambda(\cdot,\xi_0). \end{aligned} \tag{2.7.153}$$

For any $f \in \mathbf{H}^1$ we may hence compute

$$\begin{array}{rcl} \int H(\xi,\xi_0)f(\xi_0)\,d\xi_0 & = & \int e^{2\pi i\frac{1}{h}k\xi_0}e^{-2\pi i\frac{1}{h}k\xi}(L+\lambda)G_\lambda(\cdot,\xi_0)(\xi)f(\xi_0)\,d\xi_0 \\ & = & e^{-2\pi i\frac{1}{h}k\xi}[e^{2\pi i\frac{1}{h}k\xi}f(\xi)] \\ & = & f(\xi). \end{array} \tag{2.7.154}$$

Therefore, by the invertibility of $L + \lambda + 2\pi ik\frac{c}{h}$, we must have

$$G_{\lambda+2\pi ik\frac{c}{h}}(\xi,\xi_0) \quad = \quad e^{2\pi ik\frac{1}{h}(\xi_0-\xi)}G_\lambda(\xi,\xi_0). \tag{2.7.155}$$

Now recall the constants χ, χ_+, χ_- from (the proof of) Proposition 2.7.4 and define

$$\begin{array}{rcl} \overline{\lambda}^- & = & -\frac{\overline{\lambda}}{2} - i\frac{\pi c}{h} \\ \overline{\lambda}^+ & = & -\frac{\overline{\lambda}}{2} + i\frac{\pi c}{h}. \end{array} \tag{2.7.156}$$

Writing $x = hj + ct, y = hj_0 + ct_0$, we see that

$$\begin{array}{rcl} \int_{\overline{\lambda}^-}^{\chi_-} e^{\lambda(t-t_0)}G_\lambda(x,y)\,d\lambda & = & \int_{\overline{\lambda}^+}^{\chi_+} e^{(\lambda+2\pi i\frac{c}{h})(t-t_0)}e^{-2\pi i\frac{1}{h}(y-x)}G_\lambda(x,y)\,d\lambda \\ & = & \int_{\overline{\lambda}^+}^{\chi_+} e^{\lambda(t-t_0)}G_\lambda(x,y)\,d\lambda. \end{array} \tag{2.7.157}$$

Hence, if we integrate the function $e^{\lambda(t-t_0)}G_\lambda(hj+ct, hj_0+ct_0)$ along the rectangle with edges $-\frac{\overline{\lambda}}{2} - i\frac{\pi c}{h}, -\frac{\overline{\lambda}}{2} + i\frac{\pi c}{h}, \chi - i\frac{\pi c}{h}$ and $\chi + i\frac{\pi c}{h}$, then the integrals from $\chi - i\frac{\pi c}{h}$ to $-\frac{\overline{\lambda}}{2} - i\frac{\pi c}{h}$ and from $-\frac{\overline{\lambda}}{2} + i\frac{\pi c}{h}$ to $\chi + i\frac{\pi c}{h}$ cancel each other out. In particular, again writing $x = hj + ct, y = hj_0 + ct_0$, the residue theorem implies

$$\begin{array}{rcl} \mathcal{G}_j^{j_0}(t,t_0) & = & \frac{-h}{2\pi i}\int_{\chi-i\frac{\pi c}{h}}^{\chi+i\frac{\pi c}{h}} e^{\lambda(t-t_0)}G_\lambda(x,y)\,d\lambda \\ & = & \frac{h}{2\pi i}\int_{-\frac{\overline{\lambda}}{2}-i\frac{\pi c}{h}}^{-\frac{\overline{\lambda}}{2}+i\frac{\pi c}{h}} e^{\lambda(t-t_0)}G_\lambda(x,y)\,d\lambda + \frac{h}{\Omega}\begin{pmatrix} \phi^-(y)\phi^+(x) & \psi^-(y)\phi^+(x) \\ \phi^-(y)\psi^+(x) & \psi^-(y)\psi^+(x) \end{pmatrix}. \end{array} \tag{2.7.158}$$

Using Lemma 2.7.15 we also get the estimate

$$\left|\frac{h}{2\pi i}\int_{-\frac{\overline{\lambda}}{2}-i\frac{\pi c}{h}}^{-\frac{\overline{\lambda}}{2}+i\frac{\pi c}{h}} e^{\lambda(t-t_0)}G_\lambda(x,y)\,d\lambda\right| \quad \le \quad \frac{h}{2\pi}\frac{2c\pi}{h}e^{-\overline{\lambda}(t-t_0)}Ke^{-\tilde{\beta}|x-y|}, \tag{2.7.159}$$

which yields the desired bound (2.7.149). ■

For any $t \in \mathbb{R}$, we introduce the suggestive notation

$$\Pi^c(t) \quad = \quad \mathcal{E}(t,t) \tag{2.7.160}$$

together with

$$\Pi^s(t) \quad = \quad I - \Pi^c(t). \tag{2.7.161}$$

Recalling the notation introduced in (2.7.12), we set out to show that $\Pi^c(t) * \Pi^c(t) = \Pi^c(t)$ and $\Pi^s(t) * \Pi^s(t) = \Pi^s(t)$. Later on, we will view these operators as projections that correspond to the center and stable parts of the flow induced by $\mathcal{G}$ respectively.

To establish the identity $\Pi^c(t) * \Pi^c(t) = \Pi^c(t)$, it suffices to show that

$$\begin{aligned}\begin{pmatrix} \phi^-(x_{j_0})\phi^+(x_j) & \psi^-(x_{j_0})\phi^+(x_j) \\ \phi^-(x_{j_0})\psi^+(x_j) & \psi^-(x_{j_0})\psi^+(x_j) \end{pmatrix} \quad = \quad & \frac{h}{\Omega}\sum_{i\in\mathbb{Z}} \begin{pmatrix} \phi^-(x_i)\phi^+(x_j) & \psi^-(x_i)\phi^+(x_j) \\ \phi^-(x_i)\psi^+(x_j) & \psi^-(x_i)\psi^+(x_j) \end{pmatrix} \\ & \times \begin{pmatrix} \phi^-(x_{j_0})\phi^+(x_i) & \psi^-(x_{j_0})\phi^+(x_i) \\ \phi^-(x_{j_0})\psi^+(x_i) & \psi^-(x_{j_0})\psi^+(x_i) \end{pmatrix},\end{aligned} \tag{2.7.162}$$

in which $x_i = hi + ct$ for $i \in \mathbb{Z}$. We now write our linear operator in the form

$$L\Psi(\xi) \quad = \quad c\tfrac{d}{d\xi}\Psi(\xi) + \sum_{j=-\infty}^{\infty} A_j(\xi)\Psi(\xi + jh), \tag{2.7.163}$$

where

$$A_j(\xi) \quad = \quad \begin{cases} \begin{pmatrix} \frac{1}{h^2}\alpha_{|j|} & 0 \\ 0 & 0 \end{pmatrix} & \text{if } j \neq 0 \\ \begin{pmatrix} -2\frac{1}{h^2}\sum_{k>0}\alpha_k + g_u(\overline{u}(\xi)) & 1 \\ -\rho & \rho\gamma \end{pmatrix} & \text{if } j = 0. \end{cases} \tag{2.7.164}$$

Before we continue, we first prove a small lemma that will help us to relate discrete inner products with their continuous counterparts.

Lemma 2.7.17. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. For all $\xi \in \mathbb{R}$ we have the identity*

$$c\begin{pmatrix} \phi^-(\xi)\phi^+(\xi) \\ \psi^-(\xi)\psi^+(\xi) \end{pmatrix} \quad = \quad \sum_{j=-\infty}^{\infty} \int_0^{hj} B(\xi+\theta-hj)A_j(\xi+\theta-hj)\Phi^+(\xi+\theta)d\theta, \tag{2.7.165}$$

where

$$B(\xi) \quad = \quad \begin{pmatrix} \phi^-(\xi) & 0 \\ 0 & \psi^-(\xi) \end{pmatrix}. \tag{2.7.166}$$

Proof. Our strategy is to differentiate both sides of (2.7.165) and to show their derivatives are equal. Starting with the first component, we pick $N \in \mathbb{Z}_{>0} \cup \{\infty\}$ and write

$$D(N) \quad := \quad \tfrac{d}{d\xi}\sum_{j=-N}^{N}\int_0^{hj}\phi^-(\xi+\theta-hj)\Big[A_j(\xi+\theta-hj)\Phi^+(\xi+\theta)\Big]^{(1)}d\theta. \tag{2.7.167}$$

For finite N, we may compute

$$\begin{array}{rcl} D(N) & = & \frac{d}{d\xi} \sum_{j=-N}^{N} \int_{\xi-hj}^{\xi} \phi^-(\theta) \Big[A_j(\theta)\Phi^+(\theta+hj)\Big]^{(1)} d\theta \\ & = & \sum_{j=-N}^{N} \phi^-(\xi) \Big[A_j(\xi)\Phi^+(\xi+hj)\Big]^{(1)} \\ & & \quad - \sum_{j=-N}^{N} \phi^-(\xi-hj) \Big[A_j(\xi-hj)\Phi^+(\xi)\Big]^{(1)}. \end{array} \tag{2.7.168}$$

Now for $j > 0$ we have $|A_j(\xi)\Phi^+(\xi+hj)| \leq \frac{1}{h^2}|\alpha_j|$, so the partial sums converge uniformly. Hence, it follows that

$$\begin{array}{rcl} D(\infty) & = & \sum_{j=-\infty}^{\infty} \phi^-(\xi) \Big[A_j(\xi)\Phi^+(\xi+hj)\Big]^{(1)} \\ & & \quad - \sum_{j=-\infty}^{\infty} \phi^-(\xi-hj) \Big[A_j(\xi-hj)\Phi^+(\xi)\Big]^{(1)} \\ & = & \phi^-(\xi)c(\phi^+)'(\xi) + c(\phi^-)'(\xi)\phi^+(\xi) \\ & = & c(\phi^-\phi^+)'(\xi), \end{array} \tag{2.7.169}$$

since $\Phi^+ \in \ker(L)$ and $\Phi^- \in \ker(L^*)$.

We now set out to show that both sides of (2.7.165) converge to zero as $\xi \to \infty$. Pick $\varepsilon > 0$ and let $N \in \mathbb{Z}_{>0}$ be large enough to ensure that

$$\sum_{j \geq N} \frac{1}{h^2} j|\alpha_j| \quad \leq \quad \frac{\varepsilon}{4(1+\|\phi^-\|_\infty)\|\Phi^+\|_\infty}. \tag{2.7.170}$$

In addition, let Ξ be large enough to have

$$|\phi^-(\xi)| \quad \leq \quad \frac{\varepsilon}{4(1+\sum_{j=-N}^{N} |j\alpha_{|j|}|)\|\Phi^+\|_\infty} \tag{2.7.171}$$

for all $\xi \geq \Xi - N$. This Ξ exists since $\phi^- \in H^1$. For such ξ we may estimate

$$\Big|\sum_{j=-\infty}^{\infty} \int_0^{hj} \phi^-(\xi+\theta-hj) \Big[A_j(\xi+\theta-hj)\Phi^+(\xi+\theta)\Big]^{(1)} d\theta\Big| \quad < \quad \varepsilon, \tag{2.7.172}$$

which allows us to compute

$$\begin{array}{rcl} \lim_{\xi\to\infty} \sum_{j=-\infty}^{\infty} \int_0^{hj} \phi^-(\xi+\theta-hj) \Big[A_j(\xi+\theta-hj)\Phi^+(\xi+\theta)\Big]^{(1)} d\theta & = & 0 \\ & = & \lim_{\xi\to\infty} c\phi^-(\xi)\phi^+(\xi). \end{array} \tag{2.7.173}$$

With that we have proved our claim. Furthermore, we can repeat the arguments above to obtain

$$c\psi^-(\xi)\psi^+(\xi) \quad = \quad \sum_{j=-\infty}^{\infty} \int_0^{hj} \psi^-(\xi+\theta-hj) \Big[A_j(\xi+\theta-hj)\Phi^+(\xi+\theta)\Big]^{(2)} d\theta. \tag{2.7.174}$$

■

We are now ready to show that $\Pi^c(t) * \Pi^c(t) = \Pi^c(t)$ and $\Pi^s(t) * \Pi^s(t) = \Pi^s(t)$. This result is based on the first part of [109, Lem. 2.9].

Lemma 2.7.18. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. Then $\Pi^c(t) * \Pi^c(t) = \Pi^c(t)$ and $\Pi^s(t) * \Pi^s(t) = \Pi^s(t)$ for all $t \in \mathbb{R}$.*

Proof. For $k \in \mathbb{Z}$ we write $x_k = hk + ct$. In addition, for notational convenience we write $B_j(\theta) = \left[A_j(\theta)\Phi^+(\theta+hj)\right]^{(1)}$ for $j \in \mathbb{Z}$ and $\theta \in \mathbb{R}$. Using the results from Lemma 2.7.17 we may compute

$$\begin{aligned} c\sum_{k=-\infty}^{\infty} \phi^-(x_k)\phi^+(x_k) &= \sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\int_0^{hj} \phi^-(x_k+\theta-hj)B_j(x_k+\theta-hj)d\theta \\ &= \sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\int_{x_{k-j}}^{x_k} \phi^-(\theta)B_j(\theta)d\theta \\ &= \sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty} j\phi^-(\theta)B_j(\theta)d\theta, \end{aligned} \tag{2.7.175}$$

where we were allowed to interchange the two infinite sums because

$$\begin{aligned} \Big|\sum_{k=-N}^{N}\int_{x_{k-j}}^{x_k} \phi^-(\theta)B_j(\theta)d\theta\Big| &\leq \Big|\int_{-\infty}^{\infty} j\phi^-(\theta)B_j(\theta)d\theta\Big| \\ &\leq \|\phi^-\|_1\tfrac{1}{h^2}|j\alpha_{|j|}|\|\phi^+\|_\infty \end{aligned} \tag{2.7.176}$$

holds for all $N \in \mathbb{Z}_{>0}$ and $j \in \mathbb{Z}$. This expression is summable over j, allowing us to apply Lebesgue's theorem. On the other hand, we have

$$\begin{aligned} c\int_{-\infty}^{\infty} \phi^-(\xi)\phi^+(\xi)\,d\xi &= \int_{-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\int_0^{hj} \phi^-(\xi+\theta-hj)B_j(\xi+\theta-hj)d\theta\,d\xi \\ &= \sum_{j=-\infty}^{\infty}\int_0^{hj}\int_{-\infty}^{\infty} \phi^-(\xi+\theta-hj)B_j(\xi+\theta-hj)\,d\xi d\theta \\ &= \sum_{j=-\infty}^{\infty}\int_0^{hj}\int_{-\infty}^{\infty} \phi^-(\xi-hj)B_j(\xi-hj)d\xi d\theta \\ &= \sum_{j=-\infty}^{\infty} hj\int_{-\infty}^{\infty} \phi^-(\xi-hj)B_j(\xi-hj)d\xi. \end{aligned} \tag{2.7.177}$$

Interchanging the integral with the sum was allowed since ϕ^- and ϕ^+ decay exponentially, say $|\phi^-(x)| \leq \kappa e^{-\delta|x|}$ and $|\phi^+(x)| \leq \kappa e^{-\delta|x|}$. In particular, for each $N \in \mathbb{Z}_{>0}$ and each $\xi \in \mathbb{R}$ we have

$$\Big|\sum_{j=-N}^{N}\int_0^{hj} \phi^-(\xi+\theta-hj)B_j(\xi+\theta-hj)d\theta\Big| \leq \sum_{j=-\infty}^{\infty} h\kappa^2 e^{-\delta|\xi|}|j\alpha_{|j|}|\|\Phi^+\|_\infty, \tag{2.7.178}$$

which is integrable in ξ. Furthermore, the interchanging of the two integrals was allowed, since by the exponential decay of ϕ^- we also see that for each $j \in \mathbb{Z}, \xi \in \mathbb{R}$ and

$\theta \in (0, hj)$ we have

$$|\phi^-(\xi + \theta - hj)B_j(\xi + \theta - hj)| \quad \leq \quad \kappa e^{-\delta|\xi+\theta-hj|}|\alpha_{|j|}|\|\Phi^+\|_\infty. \tag{2.7.179}$$

This is an integrable function for $(\xi, \theta) \in \mathbb{R} \times (0, hj)$, allowing us to apply Fubini's theorem.

In particular, we see that

$$\int_{-\infty}^{\infty} \phi^-(\xi)\phi^+(\xi)\, d\xi \quad = \quad h \sum_{k=-\infty}^{\infty} \phi^-(x_k)\phi^+(x_k). \tag{2.7.180}$$

In the same way we obtain

$$\int_{-\infty}^{\infty} \psi^-(\xi)\psi^+(\xi)\, d\xi \quad = \quad h \sum_{k=-\infty}^{\infty} \psi^-(x_k)\psi^+(x_k). \tag{2.7.181}$$

By writing out the sums it now follows that indeed (2.7.162) holds. ■

Proof of Proposition 2.7.1. The calculations above imply that $\mathcal{E}(t, t_0) = \mathcal{E}(t, t_0) * \Pi^c(t_0)$, which means that we must also have $\mathcal{E}(t, t_0) * \Pi^s(t_0) = 0$.

Observe that for any $t_0 \in \mathbb{R}$, the function $V_j(t) := \begin{pmatrix} \phi^+(hj + ct) \\ \psi^+(hj + ct) \end{pmatrix}$ is the unique solution to (2.7.6) with $V_j(t_0) = \begin{pmatrix} \phi^+(hj + ct_0) \\ \psi^+(hj + ct_0) \end{pmatrix}$. Hence, by the definition of the Green's function $\mathcal{G}(t, t_0)$ we see that

$$V(t) \quad = \quad \mathcal{G}(t, t_0) * V(t_0) \tag{2.7.182}$$

for all $t \in \mathbb{R}$. Furthermore, we recall that

$$\mathcal{E}_j^{j_0}(t_0, t_0) \quad = \quad V_j(t_0)\Phi^-(hj_0 + ct_0). \tag{2.7.183}$$

For $j, j_0 \in \mathbb{Z}$ we may hence compute

$$\begin{array}{rcl} \Big[\mathcal{G}(t, t_0) * \Pi^c(t_0)\Big]_j^{j_0} & = & \sum_{i\in\mathbb{Z}} \mathcal{G}_j^i(t, t_0) * \mathcal{E}_i^{j_0}(t, t_0) \\ & = & \frac{h}{\Omega} \sum_{i\in\mathbb{Z}} \mathcal{G}_j^i(t, t_0)_i * V_i(t_0)\Phi^-(hj_0 + ct_0) \\ & = & \frac{h}{\Omega} V_j(t)\Phi^-(hj_0 + ct_0) \\ & = & \mathcal{E}_j^{j_0}(t, t_0). \end{array} \tag{2.7.184}$$

In particular, we obtain $\mathcal{G}(t, t_0) * \Pi^c(t_0) = \mathcal{E}(t, t_0)$ and thus

$$\begin{array}{rcl} \tilde{\mathcal{G}}(t, t_0) * \Pi^c(t_0) & = & \mathcal{G}(t, t_0) * \Pi^c(t_0) - \mathcal{E}(t, t_0) * \Pi^c(t_0) \\ & = & \mathcal{E}(t, t_0) - \mathcal{E}(t, t_0) \\ & = & 0. \end{array} \tag{2.7.185}$$

Therefore, we must have

$$\begin{array}{rcl}\mathcal{G}(t,t_0) & = & \mathcal{E}(t,t_0)+\tilde{\mathcal{G}}(t,t_0) \\ & = & \mathcal{E}(t,t_0)*\Big(\Pi^c(t_0)+\Pi^s(t_0)\Big)+\tilde{\mathcal{G}}(t,t_0)*\Big(\Pi^c(t_0)+\Pi^s(t_0)\Big) \\ & = & \mathcal{E}(t,t_0)*\Pi^c(t_0)+\tilde{\mathcal{G}}(t,t_0)*\Pi^s(t_0),\end{array} \tag{2.7.186}$$

which completes the proof. ∎

2.8 Nonlinear stability

In this section, we will finally prove Theorem 2.2.3, along the lines of the approach described in [109]. The main contribution here is that we give a detailed description of the manner in which one can account for the shift-periodicity of the underlying problem when constructing the stable manifolds for the family $\overline{U}(\cdot+\vartheta)$.

Recall from §2.2 that the space ℓ^p is defined by

$$\ell^p \quad = \quad \{V\in(\mathbb{R}^2)^{\mathbb{Z}} : \|V\|_{\ell^p} := \sum_{j\in\mathbb{Z}}|V_j|^p<\infty\} \tag{2.8.1}$$

for $1\le p<\infty$ and

$$\ell^\infty \quad = \quad \{V\in(\mathbb{R}^2)^{\mathbb{Z}} : \|V\|_{\ell^\infty} := \sup_{j\in\mathbb{Z}}|V_j|<\infty\}. \tag{2.8.2}$$

In addition, we recall the notation $(\overline{U})_j(t)=\Big(\overline{u}(hj+ct),\overline{w}(hj+ct)\Big)$ and we let $\tilde{\beta}>0$ be the constant appearing in Proposition 2.7.1.

Exploiting Lemma 2.7.8 we see that

$$\|\mathcal{E}_j^{j_0}(t,t_0)\| \quad \le \quad C_1e^{-\tilde{\beta}|hj+ct-hj_0-ct_0|} \tag{2.8.3}$$

for some constant $C_1>0$. Lemma 2.7.18 hence allows us to define $\Pi^c(t)\in\mathcal{B}(\ell^p;\ell^p)$ and $\Pi^s(t)\in\mathcal{B}(\ell^p,\ell^p)$ by writing

$$\begin{array}{rcl}\Pi^c(t)V & = & \mathcal{E}(t,t)*V, \\ \Pi^s(t)V & = & \Big[I-\Pi^c(t)\Big]V.\end{array} \tag{2.8.4}$$

The proof of our nonlinear stability result proceeds in two main steps. In particular, we first construct the stable manifolds of the solutions $(\overline{u},\overline{w})(\cdot-\tilde{\theta})$ for each $\tilde{\theta}\in\mathbb{R}$. This result is based on the first half of the proof of [109, Prop. 2.1].

Proposition 2.8.1. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. There exists a constant $\eta>0$, independent of p, such that for each $\tilde{\theta}\in\mathbb{R}$ and each $W_s\in\mathrm{Range}(\Pi^s(\tilde{\theta}))$ with $\|W_s\|_{\ell^p}<\eta$ there is a unique function $\mathcal{U}_*^{\tilde{\theta}}(W_s):[0,\infty)\to\ell^p$ such that*

1. $U(t) = \overline{U}(t+\tilde{\theta}) + \mathcal{U}_*^{\tilde{\theta}}(W_s)(t)$ *is a solution of (2.2.1) for all* $t \geq 0$,

2. $\mathcal{U}_*^{\tilde{\theta}}(W_s)(t)$ *decays exponentially to* 0 *as* $t \to \infty$,

3. $\Pi^s(\tilde{\theta})\mathcal{U}_*^{\tilde{\theta}}(W_s)(0) = W_s$.

In addition, there exist constants $C_6 > 0$ *and* $C_{13} > 0$*, independent of* p*, such that the estimate*

$$\|\Pi^c(\tilde{\theta})\mathcal{U}_*^{\tilde{\theta}}(W_s)(0)\|_{\ell^p} \quad \leq \quad C_6\|W_s\|_{\ell^p}^2 \tag{2.8.5}$$

holds for all $\tilde{\theta} \in \mathbb{R}$ *and each* $W_s \in \mathrm{Range}(\Pi^s(\tilde{\theta}))$ *with* $\|W_s\|_{\ell^p} < \eta$*, while the estimate*

$$\begin{aligned}\|\Pi^c(\tilde{\theta}_1)\mathcal{U}_*^{\tilde{\theta}_1}(W_s^1)(0) - \Pi^c(\tilde{\theta}_2)\mathcal{U}_*^{\tilde{\theta}_2}(W_s^2)(0)\|_{\ell^p} \quad \leq \quad & C_{13}\Big[\|W_s^1\|_{\ell^p} + \|W_s^2\|_{\ell^p}\Big] \\ & \times\Big[\|W_s^1 - W_s^2\|_{\ell^p} + |\theta_1 - \theta_2|\Big]\end{aligned} \tag{2.8.6}$$

holds for all $W_s^1 \in \mathrm{Range}(\Pi^s(\tilde{\theta}_1))$*, all* $W_s^2 \in \mathrm{Range}(\Pi^s(\tilde{\theta}_2))$ *and all* $\tilde{\theta}_1 \in \mathbb{R}$ *and* $\tilde{\theta}_2 \in \mathbb{R}$ *with* $\|W_s^1\|_{\ell^p} < \eta$*,* $\|W_s^2\|_{\ell^p} < \eta$ *and* $|\tilde{\theta}_2 - \tilde{\theta}_1| < \eta$*.*

It then suffices to show that the space around the family of travelling pulse solutions can be completely covered by these stable manifolds. We remark that $\tilde{\theta}$ in the result below will correspond with the asymptotic phase shift.

Proposition 2.8.2. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. Then there exists a constant* $\delta > 0$*, which does not depend on* p*, such that for all initial conditions* $U^0 \in \ell^p$ *with* $\|U^0 - \overline{U}(0)\|_{\ell^p} < \delta$ *there exists* $\tilde{\theta} \in \mathbb{R}$ *and* $W_s \in \mathrm{Range}(\Pi^s(\tilde{\theta}))$ *such that*

$$U^0 \quad = \quad \overline{U}(\tilde{\theta}) + \mathcal{U}_*^{\tilde{\theta}}(W_s). \tag{2.8.7}$$

We write the LDE (2.2.1) as

$$\tfrac{d}{dt}V(t) \quad = \quad \mathcal{F}\Big(V(t)\Big), \tag{2.8.8}$$

where

$$\mathcal{F}\Big(V(t)\Big)_j \quad = \quad \tfrac{1}{c}\left(\begin{array}{l} \frac{1}{h^2}\sum\limits_{k>0}\alpha_k\Big[V_{j+k}^{(1)}(t) + V_{j-k}^{(1)}(t) - 2V_j^{(1)}(t)\Big] + g(V_j^{(1)}(t)) - V_j^{(2)}(t) \\ \rho[V_j^{(1)}(t) - \gamma V_j^{(2)}(t)] \end{array}\right). \tag{2.8.9}$$

Then we see that $\mathcal{A}(t) = D\mathcal{F}\Big(\overline{U}(t)\Big)$, where $\mathcal{A}(t)$ is defined in (2.7.6). We now write

$$\mathcal{N}_\theta\Big(t, V(t)\Big) \quad = \quad \mathcal{F}\Big(V(t) + \overline{U}(t+\theta)\Big) - \mathcal{F}(\overline{U}(t+\theta)) - D\mathcal{F}\Big(\overline{U}(t+\theta)\Big)V(t) \tag{2.8.10}$$

and set out to solve the differential equation

$$\tfrac{d}{dt}V(t) \quad = \quad D\mathcal{F}\Big(\overline{U}(t+\theta)\Big)V(t) + \mathcal{N}_\theta\Big(t, V(t)\Big). \tag{2.8.11}$$

Indeed, if V satisfies (2.8.11), then we see that

$$\begin{array}{rcl}\frac{d}{dt}\Big(\overline{U}(t+\theta)+V(t)\Big) & = & \mathcal{F}\Big(\overline{U}(t+\theta)\Big)+D\mathcal{F}\Big(\overline{U}(t+\theta)\Big)V(t)+\mathcal{N}_\theta\Big(t,V(t)\Big) \\ & = & \mathcal{F}\Big(\overline{U}(t+\theta)+V(t)\Big),\end{array} \tag{2.8.12}$$

which means that $\overline{U}(\cdot+\theta)+V$ is indeed a solution of (2.2.1).

Our goal is to construct decaying solutions to (2.8.11) for multiple values of θ using a single Green's function. To this end, we write

$$\mathcal{M}^{\tilde{\theta}}\Big(\theta,t,V\Big) = \mathcal{N}_\theta\Big(t,V(t)\Big)+D\mathcal{F}\Big(\overline{U}(t+\theta)\Big)V(t)-D\mathcal{F}\Big(\overline{U}(t+\tilde{\theta})\Big)V(t). \tag{2.8.13}$$

This allow us to rewrite (2.8.11) as

$$\frac{d}{dt}V(t) = D\mathcal{F}\Big(\overline{U}(t+\tilde{\theta})\Big)V(t)+\mathcal{M}^{\tilde{\theta}}\Big(\theta,t,V\Big). \tag{2.8.14}$$

Lemma 2.8.3. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. Then $\mathcal{M}^{\tilde{\theta}}(\theta,t,\cdot)$ maps ℓ^p into itself and there exists a constant $C_2>0$, independent of p and $\tilde{\theta}$, so that we have the estimate*

$$\|\mathcal{M}^{\tilde{\theta}}(\theta,t,V)\|_{\ell^p} \le C_2\|V\|_{\ell^p}^2+C_2|\tilde{\theta}-\theta|\|V\|_{\ell^p}, \tag{2.8.15}$$

for $V\in\ell^p$ with $\|V\|_{\ell^p}\le 1$ and $\theta\in\mathbb{R}$ with $|\theta-\tilde{\theta}|\le 1$, together with

$$\begin{array}{rcl}\|\mathcal{M}^{\tilde{\theta}}(\theta_1,t,V_1)-\mathcal{M}^{\tilde{\theta}}(\theta_2,t,V_2)\|_{\ell^p} & \le & C_2\|V_1-V_2\|_{\ell^p}\Big[\|V_1\|_{\ell^p}+\|V_2\|_{\ell^p} \\ & & \qquad +|\tilde{\theta}-\theta_1|+|\theta_2-\tilde{\theta}|\Big] \\ & & +C_2|\tilde{\theta}_1-\tilde{\theta}_2|\Big[\|V_1\|_{\ell^p}+\|V_2\|_{\ell^p}\Big]\end{array} \tag{2.8.16}$$

for $V_1\in\ell^p$, $V_2\in\ell^p,\theta_1\in\mathbb{R}$ and $\theta_2\in\mathbb{R}$ with $\|V_1\|_{\ell^p}\le 1,\|V_2\|_{\ell^p}\le 1,|\theta_1-\tilde{\theta}|\le 1$ and $|\theta_2-\tilde{\theta}|\le 1$.

Proof. A Taylor expansion around $\overline{U}^{(1)}(t+\tilde{\theta})$ yields the pointwise identity

$$\begin{array}{rcl}\mathcal{M}^{\tilde{\theta}}(\theta,t,V)^{(1)} & = & \frac{1}{c}\left(g\Big(V^{(1)}+\overline{U}^{(1)}(t+\theta)\Big)-g_u\Big(\overline{U}^{(1)}(t+\tilde{\theta})\Big)V^{(1)}-g\Big(\overline{U}^{(1)}(t+\theta)\Big)\right) \\ & = & \frac{1}{c}\left(\frac{1}{2}g_{uu}(\xi_1)(V^{(1)})^2+\Big[g_u\Big(\overline{U}^{(1)}(t+\theta)\Big)-g_u\Big(\overline{U}^{(1)}(t+\tilde{\theta})\Big)\Big]V^{(1)}\right) \\ & = & \frac{1}{c}\left(\frac{1}{2}g_{uu}(\xi_1)(V^{(1)})^2+\frac{1}{2}g_{uu}(\xi_2)\Big[\overline{U}^{(1)}(t+\theta)-\overline{U}^{(1)}(t+\tilde{\theta})\Big]V^{(1)}\right) \\ & = & \frac{1}{c}\left(\frac{1}{2}g_{uu}(\xi_1)(V^{(1)})^2+\frac{1}{2}g_{uu}(\xi_2)\Big[\frac{d}{dt}\overline{U}^{(1)}(\xi_3)\Big]|\theta-\tilde{\theta}|V^{(1)}\right), \\ \mathcal{M}^{\tilde{\theta}}(\theta,t,V)^{(2)} & = & 0,\end{array} \tag{2.8.17}$$

where ξ_1 is between $\overline{U}^{(1)}(t+\tilde{\theta})$ and $\overline{U}^{(1)}(t+\tilde{\theta})+V$, ξ_2 is between $\overline{U}^{(1)}(t+\tilde{\theta})$ and $\overline{U}^{(1)}(t)$ and ξ_3 is between $t+\theta$ and $t+\tilde{\theta}$. For a bounded function f, we have the pointwise bound

$$\begin{array}{lcl} |g_{uu}(f)| & = & |6f + 2r_0 + 2| \\ & \leq & 6\|f\|_\infty + 2r_0 + 2. \end{array} \tag{2.8.18}$$

Therefore we get the pointwise bound

$$\begin{array}{lcl} |\mathcal{M}^{\tilde{\theta}}(\theta, t, V)| & \leq & \left| \frac{1}{c}\frac{1}{2} \begin{pmatrix} \Big(6\|\overline{u}\|_\infty + 2r_0 + 2\Big)\|\overline{u}'\|_\infty |\theta - \tilde{\theta}| V^{(1)} \\ 0 \end{pmatrix} \right| \\ & & + \left| \frac{1}{c}\frac{1}{2} \begin{pmatrix} \Big(6\|V^{(1)}\|_{\ell^\infty} + 6\|\overline{u}\|_\infty + 2r_0 + 2\Big)(V^{(1)})^2 \\ 0 \end{pmatrix} \right| \\ & \leq & \frac{1}{|c|}\frac{1}{2}\Big(6\|\overline{u}\|_\infty + 2r_0 + 2\Big)\|\overline{u}'\|_\infty |\theta - \tilde{\theta}||V| \\ & & + \frac{1}{|c|}\frac{1}{2}(6\|V\|_{\ell^\infty} + 6\|\overline{u}\|_\infty + 2r_0 + 2)|(V^{(1)})|^2. \end{array} \tag{2.8.19}$$

Furthermore, for $1 \leq p < \infty$ we see

$$\begin{array}{lcl} \left\| \begin{pmatrix} |(V^{(1)})|^2 \\ 0 \end{pmatrix} \right\|_{\ell^p} & = & \left(\sum_{j \in \mathbb{Z}} |V_j^{(1)}|^{2p} \right)^{\frac{1}{p}} \\ & \leq & \left(\left(\sum_{j \in \mathbb{Z}} |V_j^{(1)}|^p \right) \left(\sup_{j \in \mathbb{Z}} |V_j^{(1)}|^p \right) \right)^{\frac{1}{p}} \\ & \leq & \|V\|_{\ell^p}^2, \end{array} \tag{2.8.20}$$

which clearly also holds for $p = \infty$ upon skipping the intermediate two steps. We hence obtain the bound

$$\begin{array}{lcl} \|\mathcal{M}^{\tilde{\theta}}(\theta, t, V)\|_{\ell^p} & \leq & \frac{1}{|c|}\frac{1}{2}\Big(6\|\overline{u}\|_\infty + 2r_0 + 2\Big)\|\overline{u}'\|_\infty |\theta - \tilde{\theta}| \|V\|_{\ell^p} \\ & & + \frac{1}{|c|}\frac{1}{2}(6\|\overline{u}\|_\infty + 2r_0 + 8)\|V\|_{\ell^p}^2 \\ & \leq & C_2\|V\|_{\ell^p}^2 + C_2|\theta - \tilde{\theta}| \|V\|_{\ell^p}, \end{array} \tag{2.8.21}$$

for some constant $C_2 > 0$, which is independent of p and $\tilde{\theta}$.

We now write

$$\begin{array}{lcl} d\mathcal{M} & = & \mathcal{M}^{\tilde{\theta}}\Big(\theta_1, t, V_1\Big) - \mathcal{M}^{\tilde{\theta}}\Big(\theta_2, t, V_2\Big) \\ & = & \frac{1}{c} \begin{pmatrix} g\Big(V_1^{(1)} + \overline{U}^{(1)}(t+\theta_1)\Big) - g_u\Big(\overline{U}^{(1)}(t+\tilde{\theta})\Big)V_1^{(1)} - g\Big(\overline{U}^{(1)}(t+\theta_1)\Big) \\ 0 \end{pmatrix} \\ & & + \frac{1}{c} \begin{pmatrix} -g\Big(V_2^{(1)} + \overline{U}^{(1)}(t+\theta_2)\Big) + g_u\Big(\overline{U}^{(1)}(t+\tilde{\theta})\Big)V_2^{(1)} + g\Big(\overline{U}^{(1)}(t+\theta_2)\Big) \\ 0 \end{pmatrix}. \end{array} \tag{2.8.22}$$

Using Taylor expansions around $\overline{U}^{(1)}(t+\theta_1)$ and $\overline{U}^{(1)}(t+\theta_2)$, we obtain the pointwise identities

$$\begin{array}{rcl} g\Big(V_1^{(1)}+\overline{U}^{(1)}(t+\theta_1)\Big)-g\Big(\overline{U}^{(1)}(t+\theta_1)\Big) & = & \frac{1}{6}g_{uuu}(\xi_1)(V_1^{(1)})^3 \\ & & +\frac{1}{2}g_{uu}(\overline{U}^{(1)}(t+\theta_1))(V_1^{(1)})^2 \\ & & +g_u\Big(\overline{U}^{(1)}(t+\theta_1)\Big)V_1^{(1)}, \\ g\Big(V_2^{(1)}+\overline{U}^{(1)}(t+\theta_2)\Big)-g\Big(\overline{U}^{(1)}(t+\theta_2)\Big) & = & \frac{1}{6}g_{uuu}(\xi_2)(V_2^{(1)})^3 \\ & & +\frac{1}{2}g_{uu}(\overline{U}^{(1)}(t+\theta_2))(V_2^{(1)})^2 \\ & & +g_u\Big(\overline{U}^{(1)}(t+\theta_2)\Big)V_2^{(1)}, \end{array} \tag{2.8.23}$$

where ξ_1 is in between $\overline{U}^{(1)}(t+\theta_1)$ and $V_1^{(1)}+\overline{U}^{(1)}(t+\theta_1)$ and ξ_2 is in between $\overline{U}^{(1)}(t+\theta_2)$ and $\overline{U}^{(1)}(t+\theta_2)+V_2$. This allows us to collect all terms of the same order together and write

$$d\mathcal{M} \;=\; dM_1+dM_2+dM_3, \tag{2.8.24}$$

where

$$\begin{array}{rcl} dM_1 & = & \frac{1}{c}\begin{pmatrix} \frac{1}{6}g_{uuu}(\xi_1)(V_1^{(1)})^3-\frac{1}{6}g_{uuu}(\xi_2)(V_2^{(1)})^3 \\ 0 \end{pmatrix}, \\ dM_2 & = & \frac{1}{c}\begin{pmatrix} \frac{1}{2}g_{uu}\Big(\overline{U}^{(1)}(t+\theta_1)\Big)(V_1^{(1)})^2-\frac{1}{2}g_{uu}\Big(\overline{U}^{(1)}(t+\theta_2)\Big)(V_2^{(1)})^2 \\ 0 \end{pmatrix}, \\ dM_3 & = & \frac{1}{c}\begin{pmatrix} \Big[g_u\Big(\overline{U}^{(1)}(t+\theta_1)\Big)-g_u\Big(\overline{U}^{(1)}(t+\tilde{\theta})\Big)\Big]V_1^{(1)} \\ 0 \end{pmatrix} \\ & & +\frac{1}{c}\begin{pmatrix} -\Big[g_u\Big(\overline{U}^{(1)}(t+\theta_2)\Big)-g_u\Big(\overline{U}^{(1)}(t+\tilde{\theta})\Big)\Big]V_2^{(1)} \\ 0 \end{pmatrix}. \end{array} \tag{2.8.25}$$

Note that $g_{uuu}=6$ is constant. A Taylor expansion around $\overline{U}^{(1)}(t+\theta_1)$ yields the pointwise identity

$$\begin{array}{rcl} dM_2^{(1)} & = & \frac{1}{c}\Big(\frac{1}{2}\Big(g_{uu}(\overline{U}^{(1)}(t+\theta_1))-g_{uu}(\overline{U}^{(1)}(t+\theta_2))\Big)(V_1^{(1)})^2\Big) \\ & & +\frac{1}{c}\Big(-\frac{1}{2}g_{uu}(\overline{U}^{(1)}(t+\theta_2))\Big((V_2^{(1)})^2-(V_1^{(1)})^2\Big)\Big) \\ & = & \frac{1}{c}\Big(3\Big(\overline{U}^{(1)}(t+\theta_1)-\overline{U}^{(1)}(t+\theta_2)\Big)(V_1^{(1)})^2\Big) \\ & & +\frac{1}{c}\Big(-\frac{1}{2}g_{uu}(\overline{U}^{(1)}(t+\theta_2))\Big((V_2^{(1)})^2-(V_1^{(1)})^2\Big)\Big) \\ & = & \frac{1}{c}\Big(3(\overline{U}^{(1)})'(\xi_3)(\theta_1-\theta_2)(V_1^{(1)})^2\Big) \\ & & +\frac{1}{c}\Big(-\frac{1}{2}g_{uu}(\overline{U}^{(1)}(t+\theta_2))\Big((V_2^{(1)})^2-(V_1^{(1)})^2\Big)\Big), \end{array} \tag{2.8.26}$$

where ξ_3 is in between $t+\theta_1$ and $t+\theta_2$. Using Taylor expansions around $\overline{U}^{(1)}(t+\theta_2)$ and $\overline{U}^{(1)}(t+\tilde{\theta})$, we obtain the pointwise identity

$$\begin{aligned} dM_3^{(1)} &= \tfrac{1}{c}\Big[g_u\Big(\overline{U}^{(1)}(t+\theta_1)\Big) - g_u\Big(\overline{U}^{(1)}(t+\tilde{\theta})\Big)\Big]V_1^{(1)} \\ &\quad -\tfrac{1}{c}\Big[g_u\Big(\overline{U}^{(1)}(t+\theta_2)\Big) - g_u\Big(\overline{U}^{(1)}(t+\tilde{\theta})\Big)\Big]V_2^{(1)} \\ &= \tfrac{1}{c}\Big(\Big[g_{uu}(\xi_4)(\theta_1-\theta_2)\Big]V_1^{(1)} - \Big[g_{uu}(\xi_5)(\theta_2-\tilde{\theta})\Big]V_2^{(1)}\Big), \end{aligned} \tag{2.8.27}$$

where ξ_4 is in between $\overline{U}^{(1)}(t+\theta_1)$ and $\overline{U}^{(1)}(t+\theta_2)$ and ξ_5 is in between $\overline{U}^{(1)}(t)$ and $\overline{U}^{(1)}(t+\theta_2)$. We estimate

$$\begin{aligned} \|d\mathcal{M}_1\|_{\ell^p} &\leq \tfrac{1}{|c|}\|V_1^3 - V_2^3\|_{\ell^p} \\ &\leq \tfrac{1}{|c|}\Big[\|V_1\|_{\ell^\infty}\|V_1^2-V_2^2\|_{\ell^p} + \|V_1-V_2\|_{\ell^\infty}\|V_2^2\|_{\ell^p}\Big] \\ &\leq \tfrac{1}{|c|}\Big[\|V_1\|_{\ell^p}\Big[\|V_1\|_{\ell^p}+\|V_2\|_{\ell^p}\Big]\|V_1-V_2\|_{\ell^p} + \|V_1-V_2\|_{\ell^p}\|V_2\|_{\ell^p}^2\Big], \end{aligned} \tag{2.8.28}$$

together with

$$\begin{aligned} \|d\mathcal{M}_2\|_{\ell^p} &\leq \tfrac{1}{|c|}\Big[3\|(\overline{U}^{(1)})'(\xi_3)(\theta_1-\theta_2)\|_\infty\|V_1\|_{\ell^p}^2 \\ &\qquad +\Big(6\|\overline{u}\|_\infty + 2r_0 + 2\Big)\Big[\|V_1\|_{\ell^\infty}\|V_1-V_2\|_{\ell^p} + \|V_1-V_2\|_{\ell^\infty}\|V_2\|_{\ell^p}\Big]\Big] \\ &\leq \tfrac{1}{|c|}\Big[3\|\overline{u}'\|_\infty|\theta_1-\theta_2|\|V_1\|_{\ell^p}^2 \\ &\qquad +\Big(6\|\overline{u}\|_\infty + 2r_0 + 2\Big)\Big[\|V_1\|_{\ell^p}+\|V_2\|_{\ell^p}\Big]\|V_1-V_2\|_{\ell^p}\Big] \end{aligned} \tag{2.8.29}$$

and

$$\begin{aligned} \|d\mathcal{M}_3\|_{\ell^p} &\leq \tfrac{1}{|c|}\Big[\|g_{uu}(\xi_4)(\theta_1-\theta_2)\|_\infty\|V_1\|_{\ell^p} + \|g_{uu}(\xi_5)(\theta_2-\tilde{\theta})\|_\infty\|V_1-V_2\|_{\ell^p}\Big] \\ &\leq \tfrac{1}{|c|}\Big[\Big(6\|\overline{u}\|_\infty + 2r_0 + 2\Big)|\theta_1-\theta_2|\|V_1\|_{\ell^p} \\ &\qquad +\Big(6\|\overline{u}\|_\infty + 2r_0 + 2\Big)|\theta_2-\tilde{\theta}|\|V_1-V_2\|_{\ell^p}\Big]. \end{aligned} \tag{2.8.30}$$

Combining these estimates yields

$$\begin{aligned} \|d\mathcal{M}\|_{\ell^p} &\leq C_2\Big[\|V_1\|_{\ell^p}+\|V_2\|_{\ell^p}+|\tilde{\theta}-\theta_1|+|\theta_2-\tilde{\theta}|\Big]\|V_1-V_2\|_{\ell^p} \\ &\qquad +C_2|\tilde{\theta}_1-\tilde{\theta}_2|\Big[\|V_1\|_{\ell^p}+\|V_2\|_{\ell^p}\Big]. \end{aligned} \tag{2.8.31}$$

■

Lemma 2.8.4. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. Then there exists a constant $C_3 > 0$, independent of p, such that for $V \in \ell^p$ we have the bound*

$$\|\tilde{\mathcal{G}}(t,t_0)V\|_{\ell^p} \leq C_3 e^{-\tilde{\beta}(t-t_0)}\|V\|_{\ell^p}, \tag{2.8.32}$$

for all $t, t_0 \in \mathbb{R}$.

Proof. We let $f(t,t_0)_j = e^{-\tilde{\beta}|ct-ct_0+hj|}$ and write $\tilde{V}_j = |V_j|$ for $V \in \ell^p$. Using Lemma 2.7.16 and Young's inequality we obtain

$$\begin{aligned} \|\tilde{\mathcal{G}}(t,t_0)V\|_{\ell^p} &\leq 8Ce^{-\tilde{\beta}(t-t_0)}\|f(t,t_0) * \tilde{V}\|_{\ell^p(\mathbb{Z},\mathbb{R})} \\ &\leq 8Ce^{-\tilde{\beta}(t-t_0)}\|f(t,t_0)\|_{\ell^1(\mathbb{Z},\mathbb{R})}\|\tilde{V}\|_{\ell^p(\mathbb{Z},\mathbb{R})} \\ &= 8Ce^{-\tilde{\beta}(t-t_0)}\|f(t,t_0)\|_{\ell^1(\mathbb{Z},\mathbb{R})}\|V\|_{\ell^p} \\ &\leq C_3 e^{-\tilde{\beta}(t-t_0)}\|V\|_{\ell^p}, \end{aligned} \tag{2.8.33}$$

where

$$C_3 \quad = \quad 8C \sup_{x\in[0]} \sum_{j\in\mathbb{Z}} e^{-\tilde{\beta}|hj+x|} < \infty. \tag{2.8.34}$$

Note that C_3 is independent of p. ∎

From the defining system (2.7.8), it is clear that for each $\tilde{\theta} \in \mathbb{R}$ the Green's function of the linearisation of (2.2.1) around the wave $\overline{U}(\cdot + \tilde{\theta})$ is given by $\mathcal{G}(t+\tilde{\theta}, t_0+\tilde{\theta})$.

Fix $W_s \in \mathrm{Range}\Big(\Pi^s(\tilde{\theta})\Big)$ and consider the fixed point problem

$$\begin{aligned} V(t) \quad = \quad & \tilde{\mathcal{G}}(t+\tilde{\theta},\tilde{\theta})W_s + \int_0^t \tilde{\mathcal{G}}(t+\tilde{\theta}, t_0+\tilde{\theta})\Pi^s(t_0+\tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta, t_0, V(t_0)\Big)dt_0 \\ & + \int_\infty^t \mathcal{E}(t+\tilde{\theta}, t_0+\tilde{\theta})\Pi^c(t_0+\tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta, t_0, V(t_0)\Big)dt_0. \end{aligned} \tag{2.8.35}$$

We aim to construct decaying solutions to (2.8.14) by solving this fixed point problem in the space

$$BC_{-\tilde{\beta}/2}([0,\infty),\ell^p) \quad := \quad \{V \in C([0,\infty),\ell^p) : \|V\|_{-\tilde{\beta}/2} < \infty\}, \tag{2.8.36}$$

where

$$\|V\|_{-\tilde{\beta}/2} \quad = \quad \sup_{\xi\in[0,\infty)} e^{\frac{\tilde{\beta}}{2}\xi}\|V(\xi)\|_{\ell^p} \tag{2.8.37}$$

Here the integrals are taken component-wise, but we see that for each $1 \leq p \leq \infty$ this component-wise integral corresponds to the Bochner integral.

Lemma 2.8.5. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. If the function* $V \in BC_{-\tilde{\beta}/2}([0,\infty),\ell^p)$ *satisfies the fixed point problem (2.8.35), then* V *satisfies (2.8.14) and, hence,* $V(t) + \overline{U}(t+\theta)$ *is a solution of (2.2.1).*

Proof. If $V(t)$ satisfies this fixed point problem then we see that

$$\begin{array}{rcl} \frac{d}{dt}\Big(\tilde{\mathcal{G}}(t+\tilde{\theta},\tilde{\theta})W_s\Big) & = & D\mathcal{F}\Big(\overline{U}(t+\tilde{\theta})\Big)\tilde{\mathcal{G}}(t+\tilde{\theta},\tilde{\theta})W_s \\ & & +D\mathcal{F}\Big(\overline{U}(t+\tilde{\theta})\Big)\mathcal{E}(t+\tilde{\theta},\tilde{\theta})\Pi^c(\tilde{\theta})W_s \\ & & -\frac{d}{dt}\Big(\mathcal{E}(t+\tilde{\theta},\tilde{\theta})\Pi^c(\tilde{\theta})W_s\Big) \\ & = & D\mathcal{F}\Big(\overline{U}(t+\tilde{\theta})\Big)\tilde{\mathcal{G}}(t+\tilde{\theta},\theta)W_s + 0 - 0 \\ & = & D\mathcal{F}\Big(\overline{U}(t+\tilde{\theta})\Big)\tilde{\mathcal{G}}(t+\tilde{\theta},\tilde{\theta})W_s. \end{array} \tag{2.8.38}$$

Writing

$$\mathcal{D}(t) \quad = \quad \tfrac{d}{dt}V(t) - \tfrac{d}{dt}\Big(\tilde{\mathcal{G}}(t+\tilde{\theta},\tilde{\theta})W_s\Big), \tag{2.8.39}$$

we can compute

$$\begin{array}{rcl} \mathcal{D}(t) & = & \int_0^t \frac{d}{dt}\Big[\tilde{\mathcal{G}}(t+\tilde{\theta},t_0+\tilde{\theta})\Pi^s(t_0+\tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta,t_0,V(t_0)\Big)\Big]dt_0 \\ & & +\int_\infty^t \frac{d}{dt}\Big[\mathcal{E}(t+\tilde{\theta},t_0+\tilde{\theta})\Pi^c(t_0+\tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta,t_0,V(t_0)\Big)\Big]dt_0 \\ & & +\tilde{\mathcal{G}}(t+\tilde{\theta},t+\tilde{\theta})\Pi^s(t+\tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta,t,V(t)\Big) \\ & & +\mathcal{E}(t+\tilde{\theta},t+\tilde{\theta})\Pi^c(t+\tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta,t,V(t)\Big) \\ & = & \int_0^t \frac{d}{dt}\Big[\mathcal{G}(t+\tilde{\theta},t_0+\tilde{\theta})\Pi^s(t_0+\tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta,t_0,V(t_0)\Big)\Big]dt_0 \\ & & +\int_\infty^t \frac{d}{dt}\Big[\mathcal{G}(t+\tilde{\theta},t_0+\tilde{\theta})\Pi^c(t_0+\tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta,t_0,V(t_0)\Big)\Big]dt_0 \\ & & +\mathcal{G}(t+\tilde{\theta},t+\tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta,t,V(t)\Big). \end{array} \tag{2.8.40}$$

Exploiting $\mathcal{G}(t+\tilde{\theta},t+\tilde{\theta}) = I$, this yields

$$\begin{array}{rcl} \mathcal{D}(t) & = & \int_0^t D\mathcal{F}\Big(\overline{U}(t+\tilde{\theta})\Big)\mathcal{G}(t+\tilde{\theta},t_0+\tilde{\theta})\Pi^s(t_0+\tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta,t_0,V(t_0)\Big)dt_0 \\ & & +\int_\infty^t D\mathcal{F}\Big(\overline{U}(t+\tilde{\theta})\Big)\mathcal{G}(t+\tilde{\theta},t_0+\tilde{\theta})\Pi^c(t_0+\tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta,t_0,V(t_0)\Big)dt_0 \\ & & +\mathcal{M}^{\tilde{\theta}}\Big(\theta,t,V(t)\Big) \\ & = & D\mathcal{F}\Big(\overline{U}(t+\tilde{\theta})\Big)\int_0^t \tilde{\mathcal{G}}(t+\tilde{\theta},t_0+\tilde{\theta})\Pi^s(t_0+\tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta,t_0,V(t_0)\Big)dt_0 \\ & & +D\mathcal{F}\Big(\overline{U}(t+\tilde{\theta})\Big)\int_\infty^t \mathcal{E}(t+\tilde{\theta},t_0+\tilde{\theta})\Pi^c(t_0+\tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta,t_0,V(t_0)\Big)dt_0 \\ & & +\mathcal{M}^{\tilde{\theta}}\Big(\theta,t,V(t)\Big) \end{array} \tag{2.8.41}$$

and thus

$$\begin{array}{rcl} \frac{d}{dt}V(t) & = & D\mathcal{F}\Big(\overline{U}(t+\tilde{\theta})\Big)V(t) + \mathcal{M}^{\tilde{\theta}}\Big(\theta,t,V(t)\Big) \\ & = & D\mathcal{F}\Big(\overline{U}(t+\theta)\Big)V(t) + \mathcal{N}_\theta\Big(t,V(t)\Big). \end{array} \tag{2.8.42}$$

■

Lemma 2.8.6. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. There exists $\eta > 0$, independent of p and $\tilde{\theta}$, so that for all $W_s \in \mathrm{Range}\Big(\Pi^s(\tilde{\theta})\Big)$ that have $\|W_s\|_{\ell^p} \leq \eta$ and all $|\theta - \tilde{\theta}| \leq \eta$, the fixed point problem (2.8.35) has a unique solution $\mathcal{W}^{\tilde{\theta}}_{*;\theta}(W_s)$ in the space $BC_{-\tilde{\beta}/2}([0,\infty),\ell^p)$.*

Proof. We first rewrite (2.8.35) as

$$V = T(W_s, V), \tag{2.8.43}$$

where

$$\begin{aligned} T(W_s,V) \;=\; & \tilde{\mathcal{G}}(t+\tilde{\theta},\tilde{\theta})W_s + \textstyle\int_0^t \tilde{\mathcal{G}}(t+\tilde{\theta},t_0+\tilde{\theta})\Pi^s(t_0+\tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\tilde{\theta},t_0,V(t_0)\Big)dt_0 \\ & + \textstyle\int_\infty^t \mathcal{E}(t+\tilde{\theta},t_0+\tilde{\theta})\Pi^c(t_0+\tilde{\theta})\mathcal{M}^{\theta}\Big(\theta,t_0,V(t_0)\Big)dt_0. \end{aligned} \tag{2.8.44}$$

Pick $V \in BC_{-\tilde{\beta}/2}([0,\infty),\ell^p)$ with $\|V\|_{-\tilde{\beta}/2} \leq 1$. Writing

$$\mathcal{I} = \textstyle\int_0^t \tilde{\mathcal{G}}(t+\tilde{\theta},t_0+\tilde{\theta})\Pi^s(t_0+\tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta,t_0,V(t_0)\Big)dt_0, \tag{2.8.45}$$

Lemma 2.8.3 and Lemma 2.8.4 imply

$$\begin{aligned} \|\mathcal{I}\|_{\ell^p} &\leq \textstyle\int_0^t \|\tilde{\mathcal{G}}(t+\tilde{\theta},t_0+\tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta,t_0,V(t_0)\Big)\|_{\ell^p}\ dt_0 \\ &\leq \textstyle\int_0^t C_3 e^{-\tilde{\beta}(t-t_0)}\|\mathcal{M}^{\tilde{\theta}}\Big(\theta,t_0,V(t_0)\Big)\|_{\ell^p}\ dt_0 \\ &\leq \textstyle\int_0^t C_3 e^{-\tilde{\beta}(t-t_0)} C_2\|V(t_0)\|_{\ell^p}\Big[\|V(t_0)\|_{\ell^p} + |\theta-\tilde{\theta}|\Big]dt_0 \\ &\leq \textstyle\int_0^t C_3 e^{-\tilde{\beta}(t-t_0)} C_2\|V\|_{-\tilde{\beta}/2}e^{-\tilde{\beta}t_0/2}\Big[e^{-\tilde{\beta}t_0/2}\|V\|_{-\tilde{\beta}/2} + |\theta-\tilde{\theta}|\Big]dt_0 \\ &\leq C_3C_2\|V\|_{-\tilde{\beta}/2}\Big[te^{-\tilde{\beta}t}\|V\|_{-\tilde{\beta}/2} + \tfrac{2}{\tilde{\beta}}e^{-\tilde{\beta}t/2}|\theta-\tilde{\theta}|\Big]. \end{aligned} \tag{2.8.46}$$

Observe that if we multiply this final function with $e^{\tilde{\beta}t/2}$ we still have a bounded function. Since this holds for all p we see that

$$\|\mathcal{I}\|_{-\tilde{\beta}/2} \leq C_4\|V\|_{-\tilde{\beta}/2}\Big[\|V\|_{-\tilde{\beta}/2} + |\theta-\tilde{\theta}|\Big] \tag{2.8.47}$$

for some constant C_4, which is independent of p.

We write

$$\mathcal{J}(t) = \textstyle\int_t^\infty \mathcal{E}(t+\tilde{\theta},t_0+\tilde{\theta})\Pi^c(t_0+\tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta,t_0,V(t_0)\Big)dt_0. \tag{2.8.48}$$

Mimicking the computation above and using the explicit expression (2.8.3), we see that

$$\begin{aligned} \|\mathcal{J}(t)\|_{\ell^p} &\leq \textstyle\int_\infty^t C_1C_2\|V\|_{-\tilde{\beta}/2}e^{-\tilde{\beta}t_0/2}\Big[e^{-\tilde{\beta}t_0/2}\|V\|_{-\tilde{\beta}/2} + |\theta-\tilde{\theta}|\Big]dt_0 \\ &= C_1C_2\|V\|_{-\tilde{\beta}/2}\Big[\tfrac{1}{\tilde{\beta}}e^{-\tilde{\beta}t}\|V\|_{-\tilde{\beta}/2} + \tfrac{2}{\tilde{\beta}}e^{-\tilde{\beta}t/2}|\theta-\tilde{\theta}|\Big]. \end{aligned} \tag{2.8.49}$$

Observe that if we multiply this final function with $e^{\frac{\tilde{\beta}}{2}t}$ we still have a bounded function. Since this holds for all p we see that

$$\|\mathcal{J}\|_{-\tilde{\beta}/2} \leq C_5\|V\|_{-\tilde{\beta}/2}\Big[\|V\|_{-\tilde{\beta}/2} + |\theta - \tilde{\theta}|\Big] \tag{2.8.50}$$

for some constant C_5, which is independent of p.

Finally, Lemma 2.8.4 yields the bound

$$\|\tilde{\mathcal{G}}(t+\tilde{\theta},\tilde{\theta})W_s\|_{\ell^p} \leq C_3 e^{-\tilde{\beta}t}\|W_s\|_{\ell^p}, \tag{2.8.51}$$

which means

$$\|\tilde{\mathcal{G}}(t+\tilde{\theta},\tilde{\theta})W_s\|_{-\tilde{\beta}/2} \leq C_3\|W_s\|_{\ell^p}. \tag{2.8.52}$$

This yields the bound

$$\|T(W_s,V)\|_{-\tilde{\beta}/2} \leq C_3\|W_s\|_{\ell^p} + (C_4+C_5)\|V\|_{-\tilde{\beta}/2}\Big[\|V\|_{-\tilde{\beta}/2} + |\theta - \tilde{\theta}|\Big]. \tag{2.8.53}$$

Let $V_1 \in BC_{-\tilde{\beta}/2}([0,\infty),\ell^p)$ and $V_2 \in BC_{-\tilde{\beta}/2}([0,\infty),\ell^p)$ with $\|V_1\|_{-\tilde{\beta}/2} \leq 1$ and $\|V_2\|_{-\tilde{\beta}/2} \leq 1$. Again, we write

$$d\mathcal{M} = \mathcal{M}^{\tilde{\theta}}\Big(\theta,t,V_1(t)\Big) - \mathcal{M}^{\tilde{\theta}}\Big(\theta,t,V_2(t)\Big). \tag{2.8.54}$$

Using Lemma 2.8.3 it follows that

$$\|d\mathcal{M}\|_{\ell^p} \leq C_2\Big[\|V_1\|_{\ell^p} + \|V_2\|_{\ell^p} + |\theta-\tilde{\theta}|\Big]\|V_1 - V_2\|_{\ell^p}. \tag{2.8.55}$$

Mimicking the above computations, this yields

$$\begin{aligned}\|T(W_s,V_1) - T(W_s,V_2)\|_{-\tilde{\beta}/2} \leq{}& (C_4+C_5)\|V_1-V_2\|_{-\tilde{\beta}/2}\Big[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2} \\ &+|\theta-\tilde{\theta}|\Big].\end{aligned} \tag{2.8.56}$$

We now fix

$$\delta = \min\{1, \tfrac{1}{4(C_4+C_5)}\} \tag{2.8.57}$$

and

$$\eta = \min\{\tfrac{1}{4(C_4+C_5)}, \tfrac{\delta}{4C_3}\}. \tag{2.8.58}$$

For each $V \in BC_{-\tilde{\beta}/2}([0,\infty),\ell^p)$, $V_1 \in BC_{-\tilde{\beta}/2}([0,\infty),\ell^p)$ and $V_2 \in BC_{-\tilde{\beta}/2}([0,\infty),\ell^p)$ with $\|V\|_{-\tilde{\beta}/2} \leq \delta$, $\|V_1\|_{-\tilde{\beta}/2} \leq \delta$ and $\|V_2\|_{-\tilde{\beta}/2} \leq \delta$, each $\theta \in \mathbb{R}$ with $|\theta - \tilde{\theta}| < \eta$ and each $W_s \in \ell^p$ with $\|W_s\|_{\ell^p} < \eta$, we now obtain

$$\begin{aligned}\|T(W_s,V)\|_{-\tilde{\beta}/2} &\leq \tfrac{\delta}{4} + \delta\Big[\tfrac{1}{4}+\tfrac{1}{4}\Big] \\ &\leq \delta\end{aligned} \tag{2.8.59}$$

and

$$\|T(W_s,V_1) - T(W_s,V_2)\|_{-\tilde{\beta}/2} \leq \Big[\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{4}\Big]\|V_1-V_2\|_{-\tilde{\beta}/2}. \tag{2.8.60}$$

Hence we see that the equation (2.8.35) has a unique solution $\mathcal{W}^{\tilde{\theta}}_{*;\theta}(W_s)$. ■

Lemma 2.8.7. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. For each $W_s \in \mathrm{Range}\Big(\Pi^s(\tilde{\theta})\Big)$ with $\|W_s\|_{\ell^p} \leq \eta$ and each $|\theta - \tilde{\theta}| \leq \eta$ we have $\Pi^s(\tilde{\theta})\mathcal{W}^{\tilde{\theta}}_{*;\theta}(W_s)(0) = W_s$ and $\|\Pi^c(\tilde{\theta})\mathcal{W}^{\tilde{\theta}}_{*;\theta}(W_s)(0)\|_{\ell^p} \leq C_6\|W_s\|^2_{\ell^p}$ for some constant $C_6 > 0$, which is independent of p and $\tilde{\theta}$.*

Proof. It is clear that

$$\begin{array}{rcl} \Pi^s(\tilde{\theta})\tilde{\mathcal{G}}(\tilde{\theta}, t_0 + \tilde{\theta}) & = & \tilde{\mathcal{G}}(\tilde{\theta}, \tilde{\theta} + t_0), \\ \Pi^c(\tilde{\theta})\mathcal{E}(\tilde{\theta}, \tilde{\theta} + t_0) & = & \mathcal{E}(\tilde{\theta}, t_0 + \tilde{\theta}). \end{array} \tag{2.8.61}$$

This allows us to compute

$$\begin{array}{rcl} \Pi^s(\tilde{\theta})\mathcal{W}^{\tilde{\theta}}_{*;\theta}(W_s)(0) & = & \Pi^s(\tilde{\theta})\tilde{\mathcal{G}}(\tilde{\theta}, \tilde{\theta})W_s \\ & & \quad +\Pi^s(\tilde{\theta}) \int_0^0 \tilde{\mathcal{G}}(\tilde{\theta}, t_0 + \tilde{\theta})\Pi^s(t_0 + \tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta, t_0, \mathcal{W}^{\tilde{\theta}}_{*;\theta}(W_s)(t_0)\Big)dt_0 \\ & & \quad +\Pi^s(\tilde{\theta}) \int_\infty^0 \mathcal{E}(\tilde{\theta}, t_0 + \tilde{\theta})\Pi^c(t_0 + \tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta, t_0, \mathcal{W}^{\tilde{\theta}}_{*;\theta}(W_s)(t_0)\Big)dt_0 \\ & = & \Pi^s(\tilde{\theta})W_s \\ & & \quad + \int_\infty^0 \Pi^s(\tilde{\theta})\mathcal{E}(\tilde{\theta}, t_0 + \tilde{\theta})\Pi^c(t_0 + \tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta, t_0, \mathcal{W}^{\tilde{\theta}}_{*;\theta}(W_s)(t_0)\Big)dt_0 \\ & = & W_s + 0 \\ & = & W_s, \end{array} \tag{2.8.62}$$

together with

$$\begin{array}{rcl} \Pi^c(\tilde{\theta})\mathcal{W}^{\tilde{\theta}}_{*;\theta}(W_s)(0) & = & \Pi^c(\tilde{\theta})\tilde{\mathcal{G}}(\tilde{\theta}, \tilde{\theta})W_s \\ & & \quad +\Pi^c(\tilde{\theta}) \int_0^0 \tilde{\mathcal{G}}(\tilde{\theta}, t_0 + \tilde{\theta})\Pi^s(t_0 + \tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta, t_0, \mathcal{W}^{\tilde{\theta}}_{*;\theta}(W_s)(t_0)\Big)dt_0 \\ & & \quad +\Pi^c(\tilde{\theta}) \int_\infty^0 \mathcal{E}(\tilde{\theta}, t_0 + \tilde{\theta})\Pi^c(t_0 + \tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta, t_0, \mathcal{W}^{\tilde{\theta}}_{*;\theta}(W_s)(t_0)\Big)dt_0 \\ & = & \int_\infty^0 \mathcal{E}(\tilde{\theta}, t_0 + \tilde{\theta})\Pi^c(t_0 + \tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta, t_0, \mathcal{W}^{\tilde{\theta}}_{*;\theta}(W_s)(t_0)\Big)dt_0. \end{array} \tag{2.8.63}$$

We assume without loss of generality that η is small enough to ensure

$$(C_4 + C_5)\|\mathcal{W}^{\tilde{\theta}}_{*;\theta}(W_s)\|_{-\tilde{\beta}/2}\Big[\|\mathcal{W}^{\tilde{\theta}}_{*;\theta}(W_s)\|_{-\tilde{\beta}/2} + |\theta - \tilde{\theta}|\Big] \quad \leq \quad \tfrac{1}{2}\|\mathcal{W}^{\tilde{\theta}}_{*;\theta}(W_s)\|_{-\tilde{\beta}/2}. \tag{2.8.64}$$

Using (2.8.53), we obtain

$$
\begin{array}{lll}
\|\int_{\infty}^{0} \mathcal{E}(\tilde{\theta}, t_0 + \tilde{\theta})\Pi^c(t_0 + \tilde{\theta})\mathcal{M}^{\tilde{\theta}}\Big(\theta, t_0, \mathcal{W}^{\tilde{\theta}}_{*;\theta}(W_s)(t_0)\Big)dt_0\|_{\ell^p} & \leq & C_5\Big[\|\mathcal{W}^{\tilde{\theta}}_{*;\theta}(W_s)\|_{-\tilde{\beta}/2} \\
& & \qquad +|\theta - \tilde{\theta}|\Big] \\
& \leq & 4C_5C_3^2\|W_s\|^2_{\ell^p} \\
& := & C_6\|W_s\|^2_{\ell^p}.
\end{array}
\tag{2.8.65}
$$

This yields the desired estimate

$$
\|\Pi^c(\tilde{\theta})\mathcal{W}^{\tilde{\theta}}_{*;\theta}(W_s)\|_{\ell^p} \quad \leq \quad C_6\|W_s\|^2_{\ell^p}. \tag{2.8.66}
$$

■

Exploiting Lemma 2.7.8 and Lemma 2.7.9, we pick $C_7 > 0$ in such a way that

$$
|\Phi^{\pm}(\xi)| + |(\Phi^{\pm})'(\xi)| \quad \leq \quad C_7 e^{-\tilde{\beta}|\xi|} \tag{2.8.67}
$$

holds for all $\xi \in \mathbb{R}$, decreasing $\tilde{\beta}$ if necessary.

Lemma 2.8.8. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. Then there exists a constant $C_9 > 0$, independent of p such that for each $\theta \in \mathbb{R}$ we have the bound*

$$
\|\Phi^+(h \cdot + \theta)\|_{\ell^p} \quad \leq \quad C_9. \tag{2.8.68}
$$

In addition, for each $\theta \in \mathbb{R}$ and each sequence $\{\xi(j)\}$ with $\|\xi_j\|_\infty \leq 1$, we have the bound

$$
\|(\Phi^-)'(h \cdot + \xi(\cdot))\|_{\ell^1} \quad \leq \quad C_9. \tag{2.8.69}
$$

Proof. Note that for each $k \in \mathbb{Z}$ we have

$$
\begin{array}{lll}
\|\Phi^+(h \cdot + \theta_1)\|^p_{\ell^p} & = & \sum_{j\in\mathbb{Z}} |\Phi^+(hj + \theta_1)|^p \\
& = & \sum_{j\in\mathbb{Z}} |\Phi^+\Big(hj + hk + (\theta_1 - hk)\Big)|^p \\
& = & \sum_{j\in\mathbb{Z}} |\Phi^+\Big(hj + (\theta_1 - hk)\Big)|^p \\
& = & \|\Phi^+\Big(h \cdot + (\theta_1 - hk)\Big)\|^p_{\ell^p}.
\end{array}
\tag{2.8.70}
$$

Hence we assume without loss of generality that $|\theta_1| \leq 1$. We see with (2.8.67) that there is a constant $C_8 > 0$ such that

$$
\begin{array}{lll}
\|\Phi^+(h \cdot + c\theta_1)\|_{\ell^p} & \leq & C_8\|e^{-\tilde{\beta}|h\cdot+\theta_1|}|\|_{\ell^p(\mathbb{Z},\mathbb{R})} \\
& \leq & C_8\|e^{-\tilde{\beta}|h\cdot|}e^{\tilde{\beta}|\theta_1|}\|_{\ell^p(\mathbb{Z},\mathbb{R})} \\
& \leq & C_8 e^{\tilde{\beta}|\theta_1|}\|e^{-\tilde{\beta}p|h\cdot|}\|^{\frac{1}{p}}_{\ell^1(\mathbb{Z},\mathbb{R})} \\
& \leq & 2C_8\max\{1, \|e^{-\tilde{\beta}|h\cdot|}\|_{\ell^1(\mathbb{Z},\mathbb{R})}\} \\
& := & C_9,
\end{array}
\tag{2.8.71}
$$

if we assume that $\tilde{\beta}$ is small enough such that $e^{\tilde{\beta}|\theta_1|} \leq e^{\tilde{\beta}} \leq 2$. A similar calculation yields

$$\begin{array}{rcl} \|(\Phi^-)'(h\cdot+\xi(\cdot))\|_{\ell^1} & \leq & C_8\|e^{-\tilde{\beta}|h\cdot+\xi(\cdot)|}\|_{\ell^1(\mathbb{Z},\mathbb{R})} \\ & \leq & C_8\|e^{-\tilde{\beta}|h\cdot|}e^{\tilde{\beta}}\|_{\ell^1(\mathbb{Z},\mathbb{R})} \\ & \leq & C_8 e^{\tilde{\beta}}\|e^{-\tilde{\beta}|h\cdot|}\|_{\ell^1(\mathbb{Z},\mathbb{R})} \\ & \leq & 2C_8\max\{1,\|e^{-\tilde{\beta}|h\cdot|}\|_{\ell^1(\mathbb{Z},\mathbb{R})}\} \\ & = & C_9. \end{array} \tag{2.8.72}$$

■

We have

$$(\Pi^c(\theta)V)_j = \frac{1}{\Omega}\sum_{j_0\in\mathbb{Z}}\langle\Phi^-(hj_0+c\theta),V_{j_0}\rangle\Phi^+(hj+c\tilde{\theta}). \tag{2.8.73}$$

For notational compactness we write

$$\Pi^c(\theta)V = \overline{\lambda}^c(\theta)(V)\Phi^+(h\cdot+c\theta). \tag{2.8.74}$$

Lemma 2.8.9. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. For $V \in \ell^p$ and $\theta_1,\theta_2 \in \mathbb{R}$ with $|\theta_2-\theta_1| \leq 1$ we have the bounds*

$$\|\Pi^c(\theta_1)\Pi^s(\theta_2)V\|_{\ell^p} \leq C_{10}|\theta_1-\theta_2|\|V\|_{\ell^p} \tag{2.8.75}$$

and

$$\|\Pi^s(\theta_1)\Pi^c(\theta_2)V\|_{\ell^p} \leq C_{10}|\theta_1-\theta_2|\|V\|_{\ell^p} \tag{2.8.76}$$

for some constant $C_{10} > 0$ which does not depend on p.

Proof. Writing

$$\mathcal{P} = \overline{\lambda}^c(\theta_1)\Big(\Pi^s(\theta_2)V\Big), \tag{2.8.77}$$

we obtain

$$\begin{array}{rcl} \|\mathcal{P}\|_{\ell^p} & = & |\frac{1}{\Omega}\sum_{j_0\in\mathbb{Z}}\langle\Phi^-(hj_0+c\theta_1),(\Pi^s(\theta_2)V)_{j_0}\rangle| \\ & = & \Omega^{-1}\left|\sum_{j_0\in\mathbb{Z}}\langle\Phi^-(hj_0+c\theta_1)-\Phi^-(hj_0+c\theta_2),(\Pi^s(\theta_2)V)_{j_0}\rangle\right| \\ & \leq & \Omega^{-1}\sum_{j_0\in\mathbb{Z}}|\Phi^-(hj_0+c\theta_1)-\Phi^-(hj_0+c\theta_2)||(\Pi^s(\theta_2)V)_{j_0}| \\ & \leq & \Omega^{-1}\|\Phi^-(h\cdot+c\theta_1)-\Phi^-(h\cdot+c\theta_2)\|_{\ell^1}\|\Pi^s(\theta_2)V\|_{\ell^\infty} \\ & \leq & \Omega^{-1}|\theta_1-\theta_2|\|c(\Phi^-)'(h\cdot+\xi(\cdot))\|_{\ell^1}\|\Pi^s(\theta_2)V\|_{\ell^p}, \end{array} \tag{2.8.78}$$

where each $\xi(j)$ is in between $c\theta_1$ and $c\theta_2$. Thus we obtain with Lemma 2.8.4 and Lemma 2.8.8

$$\begin{array}{rcl} \|\Pi^c(\theta_1)\Pi^s(\theta_2)V\|_{\ell^p} & \leq & \frac{1}{\Omega}C_9|c||\theta_1-\theta_2|\|(\Pi^s(\theta_2)V)\|_{\ell^p}C_9 \\ & \leq & \frac{1}{\Omega}C_9|c||\theta_1-\theta_2|C_3\|V\|_{\ell^p}C_9 \\ & \leq & \frac{1}{2}C_{10}|\theta_1-\theta_2|\|V\|_{\ell^p} \end{array} \tag{2.8.79}$$

for some constant $C_{10} > 0$ which is independent of p.

Furthermore we can compute

$$\begin{array}{rcl}\Pi^s(\theta_1)\Pi^c(\theta_2)V & = & \Big[I - \Pi^c(\theta_1)\Big]\Big[I - \Pi^s(\theta_2)\Big]V \\ & = & V - \Pi^c(\theta_1)V - \Pi^s(\theta_2)V + \Pi^c(\theta_1)\Pi^s(\theta_2)V \\ & = & -\Pi^c(\theta_1)V + \Pi^c(\theta_2)V + \Pi^c(\theta_1)\Pi^s(\theta_2)V.\end{array} \tag{2.8.80}$$

This allows us to estimate

$$\begin{array}{rcl}\|{-\Pi^c(\theta_1)V} + \Pi^c(\theta_2)V\|_{\ell^p} & \leq & |\overline{\lambda}^c(\theta_1)(V) - \overline{\lambda}^c(\theta_2)(V)|\|\Phi^+(\theta_1)\|_{\ell^p} \\ & & \quad +|\lambda(\theta_2)|\|\Phi^+(\theta_1) - \Phi^+(\theta_2)\|_{\ell^p} \\ & \leq & C_9|\theta_1 - \theta_2|C_9\|V\|_{\ell^p} \\ & & \quad +C_9|\theta_2||\theta_1 - \theta_2|\|(\Phi^-)'(h\cdot + \eta(\cdot))\|_{\ell^1}\|V\|_{\ell^p} \\ & \leq & C_9|\theta_1 - \theta_2|C_9\|V\|_{\ell^p} + C_9|\theta_2||\theta_1 - \theta_2|C_9\|V\|_{\ell^p} \\ & \leq & \frac{1}{2}C_{10}|\theta_1 - \theta_2|\|V\|_{\ell^p},\end{array} \tag{2.8.81}$$

where each $\eta(j)$ is between $c\theta_1$ and $c\theta_2$. We thus obtain

$$\|\Pi^s(\theta_1)\Pi^c(\theta_2)V\|_{\ell^p} \quad \leq \quad C_{10}|\theta_1 - \theta_2|\|V\|_{\ell^p}. \tag{2.8.82}$$

■

Proof of Proposition 2.8.1. We set

$$\mathcal{U}_*^{\tilde{\theta}}(W_s) \quad = \quad \mathcal{W}^{\tilde{\theta}}_{*;\tilde{\theta}}(W_s) \tag{2.8.83}$$

for all $\tilde{\theta} \in \mathbb{R}$.

Fix $\tilde{\theta} \in \mathbb{R}$ and pick $\theta \in \mathbb{R}$ with $|\theta - \tilde{\theta}| \leq \eta$ and pick $W_s \in \mathrm{Range}(\Pi^s(\theta))$ with $\|W_s\|_{\ell^p} \leq \eta$. By uniqueness of the solution to (2.8.35) it follows that

$$\mathcal{U}_*^{\theta}(W_s) \quad = \quad \mathcal{W}^{\tilde{\theta}}_{*;\theta}(W_s^0) \tag{2.8.84}$$

for some $W_s^0 \in \mathrm{Range}\Big(\Pi^s(\tilde{\theta})\Big)$. Since $\Pi^s(\tilde{\theta})\mathcal{W}^{\tilde{\theta}}_{*;\theta}(\cdot)(0)$ is the identity map on $\mathrm{Range}\Big(\Pi^s(\tilde{\theta})\Big)$, it follows that

$$W_s^0 \quad = \quad \Pi^s(\tilde{\theta})\mathcal{U}_*^{\theta}(W_s)(0). \tag{2.8.85}$$

We now see

$$\begin{array}{rcl}W_s^0 - W_s & = & \Pi^s(\tilde{\theta})\mathcal{U}_*^{\theta}(W_s)(0) - W_s \\ & = & \Pi^s(\tilde{\theta})\Big[\mathcal{U}_*^{\theta}(W_s)(0) - W_s\Big] + \Pi^s(\tilde{\theta})W_s - W_s \\ & = & \Pi^s(\tilde{\theta})\Pi^c(\theta)\mathcal{U}_*^{\theta}(W_s)(0) + W_s - \Pi^c(\tilde{\theta})W_s - W_s \\ & = & \Pi^s(\tilde{\theta})\Pi^c(\theta)\mathcal{U}_*^{\theta}(W_s)(0) - \Pi^c(\tilde{\theta})\Pi^s(\theta)W_s.\end{array} \tag{2.8.86}$$

Lemma 2.8.9 hence implies

$$\begin{array}{lcl} \|W_s^0 - W_s\|_{\ell^p} & \leq & C_{10}|\tilde{\theta} - \theta| \|\mathcal{U}_*^{\theta}(W_s)(0)\|_{\ell^p} + C_{10}|\tilde{\theta} - \theta| \|W_s\|_{\ell^p} \\ & \leq & C_{10}|\tilde{\theta} - \theta| \Big[\|W_s\|_{\ell^p} + C_6 \|W_s\|_{\ell^p}^2 \Big] + C_{10}|\tilde{\theta} - \theta| \|W_s\|_{\ell^p} \\ & \leq & C_{11}|\tilde{\theta} - \theta| \|W_s\|_{\ell^p}. \end{array} \quad (2.8.87)$$

Now fix $W_s^{\tilde{\theta}} \in \mathrm{Range}\Big(\Pi^s(\tilde{\theta})\Big)$ with $\|W_s^{\tilde{\theta}}\|_{\ell^p} \leq \eta$. Then we can compute

$$\begin{array}{lcl} \Pi^c(\theta)\mathcal{U}_*^{\theta}(W_s)(0) - \Pi^c(\tilde{\theta})\mathcal{U}_*^{\tilde{\theta}}(W_s^{\tilde{\theta}})(0) & = & \mathcal{U}_*^{\theta}(W_s)(0) - W_s - \Pi^c(\tilde{\theta})\mathcal{W}_{*;\tilde{\theta}}^{\tilde{\theta}}(W_s^{\tilde{\theta}}) \\ & = & \mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s^0)(0) - W_s - \mathcal{W}_{*;\tilde{\theta}}^{\tilde{\theta}}(W_s^{\tilde{\theta}})(0) + W_s^{\tilde{\theta}} \\ & = & W_s^0 - W_s + \mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s^0)(0) \\ & & \qquad -W_s^0 - \mathcal{W}_{*;\tilde{\theta}}^{\tilde{\theta}}(W_s^{\tilde{\theta}})(0) + W_s^{\tilde{\theta}}. \end{array} \quad (2.8.88)$$

Writing

$$\begin{array}{lcl} V_1 & = & \mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s^0), \\ V_2 & = & \mathcal{W}_{*;\tilde{\theta}}^{\tilde{\theta}}(W_s^{\tilde{\theta}}), \end{array} \quad (2.8.89)$$

we can mimic the steps in (2.8.56) to obtain the estimate

$$\begin{array}{lcl} \|V_1(0) - W_s^0 - V_2(0) + W_s^{\tilde{\theta}}\|_{\ell^p} & \leq & C_{12}\Big[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2} + |\tilde{\theta} - \theta|\Big]\|V_1 - V_2\|_{-\tilde{\beta}/2} \\ & & +C_{12}|\tilde{\theta} - \theta|\Big[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2}\Big] \end{array} \quad (2.8.90)$$

for some constant $C_{12} > 0$, which is independent of p and $\tilde{\theta}$. Without loss of generality we can assume that η is small enough to ensure

$$C_{12}\Big[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2} + |\tilde{\theta} - \theta|\Big] \quad \leq \quad \tfrac{1}{2}. \quad (2.8.91)$$

An estimate similar to (2.8.56) therefore yields

$$\begin{array}{lcl} \|V_1 - V_2\|_{-\tilde{\beta}/2} & \leq & C_3\|W_s^0 - W_s^{\tilde{\theta}}\|_{\ell^p} \\ & & \quad +C_{12}\Big[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2} + |\tilde{\theta} - \theta|\Big]\|V_1 - V_2\|_{-\tilde{\beta}/2} \\ & & \quad +C_{12}|\tilde{\theta} - \theta|\Big[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2}\Big] \\ & \leq & C_3\|W_s - W_s^{\tilde{\theta}}\|_{\ell^p} + C_3\|W_s - W_s^0\|_{\ell^p} + \tfrac{1}{2}\|V_1 - V_2\|_{-\tilde{\beta}/2} \\ & & \quad +C_{12}|\tilde{\theta} - \theta|\Big[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2}\Big] \\ & \leq & C_3\|W_s - W_s^{\tilde{\theta}}\|_{\ell^p} + C_3C_{11}|\tilde{\theta} - \theta|\|W_s\|_{\ell^p} + \tfrac{1}{2}\|V_1 - V_2\|_{-\tilde{\beta}/2} \\ & & \quad +C_{12}|\tilde{\theta} - \theta|\Big[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2}\Big], \end{array} \quad (2.8.92)$$

and thus

$$\begin{array}{lcl} \|V_1 - V_2\|_{-\tilde{\beta}/2} & \leq & 2C_3\|W_s - W_s^{\tilde{\theta}}\|_{\ell^p} + 2C_3C_{11}|\tilde{\theta} - \theta|\|W_s\|_{\ell^p} \\ & & +2C_{12}|\tilde{\theta} - \theta|\Big[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2}\Big]. \end{array} \tag{2.8.93}$$

Exploiting (2.8.65), this yields

$$\begin{array}{rcl} dP & := & \|\Pi^c(\theta)\mathcal{U}_*^{\theta}(W_s)(0) - \Pi^c(\tilde{\theta})\mathcal{U}_*^{\tilde{\theta}}(W_s^{\tilde{\theta}})(0)\|_{\ell^p} \\ & \leq & C_{11}|\tilde{\theta} - \theta|\|W_s\|_{\ell^p} + C_{12}\Big[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2} + |\tilde{\theta} - \theta|\Big]\|V_1 - V_2\|_{-\tilde{\beta}/2} \\ & & +C_{12}|\tilde{\theta} - \theta|\Big[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2}\Big] \\ & \leq & C_{13}\Big[\|W_s\|_{\ell^p} + \|W_s^{\tilde{\theta}}\|_{\ell^p} + |\tilde{\theta} - \theta|\Big]\|W_s - W_s^{\tilde{\theta}}\|_{\ell^p} \\ & & +C_{13}|\tilde{\theta} - \theta|\Big[\|W_s\|_{\ell^p} + \|W_s^{\tilde{\theta}}\|_{\ell^p}\Big] \end{array} \tag{2.8.94}$$

for some constant C_{13}, which is independent of p and $\tilde{\theta}$. ∎

We now expand upon the ideas developed in the second half of [109, Prop. 2.1] to foliate the state space surrounding the travelling pulses $\overline{U}(\cdot + \tilde{\theta})$ by the stable manifolds constructed above. We proceed by showing that these stable manifolds depend continuously on $\tilde{\theta}$. This allows us to set up an appropriate fixed point problem to establish Proposition 2.8.2.

We write

$$\overline{U}(\tilde{\theta}) = \overline{U}(0) - \tilde{\theta}\overline{U}'(0) + \mathcal{N}_1^{\tilde{\theta}}. \tag{2.8.95}$$

Lemma 2.8.10. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. Then we have the bounds*

$$\|\mathcal{N}_1^{\tilde{\theta}}\|_{\ell^p} \leq C_{14}\tilde{\theta}^2 \tag{2.8.96}$$

and

$$\|\mathcal{N}_1^{\tilde{\theta}_2} - \mathcal{N}_1^{\tilde{\theta}_1}\|_{\ell^p} \leq C_{15}(|\tilde{\theta}_1| + |\tilde{\theta}_2|)|\tilde{\theta}_1 - \tilde{\theta}_2|, \tag{2.8.97}$$

for $\tilde{\theta}, \tilde{\theta}_1, \tilde{\theta}_2 \in [-\eta, \eta]$ *and for some constants* $C_{14} > 0$ *and* $C_{15} > 0$*, which do not depend on* p*.*

Proof. Using Lemma 2.7.9 we see that there exists a sequence $\{\xi_j\}$ with $|\xi_j| \leq |\tilde{\theta}|$ such that

$$\begin{array}{rcl} \|\mathcal{N}_1^{\tilde{\theta}}\|_{\ell^p} & = & \frac{1}{2}\|\{\tilde{\theta}^2\overline{U}''(\xi_j)\}\|_{\ell^p} \\ & \leq & \frac{1}{2}C_7\tilde{\theta}^2\|\{e^{-\tilde{\beta}|hj + c\xi_j|}\}\|_p \\ & \leq & \frac{1}{2}C_7\tilde{\theta}^2 e^{\tilde{\beta}|c\tilde{\theta}|}\|\{e^{-\tilde{\beta}|hj|}\}\|_p \\ & \leq & C_7\tilde{\theta}^2\|\{e^{-\tilde{\beta}|hj|}\}\|_p \\ & := & C_{14}\tilde{\theta}^2, \end{array} \tag{2.8.98}$$

where C_{14} does not depend on p as before. We can hence write

$$\overline{U}(\tilde{\theta}) \quad = \quad \overline{U}(0) - \tilde{\theta}\overline{U}'(0) + \mathcal{N}_1^{\tilde{\theta}} \tag{2.8.99}$$

with

$$\|\mathcal{N}_1^{\tilde{\theta}}\|_{\ell^p} \quad \leq \quad C_{14}\tilde{\theta}^2. \tag{2.8.100}$$

Furthermore, using Lemma 2.7.9, we see that we can find sequences $\{\xi_j\}$ and $\{\eta_j\}$ with ξ_j between $hj + c\tilde{\theta}_1$ and $hj + c\tilde{\theta}_2$ and η_j between hj and $hj + c\tilde{\theta}_1$ so that

$$\begin{array}{lcl} \|\mathcal{N}_1^{\tilde{\theta}_2} - \mathcal{N}_1^{\tilde{\theta}_1}\|_{\ell^p} & = & \|\overline{U}(\tilde{\theta}_1) + \tilde{\theta}_1\overline{U}'(0) - \overline{U}(\tilde{\theta}_2) - \tilde{\theta}_2\overline{U}'(0)\|_{\ell^p} \\ & \leq & |\tilde{\theta}_1 - \tilde{\theta}_2|^2\|\{\overline{U}''(\xi_j)\}\|_{\ell^p} + |\tilde{\theta}_1 - \tilde{\theta}_2|\|\overline{U}'(0) - \overline{U}'(\tilde{\theta}_1)\|_{\ell^p} \\ & \leq & |\tilde{\theta}_1 - \tilde{\theta}_2|^2\|\{\overline{U}''(\xi_j)\}\|_{\ell^p} + |\tilde{\theta}_1 - \tilde{\theta}_2||\tilde{\theta}_1|\|\{\overline{U}''(\eta_j)\}\|_{\ell^p} \\ & \leq & C_{15}(|\tilde{\theta}_1| + |\tilde{\theta}_2|)|\tilde{\theta}_1 - \tilde{\theta}_2|, \end{array} \tag{2.8.101}$$

similarly to the calculations from Lemma 2.8.3. ∎

We write

$$\mathcal{N}_2^{\tilde{\theta}}(W) \quad = \quad \mathcal{U}_*^{\tilde{\theta}}(\Pi^s(\tilde{\theta})W)(0) - \Pi^s(\tilde{\theta})W \tag{2.8.102}$$

for $W \in \mathrm{Range}(\Pi^s(0))$ with $\|\Pi^s(\tilde{\theta})W\|_{\ell^p} < \eta$. We note that Lemma 2.8.4 implies that this inequality holds if $\|W\|_{\ell^p} < \frac{\eta}{C_3}$.

Lemma 2.8.11. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. Recall the constants C_6 and C_{13} appearing in Proposition 2.8.1 and Lemma 2.8.4. Then for any $\tilde{\theta} \in [-\eta, \eta]$ and $W \in \mathrm{Range}\Big(\Pi^s(0)\Big)$ with $\|W\|_{\ell^p} < \frac{\eta}{C_3}$ we have the bound*

$$\|\mathcal{N}_2^{\tilde{\theta}}(W)\|_{\ell^p} \quad \leq \quad C_6\|V\|_{\ell^p}^2. \tag{2.8.103}$$

In addition, for any $\tilde{\theta}_1, \tilde{\theta}_2 \in [-\eta, \eta]$ and $W_1, W_2 \in \mathrm{Range}\Big(\Pi^s(0)\Big)$ with $\|W_1\|_{\ell^p} < \frac{\eta}{C_3}$ and $\|W_2\|_{\ell^p} < \frac{\eta}{C_3}$, we have

$$\begin{array}{lcl} \|\mathcal{N}_2^{\tilde{\theta}_2}(W_2) - \mathcal{N}_2^{\tilde{\theta}_1}(W_1)\|_{\ell^p} & \leq & C_{13}\Big[\|W_1\|_{\ell^p} + \|W_2\|_{\ell^p} + |\theta_1 - \theta_2|\Big]\|W_1 - W_2\|_{\ell^p} \\ & & +C_{13}|\theta_1 - \theta_2|\Big[\|W_1\|_{\ell^p} + \|W_2\|_{\ell^p}\Big]. \end{array} \tag{2.8.104}$$

Proof. Note that

$$\Pi^s(\tilde{\theta})\mathcal{N}_2^{\tilde{\theta}}(W) = 0, \tag{2.8.105}$$

so that

$$\begin{array}{lcl} \mathcal{N}_2^{\tilde{\theta}}(W) & = & \Pi^c(\tilde{\theta})U_*^{\tilde{\theta}}(W)(0) - \Pi^c(\tilde{\theta})\Pi^s(\tilde{\theta})W \\ & = & \Pi^c(\tilde{\theta})U_*^{\tilde{\theta}}(W)(0). \end{array} \tag{2.8.106}$$

Therefore, both bounds follow from Proposition 2.8.1. ∎

Let $\delta > 0$ be a small constant, which we will determine later. Pick U^0 in such a way that $\|U^0 - \overline{U}(0)\|_{\ell^p} < \delta$. We write $U^0 = \overline{U}(0) + V_0$.

Our goal is to find a small $W \in \mathrm{Range}\Big(\Pi^s(0)\Big)$ and a small $\tilde{\theta}$ in such a way that

$$V^0 + \overline{U}(0) \quad = \quad \overline{U}(\tilde{\theta}) + U_*^{\tilde{\theta}}(\Pi^s(\tilde{\theta})W)(0). \tag{2.8.107}$$

Using our notation from above we see that

$$\begin{aligned} \overline{U}(\tilde{\theta}) + U_*^{\tilde{\theta}}(\Pi^s(\tilde{\theta})W)(0) &= \overline{U}(\tilde{\theta}) + \Pi^s(\tilde{\theta})W + \mathcal{N}_2^{\tilde{\theta}}(W) \\ &= \overline{U}(0) + \tilde{\theta}\overline{U}'(0) + \mathcal{N}_1^{\tilde{\theta}} + \Pi^s(\tilde{\theta})W + \mathcal{N}_2^{\tilde{\theta}}(W) \\ &= \overline{U}(0) + \tilde{\theta}\overline{U}'(0) + \mathcal{N}_1^{\tilde{\theta}} + W - \Pi^c(\tilde{\theta})W + \mathcal{N}_2^{\tilde{\theta}}(W), \end{aligned} \tag{2.8.108}$$

which means that (2.8.107) can be written as

$$V^0 \quad = \quad \tilde{\theta}\overline{U}'(0) + \mathcal{N}_1^{\tilde{\theta}} + W - \Pi^c(\tilde{\theta})W + \mathcal{N}_2^{\tilde{\theta}}(W) \tag{2.8.109}$$

We write $\overline{\lambda}^c : \mathrm{Range}(\Pi^c(0)) \to \mathbb{R}$ for the map $\mu\overline{U}' \mapsto \mu$. This allow us to rephrase (2.8.109) as the fixed point problem

$$\begin{cases} \Pi^s(0)V^0 &= \Pi^s(0)\mathcal{N}_1^{\tilde{\theta}} + W - \Pi^s(0)\Pi^c(\tilde{\theta})(W) + \Pi^s(0)\Big(\mathcal{N}_2^{\tilde{\theta}}(W)\Big) \\ \overline{\lambda}^c\Big[\Pi^c(0)V^0\Big] &= \tilde{\theta} + \overline{\lambda}^c\Big[\Pi^c(0)\mathcal{N}_1^{\tilde{\theta}}\Big] - \overline{\lambda}^c\Big[\Pi^c(0)\Pi^c(\tilde{\theta})(W)\Big] + \overline{\lambda}^c\Big[\Pi^c(0)\Big(\mathcal{N}_2^{\tilde{\theta}}(W)\Big)\Big]. \end{cases} \tag{2.8.110}$$

We show that equation (2.8.110) has a solution in the space

$$X_{\kappa,\varepsilon_\theta} \quad := \quad \{V \in \mathrm{Range}(\Pi^s(0)) : \|V\|_{\ell^p} \leq \kappa\} \times [-\varepsilon_\theta, \varepsilon_\theta] \tag{2.8.111}$$

for some $\kappa, \varepsilon_\theta$ which we will determine later. Without loss of generality we assume that $\kappa, \varepsilon_\theta$ are small enough such that all previous inequalities hold.

Lemma 2.8.12. *Assume that (HP1),(HP2), (HS), (Hα1) and (Hα2) are satisfied. There are small constants $\delta > 0$ $\kappa > 0$ and $\varepsilon_\theta > 0$, independent of p, such that for each $V^0 \in \ell^p$ with $\|V^0\|_{\ell^p} < \delta$ the fixed point problem (2.8.110) has a unique solution $(W, \tilde{\theta}) \in X_{\kappa,\varepsilon_\theta}$. Moreover there is a constant $C_{19} > 0$ such that we have the bound*

$$\|W\|_{\ell^p} \quad \leq \quad C_{19}\|V^0\|_{\ell^p}. \tag{2.8.112}$$

Proof. We show that the map

$$\begin{aligned} T : (W, \theta) \quad \mapsto \quad &\begin{pmatrix} \Pi^s(0)V^0 - \Pi^s(0)\mathcal{N}_1^{\tilde{\theta}} + \Pi^s(0)\Pi^c(\tilde{\theta})(W) - \Pi^s(0)(\mathcal{N}_2^{\tilde{\theta}}(W)) \\ \overline{\lambda}^c\Big[\Pi^c(0)V^0\Big] - \overline{\lambda}^c\Big[\Pi^c(0)\mathcal{N}_1^{\tilde{\theta}}\Big] \end{pmatrix} \\ &+ \begin{pmatrix} 0 \\ \overline{\lambda}^c\Big[\Pi^c(0)\Pi^c(\tilde{\theta})(W)\Big] - \overline{\lambda}^c\Big[\Pi^c(0)(\mathcal{N}_2^{\tilde{\theta}}(W)\Big] \end{pmatrix} \end{aligned} \tag{2.8.113}$$

maps $X_{\kappa,\varepsilon_\theta}$ into $X_{\kappa,\varepsilon_\theta}$ and is a contraction. Recall the constant C_1 from (2.8.3). Note that

$$\begin{aligned}\Pi^c(0)\Pi^c(\tilde\theta)(W) &= \Pi^c(0)(W) - \Pi^c(0)\Pi^s(\tilde\theta)(W)\\ &= -\Pi^c(0)\Pi^s(\tilde\theta)(W)\end{aligned} \tag{2.8.114}$$

We see, using Lemma 2.8.9, Lemma 2.8.10 and Lemma 2.8.11, and setting

$$C_{16} = C_1 \sum_{j\in\mathbb{Z}} e^{-\tilde\beta|hj|}, \tag{2.8.115}$$

that

$$\begin{aligned}\|T(W,\tilde\theta)^{(1)}\|_{\ell^p} &\le (1+C_{16})\|V^0 - \mathcal{N}_1^{\tilde\theta} - \mathcal{N}_2^{\tilde\theta}(W)\|_{\ell^p} + C_{10}|\tilde\theta|\|W\|_{\ell^p}\\ &\le (1+C_{16})\Big(\|V^0\|_{\ell^p} + C_{14}\tilde\theta^2 + C_6\|W\|_{\ell^p}^2\Big) + C_{10}|\tilde\theta|\|W\|_{\ell^p}\\ &\le C_{17}\Big[\|V^0\|_{\ell^p} + \tilde\theta^2 + \|W\|_{\ell^p}^2 + |\tilde\theta|\|W\|_{\ell^p}\Big]\end{aligned} \tag{2.8.116}$$

and

$$\begin{aligned}|T(W,\tilde\theta)^{(2)}| &\le \tfrac{C_{16}(\|V^0\|_{\ell^p} + C_{14}\tilde\theta^2 + C_6\|W\|_{\ell^p}^2) + (1+C_{16})C_{10}|\tilde\theta|\|W\|_{\ell^p}}{\Omega}\\ &\le C_{17}\Big[\|V^0\|_{\ell^p} + \tilde\theta^2 + \|W\|_{\ell^p}^2 + |\tilde\theta|\|W\|_{\ell^p}\Big]\end{aligned} \tag{2.8.117}$$

for some constant $C_{17} > 0$, which is independent of p. Note that

$$\begin{aligned}\Pi^c(\tilde\theta_2)(W_2) - \Pi^c(\tilde\theta_1)(W_1) &= \Pi^c(\tilde\theta_2)W_2 - \Pi^c(\tilde\theta_1)W_2 + \Pi^c(\tilde\theta_1)(W_1 - W_2)\\ &= \Pi^c(\tilde\theta_2)W_2 - \Pi^c(\tilde\theta_1)W_2\\ &\qquad + \Pi^c(\tilde\theta_1)(W_1 - W_2) - \Pi^c(0)(W_1 - W_2).\end{aligned} \tag{2.8.118}$$

Using Lemma 2.8.9, (2.8.81), Lemma 2.8.10 and Lemma 2.8.11 we obtain

$$\begin{aligned}\|T(W_1,\tilde\theta_1)^{(1)} - T(W_2,\tilde\theta_2)^{(1)}\|_{\ell^p} &\le (1+C_{16})\Big(\|\mathcal{N}_1^{\tilde\theta_2} - \mathcal{N}_1^{\tilde\theta_1}\|_{\ell^p}\\ &\qquad + \|\Pi^c(\tilde\theta_2)(W_2) - \Pi^c(\tilde\theta_1)(W_1)\|_{\ell^p}\\ &\qquad + \|\mathcal{N}_2^{\tilde\theta_2}(W_2) - \mathcal{N}_2^{\tilde\theta_1}(W_1)\|_{\ell^p}\Big)\\ &\le (1+C_{16})\Big(C_{15}(|\tilde\theta_1| + |\tilde\theta_2|)|\tilde\theta_2 - \tilde\theta_1|\\ &\qquad + C_{10}|\tilde\theta_1 - \tilde\theta_2|\|W_2\| + C_{10}|\tilde\theta_1|\|W_1 - W_2\|\\ &\qquad + C_{13}\Big[\|W_1\|_{\ell^p} + \|W_2\|_{\ell^p} + |\tilde\theta_1 - \tilde\theta_2|\Big]\|W_1 - W_2\|_{\ell^p}\\ &\qquad + C_{13}|\tilde\theta_2 - \tilde\theta_1|\Big[\|W_1\|_{\ell^p} + \|W_2\|_{\ell^p}\Big]\Big)\\ &\le C_{18}\Big[|\tilde\theta_1| + |\tilde\theta_2| + \|W_1\|_{\ell^p} + \|W_2\|_{\ell^p}\Big]\\ &\qquad \times\Big[|\tilde\theta_1 - \tilde\theta_2| + \|W_1 - W_2\|_{\ell^p}\Big],\end{aligned} \tag{2.8.119}$$

together with

$$\begin{array}{lcl} |T(W_1,\tilde{\theta}_1)^{(2)} - T(W_2,\tilde{\theta}_2)^{(2)}| & \leq & \frac{C_{16}\|\mathcal{N}_1^{\tilde{\theta}_2} - \mathcal{N}_1^{\tilde{\theta}_1} + \Pi^c(\tilde{\theta}_2)(W_2) - \Pi^c(\tilde{\theta}_1)(W_1) + \mathcal{N}_2^{\tilde{\theta}_2}(W_2) - \mathcal{N}_2^{\tilde{\theta}_1}(W_1)\|_{\ell^p}}{\Omega} \\ & \leq & C_{18}\Big[|\tilde{\theta}_1| + |\tilde{\theta}_2| + \|W_1\|_{\ell^p} + \|W_2\|_{\ell^p}\Big] \\ & & \qquad \times\Big[|\tilde{\theta}_1 - \tilde{\theta}_2| + \|W_1 - W_2\|_{\ell^p}\Big] \end{array} \tag{2.8.120}$$

for some constant $C_{18} > 0$, which is independent of p. First we let $0 < \kappa_{\max} < 1$ and $0 < \theta_{\max} < 1$ be constants such that all inequalities above hold for all $|\kappa| \leq \kappa_{\max}$ and all $|\theta| \leq \theta_{\max}$. In particular, we demand that $\kappa_{\max} < \eta$, $\kappa_{\max} < \frac{\eta}{C_3}$ and $\theta_{\max} < \eta$. Finally we write

$$\delta \;=\; \kappa \;=\; \varepsilon_\theta \;=\; \tfrac{1}{20}\min\{\kappa_{\max}, \theta_{\max}, \tfrac{1}{C_{17}}, \tfrac{1}{C_{18}}\} > 0. \tag{2.8.121}$$

With these choices we obtain the estimate

$$\begin{array}{lcl} \|T(W,\tilde{\theta})^{(1)}\|_{\ell^p} & \leq & C_{17}\Big[\|V^0\|_{\ell^p} + \tilde{\theta}^2 + \|W\|_{\ell^p}^2 + |\tilde{\theta}|\|W\|_{\ell^p}\Big] \\ & \leq & \frac{1}{20}\kappa + \frac{1}{20}\kappa + \frac{1}{20}\kappa + \frac{1}{20}\kappa \\ & \leq & \frac{1}{2}\kappa. \end{array} \tag{2.8.122}$$

Furthermore we see that

$$|T(W,\tilde{\theta})^{(2)}| \;\leq\; \tfrac{1}{2}\varepsilon_\theta. \tag{2.8.123}$$

Hence we see that the map T indeed maps $X_{\kappa,\varepsilon_\theta}$ into $X_{\kappa,\varepsilon_\theta}$.

In addition, (2.8.119) implies

$$\begin{array}{lcl} \|T(W_1,\tilde{\theta}_1)^{(1)} - T(W_2,\tilde{\theta}_2)^{(1)}\|_{\ell^p} & \leq & C_{18}\Big[|\tilde{\theta}_1| + |\tilde{\theta}_2| + \|W_1\|_{\ell^p} + \|W_2\|_{\ell^p}\Big] \\ & & \qquad \times\Big[|\tilde{\theta}_1 - \tilde{\theta}_2| + \|W_1 - W_2\|_{\ell^p}\Big] \\ & \leq & \frac{4}{20}|\tilde{\theta}_1 - \tilde{\theta}_2| + \frac{4}{20}\|W_1 - W_2\|_{\ell^p}, \end{array} \tag{2.8.124}$$

while (2.8.120) yields

$$|T(W_1,\tilde{\theta}_1)^{(2)} - T(W_2,\tilde{\theta}_2)^{(2)}| \;\leq\; \tfrac{4}{20}|\tilde{\theta}_1 - \tilde{\theta}_2| + \tfrac{4}{20}\|W_1 - W_2\|_{\ell^p}. \tag{2.8.125}$$

Therefore the map T is a contraction and thus the fixed point problem (2.8.110) has a unique solution $(W,\tilde{\theta})$. Moreover we see that

$$\begin{array}{lcl} \|(W,\tilde{\theta})\|_{\ell^p\times\mathbb{R}} & \leq & \|T(W,\tilde{\theta}) - T(0,0)\|_{\ell^p\times\mathbb{R}} + \|T(0,0)\|_{\ell^p\times\mathbb{R}} \\ & \leq & \frac{1}{2}\|(W,\tilde{\theta})\|_{\ell^p\times\mathbb{R}} + 2C_{17}\|V^0\|_{\ell^p}, \end{array} \tag{2.8.126}$$

which yields

$$\begin{array}{lcl} \|(W,\tilde{\theta})\|_{\ell^p\times\mathbb{R}} & \leq & 4C_{17}\|V^0\|_{\ell^p} \\ & = & C_{19}\|V^0\|_{\ell^p} \end{array} \tag{2.8.127}$$

as desired. ∎

Proof of Proposition 2.8.2. If $(W, \tilde{\theta})$ satisfies (2.8.110) then we see from (2.8.108) that

$$\begin{aligned} U^0 &= \overline{U}(0) + V^0 \\ &= \overline{U}(\tilde{\theta}) + \mathcal{U}_*^{\tilde{\theta}}(W)(0), \end{aligned} \tag{2.8.128}$$

as desired. ∎

Proof of Theorem 2.2.3. Let U be the solution of (2.2.1) with an initial condition $U(0) = U^0$ for which $\|U^0 - \overline{U}(0)\|_{\ell^p} < \delta$. By Proposition 2.8.2 and by uniqueness of the solution we see that

$$U = \overline{U}(\tilde{\theta}) + \mathcal{U}_*^{\tilde{\theta}}(W) \tag{2.8.129}$$

for some small $\tilde{\theta} \in \mathbb{R}$ and $W \in \ell^p$ with

$$\|W\|_{\ell^p} \leq C_{19}\|U^0 - \overline{U}(0)\|_{\ell^p}. \tag{2.8.130}$$

Hence we obtain

$$\begin{aligned} \|U(t) - \overline{U}(t + \tilde{\theta})\|_{\ell^p} &\leq e^{-\frac{\tilde{\beta}}{2}t}\|\mathcal{U}_*^{\tilde{\theta}}\Big(\Pi^c(\tilde{\theta})(W)\Big)\|_{-\tilde{\beta}/2} \\ &\leq 2C_3 e^{-\frac{\tilde{\beta}}{2}t}\|\Pi^c(\tilde{\theta})(W)\|_{\ell^p} \\ &\leq 2C_3 e^{-\frac{\tilde{\beta}}{2}t} C_{16}\|W\|_{\ell^p} \\ &\leq 2C_3 C_{16} C_{19}\|U^0 - \overline{U}(0)\|_{\ell^p} e^{-\frac{\tilde{\beta}}{2}t}, \end{aligned} \tag{2.8.131}$$

as desired. ∎

Chapter 3

Travelling waves for spatially discrete systems of FitzHugh-Nagumo type with periodic coefficients

This chapter has been published in SIAM Journal on Mathematical Analysis 54(4) (2019) 3492–3532 as W.M. Schouten-Straatman and H.J. Hupkes "Travelling waves for spatially discrete systems of FitzHugh-Nagumo type with periodic coefficients" [151].

Abstract. We establish the existence and nonlinear stability of travelling wave solutions for a class of lattice differential equations (LDEs) that includes the discrete FitzHugh-Nagumo system with alternating scale-separated diffusion coefficients. In particular, we view such systems as singular perturbations of spatially homogeneous LDEs, for which stable travelling wave solutions are known to exist in various settings.

The two-periodic waves considered in this paper are described by singularly perturbed multicomponent functional differential equations of mixed type (MFDEs). In order to analyze these equations, we generalize the spectral convergence technique that was developed by Bates, Chen and Chmaj to analyze the scalar Nagumo LDE. This allows us to transfer several crucial Fredholm properties from the spatially homogeneous to the spatially periodic setting. Our results hence do not require the use of comparison principles or exponential dichotomies.

Key words: Lattice differential equations, FitzHugh-Nagumo system, periodic coefficients, singular perturbations.

3.1 Introduction

In this paper we consider a class of lattice differential equations (LDEs) that includes the FitzHugh-Nagumo system

$$\begin{array}{rcl} \dot{u}_j &=& d_j(u_{j+1}+u_{j-1}-2u_j)+g(u_j;a_j)-w_j, \\ \dot{w}_j &=& \rho_j[u_j-\gamma_j w_j] \end{array} \tag{3.1.1}$$

with cubic nonlinearities

$$g(u;a) \quad = \quad u(1-u)(u-a) \tag{3.1.2}$$

and two-periodic coefficients

$$(0,\infty)\times(0,1)\times(0,1)\times(0,\infty)\ni(d_j,a_j,\rho_j,\gamma_j) \quad = \quad \begin{cases} (\varepsilon^{-2},a_o,\rho_o,\gamma_o) & \text{for odd } j, \\ (1,a_e,\rho_e,\gamma_e) & \text{for even } j. \end{cases} \tag{3.1.3}$$

We assume that the diffusion coefficients are of different orders in the sense $0<\varepsilon\ll 1$. Building on the results obtained in [108, 109] for the spatially homogeneous FitzHugh-Nagumo LDE, we show that (3.1.1) admits stable travelling pulse solutions with separate waveprofiles for the even and odd lattice sites. The main ingredient in our approach is a spectral convergence argument, which allows us to transfer Fredholm properties between linear operators acting on different spaces.

Signal propagation The LDE (3.1.1) can be interpreted as a spatially inhomogeneous discretisation of the FitzHugh-Nagumo partial differential equation (PDE)

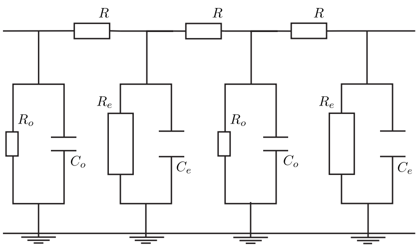
$$\begin{array}{rcl} u_t &=& u_{xx}+g(u;a)-w, \\ w_t &=& \rho\big[u-\gamma w\big], \end{array} \tag{3.1.4}$$

again with $\rho>0$ and $\gamma>0$. This PDE was proposed in the 1960s [74, 76] as a simplification of the four-component system that Hodgkin and Huxley developed to describe the propagation of spike signals through the nerve fibers of giant squids [98]. Indeed, for small $\rho>0$ (3.1.4) admits isolated pulse solutions of the form

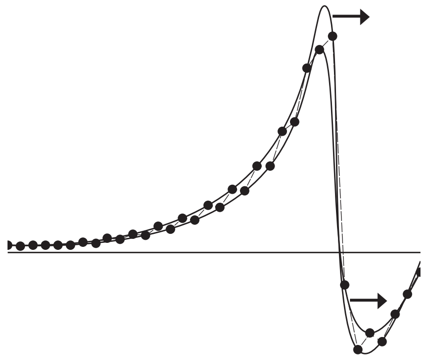
$$(u,w)(x,t) \quad = \quad (\overline{u}_0,\overline{w}_0)(x+c_0t), \tag{3.1.5}$$

in which c_0 is the wavespeed and the waveprofile $(\overline{u}_0,\overline{w}_0)$ satisfies the limits

$$\lim_{|\xi|\to\infty}(\overline{u}_0,\overline{w}_0)(\xi) \quad = \quad 0. \tag{3.1.6}$$



(a)



(b)

Figure 3.1: *(a) Simplified representation of the system (3.1.1) as an electrical circuit in a nerve fiber, analogous to [24, Fig. 1.11]. In this paper, the resistances R_o and R_e, as well as the capacitances C_o and C_e in the cell membrane alternate between the even and odd membranes. The resistivity of the intracellular fluid R is constant. (b) Schematic representation of the u-component of a travelling pulse for the system (3.1.1), which alternates between two waveprofiles.*

Such solutions were first observed numerically by FitzHugh [75], but the rigorous analysis of these pulses turned out to be a major mathematical challenge that is still ongoing. Many techniques have been developed to obtain the existence and stability of such pulse solutions in various settings, including geometric singular perturbation theory [31, 97, 117, 119], Lin's method [32, 33, 124], the variational principle [36] and the Maslov index [46, 47].

It turns out that electrical signals can only reach feasible speeds when travelling through nerve fibers that are insulated by a myelin coating. Such coatings are known to admit regularly spaced gaps at the nodes of Ranvier [143], where propagating signals can be chemically reinforced. In fact, the action potentials effectively jump from one node to the next through a process called saltatory conduction [127]. In order to include these effects, it is natural [123] to replace (3.1.4) by the FitzHugh-Nagumo LDE

$$\begin{aligned} \dot{u}_j &= \tfrac{1}{\varepsilon^2}(u_{j+1} + u_{j-1} - 2u_j) + g(u_j; a) - w_j, \\ \dot{w}_j &= \rho[u_j - \gamma w_j]. \end{aligned} \tag{3.1.7}$$

In this equation the variable u_j describes the potential at the node $j \in \mathbb{Z}$ node, while w_j describes the dynamics of the recovery variables. We remark that this LDE arises directly from (3.1.4) by using the nearest-neighbour discretisation of the Laplacian on a grid with spacing $\varepsilon > 0$.

In [108, 109], Hupkes and Sandstede studied (3.1.7) and showed that for a sufficiently far from $\frac{1}{2}$ and small $\rho > 0$, there exists a stable locally unique travelling pulse solution

$$(u_j, w_j)(t) = (\overline{u}, \overline{w})(j + ct). \tag{3.1.8}$$

The techniques relied on exponential dichotomies and Lin's method to develop an infinite-dimensional analogue of the exchange lemma. In [69], the existence part of these results was generalized to versions of (3.1.7) that feature infinite-range discretisations of the Laplacian that involve all neighbours instead of only the nearest neighbours. The stability results were also recently generalized to this setting [150], but only for small $\varepsilon > 0$ at present. Such systems with infinite-range interactions play an important role in neural field models [15, 23, 24, 142], which aim to describe the dynamics of large networks of neurons.

Our motivation here for studying the 2-periodic version (3.1.1) of the FitzHugh-Nagumo LDE (3.1.7) comes from recent developments in optical nanoscopy. Indeed, the results in [50, 51, 165] clearly show that certain proteins in the cytoskeleton of nerve fibers are organized periodically. This periodicity turns out to be a universal feature of all nerve systems, not just those which are insulated with a myelin coating. Since it also manifests itself at the nodes of Ranvier, it is natural to allow the parameters in (3.1.7) to vary in a periodic fashion. This can be understood by considering the generic circuit-models that are typically used to model nerve axons; see Figure 3.1(a).

The results in this paper are a first step in this direction. The restriction on the diffusion parameters is rather severe, but the absence of a comparison principle forces us to take a perturbative approach. We emphasize that the scale separation in the diffusion coefficients means that there is no natural continuum limit for (3.1.9) that can be recovered by sending the node separation to zero.

Periodicity Periodic patterns are frequently encountered when studying the behaviour of physical systems that have a discrete underlying spatial structure. Examples include the presence of twinning microstructures in shape memory alloys [17] and the formation of domain-wall microstructures in dielectric crystals [158].

At present, however, the mathematical analysis of such models has predominantly focused on one-component systems. For example, the results in [39] cover the bistable Nagumo LDE

$$\dot{u}_j \quad = \quad d_j(u_{j+1} + u_{j-1} - 2u_j) + g(u_j; a_j) \tag{3.1.9}$$

with spatially periodic coefficients $(d_j, a_j) \in (0, \infty) \times (0, 1)$. Exploiting the comparison principle, the authors were able to establish the existence of stable travelling wave solutions. Similar results were obtained in [89] for monostable versions of (3.1.9).

Let us also mention the results in [65, 67, 100], where the authors consider chains of alternating masses connected by identical springs (and vice versa). The dynamical behaviour of such systems can be modelled by LDEs of Fermi–Pasta–Ulam type with periodic coefficients. In certain limiting cases the authors were able to construct so-called nanopterons, which are multicomponent wave solutions that have low-amplitude oscillations in their tails.

In the examples above, the underlying periodicity is built into the spatial system itself. However, periodic patterns also arise naturally as solutions to spatially homogeneous discrete systems. As an example, systems of the form (3.1.9) with homogeneous but negative diffusion coefficients $d_j = d < 0$ have been used to describe phase transitions for grids of particles that have visco-elastic interactions [29, 30, 159]. Upon introducing separate scalings for the odd and even lattice sites, this one-component LDE can be turned into a 2-periodic system of the form

$$\begin{array}{rcl} \dot{v}_j & = & d_e\big(w_j + w_{j-1} - 2v_j\big) - f_e(v_j), \\ \dot{w}_j & = & d_o\big(v_{j+1} + v_j - 2w_j\big) - f_o(w_j) \end{array} \tag{3.1.10}$$

with positive coefficients $d_e > 0$ and $d_o > 0$. Systems of this type have been analyzed in considerable detail in [26, 160], where the authors establish the co-existence of patterns that can be both monostable and bistable in nature.

As a final example, let us mention that the LDE (3.1.9) with positive spatially homogeneous diffusion coefficients $d_j = d > 0$ can admit many periodic equilibria [129]. In [106], the authors construct bichromatic travelling waves that connect spatially homogeneous rest-states with such 2-periodic equilibria. Such waves can actually travel in parameter regimes where the standard monochromatic waves that connect zero to one are trapped. This presents a secondary mechanism by which the stable states zero and one can spread throughout the spatial domain.

Wave equations Returning to the 2-periodic FitzHugh-Nagumo LDE (3.1.1), we use the travelling wave Ansatz

$$(u, w)_j(t) = \begin{cases} (\overline{u}_o, \overline{w}_o)(j + ct) & \text{when } j \text{ is odd,} \\ (\overline{u}_e, \overline{w}_e)(j + ct) & \text{when } j \text{ is even,} \end{cases} \tag{3.1.11}$$

illustrated in Figure 3.1(b), to arrive at the coupled system

$$\begin{array}{rcl} c\overline{u}_o'(\xi) & = & \frac{1}{\varepsilon^2}\big(\overline{u}_e(\xi+1) + \overline{u}_e(\xi-1) - 2\overline{u}_o(\xi)\big) + g(\overline{u}_o(\xi); a_o) - \overline{w}_o(\xi), \\ c\overline{w}_o'(\xi) & = & \rho_o[\overline{u}_o(\xi) - \gamma_o\overline{w}_o(\xi)], \\ c\overline{u}_e'(\xi) & = & \big(\overline{u}_o(\xi+1) + \overline{u}_o(\xi-1) - 2\overline{u}_e(\xi)\big) + g(\overline{u}_e(\xi); a_e) - \overline{w}_e(\xi), \\ c\overline{w}_e'(\xi) & = & \rho_e[\overline{u}_e(\xi) - \gamma_e\overline{w}_e(\xi)]. \end{array} \tag{3.1.12}$$

Multiplying the first line by ε^2 and then taking $\varepsilon \downarrow 0$, we obtain the direct relation

$$\overline{u}_o(\xi) = \tfrac{1}{2}\big[\overline{u}_e(\xi+1) + \overline{u}_e(\xi-1)\big], \tag{3.1.13}$$

which can be substituted into the last two lines to yield

$$\begin{array}{rcl} c\overline{u}_e'(\xi) & = & \frac{1}{2}\big(\overline{u}_e(\xi+2) + \overline{u}_e(\xi-2) - 2\overline{u}_e(\xi)\big) + g(\overline{u}_e(\xi); a_e) - \overline{w}_e(\xi), \\ c\overline{w}_e'(\xi) & = & \rho_e[\overline{u}_e(\xi) - \gamma_e\overline{w}_e(\xi)]. \end{array} \tag{3.1.14}$$

All the odd variables have been eliminated from this last equation, which, in fact, describes pulse solutions to the spatially homogeneous FitzHugh-Nagumo LDE (3.1.7). Plugging these pulses into the remaining equation, we arrive at

$$c\overline{w}_o'(\xi) + \rho_o\gamma_o\overline{w}_o(\xi) \quad = \quad \tfrac{1}{2}\rho_o\big[\overline{u}_e(\xi+1) + \overline{u}_e(\xi-1)\big]. \tag{3.1.15}$$

This can be solved to yield the remaining second component of a singular pulse solution that we denote by

$$\overline{U}_0 = \big(\overline{u}_{o;0}, \overline{w}_{o;0}, \overline{u}_{e;0}, \overline{w}_{e;0}\big). \tag{3.1.16}$$

The main task in this paper is to construct stable travelling wave solutions to (3.1.1) by continuing this singular pulse into the regime $0 < \varepsilon \ll 1$. We use a functional analytic approach to handle this singular perturbation, focusing on the linear operator associated to the linearization of (3.1.12) with $\varepsilon > 0$ around the singular pulse. We show that this operator inherits several crucial Fredholm properties that were established in [109] for the linearization of (3.1.14) around the even pulse $\big(\overline{u}_{e;0}, \overline{w}_{e;0}\big)$.

Our results are not limited to the two-component system (3.1.1). Indeed, we consider general $(n+k)$-dimensional reaction diffusion systems with 2-periodic coefficients, where $n \geq 1$ is the number of components with a nonzero diffusion term and $k \geq 0$ is the number of components that do not diffuse. We can handle both travelling fronts and travelling pulses, but do impose conditions on the end-states that are stronger than the usual temporal stability requirements. Indeed, at times we will require (submatrices of) the corresponding Jacobians to be negative definite instead of merely spectrally stable. We emphasize that these distinctions disappear for scalar problems. In particular, our framework also covers the Nagumo LDE (3.1.9), but does not involve the use of a comparison principle.

Spectral convergence The main inspiration for our approach is the spectral convergence technique that was developed in [6] to establish the existence of travelling wave solutions to the homogeneous Nagumo LDE[1] (3.1.9) with diffusion coefficients $d_j = 1/\varepsilon^2 \gg 1$. The linear operator

$$\mathcal{L}_\varepsilon v(\xi) \quad = \quad c_0 v'(\xi) - \tfrac{1}{\varepsilon^2}\Big[v(\xi+\varepsilon) + v(\xi-\varepsilon) - 2v(\xi)\Big] - g_u(\overline{u}_0(\xi); a)v(\xi) \tag{3.1.17}$$

plays a crucial role in this approach, where the pair $(c_0, \overline{u}_0)$ is the travelling front solution of the Nagumo PDE

$$u_t \quad = \quad u_{xx} + g(u; a). \tag{3.1.18}$$

This front solutions satisfies the system

$$c_0\overline{u}_0'(\xi) \quad = \quad \overline{u}_0''(\xi) + g(\overline{u}(\xi); a), \qquad \overline{u}_0(-\infty) \quad = \quad 0, \qquad \overline{u}_0(+\infty) \quad = \quad 1, \tag{3.1.19}$$

[1] The power of the results in [6] is that they also apply to variants of (3.1.9) with infinite-range interactions. We describe their ideas here in a finite-range setting for notational clarity.

to which we can associate the linear operator

$$[\mathcal{L}_0 v](\xi) \quad = \quad c_0 v'(\xi) - v''(\xi) - g_u\big(\overline{u}(\xi); a\big) v(\xi), \tag{3.1.20}$$

which can be interpreted as the formal $\varepsilon \downarrow 0$ limit of (3.1.17). It is well-known that $\mathcal{L}_0 + \delta : H^2 \to L^2$ is invertible for all $\delta > 0$. By considering sequences

$$w_j \quad = \quad (\mathcal{L}_{\varepsilon_j} + \delta) v_j, \qquad \|v_j\|_{H^1} \quad = \quad 1, \qquad \varepsilon_j \quad \to \quad 0 \tag{3.1.21}$$

that converge weakly to a pair

$$w_0 \quad = \quad (\mathcal{L}_0 + \delta) v_0, \tag{3.1.22}$$

the authors show that also $\mathcal{L}_\varepsilon + \delta : H^1 \to L^2$ is invertible. To this end one needs to establish a lower bound for $\|w_0\|_{L^2}$, which can be achieved by exploiting inequalities of the form

$$\big\langle v(\cdot + \varepsilon) + v(\cdot - \varepsilon) - 2v(\cdot), v(\cdot) \big\rangle_{L^2} \quad \leq \quad 0, \qquad\qquad \langle v', v \rangle_{L^2} \quad = \quad 0 \tag{3.1.23}$$

and using the bistable structure of the nonlinearity g.

In [150], we showed that these ideas can be generalized to infinite-range versions of the FitzHugh-Nagumo LDE (3.1.7). The key issue there, which we must also face in this paper, is that problematic cross terms arise that must be kept under control when taking inner products. We are aided in this respect by the fact that the off-diagonal terms in the linearisation of (3.1.1) are constant multiples of each other.

A second key complication that we encounter here is that the scale separation in the diffusion terms prevents us from using the direct multicomponent analogue of the inequality (3.1.23). We must carefully include ε-dependent weights into our inner products to compensate for these imbalances. This complicates the fixed-point argument used to control the nonlinear terms during the construction of the travelling waves. In fact, it forces us to take an additional spatial derivative of the travelling wave equations.

This latter situation was also encountered in [112–114], where the spectral convergence method was used to construct travelling wave solutions to adaptive-grid discretisations of the Nagumo PDE (3.1.18). Further applications of this technique can be found in [111, 152], where full spatial-temporal discretisations of the Nagumo PDE (3.1.18) and the FitzHugh-Nagumo PDE (3.1.4) are considered.

Overview After stating our main results in §3.2 we apply the spectral convergence method discussed above to the system of travelling wave equations (3.1.12) in §3.3 and §3.4. This allows us to follow the spirit of [6, Thm. 1] to establish the existence of travelling waves in §3.5. In particular, we use a fixed point argument that mimics the proof of the standard implicit function theorem.

We follow the approach developed in [150] to analyze the spectral stability of these travelling waves in §3.6. In particular, we recycle the spectral convergence argument to analyze the linear operators $\overline{\mathcal{L}}_\varepsilon$ that arise after linearizing (3.1.12) around the newfound waves, instead of around the singular pulse $\overline{U}_0$ defined in (3.1.16). The key complication here is that for fixed small values of $\varepsilon > 0$ we need results on the invertibility of $\overline{\mathcal{L}}_\varepsilon + \lambda$ for all λ in a half-strip. By contrast, the spectral convergence method gives a range of admissible values for $\varepsilon > 0$ for each fixed λ. Switching between these two points of view is a delicate task, but fortunately the main ideas from [150] can be transferred to this setting.

The nonlinear stability of the travelling waves can be inferred from their spectral stability in a relatively straightforward fashion by appealing to the theory developed in [109] for discrete systems with finite range interactions. A more detailed description of this procedure in an infinite-range setting can be found in §2.7-2.8.

3.2 Main results

Our main results concern the LDE

$$\begin{aligned} \dot{u}_j(t) &= d_j \mathcal{D}\big[u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)\big] + f_j\big(u_j(t), w_j(t)\big), \\ \dot{w}_j(t) &= g_j\big(u_j(t), w_j(t)\big), \end{aligned} \tag{3.2.1}$$

posed on the one-dimensional lattice $j \in \mathbb{Z}$, where we take $u_j \in \mathbb{R}^n$ and $w_j \in \mathbb{R}^k$ for some pair of integers $n \geq 1$ and $k \geq 0$. We assume that the system is 2-periodic in the sense that there exists a set of four nonlinearities

$$f_o : \mathbb{R}^{n+k} \to \mathbb{R}^n, \qquad f_e : \mathbb{R}^{n+k} \to \mathbb{R}^n, \qquad g_o : \mathbb{R}^{n+k} \to \mathbb{R}^k, \qquad g_e : \mathbb{R}^{n+k} \to \mathbb{R}^k \tag{3.2.2}$$

for which we may write

$$(d_j, f_j, g_j) = \begin{cases} (\varepsilon^{-2}, f_o, g_o) & \text{for odd } j, \\ (1, f_e, g_e) & \text{for even } j. \end{cases} \tag{3.2.3}$$

Introducing the shorthand notation

$$F_o(u,w) = \big(f_o(u,w), g_o(u,w)\big), \qquad F_e(u,w) = \big(f_e(u,w), g_e(u,w)\big), \tag{3.2.4}$$

we impose the following structural condition on our system that concerns the roots of the nonlinearities F_o and F_e. These roots correspond with temporal equilibria of (3.2.1) that have a spatially homogeneous u-component. On the other hand, the w-component of these equilibria is allowed to be 2-periodic.

Assumption (HN1). The matrix $\mathcal{D} \in \mathbb{R}^{n \times n}$ is a diagonal matrix with strictly positive diagonal entries. In addition, the nonlinearities F_o and F_e are C^3-smooth and there exist four vectors

$$U_e^{\pm} = (u_e^{\pm}, w_e^{\pm}) \in \mathbb{R}^{n+k}, \qquad U_o^{\pm} = (u_o^{\pm}, w_o^{\pm}) \in \mathbb{R}^{n+k}, \tag{3.2.5}$$

for which we have the identities $u_o^- = u_e^-$ and $u_o^+ = u_e^+$, together with

$$F_o(U_o^\pm) \quad = \quad F_e(U_e^\pm) \quad = \quad 0. \tag{3.2.6}$$

We emphasize that any subset of the four vectors $U_o^\pm$ and $U_e^\pm$ is allowed to be identical. In order to address the temporal stability of these equilibria, we introduce two separate auxiliary conditions on triplets

$$\big(G, U^-, U^+\big) \in C^1\big(\mathbb{R}^{n+k}; \mathbb{R}^{n+k}\big) \times \mathbb{R}^{n+k} \times \mathbb{R}^{n+k}, \tag{3.2.7}$$

which are both stronger[2] than the requirement that all the eigenvalues of $DG(U^\pm)$ have strictly negative real parts. As can be seen, the block structure of this matrix plays an important role in (hβ), which is why we have chosen to state our results for arbitrary values of $n \geq 1$ and $k \geq 0$.

Assumption (hα). The matrices $-DG(U^-)$ and $-DG(U^+)$ are positive definite.

Assumption (hβ). For any $U \in \mathbb{R}^{n+k}$, write $DG(U)$ in the block form

$$DG(U) \quad = \quad \begin{pmatrix} G_{1,1}(U) & G_{1,2}(U) \\ G_{2,1}(U) & G_{2,2}(U) \end{pmatrix} \tag{3.2.8}$$

with $G_{1,1}(U) \in \mathbb{R}^{n \times n}$. Then the matrices $-G_{1,1}(U^-), -G_{1,1}(U^+), -G_{2,2}(U^-)$ and $-G_{2,2}(U^+)$ are positive definite. In addition, there exists a constant $\Gamma > 0$ so that $G_{1,2}(U) = -\Gamma G_{2,1}(U)^T$ holds for all $U \in \mathbb{R}^{n \times k}$.

As an illustration, we pick $0 < a < 1$ and write

$$G_{\mathrm{ngm}}(u) \quad = \quad u(1-u)(u-a) \tag{3.2.9}$$

for the nonlinearity associated with the Nagumo equation, together with

$$G_{\mathrm{fhn};\rho,\gamma}(u, w) \quad = \quad \begin{pmatrix} u(1-u)(u-a) - w \\ \rho\big[u - \gamma w\big] \end{pmatrix} \tag{3.2.10}$$

for its counterpart corresponding to the FitzHugh-Nagumo system. It can be easily verified that the triplet $(G_{\mathrm{ngm}}, 0, 1)$ satisfies (hα), while the triplet $(G_{\mathrm{fhn};\rho,\gamma}, 0, 0)$ satisfies (hβ) for $\rho > 0$ and $\gamma > 0$ with $\Gamma = \rho^{-1}$. When $a > 0$ is sufficiently small, the Jacobian $DG_{\mathrm{fhn};\rho,\gamma}(0, 0)$ has a pair of complex eigenvalues with negative real part. In this case (hα) may fail to hold.

The following assumption states that the even and odd subsystems must both satisfy one of the two auxiliary conditions above. We emphasize, however, that this does not necessarily need to be the same condition for both systems.

Assumption (HN2). The triplet (F_o, U_o^-, U_o^+) satisfies either (hα) or (hβ). The same holds for the triplet (F_e, U_e^-, U_e^+).

[2]See the proof of Lemma 3.4.6 for details.

We intend to find functions

$$(u_\varepsilon, w_\varepsilon) : \mathbb{R} \to \ell^\infty(\mathbb{Z}; \mathbb{R}^n) \times \ell^\infty(\mathbb{Z}; \mathbb{R}^k) \tag{3.2.11}$$

that take the form

$$(u_\varepsilon, w_\varepsilon)_j(t) = \begin{cases} (\overline{u}_{o;\varepsilon}, \overline{w}_{o;\varepsilon})(j + c_\varepsilon t), & \text{for odd } j \\ (\overline{u}_{e;\varepsilon}, \overline{w}_{e;\varepsilon})(j + c_\varepsilon t) & \text{for even } j \end{cases} \tag{3.2.12}$$

and satisfy (3.2.1) for all $t \in \mathbb{R}$. The waveprofiles are required to be C^1-smooth and satisfy the limits

$$\lim_{\xi \to \pm\infty} \big(\overline{u}_o(\xi), \overline{w}_o(\xi)\big) = (u_o^\pm, w_o^\pm), \qquad \lim_{\xi \to \pm\infty} \big(\overline{u}_e(\xi), \overline{w}_e(\xi)\big) = (u_e^\pm, w_e^\pm). \tag{3.2.13}$$

Substituting the travelling wave Ansatz (3.2.12) into the LDE (3.2.1) yields the coupled system

$$\begin{aligned} c_\varepsilon \overline{u}'_{o;\varepsilon}(\xi) &= \tfrac{1}{\varepsilon^2} \mathcal{D} \Delta_{\mathrm{mix}}[\overline{u}_{o;\varepsilon}, \overline{u}_{e;\varepsilon}](\xi) + f_o\big(\overline{u}_{o;\varepsilon}(\xi), \overline{w}_{o;\varepsilon}(\xi)\big), \\ c_\varepsilon \overline{w}'_{o;\varepsilon}(\xi) &= g_o\big(\overline{u}_{o;\varepsilon}(\xi), \overline{w}_{o;\varepsilon}(\xi)\big), \\ c_\varepsilon \overline{u}'_{e;\varepsilon}(\xi) &= \mathcal{D} \Delta_{\mathrm{mix}}[\overline{u}_{e;\varepsilon}, \overline{u}_{o;\varepsilon}](\xi) + f_e\big(\overline{u}_{e;\varepsilon}(\xi), \overline{w}_{e;\varepsilon}(\xi)\big), \\ c_\varepsilon \overline{w}'_{e;\varepsilon}(\xi) &= g_e\big(\overline{u}_{e;\varepsilon}(\xi), \overline{w}_{e;\varepsilon}(\xi)\big), \end{aligned} \tag{3.2.14}$$

in which we have introduced the shorthand

$$\Delta_{\mathrm{mix}}[\phi, \psi](\xi) = \psi(\xi + 1) + \psi(\xi - 1) - 2\phi(\xi). \tag{3.2.15}$$

Multiplying the first line of (3.2.14) by ε^2 and taking the formal limit $\varepsilon \downarrow 0$, we obtain the identity

$$0 = \mathcal{D} \Delta_{\mathrm{mix}}[\overline{u}_{o;0}, \overline{u}_{e;0}](\xi), \tag{3.2.16}$$

which can be explicitly solved to yield

$$\overline{u}_{o;0}(\xi) = \tfrac{1}{2} \overline{u}_{e;0}(\xi + 1) + \tfrac{1}{2} \overline{u}_{e;0}(\xi - 1). \tag{3.2.17}$$

In the $\varepsilon \downarrow 0$ limit, the even subsystem of (3.2.14) hence decouples and becomes

$$\begin{aligned} c_0 \overline{u}'_{e;0}(\xi) &= \tfrac{1}{2} \mathcal{D} \Big[\overline{u}_{e;0}(\xi + 2) + \overline{u}_{e;0}(\xi - 2) - 2\overline{u}_{e;0}(\xi)\Big] + f_e\big(\overline{u}_{e;0}(\xi), \overline{w}_{e;0}(\xi)\big), \\ c_0 \overline{w}'_{e;0}(\xi) &= g_e\big(\overline{u}_{e;0}(\xi), \overline{w}_{e;0}(\xi)\big). \end{aligned} \tag{3.2.18}$$

We require this limiting even system to have a travelling wave solution that connects U_e^- to U_e^+.

Assumption (HW1). There exists $c_0 \neq 0$ for which the system (3.2.18) has a C^1-smooth solution $\overline{U}_{e;0} = (\overline{u}_{e;0}, \overline{w}_{e;0})$ that satisfies the limits

$$\lim_{\xi \to \pm\infty} \big(\overline{u}_{e;0}(\xi), \overline{w}_{e;0}(\xi)\big) = (u_e^\pm, w_e^\pm). \tag{3.2.19}$$

Finally, taking $\varepsilon \downarrow 0$ in the second line of (3.2.14) and applying (3.2.17), we obtain the identity

$$c_0 \overline{w}'_{o;0}(\xi) \quad = \quad g_o\Big(\tfrac{1}{2}\overline{u}_{e;0}(\xi+1) + \tfrac{1}{2}\overline{u}_{e;0}(\xi-1), \overline{w}_{o;0}(\xi)\Big), \tag{3.2.20}$$

in which $\overline{w}_{o;0}$ is the only remaining unknown. We impose the following compatibility condition on this system.

Assumption (HW2). Equation (3.2.20) has a C^1-smooth solution $\overline{w}_{o;0}$ that satisfies the limits

$$\lim_{\xi \to \pm\infty} \overline{w}_{o;0}(\xi) \quad = \quad w_o^{\pm}. \tag{3.2.21}$$

Upon writing

$$\overline{U}_0 \quad = \quad (\overline{U}_{o;0}, \overline{U}_{e;0}) \quad = \quad (\overline{u}_{o;0}, \overline{w}_{o;0}, \overline{u}_{e;0}, \overline{w}_{e;0}), \tag{3.2.22}$$

we intend to seek a branch of solutions to (3.2.14) that bifurcates off the singular travelling wave $(\overline{U}_0, c_0)$. In view of the limits

$$\lim_{\xi \to \pm\infty} (\overline{U}_{o;0}, \overline{U}_{e;0})(\xi) \quad = \quad (U_o^{\pm}, U_e^{\pm}), \tag{3.2.23}$$

we introduce the spaces

$$\begin{aligned} \mathbf{H}^1_e \quad &= \quad \mathbf{H}^1_o \quad = \quad H^1(\mathbb{R};\mathbb{R}^n) \times H^1(\mathbb{R};\mathbb{R}^k), \\ \mathbf{L}^2_e \quad &= \quad \mathbf{L}^2_o \quad = \quad L^2(\mathbb{R};\mathbb{R}^n) \times L^2(\mathbb{R};\mathbb{R}^k) \end{aligned} \tag{3.2.24}$$

to analyze the perturbations from $\overline{U}_0$. The subscripts e and o in the spaces above are used solely for notational convenience.

Linearizing (3.2.18) around the solution $\overline{U}_{e;0}$, we obtain the linear operator $\overline{L}_e$: $\mathbf{H}^1_e \to \mathbf{L}^2_e$ that acts as

$$\overline{L}_e \quad = \quad c_0 \tfrac{d}{d\xi} - DF_e(\overline{U}_{e;0}) - \tfrac{1}{2} \begin{pmatrix} \mathcal{D}(S_2 - 2) & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.2.25}$$

in which we have introduced the notation

$$[S_2\phi](\xi) \quad = \quad \phi(\xi+2) + \phi(\xi-2). \tag{3.2.26}$$

Our perturbation argument to construct solutions of (3.2.14) requires $\overline{L}_e$ to have an isolated simple eigenvalue at the origin.

Assumption (HS1). There exists $\delta_e > 0$ so that the operator $\overline{L}_e + \delta$ is a Fredholm operator with index 0 for each $0 \leq \delta < \delta_e$. It has a simple eigenvalue in $\delta = 0$, i.e., we have $\mathrm{Ker}\big(\overline{L}_e\big) = \mathrm{span}(\overline{U}'_{e;0})$ and $\overline{U}'_{e;0} \notin \mathrm{Range}\big(\overline{L}_e\big)$.

We are now ready to formulate our first main result, which states that (3.2.14) admits a branch of solutions for small $\varepsilon > 0$ that converges to the singular wave $(\overline{U}_0, c_0)$ as $\varepsilon \downarrow 0$. Notice that the ε-scalings on the norms of Φ'_ε and Φ''_ε are considerably better than those suggested by a direct inspection of (3.2.14).

Theorem 3.2.1 (See §3.5). *Assume that (HN1), (HN2), (HW1), (HW2) and (HS1) are satisfied. There exists a constant $\varepsilon_* > 0$ so that for each $0 < \varepsilon < \varepsilon_*$, there exist $c_\varepsilon \in \mathbb{R}$ and $\Phi_\varepsilon = (\Phi_{o;\varepsilon}, \Phi_{e;\varepsilon}) \in \mathbf{H}^1_o \times \mathbf{H}^1_e$ for which the function*

$$\overline{U}_\varepsilon \quad = \quad \overline{U}_0 + \Phi_\varepsilon \tag{3.2.27}$$

is a solution of the travelling wave system (3.2.14) with wave speed $c = c_\varepsilon$. In addition, we have the limit

$$\lim_{\varepsilon \downarrow 0} \Big[\|\varepsilon \Phi''_{o;\varepsilon}\|_{\mathbf{L}^2_o} + \|\Phi''_{e;\varepsilon}\|_{\mathbf{L}^2_e} + \|\Phi'_\varepsilon\|_{\mathbf{L}^2_o \times \mathbf{L}^2_e} + \|\Phi_\varepsilon\|_{\mathbf{L}^2_o \times \mathbf{L}^2_e} + |c_\varepsilon - c_0| \Big] \quad = \quad 0 \tag{3.2.28}$$

and the function $\overline{U}_\varepsilon$ is locally unique up to translation.

In order to show that our newfound travelling wave solution is stable under the flow of the LDE (3.2.1), we need to impose the following extra assumption on the operator $\overline{L}_e$. To understand the restriction on λ, we recall that the spectrum of $\overline{L}_e$ admits the periodicity $\lambda \mapsto \lambda + 2\pi i c_0$.

Assumption (HS2). There exists a constant $\lambda_e > 0$ so that the operator $\overline{L}_e + \lambda : \mathbf{H}^1_e \to \mathbf{L}^2_e$ is invertible for all $\lambda \in \mathbb{C} \setminus 2\pi i c_0 \mathbb{Z}$ that have $\mathrm{Re}\,\lambda \geq -\lambda_e$.

Together with (HS1) this condition states that the wave $(\overline{U}_{e;0}, c_0)$ for the limiting even system (3.2.18) is spectrally stable. Our second main theorem shows that this can be generalized to a nonlinear stability result for the wave solutions (3.2.12) of the full system (3.2.1).

Theorem 3.2.2 (see §3.6). *Assume that (HN1), (HN2), (HW1), (HW2), (HS1) and (HS2) are satisfied and pick a sufficiently small $\varepsilon > 0$. Then there exist constants $\delta > 0$, $C > 0$ and $\beta > 0$ so that for all $1 \leq p \leq \infty$ and all initial conditions*

$$(u^0, w^0) \in \ell^p(\mathbb{Z}; \mathbb{R}^n) \times \ell^p(\mathbb{Z}; \mathbb{R}^k) \tag{3.2.29}$$

that admit the bound

$$E_0 \quad := \quad \|u^0 - u_\varepsilon(0)\|_{\ell^p(\mathbb{Z};\mathbb{R}^n)} + \|w^0 - w_\varepsilon(0)\|_{\ell^p(\mathbb{Z};\mathbb{R}^k)} \quad < \quad \delta, \tag{3.2.30}$$

there exists an asymptotic phase shift $\tilde{\theta} \in \mathbb{R}$ such that the solution (u, w) of (3.2.1) with the initial condition $(u, w)(0) = (u^0, w^0)$ satisfies the estimate

$$\|u(t) - u_\varepsilon(t + \tilde{\theta})\|_{\ell^p(\mathbb{Z};\mathbb{R}^n)} + \|w(t) - w_\varepsilon(t + \tilde{\theta})\|_{\ell^p(\mathbb{Z};\mathbb{R}^k)} \quad \leq \quad C e^{-\beta t} E_0 \tag{3.2.31}$$

for all $t > 0$.

Our final result shows that our framework is broad enough to cover the two-periodic FitzHugh-Nagumo system (3.1.1). We remark that the condition on γ_e ensures that $(0, 0)$ is the only spatially homogeneous equilibrium for the limiting even subsystem (3.1.14). This allows us to apply the spatially homogeneous results obtained in [108, 109].

Corollary 3.2.3. *Consider the LDE (3.1.1) and suppose that $\gamma_o > 0$ and $\rho_o > 0$ both hold. Suppose, furthermore, that a_e is sufficiently far away from $\frac{1}{2}$, that $0 < \gamma_e < 4(1-a_e)^{-2}$ and that $\rho_e > 0$ is sufficiently small. Then for each sufficiently small $\varepsilon > 0$, there exists a nonlinearly stable travelling pulse solution of the form (3.2.12) that satisfies the limits*

$$\lim_{\xi \to \pm\infty} \big(\overline{u}_o(\xi), \overline{w}_o(\xi)\big) = (0,0), \qquad \lim_{\xi \to \pm\infty} \big(\overline{u}_e(\xi), \overline{w}_e(\xi)\big) = (0,0). \tag{3.2.32}$$

Proof. Assumption (HN1) can be verified directly, while (HN2) follows from the discussion above concerning the nonlinearity $G_{\mathrm{fhn};\rho,\gamma}$ defined in (3.2.10). Assumption (HW1) follows from the existence theory developed in [108], while (HS1) and (HS2) follow from the spectral analysis in [109]. The remaining condition (HW2) can be verified by noting that the nonlinearity g_o is, in fact, linear and invertible with respect to $\overline{w}_{o;0}$ on account of Lemma 3.3.5 below. ■

3.3 The limiting system

In this section we analyze the linear operator that is associated to the limiting system that arises by combining (3.2.18) and (3.2.20). In order to rewrite this system in a compact fashion, we introduce the notation

$$[S_i\phi](\xi) = \phi(\xi+i) + \phi(\xi-i) \tag{3.3.1}$$

together with the $(n+k)\times(n+k)$-matrix $J_{\mathcal{D}}$ that has the block structure

$$J_{\mathcal{D}} = \begin{pmatrix} \mathcal{D} & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.3.2}$$

This allows us to recast (3.2.25) in the shortened form

$$\overline{L}_e = c_0 \tfrac{d}{d\xi} - \tfrac{1}{2} J_{\mathcal{D}}(S_2 - 2) - DF_e(\overline{U}_{e;0}). \tag{3.3.3}$$

One can associate a formal adjoint $\overline{L}_e^{\mathrm{adj}} : \mathbf{H}^1_e \to \mathbf{L}^2_e$ to this operator by writing

$$\overline{L}_e^{\mathrm{adj}} = -c_0 \tfrac{d}{d\xi} - \tfrac{1}{2} J_{\mathcal{D}}(S_2 - 2) - DF_e(\overline{U}_{e;0})^T. \tag{3.3.4}$$

Assumption (HS1), together with the Fredholm theory developed in [130], implies that

$$\mathrm{ind}(\overline{L}_e) = -\mathrm{ind}(\overline{L}_e^{\mathrm{adj}}) \tag{3.3.5}$$

holds for the Fredholm indices of these operators, which are defined as

$$\mathrm{ind}(L) = \dim\big(\ker(L)\big) - \mathrm{codim}\big(\mathrm{Range}(L)\big). \tag{3.3.6}$$

In particular, (HS1) implies that there exists a function

$$\overline{\Phi}_{e;0}^{\mathrm{adj}} \in \mathrm{Ker}(\overline{L}_e^{\mathrm{adj}}) \subset \mathbf{H}^1_e \tag{3.3.7}$$

that can be normalized to have

$$\langle \overline{U}'_{e;0}, \overline{\Phi}^{\mathrm{adj}}_{e;0} \rangle_{\mathbf{L}^2_e} = 1. \tag{3.3.8}$$

We also introduce the operator $\overline{L}_o : H^1(\mathbb{R};\mathbb{R}^k) \to L^2(\mathbb{R};\mathbb{R}^k)$ associated to the linearization of (3.2.20) around $\overline{U}_{o;0}$, which acts as

$$\overline{L}_o = c_0 \tfrac{d}{d\xi} - D_2 g_o(\overline{U}_{o;0}). \tag{3.3.9}$$

Here we introduced the notation $D_2 g_o$ to refer to the $k \times k$ Jacobian of g_o with respect to the final k entries. In order to couple the operator $\overline{L}_o$ with $\overline{L}_e$, we introduce the spaces

$$\mathbf{H}^1_\diamond = H^1(\mathbb{R};\mathbb{R}^k) \times \mathbf{H}^1_e, \qquad \mathbf{L}^2_\diamond = L^2(\mathbb{R};\mathbb{R}^k) \times \mathbf{L}^2_e, \tag{3.3.10}$$

together with the operator

$$\mathcal{L}_{\diamond;\delta} : \mathbf{H}^1_\diamond \to \mathbf{L}^2_\diamond \tag{3.3.11}$$

that acts as

$$\mathcal{L}_{\diamond;\delta} = \begin{pmatrix} \overline{L}_o + \delta & 0 \\ 0 & \overline{L}_e + \delta \end{pmatrix}. \tag{3.3.12}$$

Our first main result shows that $\mathcal{L}_{\diamond;\delta}$ inherits several properties of $\overline{L}_e + \delta$.

Proposition 3.3.1. *Assume that (HN1), (HN2), (HW1), (HW2) and (HS1) are satisfied. Then there exist constants $\delta_\diamond > 0$ and $C_\diamond > 0$ so that the following holds true:*

(i) For every $0 < \delta < \delta_\diamond$, the operator $\mathcal{L}_{\diamond,\delta}$ is invertible as a map from $\mathbf{H}^1_\diamond$ to $\mathbf{L}^2_\diamond$.

(ii) For any $\Theta_\diamond \in \mathbf{L}^2_\diamond$ and $0 < \delta < \delta_\diamond$ the function $\Phi_\diamond = \mathcal{L}^{-1}_{\diamond,\delta}\Theta_\diamond \in \mathbf{H}^1_\diamond$ satisfies the bound

$$\|\Phi_\diamond\|_{\mathbf{H}^1_\diamond} \le C_\diamond \Big[\|\Theta_\diamond\|_{\mathbf{L}^2_\diamond} + \tfrac{1}{\delta} \big| \langle \Theta_\diamond, (0, \overline{\Phi}^{\mathrm{adj}}_{e;0}) \rangle_{\mathbf{L}^2_\diamond} \big| \Big]. \tag{3.3.13}$$

If (HS2) also holds, then we can consider compact sets $\lambda \in M \subset \mathbb{C}$ that avoid the spectrum of $\overline{L}_e$. To formalize this, we impose the following assumption on M and state our second main result.

Assumption (hM$_{\lambda_0}$). The set $M \subset \mathbb{C}$ is compact with $2\pi i c_0 \mathbb{Z} \cap M = \emptyset$. In addition, we have $\mathrm{Re}\,\lambda \ge -\lambda_0$ for all $\lambda \in M$.

Proposition 3.3.2. *Assume that (HN1), (HN2), (HW1), (HW2), (HS1) and (HS2) are all satisfied and pick a sufficiently small constant $\lambda_\diamond > 0$. Then for any set $M \subset \mathbb{C}$ that satisfies (hM$_{\lambda_0}$) for $\lambda_0 = \lambda_\diamond$ there exists a constant $C_{\diamond;M} > 0$ so that the following holds true:*

(i) For every $\lambda \in M$, the operator $\mathcal{L}_{\diamond,\lambda}$ is invertible as a map from $\mathbf{H}^1_\diamond$ to $\mathbf{L}^2_\diamond$.

(ii) For any $\Theta_\diamond \in \mathbf{L}^2_\diamond$ and $\lambda \in M$, the function $\Phi_\diamond = \mathcal{L}^{-1}_{\diamond,\lambda}\Theta_\diamond \in \mathbf{H}^1_\diamond$ satisfies the bound

$$\|\Phi_\diamond\|_{\mathbf{H}^1_\diamond} \le C_{\diamond;M} \|\Theta_\diamond\|_{\mathbf{L}^2_\diamond}. \tag{3.3.14}$$

3.3.1 Properties of $\overline{L}_o$

The assumptions (HS1) and (HS2) already contain the information on $\overline{L}_e$ that we require to establish Propositions 3.3.1 and 3.3.2. Our task here is, therefore, to understand the operator $\overline{L}_o$. As a preparation, we show that the top-left and bottom-right corners of the limiting Jacobians $DF_o(U_o^{\pm})$ are both negative definite, which will help us to establish useful Fredholm properties.

Lemma 3.3.3. *Assume that (HN1) and (HN2) are both satisfied. Then the matrices $D_1 f_\#(U_\#^{\pm})$ and $D_2 g_\#(U_\#^{\pm})$ are all negative definite for each $\# \in \{o, e\}$.*

Proof. Note first that $D_1 f_\#$ and $D_2 g_\#$ correspond with $G_{1,1}$, respectively, $G_{2;2}$ in the block structure (3.2.8) for $DF_\#$. We hence see that the matrices $D_1 f_\#(U_\#^{\pm})$ and $D_2 g_\#(U_\#^{\pm})$ are negative definite, either directly by (hβ) or by the fact that they are principal submatrices of $DF_\#(U_\#^{\pm})$, which are negative definite if (hα) holds. ■

Lemma 3.3.4. *Assume that (HN1), (HN2), (HW1) and (HW2) are satisfied. Then there exists $\lambda_o > 0$ so that the operator $\overline{L}_o + \lambda$ is Fredholm with index zero for each $\lambda \in \mathbb{C}$ with* $\mathrm{Re}\lambda \geq -\lambda_o$.

Proof. For any $0 \leq \rho \leq 1$ and $\lambda \in \mathbb{C}$ we introduce the constant coefficient linear operator $L_{\rho,\lambda} : H^1(\mathbb{R}; \mathbb{R}^k) \to L^2(\mathbb{R}; \mathbb{R}^k)$ that acts as

$$L_{\rho,\lambda} = c_0 \tfrac{d}{d\xi} - \rho D_2 g_o(U_o^-) - (1-\rho) D_2 g_o(U_o^+) + \lambda \tag{3.3.15}$$

and has the characteristic function

$$\Delta_{L_{\rho,\lambda}}(z) = c_0 z - \rho D_2 g_o(U_o^-) - (1-\rho) D_2 g_o(U_o^+) + \lambda. \tag{3.3.16}$$

Upon introducing the matrix

$$B_\rho = -\rho D_2 g_o(U_o^-) - (1-\rho) D_2 g_o(U_o^+) - \rho D_2 g_o(U_o^-)^T - (1-\rho) D_2 g_o(U_o^+)^T, \tag{3.3.17}$$

which is positive definite by Lemma 3.3.3, we pick $\lambda_o > 0$ in such a way that $B_\rho - 2\lambda_o$ remains positive definite for each $0 \leq \rho \leq 1$. It is easy to check that the identity

$$\Delta_{L_{\rho,\lambda}}(iy) + \Delta_{L_{\rho,\lambda}}(iy)^\dagger = B_\rho + 2\mathrm{Re}\lambda \tag{3.3.18}$$

holds for any $y \in \mathbb{R}$. Here we use the symbol $\dagger$ for the conjugate transpose matrix. In particular, if we assume that $\mathrm{Re}\,\lambda \geq -\lambda_o$ and that $\Delta_{L_{\rho,\lambda}}(iy) v_o = 0$ for some nonzero $v_o \in \mathbb{C}^k$, $y \in \mathbb{R}$ and $0 \leq \rho \leq 1$, then we obtain the contradiction

$$\begin{aligned} 0 &= \mathrm{Re}\,\big[v_o^\dagger\big[\Delta_{L_\rho}(iy) + \Delta_{L_\rho}(iy)^\dagger\big]v_o\big] \\ &= \mathrm{Re}\,v_o^\dagger\big[B_\rho + 2\mathrm{Re}\lambda\big]v_o \\ &> 0. \end{aligned} \tag{3.3.19}$$

Using [130, Thm. A] together with the spectral flow principle in [130, Thm. C], this implies that $\overline{L}_o + \lambda$ is a Fredholm operator with index zero. ■

Lemma 3.3.5. *Assume that (HN1), (HN2), (HW1) and (HW2) are satisfied and pick a sufficiently small constant* $\lambda_o > 0$. *Then for any* $\lambda \in \mathbb{C}$ *with* $\mathrm{Re}\lambda \geq -\lambda_o$ *the operator* $\overline{L}_o + \lambda$ *is invertible as a map from* $H^1(\mathbb{R};\mathbb{R}^k)$ *into* $L^2(\mathbb{R};\mathbb{R}^k)$. *In addition, for each compact set*

$$M \subset \{\lambda : \mathrm{Re}\,\lambda \geq -\lambda_o\} \subset \mathbb{C} \tag{3.3.20}$$

there exists a constant $K_M > 0$ *so that the uniform bound*

$$\| [\overline{L}_o + \lambda]^{-1} \chi_o \|_{H^1(\mathbb{R};\mathbb{R}^k)} \quad \leq \quad K_M \|\chi_o\|_{L^2(\mathbb{R};\mathbb{R}^k)} \tag{3.3.21}$$

holds for any $\chi_o \in L^2(\mathbb{R};\mathbb{R}^k)$ *and any* $\lambda \in M$.

Proof. Recall the constant λ_o defined in Lemma 3.3.4 and pick any $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda \geq -\lambda_o$. On account of Lemma 3.3.4 it suffices to show that $\overline{L}_o + \lambda$ is injective. Consider therefore any nontrivial $\psi \in \mathrm{Ker}\big(\overline{L}_o + \lambda\big)$, which necessarily satisfies the ordinary differential equation (ODE)[3]

$$\psi'(\xi) \quad = \quad \tfrac{1}{c_0} D_2 g_o\big(\overline{U}_{o;0}(\xi)\big)\psi(\xi) - \tfrac{\lambda}{c_0}\psi(\xi) \tag{3.3.22}$$

posed on $\mathbb{C}^k$. Without loss of generality we may assume that $c_0 > 0$.

Since $\overline{U}_{o;0}(\xi) \to U_o^{\pm}$ as $\xi \to \pm\infty$, Lemma 3.3.3 allows us to pick a constant $m \gg 1$ in such a way that the matrix $-D_2 g_o\big(\overline{U}_{o;0}(\xi)\big) - 2\lambda_o$ is positive definite for each $|\xi| \geq m$, possibly after decreasing the size of $\lambda_o > 0$. Assuming that $\mathrm{Re}\,\lambda \geq -\lambda_o$ and picking any $\xi \leq -m$, we may hence compute

$$\begin{array}{lcl} \frac{d}{d\xi}|\psi(\xi)|^2 & = & 2\mathrm{Re}\langle \psi'(\xi), \psi(\xi)\rangle_{\mathbb{C}^k} \\ & = & \frac{2}{c_0}\mathrm{Re}\langle D_2 g_o\big(\overline{U}_{o;0}(\xi)\big)\psi(\xi), \psi(\xi)\rangle_{\mathbb{C}^k} - \frac{2\mathrm{Re}\,\lambda}{c_0}\langle \psi(\xi), \psi(\xi)\rangle_{\mathbb{C}^k} \\ & \leq & -\frac{2\lambda_o}{c_0}|\psi(\xi)|^2, \end{array} \tag{3.3.23}$$

which implies that

$$\Big(e^{\frac{2\lambda_o}{c_0}\xi} |\psi(\xi)|^2 \Big)' \quad \leq \quad 0. \tag{3.3.24}$$

Since ψ cannot vanish anywhere as a nontrivial solution to a linear ODE, we have

$$|\psi(\xi)|^2 \quad \geq \quad e^{-\frac{2\lambda_o}{c_0}(m+\xi)} |\psi(-m)|^2 \quad > \quad 0 \tag{3.3.25}$$

for $\xi \leq -m$, which means that $\psi(\xi)$ is unbounded. In particular, we see that $\psi \notin H^1(\mathbb{R};\mathbb{R}^k)$, which leads to the desired contradiction. The uniform bound (3.3.21) follows easily from continuity considerations. ∎

Proof of Proposition 3.3.1. Since the operator $\overline{L}_e$ defined in (3.2.25) has a simple eigenvalue in zero, we can follow the approach of [150, Lem. 3.1(5)] to pick two constants $\delta_\diamond > 0$ and $C > 0$ in such a way that $\overline{L}_e + \delta : \mathbf{H}^1_e \to \mathbf{L}^2_e$ is invertible with the bound

$$\| [\overline{L}_e + \delta]^{-1} (\theta_e, \chi_e) \|_{\mathbf{H}^1_e} \quad \leq \quad C\Big[\|(\theta_e, \chi_e)\|_{\mathbf{L}^2_e} + \tfrac{1}{\delta}\big| \langle (\theta_e, \chi_e), \overline{\Phi}^{\mathrm{adj}}_{e;0} \rangle_{\mathbf{L}^2_e} \big| \Big]. \tag{3.3.26}$$

[3] The discussion at https://math.stackexchange.com/questions/2668795/bounded-solution-to-general-nonautonomous-ode gave us the inspiration for this approach.

for any $0 < \delta < \delta_\diamond$ and $(\theta_e, \chi_e) \in \mathbf{L}^2_e$. Combining this estimate with Lemma 3.3.5 directly yields the desired properties. ∎

Proof of Proposition 3.3.2. These properties can be established in a fashion analogous to the proof of Proposition 3.3.1.

∎

3.4 Transfer of Fredholm properties

Our goal in this section is to lift the bounds obtained in §3.3 to the operators associated to the linearization of the full wave equation (3.2.14) around suitable functions. In particular, the arguments we develop here will be used in several different settings. In order to accommodate this, we introduce the following condition.

Assumption (hFam). For each $\varepsilon > 0$ there is a function $\tilde{U}_\varepsilon = (\tilde{U}_{o;\varepsilon}, \tilde{U}_{e;\varepsilon}) \in \mathbf{H}^1_o \times \mathbf{H}^1_e$ and a constant $\tilde{c}_\varepsilon \neq 0$ such that $\tilde{U}_\varepsilon - \overline{U}_0 \to 0$ in $\mathbf{H}^1_o \times \mathbf{H}^1_e$ and $\tilde{c}_\varepsilon \to c_0$ as $\varepsilon \downarrow 0$. In addition, there exists a constant $\tilde{K}_{\mathrm{fam}} > 0$ so that

$$|\tilde{c}_\varepsilon| + |\tilde{c}_\varepsilon^{-1}| + \left\|\tilde{U}_\varepsilon\right\|_\infty \leq \tilde{K}_{\mathrm{fam}} \tag{3.4.1}$$

holds for all $\varepsilon > 0$.

In §3.5 we will pick $\tilde{U}_\varepsilon = \overline{U}_0$ and $\tilde{c}_\varepsilon = c_0$ in (hFam) for all $\varepsilon > 0$. On the other hand, in §3.6 we will use the travelling wave solutions described in Theorem 3.2.1 to write $\tilde{U}_\varepsilon = \overline{U}_\varepsilon$ and $\tilde{c}_\varepsilon = c_\varepsilon$. We remark that (3.4.1) implies that there exists a constant $\tilde{K}_F > 0$ for which the bound

$$\|DF_o(\tilde{U}_{o;\varepsilon})\|_\infty + \|D^2F_o(\tilde{U}_{o;\varepsilon})\|_\infty + \|DF_e(\tilde{U}_{e;\varepsilon})\|_\infty + \|D^2F_e(\tilde{U}_{e;\varepsilon})\|_\infty \leq \tilde{K}_F \tag{3.4.2}$$

holds for all $\varepsilon > 0$.

For notational convenience, we introduce the product spaces

$$\mathbf{H}^1 = \mathbf{H}^1_o \times \mathbf{H}^1_e, \qquad \mathbf{L}^2 = \mathbf{L}^2_o \times \mathbf{L}^2_e. \tag{3.4.3}$$

Since we will need to consider complex-valued functions during our spectral analysis, we also introduce the spaces

$$\begin{aligned} \mathbf{L}^2_{\mathbb{C}} &= \{\Phi + i\Psi : \Phi, \Psi \in \mathbf{L}^2\}, \\ \mathbf{H}^1_{\mathbb{C}} &= \{\Phi + i\Psi : \Phi, \Psi \in \mathbf{H}^1\} \end{aligned} \tag{3.4.4}$$

and remark that any $L \in \mathcal{L}(\mathbf{H}^1; \mathbf{L}^2)$ can be interpreted as an operator in $\mathcal{L}(\mathbf{H}^1_{\mathbb{C}}; \mathbf{L}^2_{\mathbb{C}})$ by writing

$$L(\Phi + i\Psi) = L\Phi + iL\Psi. \tag{3.4.5}$$

It is well-known that taking the complexification of an operator preserves injectivity, invertibility and other Fredholm properties.

Recall the family $(\tilde{U}_\varepsilon, \tilde{c}_\varepsilon)$ introduced in (hFam). For any $\varepsilon > 0$ and $\lambda \in \mathbb{C}$ we introduce the linear operator

$$\tilde{\mathcal{L}}_{\varepsilon,\lambda} : \mathbf{H}^1_{\mathbf{C}} \to \mathbf{L}^2_{\mathbf{C}} \tag{3.4.6}$$

that acts as

$$\tilde{\mathcal{L}}_{\varepsilon,\lambda} = \begin{pmatrix} \tilde{c}_\varepsilon \frac{d}{d\xi} + \frac{2}{\varepsilon^2} J_{\mathcal{D}} - DF_o(\tilde{U}_{o;\varepsilon}) + \lambda & -\frac{1}{\varepsilon^2} J_{\mathcal{D}} S_1 \\ -J_{\mathcal{D}} S_1 & \tilde{c}_\varepsilon \frac{d}{d\xi} + 2J_{\mathcal{D}} - DF_e(\tilde{U}_{e;\varepsilon}) + \lambda \end{pmatrix}. \tag{3.4.7}$$

In order to simplify our notation, we introduce the $(2n2 + k) \times (2n + 2k)$ diagonal matrices

$$\begin{aligned} \mathcal{M}^1_\varepsilon &= \mathrm{diag}\big(\varepsilon, 1, 1, 1\big), \\ \mathcal{M}^2_\varepsilon &= \mathrm{diag}\big(1, \varepsilon, 1, 1\big), \\ \mathcal{M}^{1,2}_\varepsilon &= \mathrm{diag}\big(\varepsilon, \varepsilon, 1, 1\big). \end{aligned} \tag{3.4.8}$$

In addition, we recall the sum S_1 defined in (3.3.1) and introduce the operator

$$J_{\mathrm{mix}} = \begin{pmatrix} -2J_{\mathcal{D}} & J_{\mathcal{D}} S_1 \\ J_{\mathcal{D}} S_1 & -2J_{\mathcal{D}} \end{pmatrix}, \tag{3.4.9}$$

which allows us to restate (3.4.7) as

$$\tilde{\mathcal{L}}_{\varepsilon,\lambda} = \tilde{c}_\varepsilon \tfrac{d}{d\xi} - \mathcal{M}^1_{1/\varepsilon^2} J_{\mathrm{mix}} - DF(\tilde{U}_\varepsilon) + \lambda. \tag{3.4.10}$$

Our two main results generalize the bounds in Proposition 3.3.1 and Proposition 3.3.2 to the current setting. The scalings on the odd variables allow us to obtain certain key estimates that are required by the spectral convergence approach.

Proposition 3.4.1. *Assume that (hFam), (HN1), (HN2), (HW1), (HW2) and (HS1) are satisfied. Then there exist positive constants $C_0 > 0$ and $\delta_0 > 0$ together with a strictly positive function $\varepsilon_0 : (0, \delta_0) \to \mathbb{R}_{>0}$, so that for each $0 < \delta < \delta_0$ and $0 < \varepsilon < \varepsilon_0(\delta)$ the operator $\tilde{\mathcal{L}}_{\varepsilon,\delta}$ is invertible and satisfies the bound*

$$\|\mathcal{M}^{1,2}_\varepsilon \Phi\|_{\mathbf{H}^1} \leq C_0 \Big[\|\mathcal{M}^{1,2}_\varepsilon \Theta\|_{\mathbf{L}^2} + \tfrac{1}{\delta} \big| \langle \Theta, (0, \overline{\Phi}^{\mathrm{adj}}_{e;0}) \rangle_{\mathbf{L}^2} \big| \Big] \tag{3.4.11}$$

for any $\Phi \in \mathbf{H}^1$ and $\Theta = \tilde{\mathcal{L}}_{\varepsilon,\delta} \Phi$.

Proposition 3.4.2. *Assume that (hFam), (HN1), (HN2), (HW1), (HW2), (HS1) and (HS2) are all satisfied and pick a sufficiently small constant $\lambda_0 > 0$. Then for any set $M \subset \mathbb{C}$ that satisfies (hM_{λ_0}), there exist positive constants $C_M > 0$ and $\varepsilon_M > 0$ so that for each $\lambda \in M$ and $0 < \varepsilon < \varepsilon_M$ the operator $\tilde{\mathcal{L}}_{\varepsilon,\lambda}$ is invertible and satisfies the bound*

$$\|\Phi\|_{\mathbf{H}^1_{\mathbb{C}}} \leq C_M \|\Theta\|_{\mathbf{L}^2_{\mathbb{C}}} \tag{3.4.12}$$

for any $\Phi \in \mathbf{H}^1_{\mathbb{C}}$ and $\Theta = \tilde{\mathcal{L}}_{\varepsilon,\lambda} \Phi$.

By using bootstrapping techniques it is possible to obtain variants of the estimate in Proposition 3.4.1. Indeed, it is possible to remove the scaling on the first component of Φ (but not on the first component of Φ').

Corollary 3.4.3. *Consider the setting of Proposition 3.4.1. Then for each* $0 < \delta < \delta_0$ *and* $0 < \varepsilon < \varepsilon_0(\delta)$*, the operator* $\tilde{\mathcal{L}}_{\varepsilon,\delta}$ *satisfies the bound*

$$\|\mathcal{M}_\varepsilon^{1,2}\Phi'\|_{\mathbf{L}^2} + \|\mathcal{M}_\varepsilon^{2}\Phi\|_{\mathbf{L}^2} \quad \leq \quad C_0\Big[\|\mathcal{M}_\varepsilon^{1,2}\Theta\|_{\mathbf{L}^2} + \tfrac{1}{\delta}\big|\langle\Theta,(0,\overline{\Phi}_{e;0}^{\mathrm{adj}})\rangle_{\mathbf{L}^2}\big|\Big] \qquad (3.4.13)$$

for any $\Phi \in \mathbf{H}^1$ *and* $\Theta = \tilde{\mathcal{L}}_{\varepsilon,\delta}\Phi$*, possibly after increasing* $C_0 > 0$.

Proof. Write $\Phi = (\phi_o, \psi_o, \phi_e, \psi_e)$ and $\Theta = (\theta_o, \chi_o, \theta_e, \chi_e)$. Note that the first component of the equation $\Theta = \tilde{\mathcal{L}}_{\varepsilon,\delta}\Phi$ yields

$$2\mathcal{D}\phi_o \quad = \quad \mathcal{D}S_1\phi_e - \varepsilon^2\tilde{c}_\varepsilon\phi_o' + \varepsilon^2 D_1 f_o(\tilde{U}_{o;\varepsilon})\phi_o + \varepsilon^2 D_2 f_o(\tilde{U}_{o;\varepsilon})\psi_o - \delta\varepsilon^2\phi_o + \varepsilon^2\theta_o. \qquad (3.4.14)$$

Recall the constants $\tilde{K}_{\mathrm{fam}}$ and $\tilde{K}_F$ from (3.4.1) and (3.4.2), respectively, and write

$$d_{\min} \quad = \quad \min_{1 \leq i \leq n}\mathcal{D}_{i,i}, \qquad\qquad d_{\max} \quad = \quad \max_{1 \leq i \leq n}\mathcal{D}_{i,i}. \qquad (3.4.15)$$

We can now estimate

$$\begin{array}{rcl} 2d_{\min}\|\phi_o\|_{L^2(\mathbb{R};\mathbb{R}^n)} & \leq & 2\|\mathcal{D}\phi_o\|_{L^2(\mathbb{R};\mathbb{R}^n)} \\ & \leq & \|\mathcal{D}S_1\phi_e\|_{L^2(\mathbb{R};\mathbb{R}^n)} + \varepsilon|\tilde{c}_\varepsilon|\|\varepsilon\phi_o'\|_{L^2(\mathbb{R};\mathbb{R}^n)} \\ & & \quad +\varepsilon\|D_1 f_o(\overline{U}_{o;\varepsilon})\|_\infty\|\varepsilon\phi_o\|_{L^2(\mathbb{R};\mathbb{R}^n)} \\ & & \quad +\varepsilon\|D_2 f_o(\overline{U}_{o;\varepsilon})\|_\infty\|\varepsilon\psi_o\|_{L^2(\mathbb{R};\mathbb{R}^k)} \\ & & \quad +\varepsilon\delta\|\varepsilon\phi_o\|_{L^2(\mathbb{R};\mathbb{R}^n)} + \varepsilon\|\varepsilon\theta_o\|_{L^2(\mathbb{R};\mathbb{R}^n)} \\ & \leq & \Big[2d_{\max} + \varepsilon(\tilde{K}_{\mathrm{fam}} + 2\tilde{K}_F + \delta_0)\Big]\,\big\|\mathcal{M}_\varepsilon^{1,2}\Phi\big\|_{\mathbf{H}^1} + \varepsilon\|\mathcal{M}_\varepsilon^{1,2}\Theta\|. \end{array} \qquad (3.4.16)$$

The desired bound hence follows directly from Proposition 3.4.1. ∎

The scaling on the second components of Φ and Φ' can be removed in a similar fashion. However, in this case one also needs to remove the corresponding scaling on Θ.

Corollary 3.4.4. *Consider the setting of Proposition 3.4.1. Then for each* $0 < \delta < \delta_0$ *and* $0 < \varepsilon < \varepsilon_0(\delta)$*, the operator* $\tilde{\mathcal{L}}_{\varepsilon,\delta}$ *satisfies the bound*

$$\|\mathcal{M}_\varepsilon^{1}\Phi'\|_{\mathbf{L}^2} + \|\Phi\|_{\mathbf{L}^2} \quad \leq \quad C_0\Big[\|\mathcal{M}_\varepsilon^{1}\Theta\|_{\mathbf{L}^2} + \tfrac{1}{\delta}\big|\langle\Theta,(0,\overline{\Phi}_{e;0}^{\mathrm{adj}})\rangle_{\mathbf{L}^2}\big|\Big] \qquad (3.4.17)$$

for any $\Phi \in \mathbf{H}^1$ *and* $\Theta = \tilde{\mathcal{L}}_{\varepsilon,\delta}\Phi$*, possibly after increasing* $C_0 > 0$.

Proof. Writing $\Phi_o = (\phi_o, \psi_o)$ and $\Theta_o = (\theta_o, \chi_o)$, we can inspect the definitions (3.4.7) and (3.3.12) to obtain

$$(\overline{L}_o + \delta)\psi_o \quad = \quad D_1 g_o(\tilde{U}_{o;\varepsilon})\phi_o + \chi_o. \qquad (3.4.18)$$

Using Lemma 3.3.5 we hence obtain the estimate

$$\|\psi_o\|_{H^1(\mathbb{R};\mathbb{R}^k)} \quad \leq \quad C_1'\Big[\|D_1 g_o(\tilde{U}_{o;\varepsilon})\|_\infty\|\phi_o\|_{L^2(\mathbb{R};\mathbb{R}^n)} + \|\chi_o\|_{L^2(\mathbb{R};\mathbb{R}^k)}\Big] \qquad (3.4.19)$$

for some $C_1' > 0$. Combining this with (3.4.13) yields the desired bound (3.4.17). ∎

Our final result here provides information on the second derivatives of Φ, in the setting where Θ is differentiable. In particular, we introduce the spaces

$$\mathbf{H}^2_o \;=\; \mathbf{H}^2_e \;=\; H^2(\mathbb{R};\mathbb{R}^n) \times H^2(\mathbb{R};\mathbb{R}^k), \qquad \mathbf{H}^2 \;=\; \mathbf{H}^2_o \times \mathbf{H}^2_e. \tag{3.4.20}$$

We remark here that we have chosen to keep the scalings on the second components of Φ'' and Θ' because this will be convenient in §3.5. Note also that the stated bound on $\|\Phi\|_{\mathbf{H}^1}$ can actually be obtained by treating $\tilde{\mathcal{L}}_{\varepsilon,\delta}$ as a regular perturbation of $\mathcal{L}_{\diamond,\delta}$. The point here is that we gain an order of regularity, which is crucial for the nonlinear estimates.

Corollary 3.4.5. *Consider the setting of Proposition 3.4.1 and assume furthermore that $\|\tilde{U}_\varepsilon'\|_\infty$ is uniformly bounded for $\varepsilon > 0$. Then for each $0 < \delta < \delta_0$ and any $0 < \varepsilon < \varepsilon_0(\delta)$, the operator $\tilde{\mathcal{L}}_{\varepsilon,\delta} : \mathbf{H}^2 \to \mathbf{H}^1$ is invertible and satisfies the bound*

$$\|\mathcal{M}^{1,2}_\varepsilon \Phi''\|_{\mathbf{L}^2} + \|\Phi\|_{\mathbf{H}^1} \;\leq\; C_0\Big[\|\mathcal{M}^1_\varepsilon \Theta\|_{\mathbf{L}^2} + \|\mathcal{M}^{1,2}_\varepsilon \Theta'\|_{\mathbf{L}^2} + \tfrac{1}{\delta}\big|\langle \Theta, (0, \overline{\Phi}^{\mathrm{adj}}_{e;0})\rangle_{\mathbf{L}^2}\big|\Big] \tag{3.4.21}$$

for any $\Phi \in \mathbf{H}^2$ and $\Theta = \tilde{\mathcal{L}}_{\varepsilon,\delta}\Phi$, possibly after increasing $C_0 > 0$.

Proof. Pick two constants $0 < \delta < \delta_0$ and $0 < \varepsilon < \varepsilon_0(\delta)$ together with a function $\Phi = (\Phi_o, \Phi_e) \in \mathbf{H}^1$ and write $\Theta = \tilde{\mathcal{L}}_{\varepsilon,\delta}\Phi \in \mathbf{L}^2$. If in fact $\Phi \in \mathbf{H}^2$, then a direct differentiation shows that

$$\Theta' \;=\; \tilde{\mathcal{L}}_{\varepsilon,\delta}\Phi' - D^2F\big(\tilde{U}_\varepsilon\big)\big[\tilde{U}_\varepsilon', \Phi\big], \tag{3.4.22}$$

which due to the boundedness of Φ implies that $\Theta \in \mathbf{H}^1$. In particular, $\tilde{\mathcal{L}}_{\varepsilon,\delta}$ maps $\mathbf{H}^2$ into $\mathbf{H}^1$. Reversely, suppose that we know that $\Theta \in \mathbf{H}^1$. Rewriting (3.4.22) yields

$$\tilde{c}_\varepsilon \Phi'' \;=\; \Theta' - \delta\Phi' + \mathcal{M}^1_{1/\varepsilon^2} J_{\mathrm{mix}}\Phi' + DF(\tilde{U}_\varepsilon)\Phi' + D^2F(\tilde{U}_\varepsilon)\big[\tilde{U}_\varepsilon', \Phi\big]. \tag{3.4.23}$$

Since Φ is bounded, this allows us to conclude that $\Phi \in \mathbf{H}^2$. On account of Proposition 3.4.1 we hence see that $\tilde{\mathcal{L}}_{\varepsilon,\delta}$ is invertible as a map from $\mathbf{H}^2$ to $\mathbf{H}^1$.

Fixing $\delta_{\mathrm{ref}} = \frac{1}{2}\delta_0$, a short computation shows that

$$\tilde{\mathcal{L}}_{\varepsilon,\delta_{\mathrm{ref}}}\Phi' \;=\; \Theta' + D^2F[\tilde{U}_\varepsilon', \Phi] + (\delta_{\mathrm{ref}} - \delta)\Phi'. \tag{3.4.24}$$

By (3.4.17) we obtain the bound

$$\|\mathcal{M}^1_\varepsilon \Phi'\|_{\mathbf{L}^2} + \|\Phi\|_{\mathbf{L}^2} \;\leq\; C_0\Big[\|\mathcal{M}^1_\varepsilon \Theta\|_{\mathbf{L}^2} + \tfrac{1}{\delta}\big|\langle \Theta, (0, \overline{\Phi}^{\mathrm{adj}}_{e;0})\rangle_{\mathbf{L}^2}\big|\Big]. \tag{3.4.25}$$

On the other hand, (3.4.13) yields the estimate

$$\begin{aligned}\|\mathcal{M}^{1,2}_\varepsilon \Phi''\|_{\mathbf{L}^2} + \|\mathcal{M}^2_\varepsilon \Phi'\|_{\mathbf{L}^2} \;\leq\; & C_0\Big[\|\mathcal{M}^{1,2}_\varepsilon \Theta'\|_{\mathbf{L}^2} + \|\mathcal{M}^{1,2}_\varepsilon D^2F[\tilde{U}_\varepsilon', \Phi]\|_{\mathbf{L}^2} \\ & \quad + \|\mathcal{M}^{1,2}_\varepsilon(\delta_{\mathrm{ref}} - \delta)\Phi'\|_{\mathbf{L}^2}\Big] \\ & \quad + \tfrac{C_0}{\delta_{\mathrm{ref}}}\Big|\langle \Theta' - D^2F(\tilde{U}_\varepsilon)[\tilde{U}_\varepsilon', \Phi] \\ & \qquad - (\delta_{\mathrm{ref}} - \delta)\Phi', (0, \overline{\Phi}^{\mathrm{adj}}_{e;0})\rangle_{\mathbf{L}^2}\Big|.\end{aligned} \tag{3.4.26}$$

Since $\tilde{U}_\varepsilon$ and $\tilde{U}'_\varepsilon$ are uniformly bounded by assumption, we readily see that

$$\|\mathcal{M}_\varepsilon^{1,2} D^2 F(\tilde{U}_\varepsilon)[\tilde{U}'_\varepsilon, \Phi]\|_{\mathbf{L}^2} \quad \leq \quad \|D^2 F(\tilde{U}_\varepsilon)[\tilde{U}'_\varepsilon, \Phi]\|_{\mathbf{L}^2} \quad \leq \quad C'_1 \|\Phi\|_{\mathbf{L}^2} \tag{3.4.27}$$

for some $C'_1 > 0$. In particular, we find

$$\begin{aligned} \|\mathcal{M}_\varepsilon^{1,2}\Phi''\|_{\mathbf{L}^2} + \|\mathcal{M}_\varepsilon^2 \Phi'\|_{\mathbf{L}^2} \quad \leq \quad & C'_2 \Big[\|\mathcal{M}_\varepsilon^{1,2}\Theta'\|_{\mathbf{L}^2} + \|\Phi\|_{\mathbf{L}^2} + \|\mathcal{M}_\varepsilon^{1,2}\Phi'\|_{\mathbf{L}^2} \\ & + \|\Theta'_e\|_{\mathbf{L}^2_e} + \|\Phi'_e\|_{\mathbf{L}^2_e} \Big] \end{aligned} \tag{3.4.28}$$

for some $C'_2 > 0$. Exploiting the estimates

$$\|\Phi'_e\|_{\mathbf{L}^2_e} \quad \leq \quad \|\mathcal{M}_\varepsilon^{1,2}\Phi'\|_{\mathbf{L}^2} \quad \leq \quad \|\mathcal{M}_\varepsilon^1 \Phi'\|_{\mathbf{L}^2}, \qquad \|\Theta'_e\|_{\mathbf{L}^2_e} \quad \leq \quad \|\mathcal{M}_\varepsilon^{1,2}\Theta'\|_{\mathbf{L}^2}, \tag{3.4.29}$$

together with

$$\|\Phi'\|_{\mathbf{L}^2} \quad \leq \quad \left\|\mathcal{M}_\varepsilon^1 \Phi'\right\|_{\mathbf{L}^2} + \left\|\mathcal{M}_\varepsilon^2 \Phi'\right\|_{\mathbf{L}^2}, \tag{3.4.30}$$

the bounds (3.4.25) and (3.4.28) can be combined to arrive at the desired inequality (3.4.21). ■

3.4.1 Strategy

In this subsection we outline our broad strategy to establish Proposition 3.4.1 and Proposition 3.4.2. As a first step, we compute the Fredholm index of the operators $\tilde{\mathcal{L}}_{\varepsilon,\lambda}$ for λ in a right half-plane that includes the imaginary axis.

Lemma 3.4.6. *Assume that (hFam), (HN1), (HN2), (HW1) and (HW2) are satisfied. Then there exists a constant $\lambda_0 > 0$ so that the operators $\tilde{\mathcal{L}}_{\varepsilon,\lambda}$ are Fredholm with index zero whenever $\mathrm{Re}\,\lambda \geq -\lambda_0$ and $\varepsilon > 0$.*

Proof. Upon writing

$$\begin{aligned} F^{(1)}_{o;\rho} &= \rho DF_o(U_o^-) + (1-\rho) DF_o(U_o^+), \\ F^{(1)}_{e;\rho} &= \rho DF_e(U_e^-) + (1-\rho) DF_e(U_e^+) \end{aligned} \tag{3.4.31}$$

for any $0 \leq \rho \leq 1$, we introduce the constant coefficient operator $L_{\rho;\varepsilon,\lambda} : \mathbf{H}^1_{\mathbb{C}} \to \mathbf{L}^2_{\mathbb{C}}$ that acts as

$$L_{\rho;\varepsilon,\lambda} \quad = \quad \begin{pmatrix} \tilde{c}_\varepsilon \frac{d}{d\xi} + \frac{2}{\varepsilon^2} J_{\mathcal{D}} - F^{(1)}_{o;\rho} + \lambda & -\frac{1}{\varepsilon^2} J_{\mathcal{D}} S_1 \\ -J_{\mathcal{D}} S_1 & \tilde{c}_\varepsilon \frac{d}{d\xi} + 2J_{\mathcal{D}} - F^{(1)}_{e;\rho} + \lambda \end{pmatrix} \tag{3.4.32}$$

and has the associated characteristic function

$$\Delta_{L_{\rho;\varepsilon,\lambda}}(z) \quad = \quad \begin{pmatrix} \tilde{c}_\varepsilon z + \frac{2}{\varepsilon^2} J_{\mathcal{D}} - F^{(1)}_{o;\rho} + \lambda & -\frac{1}{\varepsilon^2} J_{\mathcal{D}} \Big[e^z + e^{-z} \Big] \\ -J_{\mathcal{D}} \Big[e^z + e^{-z} \Big] & \tilde{c}_\varepsilon z + 2J_{\mathcal{D}} - F^{(1)}_{e;\rho} + \lambda \end{pmatrix}. \tag{3.4.33}$$

Upon writing

$$F_\rho^{(1)} = \begin{pmatrix} F_{o;\rho}^{(1)} & 0 \\ 0 & F_{e;\rho}^{(1)} \end{pmatrix}, \tag{3.4.34}$$

together with

$$A(y) = \begin{pmatrix} J_{\mathcal{D}} & -J_{\mathcal{D}}\cos(y) \\ -J_{\mathcal{D}}\cos(y) & J_{\mathcal{D}} \end{pmatrix}, \tag{3.4.35}$$

we see that

$$\mathcal{M}^{1,2}_{\varepsilon^2}\Delta_{L_{\rho;\varepsilon,\lambda}}(iy) = (\tilde{c}_\varepsilon iy + \lambda)\mathcal{M}^{1,2}_{\varepsilon^2} + 2A(y) - \mathcal{M}^{1,2}_{\varepsilon^2}F^{(1)}_\rho. \tag{3.4.36}$$

For any $y \in \mathbb{R}$ and $V \in \mathbb{C}^{2(n+k)}$ we have

$$\operatorname{Re} V^\dagger \tilde{c}_\varepsilon iy \mathcal{M}^{1,2}_{\varepsilon^2} V = 0, \tag{3.4.37}$$

together with

$$\operatorname{Re} V^\dagger A(y) V \geq 0. \tag{3.4.38}$$

In particular, we see that

$$\operatorname{Re} V^\dagger \mathcal{M}^{1,2}_{\varepsilon^2}\Delta_{L_{\rho;\varepsilon,\lambda}}(iy)V \geq -\varepsilon^2 \operatorname{Re}\big[V_o^\dagger(F^{(1)}_{o;\rho} - \lambda)V_o\big] - \operatorname{Re}\big[V_e^\dagger(F^{(1)}_{e;\rho} - \lambda)V_e\big]. \tag{3.4.39}$$

Let us pick an arbitrary $\lambda_0 > 0$ and suppose that $\Delta_{L_{\rho;\varepsilon,\lambda}}(iy)V = 0$ holds for some $V \in \mathbb{C}^{2(n+k)} \setminus \{0\}$ and $\operatorname{Re}\lambda \geq -\lambda_0$. We claim that there exist constants $\vartheta_1 > 0$ and $\vartheta_2 > 0$, that do not depend on λ_0, so that

$$-\operatorname{Re} V^\dagger_\#(F^{(1)}_{\#;\rho} - \lambda)V_\# \geq (\vartheta_2 - \vartheta_1\lambda_0)|V_\#|^2 \tag{3.4.40}$$

for $\# \in \{o, e\}$. Assuming that this is indeed the case, we pick $\lambda_0 = \frac{\vartheta_2}{2\vartheta_1}$ and obtain the contradiction

$$\begin{aligned} 0 &= \operatorname{Re} V^\dagger \mathcal{M}^{1,2}_{\varepsilon^2}\Delta_{L_{\rho;\varepsilon,\lambda}}(iy)V \\ &\geq \tfrac{1}{2}\vartheta_2\big[\varepsilon^2|V_o|^2 + |V_e|^2\big] \\ &> 0. \end{aligned} \tag{3.4.41}$$

The desired Fredholm properties then follow directly from [130, Thm. C].

In order to establish the claim (3.4.40), we first assume that $F_\#$ satisfies (hα). The negative-definiteness of $F^{(1)}_{\#;\rho}$ then directly yields the bound

$$\operatorname{Re} V^\dagger_\#(F^{(1)}_{\#;\rho} - \lambda)V_\# \leq (\lambda_0 - \vartheta_2)|V_\#|^2 \tag{3.4.42}$$

for some $\vartheta_2 > 0$.

On the other hand, if $F_\#$ satisfies (hβ), then we can use the identity

$$(\tilde{c}_\varepsilon iy + \lambda)w_\# - [F^{(1)}_{\#;\rho}]_{2,2}w_\# = [F^{(1)}_{\#;\rho}]_{2,1}v_\# \tag{3.4.43}$$

to compute

$$\begin{array}{lcl}
\mathrm{Re}\, V_{\#}^{\dagger} \begin{pmatrix} 0 & [F^{(1)}_{\#;\rho}]_{1,2} \\ [F^{(1)}_{\#;\rho}]_{2,1} & 0 \end{pmatrix} V_{\#} & = & \mathrm{Re}\, V_{\#}^{\dagger} \begin{pmatrix} 0 & -\Gamma [F^{(1)}_{\#;\rho}]^{\dagger}_{2,1} \\ [F^{(1)}_{\#;\rho}]_{2,1} & 0 \end{pmatrix} V_{\#} \\
& = & \mathrm{Re}\Big[-\Gamma v_{\#}^{\dagger} [F^{(1)}_{\#;\rho}]^{\dagger}_{2,1} w_{\#} + w_{\#}^{\dagger} [F^{(1)}_{\#;\rho}]_{2,1} v_{\#} \Big] \\
& = & (1-\Gamma) \mathrm{Re}\, w_{\#}^{\dagger} [F^{(1)}_{\#;\rho}]_{2,1} v_{\#} \\
& = & (1-\Gamma) \mathrm{Re}\, w_{\#}^{\dagger} \big[\tilde{c}_{\varepsilon} iy + \lambda \big] w_{\#} \\
& & \qquad -(1-\Gamma) \mathrm{Re}\, w_{\#}^{\dagger} [F^{(1)}_{\#;\rho}]_{2,2} w_{\#} \\
& = & (1-\Gamma) \mathrm{Re}\, \lambda |w_{\#}|^2 \\
& & \qquad -(1-\Gamma) \mathrm{Re}\, w_{\#}^{\dagger} [F^{(1)}_{\#;\rho}]_{2,2} w_{\#}.
\end{array} \tag{3.4.44}$$

In particular, Lemma 3.3.3 allows us to obtain the estimate

$$\begin{array}{lcl}
\mathrm{Re}\, V_{\#}^{\dagger} (F^{(1)}_{\#;\rho} - \lambda) V_{\#} & = & -\Gamma \mathrm{Re}\, \lambda |w_{\#}|^2 + \Gamma \mathrm{Re}\, w_{\#}^{\dagger} [F^{(1)}_{\#;\rho}]_{2,2} w_{\#} \\
& & \qquad - \mathrm{Re}\, \lambda |v_{\#}|^2 + \mathrm{Re}\, v_{\#}^{\dagger} [F^{(1)}_{\#;\rho}]_{2,2} v_{\#} \\
& \leq & (\Gamma + 1) \lambda_0 |V_{\#}|^2 - \vartheta_2 |V_{\#}|^2
\end{array} \tag{3.4.45}$$

for some $\vartheta_2 > 0$, as desired. ∎

For any $\varepsilon > 0$ and $0 < \delta < \delta_{\diamond}$ we introduce the quantity

$$\Lambda(\varepsilon, \delta) \quad = \quad \inf_{\Phi \in \mathbf{H}^1, \|\mathcal{M}^{1,2}_{\varepsilon} \Phi\|_{\mathbf{H}^1} = 1} \Big[\|\mathcal{M}^{1,2}_{\varepsilon} \tilde{\mathcal{L}}_{\varepsilon,\delta} \Phi\|_{\mathbf{L}^2} + \tfrac{1}{\delta} \big| \langle \tilde{\mathcal{L}}_{\varepsilon,\delta} \Phi, (0, \overline{\Phi}^{\mathrm{adj}}_{e;0}) \rangle_{\mathbf{L}^2} \big| \Big], \tag{3.4.46}$$

which allows us to define

$$\Lambda(\delta) \quad = \quad \liminf_{\varepsilon \downarrow 0} \Lambda(\varepsilon, \delta). \tag{3.4.47}$$

Similarly, for any $\varepsilon > 0$ and any subset $M \subset \mathbb{C}$ we write

$$\Lambda(\varepsilon, M) \quad = \quad \inf_{\Phi \in \mathbf{H}^1, \lambda \in M, \|\mathcal{M}^{1,2}_{\varepsilon} \Phi\|_{\mathbf{H}^1} = 1} \|\mathcal{M}^{1,2}_{\varepsilon} \tilde{\mathcal{L}}_{\varepsilon,\lambda} \Phi\|_{\mathbf{L}^2}, \tag{3.4.48}$$

together with

$$\Lambda(M) \quad = \quad \liminf_{\varepsilon \downarrow 0} \Lambda(\varepsilon, M). \tag{3.4.49}$$

The following proposition forms the key ingredient for proving Proposition 3.4.1 and Proposition 3.4.2. It is the analogue of [6, Lem. 3.2].

Proposition 3.4.7. *Assume that (hFam), (HN1), (HN2), (HW1), (HW2) and (HS1) are satisfied. Then there exist constants $\delta_0 > 0$ and $C_0 > 0$ so that*

$$\Lambda(\delta) \quad \geq \quad \tfrac{2}{C_0} \tag{3.4.50}$$

holds for all $0 < \delta < \delta_0$.

Assume furthermore that (HS2) holds and pick a sufficiently small $\lambda_0 > 0$. *Then for any subset* $M \subset \mathbb{C}$ *that satisfies* (hM_{λ_0}), *there exists a constant* C_M *so that*

$$\Lambda(M) \quad \geq \quad \frac{2}{C_M}. \tag{3.4.51}$$

Proof of Proposition 3.4.1. Fix $0 < \delta < \delta_0$. Proposition 3.4.7 implies that we can pick $\varepsilon_0(\delta) > 0$ in such a way that $\Lambda(\varepsilon, \delta) \geq \frac{1}{C_0}$ for each $0 < \varepsilon < \varepsilon_0(\delta)$. This means that $\tilde{\mathcal{L}}_{\varepsilon,\delta}$ is injective for each such ε and that the bound (3.4.11) holds for any $\Phi \in \mathbf{H}^1$. Since $\tilde{\mathcal{L}}_{\varepsilon,\delta}$ is also a Fredholm operator with index zero by Lemma 3.4.6, it must be invertible. ∎

Proof of Proposition 3.4.2. The result can be established by repeating the arguments used in the proof of Proposition 3.4.1, noting that the operator $\mathcal{M}^{1,2}_{\varepsilon}$ is invertible. ∎

3.4.2 Proof of Proposition 3.4.7

We now set out to prove Proposition 3.4.7. In Lemma 3.4.8 and Lemma 3.4.9 we construct weakly converging sequences that realize the infima in (3.4.46)–(3.4.49). In Lemmas 3.4.10-3.4.15 we exploit the structure of our operator (3.4.10) to recover lower bounds on the norms of the derivatives of these sequences that are typically lost when taking weak limits. First recall the constant $\delta_\diamond$ from Proposition 3.3.1.

Lemma 3.4.8. *Consider the setting of Proposition 3.4.7 and pick* $0 < \delta < \delta_\diamond$. *Then there exists a sequence*

$$\{(\varepsilon_j, \Phi_j, \Theta_j)\}_{j\geq 1} \quad \subset \quad (0,1) \times \mathbf{H}^1 \times \mathbf{L^2} \tag{3.4.52}$$

together with a pair of functions

$$\Phi \in \mathbf{H}^1, \qquad \Theta \in \mathbf{L}^2 \tag{3.4.53}$$

that satisfy the following properties.

(i) We have $\lim_{j\to\infty} \varepsilon_j = 0$ *together with*

$$\lim_{j\to\infty} \Big[\|\mathcal{M}^{1,2}_{\varepsilon_j}\Theta_j\|_{\mathbf{L}^2} + \tfrac{1}{\delta}\big|\langle \Theta_j, (0, \overline{\Phi}^{\mathrm{adj}}_{e;0})\rangle_{\mathbf{L}^2}\big| \Big] \quad = \quad \Lambda(\delta). \tag{3.4.54}$$

(ii) For every $j \geq 1$ *we have the identity*

$$\tilde{\mathcal{L}}_{\varepsilon_j,\delta}\Phi_j \quad = \quad \Theta_j \tag{3.4.55}$$

together with the normalization

$$\|\mathcal{M}^{1,2}_{\varepsilon_j}\Phi_j\|_{\mathbf{H}^1} \quad = \quad 1. \tag{3.4.56}$$

(iii) Writing $\Phi = (\phi_o, \psi_o, \phi_e, \psi_e)$, *we have* $\phi_o = 0$.

(iv) *The sequence* $\mathcal{M}^{1;2}_{\varepsilon_j}\Phi_j$ *converges to* Φ *strongly in* $\mathbf{L}^2_{\mathrm{loc}}$ *and weakly in* $\mathbf{H}^1$*. In addition, the sequence* $\mathcal{M}^{1;2}_{\varepsilon_j}\Theta_j$ *converges weakly to* Θ *in* $\mathbf{L}^2$*.*

Proof. Items (i) and (ii) follow directly from the definition of $\Lambda(\delta)$. The normalization (3.4.56) and the limit (3.4.54) ensure that $\|\mathcal{M}^{1;2}_{\varepsilon_j}\Phi_j\|_{\mathbf{H}^1}$ and $\|\mathcal{M}^{1;2}_{\varepsilon_j}\Theta_j\|_{\mathbf{L}^2}$ are bounded, which allows us to obtain the weak limits (iv) after passing to a subsequence.

In order to obtain (iii), we write $\Phi_j = (\phi_{o,j}, \psi_{o,j}, \phi_{e,j}, \psi_{e,j})$ together with $\Theta_j = (\theta_{o,j}, \chi_{o,j}, \theta_{e,j}, \chi_{e,j})$ and note that the first component of (3.4.55) yields

$$\begin{aligned} 2\mathcal{D}\phi_{o,j} - \mathcal{D}S_1\phi_{e,j} \quad = \quad & -\varepsilon_j^2\tilde{c}_{\varepsilon_j}\phi'_{o,j} + \varepsilon_j^2 D_1 f_o(\tilde{U}_{o;\varepsilon_j})\phi_{o,j} \\ & + \varepsilon_j^2 D_2 f_o(\tilde{U}_{o;\varepsilon_j})\psi_{o,j} - \delta\varepsilon_j^2\phi_{o,j} + \varepsilon_j^2\theta_{o,j}. \end{aligned} \tag{3.4.57}$$

The normalization condition (3.4.56) and the limit (3.4.54) hence imply that

$$\lim_{j\to\infty} \|2\mathcal{D}\phi_{o;j} - \mathcal{D}S_1\phi_{e,j}\|_{L^2(\mathbb{R};\mathbb{R}^n)} \quad = \quad 0. \tag{3.4.58}$$

In particular, we see that $\{\phi_{o;j}\}_{j\geq 1}$ is a bounded sequence. This yields the desired identity
$\phi_o = \lim_{j\to\infty} \varepsilon_j\phi_{o,j} = 0.$

■

Lemma 3.4.9. *Consider the setting of Proposition 3.4.7 and pick a sufficiently small* $\lambda_0 > 0$*. Then for any* $M \subset \mathbb{C}$ *that satisfies (h*M_{λ_0}*), there exists a sequence*

$$\{(\lambda_j, \varepsilon_j, \Phi_j, \Theta_j)\}_{j\geq 1} \quad \subset \quad M \times (0,1) \times \mathbf{H}^1 \times \mathbf{L}^2, \tag{3.4.59}$$

together with a triplet

$$\Phi \in \mathbf{H}^1, \qquad \Theta \in \mathbf{L}^2, \qquad \lambda \in M, \tag{3.4.60}$$

that satisfy the limits

$$\varepsilon_j \to 0, \qquad \lambda_j \to \lambda, \qquad \|\mathcal{M}^{1;2}_{\varepsilon_j}\Theta_j\|_{\mathbf{L}^2} \to \Lambda(M) \tag{3.4.61}$$

as $j \to \infty$*, together with the properties (ii)–(iv) from Lemma 3.4.8, with* δ *replaced by* λ_j *in (3.4.55).*

Proof. These properties can be obtained by following the proof of Lemma 3.4.8 in an almost identical fashion. ■

In the remainder of this section, we will often treat the settings of Lemma 3.4.8 and Lemma 3.4.9 in a parallel fashion. In order to streamline our notation, we use the value λ_0 stated in Lemma 3.4.6 and interpret $\{\lambda_j\}_{j\geq 1}$ as the constant sequence $\lambda_j = \delta$ when working in the context of Lemma 3.4.8. In addition, we write $\lambda_{\max} = \delta_\diamond$ in the setting of Lemma 3.4.8 or $\lambda_{\max} = \max\{|\lambda| : \lambda \in M\}$ in the setting of Lemma 3.4.9.

Lemma 3.4.10. *Consider the setting of Lemma 3.4.8 or Lemma 3.4.9. Then the function* Φ *from Lemma 3.4.8 satisfies*

$$\|\Phi\|_{\mathbf{H}^1} \quad \leq \quad C_\diamond\Lambda(\delta), \tag{3.4.62}$$

while the function Φ *from Lemma 3.4.9 satisfies*

$$\|\Phi\|_{\mathbf{H}^1} \quad \leq \quad C_{\diamond;M}\Lambda(M). \tag{3.4.63}$$

Proof. In order to take the $\varepsilon \downarrow 0$ limit in a controlled fashion, we introduce the operator

$$\tilde{L}_{0;\lambda} = \lim_{j \to \infty} \mathcal{M}^{1}_{\varepsilon_j^2} \tilde{\mathcal{L}}_{\varepsilon_j, \lambda_j}. \tag{3.4.64}$$

Upon introducing the top-left block

$$[\tilde{L}_{0;\lambda}]_{1,1} = \begin{pmatrix} 2\mathcal{D} & 0 \\ -D_1 g_o(\overline{U}_{o;0}) & \overline{L}_o + \lambda \end{pmatrix}, \tag{3.4.65}$$

we can explicitly write

$$\tilde{L}_{0;\lambda} = \begin{pmatrix} [\tilde{L}_{0;\lambda}]_{1,1} & -J\mathcal{D}S_1 \\ -J\mathcal{D}S_1 & c_0 \frac{d}{d\xi} + 2J\mathcal{D} - DF_e(\overline{U}_{e;0}) + \lambda \end{pmatrix}. \tag{3.4.66}$$

Note that $\tilde{L}_{0;\lambda}$ and its adjoint $\tilde{L}^{\mathrm{adj}}_{0;\lambda}$ are both bounded operators from $\mathbf{H}^1$ to $\mathbf{L}^2$.

In addition, we introduce the commutators

$$B_j = \tilde{\mathcal{L}}_{\varepsilon_j, \lambda_j} M^{1,2}_{\varepsilon_j} - M^{1,2}_{\varepsilon_j} \tilde{\mathcal{L}}_{\varepsilon_j, \lambda_j}. \tag{3.4.67}$$

A short computation shows that

$$B_j = \begin{pmatrix} [B_j]_{1,1} & (\frac{1}{\varepsilon_j} - \frac{1}{\varepsilon_j^2}) J_{\mathcal{D}} S_1 \\ (1 - \varepsilon_j) J_{\mathcal{D}} S_1 & 0 \end{pmatrix}, \tag{3.4.68}$$

in which the top-left block is given by

$$[B_j]_{1,1} = (1 - \varepsilon_j) \begin{pmatrix} 0 & D_2 f_o(\tilde{U}_{o;\varepsilon_j}) \\ -D_1 g_o(\tilde{U}_{o;\varepsilon_j}) & 0 \end{pmatrix}. \tag{3.4.69}$$

Pick any test-function $Z \in C^\infty(\mathbb{R}; \mathbb{R}^{2n+2k})$ and write

$$\mathcal{I}_j = \langle \mathcal{M}^{1}_{\varepsilon_j^2} \tilde{\mathcal{L}}_{\varepsilon_j, \lambda_j} \mathcal{M}^{1,2}_{\varepsilon_j} \Phi_j, Z \rangle_{\mathbf{L}^2}. \tag{3.4.70}$$

Using the strong convergence

$$\tilde{\mathcal{L}}^{\mathrm{adj}}_{\varepsilon_j, \lambda_j} \mathcal{M}^{1}_{\varepsilon_j^2} Z \to \tilde{L}^{\mathrm{adj}}_{0;\lambda} Z \in \mathbf{L^2}, \tag{3.4.71}$$

we obtain the limit

$$\begin{aligned} \mathcal{I}_j &= \langle \mathcal{M}^{1,2}_{\varepsilon_j} \Phi_j, \tilde{\mathcal{L}}^{\mathrm{adj}}_{\varepsilon_j, \lambda_j} \mathcal{M}^{1}_{\varepsilon_j^2} Z \rangle_{\mathbf{L}^2} \\ &\to \langle \Phi, \tilde{L}^{\mathrm{adj}}_{0;\lambda} Z \rangle_{\mathbf{L}^2} \\ &= \langle \tilde{L}_{0;\lambda} \Phi, Z \rangle_{\mathbf{L}^2} \end{aligned} \tag{3.4.72}$$

as $j \to \infty$.

In particular, we see that

$$\begin{array}{rcl}\mathcal{I}_j & = & \langle \mathcal{M}^1_{\varepsilon_j^2} \mathcal{M}^{1,2}_{\varepsilon_j} \tilde{\mathcal{L}}_{\varepsilon_j,\lambda_j} \Phi_j, Z \rangle_{\mathbf{L}^2} + \langle \mathcal{M}^1_{\varepsilon_j^2} B_j \Phi_j, Z \rangle_{\mathbf{L}^2} \\ & = & \langle \mathcal{M}^1_{\varepsilon_j^2} \mathcal{M}^{1,2}_{\varepsilon_j} \Theta_j, Z \rangle_{\mathbf{L}^2} + \langle \mathcal{M}^1_{\varepsilon_j^2} B_j \Phi_j, Z \rangle_{\mathbf{L}^2} \\ & \to & \langle \mathcal{M}^1_0 \Theta, Z \rangle_{\mathbf{L}^2} + \big\langle \big(- \mathcal{D} S_1 \phi_e, -D_1 g_o(\overline{U}_{o;0}) \phi_o, \mathcal{D} S_1 \phi_o, 0 \big), Z \big\rangle_{\mathbf{L}^2}. \end{array} \quad (3.4.73)$$

It hence follows that

$$\tilde{L}_{0;\delta} \Phi \;=\; \mathcal{M}^1_0 \Theta + \big(- \mathcal{D} S_1 \phi_e, -D_1 g_o(\overline{U}_{o;0}) \phi_o, \mathcal{D} S_1 \phi_o, 0 \big). \quad (3.4.74)$$

Introducing the functions

$$\Phi_\diamond \;=\; (\psi_0, \phi_e, \psi_e), \qquad \Theta_\diamond \;=\; (\chi_o, \theta_e, \chi_e), \quad (3.4.75)$$

the identity $\phi_o = 0$ implies that

$$\mathcal{L}_{\diamond,\lambda} \Phi_\diamond \;=\; \Theta_\diamond. \quad (3.4.76)$$

In the setting of Lemma 3.4.8, we may hence use Proposition 3.3.1 to compute

$$\begin{array}{rcl}\|\Phi_\diamond\|_{\mathbf{H}^1_\diamond} & \le & C_\diamond \Big[\|\Theta_\diamond\|_{\mathbf{L}^2_\diamond} + \frac{1}{\delta} \big| \langle \Theta_\diamond, (0, \overline{\Phi}^{\mathrm{adj}}_{e;0}) \rangle_{\mathbf{L}^2_\diamond} \big| \Big] \\ & \le & C_\diamond \Big[\|\Theta\|_{\mathbf{L}^2} + \frac{1}{\delta} \big| \langle \Theta, (0, \overline{\Phi}^{\mathrm{adj}}_{e;0}) \rangle_{\mathbf{L}^2} \big| \Big]. \end{array} \quad (3.4.77)$$

The lower semi-continuity of the L^2-norm and the convergence in (iv) of Lemma 3.4.8 imply that

$$\|\Theta\|_{\mathbf{L}^2} + \tfrac{1}{\delta} \big| \langle \Theta, (0, \overline{\Phi}^{\mathrm{adj}}_{e;0}) \rangle_{\mathbf{L}^2} \big| \;\le\; \Lambda(\delta). \quad (3.4.78)$$

In particular, we find

$$\|\Phi\|_{\mathbf{H}^1} \;=\; \|\Phi_\diamond\|_{\mathbf{H}^1_\diamond} \;\le\; C_\diamond \Lambda(\delta), \quad (3.4.79)$$

as desired. In the setting of Lemma 3.4.9 the bound follows in a similar fashion. ■

We note that

$$\mathcal{M}^{1,2}_{\varepsilon_j^2} \Theta_j \;=\; \tilde{c}_{\varepsilon_j} \mathcal{M}^{1,2}_{\varepsilon_j^2} \Phi_j' + \mathcal{M}^{1,2}_{\varepsilon_j^2} \big(- DF(\tilde{U}_{\varepsilon_j}) + \lambda_j \big) \Phi_j - J_{\mathrm{mix}} \Phi_j, \quad (3.4.80)$$

in which J_{mix} is given by (3.4.9) and in which

$$DF(\tilde{U}_\varepsilon) \;=\; \begin{pmatrix} DF_o(\tilde{U}_{o;\varepsilon}) & 0 \\ 0 & DF_e(\tilde{U}_{e;\varepsilon}) \end{pmatrix}. \quad (3.4.81)$$

Lemma 3.4.11. *Assume that (HN1) is satisfied. Then the bounds*

$$\begin{array}{rcl}\mathrm{Re}\, \langle -J_{\mathrm{mix}} \Phi, \Phi' \rangle_{\mathbf{L}^2} & = & 0, \\ \mathrm{Re}\, \langle -J_{\mathrm{mix}} \Phi, \Phi \rangle_{\mathbf{L}^2} & \ge & 0 \end{array} \quad (3.4.82)$$

hold for all $\Phi \in \mathbf{H}^1_{\mathbb{C}}$.

Proof. Pick $\Phi \in \mathbf{H}^1_{\mathbb{C}}$ and write $\Phi = (\Phi_o, \Phi_e)$. We can compute

$$\begin{array}{lcl} \mathrm{Re}\,\langle -J_{\mathrm{mix}}\Phi, \Phi' \rangle_{\mathbf{L}^2} & = & \mathrm{Re}\,\langle 2J_{\mathcal{D}}\Phi_o, \Phi_o' \rangle_{\mathbf{L}^2_o} - \mathrm{Re}\,\langle J_{\mathcal{D}} S_1 \Phi_e, \Phi_o' \rangle_{\mathbf{L}^2_o} \\ & & \quad -\mathrm{Re}\,\langle J_{\mathcal{D}} S_1 \Phi_o, \Phi_e' \rangle_{\mathbf{L}^2_e} + 2\mathrm{Re}\,\langle J_{\mathcal{D}}\Phi_e, \Phi_e' \rangle_{\mathbf{L}^2_e} \\ & = & 0, \end{array} \tag{3.4.83}$$

since we have $\mathrm{Re}\,\langle J_{\mathcal{D}} S_1 \Phi_e, \Phi_o' \rangle_{\mathbf{L}^2_o} = -\mathrm{Re}\,\langle J_{\mathcal{D}} S_1 \Phi_o, \Phi_e' \rangle_{\mathbf{L}^2_e}$. Moreover, we can estimate

$$\begin{array}{lcl} \mathrm{Re}\,\langle -J_{\mathrm{mix}}\Phi, \Phi \rangle_{\mathbf{L}^2} & = & \mathrm{Re}\,\langle 2J_{\mathcal{D}}\Phi_o, \Phi_o \rangle_{\mathbf{L}^2_o} - \mathrm{Re}\,\langle J_{\mathcal{D}} S_1 \Phi_e, \Phi_o \rangle_{\mathbf{L}^2_o} \\ & & \quad -\mathrm{Re}\,\langle J_{\mathcal{D}} S_1 \Phi_o, \Phi_e \rangle_{\mathbf{L}^2_e} + 2\mathrm{Re}\,\langle J_{\mathcal{D}}\Phi_e, \Phi_e \rangle_{\mathbf{L}^2_e} \\ & \geq & 2\|\sqrt{J_{\mathcal{D}}}\Phi_o\|^2_{\mathbf{L}^2_o} + 2\|\sqrt{J_{\mathcal{D}}}\Phi_e\|^2_{\mathbf{L}^2_e} - 4\|\sqrt{J_{\mathcal{D}}}\Phi_o\|_{\mathbf{L}^2_o}\|\sqrt{J_{\mathcal{D}}}\Phi_e\|_{\mathbf{L}^2_e} \\ & \geq & 2\|\sqrt{J_{\mathcal{D}}}\Phi_o\|^2_{\mathbf{L}^2_o} + 2\|\sqrt{J_{\mathcal{D}}}\Phi_e\|^2_{\mathbf{L}^2_e} \\ & & \quad -4\Big(\frac{1}{2}\|\sqrt{J_{\mathcal{D}}}\Phi_o\|^2_{\mathbf{L}^2_o} + \frac{1}{2}\|\sqrt{J_{\mathcal{D}}}\Phi_e\|^2_{\mathbf{L}^2_e}\Big) \\ & = & 0. \end{array} \tag{3.4.84}$$

∎

Lemma 3.4.12. *Consider the setting of Lemma 3.4.8 or Lemma 3.4.9. Then the bound*

$$\big|\mathrm{Re}\big\langle \mathcal{M}^{1,2}_{\varepsilon_j^2}\big(-DF(\tilde{U}_{\varepsilon_j}) + \lambda_j\big)\Phi_j, \Phi_j' \big\rangle_{\mathbf{L}^2}\big| \quad \leq \quad 2(\tilde{K}_F + \lambda_{\max})\|\mathcal{M}^{1,2}_{\varepsilon_j}\Phi\|_{\mathbf{L}^2}\|\mathcal{M}^{1,2}_{\varepsilon_j}\Phi_j'\|_{\mathbf{L}^2} \tag{3.4.85}$$

holds for all $j \geq 1$.

Proof. We first note that

$$\begin{array}{lcl} \mathrm{Re}\big\langle \mathcal{M}^{1,2}_{\varepsilon_j^2}\big(-DF(\tilde{U}_{\varepsilon_j}) + \lambda_j\big)\Phi_j, \Phi_j' \big\rangle_{\mathbf{L}^2} & = & \mathrm{Re}\langle \varepsilon_j(-DF_o(\tilde{U}_{o;\varepsilon_j}) + \lambda_j)\Phi_{o,j}, \varepsilon_j\Phi'_{o,j} \rangle_{\mathbf{L}^2_o} \\ & & \quad +\mathrm{Re}\langle (-DF_e(\tilde{U}_{e;\varepsilon_j}) + \lambda_j)\Phi_{e,j}, \Phi'_{e,j} \rangle_{\mathbf{L}^2_e}. \end{array} \tag{3.4.86}$$

Using Cauchy-Schwarz we compute

$$\begin{array}{lcl} \big|\mathrm{Re}\big\langle \mathcal{M}^{1,2}_{\varepsilon_j^2}\big(-DF(\tilde{U}_{\varepsilon_j}) + \lambda_j\big)\Phi_j, \Phi_j' \big\rangle_{\mathbf{L}^2}\big| & \leq & \big(\tilde{K}_F + \lambda_{\max}\big)\|\varepsilon_j\Phi_{o,j}\|_{\mathbf{L}^2_o}\|\varepsilon_j\Phi'_{o,j}\|_{\mathbf{L}^2_o} \\ & & \quad +\big(\tilde{K}_F + \lambda_{\max}\big)\|\Phi_{e,j}\|_{\mathbf{L}^2_e}\|\Phi'_{e,j}\|_{\mathbf{L}^2_e} \\ & \leq & 2\big(\tilde{K}_F + \lambda_{\max}\big)\|\mathcal{M}^{1,2}_{\varepsilon_j}\Phi_j\|_{\mathbf{L}^2}\|\mathcal{M}^{1,2}_{\varepsilon_j}\Phi_j'\|_{\mathbf{L}^2}, \end{array} \tag{3.4.87}$$

as desired. ∎

Lemma 3.4.13. *Consider the setting of Lemma 3.4.8 or Lemma 3.4.9, possibly decreasing the size of $\lambda_0 > 0$. Then there exist strictly positive constants (a, m, g) together with a constant $\beta \geq 0$ so that the bound*

$$\begin{array}{lcl} \mathrm{Re}\big\langle \mathcal{M}^{1,2}_{\varepsilon_j^2}\big(-DF(\tilde{U}_{\varepsilon_j}) + \lambda_j\big)\Phi_j, \Phi_j \big\rangle_{\mathbf{L}^2} & \geq & a\|\mathcal{M}^{1,2}_{\varepsilon_j}\Phi_j\|^2_{\mathbf{L}^2} - g\displaystyle\int_{|x|\leq m}|\mathcal{M}^{1,2}_{\varepsilon_j}\Phi_j|^2 \\ & & \quad -\beta\|\mathcal{M}^{1,2}_{\varepsilon_j}\Theta_j\|^2_{\mathbf{L}^2} \end{array} \tag{3.4.88}$$

holds for all $j \geq 1$.

Proof. We first note that

$$\mathrm{Re}\big\langle \mathcal{M}^{1,2}_{\varepsilon_j^2}\big(- DF(\tilde{U}_{\varepsilon_j}) + \lambda_j\big)\Phi_j , \Phi_j \big\rangle_{\mathbf{L}^2} \quad = \quad \varepsilon^2 \mathcal{N}_{o;j} + \mathcal{N}_{e;j}, \tag{3.4.89}$$

in which we have defined

$$\mathcal{N}_{\#,j} \quad = \quad \mathrm{Re}\big\langle \big(- DF_{\#}(\tilde{U}_{\#;\varepsilon_j}) + \lambda_j\big)\Phi_{\#,j} , \Phi_{\#,j} \big\rangle_{\mathbf{L}^2_{\#}} \tag{3.4.90}$$

for $\# \in \{o, e\}$.

Let us first suppose that $F_{\#}$ satisfies (hβ) and let $\Gamma_{\#}$ be the proportionality constant from that assumption. We start by studying the cross-term

$$\begin{array}{lcl} \mathcal{C}_{\#,j} & = & -\mathrm{Re}\big\langle D_2 f_{\#}\big(\tilde{U}_{\#;\varepsilon_j}\big)\psi_{\#,j} , \phi_{\#,j} \big\rangle_{L^2(\mathbb{R};\mathbb{R}^n)} \\ & & \qquad -\mathrm{Re}\big\langle D_1 g_{\#}\big(\tilde{U}_{\#;\varepsilon_j}\big)\phi_{\#,j} , \psi_{\#,j} \big\rangle_{L^2(\mathbb{R};\mathbb{R}^k)}. \end{array} \tag{3.4.91}$$

Recalling that

$$\chi_{\#,j} \quad = \quad \tilde{c}_{\varepsilon_j}\psi'_{\#,j} - Dg_{\#;1}(\tilde{U}_{\#;\varepsilon_j})\phi_{\#,j} - Dg_{\#;2}(\tilde{U}_{\#;\varepsilon_j})\psi_{\#,j} + \lambda_j \psi_{\#,j}, \tag{3.4.92}$$

we obtain the identity

$$\begin{array}{lcl} \mathcal{C}_{\#,j} & = & (\Gamma_{\#} - 1)\mathrm{Re}\langle D_1 g_{\#}(\tilde{U}_{\#;\varepsilon_j})\phi_{\#,j} , \psi_{\#,j}\rangle_{L^2(\mathbb{R};\mathbb{R}^k)} \\ & = & (\Gamma_{\#} - 1)\mathrm{Re}\langle \tilde{c}_{\varepsilon_j}\psi'_{\#,j} - D_2 g_{\#}(\tilde{U}_{\#;\varepsilon_j})\psi_{\#,j} + \lambda_j\psi_{\#,j} - \chi_{\#,j} , \psi_{\#,j}\rangle_{L^2(\mathbb{R};\mathbb{R}^k)} \\ & = & \tilde{c}_{\varepsilon_j}(\Gamma_{\#} - 1)\mathrm{Re}\langle \psi'_{\#,j} , \psi_{\#,j}\rangle_{L^2(\mathbb{R};\mathbb{R}^k)} \\ & & \qquad +(\Gamma_{\#} - 1)\mathrm{Re}\,\langle -D_2 g_{\#}(\tilde{U}_{\#;\varepsilon_j})\psi_{\#,j} + \lambda_j\psi_{\#,j} - \chi_{\#,j} , \psi_{\#,j}\rangle_{L^2(\mathbb{R};\mathbb{R}^k)} \\ & = & (1 - \Gamma_{\#})\mathrm{Re}\,\langle D_2 g_{\#}(\tilde{U}_{\#;\varepsilon_j})\psi_{\#,j} , \psi_{\#,j}\rangle_{L^2(\mathbb{R};\mathbb{R}^k)} \\ & & \qquad +(\Gamma_{\#} - 1)\Big[\mathrm{Re}\,\lambda \, \|\psi_{\#,j}\|^2_{\mathbf{L}_{\#}} - \langle \chi_{\#,j} , \psi_{\#,j}\rangle_{L^2(\mathbb{R};\mathbb{R}^k)}\Big]. \end{array} \tag{3.4.93}$$

In particular, we see that

$$\begin{array}{lcl} \mathcal{N}_{\#,j} & = & \Gamma_{\#}\mathrm{Re}\,\lambda\langle \psi_{\#,j} , \psi_{\#,j}\rangle_{L^2(\mathbb{R};\mathbb{R}^k)} - \Gamma_{\#}\mathrm{Re}\,\langle D_2 g_{\#}(\tilde{U}_{\#;\varepsilon_j})\psi_{\#,j} , \psi_{\#,j}\rangle_{L^2(\mathbb{R};\mathbb{R}^k)} \\ & & \qquad +\mathrm{Re}\,\lambda\langle \phi_{\#,j} , \phi_{\#,j}\rangle_{L^2(\mathbb{R};\mathbb{R}^n)} - \mathrm{Re}\,\langle D_1 f_{\#}(\tilde{U}_{\#;\varepsilon_j})\phi_{\#,j} , \phi_{\#,j}\rangle_{L^2(\mathbb{R};\mathbb{R}^n)} \\ & & \qquad -(\Gamma_{\#} - 1)\langle \chi_{\#,j} , \psi_{\#,j}\rangle_{L^2(\mathbb{R};\mathbb{R}^k)}. \end{array} \tag{3.4.94}$$

Recall that $\tilde{U}_{\varepsilon} \to \overline{U}_0$ in L^{∞}, $\tilde{U}_{o;\varepsilon_j}(\xi) \to U_o^{\pm}$ and $\tilde{U}_{e;\varepsilon_j}(\xi) \to U_e^{\pm}$ for $\xi \to \pm\infty$. Using Lemma 3.3.3 and decreasing λ_0 if necessary, we see that there exist $a > (\Gamma_{\#} + 1)\lambda_0 > 0$ and $m \gg 1$ so that

$$\begin{array}{lcl} 3a|\Phi_{\#,j}(\xi)|^2 & \leq & -\mathrm{Re}\big\langle D_1 f_{\#}\big(\tilde{U}_{\#;\varepsilon_j}(\xi)\big)\phi_{\#,j}(\xi) , \phi_{\#,j}(\xi)\big\rangle_{\mathbb{R}^n} \\ & & \qquad -\Gamma_{\#}\mathrm{Re}\big\langle D_2 g_{\#}\big(\tilde{U}_{\#;\varepsilon_j}(\xi)\big)\psi_{\#,j}(\xi) , \psi_{\#,j}(\xi)\big\rangle_{\mathbb{R}^k} \end{array} \tag{3.4.95}$$

for all $|\xi| \geq m$. We hence obtain

$$\begin{array}{lcl} \mathcal{N}_{\#,j} & \geq & 2a \int_{|\xi| \geq m} |\Phi_{\#,j}(\xi)|^2 \, d\xi - (\Gamma_{\#} + 1)\big(\tilde{K}_F + \lambda_{\max}\big) \int_{|\xi| \leq m} |\Phi_{\#,j}(\xi)|^2 \, d\xi \\ & & \qquad -(\Gamma_{\#} + 1) \|\chi_{\#,j}\|_{L^2(\mathbb{R};\mathbb{R}^k)} \|\psi_{\#,j}\|_{L^2(\mathbb{R};\mathbb{R}^k)} \\ & \geq & 2a \|\Phi_{\#,j}\|^2_{\mathbf{L}^2_{\#}} - (\Gamma_{\#} + 1)\big(2a + \tilde{K}_F + \lambda_{\max}\big) \int_{|\xi| \leq m} |\Phi_{\#,j}(\xi)|^2 \, d\xi \\ & & \qquad -(\Gamma_{\#} + 1) \|\chi_{\#,j}\|_{L^2(\mathbb{R};\mathbb{R}^k)} \|\psi_{\#,j}\|_{L^2(\mathbb{R};\mathbb{R}^k)}. \end{array} \tag{3.4.96}$$

Using the standard identity $xy \leq \frac{1}{4z}x^2 + zy^2$ for $x, y \in \mathbb{R}$ and $z > 0$, we now find

$$\begin{array}{lcl} \mathcal{N}_{\#,j} & \geq & a \|\Phi_{\#,j}\|^2_{\mathbf{L}^2_{\#}} - (\Gamma_{\#} + 1)\big(2a + \tilde{K}_F + \lambda_{\max}\big) \int_{|\xi| \leq m} |\Phi_{\#,j}(\xi)|^2 \, d\xi \\ & & \qquad -\frac{1}{4a}(\Gamma_{\#} + 1)^2 \|\chi_{\#,j}\|^2_{L^2(\mathbb{R};\mathbb{R}^k)}, \end{array} \tag{3.4.97}$$

which has the desired form.

In the case where $F_{\#}$ satisfies (hα), a similar bound can be obtained in an analogous, but far easier fashion. ∎

Lemma 3.4.14. *Consider the setting of Lemma 3.4.8 or Lemma 3.4.9. Then there exists a constant $\kappa > 0$ so that the bound*

$$\kappa \|\mathcal{M}^{1,2}_{\varepsilon_j} \Phi_j\|^2_{\mathbf{L}^2} \quad \geq \quad \|\mathcal{M}^{1,2}_{\varepsilon_j} \Phi'_j\|^2_{\mathbf{L}^2} - 2\tilde{K}^2_{\mathrm{fam}} \|\mathcal{M}^{1,2}_{\varepsilon_j} \Theta_j\|^2_{\mathbf{L}^2} \tag{3.4.98}$$

holds for all $j \geq 1$.

Proof. For convenience, we assume that $\tilde{c}_{\varepsilon_j} > 0$ for all $j \geq 1$. Recalling the decomposition (3.4.80), we can use Lemma 3.4.11 and Lemma 3.4.12 to compute

$$\begin{array}{lcl} \mathrm{Re}\langle \mathcal{M}^{1,2}_{\varepsilon_j} \Theta_j, \mathcal{M}^{1,2}_{\varepsilon_j} \Phi'_j \rangle_{\mathbf{L}^2} & = & \tilde{c}_{\varepsilon_j} \mathrm{Re}\langle \mathcal{M}^{1,2}_{\varepsilon_j} \Phi'_j, \mathcal{M}^{1,2}_{\varepsilon_j} \Phi'_j \rangle_{\mathbf{L}^2} + \mathrm{Re}\langle -J_{\mathrm{mix}} \Phi_j, \Phi'_j \rangle_{\mathbf{L}^2} \\ & & \qquad + \mathrm{Re}\big\langle \mathcal{M}^{1,2}_{\varepsilon_j^2} \big(- DF(\tilde{U}_{\varepsilon_j}) + \lambda_j \big) \Phi_j, \Phi'_j \big\rangle_{\mathbf{L}^2} \\ & \geq & -2\big(\tilde{K}_F + \lambda_{\max}\big) \|\mathcal{M}^{1,2}_{\varepsilon_j} \Phi_j\|_{\mathbf{L}^2} \|\mathcal{M}^{1,2}_{\varepsilon_j} \Phi'_j\|_{\mathbf{L}^2} \\ & & \qquad + \tilde{c}_{\varepsilon_j} \|\mathcal{M}^{1,2}_{\varepsilon_j} \Phi'_j\|^2_{\mathbf{L}^2}. \end{array} \tag{3.4.99}$$

We hence see that

$$\begin{array}{lcl} \tilde{c}_{\varepsilon_j} \|\mathcal{M}^{1,2}_{\varepsilon_j} \Phi'_j\|^2_{\mathbf{L}^2} & \leq & 2\big(\tilde{K}_F + \lambda_{\max}\big) \|\mathcal{M}^{1,2}_{\varepsilon_j} \Phi_j\|_{\mathbf{L}^2} \|\mathcal{M}^{1,2}_{\varepsilon_j} \Phi'_j\|_{\mathbf{L}^2} \\ & & \qquad + \|\mathcal{M}^{1,2}_{\varepsilon_j} \Theta_j\|_{\mathbf{L}^2} \|\mathcal{M}^{1,2}_{\varepsilon_j} \Phi'_j\|_{\mathbf{L}^2}. \end{array} \tag{3.4.100}$$

Dividing by $\|\mathcal{M}^{1,2}_{\varepsilon_j} \Phi'_j\|_{\mathbf{L}^2}$ and squaring, we find

$$\tilde{c}^2_{\varepsilon_j} \|\mathcal{M}^{1,2}_{\varepsilon_j} \Phi'_j\|^2_{\mathbf{L}^2} \quad \leq \quad 8\big(\tilde{K}_F + \lambda_{\max}\big)^2 \|\mathcal{M}^{1,2}_{\varepsilon_j} \Phi_j\|^2_{\mathbf{L}^2} + 2\|\mathcal{M}^{1,2}_{\varepsilon_j} \Theta_j\|^2_{\mathbf{L}^2}, \tag{3.4.101}$$

as desired. ∎

Recall the constants (g, m, a, β) introduced in Lemma 3.4.13. Throughout the remainder of this section, we set out to obtain a lower bound for the integral

$$\mathcal{I}_j \quad = \quad g \int_{|\xi| \le m} |\mathcal{M}^{1,2}_{\varepsilon_j} \Phi_j(\xi)|^2 \, d\xi. \tag{3.4.102}$$

Lemma 3.4.15. *Consider the setting of Lemma 3.4.8 or Lemma 3.4.9. Then the bound*

$$\mathcal{I}_j \quad \ge \quad \tfrac{a}{2} \|\mathcal{M}^{1,2}_{\varepsilon_j} \Phi_j\|^2_{\mathbf{L}^2} - \Big(\tfrac{1}{2a} + \beta\Big) \|\mathcal{M}^{1,2}_{\varepsilon_j} \Theta_j\|^2_{\mathbf{L}^2} \tag{3.4.103}$$

holds for all $j \ge 1$.

Proof. Recall the decomposition (3.4.80). Combining the estimates in Lemma 3.4.11 and Lemma 3.4.13 and remembering that $\mathrm{Re}\langle \mathcal{M}^{1,2}_{\varepsilon_j} \Phi'_j, \mathcal{M}^{1,2}_{\varepsilon_j} \Phi_j \rangle_{\mathbf{L}^2} = 0$, we find

$$\begin{array}{lll} \mathcal{I}_j & \ge & a \|\mathcal{M}^{1,2}_{\varepsilon_j} \Phi_j\|^2_{\mathbf{L}^2} - \mathrm{Re}\langle \mathcal{M}^{1,2}_{\varepsilon_j} \Theta_j, \mathcal{M}^{1,2}_{\varepsilon_j} \Phi_j \rangle_{\mathbf{L}^2} - \beta \|\mathcal{M}^{1,2}_{\varepsilon_j} \Theta_j\|^2_{\mathbf{L}^2} \\ & \ge & a \|\mathcal{M}^{1,2}_{\varepsilon_j} \Phi_j\|^2_{\mathbf{L}^2} - \|\mathcal{M}^{1,2}_{\varepsilon_j} \Theta_j\|_{\mathbf{L}^2} \|\mathcal{M}^{1,2}_{\varepsilon_j} \Phi_j\|_{\mathbf{L}^2} - \beta \|\mathcal{M}^{1,2}_{\varepsilon_j} \Theta_j\|^2_{\mathbf{L}^2}. \end{array} \tag{3.4.104}$$

Using the standard identity $xy \le \frac{z}{2} x^2 + \frac{1}{2z} y^2$ for $x, y \in \mathbb{R}$ and $z > 0$ we can estimate

$$\mathcal{I}_j \quad \ge \quad \tfrac{a}{2} \|\mathcal{M}^{1,2}_{\varepsilon_j} \Phi_j\|^2_{\mathbf{L}^2} - \Big(\tfrac{1}{2a} + \beta\Big) \|\mathcal{M}^{1,2}_{\varepsilon_j} \Theta_j\|^2_{\mathbf{L}^2}, \tag{3.4.105}$$

as desired. ∎

Proof of Proposition 3.4.7. Introducing the constant $\gamma = \frac{a}{2(\kappa+1)}$, we add γ times (3.4.98) to (3.4.103) and find

$$\begin{array}{lll} \mathcal{I}_j + \frac{a\kappa}{2(\kappa+1)} \|\mathcal{M}^{1,2}_{\varepsilon_j} \Phi_j\|^2_{\mathbf{L}^2} & \ge & \frac{a}{2} \|\mathcal{M}^{1,2}_{\varepsilon_j} \Phi_j\|^2_{\mathbf{L}^2} - \Big(\frac{1}{2a} + \beta\Big) \|\mathcal{M}^{1,2}_{\varepsilon_j} \Theta_j\|^2_{\mathbf{L}^2} \\ & & \qquad + \frac{a}{2(\kappa+1)} \|\mathcal{M}^{1,2}_{\varepsilon_j} \Phi'_j\|_{\mathbf{L}^2} - \frac{a\tilde{K}^2_{\mathrm{fam}}}{2(\kappa+1)} \|\mathcal{M}^{1,2}_{\varepsilon_j} \Theta_j\|^2_{\mathbf{L}^2}. \end{array} \tag{3.4.106}$$

We hence obtain

$$\begin{array}{lll} \mathcal{I}_j & \ge & \frac{a}{2(\kappa+1)} \|\mathcal{M}^{1,2}_{\varepsilon_j} \Phi_j\|_{\mathbf{H}^1} - \Big(\frac{1}{2a} + \beta + \frac{a\tilde{K}^2_{\mathrm{fam}}}{2(\kappa+1)}\Big) \|\mathcal{M}^{1,2}_{\varepsilon_j} \Theta_j\|^2_{\mathbf{L}^2} \\ & := & C_1 - C_2 \|\mathcal{M}^{1,2}_{\varepsilon_j} \Theta_j\|^2_{\mathbf{L}^2}. \end{array} \tag{3.4.107}$$

Letting $j \to \infty$ in the setting of Lemma 3.4.8 yields

$$C_1 - C_2 \Lambda(\delta)^2 \quad \le \quad g \int_{|\xi| \le m} |\Phi(\xi)|^2 \, d\xi \quad \le \quad g C_\diamond^2 \Lambda(\delta)^2. \tag{3.4.108}$$

As such, we can conclude that

$$\Lambda(\delta) \quad \ge \quad \tfrac{2}{C_0} \tag{3.4.109}$$

for some $C_0 > 0$, as required. An analogous computation can be used for the setting of Lemma 3.4.9. ∎

3.5 Existence of travelling waves

In this section we follow the spirit of [6, Thm. 1] and develop a fixed point argument to show that (3.2.1) admits travelling wave solutions of the form (3.2.12). The main complication is that we need ε-uniform bounds on the supremum norm of the wave-profiles in order to control the nonlinear terms. This can be achieved by bounding the $\mathbf{H}^1$-norm of the perturbation, but the estimates in Proposition 3.4.1 feature a problematic scaling factor on the odd component. Fortunately, Corollary 3.4.5 does provide uniform $\mathbf{H}^1$-bounds, but it requires us to take a derivative of the travelling wave system.

Throughout this section we will apply the results from §3.4 to the constant family

$$\big(\tilde{U}_\varepsilon, \tilde{c}_\varepsilon\big) \quad = \quad \big(\overline{U}_0, c_0\big), \tag{3.5.1}$$

which clearly satisfies (hFam). In particular, we fix a small constant $\delta > 0$ and write $\mathcal{L}_{\varepsilon,\delta}$ for the operators given by (3.4.7) in this setting. We set out to construct a branch of wavespeeds c_ε and small functions

$$\Phi_\varepsilon \quad = \quad (\Phi_{o;\varepsilon}, \Phi_{e;\varepsilon}) \in \mathbf{H}^2 \tag{3.5.2}$$

in such a way that $\overline{U}_0 + \Phi_\varepsilon$ is a solution to (3.2.14). A short computation shows that this is equivalent to the system

$$\mathcal{L}_{\varepsilon,\delta}(\Phi_\varepsilon) \quad = \quad \mathcal{F}_\delta(c_\varepsilon, \Phi_\varepsilon), \tag{3.5.3}$$

which we split up by introducing the expressions

$$\begin{array}{lcl} \mathcal{R}(c, \Phi) & = & (c_0 - c)\partial_\xi\big(\overline{U}_0 + \Phi\big), \\ \mathcal{E}_0 & = & \Big(- Jc_0\overline{U}'_{o;0} + JF_o(\overline{U}_{o;0}), 0\Big), \\ \mathcal{N}_\#(\Phi_\#) & = & F_\#(\overline{U}_{\#;0} + \Phi_\#) - DF_\#(\overline{U}_{\#;0})\Phi_\# - F_\#(\overline{U}_{\#;0}) \end{array} \tag{3.5.4}$$

for $\# \in \{o, e\}$ and writing

$$\mathcal{F}_\delta(c_\varepsilon, \Phi_\varepsilon) \quad = \quad \mathcal{R}(c_\varepsilon, \Phi_\varepsilon) + \mathcal{E}_0 + \big(\mathcal{N}_o(\Phi_{o;\varepsilon}), \mathcal{N}_e(\Phi_{e;\varepsilon})\big) + \delta\Phi. \tag{3.5.5}$$

Notice that $\mathcal{R}$ contains a derivative of Φ. It is hence crucial that $\mathcal{L}_{\varepsilon,\delta}^{-1}$ gains an order of regularity, which we obtained by the framework developed in §3.4.

For any $\varepsilon > 0$ and $\Phi \in \mathbf{H}^2$ we introduce the norm

$$\|\Phi\|^2_{\mathbf{X}_\varepsilon} \quad = \quad \Big\|\mathcal{M}_\varepsilon^{1,2}\partial_\xi^2\Phi\Big\|^2_{\mathbf{L}^2} + \|\Phi\|^2_{\mathbf{H}^1}\,, \tag{3.5.6}$$

which is equivalent to the standard norm on $\mathbf{H}^2$. For any $\eta > 0$, this allows us to introduce the set

$$\mathbf{X}_{\eta;\varepsilon} \quad = \quad \{\Phi \in \mathbf{H}^2 : \|\Phi\|_{\mathbf{X}_\varepsilon} \leq \eta\}. \tag{3.5.7}$$

For convenience, we introduce the constant $\eta_* = \big[2\|\Phi^{\mathrm{adj}}_{e;0}\|_{\mathbf{L}^2_e}\big]^{-1}$, together with the formal expression

$$c_\delta(\Phi_e) = c_0 + \big[1 + \langle \partial_\xi \Phi_e, \overline{\Phi}^{\mathrm{adj}}_{e;0}\rangle_{\mathbf{L}^2_e}\big]^{-1} \Big[\delta \langle \Phi_e, \overline{\Phi}^{\mathrm{adj}}_{e;0}\rangle_{\mathbf{L}^2_e} + \langle \mathcal{N}_e(\Phi_e), \overline{\Phi}^{\mathrm{adj}}_{e;0}\rangle_{\mathbf{L}^2_e}\Big]. \tag{3.5.8}$$

Lemma 3.5.1. *Assume that (HN1), (HN2), (HW1), (HW2) and (HS1) are satisfied and pick a constant $0 < \eta \leq \eta_*$. Then the expression (3.5.8) is well-defined for any $\varepsilon > 0$ and any $\Phi = (\Phi_o, \Phi_e) \in \mathbf{X}_{\eta;\varepsilon}$. In addition, the equation*

$$\big\langle \mathcal{F}_\delta(c, \Phi), (0, \overline{\Phi}^{\mathrm{adj}}_{e;0})\big\rangle_{\mathbf{L}^2} = 0 \tag{3.5.9}$$

has the unique solution $c = c_\delta(\Phi_e)$.

Proof. We first note that

$$\langle \partial_\xi \Phi_e, \overline{\Phi}^{\mathrm{adj}}_{e;0}\rangle_{\mathbf{L}^2_e} \geq -\|\partial_\xi \Phi_e\|_{\mathbf{L}^2_e} \Big\|\overline{\Phi}^{\mathrm{adj}}_{e;0}\Big\|_{\mathbf{L}^2_e} \geq -\tfrac{1}{2}, \tag{3.5.10}$$

which implies that (3.5.8) is well-defined. The result now follows by noting that $\langle \mathcal{E}_0, (0, \overline{\Phi}^{\mathrm{adj}}_{e;0})\rangle_{\mathbf{L}^2} = 0$ and that

$$\begin{aligned} \big\langle \mathcal{R}(c, \Phi), (0, \overline{\Phi}^{\mathrm{adj}}_{e;0})\big\rangle_{\mathbf{L}^2} &= (c_0 - c)\Big(\langle \overline{U}'_{0;e}, \overline{\Phi}^{\mathrm{adj}}_{e;0}\rangle_{\mathbf{L}^2_e} + \langle \partial_\xi \Phi_e, \overline{\Phi}^{\mathrm{adj}}_{e;0}\rangle_{\mathbf{L}^2_e}\Big) \\ &= (c_0 - c)\Big(1 + \langle \partial_\xi \Phi_e, \overline{\Phi}^{\mathrm{adj}}_{e;0}\rangle_{\mathbf{L}^2_e}\Big), \end{aligned} \tag{3.5.11}$$

which implies that

$$\begin{aligned} \big\langle \mathcal{F}_\delta(c, \Phi), (0, \overline{\Phi}^{\mathrm{adj}}_{e;0})\big\rangle_{\mathbf{L}^2} &= (c_0 - c)\Big(1 + \langle \partial_\xi \Phi_e, \overline{\Phi}^{\mathrm{adj}}_{e;0}\rangle_{\mathbf{L}^2_e}\Big) + \delta \langle \Phi_e, \overline{\Phi}^{\mathrm{adj}}_{e;0}\rangle_{\mathbf{L}^2_e} \\ &\quad + \langle \mathcal{N}_e(\Phi_e), \overline{\Phi}^{\mathrm{adj}}_{e;0}\rangle_{\mathbf{L}^2_e}. \end{aligned} \tag{3.5.12}$$

■

Consider the setting of Corollary 3.4.5 and pick $0 < \delta < \delta_0$ and $0 < \varepsilon < \varepsilon_0(\delta)$. Our goal here is to find solutions to (3.5.3) by showing that the map $T_{\varepsilon,\delta} : \mathbf{X}_{\eta;\varepsilon} \to \mathbf{H}^2$ that acts as

$$T_{\varepsilon,\delta}(\Phi) = (\mathcal{L}_{\varepsilon,\delta})^{-1} \mathcal{F}_\delta\big(c_\delta(\Phi_e), \Phi\big) \tag{3.5.13}$$

admits a fixed point. For any triplet $(\Phi, \Phi^A, \Phi^B) \in \mathbf{X}^3_{\eta;\varepsilon}$, the bounds in Corollary 3.4.5 imply that

$$\|T_{\varepsilon,\delta}(\Phi)\|_{\mathbf{X}_\varepsilon} \leq C_0 \big[\big\|\mathcal{M}^1_\varepsilon \mathcal{F}_\delta\big(c_\delta(\Phi_e), \Phi\big)\big\|_{\mathbf{L}^2} + \big\|\mathcal{M}^{1,2}_\varepsilon \partial_\xi \mathcal{F}_\delta\big(c_\delta(\Phi_e), \Phi\big)\big\|_{\mathbf{L}^2} \big], \tag{3.5.14}$$

together with

$$\begin{aligned} \big\|T_{\varepsilon,\delta}(\Phi^A) - T_{\varepsilon,\delta}(\Phi^B)\big\|_{\mathbf{X}_\varepsilon} &\leq C_0 \Big\|\mathcal{M}^1_\varepsilon \Big(\mathcal{F}_\delta\big(c_\delta(\Phi^A_e), \Phi^A\big) - \mathcal{F}_\delta\big(c_\delta(\Phi^B_e), \Phi^B\big)\Big)\Big\|_{\mathbf{L}^2} \\ &\quad + C_0 \Big\|\mathcal{M}^{1,2}_\varepsilon \partial_\xi \Big(\mathcal{F}_\delta\big(c_\delta(\Phi^A_e), \Phi^A\big) - \mathcal{F}_\delta\big(c_\delta(\Phi^B_e), \Phi^B\big)\Big)\Big\|_{\mathbf{L}^2}. \end{aligned} \tag{3.5.15}$$

In order to show that $T_{\varepsilon,\delta}$ is a contraction mapping, it hence suffices to obtain suitable bounds for the terms appearing on the right-hand side of these estimates.

We start by obtaining pointwise bounds on the nonlinear terms. To this end, we compute

$$\begin{array}{rcl} \partial_\xi \mathcal{N}_o(\Phi_o) & = & \Big(DF_o(\overline{U}_{o;0} + \Phi_o) - DF_o(\overline{U}_{o;0}) - D^2 F_o(\overline{U}_{o;0})\Phi_o \Big) \overline{U}'_{o;0} \\ & & + \Big(DF_o(\overline{U}_{o;0} + \Phi_o) - DF_o(\overline{U}_{o;0}) \Big) \partial_\xi \Phi_o \end{array} \tag{3.5.16}$$

and note that a similar identity holds for $\partial_\xi \mathcal{N}_e(\Phi_e)$. In addition, we remark that there is a constant $K_F > 0$ for which the bounds

$$\|DF_\#(\overline{U}_{\#;0} + \Phi_\#)\|_\infty + \|D^2 F_\#(\overline{U}_{\#;0} + \Phi_\#)\|_\infty + \|D^3 F_\#(\overline{U}_{\#;0} + \Phi_\#)\|_\infty \quad < \quad K_F \tag{3.5.17}$$

hold for $\# \in \{o, e\}$ and all $\Phi = (\Phi_o, \Phi_e)$ that have $\|\Phi\|_{\mathbf{H}^1} \leq \eta_*$.

Lemma 3.5.2. *Assume that (HN1), (HN2), (HW1) and (HW2) are satisfied. There exists a constant $K_p > 0$ so that for each $\Phi = (\Phi_o, \Phi_e) \in \mathbf{H}^1$ with $\|\Phi\|_{\mathbf{H}^1} \leq \eta_*$, we have the pointwise estimates*

$$\begin{array}{rcl} |\mathcal{N}_o(\Phi_o)| & \leq & K_p |\Phi_o|^2, \\ |\mathcal{N}_e(\Phi_e)| & \leq & K_p |\Phi_e|^2. \end{array} \tag{3.5.18}$$

Proof. Using [55, Thm. 2.8.3] we obtain

$$|\mathcal{N}_o(\Phi_o)| \quad \leq \quad \tfrac{1}{2} K_F |\Phi_o|^2. \tag{3.5.19}$$

The estimate for $\mathcal{N}_e$ follows similarly. ∎

Lemma 3.5.3. *Assume that (HN1), (HN2), (HW1) and (HW2) are satisfied. There exists a constant $K_p > 0$ so that for each $\Phi = (\Phi_o, \Phi_e) \in \mathbf{H}^1$ with $\|\Phi\|_{\mathbf{H}^1} \leq \eta_*$, we have the pointwise estimates*

$$\begin{array}{rcl} |\partial_\xi \mathcal{N}_o(\Phi_o)| & \leq & K_p \big(|\partial_\xi \Phi_o| |\Phi_o| + |\Phi_o|^2 \big), \\ |\partial_\xi \mathcal{N}_e(\Phi_e)| & \leq & K_p \big(|\partial_\xi \Phi_e| |\Phi_e| + |\Phi_e|^2 \big). \end{array} \tag{3.5.20}$$

Proof. We rewrite (3.5.16) to obtain

$$\begin{array}{rcl} \partial_\xi \mathcal{N}_o(\Phi_o) & = & DF_o(\overline{U}_{o;0} + \Phi_o) \partial_\xi (\overline{U}_{o;0} + \Phi_o) - DF_o(\overline{U}_{o;0}) \partial_\xi (\overline{U}_{o;0} + \Phi_o) \\ & & - D^2 F_o(\overline{U}_{o;0})[\Phi_o, \partial_\xi(\overline{U}_{o;0} + \Phi_o)] + D^2 F_o(\overline{U}_{o;0})[\Phi_o, \partial_\xi \Phi_o]. \end{array} \tag{3.5.21}$$

This allows us to use [55, Thm. 2.8.3] and obtain the pointwise estimate

$$\begin{array}{rcl} |\partial_\xi \mathcal{N}_o(\Phi_o)| & \leq & \tfrac{1}{2} K_F |\Phi_o|^2 \big(|\overline{U}'_{o;0}| + |\partial_\xi \Phi_o| \big) + K_F |\Phi_o| |\partial_\xi \Phi_o| \\ & \leq & K_p \big(|\partial_\xi \Phi_o| |\Phi_o| + |\Phi_o|^2 \big). \end{array} \tag{3.5.22}$$

The estimate for $\mathcal{N}_e$ follows similarly. ∎

Lemma 3.5.4. *Assume that (HN1), (HN2), (HW1) and (HW2) are satisfied. There exists a constant $K_p > 0$ so that for each pair*

$$\Phi^A = (\Phi_o^A, \Phi_e^A) \in \mathbf{H}^1, \qquad \Phi^B = (\Phi_o^B, \Phi_e^B) \in \mathbf{H}^1 \tag{3.5.23}$$

that satisfies $\|\Phi^A\|_{\mathbf{H}^1} \le \eta_*$ and $\|\Phi^B\|_{\mathbf{H}^1} \le \eta_*$, we have the pointwise estimates

$$\begin{array}{lcl} |\mathcal{N}_o(\Phi_o^A) - \mathcal{N}_o(\Phi_o^B)| & \le & K_p\big[|\Phi_o^A| + |\Phi_o^B|\big]|\Phi_o^A - \Phi_o^B|, \\ |\mathcal{N}_e(\Phi_e^A) - \mathcal{N}_e(\Phi_e^B)| & \le & K_p\big[|\Phi_e^A| + |\Phi_e^B|\big]|\Phi_e^A - \Phi_e^B|. \end{array} \tag{3.5.24}$$

Proof. We first compute

$$\begin{array}{lcl} \mathcal{N}_o(\Phi_o^A) - \mathcal{N}_o(\Phi_o^B) & = & F_o\big(\overline{U}_{o;0} + \Phi_o^B + (\Phi_o^A - \Phi_o^B)\big) - F_o\big(\overline{U}_{o;0} + \Phi_o^B\big) \\ & & -DF_o\big(\overline{U}_{o;0} + \Phi_o^B\big)\big(\Phi_o^A - \Phi_o^B\big) \\ & & +\big[DF_o\big(\overline{U}_{o;0} + \Phi_o^B\big) - DF_o(\overline{U}_{o;0})\big]\big(\Phi_o^A - \Phi_o^B\big). \end{array} \tag{3.5.25}$$

Applying [55, Thm. 2.8.3] twice yields the pointwise estimate

$$\begin{array}{lcl} |\mathcal{N}_o(\Phi_o^A) - \mathcal{N}_o(\Phi_o^B)| & \le & K_F\Big[\frac{1}{2}|\Phi_o^A - \Phi_o^B|^2 + |\Phi_o^B||\Phi_o^A - \Phi_o^B|\Big] \\ & \le & 2K_F\big[|\Phi_o^A| + |\Phi_o^B|\big]|\Phi_o^A - \Phi_o^B|. \end{array} \tag{3.5.26}$$

The estimate for $\mathcal{N}_e$ follows similarly. ∎

Lemma 3.5.5. *Assume that (HN1), (HN2), (HW1) and (HW2) are satisfied. There exists a constant $K_p > 0$ so that for each pair*

$$\Phi^A = (\Phi_o^A, \Phi_e^A) \in \mathbf{H}^1, \qquad \Phi^B = (\Phi_o^B, \Phi_e^B) \in \mathbf{H}^1 \tag{3.5.27}$$

that satisfies $\|\Phi^A\|_{\mathbf{H}^1} \le \eta_*$ and $\|\Phi^B\|_{\mathbf{H}^1} \le \eta_*$ we have the pointwise estimates

$$\begin{array}{lcl} |\partial_\xi \mathcal{N}_\#(\Phi_\#^A) - \partial_\xi \mathcal{N}_\#(\Phi_\#^B)| & \le & K_p\Big[|\partial_\xi \Phi_\#^A| + |\Phi_\#^A| + |\partial_\xi \Phi_\#^B| + |\Phi_\#^B|\Big]|\Phi_\#^A - \Phi_\#^B| \\ & & +K_p\Big[|\Phi_\#^A| + |\Phi_\#^B|\Big]|\partial_\xi(\Phi_\#^A - \Phi_\#^B)| \end{array} \tag{3.5.28}$$

for $\# \in \{o, e\}$.

Proof. Differentiating (3.5.25) line by line, we obtain

$$\partial_\xi \mathcal{N}_o(\Phi_o^A) - \partial_\xi \mathcal{N}_o(\Phi_o^B) \quad = \quad d_1 + d_2 + d_3 \tag{3.5.29}$$

with

$$\begin{array}{rcl} d_1 & = & DF_o\big(\overline{U}_{o;0} + \Phi_o^B + (\Phi_o^A - \Phi_o^B)\big)\big(\overline{U}'_{o;0} + \partial_\xi \Phi_o^B + \partial_\xi(\Phi_o^A - \Phi_o^B)\big) \\ & & \quad - DF_o\big(\overline{U}_{o;0} + \Phi_o^B\big)\partial_\xi\big(\overline{U}_{o;0} + \Phi_o^B\big), \\ d_2 & = & -D^2F_o\big(\overline{U}_{o;0} + \Phi_o^B\big)\big[\Phi_o^A - \Phi_o^B, \partial_\xi(\overline{U}_{o;0} + \Phi_o^B)\big] \\ & & \quad - DF_o\big(\overline{U}_{o;0} + \Phi_o^B\big)\partial_\xi(\Phi_o^A - \Phi_o^B), \\ d_3 & = & \big[DF_o\big(\overline{U}_{o;0} + \Phi_o^B\big) - DF_o(\overline{U}_{o;0})\big]\partial_\xi\big(\Phi_o^A - \Phi_o^B\big) \\ & & \quad + D^2F_o\big(\overline{U}_{o;0} + \Phi_o^B\big)\big[\partial_\xi(\overline{U}_{o;0} + \Phi_o^B), \Phi_o^A - \Phi_o^B\big] \\ & & \quad - D^2F_o(\overline{U}_{o;0})[\overline{U}'_{o;0}, \Phi_o^A - \Phi_o^B]. \end{array} \tag{3.5.30}$$

Upon introducing the expressions

$$\begin{array}{rcl} d_I & = & DF_o\big(\overline{U}_{o;0} + \Phi_o^B + (\Phi_o^A - \Phi_o^B)\big)\partial_\xi\big(\overline{U}_{o;0} + \Phi_o^B\big) \\ & & \quad - DF_o\big(\overline{U}_{o;0} + \Phi_o^B\big)\partial_\xi\big(\overline{U}_{o;0} + \Phi_o^B\big) \\ & & \quad - D^2F_o\big(\overline{U}_{o;0} + \Phi_o^B\big)\big[\Phi_o^A - \Phi_o^B, \partial_\xi(\overline{U}_{o;0} + \Phi_o^B)\big], \\ d_{II} & = & \Big[DF_o\big(\overline{U}_{o;0} + \Phi_o^B + (\Phi_o^A - \Phi_o^B)\big) - DF_o\big(\overline{U}_{o;0} + \Phi_o^B\big)\Big]\partial_\xi(\Phi_o^A - \Phi_o^B), \end{array} \tag{3.5.31}$$

we see that

$$d_1 + d_2 \quad = \quad d_I + d_{II}. \tag{3.5.32}$$

Applying [55, Thm. 2.8.3] we obtain the bounds

$$\begin{array}{rcl} |d_I| & \leq & \frac{1}{2}K_F|\Phi_o^A - \Phi_o^B|^2\big[|\overline{U}'_{o;0}| + |\partial_\xi \Phi_o^B|\big], \\ |d_{II}| & \leq & K_F|\Phi_o^A - \Phi_o^B||\partial_\xi(\Phi_o^A - \Phi_o^B)|. \end{array} \tag{3.5.33}$$

In addition, the expressions

$$\begin{array}{rcl} d_{III} & = & \big[DF_o\big(\overline{U}_{o;0} + \Phi_o^B\big) - DF_o(\overline{U}_{o;0})\big]\partial_\xi\big(\Phi_o^A - \Phi_o^B\big), \\ d_{IV} & = & D^2F_o\big(\overline{U}_{o;0} + \Phi_o^B\big)[\overline{U}'_{o;0}, \Phi_o^A - \Phi_o^B] - D^2F_o(\overline{U}_{o;0})[\overline{U}'_{o;0}, \Phi_o^A - \Phi_o^B], \\ d_V & = & D^2F_o\big(\overline{U}_{o;0} + \Phi_o^B\big)[\partial_\xi \Phi_o^B, \Phi_o^A - \Phi_o^B] \end{array} \tag{3.5.34}$$

allow us to write

$$d_3 \quad = \quad d_{III} + d_{IV} + d_V. \tag{3.5.35}$$

Applying [55, Thm. 2.8.3] we may estimate

$$\begin{array}{rcl} |d_{III}| & \leq & K_F|\Phi_o^B||\partial_\xi(\Phi_o^A - \Phi_o^B)|, \\ |d_{IV}| & \leq & K_F|\Phi_o^B||\Phi_o^A - \Phi_o^B|, \\ |d_V| & \leq & K_F|\partial_\xi \Phi_o^B||\Phi_o^A - \Phi_o^B|. \end{array} \tag{3.5.36}$$

These bounds can all be absorbed into (3.5.28). The estimate for $\mathcal{N}_e$ follows similarly. ∎

With the above pointwise bounds in hand, we are ready to estimate the nonlinearities in the appropriate scaled function spaces. To this end, we introduce the notation

$$\mathcal{N}(\Phi) \quad = \quad \big(\mathcal{N}_o(\Phi_o), \mathcal{N}_e(\Phi_e)\big) \tag{3.5.37}$$

for any $\Phi = (\Phi_o, \Phi_e) \in \mathbf{H}^1$.

Lemma 3.5.6. *Assume that (HN1), (HN2), (HW1) and (HW2) are satisfied. There exists a constant $K_{\mathcal{N}} > 0$ so that for each $0 < \eta \leq \eta_*$, each $\varepsilon > 0$ and each triplet $(\Phi, \Phi^A, \Phi^B) \in \mathbf{X}^3_{\eta;\varepsilon}$ we have the bounds*

$$\begin{array}{lcl} \|\mathcal{M}^1_\varepsilon \mathcal{N}(\Phi)\|_{\mathbf{L}^2} & \leq & K_{\mathcal{N}}\eta^2, \\ \|\mathcal{M}^{1,2}_\varepsilon \partial_\xi \mathcal{N}(\Phi)\|_{\mathbf{L}^2} & \leq & K_{\mathcal{N}}\eta^2, \\ \|\mathcal{M}^1_\varepsilon \big(\mathcal{N}(\Phi^A) - \mathcal{N}(\Phi^B)\big)\|_{\mathbf{L}^2} & \leq & K_{\mathcal{N}}\eta \|\Phi^A - \Phi^B\|_{\mathbf{L}^2}, \\ \|\mathcal{M}^{1,2}_\varepsilon \partial_\xi \big(\mathcal{N}(\Phi^A) - \mathcal{N}(\Phi^B)\big)\|_{\mathbf{L}^2} & \leq & K_{\mathcal{N}}\eta \Big(\|\Phi^A - \Phi^B\|_{\mathbf{L}^2} + \|\partial_\xi(\Phi^A - \Phi^B)\|_{\mathbf{L}^2}\Big). \end{array} \tag{3.5.38}$$

Proof. All bounds follow immediately from Lemma 3.5.2-Lemma 3.5.5 upon using the Sobolev estimate $\|\phi\|_\infty \leq C_1' \|\phi\|_{H^1}$ to write

$$\begin{array}{lclclcl} \|\Phi_o\|_\infty & \leq & C_1'\eta, & \quad & \|\partial_\xi \Phi_o\|_\infty & \leq & C_1' \frac{\eta}{\varepsilon}, \\ \|\Phi_e\|_\infty & \leq & C_1'\eta, & & \|\partial_\xi \Phi_e\|_\infty & \leq & C_1'\eta, \end{array} \tag{3.5.39}$$

with identical bounds for Φ^A and Φ^B. ∎

Lemma 3.5.7. *Assume that (HN1), (HN2), (HW1) and (HW2) are satisfied. Then there exists a constant $K_{\mathcal{E}} > 0$ so that for each $\varepsilon > 0$ we have the bound*

$$\|\mathcal{M}^1_\varepsilon \mathcal{E}_0\|_{\mathbf{L}^2} + \|\mathcal{M}^{1,2}_\varepsilon \partial_\xi \mathcal{E}_0\|_{\mathbf{L}^2} \quad \leq \quad \varepsilon K_{\mathcal{E}}. \tag{3.5.40}$$

Proof. The structure of the matrix J allows us to bound

$$\big\|\mathcal{M}^1_\varepsilon \mathcal{E}_0\big\|_{\mathbf{L}^2} \quad \leq \quad \varepsilon \left\|\mathcal{E}_0\right\|_{\mathbf{L}^2}, \qquad \big\|\mathcal{M}^{1,2}_\varepsilon \partial_\xi \mathcal{E}_0\big\|_{\mathbf{L}^2} \quad \leq \quad \varepsilon \left\|\partial_\xi \mathcal{E}_0\right\|_{\mathbf{L}^2}. \tag{3.5.41}$$

The result hence follows from the inclusions

$$\overline{U}'_{o;0} \in \mathbf{H}^1_o, \qquad F_o(\overline{U}_{o;0}) \in \mathbf{H}^1_o. \tag{3.5.42}$$

The first of these can be obtained by differentiating (3.2.18) and (3.2.20). The second inclusion follows from the fact that $\overline{U}_{o;0}$ converges exponentially fast to its limiting values, which are zeroes of F_o. ∎

Lemma 3.5.8. *Assume that (HN1), (HN2), (HW1), (HW2) and (HS1) are satisfied. Then there exists a constant $K_c > 0$ in such a way that for each $0 < \eta \leq \eta_*$, each $\varepsilon > 0$, each $\delta > 0$ and each triplet $(\Phi, \Phi^A, \Phi^B) \in \mathbf{X}^3_{\eta;\varepsilon}$ we have the bounds*

$$\begin{array}{lcl} |c_\delta(\Phi_e) - c_0| & \leq & K_c\big[\delta\eta + \eta^2\big], \\ |c_\delta(\Phi^A_e) - c_\delta(\Phi^B_e)| & \leq & K_c\big(\delta + \eta\big)\|\Phi^A - \Phi^B\|_{\mathbf{L}^2}. \end{array} \tag{3.5.43}$$

Proof. Since we only need to use regular L^2-norms for these estimates, the proof of Lemma 2.4.4 also applies here. ∎

Lemma 3.5.9. *Assume that (HN1), (HN2), (HW1), (HW2) and (HS1) are satisfied. Then there exists a constant $K_{\mathcal{R}} > 0$ in such a way that for each $0 < \eta \le \eta_*$, each $0 < \varepsilon < 1$, each $\delta > 0$ and each triplet $(\Phi, \Phi^A, \Phi^B) \in \mathbf{X}^3_{\eta;\varepsilon}$ we have the bound*

$$\|\mathcal{M}^1_\varepsilon \mathcal{R}(c_\delta(\Phi_e), \Phi)\|_{\mathbf{L}^2} + \|\mathcal{M}^{1,2}_\varepsilon \partial_\xi \mathcal{R}(c_\delta(\Phi_e), \Phi)\|_{\mathbf{L}^2} \quad \le \quad K_{\mathcal{R}}[\delta\eta + \eta^2]. \tag{3.5.44}$$

Writing

$$\Delta_{AB}\mathcal{R} \quad := \quad \mathcal{R}(c_\delta(\Phi^A_e), \Phi^A) - \mathcal{R}(c_\delta(\Phi^B_e), \Phi^B), \tag{3.5.45}$$

we also have the bound

$$\begin{aligned} \|\mathcal{M}^1_\varepsilon \Delta_{AB}\mathcal{R}\|_{\mathbf{L}^2} + \|\mathcal{M}^{1,2}_\varepsilon \partial_\xi \Delta_{AB}\mathcal{R}\|_{\mathbf{L}^2} \quad &\le \quad K_{\mathcal{R}}(\delta + \eta)\|\Phi^A - \Phi^B\|_{\mathbf{L}^2} \\ &\qquad + \eta K_{\mathcal{R}}(\eta + \delta)\|\partial_\xi(\Phi^A - \Phi^B)\|_{\mathbf{L}^2} \\ &\qquad + \eta K_{\mathcal{R}}(\eta + \delta)\|\mathcal{M}^{1,2}_\varepsilon \partial^2_\xi(\Phi^A - \Phi^B)\|_{\mathbf{L}^2}. \end{aligned} \tag{3.5.46}$$

Proof. Using Lemma 3.5.8 we immediately obtain the bound

$$\begin{aligned} \|\mathcal{M}^1_\varepsilon \mathcal{R}(c_\delta(\Phi_e), \Phi)\|_{\mathbf{L}^2} \quad &\le \quad K_c\big[\delta\eta + \eta^2\big]\Big(\|\mathcal{M}^1_\varepsilon \partial_\xi \Phi\|_{\mathbf{L}^2} + \|\mathcal{M}^1_\varepsilon \overline{U}'_0\|_{\mathbf{L}^2}\Big) \\ &\le \quad K_c\big[\delta\eta + \eta^2\big]\Big(\eta + \|\overline{U}'_0\|_{\mathbf{L}^2}\Big), \end{aligned} \tag{3.5.47}$$

together with

$$\begin{aligned} \|\mathcal{M}^{1,2}_\varepsilon \partial_\xi \mathcal{R}(c_\delta(\Phi_e), \Phi)\|_{\mathbf{L}^2} \quad &\le \quad K_c\big[\delta\eta + \eta^2\big]\Big(\|\mathcal{M}^{1,2}_\varepsilon \partial^2_\xi \Phi\|_{\mathbf{L}^2} + \|\mathcal{M}^{1,2}_\varepsilon \overline{U}''_0\|_{\mathbf{L}^2}\Big) \\ &\le \quad K_c\big[\delta\eta + \eta^2\big]\Big(\eta + \|\overline{U}''_0\|_{\mathbf{L}^2}\Big). \end{aligned} \tag{3.5.48}$$

In addition, we may compute

$$\begin{aligned} \Delta_{AB}\mathcal{R} \quad &= \quad \big(c_\delta(\Phi^B_e) - c_\delta(\Phi^A_e)\big)\partial_\xi\big(\overline{U}_0 + \Phi^A\big) \\ &\qquad + \big(c_0 - c_\delta(\Phi^B_e)\big)\partial_\xi(\Phi^A - \Phi^B), \end{aligned} \tag{3.5.49}$$

which allows us to estimate

$$\begin{aligned} \|\mathcal{M}^1_\varepsilon \Delta_{AB}\mathcal{R}\|_{\mathbf{L}^2} \quad &\le \quad K_c\big(\delta + \eta\big)\|\Phi^A - \Phi^B\|_{\mathbf{L}^2}\big(\|\mathcal{M}^1_\varepsilon \overline{U}'_0\|_{\mathbf{L}^2} + \|\mathcal{M}^1_\varepsilon \partial_\xi \Phi^A\|_{\mathbf{L}^2}\big) \\ &\qquad + K_c\big[\delta\eta + \eta^2\big]\|\mathcal{M}^1_\varepsilon \partial_\xi(\Phi^A - \Phi^B)\|_{\mathbf{L}^2} \\ &\le \quad K_c\big(\delta + \eta\big)\|\Phi^A - \Phi^B\|_{\mathbf{L}^2}\big(\|\overline{U}'_0\|_{\mathbf{L}^2} + \eta\big) \\ &\qquad + K_c\big[\delta\eta + \eta^2\big]\|\partial_\xi(\Phi^A - \Phi^B)\|_{\mathbf{L}^2}, \end{aligned} \tag{3.5.50}$$

together with

$$\begin{array}{lcl} \|\mathcal{M}_\varepsilon^{1,2}\partial_\xi \Delta_{AB}\mathcal{R}\|_{\mathbf{L}^2} & \leq & K_c\big(\delta+\eta\big)\|\Phi^A-\Phi^B\|_{\mathbf{L}^2}\big(\|\mathcal{M}_\varepsilon^{1,2}\overline{U}_0''\|_{\mathbf{L}^2}+\|\mathcal{M}_\varepsilon^{1,2}\partial_\xi\Phi^A\|_{\mathbf{L}^2}\big) \\ & & \quad +K_c\big[\delta\eta+\eta^2\big]\|\mathcal{M}_\varepsilon^{1,2}\partial_\xi^2(\Phi^A-\Phi^B)\|_{\mathbf{L}^2} \\ & \leq & K_c\big(\delta+\eta\big)\|\Phi^A-\Phi^B\|_{\mathbf{L}^2}\big(\|\overline{U}_0''\|_{\mathbf{L}^2}+\eta\big) \\ & & \quad +K_c\big[\delta\eta+\eta^2\big]\|\mathcal{M}_\varepsilon^{1,2}\partial_\xi(\Phi^A-\Phi^B)\|_{\mathbf{L}^2}. \end{array} \tag{3.5.51}$$

These terms can all be absorbed into (3.5.46). ■

Proof of Theorem 3.2.1. Using Lemma 3.5.6, Lemma 3.5.7 and Lemma 3.5.9, together with the decomposition (3.5.5) and the estimates (3.5.14)-(3.5.15), we find that there exists a constant $K_T > 0$ for which the bounds

$$\begin{array}{lcl} \|T_{\varepsilon,\delta}(\Phi)\|_{\mathbf{X}_\varepsilon} & \leq & K_T\Big[\delta\eta+\eta^2+\varepsilon\Big], \\ \Big\|T_{\varepsilon,\delta}(\Phi^A)-T_{\varepsilon,\delta}(\Phi^B)\Big\|_{\mathbf{X}_\varepsilon} & \leq & K_T\Big[\delta+\eta\Big]\|\Phi^A-\Phi^B\|_{\mathbf{X}_\varepsilon} \end{array} \tag{3.5.52}$$

hold for any $\eta \leq \eta_*$, any $0 < \varepsilon < \varepsilon_0(\delta)$ and any triplet $(\Phi, \Phi^A, \Phi^B) \in \mathbf{X}^3_{\eta;\varepsilon}$. As such, we fix

$$\delta \;=\; \tfrac{1}{3K_T}, \quad \eta = \min\{\eta_*, \tfrac{1}{3K_T}\}. \tag{3.5.53}$$

Finally, we select a small positive ε_* such that $\varepsilon_* \leq \varepsilon_0(\delta)$ and $\varepsilon_* \leq \frac{1}{3K_T}\eta$. We conclude that for each $0 < \varepsilon \leq \varepsilon_*$, $T_{\varepsilon,\delta}$ maps $\mathbf{X}_{\eta;\varepsilon}$ into itself and is a contraction. This completes the proof. ■

3.6 Stability of travelling waves

Introducing the family

$$\big(\tilde{U}_\varepsilon, \tilde{c}_\varepsilon\big) \;=\; \big(\overline{U}_\varepsilon, c_\varepsilon\big), \tag{3.6.1}$$

which satisfies (hFam) on account of Theorem 3.2.1, we see that the theory developed in §3.4 applies to the operators

$$\overline{\mathcal{L}}_{\varepsilon,\lambda} : \mathbf{H}^1 \to \mathbf{L}^2 \tag{3.6.2}$$

that act as

$$\overline{\mathcal{L}}_{\varepsilon,\lambda} \;=\; c_\varepsilon \tfrac{d}{d\xi} - \mathcal{M}^1_{1/\varepsilon^2} J_{\text{mix}} - DF(\overline{U}_\varepsilon) + \lambda. \tag{3.6.3}$$

We emphasize that these operators are associated to the linearization of the travelling wave system (3.2.14) around the wave solutions $(\overline{U}_\varepsilon, c_\varepsilon)$. For convenience, we also introduce the shorthand

$$\overline{\mathcal{L}}_\varepsilon \;=\; \overline{\mathcal{L}}_{\varepsilon,0} \;=\; c_\varepsilon \tfrac{d}{d\xi} - \mathcal{M}^1_{1/\varepsilon^2} J_{\text{mix}} - DF(\overline{U}_\varepsilon). \tag{3.6.4}$$

We remark that the spectrum of $\overline{\mathcal{L}}_\varepsilon$ is $2\pi i c_\varepsilon$-periodic on account of the identity

$$\big(\overline{\mathcal{L}}_\varepsilon + \lambda\big)e^{2\pi i \cdot} \;=\; e^{2\pi i \cdot}\big(\overline{\mathcal{L}}_\varepsilon + \lambda + 2\pi i c_\varepsilon\big). \tag{3.6.5}$$

As a final preparation, we note that there exists a constant $\overline{K}_F > 0$ for which the bound

$$\|DF_o(\overline{U}_{o;\varepsilon})\|_\infty + \|D^2F_o(\overline{U}_{o;\varepsilon})\|_\infty + \|DF_e(\overline{U}_{e;\varepsilon})\|_\infty + \|D^2F_e(\overline{U}_{e;\varepsilon})\|_\infty \quad \leq \quad \overline{K}_F \tag{3.6.6}$$

holds for all $0 < \varepsilon < \varepsilon_*$.

Our main task here is to reverse the parameter dependency used in §3.4. In particular, for a fixed small value of $\varepsilon > 0$ we study the behaviour of the map $\lambda \mapsto \overline{\mathcal{L}}_{\varepsilon,\lambda}$. This allows us to obtain the main result of this section, which lifts the spectral stability assumptions (HS1) and (HS2) to the full system (3.2.14).

Proposition 3.6.1. *Assume that (HN1), (HN2), (HW1), (HW2), (HS1) and (HS2) are satisfied. Then there exists a constant $\varepsilon_{**} > 0$ so that the following properties hold for all $0 < \varepsilon < \varepsilon_{**}$.*

(i) We have

$$\mathrm{Ker}\big(\overline{\mathcal{L}}_\varepsilon\big) \quad = \quad \mathrm{span}\big(\overline{U}_\varepsilon'\big) \tag{3.6.7}$$

together with $\overline{U}_\varepsilon' \notin \mathrm{Range}\big(\overline{\mathcal{L}}_{\varepsilon,0}\big)$.

(ii) For each $\lambda \in \mathbb{C} \setminus 2\pi i c_\varepsilon \mathbb{Z}$ with $\mathrm{Re}\lambda \geq -\lambda_*$, the operator $\overline{\mathcal{L}}_{\varepsilon,\lambda}$ is invertible.

These spectral stability properties can be turned into a nonlinear stability result by applying the theory developed in [109]. The main idea is to consider a temporal Green's function for the LDE (3.2.1) and spatial Green's functions for the travelling wave equation (3.2.14). These Green's functions can be related to each other using an integral representation. Our detailed knowledge of the spectrum of the operator $\overline{\mathcal{L}}_\varepsilon$ allows us to shift the integration path and split the temporal Green's function for the linearization of (3.2.1) around the wave $\overline{U}_\varepsilon$ into two components. The first corresponds to the neutral part of the flow along the eigenfunction $\overline{U}_\varepsilon'$, while the second encodes the exponentially decaying stable part of the flow. The full orbital neighbourhood of the travelling wave $\overline{U}_\varepsilon$ can now be spanned by the family of stable manifolds for the shifted waves $\overline{U}_\varepsilon(\cdot + \vartheta)$, which all have codimension one. In particular, every initial condition in this neighbourhood converges exponentially to a shifted version of $\overline{U}_\varepsilon$.

Proof of Theorem 3.2.2. For $j \in \mathbb{Z}$ we introduce the new variables

$$\big(u_{j;o}, w_{j;o}, u_{j;e}, w_{j;e}\big) \quad = \quad \big(u_{2j+1}, w_{2j+1}, u_{2j}, w_{2j}\big), \tag{3.6.8}$$

which allows us to reformulate the 2-periodic system (3.2.1) as the equivalent $2(n+k)$-component system

$$\begin{array}{lcl} \dot{u}_{j;o}(t) & = & \frac{1}{\varepsilon^2}\mathcal{D}\big[u_{j+1;e}(t) + u_{j;e}(t) - 2u_{j;o}(t)\big] + f_o\big(u_{j;o}(t), w_{j;o}(t)\big), \\ \dot{u}_{j;o}(t) & = & g_o\big(u_{j;o}(t), w_{j;o}(t)\big), \\ \dot{u}_{j;e}(t) & = & \mathcal{D}\big[u_{j;o}(t) + u_{j-1;o}(t) - 2u_{j;e}(t)\big] + f_e\big(u_{j;e}(t), w_{j;e}(t)\big), \\ \dot{w}_{j;e}(t) & = & g_e\big(u_{j;e}(t), w_{j;e}(t)\big), \end{array} \tag{3.6.9}$$

which is spatially homogeneous.

On account of Theorem 3.2.1 and Proposition 3.6.1, it is clear that 3.6.9 satisfies the conditions (HV), (HS1)-(HS3) from [109]. An application of [109, Prop. 2.1] immediately yields the desired result. ■

3.6.1 The operator $\overline{\mathcal{L}}_\varepsilon$

Observe first that $\overline{\mathcal{L}}_\varepsilon$ is a Fredholm operator with index zero on account of Lemma 3.4.6. Our goal in this subsection is to establish the characterization of the kernel and range of this operator given in item (i) of Proposition 3.6.1. We note that this statement implies that the zero eigenvalue of $\overline{\mathcal{L}}_\varepsilon$ is simple.

At times, our discussion closely follows the lines of [150, sects. 4–5]. The novel ingredient here, however, is that we do not need to modify the spectral convergence argument from §3.4 to ensure that it also applies to the adjoint operator. Indeed, we show that all the essential properties can be obtained from the following quasi-inverse for $\overline{\mathcal{L}}_\varepsilon$, which can be constructed by mimicking the approach of [111, Prop. 3.2].

Lemma 3.6.2. *Assume that (HN1), (HN2), (HW1), (HW2), (HS1) and (HS2) are satisfied and pick a sufficiently small constant $\varepsilon_{**} > 0$. Then for every $0 < \varepsilon < \varepsilon_{**}$ there exist linear maps*

$$\begin{array}{lcl} \overline{\gamma}_\varepsilon & : & \mathbf{L}^2 \to \mathbb{R} \\ \overline{\mathcal{L}}_\varepsilon^{\mathrm{qinv}} & : & \mathbf{L}^2 \to \mathbf{H}^1, \end{array} \tag{3.6.10}$$

so that for all $\Theta \in \mathbf{L}^2$ the pair

$$(\gamma, \Psi) \quad = \quad (\overline{\gamma}_\varepsilon \Theta, \overline{\mathcal{L}}_\varepsilon^{\mathrm{qinv}} \Theta) \tag{3.6.11}$$

is the unique solution to the problem

$$\overline{\mathcal{L}}_\varepsilon \Psi \quad = \quad \Theta + \gamma \overline{U}_0' \tag{3.6.12}$$

that satisfies the normalisation condition

$$\langle (0, \Phi_{e;0}^{\mathrm{adj}}), \Psi \rangle_{\mathbf{L}^2} \quad = \quad 0. \tag{3.6.13}$$

In addition, there exists $C > 0$ such that for all $0 < \varepsilon < \varepsilon_{**}$ and all $\Theta \in \mathbf{L}^2$ we have the bound

$$|\overline{\gamma}_\varepsilon \Theta| + \|\mathcal{M}_\varepsilon^1 (\overline{\mathcal{L}}_\varepsilon^{\mathrm{qinv}} \Theta)'\|_{\mathbf{L}^2} + \|\overline{\mathcal{L}}_\varepsilon^{\mathrm{qinv}} \Theta\|_{\mathbf{L}^2} \quad \leq \quad C \|\mathcal{M}_\varepsilon^1 \Theta\|_{\mathbf{L}^2}. \tag{3.6.14}$$

Proof. The proof of [150, Lem. 4.9] remains valid in this setting.

■

We can now concentrate on the kernel of $\overline{\mathcal{L}}_\varepsilon$. The quasi-inverse constructed above allows us to develop a Liapunov-Schmidt argument to exclude kernel elements other than $\overline{U}_\varepsilon'$.

Lemma 3.6.3. *Assume that (HN1), (HN2), (HW1), (HW2), (HS1) and (HS2) are satisfied. Then for all sufficiently small $\varepsilon > 0$ we have*

$$\ker(\overline{\mathcal{L}}_\varepsilon) = \mathrm{span}\{\overline{U}'_\varepsilon\}. \tag{3.6.15}$$

Proof. This result can be obtained by following the procedure used in the proof of [150, Lem. 4.10–4.11]. ∎

We now set out to show that the eigenfunction $\overline{U}'_\varepsilon$ is, in fact, simple. As a technical preparation, we obtain a lower bound on $\overline{\gamma}_\varepsilon(\overline{U}'_\varepsilon)$, which will help us to exploit the quasi-inverse constructed in Lemma 3.6.2.

Lemma 3.6.4. *Assume that (HN1), (HN2), (HW1), (HW2), (HS1) and (HS2) are satisfied. Then there exists a constant $\gamma_* > 0$ so that the inequality*

$$|\overline{\gamma}_\varepsilon \overline{U}'_\varepsilon| \geq \gamma_* \tag{3.6.16}$$

holds for all sufficiently small $\varepsilon > 0$.

Proof. We note first that the limit $\overline{U}'_\varepsilon \to \overline{U}'_0$ in $\mathbf{L}^2$ and the inequality $\langle \overline{U}'_{e;0}, \Phi^{\mathrm{adj}}_{e;0}\rangle_{\mathbf{L}^2_e} \neq 0$ imply that there exists a constant $\nu_* > 0$ so that

$$|\langle \overline{U}'_\varepsilon, (0, \Phi^{\mathrm{adj}}_{e;0})\rangle_{\mathbf{L}^2}| \geq \nu_* \tag{3.6.17}$$

for all small $\varepsilon > 0$.

We now introduce the function

$$\Psi_\varepsilon = \overline{\mathcal{L}}^{\mathrm{qinv}}_\varepsilon \overline{U}'_\varepsilon. \tag{3.6.18}$$

The uniform bound (3.6.14) shows that we may assume an a-priori bound of the form

$$\|\Psi_\varepsilon\|_{\mathbf{L}^2} \leq C'_1 \tag{3.6.19}$$

for some $C'_1 > 0$.

For any sufficiently small $\delta > 0$ and $0 < \varepsilon < \varepsilon_0(\delta)$, the explicit form of $\overline{\gamma}_\varepsilon$ given in [150, eq. (4.47)] implies that

$$\begin{aligned} \overline{\gamma}_\varepsilon \overline{U}'_\varepsilon &= \frac{\left\langle (0,\Phi^{\mathrm{adj}}_{e;0}), \left(\overline{\mathcal{L}}_\varepsilon + \delta\right)^{-1}\left(\overline{U}'_\varepsilon + \delta\Psi_\varepsilon\right)\right\rangle_{\mathbf{L}^2}}{\left\langle (0,\Phi^{\mathrm{adj}}_{e;0}), \left(\overline{\mathcal{L}}_\varepsilon + \delta\right)^{-1}\overline{U}'_0\right\rangle_{\mathbf{L}^2}} \\ &= \frac{\left\langle (0,\Phi^{\mathrm{adj}}_{e;0}), \delta^{-1}\overline{U}'_\varepsilon + \left(\overline{\mathcal{L}}_\varepsilon + \delta\right)^{-1}\delta\Psi_\varepsilon\right\rangle_{\mathbf{L}^2}}{\left\langle (0,\Phi^{\mathrm{adj}}_{e;0}), \left(\overline{\mathcal{L}}_\varepsilon + \delta\right)^{-1}\overline{U}'_0\right\rangle_{\mathbf{L}^2}}. \end{aligned} \tag{3.6.20}$$

Since $\left(\overline{\mathcal{L}}_\varepsilon + \delta\right)^{-1}\delta\Psi_\varepsilon$ is uniformly bounded in $\mathbf{L}^2$ for all sufficiently small $\delta > 0$ and $0 < \varepsilon < \varepsilon_0(\delta)$ on account of Corollary 3.4.4 and (3.6.19), we can use the lower bound (3.6.17) to assume that $\delta > 0$ is small enough to have

$$\left|\left\langle (0, \Phi^{\mathrm{adj}}_{e;0}), \delta^{-1}\overline{U}'_\varepsilon + \left(\overline{\mathcal{L}}_\varepsilon + \delta\right)^{-1}\delta\Psi_\varepsilon\right\rangle_{\mathbf{L}^2}\right| \geq C'_2\delta^{-1} \tag{3.6.21}$$

for all such (ε, δ). Moreover, the uniform bound in Corollary 3.4.4 also yields the upper bound

$$\left|\left\langle (0, \Phi_{e;0}^{\mathrm{adj}}), \left(\overline{\mathcal{L}}_\varepsilon + \delta\right)^{-1} \overline{U}_0' \right\rangle_{\mathbf{L}^2}\right| \quad \leq \quad C_3'(1 + \delta^{-1}) \tag{3.6.22}$$

for all such (ε, δ). This gives us the lower bound

$$|\overline{\gamma}_\varepsilon \overline{U}_\varepsilon'| \quad \geq \quad \tfrac{C_2'}{C_3'} \tfrac{\delta^{-1}}{1+\delta^{-1}} \quad \geq \quad \gamma_* \tag{3.6.23}$$

for some $\gamma_* > 0$ that can be chosen independently of $\delta > 0$. ∎

Lemma 3.6.5. *Assume that (HN1), (HN2), (HW1), (HW2), (HS1) and (HS2) are satisfied. Then for all sufficiently small $\varepsilon > 0$ we have $\overline{U}_\varepsilon' \notin \mathrm{Range}(\overline{\mathcal{L}}_\varepsilon)$.*

Proof. Arguing by contradiction, let us suppose that there exists $\Psi_\varepsilon \in \mathbf{H}^1$ for which the identity

$$\overline{\mathcal{L}}_\varepsilon \Psi_\varepsilon \quad = \quad \overline{U}_\varepsilon' \tag{3.6.24}$$

holds. The observation above allows us to add an appropriate multiple of $\overline{U}_\varepsilon'$ to Ψ_ε to ensure that $\langle \Psi_\varepsilon, (0, \Phi_{e;0}^{\mathrm{adj}}) \rangle_{\mathbf{L}^2} = 0$. In particular, Lemma 3.6.2 implies that

$$\overline{\gamma}_\varepsilon \overline{U}_\varepsilon' \quad = \quad 0, \qquad \qquad \overline{\mathcal{L}}_\varepsilon^{\mathrm{qinv}} \overline{U}_\varepsilon' \quad = \quad \Psi_\varepsilon, \tag{3.6.25}$$

which immediately contradicts Lemma 3.6.4. ∎

3.6.2 Spectral stability

Here we set out to establish the statements in Proposition 3.6.1 for $\lambda \notin 2\pi i c_\varepsilon \mathbb{Z}$. In contrast to the setting in [150], the period $2\pi i c_\varepsilon$ can be uniformly bounded for $\varepsilon \downarrow 0$. In particular, we will only consider values of $\varepsilon > 0$ that are sufficiently small to ensure that

$$\tfrac{3}{4} c_0 \quad < \quad c_\varepsilon \quad < \quad \tfrac{3}{2} c_0 \tag{3.6.26}$$

holds. Recalling the constant λ_0 introduced in Proposition 3.4.2, this allows us to restrict our spectral analysis to the set

$$\mathcal{R} \quad := \quad \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \geq -\lambda_0, |\mathrm{Im}\lambda| \leq \tfrac{3}{2}\pi c_0\} \setminus \{0\}. \tag{3.6.27}$$

On account of Lemma 3.4.6, the operators $\overline{\mathcal{L}}_{\varepsilon,\lambda}$ are all Fredholm with index 0 on this set. We hence only need to establish their injectivity.

It turns out to be convenient to partition this strip into three ε-independent parts, which we illustrate in Figure 3.2. The first part (red) contains values of λ that are close to 0, which can be analyzed using the theory developed in §3.6.1. The second part (blue) contains all values of λ for which $\mathrm{Re}\lambda$ is sufficiently large. Such values can be excluded from the spectrum by straightforward norm estimates. The remaining part (green) is compact, which allows us to appeal to Proposition 3.4.2.

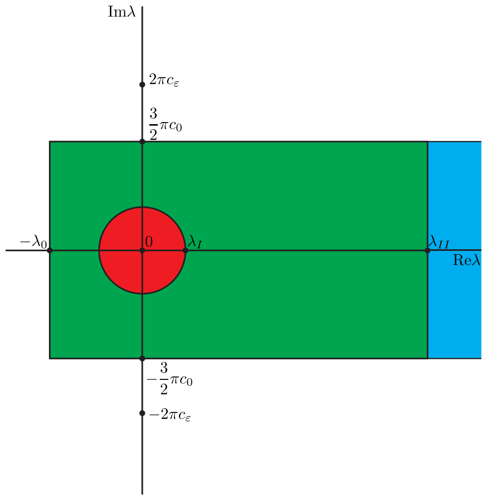


Figure 3.2: *Illustration of the decomposition of the spectrum into ε-independent regions.*

Lemma 3.6.6. *Assume that (HN1), (HN2), (HW1), (HW2), (HS1) and (HS2) are satisfied. There exists constants $\lambda_I > 0$ and $\varepsilon_I > 0$ so that the operator $\overline{\mathcal{L}}_{\varepsilon,\lambda} : \mathbf{H}^1 \to \mathbf{L}^2$ is injective for all $\lambda \in \mathbb{C}$ with $0 < |\lambda| < \lambda_I$ and $0 < \varepsilon < \varepsilon_I$.*

Proof. We argue by contradiction. Pick a small $\lambda_I > 0$ and $0 < \varepsilon < \varepsilon_{**}$ and assume that there exists $\Psi \in \mathbf{H}^1$ and $0 < |\lambda| < \lambda_I$ with $\Psi \neq 0$ and

$$\overline{\mathcal{L}}_\varepsilon \Psi = \lambda \Psi. \tag{3.6.28}$$

Aiming to exploit the quasi-inverse in Lemma 3.6.2, we use (3.6.17) to decompose Ψ as

$$\Psi = \kappa \overline{U}'_\varepsilon + \Psi^\perp \tag{3.6.29}$$

for some $\kappa \in \mathbb{R}$ and $\Psi^\perp \in \mathbf{H}^1$ that satisfies the normalisation condition

$$\langle (0, \Phi^{\mathrm{adj}}_{e;0}), \Psi^\perp \rangle_{\mathbf{L}^2} = 0. \tag{3.6.30}$$

In view of Lemma 3.6.2, the identity (3.6.28) implies that

$$\overline{\gamma}_\varepsilon \big[\kappa \lambda \overline{U}'_\varepsilon + \lambda \Psi^\perp \big] = 0, \qquad \overline{\mathcal{L}}^{\mathrm{qinv}}_\varepsilon \big[\kappa \lambda \overline{U}'_\varepsilon + \lambda \Psi^\perp \big] = \Psi^\perp. \tag{3.6.31}$$

On account of the uniform bound (3.6.14), we can assume that λ_I is small enough to have

$$\lambda_I \| \overline{\mathcal{L}}^{\mathrm{qinv}}_\varepsilon \|_{\mathcal{B}(\mathbf{L}^2;\mathbf{L}^2)} < \tfrac{1}{2}. \tag{3.6.32}$$

Since $|\lambda| < \lambda_I$, this means that we can rewrite (3.6.31) to obtain

$$\Psi^\perp = \big[I - \lambda \overline{\mathcal{L}}^{\mathrm{qinv}}_\varepsilon \big]^{-1} \overline{\mathcal{L}}^{\mathrm{qinv}}_\varepsilon \big[\kappa \lambda \overline{U}'_\varepsilon \big]. \tag{3.6.33}$$

In particular, the first identity in (3.6.31) allows us to write

$$\begin{aligned} 0 &= \overline{\gamma}_\varepsilon\Big[\kappa\lambda\overline{U}'_\varepsilon + \lambda\big[I - \lambda\overline{\mathcal{L}}^{\mathrm{qinv}}_\varepsilon\big]^{-1}\overline{\mathcal{L}}^{\mathrm{qinv}}_\varepsilon\big[\kappa\lambda\overline{U}'_\varepsilon\big]\Big] \\ &= \kappa\lambda\overline{\gamma}_\varepsilon\Big[\overline{U}'_\varepsilon + \lambda\big[I - \lambda\overline{\mathcal{L}}^{\mathrm{qinv}}_\varepsilon\big]^{-1}\overline{\mathcal{L}}^{\mathrm{qinv}}_\varepsilon\big[\overline{U}'_\varepsilon\big]\Big]. \end{aligned} \tag{3.6.34}$$

We note that the restriction (3.6.32) ensures that the second identity in (3.6.31) has no nonzero solutions $\Psi^\perp$ for $\kappa = 0$. In particular, (3.6.34) implies that we must have

$$\overline{\gamma}_\varepsilon\overline{U}'_\varepsilon = -\lambda\overline{\gamma}_\varepsilon\Big[\big[I - \lambda\overline{\mathcal{L}}^{\mathrm{qinv}}_\varepsilon\big]^{-1}\overline{\mathcal{L}}^{\mathrm{qinv}}_\varepsilon\big[\overline{U}'_\varepsilon\big]\Big]. \tag{3.6.35}$$

On account of (3.6.14) we hence obtain the estimate

$$|\overline{\gamma}_\varepsilon\overline{U}'_\varepsilon| \le C'_1|\lambda| \le C'_1\lambda_I \tag{3.6.36}$$

for some $C'_1 > 0$. However, Lemma 3.6.4 shows that the left-hand side remains bounded away from zero, which yields the desired contradiction after restricting the size of λ_I. ∎

Lemma 3.6.7. *Assume that (HN1), (HN2), (HW1), (HW2), (HS1) and (HS2) are satisfied. There exist constants $\lambda_{II} > 0$ and $\varepsilon_{II} > 0$ so that the operator $\overline{\mathcal{L}}_{\varepsilon,\lambda} : \mathbf{H}^1 \to \mathbf{L}^2$ is injective for all $\lambda \in \mathcal{R}$ with $\operatorname{Re}\lambda \ge \lambda_{II}$ and $0 < \varepsilon < \varepsilon_{II}$.*

Proof. The identity $\overline{\mathcal{L}}_{\varepsilon,\lambda}\Phi = 0$ implies that

$$c_\varepsilon\Phi' = \mathcal{M}^1_{1/\varepsilon^2}J_{\mathrm{mix}}\Phi + DF(\overline{U}_\varepsilon)\Phi - \lambda\Phi. \tag{3.6.37}$$

Taking the inner product with $\mathcal{M}^{1,2}_{\varepsilon^2}\Phi$, we may use Lemma 3.4.11 to obtain

$$\begin{aligned} 0 &\le -\operatorname{Re}\langle J_{\mathrm{mix}}\Phi, \Phi\rangle_{\mathbf{L}^2} \\ &= \operatorname{Re}\langle DF(\overline{U}_\varepsilon)\Phi, \mathcal{M}^{1,2}_{\varepsilon^2}\Phi\rangle_{\mathbf{L}^2} - \operatorname{Re}\lambda\left\|\mathcal{M}^{1,2}_{\varepsilon}\Phi\right\|_{\mathbf{L}^2} \\ &\le (K_F - \operatorname{Re}\lambda)\left\|\mathcal{M}^{1,2}_{\varepsilon}\Phi\right\|_{\mathbf{L}^2}. \end{aligned} \tag{3.6.38}$$

For $\operatorname{Re}\lambda \ge K_F$ this hence implies $\Phi = 0$, as desired. ∎

Proof of Proposition 3.6.1. On account of Lemmas 3.6.3, 3.6.5-3.6.7, it remains to consider the set

$$M = \{\lambda \in \mathcal{R} : |\lambda| \ge \lambda_I, \mathrm{Re}\lambda \le \lambda_{II}\}. \tag{3.6.39}$$

Since this set satisfies $(\mathrm{hM}_{\lambda_0})$, we can apply Proposition 3.4.2 to show that for each sufficiently small $\varepsilon > 0$, the operators $\overline{\mathcal{L}}_{\varepsilon,\lambda}$ are invertible for all $\lambda \in M$. ∎

Chapter 4

Travelling wave solutions for fully discrete FitzHugh-Nagumo type equations with infinite-range interactions

Sections 4.1-4.5 and 4.A have been submitted as W.M. Schouten-Straatman and H.J. Hupkes "Travelling wave solutions for fully discrete FitzHugh-Nagumo type equations with infinite-range interactions" [152].

Abstract. We investigate the impact of spatial-temporal discretisation schemes on the dynamics of a class of reaction-diffusion equations that includes the FitzHugh-Nagumo system. For the temporal discretisation we consider the family of six backward differential formula (BDF) methods, which includes the well-known backward-Euler scheme. The spatial discretisations can feature infinite-range interactions, allowing us to consider neural field models. We construct travelling wave solutions to these fully discrete systems in the small time-step regime by viewing them as singular perturbations of the corresponding spatially discrete system. In particular, we refine the previous approach by Hupkes and Van Vleck for scalar fully discretised systems, which is based on a spectral convergence technique that was developed by Bates, Chen and Chmaj.

Key words: Travelling waves, FitzHugh-Nagumo system, singular perturbation, spatial-temporal discretisation.

4.1 Introduction

In this paper, we consider spatial-temporal discretisations of a class of reaction-diffusion systems that contains the FitzHugh-Nagumo partial differential equation (PDE). This PDE is given by

$$\begin{aligned} u_t &= u_{xx} + g(u;r) - w \\ w_t &= \rho(u - \gamma w). \end{aligned} \tag{4.1.1}$$

Here g is the bistable, cubic nonlinearity $g(u;r) = u(1-u)(u-r)$ with $r \in (0,1)$, while $\rho > 0$ and $\gamma > 0$ are positive constants. In particular, our goal is to show that travelling waves for the system (4.1.1) persist under these spatial-temporal discretisations. As such, we contribute to the broad study of numerical schemes and their impact on the solutions under consideration, which has produced an immense quantity of literature. The main distinguishing feature is that we are interested in structures that persist for all time, while almost all of the studies in this area focus on finite time estimates.

Pulse propagation The system (4.1.1) was introduced in the 1960s [74, 76] as a simplification of the Hodgkin-Huxley equations, which were used to describe the propagation of spike signals through the nerve fibers of giant squids [98]. After observing similar pulse solutions for the system (4.1.1) numerically [75], a more rigorous, analytical approach to understanding these pulse solutions turned out to be rather delicate. Indeed, many new tools have been developed, some even very recently, to construct these pulses and analyse their stability in various settings. These techniques include geometric singular perturbation theory [31, 97, 117, 119], the variational principle [36], Lin's method [32, 33, 124], and the Maslov index [46, 47]. Pulse solutions for the system (4.1.1) take the form

$$(u,w)(x,t) = (u_0,w_0)(x + c_0 t) \tag{4.1.2}$$

for some wavespeed c_0 and smooth wave profiles u_0, w_0 that satisfy the limits

$$\lim_{|\xi|\to\infty} (u_0,w_0)(\xi) = 0. \tag{4.1.3}$$

Spatially discrete systems It is well-known that electrical pulses can only move through nerve fibres at appropriate speeds if the nerves are insulated with a myelin coating. This coating admits regularly spaced gaps at the so-called nodes of Ranvier [143]. In fact, through a process called saltatory conduction, excitations of these nerves appear to jump from one node to the next [127]. Since the FitzHugh-Nagumo PDE (4.1.1) does not take this discrete structure into account directly, it has been proposed [123] to, instead, model these phenenomena using a so-called lattice differential equation (LDE). For example, by applying a nearest-neighbour spatial discretisation to (4.1.1), we arrive at

$$\begin{aligned} \dot{u}_j &= \tau(u_{j+1} + u_{j-1} - 2u_j) + g(u_j;r) - w_j \\ \dot{w}_j &= \rho[u_j - \gamma w_j], \end{aligned} \tag{4.1.4}$$

where the variable j ranges over the lattice $\mathbb{Z}$. In the system (4.1.4), the variable u_j represents the potential at the jth node of the nerve fibre, while the variable w_j describes a recovery component. Finally, we have $\tau \sim h^{-2}$, where $h > 0$ is the distance between subsequent nodes. We emphasize that the time variable remains continuous.

Spatialy discrete travelling pulses for the system (4.1.4) take the form

$$(u, w)_j(t) \quad = \quad (\overline{u}_0, \overline{w}_0)(j + \overline{c}_0 t), \tag{4.1.5}$$

for some wavespeed $\overline{c}_0$, again with the limits (4.1.3). Plugging the Ansatz (4.1.5) into the LDE (4.1.4) yields the functional differential equation of mixed type (MFDE)

$$\begin{array}{lcl} \overline{c}_0 \overline{u}_0'(\xi) & = & \tau[\overline{u}_0(\xi+1) + \overline{u}_0(\xi-1) - 2\overline{u}_0(\xi)] + g(\overline{u}_0(\xi); r) - \overline{w}_0(\xi) \\ \overline{c}_0 \overline{w}_0'(\xi) & = & \rho[\overline{u}_0(\xi) - \gamma \overline{w}_0(\xi)] \end{array} \tag{4.1.6}$$

in which $\xi = j + \overline{c}_0 t$. In [108, 109], Hupkes and Sandstede developed an infinite dimensional version of the exchange lemma to show that the system (4.1.4) admits nonlinearly stable travelling pulse solutions. They relied heavily on the existence of exponential dichotomies for MFDEs, which were established in [96, 133]. In addition, we established the existence and nonlinear stability of pulse solutions for a spatially periodic version of (4.1.4) [151] by building on a spectral convergence method developed by Bates, Chen and Chmaj [6]. The spectral convergence method plays an important role in this paper as well and will be treated in more detail later on.

Infinite-range interactions Neural field models aim to describe the dynamic behaviour of large networks of neurons. In neural networks, neurons interact with each other over large distances through their interconnecting nerve axons [15, 23, 24, 142]. It has been proposed [23, Eq. (3.31)] to capture these long distance interactions using an infinite-range version of the system (4.1.4). To be concrete, we focus our discussion on the prototype system

$$\begin{array}{lcl} \dot{u}_j & = & \tau \sum_{m \in \mathbb{Z}_{>0}} e^{-m^2} [u_{j+m} + u_{j-m} - 2u_j] + g(u_j; r) - w_j \\ \dot{w}_j & = & \rho[u_j - \gamma w_j]. \end{array} \tag{4.1.7}$$

This system can also be obtained directly from the PDE (4.1.1) by using an infinite-range spatial discretisation.

We emphasize that infinite-range interactions also arise naturally when considering discretisations of fractional Laplacians [43]. Indeed, such operators are intrinsically nonlocal and are used in many physical systems that feature nonstandard diffusion processes, such as amorphous semiconductors [87] and liquid crystals [44].

Substituting the travelling pulse Ansatz (4.1.5) into (4.1.7) now yields the MFDE

$$\begin{array}{lcl} \overline{c}_0 \overline{u}_0'(\xi) & = & \tau \sum_{m \in \mathbb{Z}_{>0}} e^{-m^2} [\overline{u}_0(\xi+m) + \overline{u}_0(\xi-m) - 2\overline{u}_0(\xi)] + g(\overline{u}_0(\xi); r) - \overline{w}_0(\xi) \\ \overline{c}_0 \overline{w}_0'(\xi) & = & \rho[\overline{u}_0(\xi) - \gamma \overline{w}_0(\xi)], \end{array} \tag{4.1.8}$$

which features infinitely many shifts. Since exponential dichotomies for MFDEs with infinitely many shifts have only been established very recently [149], the techniques used by Hupkes and Sandstede for the LDE (4.1.4) have not yet been fully developed for the system (4.1.8). Instead, Faye and Scheel [69] used a functional analytic approach to construct pulse solutions for the system (4.1.7). In addition, by applying the previously mentioned spectral convergence method, we were able to show that these pulses are nonlinearly stable [150] for $\tau \gg 1$, which corresponds to fine discretisations of the PDE (4.1.1). As of now, no comprehensive result has been found for the system (4.1.7).

Spatial-temporal discretisations Our main goal here is to understand the impact of temporal discretisation schemes on the behaviour of travelling wave solutions of the system (4.1.7). This is a relatively novel area of study, although a handful of results have been established for scalar problems. For example, Bambusi, Faou, Greébert and Jézéquel constructed solutions to fully discrete Schrödinger equations with Dirichlet or periodic spatial boundary conditions in [4, 64]. Most other studies have focused on spatial-temporal discretisations of the Nagumo PDE

$$u_t = u_{xx} + g(u;r), \tag{4.1.9}$$

or, equivalently, temporal discretisations of the Nagumo LDE

$$\dot{u}_j = \tau(u_{j+1} + u_{j-1} - 2u_j) + g(u_j;r). \tag{4.1.10}$$

The PDE (4.1.9) and the LDE (4.1.10) can be seen as scalar versions of the FitzHugh-Nagumo PDE (4.1.1) and LDE (4.1.4) respectively.

The early works by Elmer and Van Vleck [58–60] provided ad-hoc techniques to understand the impact of spatial-, temporal- and spatial-temporal discretisations of the PDE (4.1.9) on the dynamics of travelling waves. In addition, Chow, Mallet-Paret and Shen [42] established the existence of travelling wave solutions to temporal discretisations of the LDE (4.1.10) by considering Poincare return maps for the dynamics of this LDE. These results were later expanded by Hupkes and Van Vleck [111], whose methods allowed them to address issues of uniqueness and parameter-dependence. Let us also mention the recent series of papers [112–114] by Hupkes and Van Vleck, who study spatial discretisation schemes with an adaptive grid. That is, the authors consider a time dependent moving mesh method which aims to equidistribute the arclength of the solution under consideration.

In order to introduce the temporal discretisation schemes that we study in this paper, we briefly discuss the test problem

$$\dot{v} = \lambda v \tag{4.1.11}$$

with $\lambda < 0$. Applying the forward-Euler discretisation scheme with time-step $\Delta t > 0$ yields

$$v_{n+1} = v_n + \lambda \Delta t v_n = (1 + \lambda \Delta t) v_n, \tag{4.1.12}$$

where $n \in \mathbb{Z}$. Since a nontrivial solution of the test problem (4.1.11) converges to zero as $t \to \infty$, the convergence $v_n \to 0$ should also be enforced. However, this yields the restriction $0 < \Delta t < 2|\lambda|^{-1}$, which cannot be satisfied for all $\lambda < 0$ for a fixed time-step $\Delta t > 0$. In contrast, these issues do not occur for the backward-Euler discretisation scheme. For the test problem (4.1.11), this scheme yields

$$v_{n+1} \quad = \quad v_n + \lambda \Delta t v_{n+1}, \tag{4.1.13}$$

or equivalently

$$v_{n+1} \quad = \quad (1 - \lambda \Delta t)^{-1} v_n. \tag{4.1.14}$$

In particular, we see that $v_n \to 0$ for any value of $\lambda < 0$ and time-step $\Delta t > 0$. A numerical scheme is called $A(\alpha)$ stable if this property holds for all λ in the wedge $\{z \in \mathbb{C} \setminus \{0\} : \mathrm{Arg}(-z) < \alpha\}$. We note that the backward-Euler discretisation is $A(\frac{\pi}{2})$ stable.

In fact, the backward-Euler discretisation scheme is one of six so-called backwards differentiation formula (BDF) methods. These BDF methods are all $A(\alpha)$ stable for various coefficients $0 < \alpha \leq \frac{\pi}{2}$ and have several convenient analytical properties. For this reason, we have to chosen to focus on these temporal discretisation schemes in this paper. We do, however, emphasize that there are other stable discretisation schemes which we could have used, see for example [90].

Applied to the Nagumo system, the backward-Euler discretisation scheme yields the evolution

$$\tfrac{1}{\Delta t}\Big[U_j(n\Delta t) - U_j\big((n-1)\Delta t\big)\Big] \quad = \quad \tau\big[U_{j+1} + U_{j-1} - 2U_j\big](n\Delta t) + g\big(U_j(n\Delta t); r\big). \tag{4.1.15}$$

A travelling wave solution for the system (4.1.15) with wavespeed c takes the form

$$U_j(n\Delta t) \quad = \quad \Phi(j + nc\Delta t), \tag{4.1.16}$$

with the limits

$$\lim_{\xi \to -\infty} \Phi(\xi) \quad = \quad 0, \qquad \lim_{\xi \to \infty} \Phi(\xi) \quad = \quad 1. \tag{4.1.17}$$

As such, the travelling waves need to satisfy the system

$$\tfrac{1}{\Delta t}\big[\Phi(\xi) - \Phi(\xi - c\Delta t)\big] \quad = \quad \tau\big[\Phi(\xi+1) + \Phi(\xi-1) - 2\Phi(\xi)\big] + g(\Phi(\xi); r). \tag{4.1.18}$$

Hupkes and Van Vleck showed [111] that, for sufficiently large, rational values of $M = (c\Delta t)^{-1}$, the system (4.1.15) admits travelling wave solutions with wavespeed c. These travelling waves are constructed as perturbations of travelling wave solutions of the LDE (4.1.10). The corresponding transition from the semi-discrete setting to the fully discrete setting is highly singular, since a derivative is replaced by a difference. The rationality of M plays a key role here, as it ensures that the domain of the variable ξ in the system (4.1.18) is a discrete subset of the real line. This restriction arises naturally in the analysis, since it ensures we can use finitely many interpolations to go from a fully discrete to a spatially discrete setting.

Spectral convergence In order to analyse this singular perturbation, Hupkes and Van Vleck relied heavily on the previously mentioned spectral convergence method, which also plays an important role in [9, 112–114, 150, 151]. This method was introduced in [6] to construct travelling wave solutions to an infinite-range version of the Nagumo LDE (4.1.10) in the near-continuum regime, i.e. when the discretisation distance $h \sim \tau^{-\frac{1}{2}}$ is sufficiently small. A key role in [6] is reserved for the family of operators

$$\mathcal{L}_h v(\xi) \quad = \quad c_0 v'(\xi) - \tfrac{1}{h^2}\big[v(\xi+h) + v(\xi-h) - 2v(\xi)\big] - g_U(\overline{u}_0(\xi);r)v(\xi), \quad (4.1.19)$$

which arise as the linearization of the travelling wave MFDE corresponding to the LDE (4.1.10) around the travelling wave solution $(c_0, \overline{u}_0)$ to the PDE (4.1.9). The main question is what properties these operators inherit from their continuous counterpart

$$\mathcal{L}_0 v(\xi) \quad = \quad c_0 v'(\xi) - v''(\xi) - g_U(\overline{u}_0(\xi);r)v(\xi). \quad (4.1.20)$$

In particular, the authors in [6] fixed a constant $\delta > 0$ and used the invertibility of the operator $\mathcal{L}_0 + \delta$ to establish the invertibility of the operator $\mathcal{L}_h + \delta$ for $h > 0$ sufficiently small. Indeed, they considered weakly converging sequences $\{v_n\}$ and $\{w_n\}$ with $\mathcal{L}_h v_n + \delta v_n = w_n$ and tried to find a uniform (in h and δ) lower bound on the norm of v_n' in terms of the norm of w_n. Such a lower bound prevents the limitless transfer of energy into oscillatory modes, a common concern when dealing with weakly converging sequences. The bistable nature of the nonlinearity g was used to control the behaviour at $\pm\infty$, while the local L^2-norm can be bounded on the remaining compact set. We emphasize that this method requires a detailed understanding of the limiting operator $\mathcal{L}_0$.

In [111], this method was lifted to the fully discrete Nagumo equation (4.1.18). Writing $M = \frac{p}{q}$ with $\gcd(p,q) = 1$, the corresponding limiting operator resembles a q times coupled version of the operator $\mathcal{L}_h$ given by (4.1.19). For $q = 2$, this limiting operator takes the form

$$\begin{aligned} \mathcal{K}_q v(\zeta,\xi) \quad = \quad & \overline{c}v'(\zeta,\xi) - \tau\big[v(\zeta+\tfrac{1}{2},\xi+1) + v(\zeta-\tfrac{1}{2},\xi-1) - 2v(\zeta,\xi)\big] \\ & -g_U(\overline{u}(\xi);r)v(\zeta,\xi), \end{aligned} \quad (4.1.21)$$

where $\overline{u}$ is the travelling wave solution of the LDE (4.1.10) with wavespeed $\overline{c}$. Here the domain of the variables ζ and ξ is given by $\zeta \in \{0, \frac{1}{2}\}$ and $\xi \in \mathbb{R}$, with the convention that $v(\zeta+1,\xi) = v(\zeta,\xi)$. Since the MFDE corresponding to (4.1.21) admits a comparison principle, the Fredholm properties of the operator $\mathcal{K}_q$ follow directly from the general results in [110]. Hupkes and Van Vleck generalized the spectral convergence method to lift the Fredholm properties of the operator $\mathcal{K}_q$ to the operator

$$\begin{aligned} \mathcal{K}_M v(\zeta,\xi) \quad = \quad & \overline{c}M\big[v(\zeta,\xi) - v(\zeta,\xi-M^{-1})\big] \\ & -\tau\big[v(\zeta+\tfrac{1}{2},\xi+1-\tfrac{1}{2}M^{-1}) + v(\zeta-\tfrac{1}{2},\xi-1+\tfrac{1}{2}M^{-1}) - 2v(\zeta,\xi)\big] \\ & -g_U\big(\overline{u}(\xi);r\big)v(\zeta,\xi), \end{aligned} \quad (4.1.22)$$

in the regime $M \gg 1$, again with $\zeta \in \{0, \frac{1}{2}\}$ and $\xi \in \frac{1}{2}M^{-1}\mathbb{Z}$. The operator $\mathcal{K}_M$ arises as the linearisation of the fully discrete system (4.1.18) around the travelling wave $\overline{u}$, using the additional ζ variable to ensure that all ξ-shifted arguments are multiples of M^{-1} .

Results In this paper, we consider reaction-diffusion LDEs such as (4.1.7) and replace the temporal derivative by one of the six BDF discretisation schems. For example, applying the backward-Euler method to (4.1.7), we arrive at the prototype system

$$\begin{aligned} \tfrac{1}{\Delta t}[U_j(n\Delta t) - U_j((n-1)\Delta t)] &= \tau \sum_{m=1}^{\infty} e^{-m^2}\big[U_{j+m} + U_{j-m} - 2U_j\big](n\Delta t) \\ &\quad + g(U_j(n\Delta t); r) - W_j(n\Delta t) \\ \tfrac{1}{\Delta t}[W_j(n\Delta t) - W_j((n-1)\Delta t)] &= \rho[U_j(n\Delta t) - \gamma W_j(n\Delta t)]. \end{aligned} \tag{4.1.23}$$

Our main result states that systems such as (4.1.23) admit travelling wave solutions. To achieve this, we extend the spectral convergence method that was developed in [111] for scalar LDEs with finite-range spatial interactions to the current setting, which features multi-component systems with infinite-range interactions. This generalisation is far from trivial and requires several technical obstructions to be resolved.

The first main obstacle is that the spectral convergence method hinges on the understanding of the corresponding limiting operator. Indeed, the analog of the operator $\mathcal{K}_q$ from (4.1.21) for our system (4.1.23) does not admit a comparison principle, since this is not available for FitzHugh-Nagumo type systems. As such, very limited a-priori knowledge is available for this limiting operator, which forces us to prove many of its properties from scratch. For this, we mainly employ techniques from harmonic analysis.

The second main obstacle is that the system setting introduces several cross-terms that need to be controlled. Several key techniques from our earlier works [150, 151] concerning spatially discrete systems can be adjusted to handle these cross-terms in the present fully-discrete setting. However, several crucial points in the analysis still require these terms to be handled with special care.

The remaining obstacles are directly related to the infinite-range interactions, which introduce several convergence issues that need to be overcome. It also requires us to establish more refined estimates on the decay rates of solutions to our limiting MFDE. We achieve this by employing an explicit representation of the corresponding inverse linear operator that was first introduced in [150].

Loss of uniqueness In [111], Hupkes and Van Vleck extensively studied the uniqueness and parameter-dependence of the travelling wave solutions of (4.1.15). The key observation is that the rationality of the variable $M = (c\Delta t)^{-1}$ breaks the translational symmetry in the travelling wave problem, potentially allowing a *family* of solutions to exist. For example, one can apply an irrational phase shift to the continuous wave-profiles for (4.1.10) that underlies the perturbation argument discussed above. In this fashion, one could construct a different fully discrete wave for the same detuning parameter value r in the nonlinearity $g(\cdot;r)$. However, this is a very delicate issue. In particular, $M = (c\Delta t)^{-1}$ is fixed in the analysis, so additional work is required to obtain results for fixed time-steps $\Delta t > 0$.

For the backward-Euler discretisation scheme, this nonuniqueness can be made fully rigorous. In particular, Hupkes and Van Vleck showed that, for a fixed time step $\Delta t > 0$ both the $r(c)$ relation and the $c(r)$ relation can be multi-valued. In particular, for a fixed value of c there can be multiple values of r for which a solution to the system (4.1.15) exists and vice-versa. This can be achieved by embedding the system (4.1.18) into an MFDE that admits a comparison principle, allowing it to be analysed using the techniques developed by Keener [122] and Mallet-Paret [131].

By contrast, the $c(r)$ relation for travelling wave solutions to the PDE (4.1.9) and the LDE (4.1.10) are both single-valued. The same holds for the $r(c)$ relation, with the single exception that it can be multi-valued for (4.1.10) in the special case $c = 0$ [57, 99]. This reflects the well-known wave-pinning phenomenon caused by the broken translational symmetry of the lattice [16, 56, 62, 99, 122, 132].

In this paper we study the $r(c)$ and the $c(r)$ relation for a fully-discrete version of the FitzHugh-Nagumo system. For the corresponding PDE (4.1.1) and LDE (4.1.4), numerical evidence [34, 125] suggests that both these relations are at most 2-valued. In addition, theoretical results [32] for this PDE usually yield a locally unique $r(c)$ relation. For the system (4.1.23) a comparison principle is not available, rendering a direct analysis similar to the one in [111] infeasible. Instead, we run several numerical simulations to investigate these issues. These computations indicate that both the $r(c)$ and the $c(r)$ relation are typically multi-valued. Indeed, the points (r, c) points at which we were able to find solutions appear to map onto a surface instead of a curve. That is, there exists an entire spectrum of travelling wave solutions with different wavespeeds to the same fully discrete system.

4.2 Main result

Our main goal is to study the impact of several important temporal discretisation schemes on travelling wave solutions of reaction-diffusion LDEs of the form

$$\dot{U}_j = \tau \sum_{m>0} \alpha_m [U_{j+m} + U_{j-m} - 2U_j] + \mathcal{G}(U_j; r). \tag{4.2.1}$$

This LDE is posed on the one-dimensional lattice $j \in \mathbb{Z}$, but may have multiple components in the sense that $U_j \in \mathbb{R}^d$ for some integer $d \geq 1$. We start by discussing the structural conditions that we impose on the LDE (4.2.1) and its travelling wave solutions in §4.2.1 respectively §4.2.2. In §4.2.3 we introduce the appropriate temporal discretisation schemes and formulate our main result. Finally, we discuss some numerical results concerning the nonuniqueness of the fully discrete travelling waves in §4.2.4.

4.2.1 The spatially discrete system

Besides a handful of exceptions [6, 68, 69, 88, 149, 150], almost all results concerning LDEs of the form (4.2.1) assume that only finitely many of the coefficients α_m in (4.2.1) are nonzero. However, following [6, 150], we will impose the following much weaker conditions.

Assumption (HS1). The coefficients $\{\alpha_m\}_{m\in\mathbb{Z}_{>0}}$ are diagonal $d\times d$ matrices and $\tau>0$ is a positive constant. There exists $1\le d_{\mathrm{diff}}\le d$ so that for each $1\le i\le d_{\mathrm{diff}}$ we have $\alpha_m^{(i,i)}\neq 0$ for some $m\in\mathbb{Z}_{>0}$, while $\alpha_n^{(j,j)}=0$ for all $n\in\mathbb{Z}_{>0}$ and all $d_{\mathrm{diff}}<j\le d$. The coefficients $\{\alpha_m\}_{m\in\mathbb{Z}_{>0}}$ satisfy the bound

$$\sum_{m>0}|\alpha_m|e^{m\nu} \quad < \quad \infty \tag{4.2.2}$$

for some constant $\nu>0$, as well as the identity

$$\sum_{m>0}\alpha_m^{(i,i)}m^2 \quad = \quad 1 \tag{4.2.3}$$

for each $1\le i\le d_{\mathrm{diff}}$. Finally, the inequality

$$A_i(z):=\sum_{m>0}\alpha_m^{(i,i)}\Big(1-\cos(mz)\Big) \quad > \quad 0 \tag{4.2.4}$$

holds for all $z\in(0,2\pi)$ and all $1\le i\le d_{\mathrm{diff}}$.

In particular, the diffusion matrices $\{\alpha_m\}_{m\in\mathbb{Z}_{>0}}$ only act directly on the first d_{diff} components of U_j. For example, for the FitzHugh-Nagumo LDE

$$\begin{array}{rcl}\dot{u}_j &=& \tau\sum_{m>0}\alpha_m[u_{j+m}+u_{j-m}-2u_j]+u_j(1-u_j)(u_j-r)-w_j\\ \dot{w}_j &=& \rho\big[u_j-\gamma w_j\big],\end{array} \tag{4.2.5}$$

we have $d=2$ and $d_{\mathrm{diff}}=1$, while for the Nagumo LDE

$$\dot{u}_j \quad = \quad \tau\sum_{m>0}\alpha_m[u_{j+m}+u_{j-m}-2u_j]+u_j(1-u_j)(u_j-r) \tag{4.2.6}$$

we have $d=d_{\mathrm{diff}}=1$.

We note that (4.2.4) is automatically satisfied if $\alpha_m^{(i,i)}\ge 0$ for all $m\in\mathbb{Z}_{>0}$ and $\alpha_1^{(i,i)}\neq 0$. The conditions in (HS1) ensure that for $\phi\in L^\infty(\mathbb{R};\mathbb{R})$ with $\phi''\in L^2(\mathbb{R};\mathbb{R})$ and $1\le i\le d_{\mathrm{diff}}$, we have the limit

$$\lim_{h\downarrow 0}\ \|\tfrac{1}{h^2}\sum_{m>0}\alpha_m^{(i,i)}\big[\phi(\cdot+hm)+\phi(\cdot-hm)-2\phi(\cdot)\big]-\phi''\|_{L^2(\mathbb{R};\mathbb{R})}=0; \tag{4.2.7}$$

see [6, Lem. 2.1]. In particular, (HS1) ensures that (4.2.5) can be interpreted as the spatial discretisation of the FitzHugh-Nagumo PDE (4.1.1) on a grid with distance h, where $\tau = \frac{1}{h^2}$. Additional remarks concerning this assumption in the scalar case $d = 1$ can be found in [6, §1].

We now turn to the spatially homogeneous equilibrium solutions to (4.2.1), which are roots of the nonlinearity $\mathcal{G}$. We will assume that there are two r-independent equilibria $P^{\pm}$, but emphasize that they are allowed to be identical.

Assumption (HS2). The parameter dependent nonlinearity $\mathcal{G} : \mathbb{R}^d \times (0,1) \to \mathbb{R}^d$ is C^2-smooth. There exist $P^{\pm} \in \mathbb{R}^d$ so that $\mathcal{G}(P^{\pm}; r) = 0$ holds for all $r \in (0,1)$.

The temporal stability of these two equilibria $P^{\pm}$ plays an essential and delicate role in our analysis. Indeed, it does not suffice to simply require that the eigenvalues of $DG(P^{\pm})$ have strictly negative real parts, see the proof of [151, Lem. 4.6] for details. Following [151], we consider two auxiliary assumptions on the triplet $(\mathcal{G}, P^-, P^+)$ to address this issue. Recalling the constant $1 \le d_{\text{diff}} \le d$ from (HS1), we first write $D\mathcal{G}(U;r)$ in the block form

$$D\mathcal{G}(U;r) \quad = \quad \begin{pmatrix} \mathcal{G}^{[1,1]}(U;r) & \mathcal{G}^{[1,2]}(U;r) \\ \mathcal{G}^{[2,1]}(U;r) & \mathcal{G}^{[2,2]}(U;r) \end{pmatrix} \tag{4.2.8}$$

for any $U \in \mathbb{R}^d$ and $r \in (0,1)$, taking $D\mathcal{G}^{[1,1]}(U;r) \in \mathbb{R}^{d_{\text{diff}} \times d_{\text{diff}}}$.

Assumption (HS3$_{\overline{r}}$). The triplet $(\mathcal{G}, P^-, P^+)$ satisfies at least one of the following conditions.

(a) The matrices $-D\mathcal{G}(P^-; \overline{r})$ and $-D\mathcal{G}(P^+; \overline{r})$ are positive definite.

(b) The matrices $-\mathcal{G}^{[1,1]}(P^-; \overline{r}), -\mathcal{G}^{[1,1]}(P^+; \overline{r}), -\mathcal{G}^{[2,2]}(P^-; \overline{r})$ and $-\mathcal{G}^{[2,2]}(P^+; \overline{r})$ are positive definite. In addition, there exists a constant $\Gamma > 0$ so that $\mathcal{G}^{[1,2]}(U; \overline{r}) = -\Gamma \mathcal{G}^{[2,1]}(U; \overline{r})^T$ holds for all $U \in \mathbb{R}^d$.

To illustrate these assumptions, we consider the nonlinearity

$$G_{\text{fhn}}(u, w; r) \quad = \quad \begin{pmatrix} u(1-u)(u-r) - w \\ \rho\big[u - \gamma w\big] \end{pmatrix} \tag{4.2.9}$$

corresponding to the FitzHugh-Nagumo LDE (4.2.5). The triplet $(G_{\text{fhn}}, 0, 0)$ can easily be seen to satisfy (HS3$_{\overline{r}}$(b)) with $\Gamma = \frac{1}{\rho}$. However, when $r > 0$ is sufficiently small the Jacobian $DG_{\text{fhn}}(0; r)$ has a pair of complex eigenvalues with negative real part. In this case, the condition (HS3$_{\overline{r}}$(a)) may fail to hold.

4.2.2 Spatially discrete travelling waves

Our final two assumptions for (4.2.1) concern the existence and stability of travelling wave solutions that connect the equilibria P^- and P^+. These solutions take the form

$$U_j(t) \quad = \quad \overline{U}_0(j + \overline{c}_0 t) \tag{4.2.10}$$

for some smooth profile $\overline{U}_0$ and nonzero wavespeed $\overline{c}_0$. Substituting the Ansatz (4.2.10) into (4.2.1) and writing $\xi = j + \overline{c}_0 t$, we see that the pair $(\overline{c}_0, \overline{U}_0)$ must satisfy the travelling wave MFDE

$$\overline{c}_0 \overline{U}_0'(\xi) \quad = \quad \tau \sum_{m>0} \alpha_m \Big[\overline{U}_0(\xi + m) + \overline{U}_0(\xi - m) - 2\overline{U}_0(\xi) \Big] + \mathcal{G}\big(\overline{U}_0(\xi); r\big), \qquad (4.2.11)$$

together with the boundary conditions

$$\lim_{\xi \to \pm\infty} \overline{U}_0(\xi) \quad = \quad P^{\pm}. \qquad (4.2.12)$$

Assumption (HW1$_{\overline{r}}$). There exists a waveprofile $\overline{U}_0$ and a wavespeed $\overline{c}_0 \neq 0$ that solve the travelling wave MFDE (4.2.11) for $r = \overline{r}$, together with the boundary conditions (4.2.12).

We now turn to the spectral stability of these travelling wave solutions. To this end, we introduce the operator $L_0 : H^1(\mathbb{R}; \mathbb{R}^d) \to L^2(\mathbb{R}; \mathbb{R}^d)$ for the linearisation of (4.2.11) around the travelling wave $\overline{U}_0$, which acts as

$$L_0 \quad = \quad \overline{c}_0 \partial_\xi - \Delta_0 - D_U \mathcal{G}\big(\overline{U}_0; \overline{r}\big). \qquad (4.2.13)$$

Here the operator $\Delta_0 : L^2(\mathbb{R}; \mathbb{R}^d) \to L^2(\mathbb{R}; \mathbb{R}^d)$ is given by

$$\Delta_0 \quad = \quad \tau \sum_{m>0} \alpha_m \Big[T_0^m + T_0^{-m} - 2 \Big], \qquad (4.2.14)$$

where

$$(T_0 \Phi)(\xi) \quad = \quad \Phi(\xi + 1). \qquad (4.2.15)$$

In addition, we introduce the formal adjoint $L_0^* : H^1(\mathbb{R}; \mathbb{R}^d) \to L^2(\mathbb{R}; \mathbb{R}^d)$ of L_0 that acts as

$$L_0^* \quad = \quad -\overline{c}_0 \partial_\xi - \Delta_0 - D_U \mathcal{G}\big(\overline{u}_0; \overline{r}\big)^T. \qquad (4.2.16)$$

We remark that the spectrum of L_0 is $2\pi i c_0$-periodic on account of the identity

$$\big(L_0 + \lambda\big) e^{2\pi i \cdot} \quad = \quad e^{2\pi i \cdot} \big(L_0 + \lambda + 2\pi i c_0\big), \qquad (4.2.17)$$

see [150, Lem. 5.1]. We impose the following condition on the spectral properties of this operator L_0.

Assumption (HW2$_{\overline{r}}$). There exist functions $\Phi_0^{\pm} \in H^1(\mathbb{R}; \mathbb{R}^d)$, together with a constant $\tilde{\lambda} > 0$ so that the following properties hold for the LDE (4.2.1) with $r = \overline{r}$.

(i) We have the identity

$$\Phi_0^+ \quad = \quad \overline{U}_0', \qquad (4.2.18)$$

together with the normalisation

$$\langle \Phi_0^+, \Phi_0^- \rangle_{L^2(\mathbb{R}; \mathbb{R}^d)} \quad = \quad 1. \qquad (4.2.19)$$

(ii) The spectrum of the operator $-L_0$ in the half-plane $\{z \in \mathbb{C} : \mathrm{Re}\, z \geq -\tilde{\lambda}\}$ consists precisely of the points $2\pi i m \bar{c}_0$ with $m \in \mathbb{Z}$, which are all eigenvalues of L_0. Moreover, we have the identities

$$\begin{aligned}\ker(L_0) &= \mathrm{span}\{\Phi_0^+\} \\ &= \{g \in L^2(\mathbb{R};\mathbb{R}^d) : \langle g, \Psi\rangle_{L^2(\mathbb{R};\mathbb{R}^d)} = 0 \text{ for all } \Psi \in \mathrm{Range}(L_0^*)\}\end{aligned} \tag{4.2.20}$$

and

$$\begin{aligned}\ker(L_0^*) &= \mathrm{span}\{\Phi_0^-\} \\ &= \{g \in L^2(\mathbb{R};\mathbb{R}^d) : \langle g, \Psi\rangle_{L^2(\mathbb{R};\mathbb{R}^d)} = 0 \text{ for all } \Psi \in \mathrm{Range}(L_0)\}.\end{aligned} \tag{4.2.21}$$

Recall that an eigenvalue λ of a Fredholm operator L is said to be *simple* if the kernel of $L-\lambda$ is spanned by one vector v and the equation $(L-\lambda)w = v$ does not have a solution w. Note that if L has a formal adjoint L^*, this is equivalent to the condition that $\langle v, w\rangle \neq 0$ for all nontrivial $w \in \ker(L^* - \overline{\lambda})$. In particular, the normalisation (4.2.19) implies that the eigenvalues $2\pi i\bar{c}_0\mathbb{Z}$ are all simple eigenvalues of $-L_0$.

For the FitzHugh-Nagumo system (4.2.5), the assumptions (HW1$_{\overline{r}}$) and (HW2$_{\overline{r}}$) are both satisfied for all sufficiently small discretisation distances $h > 0$ and sufficiently small $\rho > 0$, see [150, Thm. 2.1, Thm. 2.2, Prop. 4.2]. If the shifts have finite-range, i.e. $\alpha_m = 0$ for all sufficiently large m, then these assumptions are satisfied [108, Thm. 1]-[109, Prop. 5.1] for sufficiently small $\rho > 0$ without any restriction on the discretisation distance h. There are, however, conditions on r and γ in both cases.

4.2.3 The fully discrete system

We aim to approximate solutions to (4.2.1) at discrete time intervals $t = n\Delta t$ by

$$U_j(n\Delta t) \sim W_j(n\Delta t). \tag{4.2.22}$$

We need to apply an appropriate discretisation scheme to the temporal derivative in (4.2.1). Although there are many different approximation schemes available, we mainly focus on the six so-called BDF methods. These methods are based on interpolation polynomials of different degrees. In particular, the BDF method of order $k \in \{1, 2, ..., 6\}$ approximates U' in (4.2.1) at $t = n\Delta t$ by first constructing an interpolating polynomial of degree k through the $k+1$ points $\{W((n-n')\Delta t)\}_{n'=0}^{k}$ and then computing the derivative of this polynomial at $W(n\Delta t)$. As such, the temporal discretisations of the LDE (4.2.1) under consideration are of the form

$$\begin{aligned}\beta_k^{-1}\tfrac{1}{\Delta t}\sum_{n'=0}^{k}\mu_{n';k}W_j\big(n\Delta t-(k-n')\Delta t\big) &= \tau\sum_{m>0}\alpha_m[W_{j+m}(n\Delta t)+W_{j-m}(n\Delta t) \\ &\qquad -2W_j(n\Delta t)] \\ &\quad +\mathcal{G}\big(W_j(n\Delta t);r\big).\end{aligned} \tag{4.2.23}$$

$\mu_{n;k}$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$n=0$	-1	$\frac{1}{3}$	$-\frac{2}{11}$	$\frac{3}{25}$	$-\frac{12}{137}$	$\frac{10}{147}$
$n=1$	1	$-\frac{4}{3}$	$\frac{9}{11}$	$-\frac{16}{25}$	$\frac{75}{137}$	$-\frac{72}{147}$
$n=2$		1	$-\frac{18}{11}$	$\frac{36}{25}$	$-\frac{200}{137}$	$\frac{225}{147}$
$n=3$			1	$-\frac{48}{25}$	$\frac{300}{137}$	$-\frac{400}{147}$
$n=4$				1	$-\frac{300}{137}$	$\frac{450}{147}$
$n=5$					1	$-\frac{360}{147}$
$n=6$						1
β_k	1	$\frac{2}{3}$	$\frac{6}{11}$	$\frac{12}{25}$	$\frac{60}{137}$	$\frac{60}{147}$

Table 4.1: *The coefficients $\mu_{n;k}$ and β_k associated to the BDF discretisation schemes as given by (4.2.24).*

The coefficients β_k and $\{\mu_{n;k}\}$ in (4.2.23) are given implicitly by the identities

$$\begin{aligned} \sum_{n=0}^{k} \mu_{n;k} v\big((n-k)\Delta t\big) &= \sum_{n'=1}^{k} [\partial^{n'} v](0), \\ \beta_k &= \sum_{n=0}^{k} \mu_{n;k}(n-k), \end{aligned} \tag{4.2.24}$$

which must hold for any scalar function v. Here we have introduced the notation

$$[\partial v](n\Delta t) \quad = \quad v\big(n\Delta t\big) - v\big((n-1)\Delta t\big). \tag{4.2.25}$$

This definition yields that $\sum_{n=0}^{k} \mu_{n;k} = 0$, which allows us to identify

$$\beta_k \quad = \quad \sum_{n=0}^{k} \mu_{n;k}(n-m) \quad = \quad \sum_{n=1}^{k} \mu_{n;k} n. \tag{4.2.26}$$

For convenience, the values of the coefficients β_k and $\mu_{n;k}$ can be found in Table 4.1. We note that the BDF method of order 1 is the well-known backward-Euler method.

Our main goal is to study travelling wave solutions to the fully discrete system (4.2.23), utilizing our assumptions for the spatially discrete system (4.2.1). Such solutions are given by the Ansatz

$$W_j(n\Delta t) \quad = \quad \Phi(j + nc\Delta t), \tag{4.2.27}$$

for some wave speed c and profile Φ with the boundary conditions

$$\Phi(\pm\infty) \quad = \quad P^{\pm}, \tag{4.2.28}$$

in a sense that we make precise below.

For notational convenience, we introduce the quantity $M = (c\Delta t)^{-1}$. Substituting the Ansatz (4.2.27) into (4.2.23) yields the system

$$c[\mathcal{D}_{k,M}\Phi](\xi) \quad = \quad \tau \sum_{m>0} \alpha_m[\Phi(\xi+m) + \Phi(\xi-m) - 2\Phi(\xi)] + \mathcal{G}\big(\Phi(\xi);r\big), \tag{4.2.29}$$

for all ξ that can be written as $\xi = n + jM^{-1}$ for $(j,n) \in \mathbb{Z}^2$. Here we have introduced the discrete derivatives

$$[\mathcal{D}_{k,M}\Phi](\xi) \quad = \quad \beta_k^{-1} M \sum_{n'=0}^{k} \mu_{n';k} \Phi\big(\xi - (k-n')M^{-1}\big), \tag{4.2.30}$$

for $k \in \{1, 2, ..., 6\}$. From [111, eq. (2.13)] we obtain the useful estimate

$$|[\mathcal{D}_{k,M}\Phi](\xi) - \Phi'(\xi)| \quad \leq \quad C_l M^{-l} \sup_{-kM^{-1} \leq \theta \leq 0} |\Phi^{(l+1)}(\xi+\theta)|, \tag{4.2.31}$$

for all integers $1 \leq l \leq k$ and all $\Phi \in C^{l+1}(\mathbb{R};\mathbb{R}^d)$, in which the constant $C_l \geq 1$ is independent of k, Φ and M. Indeed, this estimate shows that the regular derivative can be approximated by the discrete derivatives as the time step Δt shrinks to zero. We emphasize that BDF discretisation schemes of order $k \geq 2$ do not allow for a comparison principle, even when the original LDE does allow for one. This is a consequence of the existence of coefficients $\mu_{n;k} > 0$ that have $n < k$.

Most of our results, including our main theorem, require a restriction on the values of M that are allowed. In particular, upon fixing an integer $q \geq 1$, we introduce the set

$$\mathcal{M}_q \quad = \quad \{\tfrac{p}{q} : p \in \mathbb{N} \text{ has } \gcd(p,q) = 1 \text{ and } p \geq q\}. \tag{4.2.32}$$

Often, we introduce $M = \frac{p}{q} \in \mathcal{M}_q$, which implicitly defines the integer $p = p(M) = qM$. Moreover, we see that the natural domain for the values of ξ in the system (4.2.29), as well as in the boundary conditions (4.2.28), is precisely the set $p^{-1}\mathbb{Z}$.

Theorem 4.2.1. *Assume that (HS1) and (HS2) are satisfied and pick $\overline{r}$ in such a way that (HS3$_{\overline{r}}$), (HW1$_{\overline{r}}$) and (HW2$_{\overline{r}}$) are satisfied. Fix a pair of integers $1 \leq k \leq 6$ and $q \geq 1$. Then there exist constants $M_* \gg 1$ and $\delta_r > 0$ so that for any $M = \frac{p}{q} \in \mathcal{M}_q$ with $M \geq M_*$, there exist continuous functions*

$$\begin{aligned} c_M : \mathbb{R} \times [\overline{r} - \delta_r, \overline{r} + \delta_r] \quad &\to \quad \mathbb{R}, \\ \overline{U}_M : \mathbb{R} \times [\overline{r} - \delta_r, \overline{r} + \delta_r] \quad &\to \quad \ell^\infty(p^{-1}\mathbb{Z};\mathbb{R}^d) \end{aligned} \tag{4.2.33}$$

that satisfy the following properties.

(i) For any $(\theta, r) \in \mathbb{R} \times [\overline{r} - \delta_r, \overline{r} + \delta_r]$, the pair $c = c_M(\theta, r)$ and $\overline{U} = \overline{U}_M(\theta, r)$ satisfies the system

$$c[\mathcal{D}_{k,M}\overline{U}](\xi) \quad = \quad \tau \sum_{m>0} \alpha_m[\overline{U}(\xi+m) + \overline{U}(\xi-m) - 2\overline{U}(\xi)] + \mathcal{G}\big(\overline{U}(\xi);r\big) \tag{4.2.34}$$

for $\xi \in p^{-1}\mathbb{Z}$, together with the boundary conditions

$$\lim_{\xi \to \pm\infty, \xi \in p^{-1}\mathbb{Z}} \overline{U}(\xi) \quad = \quad P^{\pm}. \tag{4.2.35}$$

(ii) For any $(\theta, r) \in \mathbb{R} \times [\overline{r} - \delta_r, \overline{r} + \delta_r]$, the solution $\overline{U} = \overline{U}_M(\theta, r)$ admits the normalisation

$$\sum_{\xi \in p^{-1}\mathbb{Z}} \Big[\Big\langle \Phi_0^-(\xi + \theta), \overline{U}(\xi) - \overline{U}_0(\xi + \theta) \Big\rangle_{\mathbb{R}^d} \Big] \quad = \quad 0. \tag{4.2.36}$$

(iii) For any $(\theta, r) \in \mathbb{R} \times [\overline{r} - \delta_r, \overline{r} + \delta_r]$, we have the shift-periodicity

$$\begin{array}{rcl} c_M(\theta + p^{-1}, r) & = & c_M(\theta, r), \\ \overline{U}_M(\theta + p^{-1}, r)(\xi) & = & \overline{U}_M(\theta, r)(\xi + p^{-1}). \end{array} \tag{4.2.37}$$

In addition, there exists $\delta > 0$ such that the following holds true. Any triplet $(c, \overline{U}, \theta) \in \mathbb{R} \times \ell^\infty(p^{-1}\mathbb{Z}; \mathbb{R}^d) \times \mathbb{R}$ that satisfies (4.2.34) for some pair $(r, M) \in (0,1) \times \mathcal{M}_q$ with

$$|r - \overline{r}| \quad < \quad \delta, \qquad M \quad = \quad \tfrac{p}{q} \quad > \quad \delta^{-1} \quad \geq \quad M_* \tag{4.2.38}$$

and also enjoys the estimate

$$p^{-1} \sum_{\xi \in p^{-1}\mathbb{Z}} \Big[|\overline{U}(\xi) - \overline{U}_0(\xi + \theta)|^2 + |\mathcal{D}_{k,M}\overline{U}(\xi) - \mathcal{D}_{k,M}\overline{U}_0(\xi + \theta)|^2 \Big] \quad < \quad \delta^2, \tag{4.2.39}$$

must actually satisfy $c = c_M(\tilde{\theta}, r)$ and $\overline{U} = \overline{U}_M(\tilde{\theta}, r)$ for some $\tilde{\theta} \in \mathbb{R}$.

The factor p^{-1} in (4.2.39) is used to compensate the growing number of terms as $p \to \infty$. In particular, we can view this as a uniqueness result with respect to a scaled L^2-norm that will be specified later.

4.2.4 Nonuniqueness and numerical examples

Fixing $r \in [\overline{r} - \delta_r, \overline{r} + \delta_r]$, $M = \frac{p}{q} \geq M_*$ and $\theta \in \mathbb{R}$, the travelling wave $(c_M(\theta, r), \overline{U}_M(\theta, r))$ is constructed as a perturbation of the travelling wave $(\overline{c}_0, \overline{U}_0(\cdot + \theta))$ on the domain $p^{-1}\mathbb{Z}$. Since the wave profiles $\overline{U}_0(\cdot + \theta)$ and $\overline{U}_0(\cdot + \theta + p^{-1})$ are simply translates of each other on this domain, the shift-periodicity (4.2.37) follows easily. However, it is not clear how, specifically, the travelling wave depends on θ. Indeed, in [111, §5], Hupkes and Van Vleck show that it is reasonable to expect that the derivative $\partial_\theta c_M(\theta, \overline{r})$ is exponentially small in M. As such, it is unclear how to further analyse this dependence.

We emphasize that in general the travelling wave solution will not necessarily be unique, even up to translation. In particular, fixing $\theta \in (0, p^{-1})$, we note that the waves $\overline{U}_0$ and $\overline{U}_0(\cdot + \theta)$ are different on the domain $p^{-1}\mathbb{Z}$. One might be tempted to conclude that if M is sufficiently large, the wave profiles $\overline{U}_M(0, r)$ and $\overline{U}_M(\theta, r)$ are different as well. However, a larger value of M means that the grid $p^{-1}\mathbb{Z}$ becomes finer. In particular, since the travelling waves $\overline{U}_M(0, r)$ and $\overline{U}_M(\theta, r)$ are perturbations of the waves $\overline{U}_0$ and $\overline{U}_0(\cdot + \theta)$, it could be that these perturbations cancel out the difference between $\overline{U}_0$ and $\overline{U}_0(\cdot + \theta)$.

In addition, since the constant $M = (c\Delta t)^{-1}$ is fixed in the statement of Theorem 4.2.1, fluctuations in c automatically lead to changes in Δt. This complicates our understanding of the fully discrete system for a fixed timestep $\Delta t > 0$. Our main goal here is to show that the wavespeed c and the detuning parameter r do not depend on each other in a locally unique fashion, which is in major contrast to the corresponding continuous and semi-discrete systems.

However, the lack of a comparison principle for FitzHugh-Nagumo systems heavily complicates a direct analysis. As such, we have chosen to, instead, use numerical simulations to illustrate these phenomena. In particular, we focus on the backward-Euler discretisation of the FitzHugh-Nagumo MFDE, which takes the form

$$\begin{array}{rcl} (h\Delta t)^{-1}[u(\xi) - u(\xi - c\Delta t)] & = & h^{-2}[u(\xi+1) + u(\xi-1) - 2u(\xi)] + g(u(\xi); r) - w(\xi) \\ (h\Delta t)^{-1}[u(\xi) - u(\xi - c\Delta t)] & = & \rho\big[u(\xi) - \gamma w(\xi)\big]. \end{array} \tag{4.2.40}$$

Here we fix $\rho = 0.01$, $\gamma = 5$, $h = \frac{5}{8}$ and we let g be the bistable nonlinearity

$$g(u; r) \quad = \quad u(1-u)(u-r). \tag{4.2.41}$$

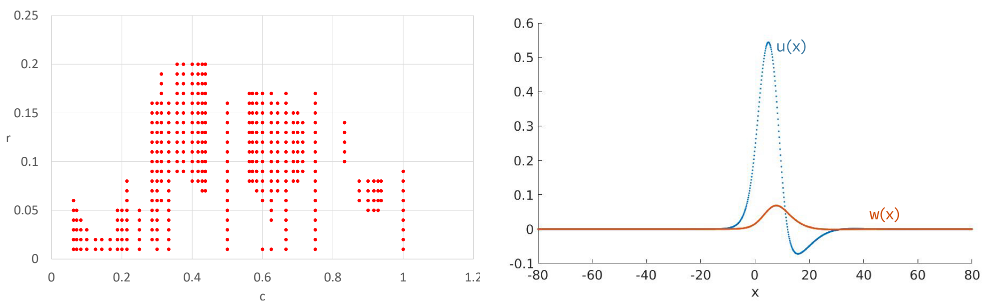
Upon fixing the timestep $\Delta t = 2$, we repeatedly solved the system (4.2.40) with Neumann boundary conditions on the interval $[-80, 80]$ for different values of the parameters $(c, r) \in \mathbb{Q} \times (0, 1)$.

These simulations turned out to be rather delicate, since the quality of the initial condition heavily influenced whether a solution could be found. In many cases, the simulation returned the zero solution. Simply augmenting an extra nontriviality condition often produced no solution at all. In addition, the value of c greatly determines the number of points $\xi \in \mathbb{R}$ for which the values $(u, v)(\xi)$ need to be determined. In particular, upon writing

$$c \quad = \quad \tfrac{q\Delta t}{p}, \tag{4.2.42}$$

we needed to consider the points in the set $p^{-1}\mathbb{Z} \cap [-80, 80]$, which rapidly grows in number as p increases. We considered values of c of the form (4.2.42) for values of $p \in \{1, 2, ..., 8\}$ and $q \in \{1, 2, ..., 2p\}$ with $\gcd(p, q) = 1$, while the values of r were taken in $\frac{1}{100}\mathbb{Z} \cap (0, \frac{1}{5})$.

Figure 4.1(a) depicts the pairs (c, r) for which such a numerical solution could be found. It is highly likely that a solution still exists at some of the other parameter values that we investigated. In any case, our simulations clearly show that the parameters c and r depend on each other in an intricate fashion. In particular, our results suggest that travelling wave solutions to the system (4.2.40) are not unique, since we were able to find solutions with a range of different wavespeeds at the same value for r. We refer to [34] and [125] for the corresponding dependence for the FitzHugh-Nagumo PDE and LDE respectively. In both cases, this dependence is given by a curve in the (c, r)-plane that resembles the symbol $\cap$.



(a) (b)

Figure 4.1: *(a) Numerical computations of the pairs* (c, r) *for which travelling wave solutions to the system (4.2.40) exist. We emphasize that there may be parameter values where we could not find a solution, but where a solution exists nonetheless. These simulations clearly show that the relationship* $r(c)$ *is multi-valued. (b) A plot of one of the travelling waves found in this numerical procedure with* $r = 0.11$ *and* $c = 0.3125$.

4.3 Setup

The fully discrete travelling wave equation (4.2.29) is a highly singular perturbation of the semi-discrete travelling wave MFDE (4.2.11), which is the key complication for our analysis. In order to tackle this issue, we start by studying the linear operators that arise when linearizing the fully discrete travelling wave equation (4.2.29) around the semi-discrete travelling wave $(\overline{c}_0, \overline{U}_0)$. In particular, we define the linear expressions

$$L_{k,M}\Phi(\xi) \quad = \quad \overline{c}_0[\mathcal{D}_{k,M}\Phi](\xi) - \Delta_0\Phi(\xi) - D_U\mathcal{G}\big(\overline{U}_0(\xi)\big)\Phi(\xi). \tag{4.3.1}$$

Our aim is to establish that the operators $L_{k,M}$ inherit several useful properties from the operator L_0 defined in (4.2.13) in the small timestep regime $\Delta t \ll 1$.

In this section we summarize and adept the setup from [111], sticking to the same notation as much as possible. In order to formulate our results, we need to define several function spaces. For any $\eta \in \mathbb{R}$, we write

$$\begin{array}{lcl} BC_\eta(\mathbb{R};\mathbb{R}^d) & = & \{F \in C(\mathbb{R};\mathbb{R}^d) \mid \sup_{\xi\in\mathbb{R}} e^{-\eta|\xi|}|F(\xi)| < \infty\}, \\ BC^1_\eta(\mathbb{R};\mathbb{R}^d) & = & \{F \in C^1(\mathbb{R};\mathbb{R}^d) \mid \sup_{\xi\in\mathbb{R}} e^{-\eta|\xi|}[|F(\xi)| + |F'(\xi)|] < \infty\}. \end{array} \tag{4.3.2}$$

In addition, given a Hilbert space H and any $\mu > 0$, we define the corresponding sequence space

$$\ell^2_\mu(H) \quad = \quad \{v : \mu^{-1}\mathbb{Z} \to H \mid \|v\|_{\ell^2_\mu(H)} := \langle v, v\rangle^{\frac{1}{2}}_{\ell^2_\mu(H)} < \infty\}, \tag{4.3.3}$$

which is a Hilbert space equipped with the inner product

$$\langle v, v\rangle_{\ell^2_\mu(H)} \quad = \quad \mu^{-1}\sum_{\xi\in\mu^{-1}\mathbb{Z}} \langle v(\xi), w(\xi)\rangle_H. \tag{4.3.4}$$

For now, we fix two integers $q \geq 1$ and $1 \leq k \leq 6$, together with a constant $M = \frac{p}{q} \in \mathcal{M}_q$. To streamline our notation, we write $\mathcal{Y}_M$ to refer to the space $\ell^2_p(\mathbb{R}^d)$, i.e.,

$$\mathcal{Y}_M = \ell^2_p(\mathbb{R}^d), \qquad \langle \Phi, \Psi \rangle_{\mathcal{Y}_M} = \langle \Phi, \Psi \rangle_{\ell^2_p(\mathbb{R}^d)}. \tag{4.3.5}$$

Moreover, we introduce the space $\mathcal{Y}^1_{k,M}$, which differs from $\mathcal{Y}_M$ only by its inner product. To be more precise, we write

$$\begin{array}{lcl} \mathcal{Y}^1_{k,M} & = & \ell^2_p(\mathbb{R}^d), \\ \langle \Phi, \Psi \rangle_{\mathcal{Y}^1_{k,M}} & = & \langle \Phi, \Psi \rangle_{\ell^2_p(\mathbb{R}^d)} + \langle \mathcal{D}_{k,M} \Phi, \mathcal{D}_{k,M} \Psi \rangle_{\ell^2_p(\mathbb{R}^d)}. \end{array} \tag{4.3.6}$$

In addition, for $f \in BC_{-\eta}(\mathbb{R};\mathbb{R}^d)$ with $\eta > 0$, we write $\pi_{\mathcal{Y}_M}$ for the sequence

$$\big[\pi_{\mathcal{Y}_M} f\big](\xi) = f(\xi), \qquad \xi \in p^{-1}\mathbb{Z}. \tag{4.3.7}$$

If moreover $f \in BC^1_{-\eta}(\mathbb{R};\mathbb{R}^d)$ and we wish to be explicit, we often write $\pi_{\mathcal{Y}^1_{k,M}} f$ to refer to the restriction (4.3.7). The restriction operators $\pi_{\mathcal{Y}_M}$ and $\pi_{\mathcal{Y}^1_{k,M}}$ are bounded, see Lemma 4.A.1.

We can now consider the operators $L_{k,M}$ appearing in (4.3.1) as bounded linear maps

$$L_{k,M} : \ \mathcal{Y}^1_{k,M} \ \to \ \mathcal{Y}_M. \tag{4.3.8}$$

Our goal is to define new sequence spaces, which allow us to pass to the limit $M \to \infty$ in a controlled fashion. The basic idea is to use L^2-interpolants for functions in $\mathcal{Y}_M$ and H^1-interpolants for functions in $\mathcal{Y}^1_{k,M}$, so that the sequences in these spaces can be compared regardless of the different values of M. The main difficulty is to control terms of the form $v(\xi + p^{-1}) - v(\xi)$ for $v \in \mathcal{Y}^1_{k,M}$ with $M = \frac{p}{q}$, which is impossible to extract solely from the behaviour of $\mathcal{D}_{k,M} v$.

To tackle this issue, we need to perform q separate interpolations. Each of these interpolations must bridge a gap of size $M^{-1} = \frac{q}{p}$. In particular, upon fixing an integer $q \geq 1$ and writing

$$\begin{array}{lcl} \mathbb{Z}_q & = & \{0, 1, 2, ..., q\}, \\ \mathbb{Z}^\circ_q & = & \{1, 2, ..., q-1\}, \end{array} \tag{4.3.9}$$

we introduce the space

$$\ell^2_{q,\perp} = \{\Phi : q^{-1}\mathbb{Z}_q \to \mathbb{R}^d\}, \tag{4.3.10}$$

equipped with the inner product

$$\langle \Phi, \Psi \rangle_{\ell^2_{q,\perp}} = q^{-1}\Big[\tfrac{1}{2}\Phi(0)\Psi(0) + \tfrac{1}{2}\Phi(1)\Psi(1) + \sum_{\zeta \in q^{-1}\mathbb{Z}^\circ_q} \Phi(\zeta)\Psi(\zeta)\Big]. \tag{4.3.11}$$

Upon introducing the notation $\Phi(\zeta, \xi) = [\Phi(\xi)](\zeta)$ for $\Phi \in \ell^2_M(\ell^2_{q,\perp})$ with $\zeta \in q^{-1}\mathbb{Z}_q$ and $\xi \in M^{-1}\mathbb{Z}$, we define the space

$$\mathcal{H}_M = \{\phi \in \ell^2_M(\ell^2_{q,\perp}) : \Phi(1,\xi) = \Phi(0, \xi + M^{-1}) \text{ for all } \xi \in M^{-1}\mathbb{Z}\}, \tag{4.3.12}$$

equipped with the inner product

$$\langle \Phi, \Psi \rangle_{\mathcal{H}_M} = M^{-1} \sum_{\xi \in M^{-1}\mathbb{Z}} \langle \Phi(\cdot, \xi), \Psi(\cdot, \xi) \rangle_{\ell^2_{q,\perp}}. \tag{4.3.13}$$

For any $\eta > 0$ and any $f \in BC_{-\eta}(\mathbb{R}; \mathbb{R}^d)$, we now write $\pi_{\mathcal{H}_M} f \in \mathcal{H}_M$ for the function

$$[\pi_{\mathcal{H}_M} f](\zeta, \xi) = f(\xi + \zeta M^{-1}), \qquad \zeta \in q^{-1}\mathbb{Z}_q, \qquad \xi \in M^{-1}\mathbb{Z}. \tag{4.3.14}$$

We extend the operators $\mathcal{D}_{k,M}$ to $\mathcal{H}_M$ by writing

$$[\mathcal{D}_{k,M}\Phi](\zeta, \xi) = [\mathcal{D}_{k,M}\Phi(\zeta, \cdot)](\xi). \tag{4.3.15}$$

Note that these operators act only on the second component of Φ. This allows us to define our final space

$$\mathcal{H}^1_{k,M} = \mathcal{H}_M, \tag{4.3.16}$$

equipped with the inner product

$$\langle \Phi, \Psi \rangle_{\mathcal{H}^1_{k,M}} = \langle \Phi, \Psi \rangle_{\mathcal{H}_M} + \langle \mathcal{D}_{k,M}\Phi, \mathcal{D}_{k,M}\Psi \rangle_{\mathcal{H}_M}. \tag{4.3.17}$$

In fact, we can relate the spaces $\mathcal{H}_M$ and $\mathcal{H}^1_{k,M}$ to the spaces defined earlier. To see this, we define the isometries

$$\mathcal{J}_M : \mathcal{Y}_M \to \mathcal{H}_M, \qquad \mathcal{J}^1_{k,M} : \mathcal{Y}^1_{k,M} \to \mathcal{H}^1_{k,M}, \tag{4.3.18}$$

for $M = \frac{p}{q} \in \mathcal{M}_q$, which both act as

$$[\mathcal{J}_M\Phi](\zeta, \xi) = [\mathcal{J}^1_{k,M}\Phi](\zeta, \xi) = \Phi(\xi + M^{-1}\zeta), \tag{4.3.19}$$

for $\zeta \in q^{-1}\mathbb{Z}_q$ and $\xi \in M^{-1}\mathbb{Z}$, see Lemma 4.A.3. Note that $\pi_{\mathcal{H}_M} = \mathcal{J}_M \pi_{\mathcal{Y}_M}$.

Our goal is to interpret $L_{k,M}$ as a map from $\mathcal{H}^1_{k,M}$ into $\mathcal{H}_M$. To this end, we pick $n \in \mathbb{Z}$ and $0 < \vartheta \leq 1$ in such a way that

$$1 = (n + \vartheta)M^{-1}. \tag{4.3.20}$$

Since $M = \frac{p}{q} \in \mathcal{M}_q$, we see that $\vartheta = \frac{p - nq}{q}$, which yields

$$nM^{-1} = 1 - \vartheta M^{-1}, \qquad \vartheta \in q^{-1}\mathbb{Z}_q \setminus \{0\}. \tag{4.3.21}$$

In fact, because $\gcd(p, q) = 1$, it follows that $\gcd(p, \vartheta q) = 1$.

With these preparations in hand, we now write $\mathcal{K}_{k,M} : \mathcal{H}^1_{k,M} \to \mathcal{H}_M$ for the linear operator that acts as

$$[\mathcal{K}_{k,M}\Phi](\zeta, \xi) = \overline{c}_0[\mathcal{D}_{k,M}\Phi](\zeta, \xi) - [\Delta_M\Phi](\zeta, \xi) - D_U\mathcal{G}\big(\overline{U}_0(\xi + \zeta M^{-1}); \overline{r}\big)\Phi(\zeta, \xi), \tag{4.3.22}$$

for $\zeta \in q^{-1}\mathbb{Z}_q$ and $\xi \in M^{-1}\mathbb{Z}$. Here the operator Δ_M is given by

$$\Delta_M = \tau \sum_{m>0} \alpha_m \Big[T_M^m + T_M^{-m} - 2 \Big], \tag{4.3.23}$$

where we have introduced the twist operator $T_M : \mathcal{H}_M \to \mathcal{H}_M$ that acts as

$$[T_M \Phi](\zeta, \xi) = \Phi(\zeta + \vartheta, \xi + nM^{-1}), \tag{4.3.24}$$

taking into account the convention

$$\Phi(\zeta \pm 1, \xi) = \Phi(\zeta, \xi \pm M^{-1}). \tag{4.3.25}$$

In particular, we see that the shift ϑ acts as a rotation number, connecting the different components of Φ in the ζ-direction. The inequality

$$\langle \Delta_M \Phi, \Phi \rangle_{\mathcal{H}_M} \leq 0 \tag{4.3.26}$$

for $\Phi \in \mathcal{H}_M$ is almost trivial to verify in the finite-range setting, but turns out to be much harder to establish when dealing with infinite-range interactions; see Lemma 4.A.5.

Finally, we introduce the notation

$$D\mathcal{G}\big(\pi_{\mathcal{H}_M} \overline{U}_0; \overline{r}\big) : \mathcal{H}_M \to \mathcal{H}_M \tag{4.3.27}$$

to refer to the multiplication operator

$$[D\mathcal{G}\big(\pi_{\mathcal{H}_M} \overline{U}_0; \overline{r}\big)\Phi](\zeta, \xi) = D_U \mathcal{G}\big(\overline{U}_0(\xi + \zeta M^{-1}); \overline{r}\big)\Phi(\zeta, \xi). \tag{4.3.28}$$

In fact, it is easy to see that

$$\mathcal{K}_{k,M} \mathcal{J}^1_{k,M} = \mathcal{J}_M L_{k,M}, \tag{4.3.29}$$

which shows that $\mathcal{K}_{k,M}$ and $L_{k,M}$ are equivalent.

Since the operator $\mathcal{K}_{k,M}$ is not self-adjoint, we need to introduce the formal adjoint $\mathcal{K}^*_{k,M} : \mathcal{H}^1_{k,M} \to \mathcal{H}_M$ of $\mathcal{K}_{k,M}$ by writing

$$\mathcal{K}^*_{k,M}\Phi = \overline{c}_0[\mathcal{D}^*_{k,M}\Phi] - \Delta_M \Phi - D\mathcal{G}\big(\pi_{\mathcal{H}_M} \overline{U}_0; \overline{r}\big)^T \Phi, \tag{4.3.30}$$

in which we have defined

$$[\mathcal{D}^*_{k,M}\Phi](\zeta, \xi) = \beta_k^{-1} M \sum_{n'=0}^{k} \mu_{n';k} \Phi(\xi + (k - n')M^{-1}). \tag{4.3.31}$$

Moreover, we introduce the space

$$\ell^2_{q,\perp;\infty} = \{\phi \in \ell^2_{q,\perp} : \phi(1) = \phi(0)\}, \tag{4.3.32}$$

together with the map

$$[\pi_\perp f](\zeta,\xi) \quad = \quad f(\xi), \qquad \zeta \in q^{-1}\mathbb{Z}_q, \qquad \xi \in \mathbb{R}, \tag{4.3.33}$$

which constructs a function $\pi_\perp f \in L^2(\mathbb{R}, \ell^2_{q,\perp;\infty})$ from a function $f \in L^2(\mathbb{R};\mathbb{R}^d)$.

Taking the limit $M \to \infty$, while keeping ϑ and q fixed as in (4.3.20), we see that $\mathcal{K}_{k,M}$ and $\mathcal{K}^*_{k,M}$ formally approach the limiting operators

$$\begin{aligned} \overline{\mathcal{K}}_{q,\vartheta} : H^1(\mathbb{R}, \ell^2_{q,\perp;\infty}) &\quad\to\quad L^2(\mathbb{R}, \ell^2_{q,\perp;\infty}), \\ \overline{\mathcal{K}}^*_{q,\vartheta} : H^1(\mathbb{R}, \ell^2_{q,\perp;\infty}) &\quad\to\quad L^2(\mathbb{R}, \ell^2_{q,\perp;\infty}), \end{aligned} \tag{4.3.34}$$

that act as

$$\begin{aligned} \overline{\mathcal{K}}_{q,\vartheta}\Theta &\quad=\quad \overline{c}_0 \partial_\xi \Theta - \Delta_{q,\vartheta}\Theta - D\mathcal{G}\big(\pi_{\mathcal{H}_M}\overline{U}_0; \overline{r}\big)\Theta, \\ \overline{\mathcal{K}}^*_{q,\vartheta}\Theta &\quad=\quad -\overline{c}_0 \partial_\xi \Theta - \Delta_{q,\vartheta}\Theta - D\mathcal{G}\big(\pi_{\mathcal{H}_M}\overline{U}_0; \overline{r}\big)^T\Theta. \end{aligned} \tag{4.3.35}$$

Here the operator $\Delta_{q,\vartheta}$ is given by

$$\Delta_{q,\vartheta} \quad = \quad \tau \sum_{m>0} \alpha_m \Big[T^m_{q,\vartheta} + T^{-m}_{q,\vartheta} - 2\Big], \tag{4.3.36}$$

in which we have introduced the twist operator

$$\big[T_{q,\vartheta}\Theta\big](\zeta,\xi) \quad = \quad \Theta(\zeta+\vartheta, \xi+1), \tag{4.3.37}$$

for $\zeta \in q^{-1}\mathbb{Z}_q$ and $\xi \in \mathbb{R}$. In the same spirit as (4.3.25), we here make the convention $\Phi(\zeta+1,\xi) = \Phi(\zeta,\xi)$. Notice that the limiting operator $\overline{\mathcal{K}}_{q,\vartheta}$ reduces to the operator L_0 defined in (4.2.13) for ζ-independent functions.

4.4 The limiting system

Our goal here is to exploit our understanding of the operator L_0 in order to determine the Fredholm properties of the limiting operator $\overline{\mathcal{K}}_{q,\vartheta}$. Due to the lack of a comparison principle we cannot immediately appeal to a general Frobenius-Peron-type result as was possible in [111]. The theory in this section aims to fill these gaps and can be considered the key technical contribution of this paper. We collect the main results in the following Proposition, which plays an essential role in Lemma 4.5.3 below.

Proposition 4.4.1 (cf. [111, Lem. 3.6]). *Assume that (HS1) and (HS2) are satisfied and pick $\overline{r}$ in such a way that (HS3$_{\overline{r}}$), (HW1$_{\overline{r}}$) and (HW2$_{\overline{r}}$) are satisfied. Fix an integer $q \geq 1$, together with a constant $\vartheta \in q^{-1}\mathbb{Z}_q$ that has $\gcd(\vartheta q, q) = 1$. Then the operators $\overline{\mathcal{K}}_{q,\vartheta}$ and $\overline{\mathcal{K}}^*_{q,\vartheta}$ are both Fredholm operators with index 0 and we have the identities*

$$\ker(\overline{\mathcal{K}}_{q,\vartheta}) \quad = \quad \mathrm{span}\{\pi_\perp \Phi_0^+\}, \qquad \ker(\overline{\mathcal{K}}^*_{q,\vartheta}) \quad = \quad \mathrm{span}\{\pi_\perp \Phi_0^-\}. \tag{4.4.1}$$

Moreover, recalling the constant $\tilde{\lambda}$ appearing in (HW2$_{\overline{r}}$), the operator $\overline{\mathcal{K}}_{q,\vartheta} + \lambda$ is invertible for each $\lambda \in \mathbb{C}$ that has $\mathrm{Re}\,\lambda \geq -\tilde{\lambda}$ and $\lambda \notin 2\pi i \overline{c}_0 q^{-1}\mathbb{Z}$. Finally, there exists *constants $C > 0$ and $\delta_0 > 0$ so that for each $0 < \delta < \delta_0$ and each $\Theta \in L^2(\mathbb{R}, \ell^2_{q,\perp;\infty})$ we have the bound*

$$\|[\overline{\mathcal{K}}_{q,\vartheta} + \delta]^{-1}\Theta\|_{H^1(\mathbb{R},\ell^2_{q,\perp;\infty})} \quad \leq \quad C\Big[\|\Theta\|_{L^2(\mathbb{R},\ell^2_{q,\perp;\infty})} + \tfrac{1}{\delta}|\langle\Theta, \pi_\perp\Phi_0^-\rangle_{L^2(\mathbb{R},\ell^2_{q,\perp;\infty})}|\Big]. \tag{4.4.2}$$

The first step towards proving Proposition 4.4.1 is to find the eigenvalues of the operator $\overline{\mathcal{K}}_{q,\vartheta}$. After that, we will focus on the essential spectrum of this operator. The idea behind the proof of Lemma 4.4.2 below can best be illustrated by considering the case $q = 2$. In this case, we have $\vartheta = \frac{1}{2}$, together with

$$\big[T_{2,\frac{1}{2}}\Theta\big](\zeta, \xi) \quad = \quad \Theta\big(\zeta + \tfrac{1}{2}, \xi + 1\big). \tag{4.4.3}$$

Upon writing

$$[\Pi_0\Theta](\xi) \quad := \quad \Theta(0, \xi) + \Theta(\tfrac{1}{2}, \xi), \qquad [\Pi_1\Theta](\xi) \quad := \quad \Theta(0, \xi) - \Theta(\tfrac{1}{2}, \xi), \tag{4.4.4}$$

one may verify the commutation relations

$$\big[T_0\Pi_0\Theta\big](\xi) \quad = \quad \big[\Pi_0 T_{2,\frac{1}{2}}\Theta\big](\xi), \qquad \big[T_0\Pi_1\Theta\big](\xi) \quad = \quad -\big[\Pi_1 T_{2,\frac{1}{2}}\Theta\big](\xi). \tag{4.4.5}$$

In particular, if Θ is in the kernel of $\overline{\mathcal{K}}_{2,\frac{1}{2}} + \lambda$, the functions

$$X_0(\xi) \quad = \quad [\Pi_0\Theta](\xi), \qquad X_1(\xi) \quad = \quad e^{-\pi i\xi}[\Pi_1\Theta](\xi) \tag{4.4.6}$$

are eigenfunctions of the operator L_0 with eigenvalues $-\lambda$ and $-\lambda - \overline{c}_0\pi i$ respectively. Since $-\lambda$ and $-\lambda - \overline{c}_0\pi i$ cannot both be eigenvalues of L_0 at the same time in view of (HW2$_{\overline{r}}$), this means that at least one of the functions X_0 or X_1 is identically 0.

Without loss, we assume that $X_0 = 0$. In this case, the function Θ can explicitly be identified as

$$\Theta(0, \xi) \quad = \quad \tfrac{1}{2}e^{\pi i\xi}X_1(\xi), \qquad \Theta(\tfrac{1}{2}, \xi) \quad = \quad -\tfrac{1}{2}e^{\pi i\xi}X_1(\xi). \tag{4.4.7}$$

As such, the eigenfunctions of $\overline{\mathcal{K}}_{q,\vartheta}$ can be expressed in terms of those of L_0, thus providing an upper bound on the dimension of the corresponding eigenspace.

Lemma 4.4.2. *Consider the setting of Proposition 4.4.1. Then for any $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq -\tilde{\lambda}$ and $\lambda \notin \overline{c}_0 2\pi i q^{-1}\mathbb{Z}$, we have the identity*

$$\ker(\overline{\mathcal{K}}_{q,\vartheta} + \lambda) \quad = \quad \{0\}. \tag{4.4.8}$$

In addition, we have the identity

$$\ker(\overline{\mathcal{K}}_{q,\vartheta}) \quad = \quad \operatorname{span}\{\pi_\perp\Phi_0^+\}. \tag{4.4.9}$$

Proof. Fix $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq -\tilde{\lambda}$. Suppose that Θ is in the kernel of the operator $\overline{\mathcal{K}}_{q,\vartheta} + \lambda$. For $n \in \{0, ..., q-1\}$ we set

$$[\Pi_n\Theta](\xi) \quad = \quad \sum_{n'=0}^{q-1} \zeta_q^{n\cdot n'}\Theta\big(n'\vartheta, \xi\big), \tag{4.4.10}$$

together with

$$X_n(\xi) = e^{-\frac{2\pi i n}{q}\xi}[\Pi_n \Theta](\xi) = \zeta_q^{-n\xi}[\Pi_n \Theta](\xi), \tag{4.4.11}$$

with $\zeta_q = \exp[2\pi i/q]$ the q-th root of unity. Recalling that $\gcd(\vartheta q, q) = 1$, it follows that this sum contains each of the functions $\Theta\big(0, \xi\big), ..., \Theta\big((q-1)q^{-1}, \xi\big)$ exactly once. Recalling the definitions of the operators T_0 and $T_{q,\vartheta}$ from (4.2.15) and (4.3.37), we can compute

$$\begin{array}{lcl} [T_0 \Pi_n \Theta](\xi) & = & [\Pi_n \Theta](\xi + 1) \\ & = & \sum_{n'=0}^{q-1} \zeta_q^{nn'} \Theta\big(n'\vartheta, \xi + 1\big) \\ & = & \sum_{n'=0}^{q-1} \zeta_q^{nn'} (T_{q,\vartheta}\Theta)\big((n'-1)\vartheta, \xi\big) \\ & = & \zeta_q^n \sum_{n'=0}^{q-1} \zeta_q^{n(n'-1)} (T_{q,\vartheta}\Theta)\big((n'-1)\vartheta, \xi\big) \\ & = & \zeta_q^n [\Pi_n T_{q,\vartheta}\Theta](\xi), \end{array} \tag{4.4.12}$$

which implies

$$\begin{array}{lcl} T_0 X_n(\xi) & = & \zeta_q^{-n(\xi+1)}[T_0 \Pi_n \Theta](\xi + 1) \\ & = & \zeta_q^{-n(\xi+1)} \zeta_q^n [\Pi_n T_{q,\vartheta}\Theta](\xi) \\ & = & \zeta_q^{-n\xi}[\Pi_n T_{q,\vartheta}\Theta](\xi). \end{array} \tag{4.4.13}$$

This allows us to obtain the identity

$$\begin{array}{lcl} (L_0 + \lambda) X_n(\xi) & = & \overline{c}_0 X_n'(\xi) - \Delta_0 X_n(\xi) - D_U \mathcal{G}\big(\overline{U}_0(\xi); \overline{r}\big) X_n(\xi) + \lambda X_n(\xi) \\ & = & \overline{c}_0 \zeta_q^{-n\xi}[\Pi_n \Theta]'(\xi) - \overline{c}_0 \frac{2\pi i n}{q} X_n(\xi) - \zeta_q^{-n\xi}[\Pi_n \Delta_{q,\vartheta}\Theta](\xi) \\ & & -\zeta_q^{-n\xi} D_U \mathcal{G}\big(\overline{U}_0(\xi); \overline{r}\big)[\Pi_n \Theta](\xi) + \zeta_q^{-n\xi}\lambda[\Pi_n \Theta](\xi) \\ & = & \zeta_q^{-n\xi}\Big[\Pi_n\big(\overline{\mathcal{K}}_{q,\vartheta} + \lambda\big)\Theta\Big](\xi) - \overline{c}_0 \frac{2\pi i n}{q} X_n(\xi) \\ & = & -\overline{c}_0 \frac{2\pi i n}{q} X_n(\xi). \end{array} \tag{4.4.14}$$

Suppose first that $\lambda \notin 2\overline{c}_0 \pi i q^{-1}\mathbb{Z}$. Then it follows from (HW2$_{\overline{r}}$) that $-2\overline{c}_0 \pi i n q^{-1} - \lambda$ is no eigenvalue of L_0 for all $0 \le n \le q-1$. In particular, we must have $X_n = 0$ for all $0 \le n \le q-1$. This means that the functions $\Pi_n \Theta$ for $0 \le n \le q-1$ are also identically 0. Since the $q \times q$ Vandermonde matrix Z given by $Z_{n,n'} = \zeta_q^{n \cdot n'}$ is invertible, we obtain $\Theta(n\vartheta, \cdot) = 0$ for all $0 \le n \le q-1$ from which (4.4.8) follows.

Turning to the case $\lambda = 0$, we see that $-2\overline{c}_0 \pi i n q^{-1} - \lambda = -2\overline{c}_0 \pi i n q^{-1}$ can only be an eigenvalue of L_0 when $nq^{-1} \in \mathbb{Z}$ on account of (HW2$_{\overline{r}}$). Since $nq^{-1} \notin \mathbb{Z}$ for $1 \le n \le q-1$, we have $X_n = 0$ for those values of n. In addition, we have $X_0 = \mu\Phi_0^+$ for some $\mu \in \mathbb{C}$. Recalling the invertible matrix Z given by $Z_{n,n'} = \zeta_q^{n \cdot n'}$, we obtain the identity

$$\Big(\Theta\big(0, \cdot\big), \Theta\big(\vartheta, \cdot\big), ..., \Theta\big((q-1)\vartheta, \cdot\big)\Big)^T = Z^{-1}\big(\mu\Phi_0^+, 0, ..., 0\big)^T. \tag{4.4.15}$$

In particular, the kernel $\ker(\overline{\mathcal{K}}_{q,\vartheta})$ is one-dimensional. Since $L_0\Phi_0^+ = 0$ by (HW2$_{\overline{r}}$), it follows immediately that $\overline{\mathcal{K}}_{q,\vartheta}\pi_\perp\Phi_0^+ = 0$, which implies (4.4.9). ■

We now shift our attention to the Fredholm properties of $\overline{\mathcal{K}}_{q,\vartheta}$, which we aim to extract from those of L_0 in a similar fashion. The results in [68, 130] show that it suffices to consider the limiting operators

$$\begin{array}{lcl} \overline{\mathcal{K}}_{q,\vartheta,\pm\infty}\Theta & = & \overline{c}_0\partial_\xi\Theta - \Delta_{q,\vartheta}\Theta - D\mathcal{G}\big(P^\pm;\overline{r}\big)\Theta, \\ L_{\pm\infty}\Theta & = & \overline{c}_0\partial_\xi\Theta - \Delta_0\Theta - D\mathcal{G}\big(P^\pm;\overline{r}\big)\Theta, \end{array} \tag{4.4.16}$$

which have constant coefficients. For $\lambda \in \mathbb{C}$ and $0 \le \rho \le 1$ we introduce the notation

$$\begin{array}{lcl} \overline{\mathcal{K}}_{q,\vartheta,\rho;\lambda} & = & \rho\overline{\mathcal{K}}_{q,\vartheta,-\infty} + (1-\rho)\overline{\mathcal{K}}_{q,\vartheta,\infty} + \lambda, \\ L_{\rho;\lambda} & = & \rho L_{-\infty} + (1-\rho)L_\infty + \lambda. \end{array} \tag{4.4.17}$$

We set out to show that for λ in a suitable right half-plane and $0 \le \rho \le 1$, the operators $\overline{\mathcal{K}}_{q,\vartheta,\rho;\lambda}$ and $L_{\rho;\lambda}$ are hyperbolic in the sense of [68, 130]. In particular, we write

$$\Delta_{q,\vartheta,\rho;\lambda}(z) \;=\; \Big[\overline{\mathcal{K}}_{q,\vartheta,\rho;\lambda}e^{z\xi}\Big](0), \qquad \Delta_{\rho;\lambda}(z) \;=\; \Big[L_{\rho;\lambda}e^{z\xi}\Big](0) \tag{4.4.18}$$

and establish that $\det\big(\Delta_{q,\vartheta,\rho;\lambda}(iy)\big) \neq 0$ for all $y \in \mathbb{R}$ by first showing that $\det\big(\Delta_{\rho;\lambda}(iy)\big) \neq 0$. We can subsequently use the spectral flow principle to compute the Fredholm index of $\overline{\mathcal{K}}_{q,\vartheta} + \lambda$.

We start by considering the characteristic function $\Delta_{\rho;\lambda}$ from (4.4.18). For notational convenience we set

$$D\mathcal{G}_\rho \;=\; \rho D\mathcal{G}\big(P^-;\overline{r}\big) + (1-\rho)D\mathcal{G}\big(P^+;\overline{r}\big) \tag{4.4.19}$$

for $0 \le \rho \le 1$ and use the definition (4.2.4) to write

$$\begin{array}{lcl} \Delta_{\rho;\lambda}(iy) & = & \overline{c}_0 iy - \tau\sum_{m>0}\alpha_m\Big[e^{miy} + e^{-miy} - 2\Big] - D\mathcal{G}_\rho + \lambda \\ & = & \overline{c}_0 iy + \tau\sum_{m>0}\alpha_m\Big[2 - 2\cos(my)\Big] - D\mathcal{G}_\rho + \lambda \\ & = & \overline{c}_0 iy + 2\tau A(y) - D\mathcal{G}_\rho + \lambda. \end{array} \tag{4.4.20}$$

For any $V = (v_1, ..., v_d) \in \mathbb{C}^d$ we may exploit the inequality (4.2.4) to obtain

$$\tau V^\dagger A(y)V \;=\; 2\tau\sum_{j=1}^{d}|v_j|^2 A_j(y) \;\ge\; 0. \tag{4.4.21}$$

Here we introduced † for the conjugate transpose.

In order to prove that $L_{\pm\infty}+\lambda$ is hyperbolic, we need to distinguish between the setting where the triplet $(\mathcal{G}, P^-, P^+)$ satisfies (HS3$_{\overline{r}}$(a)) and where it satisfies (HS3$_{\overline{r}}$(b)). A similar computation was performed in [151, Lem. 4.6].

Lemma 4.4.3. *Assume that (HS1) and (HS2) are satisfied and pick $\overline{r}$ in such a way that (HW1$_{\overline{r}}$) and (HW2$_{\overline{r}}$) are satisfied. Assume that the triplet $(\mathcal{G}, P^-, P^+)$ satisfies (HS3$_{\overline{r}}$(a)). Pick $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > -\tilde{\lambda}$ and $0 \le \rho \le 1$. Then we have $\det\big(\Delta_{\rho;\lambda}(iy)\big) \neq 0$ for all $y \in \mathbb{R}$.*

Proof. For fixed $y \in \mathbb{R}$ we introduce the matrix

$$\begin{array}{lcl} X & = & \frac{1}{2}\big[\Delta_{\rho;\lambda}(iy) + \Delta_{\rho;\lambda}(iy)^{\dagger}\big] \\ & = & \tau A(y) - D\mathcal{G}_\rho - D\mathcal{G}_\rho^T + \mathrm{Re}\,\lambda. \end{array} \tag{4.4.22}$$

By decreasing $\tilde{\lambda}$ if necessary, we can assume that $-D\mathcal{G}_\rho - D\mathcal{G}_\rho^T + \mathrm{Re}\,\lambda$ is positive definite. It follows that X is the sum of a positive semi-definite matrix and a positive definite matrix and as such, it is positive definite itself. As a consequence, $\Delta_{\rho;\lambda}$ is positive definite as well and hence we obtain $\det\big(\Delta_{\rho;\lambda}(iy)\big) \neq 0$. ∎

Lemma 4.4.4. *Assume that (HS1) and (HS2) are satisfied and pick $\overline{r}$ in such a way that (HW1$_{\overline{r}}$) and (HW2$_{\overline{r}}$) are satisfied. Assume that the triplet $(\mathcal{G}, P^-, P^+)$ satisfies (HS3$_{\overline{r}}$(b)). Pick $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > -\tilde{\lambda}$ and $0 \le \rho \le 1$. Then we have $\det\big(\Delta_{\rho;\lambda}(iy)\big) \neq 0$ for all $y \in \mathbb{R}$.*

Proof. We recall the proportionality constant $\Gamma > 0$ from (HS3$_{\overline{r}}$(b)). In particular, upon writing

$$D\mathcal{G}_\rho \;=\; \begin{pmatrix} D\mathcal{G}_\rho^{[1,1]} & D\mathcal{G}_\rho^{[1,2]} \\ D\mathcal{G}_\rho^{[2,1]} & D\mathcal{G}_\rho^{[2,2]} \end{pmatrix}, \tag{4.4.23}$$

we have $D\mathcal{G}_\rho^{[1,2]} = -\Gamma(D\mathcal{G}_\rho^{[2,1]})^T$. Suppose that $\Delta_{\rho;\lambda}(iy)V = 0$ for some $V \in \mathbb{C}^d$. Write $V = (u, w)$ where u contains the first d_{diff} components of V. Then we can compute

$$\begin{array}{lcl} 0 & = & \mathrm{Re}\,V^\dagger \Delta_{\rho;\lambda}(iy)V \\ & = & \mathrm{Re}\Big[-\tau V^\dagger A(y)V - V^\dagger D\mathcal{G}_\rho V + \lambda|V|^2\Big] \\ & = & \mathrm{Re}\Big[-\tau V^\dagger A(y)V - u^\dagger D\mathcal{G}_\rho^{[1,1]}u - u^\dagger D\mathcal{G}_\rho^{[1,2]}w \\ & & \qquad -w^\dagger D\mathcal{G}_\rho^{[2,1]}u - w^\dagger D\mathcal{G}_\rho^{[2,2]}w + \lambda|u|^2 + \lambda|w|^2\Big]. \end{array} \tag{4.4.24}$$

The second component of the equation $\Delta_{\rho;\lambda}(iy)V = 0$ is equivalent to

$$D\mathcal{G}_\rho^{[2,1]}u \;=\; -D\mathcal{G}_\rho^{[2,2]}w + \lambda w. \tag{4.4.25}$$

As such, we can rewrite the cross-terms in (4.4.24) to obtain

$$\begin{array}{lcl} \mathrm{Re}\big[-u^\dagger D\mathcal{G}_\rho^{[1,2]}w - w^\dagger D\mathcal{G}_\rho^{[2,1]}u\big] & = & \mathrm{Re}\,(1-\Gamma)\Big[-w^\dagger D\mathcal{G}_\rho^{[2,1]}u\Big] \\ & = & \mathrm{Re}\,(\Gamma-1)\Big[-w^\dagger D\mathcal{G}_\rho^{[2,2]}w + \lambda|w|^2\Big]. \end{array} \tag{4.4.26}$$

As a consequence, (4.4.24) reduces to

$$0 = \mathrm{Re}\Big[-\tau V^\dagger A(y) V - u^\dagger D\mathcal{G}_\rho^{[1,1]} u + \lambda |u|^2 - \Gamma w^\dagger D\mathcal{G}_\rho^{[2,2]} w + \Gamma\lambda |w|^2 \Big]. \quad (4.4.27)$$

By decreasing $\tilde{\lambda}$ if necessary, we can assume that $-D\mathcal{G}_\rho^{[1,1]} + \mathrm{Re}\,\lambda$ and $-\Gamma D\mathcal{G}_\rho^{[2,2]} + \Gamma\mathrm{Re}\,\lambda$ are positive definite. Therefore, we must have $V = 0$, from which it follows that $\det\big(\Delta_{\rho;\lambda}(iy)\big) \neq 0$. ■

Lemma 4.4.5. *Consider the setting of Proposition 4.4.1. Pick $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > -\tilde{\lambda}$ and $0 \le \rho \le 1$. Then we have $\det\big(\Delta_{q,\vartheta,\rho;\lambda}(iy)\big) \neq 0$ for all $y \in \mathbb{R}$.*

Proof. Suppose there exists $V \in \ell^2_{q,\perp;\infty}$ and $y \in \mathbb{R}$ for which

$$\Delta_{q,\vartheta,\rho;\lambda}(iy) V = 0. \quad (4.4.28)$$

We then write

$$W\big(\tfrac{n}{q}, \xi\big) = e^{iy\xi} V\big(\tfrac{n}{q}\big) \quad (4.4.29)$$

for $0 \le n \le q-1$. The definition of the characteristic function yields

$$\begin{aligned} \overline{\mathcal{K}}_{q,\vartheta,\rho;\lambda} W &= e^{iy\xi}\left[\overline{\mathcal{K}}_{q,\vartheta,\rho;\lambda} e^{iy\xi} V\right](0) \\ &= e^{iy\xi} \Delta_{q,\vartheta,\rho;\lambda}(iy) V \\ &= 0. \end{aligned} \quad (4.4.30)$$

Recalling the projections (4.4.10), we write

$$X_n(\xi) = e^{-\frac{2\pi i n}{q}\xi} [\Pi_n W](\xi) \quad (4.4.31)$$

and use a computation similar to (4.4.14) to find

$$\begin{aligned} L_{\rho;\lambda} X_n(\xi) &= e^{-\frac{2\pi i n}{q}\xi}\left[\Pi_n \overline{\mathcal{K}}_{q,\vartheta,\rho;\lambda} W\right](\xi) - \overline{c}_0 \tfrac{2\pi i n}{q} X_n(\xi) \\ &= -\overline{c}_0 \tfrac{2\pi i n}{q} X_n(\xi). \end{aligned} \quad (4.4.32)$$

On account of Lemmas 4.4.3-4.4.4, it follows from the spectral flow theorem [68, Thm. 1.6] and [68, Thm. 1.7] that $L_{\rho;\lambda - \overline{c}_0 2\pi i n q^{-1}}$ is hyperbolic. Applying [150, Lem. 6.3], which is a generalization of [130, Thm. 4.1], yields that $L_{\rho;\lambda - \overline{c}_0 2\pi i n \vartheta}$ is invertible as a map from $W^{1,\infty}(\mathbb{R};\mathbb{R}^d)$ to $L^\infty(\mathbb{R};\mathbb{R}^d)$. Therefore, we must have $X_n = 0$ for all $0 \le n \le q-1$. This implies that $W(\frac{n}{q}, \xi) = 0$ for all $0 \le n \le q-1$ and thus that $V = 0$, which yields the desired result. ■

Proof of Proposition 4.4.1. These results, except the bound (4.4.2), follow from combining Lemma 4.4.2, Lemma 4.4.5 and the spectral flow theorem [68, Thm. 1.6-1.7]. The bound (4.4.2) can be obtained by following the proof of [6, Lem. 3.1]. ■

4.5 Linear theory for $\Delta t \to 0$

In this section, we apply the spectral convergence method to lift the Fredholm properties of the semi-discrete system to the fully discrete system in the small timestep regime $\Delta t \ll 1$. In particular, we establish the main result below, which gives a quasi-inverse for the operators $L_{k,M}$. This turns out to be the key ingredient in the construction of the discrete waves, which can subsequently be proved by means of a standard fixed point argument.

Proposition 4.5.1 (cf. [111, Prop. 3.2]). *Assume that (HS1) and (HS2) are satisfied and pick $\overline{r}$ in such a way that (HS3$_{\overline{r}}$), (HW1$_{\overline{r}}$) and (HW2$_{\overline{r}}$) are satisfied. Fix a pair of integers $1 \le k \le 6$ and $q \ge 1$, together with a sufficiently small $\eta > 0$ and sufficiently large constants $M_* \in \mathcal{M}_q$ and $C > 0$. Then for each $M \in \mathcal{M}_q$ with $M \ge M_*$ there exist linear maps*

$$\gamma^*_{k,M} : \mathcal{Y}_M \to \mathbb{R}, \qquad \mathcal{V}^*_{k,M} : \mathcal{Y}_M \to \mathcal{Y}^1_{k,M}, \tag{4.5.1}$$

so that for all $\Psi \in \mathcal{Y}_M$ the pair

$$(\gamma, V) = (\gamma^*_{k,M}\Psi, \mathcal{V}^*_{k,M}\Psi) \tag{4.5.2}$$

is the unique solution to the problem

$$L_{k,M}V = \Psi + \gamma \pi_{\mathcal{Y}_M} \mathcal{D}_{k,M}\overline{U}_0 \tag{4.5.3}$$

that satisfies the normalisation condition

$$\langle \pi_{\mathcal{Y}_M}\Phi_0^-, V\rangle_{\mathcal{Y}_M} = 0. \tag{4.5.4}$$

In addition, for all $\Psi \in \mathcal{Y}_M$ we have the bound

$$|\gamma^*_{k,M}\Psi| + \|\mathcal{V}^*_{k,M}\Psi\|_{\mathcal{Y}^1_{k,M}} \le C\|\Psi\|_{\mathcal{Y}_M}. \tag{4.5.5}$$

In order to facilitate the reading, we first outline our strategy and formulate two intermediate results in §4.5.1. This strategy heavily follows the program in [111], allowing us to simply refer to these results in many cases. However, due to the lack of a comparison principle and the many cross-terms we need to control, there are several key points in the analysis that need a fully new approach, which we develop in §4.5.2. In addition, the infinite-range setting forces us to obtain an extra order of regularity on the operator $(L_0 + \delta)^{-1}$, which we achieve in §4.5.3.

4.5.1 Strategy

Recalling the spaces $\mathcal{H}_M$ and $\mathcal{H}^1_{k,M}$ from (4.3.12) and (4.3.15), we introduce the quantities

$$\begin{aligned} \mathcal{E}_{k,M}(\delta) &= \inf_{\|\Phi\|_{\mathcal{H}^1_{k,M}}=1} \Big[\|\mathcal{K}_{k,M}\Phi + \delta\Phi\|_{\mathcal{H}_M} + \delta^{-1}\Big|\langle \pi_{\mathcal{H}_M}\Phi_0^-, \mathcal{K}_{k,M}\Phi + \delta\Phi\rangle_{\mathcal{H}_M}\Big|\Big], \\ \mathcal{E}^*_{k,M}(\delta) &= \inf_{\|\Phi\|_{\mathcal{H}^1_{k,M}}=1} \Big[\|\mathcal{K}^*_{k,M}\Phi + \delta\Phi\|_{\mathcal{H}_M} + \delta^{-1}\Big|\langle \pi_{\mathcal{H}_M}\Phi_0^+, \mathcal{K}^*_{k,M}\Phi + \delta\Phi\rangle_{\mathcal{H}_M}\Big|\Big], \end{aligned} \tag{4.5.6}$$

together with

$$\begin{array}{rcl}\kappa(\delta) & = & \liminf_{M\to\infty, M\in\mathcal{M}_q} \mathcal{E}_{k,M}(\delta), \\ \kappa^*(\delta) & = & \liminf_{M\to\infty, M\in\mathcal{M}_q} \mathcal{E}^*_{k,M}(\delta)\end{array} \tag{4.5.7}$$

for $\delta \in (0, \delta_0)$.

The key step towards proving Proposition 4.5.1 is the establishment of lower bounds for these quantities. This procedure is based on [6, Lem. 3.2]. Our strategy to prove it is essentially the same, but some major modifications are needed to incorporate the difficulties arising from the discrete derivatives.

Proposition 4.5.2 (cf. [111, Prop. 3.7]). *Assume that (HS1) and (HS2) are satisfied and pick $\overline{r}$ in such a way that (HS3$_{\overline{r}}$), (HW1$_{\overline{r}}$) and (HW2$_{\overline{r}}$) are satisfied. Fix a pair of integers $1 \le k \le 6$ and $q \ge 1$. Then there exists $\kappa > 0$ such that for all $0 < \delta < \delta_0$ we have*

$$\kappa(\delta) \;\ge\; \kappa, \qquad \kappa^*(\delta) \;\ge\; \kappa. \tag{4.5.8}$$

We are now ready to start our interpolation procedure. For any $\xi \in \mathbb{R}$, we pick two quantities $\xi^{\pm}_M(\xi) \in M^{-1}\mathbb{Z}$ in such a way that

$$\xi^-_M(\xi) \;\le\; \xi \;<\; \xi^+_M(\xi), \qquad \xi^+_M(\xi) - \xi^-_M(\xi) \;=\; M^{-1}. \tag{4.5.9}$$

Using these quantities, we can define two interpolation operators

$$\begin{array}{rcccl}\mathcal{I}^0_M & : & \mathcal{H}_M & \to & L^2(\mathbb{R}, \ell^2_{q,\perp;\infty}), \\ \mathcal{I}^1_{k,M} & : & \mathcal{H}^1_{k,M} & \to & H^1(\mathbb{R}, \ell^2_{q,\perp;\infty}),\end{array} \tag{4.5.10}$$

that act as

$$\begin{array}{rcl}[\mathcal{I}^0_M\phi](\zeta,\xi) & = & \phi\Big(\zeta, \xi^-_M(\xi)\Big), \\ [\mathcal{I}^1_{k,M}\phi](\zeta,\xi) & = & M\Big[(\xi - \xi^-_M(\xi))\phi\Big(\zeta, \xi^+_M(\xi)\Big) + (\xi^+_M(\xi) - \xi)\phi\Big(\zeta, \xi^-_M(\xi)\Big)\Big],\end{array} \tag{4.5.11}$$

for all $\zeta \in q^{-1}\mathbb{Z}_q$ and all $\xi \in \mathbb{R}$. These operators can be seen as interpolations of order zero and one respectively, both acting only on the second coordinate of ϕ. We refer to [111, Lem. 3.10-3.12] for some useful estimates involving these interpolations.

With these preparations in hand, we start the proof of Proposition 4.5.2 using the methods described in the proof of [6, Lem. 3.2]. We focus on the quantity $\kappa(\delta)$ defined in (4.5.7), noting that $\kappa^*(\delta)$ can be treated in a similar fashion. In particular, we find a lower bound for $\kappa(\delta)$ by constructing sequences that minimize this quantity. At this point it becomes clear why we work on the spaces $H^1(\mathbb{R}, \ell^2_{q,\perp})$ and $L^2(\mathbb{R}, \ell^2_{q,\perp})$, as we exploit the fact that bounded closed subsets of these spaces are weakly compact.

Lemma 4.5.3 (cf. [111, Lem. 3.16-3.17]). *Assume that (HS1) and (HS2) are satisfied and pick $\overline{r}$ in such a way that (HS3$_{\overline{r}}$), (HW1$_{\overline{r}}$) and (HW2$_{\overline{r}}$) are satisfied. Fix a pair of integers $1 \le k \le 6$ and $q \ge 1$, as well as $0 < \delta < \delta_0$. Then there exist two functions*

$$\Phi_* \;\in\; H^1(\mathbb{R}, \ell^2_{q,\perp;\infty}), \qquad \Psi_* \;\in\; L^2(\mathbb{R}, \ell^2_{q,\perp;\infty}), \tag{4.5.12}$$

together with three sequences

$$\{M_j\}_{j\in\mathbb{N}} \subset \mathcal{M}_q, \; \{\Phi_j\}_{j\in\mathbb{N}} \subset \mathcal{H}^1_{k,M_j}, \; \{\Psi_j\}_{j\in\mathbb{N}} \subset \mathcal{H}_{M_j} \tag{4.5.13}$$

and two constants $\vartheta \in q^{-1}\mathbb{Z}_q \setminus \{0\}$ *and* $K_1 > 0$ *that satisfy the following properties.*

(i) We have $\lim_{j\to\infty} M_j = \infty$ *and* $\|\Phi_j\|_{\mathcal{H}^1_{k,M_j}} = 1$ *for all* $j \in \mathbb{N}$.

(ii) The identity

$$\Psi_j = \mathcal{K}_{k,M_j}\Phi_j + \delta\Phi_j \tag{4.5.14}$$

holds for all $j \in \mathbb{N}$.

(iii) Recalling the constant $\kappa(\delta)$ *defined in (4.5.7), we have the limit*

$$\kappa(\delta) = \lim_{j\to\infty} \Big[\|\mathcal{K}_{k,M}\Phi_j + \delta\Phi_j\|_{\mathcal{H}_{M_j}} + \delta^{-1}\Big|\langle \pi_{\mathcal{H}_{M_j}}\Phi_0^-, \mathcal{K}_{k,M_j}\Phi_j + \delta\Phi_j\rangle_{\mathcal{H}_{M_j}}\Big|\Big]. \tag{4.5.15}$$

(iv) As $j \to \infty$, *we have the weak convergences*

$$\begin{aligned} \mathcal{I}^1_{k,M_j}\Phi_j &\rightharpoonup \Phi_* \in H^1(\mathbb{R}, \ell^2_{q,\perp}), \\ \mathcal{I}^0_{M_j}\Psi_j &\rightharpoonup \Psi_* \in L^2(\mathbb{R}, \ell^2_{q,\perp}). \end{aligned} \tag{4.5.16}$$

(v) For any compact interval $\mathcal{I} \subset \mathbb{R}$, *we have the strong convergences*

$$\begin{aligned} (\mathcal{I}^1_{k,M_j}, \mathcal{I}^1_{k,M_j})\Phi_j &\to \Phi_* \in L^2(\mathcal{I}, \ell^2_{q,\perp}), \\ (\mathcal{I}^0_{M_j}, \mathcal{I}^0_{M_j})\Psi_j &\to \Psi_* \in L^2(\mathcal{I}, \ell^2_{q,\perp}) \end{aligned} \tag{4.5.17}$$

as $j \to \infty$.

(vi) The function Φ_* *is a weak solution to* $(\overline{\mathcal{K}}_{q,\vartheta} + \delta)\Phi_* = \Psi_*$ *and we have the bound*

$$\|\Phi_*\|_{H^1(\mathbb{R},\ell^2_{q,\perp;\infty})} \le K_1\kappa(\delta). \tag{4.5.18}$$

Proof. In view of Proposition 4.4.1 and Lemma 4.A.6, we can follow the proof of [111, Lem. 3.16-3.17] almost verbatim. ∎

In order to prove Proposition 4.5.2, we need to establish a lower bound on the norm $\|\Phi_*\|_{H^1(\mathbb{R},\ell^2_{q,\perp;\infty})}$ on account of (4.5.18). In Proposition 4.5.4 we follow the approach of [111, Lem. 3.18] in order to obtain this lower bound. Here we have to deal with both the cross-terms arising from the system setting as well as the infinite-range interactions.

Proposition 4.5.4 (see §4.5.2)**.** *Consider the setting of Lemma 4.5.3. Then there exist constants* $K_2 > 1$ *and* $K_3 > 1$ *so that for any* $0 < \delta < \delta_0$, *the function* Φ_* *satisfies the bound*

$$\|\Phi_*\|^2_{H^1(\mathbb{R},\ell^2_{q,\perp;\infty})} \ge K_2 - K_3\kappa(\delta)^2. \tag{4.5.19}$$

Proof of Proposition 4.5.2. Combining the bounds (4.5.18) and (4.5.19) immediately yields

$$K_2 - K_3\kappa(\delta)^2 \quad \leq \quad K_1^2\kappa(\delta)^2. \tag{4.5.20}$$

Solving this quadratic inequality, we obtain

$$\kappa(\delta) \quad \geq \quad \sqrt{\tfrac{K_2}{K_1^2+K_3}} \quad := \quad \kappa. \tag{4.5.21}$$

The lower bound on $\kappa^*(\delta)$ follows in a similar fashion. ∎

In order to establish Proposition 4.5.1, we need more control on the operator L_0 than in [150]. In particular, due to the infinite-range interactions it is not immediately clear that this operator preserves the exponential decay properties of the function spaces (4.3.2).

Proposition 4.5.5. *Assume that (HS1) and (HS2) are satisfied and pick $\overline{r}$ in such a way that (HS3$_{\overline{r}}$), (HW1$_{\overline{r}}$) and (HW2$_{\overline{r}}$) are satisfied. Fix a sufficiently small constant $\eta > 0$. Then there exist constants $\delta_* > 0$ and $K > 0$, so that for each $0 < \delta < \delta_*$ and each $G \in BC^1_{-\eta}(\mathbb{R};\mathbb{R}^d)$ we have the bounds*

$$\begin{array}{lcl} \|(L_0+\delta)^{-1}G\|_{BC_{-\eta}(\mathbb{R};\mathbb{R}^d)} & \leq & K\delta^{-1}\|G\|_{BC_{-\eta}(\mathbb{R};\mathbb{R}^d)} \\ \|[(L_0+\delta)^{-1}G]'\|_{BC_{-\eta}(\mathbb{R};\mathbb{R}^d)} & \leq & K\delta^{-1}\|G\|_{BC_{-\eta}(\mathbb{R};\mathbb{R}^d)} \\ \|[(L_0+\delta)^{-1}G]''\|_{BC_{-\eta}(\mathbb{R};\mathbb{R}^d)} & \leq & K\delta^{-1}\|G\|_{BC^1_{-\eta}(\mathbb{R};\mathbb{R}^d)}. \end{array} \tag{4.5.22}$$

Proof of Proposition 4.5.1. On account of Proposition 4.5.5, we can follow the procedure developed in [111, §3.3] to arrive at the desired result. ∎

4.5.2 Spectral convergence

In this section we set out to prove Proposition 4.5.4 using the spectral convergence method. The main idea is to derive an upper bound for the discrete derivative $\mathcal{D}_{k,M_j}\Phi_j$, together with a lower bound for Φ_j restricted to a large—but finite—interval. This prevents the $\mathcal{H}^1_{k,M_j}$-norm of Φ_j from leaking away into oscillations or tail effects, providing the desired control on the limit (4.5.17). All constants introduced in Lemmas 4.5.6-4.5.8 and Proposition 4.5.4 are independent of $0 < \delta < \delta_0$.

Lemma 4.5.6. *Consider the setting of Lemma 4.5.3. Then there exists a constant $C_1 > 0$ so that the bound*

$$2\|\Psi_j\|^2_{\mathcal{H}_{M_j}} + 2C_1\|\Phi_j\|^2_{\mathcal{H}_{M_j}} \quad \geq \quad \overline{c}_0^2\|\mathcal{D}_{k,M_j}\Phi_j\|^2_{\mathcal{H}_{M_j}} \tag{4.5.23}$$

holds for all $j \in \mathbb{N}$.

Proof. We will assume $\overline{c}_0 > 0$, noting that the case where $\overline{c}_0 < 0$ can be treated in a similar fashion. In view of the identity

$$\mathcal{K}_{k,M_j}\Phi_j + \delta\Phi_j \quad = \quad \Psi_j, \tag{4.5.24}$$

we can compute

$$\begin{aligned}\langle \Psi_j, \mathcal{D}_{k,M_j}\Phi_j\rangle_{\mathcal{H}_{M_j}} &= \overline{c}_0 \|\mathcal{D}_{k,M_j}\Phi_j\|^2_{\mathcal{H}_{M_j}} - \langle \Delta_{M_j}\Phi_j, \mathcal{D}_{k,M_j}\Phi_j\rangle_{\mathcal{H}_{M_j}} \\ &\quad - \langle D\mathcal{G}\big(\pi_{\mathcal{H}_{M_j}}\overline{U}_0;\overline{r}\big)\Phi_j, \mathcal{D}_{k,M_j}\Phi_j\rangle_{\mathcal{H}_{M_j}} + \delta\langle \Phi_j, \mathcal{D}_{k,M_j}\Phi_j\rangle_{\mathcal{H}_{M_j}}.\end{aligned} \tag{4.5.25}$$

Writing

$$K = \|D\mathcal{G}\big(\overline{U}_0;\overline{r}\big)\|_\infty + 4\tau\sum_{m>0}|\alpha_m| \tag{4.5.26}$$

and remembering that $0 < \delta < \delta_0 < 1$, we may use the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}K\|\Phi_j\|_{\mathcal{H}_{M_j}}\|\mathcal{D}_{k,M_j}\Phi_j\|_{\mathcal{H}_{M_j}} &\geq \langle \Delta_{M_j}\Phi_j, \mathcal{D}_{k,M_j}\Phi_j\rangle_{\mathcal{H}_{M_j}} \\ &\quad + \langle D\mathcal{G}\big(\pi_{\mathcal{H}_{M_j}}\overline{U}_0;\overline{r}\big)\Phi_j, \mathcal{D}_{k,M_j}\Phi_j\rangle_{\mathcal{H}_{M_j}} \\ &\quad - \delta\langle \Phi_j, \mathcal{D}_{k,M_j}\Phi_j\rangle_{\mathcal{H}_{M_j}} \\ &= \overline{c}_0\|\mathcal{D}_{k,M_j}\Phi_j\|^2_{\mathcal{H}_{M_j}} - \langle \Psi_j, \mathcal{D}_{k,M_j}\Phi_j\rangle_{\mathcal{H}_{M_j}} \\ &\geq \overline{c}_0\|\mathcal{D}_{k,M_j}\Phi_j\|^2_{\mathcal{H}_{M_j}} - \|\Psi_j\|_{\mathcal{H}_{M_j}}\|\mathcal{D}_{k,M_j}\Phi_j\|_{\mathcal{H}_{M_j}}.\end{aligned} \tag{4.5.27}$$

This yields the bound

$$\|\Psi_j\|_{\mathcal{H}_{M_j}} + K\|\Phi_j\|_{\mathcal{H}_{M_j}} \geq \overline{c}_0\|\mathcal{D}_{k,M_j}\Phi_j\|_{\mathcal{H}_{M_j}}. \tag{4.5.28}$$

Squaring this inequality gives the desired estimate (4.5.23). ∎

Lemma 4.5.7. *Consider the setting of Lemma 4.5.3 and assume that the triplet $(\mathcal{G}, P^-, P^+)$ satisfies (HS3$_{\overline{r}}$(a)). There exist positive constants μ, C_3, C_4 and C_5 so that the bound*

$$M_j^{-1}\sum_{\xi\in M_j^{-1}\mathbb{Z}:|\xi|\leq\mu}|\Phi_j(\cdot,\xi)|^2_{\ell^2_{q,\perp}} \geq C_3\|\Phi_j\|^2_{\mathcal{H}_{M_j}} - C_4\|\Psi_j\|^2_{\mathcal{H}_{M_j}} - C_5M_j^{-1}\|\mathcal{D}_{k,M_j}\Phi_j\|^2_{\mathcal{H}_{M_j}} \tag{4.5.29}$$

holds for all $j \in \mathbb{N}$.

Proof. Invoking Lemma 4.A.4 and Lemma 4.A.5, we can estimate

$$\begin{aligned}\langle \Psi_j, \Phi_j\rangle_{\mathcal{H}_{M_j}} &= \langle[\mathcal{K}_{k,M_j} + \delta]\Phi_j, \Phi_j\rangle_{\mathcal{H}_{M_j}} \\ &= \overline{c}_0\langle \mathcal{D}_{k,M_j}\Phi_j, \Phi_j\rangle_{\mathcal{H}_{M_j}} - \langle \Delta_{M_j}\Phi_j, \Phi_j\rangle_{\mathcal{H}_{M_j}} \\ &\quad - \langle D\mathcal{G}\big(\pi_{\mathcal{H}_{M_j}}\overline{U}_0;\overline{r}\big)\Phi_j, \Phi_j\rangle_{\mathcal{H}_{M_j}} + \delta\|\Phi_j\|^2_{\mathcal{H}_{M_j}} \\ &\geq \overline{c}_0\langle \mathcal{D}_{k,M_j}\Phi_j, \Phi_j\rangle_{\mathcal{H}_{M_j}} - \langle D\mathcal{G}\big(\pi_{\mathcal{H}_{M_j}}\overline{U}_0;\overline{r}\big)\Phi_j, \Phi_j\rangle_{\mathcal{H}_{M_j}} \\ &\geq -C_2M_j^{-1}\|\mathcal{D}_{k,M_j}\Phi_j\|^2_{\mathcal{H}_{M_j}} - \langle D\mathcal{G}\big(\pi_{\mathcal{H}_{M_j}}\overline{U}_0;\overline{r}\big)\Phi_j, \Phi_j\rangle_{\mathcal{H}_{M_j}}\end{aligned} \tag{4.5.30}$$

for some $C_2 > 1$. Since $-D\mathcal{G}\big(P^\pm;\overline{r}\big)$ is positive definite and $-D\mathcal{G}$ is continuous, we can choose $\mu > 0$ and $a > 0$ in such a way that the matrix

$$B(\xi) = -D\mathcal{G}\big(\overline{U}_0(\xi);\overline{r}\big) - a \tag{4.5.31}$$

is positive definite for all $|\xi| \geq \mu$. Using the definition of this matrix and writing

$$\mathcal{I} \;=\; (\|D\mathcal{G}\big(\overline{U}_0;\overline{r}\big)\|_\infty + a)M_j^{-1} \sum_{\xi \in M_j^{-1}\mathbb{Z}: |\xi| \leq \mu} |\Phi_j(\cdot,\xi)|^2_{\ell^2_{q,\perp}}, \tag{4.5.32}$$

we can estimate

$$\begin{array}{rcl} -\langle D\mathcal{G}\big(\pi_{\mathcal{H}_{M_j}}\overline{U}_0;\overline{r}\big)\Phi_j,\Phi_j\rangle_{\mathcal{H}_{M_j}} & = & a\|\Phi_j\|^2_{\mathcal{H}_{M_j}} - \langle B\Phi_j,\Phi_j\rangle_{\mathcal{H}_{M_j}} \\ & \geq & a\|\Phi_j\|^2_{\mathcal{H}_{M_j}} - M_j^{-1}\sum_{\xi\in M_j^{-1}\mathbb{Z}} |B(\xi)\Phi_j(\cdot,\xi)|^2_{\ell^2_{q,\perp}} \\ & \geq & a\|\Phi_j\|^2_{\mathcal{H}_{M_j}} - \mathcal{I}. \end{array} \tag{4.5.33}$$

In particular, we can combine (4.5.30) and (4.5.33) to obtain

$$\langle\Psi_j,\Phi_j\rangle_{\mathcal{H}_{M_j}} \;\geq\; a\|\Phi_j\|^2_{\mathcal{H}_{M_j}} - \mathcal{I} - C_2 M_j^{-1}\|\mathcal{D}_{k,M_j}\Phi_j\|^2_{\mathcal{H}_{M_j}}. \tag{4.5.34}$$

We can hence rearrange (4.5.34) and estimate

$$\begin{array}{rcl} \mathcal{I} & \geq & a\|\Phi_j\|^2_{\mathcal{H}_{M_j}} - C_2M_j^{-1}\|\mathcal{D}_{k,M_j}\Phi_j\|^2_{\mathcal{H}_{M_j}} - \langle\Psi_j,\Phi_j\rangle_{\mathcal{H}_{M_j}} \\ & \geq & \frac{a}{2}\|\Phi_j\|^2_{\mathcal{H}_{M_j}} - \frac{2}{a}\|\Psi_j\|^2_{\mathcal{H}_{M_j}} - C_2M_j^{-1}\|\mathcal{D}_{k,M_j}\Phi_j\|^2_{\mathcal{H}_{M_j}}, \end{array} \tag{4.5.35}$$

which yields the desired bound.

Lemma 4.5.8. *Consider the setting of Lemma 4.5.3 and assume that the triplet $(\mathcal{G}, P^-, P^+)$ satisfies (HS3$_{\overline{r}}$(b)). Then there exist positive constants μ, C_3, C_4 and C_5 so that the bound*

$$\begin{array}{rcl} M_j^{-1}\sum_{\xi\in M_j^{-1}\mathbb{Z}:|\xi|\leq\mu} |\Phi_j(\cdot,\xi)|^2_{\ell^2_{q,\perp}} & \geq & C_3\|\Phi_j\|^2_{\mathcal{H}_{M_j}} - C_4\|\Psi_j\|^2_{\mathcal{H}_{M_j}} \\ & & -C_5M_j^{-1}\|\mathcal{D}_{k,M_j}\Phi_j\|^2_{\mathcal{H}_{M_j}} \end{array} \tag{4.5.36}$$

holds for all $j \in \mathbb{N}$.

Proof. Recall the proportionality constant $\Gamma > 0$ from (HS3$_{\overline{r}}$(b)). In particular, upon writing

$$D\mathcal{G} \;=\; \begin{pmatrix} D\mathcal{G}^{[1,1]} & D\mathcal{G}^{[1,2]} \\ D\mathcal{G}^{[2,1]} & D\mathcal{G}^{[2,2]} \end{pmatrix}, \tag{4.5.37}$$

we have $D\mathcal{G}^{[1,2]} = -\Gamma(D\mathcal{G}^{[2,1]})^T$. For each $M \in \mathcal{M}_q$, we introduce the decomposition

$$\mathcal{H}_M \;=\; \mathcal{H}_M^{[1]} \times \mathcal{H}_M^{[2]}, \tag{4.5.38}$$

which splits every $\Phi = (\phi,\theta) \in \mathcal{H}_M$ in such a way that $\phi \in \mathcal{H}_M^{[1]}$ contains the first d_{diff} components of Φ, while $\theta \in \mathcal{H}_M^{[2]}$ contains the other $d - d_{\mathrm{diff}}$ components. For each $j \geq 0$ we write $\Phi_j = (\phi_j,\theta_j)$ and $\Psi_j = (\psi_j,\chi_j)$ with $\phi_j,\psi_j \in \mathcal{H}_{M_j}^{[1]}$ and $\theta_j,\chi_j \in \mathcal{H}_{M_j}^{[2]}$.

Using this decomposition, we can expand the inner product as

$$\begin{aligned} -\langle D\mathcal{G}\big(\pi_{\mathcal{H}_{M_j}}\overline{U}_0;\overline{r}\big)\Phi_j,\Phi_j\rangle_{\mathcal{H}_{M_j}} &= -\langle D\mathcal{G}^{[1,1]}(\pi_{\mathcal{H}_{M_j}}\overline{U}_0)\phi_j,\phi_j\rangle_{\mathcal{H}^{[1]}_{M_j}} + \mathcal{C} \\ &\quad -\langle D\mathcal{G}^{[2,2]}(\pi_{\mathcal{H}_{M_j}}\overline{U}_0)\theta_j,\theta_j\rangle_{\mathcal{H}^{[2]}_{M_j}}, \end{aligned} \tag{4.5.39}$$

where we have introduced the cross-terms

$$\mathcal{C} := -\langle D\mathcal{G}^{[1,2]}(\pi_{\mathcal{H}_{M_j}}\overline{U}_0)\theta_j,\phi_j\rangle_{\mathcal{H}^{[1]}_{M_j}} - \langle D\mathcal{G}^{[2,1]}(\pi_{\mathcal{H}_{M_j}}\overline{U}_0)\phi_j,\theta_j\rangle_{\mathcal{H}^{[2]}_{M_j}}. \tag{4.5.40}$$

Recalling $D\mathcal{G}^{[1,2]} = -\Gamma(D\mathcal{G}^{[2,1]})^T$ and exploiting the identity

$$\chi_j = \overline{c}_0\mathcal{D}_{k,M_j}\theta_j - D\mathcal{G}^{[2,1]}(\pi_{\mathcal{H}_{M_j}}\overline{U}_0)\phi_j - D\mathcal{G}^{[2,2]}(\pi_{\mathcal{H}_{M_j}}\overline{U}_0)\theta_j + \delta\theta_j, \tag{4.5.41}$$

we can rewrite the cross-terms to obtain

$$\begin{aligned} \mathcal{C} &= -\langle D\mathcal{G}^{[1,2]}(\pi_{\mathcal{H}_{M_j}}\overline{U}_0)\theta_j,\phi_j\rangle_{\mathcal{H}^{[1]}_{M_j}} - \langle D\mathcal{G}^{[2,1]}(\pi_{\mathcal{H}_{M_j}}\overline{U}_0)\phi_j,\theta_j\rangle_{\mathcal{H}^{[2]}_{M_j}} \\ &= -(1-\Gamma)\langle D\mathcal{G}^{[2,1]}(\pi_{\mathcal{H}_{M_j}}\overline{U}_0)\phi_j,\theta_j\rangle_{\mathcal{H}^{[2]}_{M_j}} \\ &= (\Gamma-1)\langle \overline{c}_0\mathcal{D}_{k,M_j}\theta_j - D\mathcal{G}^{[2,2]}(\pi_{\mathcal{H}_{M_j}}\overline{U}_0)\theta_j + \delta\theta_j - \chi_j,\theta_j\rangle_{\mathcal{H}^{[2]}_{M_j}}. \end{aligned} \tag{4.5.42}$$

The identities (4.5.39) and (4.5.42) allow us to expand the inner product

$$\begin{aligned} \langle\Psi_j,\Phi_j\rangle_{\mathcal{H}_{M_j}} &= \langle[\mathcal{K}_{k,M_j}+\delta]\Phi_j,\Phi_j\rangle_{\mathcal{H}_{M_j}} \\ &= \overline{c}_0\langle\mathcal{D}_{k,M_j}\Phi_j,\Phi_j\rangle_{\mathcal{H}_{M_j}} - \langle\Delta_{M_j}\Phi_j,\Phi_j\rangle_{\mathcal{H}_{M_j}} \\ &\qquad -\langle D\mathcal{G}\big(\pi_{\mathcal{H}_{M_j}}\overline{U}_0;\overline{r}\big)\Phi_j,\Phi_j\rangle_{\mathcal{H}_{M_j}} + \delta\|\Phi_j\|^2_{\mathcal{H}_{M_j}} \\ &= \overline{c}_0\langle\mathcal{D}_{k,M_j}\phi_j,\phi_j\rangle_{\mathcal{H}^{[1]}_{M_j}} + \Gamma\overline{c}_0\langle\mathcal{D}_{k,M_j}\theta_j,\theta_j\rangle_{\mathcal{H}^{[2]}_{M_j}} - \langle\Delta_{M_j}\Phi_j,\Phi_j\rangle_{\mathcal{H}_{M_j}} \\ &\qquad -\langle D\mathcal{G}^{[1,1]}(\pi_{\mathcal{H}_{M_j}}\overline{U}_0)\phi_j,\phi_j\rangle_{\mathcal{H}^{[1]}_{M_j}} - \Gamma\langle D\mathcal{G}^{[2,2]}(\pi_{\mathcal{H}_{M_j}}\overline{U}_0)\theta_j,\theta_j\rangle_{\mathcal{H}^{[2]}_{M_j}} \\ &\qquad -(\Gamma-1)\langle\chi_j,\theta_j\rangle_{\mathcal{H}^{[2]}_{M_j}} + \delta\|\phi_j\|^2_{\mathcal{H}^{[1]}_{M_j}} + \delta\Gamma\|\theta_j\|_{\mathcal{H}^{[2]}_{M_j}}. \end{aligned} \tag{4.5.43}$$

As such, we can use Lemma 4.A.4 and Lemma 4.A.5 to estimate

$$\begin{aligned} \langle\Psi_j,\Phi_j\rangle_{\mathcal{H}_{M_j}} &\geq \overline{c}_0\langle\mathcal{D}_{k,M_j}\phi_j,\phi_j\rangle_{\mathcal{H}^{[1]}_{M_j}} + \Gamma\overline{c}_0\langle\mathcal{D}_{k,M_j}\theta_j,\theta_j\rangle_{\mathcal{H}^{[2]}_{M_j}} \\ &\qquad -\langle D\mathcal{G}^{[1,1]}(\pi_{\mathcal{H}_{M_j}}\overline{U}_0)\phi_j,\phi_j\rangle_{\mathcal{H}^{[1]}_{M_j}} - \Gamma\langle D\mathcal{G}^{[2,2]}(\pi_{\mathcal{H}_{M_j}}\overline{U}_0)\theta_j,\theta_j\rangle_{\mathcal{H}^{[2]}_{M_j}} \\ &\qquad -(\Gamma+1)\|\chi_j\|_{\mathcal{H}^{[2]}_{M_j}}\|\theta_j\|_{\mathcal{H}^{[2]}_{M_j}} \\ &\geq -(1+\Gamma)C_2M_j^{-1}\|\mathcal{D}_{k,M_j}\Phi_j\|^2_{\mathcal{H}_{M_j}} \\ &\qquad -\langle D\mathcal{G}^{[1,1]}(\pi_{\mathcal{H}_{M_j}}\overline{U}_0)\phi_j,\phi_j\rangle_{\mathcal{H}^{[1]}_{M_j}} - \Gamma\langle D\mathcal{G}^{[2,2]}(\pi_{\mathcal{H}_{M_j}}\overline{U}_0)\theta_j,\theta_j\rangle_{\mathcal{H}^{[2]}_{M_j}} \\ &\qquad -(\Gamma+1)\|\chi_j\|_{\mathcal{H}^{[2]}_{M_j}}\|\theta_j\|_{\mathcal{H}^{[2]}_{M_j}} \end{aligned} \tag{4.5.44}$$

for some $C_2 > 1$.

Since $-D\mathcal{G}^{[1,1]}(P^{\pm})$ and $-D\mathcal{G}^{[2,2]}(P^{\pm})$ are positive definite and $-D\mathcal{G}$ is continuous, we can choose $\mu > 0$ and $a > 0$ to be positive constants such that the matrices

$$B_1(\xi) \;=\; -D\mathcal{G}^{[1,1]}\big(\overline{U}_0(\xi)\big) - a, \qquad B_2(\xi) \;=\; -\Gamma D\mathcal{G}^{[2,2]}\big(\overline{U}_0(\xi)\big) - a \tag{4.5.45}$$

are positive definite for all $|\xi| \geq \mu$. Defining $\mathcal{I}$ as in (4.5.32), this allows us to estimate

$$\begin{array}{rcl} -\langle D\mathcal{G}^{[1,1]}(\pi_{\mathcal{H}_{M_j}}\overline{U}_0)\phi_j, \phi_j\rangle_{\mathcal{H}^{[1]}_{M_j}} & = & a\|\phi_j\|^2_{\mathcal{H}^{[1]}_{M_j}} + \langle B_1\phi_j, \phi_j\rangle_{\mathcal{H}^{[1]}_{M_j}} \\ & \geq & a\|\Phi_j\|^2_{\mathcal{H}_{M_j}} - M_j^{-1}\sum_{\xi\in M_j^{-1}\mathbb{Z}} |B_1(\xi)\phi_j(\cdot,\xi)|^2_{\ell^2_{q,\perp}} \\ & \geq & a\|\Phi_j\|^2_{\mathcal{H}_{M_j}} - \mathcal{I}, \end{array} \tag{4.5.46}$$

together with

$$-\Gamma\langle D\mathcal{G}^{[2,2]}(\pi_{\mathcal{H}_{M_j}}\overline{U}_0)\theta_j, \theta_j\rangle_{\mathcal{H}^{[2]}_{M_j}} \;\geq\; a\|\theta_j\|^2_{\mathcal{H}^{[2]}_{M_j}} - \Gamma\mathcal{I}. \tag{4.5.47}$$

Combining the estimates (4.5.44), (4.5.46) and (4.5.47) yields the bound

$$\begin{array}{rcl} \langle\Psi_j, \Phi_j\rangle_{\mathcal{H}_{M_j}} & \geq & a\|\Phi_j\|^2_{\mathcal{H}_{M_j}} - (1+\Gamma)\mathcal{I} - (1+\Gamma)C_2M_j^{-1}\|\mathcal{D}_{k,M_j}\Phi_j\|^2_{\mathcal{H}_{M_j}} \\ & & -(\Gamma+1)\|\chi_j\|_{\mathcal{H}^{[2]}_{M_j}}\|\theta_j\|_{\mathcal{H}^{[2]}_{M_j}}. \end{array} \tag{4.5.48}$$

Hence we obtain

$$\begin{array}{rcl} (1+\Gamma)\mathcal{I} & \geq & a\|\Phi_j\|^2_{\mathcal{H}_{M_j}} - (1+\Gamma)C_2M_j^{-1}\|\mathcal{D}_{k,M_j}\Phi_j\|^2_{\mathcal{H}_{M_j}} \\ & & -\langle\Psi_j, \Phi_j\rangle_{\mathcal{H}_{M_j}} - (\Gamma+1)\|\chi_j\|_{\mathcal{H}^{[2]}_{M_j}}\|\theta_j\|_{\mathcal{H}^{[2]}_{M_j}} \\ & \geq & \frac{a}{2}\|\Phi_j\|^2_{\mathcal{H}_{M_j}} - \big(\frac{1}{a} + \frac{\Gamma+1}{a}\big)\|\Psi_j\|^2_{\mathcal{H}_{M_j}} - (1+\Gamma)C_2M_j^{-1}\|\mathcal{D}_{k,M_j}\Phi_j\|^2_{\mathcal{H}_{M_j}}, \end{array} \tag{4.5.49}$$

which yields the desired bound. ∎

Proof of Proposition 4.5.4. Rescaling (4.5.23) yields

$$0 \;\geq\; \frac{C_3}{\overline{c}_0^2+2C_1}\Big[\overline{c}_0^2\|\mathcal{D}_{k,M_j}\Phi_j\|^2_{\mathcal{H}_{M_j}} - 2C_1\|\Phi_j\|^2_{\mathcal{H}_{M_j}} - 2\|\Psi_j\|^2_{\mathcal{H}_{M_j}}\Big], \tag{4.5.50}$$

which can be added to (4.5.29) or (4.5.36) to obtain

$$\begin{array}{rcl} M_j^{-1}\sum_{\xi\in M_j^{-1}\mathbb{Z}:|\xi|\leq\mu} |\Phi_j(\cdot,\xi)|^2_{\ell^2_{q,\perp}} & \geq & C_3\|\Phi_j\|^2_{\mathcal{H}_{M_j}} - C_4\|\Psi_j\|^2_{\mathcal{H}_{M_j}} - C_5M_j^{-1}\|\mathcal{D}_{k,M_j}\Phi_j\|^2_{\mathcal{H}_{M_j}} \\ & & +\frac{C_3}{\overline{c}_0^2+2C_1}\Big[\overline{c}_0^2\|\mathcal{D}_{k,M_j}\Phi_j\|^2_{\mathcal{H}_{M_j}} - 2C_1\|\Phi_j\|^2_{\mathcal{H}_{M_j}} \\ & & \qquad -2\|\Psi_j\|^2_{\mathcal{H}_{M_j}}\Big] \\ & = & \frac{\overline{c}_0^2C_3}{\overline{c}_0^2+2C_1}\Big[\|\mathcal{D}_{k,M_j}\Phi_j\|^2_{\mathcal{H}_{M_j}} + \|\Phi_j\|^2_{\mathcal{H}_{M_j}}\Big] \\ & & -\Big[C_4 + \frac{C_3}{\overline{c}_0^2+2C_1}\Big]\|\Psi_j\|^2_{\mathcal{H}_{M_j}} \\ & & -C_5M_j^{-1}\|\mathcal{D}_{k,M_j}\Phi_j\|^2_{\mathcal{H}_{M_j}}. \end{array} \tag{4.5.51}$$

Remembering that $\|\Phi_j\|_{\mathcal{H}^1_{k,M_j}} = 1$, we can pick constants $C_6 > 0$, $C_7 > 0$ and $C_8 > 0$, which all are independent of $0 < \delta < \delta_0$, in such a way that

$$M_j^{-1} \sum_{\xi \in M_j^{-1}\mathbb{Z}: |\xi| \leq \mu} |\Phi_j(\cdot, \xi)|^2_{\ell^2_{q,\perp}} \geq C_6 - C_7 \|\Psi_j\|^2_{\mathcal{H}_{M_j}} - C_8 M_j^{-1}. \tag{4.5.52}$$

The strong convergence $\mathcal{I}^0_{M_j}\Phi_j \to \Phi_* \in L^2([-\mu - 1, \mu + 1]; \ell^2_{q,\perp})$ now yields the limiting behaviour

$$\begin{aligned} M_j^{-1} \sum_{\xi \in M_j^{-1}\mathbb{Z}: |\xi| \leq \mu} |\Phi_j(\cdot, \xi)|^2_{\ell^2_{q,\perp}} &= \int_{-\mu}^{\mu + M_j^{-1}} \left| [\mathcal{I}^0_{M_j}\Phi_j](\cdot, \xi) \right|^2_{\ell^2_{q,\perp}} d\xi \\ &\leq \int_{-\mu}^{\mu+1} \left| [\mathcal{I}^0_{M_j}\Phi_j](\cdot, \xi) \right|^2_{\ell^2_{q,\perp}} d\xi \\ &\to \int_{-\mu}^{\mu+1} \left| \Phi_*(\cdot, \xi) \right|^2_{\ell^2_{q,\perp}} d\xi, \end{aligned} \tag{4.5.53}$$

as $j \to \infty$. In view of the bound $\limsup_{j \to \infty} \|\Psi_j\|^2_{\mathcal{H}_{M_j}} \leq \kappa(\delta)^2$, this gives the desired inequality

$$\|\Phi_*\|^2_{H^1(\mathbb{R}, \ell^2_{q,\perp})} \geq \int_{-\mu}^{\mu+1} \left| \Phi_*(\cdot, \xi) \right|^2_{\ell^2_{q,\perp}} d\xi \geq C_6 - C_7 \kappa(\delta)^2. \tag{4.5.54}$$

■

4.5.3 Exponential decay

In this section we set out to prove Proposition 4.5.5. The main ingredient to establish this result is to show that for $0 < \delta < \delta_0$ the map $(L_0 + \delta)^{-1}$ maps $BC^1_{-\eta}(\mathbb{R}; \mathbb{R}^d)$ into the space

$$BC^2_{-\eta}(\mathbb{R}; \mathbb{R}^d) = \{F \in BC_{-\eta}(\mathbb{R}; \mathbb{R}^d) : \sup_{\xi \in \mathbb{R}} e^{-\eta|\xi|} [|F(\xi)| + |F'(\xi)| + |F''(\xi)|] < \infty\}. \tag{4.5.55}$$

This is not immediately clear, since if we have

$$F = (L_0 + \delta)^{-1} G \tag{4.5.56}$$

with $G \in BC^1_{-\eta}(\mathbb{R}; \mathbb{R}^d)$, it is impossible to express F as a local function of G due to the infinite-range interactions. We first establish this result for the subspaces $H^1(\mathbb{R}; \mathbb{R}^d)$ and $H^2(\mathbb{R}; \mathbb{R}^d)$.

Lemma 4.5.9. *Assume that (HS1) and (HS2) are satisfied and pick $\overline{r}$ in such a way that (HS3$_{\overline{r}}$), (HW1$_{\overline{r}}$) and (HW2$_{\overline{r}}$) are satisfied. Then for each $0 < \delta < \delta_0$ and each $G \in H^1(\mathbb{R}; \mathbb{R}^d)$ we have*

$$(L_0 + \delta)^{-1} G \in H^2(\mathbb{R}; \mathbb{R}^d). \tag{4.5.57}$$

Proof. Fix $0 < \delta < \delta_0$ and $G \in H^1(\mathbb{R};\mathbb{R}^d)$. Write $F = (L_0 + \delta)^{-1}G \in H^1(\mathbb{R};\mathbb{R}^d)$. Then we can rewrite the equation $(L_0 + \delta)F = G$ in the form

$$\overline{c}_0 F' = G + \Delta_0 F + D_U \mathcal{G}\big(\overline{U}_0;\overline{r}\big)F - \delta F. \tag{4.5.58}$$

From this representation it immediately follows that $F' \in L^\infty(\mathbb{R};\mathbb{R}^d)$. Differentiating both sides yields

$$\overline{c}_0 F'' = G' + (\Delta_0 F)' + D_U \mathcal{G}\big(\overline{U}_0;\overline{r}\big)F' + D^2\mathcal{G}\big(\overline{U}_0;\overline{r}\big)[\overline{U}_0', F] - \delta F'. \tag{4.5.59}$$

Writing

$$F_n(x) = \tau \sum_{m=1}^{n} \alpha_m \Big[F(x+m) + F(x-m) - 2F(x)\Big] \tag{4.5.60}$$

for $n \in \mathbb{Z}_{>0}$, we can compute

$$F_n'(x) = \tau \sum_{m=1}^{n} \alpha_m \Big[F'(x+m) + F'(x-m) - 2F'(x)\Big]. \tag{4.5.61}$$

This allows us to estimate

$$|F_n'(x) - (\Delta_0 F')(x)| \leq 4\tau \sum_{m=n+1}^{\infty} |\alpha_m| \|F'\|_\infty. \tag{4.5.62}$$

In particular, the sequence $\{F_n'\}$ converges uniformly to $\Delta_0 F'$, from which it follows that

$$(\Delta_0 F)'(x) = \tau \sum_{m=1}^{\infty} \alpha_m \Big[F'(x+m) + F'(x-m) - 2F'(x)\Big] = (\Delta_0 F')(x). \tag{4.5.63}$$

Since $F, G \in H^1(\mathbb{R};\mathbb{R}^d)$, these considerations yield that $F'' \in L^2(\mathbb{R};\mathbb{R}^d)$, from which the desired result follows. ∎

We now turn to the desired exponential decay. The assumptions (HW1$_{\overline{r}}$) and (HW2$_{\overline{r}}$) yield the following useful properties of the operator L_0.

Lemma 4.5.10. *Assume that (HS1) and (HS2) are satisfied and pick $\overline{r}$ in such a way that (HS3$_{\overline{r}}$), (HW1$_{\overline{r}}$) and (HW2$_{\overline{r}}$) are satisfied. Then the following properties hold for the LDE (4.2.1) with $r = \overline{r}$.*

(i) The functions Φ_0^+ and Φ_0^- together with their derivatives decay exponentially.

(ii) Upon introducing the spaces

$$X_0 = \{F \in H^1(\mathbb{R};\mathbb{R}^d) : \langle \Phi_0^-, F\rangle_{L^2(\mathbb{R};\mathbb{R}^d)} = 0\} \tag{4.5.64}$$

and

$$Y_0 = \{G \in L^2(\mathbb{R};\mathbb{R}^d) : \langle \Phi_0^-, G\rangle_{L^2(\mathbb{R};\mathbb{R}^d)} = 0\}, \tag{4.5.65}$$

the operator $L_0 : X_0 \to Y_0$ is invertible.

In addition, there exists a constant $\tilde{\eta} > 0$ in such a way that for each $0 < \eta < \tilde{\eta}$ the map L_0 maps $BC^1_{-\eta}(\mathbb{R};\mathbb{R}^d)$ into $BC_{-\eta}(\mathbb{R};\mathbb{R}^d)$.

Proof. The proof of the statements (i)-(ii) follows the procedure described [150, Lem. 4.15, 6.8, 6.9] and will hence be omitted. It hence suffices to prove that Δ_0 maps $BC_{-\eta}(\mathbb{R};\mathbb{R}^{d_{\mathrm{diff}}})$ into itself for η small enough. Upon picking $F \in BC_{-\eta}(\mathbb{R};\mathbb{R}^{d_{\mathrm{diff}}})$ and $K \in \mathbb{R}_{>0}$ in such a way that the bound

$$|F(\xi)| \quad \leq \quad K e^{-\eta|\xi|} \tag{4.5.66}$$

holds, we estimate

$$\begin{array}{lcl} |\Delta_0 F(\xi)| & \leq & \tau \sum_{m>0} |\alpha_m| K\Big(e^{-\eta|\xi+m|} + e^{-\eta|\xi-m|} + 2e^{-\eta|\xi|}\Big) \\ & \leq & \tau \sum_{m>0} |\alpha_m| K e^{-\eta|\xi|}\Big(2e^{\eta m} + 2\Big). \end{array} \tag{4.5.67}$$

We can hence set $\tilde{\eta} = \nu$, where ν is defined in (HS1). A computation similar to the proof of [150, Lem. 6.5] yields the continuity of $\Delta_0 f$, from which the desired result follows. ∎

We now recall the notation $L_0^{\mathrm{qinv}} G$ that was introduced in [150, Cor. 4.4] for the unique solution F of the equation

$$L_0 F \quad = \quad G - \frac{\langle \Phi_0^-, G\rangle_{L^2(\mathbb{R};\mathbb{R}^d)}}{\langle \Phi_0^-, \Phi_0^+\rangle_{L^2(\mathbb{R};\mathbb{R}^d)}} \Phi_0^+ \tag{4.5.68}$$

in the space X_0, which is given explicitly by

$$L_0^{\mathrm{qinv}} G \quad = \quad L_0^{-1}\Big[G - \frac{\langle \Phi_0^-, G\rangle_{L^2(\mathbb{R};\mathbb{R}^d)}}{\langle \Phi_0^-, \Phi_0^+\rangle_{L^2(\mathbb{R};\mathbb{R}^d)}} \Phi_0^+\Big]. \tag{4.5.69}$$

The proof of [150, Prop. 5.2] provides the representation

$$(L_0 + \delta)^{-1} G \quad = \quad \delta^{-1} \frac{\langle \Phi_0^-, G\rangle_{L^2(\mathbb{R};\mathbb{R}^d)}}{\langle \Phi_0^-, \Phi_0^+\rangle_{L^2(\mathbb{R};\mathbb{R}^d)}} \Phi_0^+ + [I + \delta L_0^{-1}]^{-1} L_0^{\mathrm{qinv}} G \tag{4.5.70}$$

for each $0 < \delta < \delta_0$ and each $G \in L^2(\mathbb{R};\mathbb{R}^d)$. In addition, we can use Lemma 4.5.10 to pick constants $\tilde{K} > 0$ and $\tilde{\alpha} > 0$ in such a way that

$$|\Phi_0^+(x)| \quad \leq \quad \tilde{K} e^{-\tilde{\alpha}|x|} \tag{4.5.71}$$

holds for all $x \in \mathbb{R}$. Let $\tilde{\eta} > 0$ be the constant from Lemma 4.5.10. Using [150, Lem. 6.6], which is a generalization of [130, Prop. 5.3], we can pick constants $K_1 > 0$ and $0 < \alpha \leq \min\{\tilde{\eta}, \tilde{\alpha}\}$ in such a way that

$$\begin{array}{lcl} |L_0^{\mathrm{qinv}} G(x)| & \leq & K_1 e^{-\alpha|x|} \|L_0^{\mathrm{qinv}} G\|_\infty \\ & & + K_1 \int_{-\infty}^{\infty} e^{-\alpha|x-y|} \Big| G(y) - \frac{\langle \Phi_0^-, G\rangle_{L^2(\mathbb{R};\mathbb{R}^d)}}{\langle \Phi_0^-, \Phi_0^+\rangle_{L^2(\mathbb{R};\mathbb{R}^d)}} \Phi_0^+(y) \Big| dy \end{array} \tag{4.5.72}$$

holds for each $G \in L^2(\mathbb{R};\mathbb{R}^d)$. The following three results use (4.5.70) and (4.5.72) to establish the desired pointwise bound for $(L_0 + \delta)^{-1}$.

Lemma 4.5.11. *Assume that (HS1) and (HS2) are satisfied and pick $\overline{r}$ in such a way that (HS3$_{\overline{r}}$), (HW1$_{\overline{r}}$) and (HW2$_{\overline{r}}$) are satisfied. Recall the constant $\alpha > 0$ from (4.5.72) and fix $0 < \eta \leq \alpha$. Then there exists a constant $K > 0$ so that for each $G \in BC^1_{-\eta}(\mathbb{R};\mathbb{R}^d)$ we have the bound*

$$|L_0^{\mathrm{qinv}} G(x)| \quad \leq \quad K\|G\|_{BC_{-\eta}(\mathbb{R};\mathbb{R}^d)} e^{-\eta|x|}. \tag{4.5.73}$$

Proof. Pick $0 < \eta \leq \alpha$ and $G \in BC^1_{-\eta}(\mathbb{R};\mathbb{R}^d)$. Recalling (4.5.72), we can estimate

$$\begin{array}{rcl} \|L_0^{\mathrm{qinv}} G\|_\infty & \leq & \|L_0^{\mathrm{qinv}} G\|_{H^1(\mathbb{R};\mathbb{R}^d)} \\ & \leq & \|L_0^{-1}\|_{\mathcal{L}(Y_0,X_0)} \|G - \frac{\langle \Phi_0^-, G\rangle_{L^2(\mathbb{R};\mathbb{R}^d)}}{\langle \Phi_0^-, \Phi_0^+\rangle_{L^2(\mathbb{R};\mathbb{R}^d)}} \Phi_0^+ \|_{L^2(\mathbb{R};\mathbb{R}^d)} \\ & \leq & \|L_0^{-1}\|_{\mathcal{L}(Y_0,X_0)} \Big(1 + \frac{1}{|\langle \Phi_0^-, \Phi_0^+\rangle_{L^2(\mathbb{R};\mathbb{R}^d)}|}\Big) \|G\|_{L^2(\mathbb{R};\mathbb{R}^d)}. \end{array} \tag{4.5.74}$$

Combining these estimates yields the bound

$$\begin{array}{rcl} |L_0^{\mathrm{qinv}} G(x)| & \leq & K_1 e^{-\alpha|x|} \|L_0^{-1}\|_{\mathcal{L}(Y_0,X_0)} \Big(1 + \frac{1}{|\langle \Phi_0^-, \Phi_0^+\rangle_{L^2(\mathbb{R};\mathbb{R}^d)}|}\Big) \|G\|_{L^2(\mathbb{R};\mathbb{R}^d)} \\ & & \quad + K_1 \Big(\|G\|_{BC_{-\eta}(\mathbb{R},\mathbb{R}^d)} + \tilde{K} \frac{1}{|\langle \Phi_0^-, \Phi_0^+\rangle_{L^2(\mathbb{R};\mathbb{R}^d)}|} \|G\|_{L^2(\mathbb{R};\mathbb{R}^d)} \Big) \\ & & \qquad \times \int_{-\infty}^{\infty} e^{-\alpha|x-y|} e^{-\eta|x|} dy \\ & \leq & K_1 e^{-\alpha|x|} \|L_0^{-1}\|_{\mathcal{L}(Y_0,X_0)} \Big(1 + \frac{1}{|\langle \Phi_0^-, \Phi_0^+\rangle_{L^2(\mathbb{R};\mathbb{R}^d)}|}\Big) \|G\|_{L^2(\mathbb{R};\mathbb{R}^d)} \\ & & \quad + K_1 \Big(\|G\|_{BC_{-\eta}(\mathbb{R},\mathbb{R}^d)} + \tilde{K} \frac{1}{|\langle \Phi_0^-, \Phi_0^+\rangle_{L^2(\mathbb{R};\mathbb{R}^d)}|} \|G\|_{L^2(\mathbb{R};\mathbb{R}^d)} \Big) e^{-\eta|x|} \\ & \leq & K_2 \Big(\|G\|_{BC_{-\eta}(\mathbb{R};\mathbb{R}^d)} + \|G\|_{L^2(\mathbb{R};\mathbb{R}^d)} \Big) e^{-\eta|x|}. \end{array} \tag{4.5.75}$$

Finally, we note that $\|G\|_{L^2(\mathbb{R};\mathbb{R}^d)} \leq K_3 \|G\|_{BC_{-\eta}(\mathbb{R};\mathbb{R}^d)}$ for some constant $K_3 > 0$, which implies the desired result. ∎

Lemma 4.5.12. *Consider the setting of Lemma 4.5.11. Then there exist constants $0 < \delta_* \leq \delta_0$ and $K > 0$ so that for each $0 < \delta < \delta_*$ and each $G \in BC^1_{-\eta}(\mathbb{R};\mathbb{R}^d)$ we have the bound*

$$|[I + \delta L_0^{-1}]^{-1} L_0^{\mathrm{qinv}} G(x)| \quad \leq \quad K \|G\|_{BC_{-\eta}(\mathbb{R};\mathbb{R}^d)} e^{-\eta|x|}. \tag{4.5.76}$$

Proof. Pick $G \in BC^1_{-\eta}(\mathbb{R};\mathbb{R}^d)$. For $n \in \mathbb{Z}_{>0}$ a calculation similar to (4.5.74) yields

$$\|(L_0^{-1})^n L_0^{\mathrm{qinv}} G\|_\infty \quad \leq \quad \|L_0^{-1}\|^{n+1}_{\mathcal{L}(Y_0,X_0)} \Big(1 + \frac{1}{|\langle \Phi_0^-, \Phi_0^+\rangle_{L^2(\mathbb{R};\mathbb{R}^d)}|}\Big) \|G\|_{L^2(\mathbb{R};\mathbb{R}^d)}. \tag{4.5.77}$$

Using [150, Lem. 6.6] and Lemma 4.5.11, we obtain

$$\begin{array}{rcl} |L_0^{-1} L_0^{\mathrm{qinv}} G(x)| & \leq & K_1 e^{-\alpha|x|} \|L_0^{-1} L_0^{\mathrm{qinv}} G\|_\infty + K_1 \int_{-\infty}^{\infty} e^{-\alpha|x-y|} \Big| L_0^{\mathrm{qinv}} G \Big| dy \\ & \leq & K_1 \Big(1 + \|L_0^{-1}\|_{\mathcal{L}(Y_0,X_0)}\Big) K \|G\|_{BC_{-\eta}(\mathbb{R};\mathbb{R}^d)} e^{-\eta|x|}. \end{array} \tag{4.5.78}$$

Continuing in this fashion we see that the estimate

$$|(L_0^{-1})^n L_0^{\mathrm{qinv}} G(x)| \quad \leq \quad K_2^n K \|G\|_{BC_{-\eta}(\mathbb{R};\mathbb{R}^d)} e^{-\eta|x|} \tag{4.5.79}$$

holds for all $n \in \mathbb{Z}_{>0}$ and for some constant $K_2 > 0$. If we set

$$\delta_* \quad = \quad \min\Big\{ \delta_0, \frac{1}{\|L_0^{-1}\|_{\mathcal{L}(Y_0, X_0)} \Big(1 + \frac{1}{|\langle \Phi_0^-, \Phi_0^+ \rangle_{L^2(\mathbb{R};\mathbb{R}^d)}|}\Big)}, \frac{1}{K_2} \Big\}, \tag{4.5.80}$$

then for each $n \in \mathbb{Z}_{>0}$ and each $0 < \delta < \delta_*$ we have

$$\|(-\delta)^n (L_0^{-1})^n L_0^{\mathrm{qinv}} G\|_\infty \quad \leq \quad \tfrac{1}{2} \|G\|_{BC_{-\eta}(\mathbb{R};\mathbb{R}^d)}. \tag{4.5.81}$$

In particular, it follows that

$$\sum_{n=0}^{\infty} (-\delta)^n (L_0^{-1})^n L_0^{\mathrm{qinv}} G \quad \to \quad [I + \delta L_0^{-1}]^{-1} L_0^{\mathrm{qinv}} G \tag{4.5.82}$$

in $H^1(\mathbb{R};\mathbb{R}^d)$. Since $H^1(\mathbb{R};\mathbb{R}^d)$-convergence implies pointwise convergence we see that

$$\begin{aligned} |[I + \delta L_0^{-1}]^{-1} L_0^{\mathrm{qinv}} G(x)| \quad & = \quad |\sum_{n=0}^{\infty} (-\delta)^n (L_0^{-1})^n L_0^{\mathrm{qinv}} G(x)| \\ & \leq \quad \sum_{n=0}^{\infty} \delta_*^n K K_2^n \|G\|_{BC_{-\eta}(\mathbb{R};\mathbb{R}^d)} e^{-\eta|x|} \\ & \leq \quad K_3 \|G\|_{BC_{-\eta}(\mathbb{R};\mathbb{R}^d)} e^{-\eta|x|}. \end{aligned} \tag{4.5.83}$$

■

Corollary 4.5.13. *Consider the setting of Lemma 4.5.12. There exists a constant $K > 0$ so that for each $0 < \delta < \delta_*$ and each $G \in BC^1_{-\eta}(\mathbb{R};\mathbb{R}^d)$ we have the bound*

$$\begin{aligned} |(L_0 + \delta)^{-1} G(x)| \quad & \leq \quad K \delta^{-1} \|G\|_{BC_{-\eta}(\mathbb{R};\mathbb{R}^d)} e^{-\eta|x|}, \\ |[(L_0 + \delta)^{-1} G]'(x)| \quad & \leq \quad K \delta^{-1} \|G\|_{BC_{-\eta}(\mathbb{R};\mathbb{R}^d)} e^{-\eta|x|}, \\ |[(L_0 + \delta)^{-1} G]''(x)| \quad & \leq \quad K \delta^{-1} \|G\|_{BC^1_{-\eta}(\mathbb{R};\mathbb{R}^d)} e^{-\eta|x|}. \end{aligned} \tag{4.5.84}$$

Proof. Fix $0 < \delta < \delta_*$ and $G \in BC^1_{-\eta}(\mathbb{R};\mathbb{R}^d)$. Write $F = (L_0 + \delta)^{-1} G$. The representation (4.5.70) together with Lemma 4.5.12 immediately yields the bound

$$\begin{aligned} |F(x)| \quad & \leq \quad \delta^{-1} \frac{1}{|\langle \Phi_0^-, \Phi_0^+ \rangle_{L^2(\mathbb{R};\mathbb{R}^d)}|} \|G\|_{L^2(\mathbb{R};\mathbb{R}^d)} \tilde{K} e^{-\tilde{\alpha}|x|} + K \|G\|_{BC_{-\eta}(\mathbb{R};\mathbb{R}^d)} e^{-\eta|x|} \\ & \leq \quad \delta^{-1} K_2 \|G\|_{BC_{-\eta}(\mathbb{R};\mathbb{R}^d)} e^{-\eta|x|}. \end{aligned} \tag{4.5.85}$$

In addition, the representation (4.5.58) together with the bounds (4.5.67) and (4.5.85) yields that

$$|F'(x)| \quad \leq \quad K \delta^{-1} \|G\|_{BC_{-\eta}(\mathbb{R};\mathbb{R}^d)} e^{-\eta|x|} \tag{4.5.86}$$

for some constant $K > 0$. Similarly, the representation (4.5.59) yields the bound

$$|F''(x)| \quad \leq \quad K \delta^{-1} \|G\|_{BC^1_{-\eta}(\mathbb{R};\mathbb{R}^d)} e^{-\eta|x|}. \tag{4.5.87}$$

■

Proof of Proposition 4.5.5. Corollary 4.5.13 implies the desired result. ■

4.6 Proof of main result

In this section we mainly follow the approach of [111, §4.1]. We lift the computations from [111] in a more detailed fashion, in order to ensure that the multi-component nature of the nonlinearity $\mathcal{G}$ does not cause any issues. Luckily, we only need to take care of some minor, technical difficulties. For example, due to the higher generality of the nonlinearity $\mathcal{G}$, we can no longer refer to [52, Lem. App.IV.1.1] to conclude that $\mathcal{G}$ depends continuously on the perturbation. Instead, we need to prove this continuity in a direct fashion, carefully employing the exponential decay of the travelling wave $\overline{U}$ and the perturbations involved.

Let us fix an integer $q \geq 1$, together with a constant $M = \frac{p}{q} \in \mathcal{M}_q$. Our goal is to construct a solution $\overline{U}$ to the nonlinear problem

$$c[\mathcal{D}_{k,M}\overline{U}](\xi) \quad = \quad \tau \sum_{m>0} \alpha_m[\overline{U}(\xi+m) + \overline{U}(\xi-m) - 2\overline{U}(\xi)] + \mathcal{G}\big(\overline{U}(\xi); r\big), \qquad (4.6.1)$$

where $\xi \in p^{-1}\mathbb{Z}$, that has the form

$$\overline{U}(\xi) \quad = \quad \overline{U}_0(\xi+\theta) + \Phi(\xi), \qquad \xi \in p^{-1}\mathbb{Z}, \qquad (4.6.2)$$

for some $\theta \in \mathbb{R}$ and some $\Phi \in \mathcal{Y}_M$. Note that this form automatically ensures that $\overline{U}$ satisfies the boundary conditions

$$\lim_{\xi \to \pm\infty, \xi \in p^{-1}\mathbb{Z}} \overline{U}(\xi) \quad = \quad P^{\pm}. \qquad (4.6.3)$$

For notational compactness, we introduce the functions

$$\overline{U}_\theta(\xi) \; = \; \overline{U}_0(\xi+\theta), \qquad \Phi^+_\theta(\xi) \; = \; \Phi^+_0(\xi+\theta), \qquad \Phi^-_\theta(\xi) \; = \; \Phi^-_0(\xi+\theta), \qquad (4.6.4)$$

together with the linear operators

$$L_{k,M;\theta} : \mathcal{Y}^1_{k,M} \to \mathcal{Y}_M, \qquad (4.6.5)$$

that act as

$$L_{k,M;\theta}\Phi(\xi) \quad = \quad \overline{c}_0[\mathcal{D}_{k,M}\Phi](\xi) - \Delta_0 V(\xi) - D_U\mathcal{G}\big(\overline{U}_0(\xi+\theta); \overline{r}\big)\Phi(\xi), \qquad (4.6.6)$$

where $\xi \in p^{-1}\mathbb{Z}$. Naturally, these operators satisfy the properties described in Proposition 4.5.1 provided all occurrences of Φ^+_0 and Φ^-_0 are replaced by Φ^+_θ and Φ^-_θ respectively. In particular, we write

$$\begin{array}{rccl} \gamma^*_{k,M;\theta} : & \mathcal{Y}_M & \to & \mathbb{R}, \\ \mathcal{V}^*_{k,M;\theta} : & \mathcal{Y}_M & \to & \mathcal{Y}^1_{k,M} \end{array} \qquad (4.6.7)$$

for the maps appearing in that result, as well as $M_{*;\theta}$ and C_θ for the corresponding constants. Since the nonlinearity $\mathcal{G}(\cdot; \overline{r})$ and the travelling wave $\overline{U}_0$ are continuous, it is clear that the map

$$\theta \quad \mapsto \quad L_{k,M;\theta} \in \mathcal{L}(\mathcal{Y}^1_{k,M}, \mathcal{Y}_M) \qquad (4.6.8)$$

is continuous. The representations [111, Eq. (3.149)] and [111, Eq. (3.150)], therefore, imply that the maps

$$\begin{array}{rcl} \theta & \mapsto & \gamma^*_{k,M;\theta} \in \mathcal{L}(\mathcal{Y}_M, \mathbb{R}) \\ \theta & \mapsto & \mathcal{V}^*_{k,M;\theta} \in \mathcal{L}(\mathcal{Y}_M, \mathcal{Y}^1_{k,M}) \end{array} \tag{4.6.9}$$

are continuous as well. As such, the constants $M_{*;\theta}$ and C_θ can be chosen to depend continuously on θ. Our goal is to find a lower bound for these constants. For $\theta \in \mathbb{R}$ we write S_θ for the shift operator $f \mapsto f(\cdot + \theta)$. For any $M \in \mathcal{M}_q$ we observe that S_1 and S_{-1} map $\mathcal{Y}^1_{k,M}$ and $\mathcal{Y}_M$ into themselves and that these maps are isometric isomorphisms. Moreover, we observe that for each $\theta \in \mathbb{R}$ we have the identity

$$L_{k,M;\theta} = S_1 L_{k,M;\theta-1} S_{-1}. \tag{4.6.10}$$

As such, we can restrict ourselves to the values of $M_{*;\theta}$ and C_θ for $\theta \in [0,1]$. Since $[0,1]$ is compact and $M_{*;\theta}$ and C_θ depend continuously on θ, we can pick an uniform quantities $C_{\mathrm{unif}} > 0$ and M_{unif} in such a way that the bounds

$$|\gamma^*_{k,M;\theta} f| + \|\mathcal{V}^*_{k,M;\theta} f\|_{\mathcal{Y}^1_{k,M}} \leq C_{\mathrm{unif}} \|f\|_{\mathcal{Y}_M} \tag{4.6.11}$$

and

$$\left|\gamma^*_{k,M;\theta} f - \langle \pi_{\mathcal{Y}_M} \Phi_0^-, \pi_{\mathcal{Y}_M} f \rangle_{\mathcal{Y}_M}\right| \leq C_{\mathrm{unif}} M^{-1} \|f\|_{L^2(\mathbb{R};\mathbb{R}^d)} \tag{4.6.12}$$

hold for all sufficiently small $\eta > 0$, all $M \in \mathcal{M}_q$ with $M \geq M_{\mathrm{unif}}$, all $f \in BC^1_{-\eta}(\mathbb{R};\mathbb{R}^d)$ and all $\theta \in \mathbb{R}$.

4.6.1 Existence of solutions

Substituting the Ansatz (4.6.2) into (4.6.1), we obtain

$$c[\mathcal{D}_{k,M}\Phi](\xi) + c[\mathcal{D}_{k,M}\overline{U}_\theta](\xi) = \Delta_0 \overline{U}_\theta(\xi) + \Delta_0 \Phi(\xi) + \mathcal{G}\Big(\overline{U}_\theta(\xi) + \Phi(\xi); r\Big). \tag{4.6.13}$$

The proof of Theorem 4.2.1 proceeds in two main steps. In particular, we first show the existence of wave solutions to (4.6.1) before we turn to the uniqueness. The existence results are summarized in the following proposition.

Proposition 4.6.1. *Assume that (HS1) and (HS2) are satisfied and pick $\overline{r}$ in such a way that (HS3$_{\overline{r}}$), (HW1$_{\overline{r}}$) and (HW2$_{\overline{r}}$) are satisfied. Fix a pair of integers $1 \leq k \leq 6$ and $q \geq 1$. Then there exist constants $M_* \gg 1$ and $\delta_r > 0$ so that for any $M = \frac{p}{q} \in \mathcal{M}_q$ with $M \geq M_*$, there exist continuous functions*

$$\begin{array}{rcl} c_M : \mathbb{R} \times [\overline{r} - \delta_r, \overline{r} + \delta_r] & \to & \mathbb{R} \\ \overline{U}_M : \mathbb{R} \times [\overline{r} - \delta_r, \overline{r} + \delta_r] & \to & \ell^\infty(p^{-1}\mathbb{Z}, \mathbb{R}^d), \end{array} \tag{4.6.14}$$

that satisfy properties (i)-(iii) of Theorem 4.2.1.

For any $V \in \mathbb{R}^d$ and $(\xi, \theta, r) \in \mathbb{R} \times \mathbb{R} \times (0,1)$ we consider the nonlinear expression

$$\mathcal{N}_0(V; \xi, \theta, r) = \mathcal{G}\Big(\overline{U}_\theta(\xi) + V; r\Big) - \mathcal{G}\Big(\overline{U}_\theta(\xi); r\Big) - D_U \mathcal{G}\Big(\overline{U}_\theta(\xi); r\Big) V. \tag{4.6.15}$$

Plugging this expression into (4.6.13) we arrive at

$$\begin{array}{rcl} c[\mathcal{D}_{k,M}\Phi](\xi) + c[\mathcal{D}_{k,M}\overline{U}_\theta](\xi) & = & \Delta_0\overline{U}_\theta(\xi) + \Delta_0\Phi(\xi) + D_U\mathcal{G}\Big(\overline{U}_\theta(\xi);\overline{r}\Big)\Phi(\xi) \\ & & +\mathcal{G}\Big(\overline{U}_\theta(\xi);r\Big) + \mathcal{N}_0(\Phi(\xi);\xi,\theta,r) \\ & & +D_U\mathcal{G}\Big(\overline{U}_\theta(\xi);r\Big)\Phi(\xi) - D_U\mathcal{G}\Big(\overline{U}_\theta(\xi);\overline{r}\Big)\Phi(\xi). \end{array} \tag{4.6.16}$$

Exploiting that $\overline{U}_\theta$ is a wave solution of the semi-discrete equation, i.e. that

$$\overline{c}_0\overline{U}'_\theta(\xi) \quad = \quad \Delta_0\overline{U}_\theta(\xi) + \mathcal{G}\Big(\overline{U}_\theta(\xi);\overline{r}\Big), \tag{4.6.17}$$

we find that the pair (c,Φ) must satisfy the equation

$$\begin{array}{rcl} L_{k,M;\theta}\Phi & = & (\overline{c}_0 - c)[\mathcal{D}_{k,M}\overline{U}_\theta](\xi) + [\mathcal{R}_A(c,\Phi)](\xi) \\ & & +[\mathcal{R}_B(\Phi;\theta,r)](\xi) + [\mathcal{R}_C(\theta,M)](\xi). \end{array} \tag{4.6.18}$$

Here we have introduced the quantities

$$\begin{array}{rcl} [\mathcal{R}_A(c,\Phi)](\xi) & = & (\overline{c}_0 - c)[\mathcal{D}_{k,M}\Phi](\xi) \\ {[\mathcal{R}_B(v;\theta,r)](\xi)} & = & D_U\mathcal{G}\Big(\overline{U}_\theta(\xi);r\Big)\Phi(\xi) - D_U\mathcal{G}\Big(\overline{U}_\theta(\xi);\overline{r}\Big)\Phi(\xi) \\ & & +\mathcal{G}\Big(\overline{U}_\theta(\xi);r\Big) - \mathcal{G}\Big(\overline{U}_\theta(\xi);\overline{r}\Big) + \mathcal{N}_0(\Phi(\xi);\xi,\theta,r) \\ & = & \mathcal{G}\Big(\overline{U}_\theta(\xi) + \Phi(\xi);r\Big) - \mathcal{G}\Big(\overline{U}_\theta(\xi) + \Phi(\xi);\overline{r}\Big) + \mathcal{N}_0(\Phi(\xi);\xi,\theta,\overline{r}), \end{array} \tag{4.6.19}$$

together with

$$\begin{array}{rcl} [\mathcal{R}_C(\theta,M)](\xi) & = & -\overline{c}_0[\mathcal{D}_{k,M}\overline{U}_\theta](\xi) + \Delta_0\overline{U}_\theta(\xi) + \mathcal{G}\Big(\overline{U}_\theta(\xi);\overline{r}\Big) \\ & = & \overline{c}_0\Big[\overline{U}'_\theta - \mathcal{D}_{k,M}\overline{U}_\theta\Big](\xi). \end{array} \tag{4.6.20}$$

Note that the term $\mathcal{R}_B$ incorporates the effects caused by varying the parameters in our equation, while the term $\mathcal{R}_C$ describes the effect of moving from the regular derivative to the discrete derivative.

Note that in our current notation the normalization condition (4.2.36) reduces to

$$\langle \pi_{\mathcal{Y}_M}\Phi_0^-(\cdot+\theta), \overline{U} - \pi_{\mathcal{Y}_M}\overline{U}_0(\cdot+\theta)\rangle_{\mathcal{Y}_M} \quad = \quad 0. \tag{4.6.21}$$

Proposition 4.5.1 and our considerations above show that solutions (c,Φ) to (4.6.18) must satisfy the fixed point problem

$$\begin{array}{rcl} \overline{c}_0 - c & = & \gamma^*_{k,M;\theta}\Big[\mathcal{R}_A(c,\Phi) + \mathcal{R}_B(\Phi;\theta,r) + \mathcal{R}_C(\theta,M)\Big] \\ \Phi & = & \mathcal{V}^*_{k,M;\theta}\Big[\mathcal{R}_A(c,\Phi) + \mathcal{R}_B(v;\theta,r) + \mathcal{R}_C(\theta,M)\Big]. \end{array} \tag{4.6.22}$$

Lemma 4.6.2 ([111, Lem. 4.1]). *Assume that (HS1) and (HS2) are satisfied and pick $\overline{r}$ in such a way that (HS3$_{\overline{r}}$), (HW1$_{\overline{r}}$) and (HW2$_{\overline{r}}$) are satisfied. Fix a pair of integers $1 \le k \le 6$ and $q \ge 1$. There exists a constant $C > 1$ so that for all $M = \frac{p}{q} \in \mathcal{M}_q$ and $\Phi \in \mathcal{Y}^1_{k,M}$ we have the bound*

$$\|\Phi\|_\infty \quad := \quad \sup_{\xi \in p^{-1}\mathbb{Z}} |\Phi(\xi)| \quad \le \quad C\|\Phi\|_{\mathcal{Y}^1_{k,M}}. \tag{4.6.23}$$

Lemma 4.6.3 (cf. [111, Lem. 4.2]). *Assume that (HS1) and (HS2) are satisfied and pick $\overline{r}$ in such a way that (HS3$_{\overline{r}}$), (HW1$_{\overline{r}}$) and (HW2$_{\overline{r}}$) are satisfied. Fix a pair of integers $1 \le k \le 6$ and $q \ge 1$. There exists a constant $C > 1$ so that for all $M = \frac{p}{q} \in \mathcal{M}_q$, all $(\theta, r) \in \mathbb{R} \times (0,1)$ and $\Phi \in \mathcal{Y}^1_{k,M}$ with $\|\Phi\|_{\mathcal{Y}^1_{k,M}} \le 1$ we have the bound*

$$\|\mathcal{R}_B(\Phi;\theta,r)\|_{\mathcal{Y}^1_{k,M}} \quad \le \quad C|r-\overline{r}| + C\|\Phi\|_{\mathcal{Y}_M}\|\Phi\|_{\mathcal{Y}^1_{k,M}}. \tag{4.6.24}$$

Proof. The restriction on Φ, together with Lemma 4.6.2 yields the bound

$$\|\Phi\|_\infty \quad \le \quad C_1 \tag{4.6.25}$$

for some $C_1 > 0$. For each $\xi \in p^{-1}\mathbb{Z}$ we get using a Taylor expansion the uniform estimate

$$\begin{aligned} |\mathcal{N}_0(\Phi(\xi);\xi,\theta,r)| &= |\mathcal{R}_1\big(\overline{U}_0(\xi),\Phi(\xi)\big)| \\ &\le C_2|\Phi(\xi)|^2, \end{aligned} \tag{4.6.26}$$

for some remainder term $\mathcal{R}_1\big(\overline{U}_0(\xi),\Phi(\xi)\big)$. Note that $C_2 > 0$ can be chosen independent of ξ, Φ, θ, M and r, see for example [55, Thm. 2.8.3]. This allows us to estimate

$$\begin{aligned} \|\mathcal{N}_0(\Phi(\cdot);\cdot,\theta,r)\|^2_{\mathcal{Y}_M} &= p^{-1}\sum_{\xi \in p^{-1}\mathbb{Z}} |\mathcal{N}_0(\Phi(\xi);\xi,\theta,r)|^2 \\ &\le [C_2]^2 p^{-1}\sum_{\xi\in p^{-1}\mathbb{Z}} |\Phi(\xi)|^4 \\ &\le [C_2]^2 p^{-1}\|v\|^2_\infty \sum_{\xi\in p^{-1}\mathbb{Z}} |\Phi(\xi)|^2 \\ &\le C_3\|\Phi\|^2_{\mathcal{Y}^1_{k,M}}\|\Phi\|^2_{\mathcal{Y}_M} \end{aligned} \tag{4.6.27}$$

for some $C_3 > 0$.

Using a Taylor expansion we can write

$$\mathcal{G}\Big(\overline{U}_\theta(\xi)+\Phi(\xi);r\Big) - \mathcal{G}\Big(\overline{U}_\theta(\xi)+\Phi(\xi);\overline{r}\Big) \quad = \quad D_2\mathcal{G}\Big(\overline{U}_\theta(\xi)+\Phi(\xi),\zeta(\xi)\Big)(r-\overline{r}) \tag{4.6.28}$$

where $\zeta(\xi)$ is in between $\overline{r}$ and r.

With Lemma 4.5.10 we pick $C_4 > 0$ and $\alpha > 0$ in such a way that

$$|\overline{U}_0'(\xi)| \quad \le \quad C_4 e^{-\alpha|\xi|} \tag{4.6.29}$$

holds for all $\xi \in \mathbb{R}$. The limiting value $\lim_{\xi\to-\infty} \overline{U}_0(\xi) = P^-$ implies that for $\xi < 0$ we can write

$$\overline{U}_0(\xi) - P^- \quad = \quad \int_{-\infty}^{\xi} \overline{U}_0'(\xi')d\xi'. \tag{4.6.30}$$

This allows us to compute

$$\begin{array}{rcl} |\overline{U}_0(\xi) - P^-| & \leq & \int_{-\infty}^{\xi} C_4 e^{\alpha\xi'} d\xi' \\ & = & \frac{1}{\alpha} C_4 e^{\alpha\xi} \\ & = & \frac{1}{\alpha} C_4 e^{-\alpha|\xi|} \end{array} \tag{4.6.31}$$

for $\xi < 0$. The limiting value $\lim_{\xi\to\infty} \overline{U}_0(\xi) = P^+$ implies that for $\xi \geq 0$ we can write

$$\overline{U}_0(\xi) - P^+ \;\; = \;\; \int_{\xi}^{\infty} \overline{U}_0'(\xi') d\xi', \tag{4.6.32}$$

which allows us to do the analogous computation to obtain

$$|\overline{U}_0(\xi) - P^+| \;\; \leq \;\; \frac{1}{\alpha} C_4 e^{-\alpha|\xi|} \tag{4.6.33}$$

for $\xi \geq 0$.

Note that $D_2\mathcal{G}(P^{\pm}, \rho) = 0$ for all $0 < \rho < 1$ and that $D_1 D_2 \mathcal{G}(V, \rho)$ is bounded for $|V| \leq \|\overline{U}_0\|_\infty + C_1$ and $0 < \rho < 1$. Therefore, we can pick a constant $C_5 > 0$ in such a way that

$$|D_2\mathcal{G}(V,\rho)| \;\; \leq \;\; C_5 \min\{|V - P^-|, |V - P^+|\}, \tag{4.6.34}$$

for $|V| \leq \|\overline{U}_0\|_\infty + C_1$ and $0 < \rho < 1$. As such, we can estimate

$$\begin{array}{rcl} d & := & \|D_2\mathcal{G}\Big(\overline{U}_\theta(\xi) + \Phi(\xi), \zeta(\xi)\Big)\|_{\mathcal{Y}_M}^2 \\ & \leq & p^{-1} \sum_{\xi \in p^{-1}\mathbb{Z}} C_5 \min\Big\{|\overline{U}_\theta(\xi) + \Phi(\xi) - P^-|^2, |\overline{U}_\theta(\xi) + \Phi(\xi) - P^+|^2\Big\} \\ & \leq & 2C_5 p^{-1} \sum_{\xi \in p^{-1}\mathbb{Z}} \Big[\min\Big\{|\overline{U}_\theta(\xi) - P^-|^2, |\overline{U}_\theta(\xi) - P^+|^2\Big\} + |\Phi(\xi)|^2\Big] \\ & \leq & 2C_5 \Big[\|\Phi\|_{Y_M} + p^{-1} \sum_{\xi \in p^{-1}\mathbb{Z}} \frac{1}{\alpha} C_4 e^{-\alpha|\xi|}\Big] \\ & \leq & C_6 \end{array} \tag{4.6.35}$$

for some constant $C_6 > 0$. The desired bound on $\mathcal{R}_B$ now follows from combining (4.6.27) with the representation (4.6.28) and the bound (4.6.35). ■

Lemma 4.6.4 ([111, Lem. 4.2]). *Assume that (HS1) and (HS2) are satisfied and pick $\overline{r}$ in such a way that (HS3$_{\overline{r}}$), (HW1$_{\overline{r}}$) and (HW2$_{\overline{r}}$) are satisfied. Fix a pair of integers $1 \leq k \leq 6$ and $q \geq 1$. There exists a constant $C > 1$ so that for all $M = \frac{p}{q} \in \mathcal{M}_q$, all $(c, \theta) \in \mathbb{R} \times \mathbb{R}$ and $\Phi \in \mathcal{Y}^1_{k,M}$ with $\|\Phi\|_{\mathcal{Y}^1_{k,M}} \leq 1$ we have the bounds*

$$\begin{array}{rcl} \|\mathcal{R}_A(c, \Phi)\|_{\mathcal{Y}^1_{k,M}} & \leq & |c - \overline{c}_0| \|\mathcal{D}_{k,M}\Phi\|_{\mathcal{Y}_M}, \\ \|\mathcal{R}_C(\theta, M)\|_{\mathcal{Y}^1_{k,M}} & \leq & CM^{-1}. \end{array} \tag{4.6.36}$$

Lemma 4.6.5 (cf. [111, Lem. 4.3]). *Assume that (HS1) and (HS2) are satisfied and pick $\overline{r}$ in such a way that (HS3$_{\overline{r}}$), (HW1$_{\overline{r}}$) and (HW2$_{\overline{r}}$) are satisfied. Fix a pair of* *integers $1 \le k \le 6$ and $q \ge 1$. Then there exists a constant $C > 1$ such that for any pair of constants $0 < \delta_c < 1$ and $0 < \delta_\phi < 1$ and any multiplet*

$$(\Phi_1, \Phi_2, c_1, c_2, r, \theta) \in \mathcal{Y}^1_{k,M} \times \mathcal{Y}^1_{k,M} \times \mathbb{R} \times \mathbb{R} \times (0,1) \times \mathbb{R} \tag{4.6.37}$$

with

$$\begin{array}{lcl} \|\Phi_1\|_{\mathcal{Y}^1_{k,M}} + \|\Phi_2\|_{\mathcal{Y}^1_{k,M}} & \le & \delta_\phi \\ |c_1 - \overline{c}_0| + |\overline{c}_0 - c_2| & \le & \delta_c, \end{array} \tag{4.6.38}$$

we have the bounds

$$\begin{array}{lcl} \|\mathcal{R}_A(c_1, \Phi_1) - \mathcal{R}_A(c_2, \Phi_2)\|_{\mathcal{Y}^1_{k,M}} & \le & \delta_\phi |c_1 - c_2| + \delta_c \|\mathcal{D}_{k,M}[\Phi_1 - \Phi_2]\|_{\mathcal{Y}_M}, \\ \|\mathcal{R}_B(\Phi_1; \theta, r) - \mathcal{R}_B(\Phi_2; \theta, r)\|_{\mathcal{Y}^1_{k,M}} & \le & C|r - \overline{r}| \|\Phi_1 - \Phi_2\|_{\mathcal{Y}_M} + C\delta_\phi \|\Phi_1 - \Phi_2\|_{\mathcal{Y}_M}. \end{array} \tag{4.6.39}$$

Proof. The estimate for $\mathcal{R}_A$ is immediate. Lemma 4.6.2 implies that $\|\Phi_1\|_\infty + \|\Phi_2\|_\infty \le C_1 \delta_\phi$ for some $C_1 > 0$. Using two Taylor approximations, we write

$$\begin{array}{rcl} d\mathcal{N} & := & |\mathcal{N}_0(\Phi_1(\xi); \xi, \theta, r) - \mathcal{N}_0(\Phi_2(\xi); \xi, \theta, r)| \\ & = & \Big| \mathcal{G}\Big(\overline{U}_\theta(\xi) + \Phi_2(\xi) + \big(\Phi_1(\xi) - \Phi_2(\xi)\big); r\Big) - \mathcal{G}\Big(\overline{U}_\theta(\xi) + \Phi_2(\xi); r\Big) \\ & & \quad - D_U \mathcal{G}\Big(\overline{U}_\theta(\xi) + \Phi_2(\xi); r\Big)\big(\Phi_1(\xi) - \Phi_2(\xi)\big) \\ & & \quad + \Big[D_U \mathcal{G}\Big(\overline{U}_\theta(\xi) + \Phi_2(\xi); r\Big) - D_U \mathcal{G}\Big(\overline{U}_\theta(\xi); r\Big)\Big]\big(\Phi_1(\xi) - \Phi_2(\xi)\big)\Big| \\ & = & \Big| \mathcal{R}_1\Big(\overline{U}_\theta(\xi) + \Phi_2(\xi), \Phi_1(\xi) - \Phi_2(\xi)\Big)\big(\Phi_1(\xi) - \Phi_2(\xi)\big) \\ & & \quad + \mathcal{R}_2\Big(\overline{U}_\theta(\xi), \Phi_2(\xi)\Big)\big(\Phi_1(\xi) - \Phi_2(\xi)\big)\Big|, \end{array} \tag{4.6.40}$$

for some remainder terms $\mathcal{R}_1\Big(\overline{U}_\theta(\xi) + \Phi_2(\xi), \Phi_1(\xi) - \Phi_2(\xi)\Big)$ and $\mathcal{R}_2\Big(\overline{U}_\theta(\xi), \Phi_2(\xi)\Big)$. Using [55, Thm. 2.8.3] we can pick a constant $C_1 > 0$ in such a way that

$$\begin{array}{lclcl} \Big|\mathcal{R}_1\Big(\overline{U}_\theta(\xi) + \Phi_2(\xi), \Phi_1(\xi) - \Phi_2(\xi)\Big)\Big| & \le & C_1 \big|\Phi_1(\xi) - \Phi_2(\xi)\big| & \le & 2C_1 \delta_\phi, \\ \Big|\mathcal{R}_2\Big(\overline{U}_\theta(\xi), \Phi_2(\xi)\Big)\Big| & \le & C_1 |\Phi_2(\xi)| & \le & C_1 \delta_\phi. \end{array} \tag{4.6.41}$$

We, therefore, obtain the pointwise inequality

$$d\mathcal{N} \le 3C_1 \delta_\phi \big|\Phi_1(\xi) - \Phi_2(\xi)\big|, \tag{4.6.42}$$

which allows us to compute

$$\begin{array}{rcl} \|\mathcal{N}_0(\Phi_1(\xi); \xi, \theta, r) - \mathcal{N}_0(\Phi_2(\xi); \xi, \theta, r)\|^2_{\mathcal{Y}_M} & \le & p^{-1} \sum_{\xi \in p^{-1}\mathbb{Z}} [C_2]^2 \delta_\phi^2 |\Phi_1(\xi) - \Phi_2(\xi)|^2 \\ & = & [3C_1]^2 \delta_\phi^2 \|\Phi_1 - \Phi_2\|^2_{\mathcal{Y}_M}. \end{array} \tag{4.6.43}$$

Similarly to (4.6.28), we can now write

$$\begin{aligned} dg &:= \mathcal{G}\Big(\overline{U}_\theta(\xi) + \Phi_1(\xi); r\Big) - \mathcal{G}\Big(\overline{U}_\theta(\xi) + \Phi_1(\xi); \overline{r}\Big) \\ &\qquad -\mathcal{G}\Big(\overline{U}_\theta(\xi) + \Phi_2(\xi); r\Big) + \mathcal{G}\Big(\overline{U}_\theta(\xi) + \Phi_2(\xi); \overline{r}\Big) \\ &= D_2\mathcal{G}\Big(\overline{U}_\theta(\xi) + \Phi_2(\xi), \zeta_1(\xi)\Big) - D_2\mathcal{G}\Big(\overline{U}_\theta(\xi) + \Phi_1(\xi), \zeta_2(\xi)\Big), \end{aligned} \tag{4.6.44}$$

where $\zeta_1(\xi)$ and $\zeta_2(\xi)$ are both in between r and $\overline{r}$. Similarly to (4.6.34) we can pick a constant $C_2 > 0$ in such a way that

$$|D_2\mathcal{G}(U_1, \rho) - D_2\mathcal{G}(U_2, \rho)| \quad \le \quad C_2|U_1 - U_2|, \tag{4.6.45}$$

for $|U_1|, |U_2| \le \|\overline{U}_0\|_\infty + C_1\delta_\phi$ and $0 < \rho < 1$. Thus we can immediately estimate

$$\|dg\|_{\mathcal{Y}_M} \quad \le \quad C_2\|\Phi_1 - \Phi_2\|_{\mathcal{Y}_M}, \tag{4.6.46}$$

which yields the desired bound for $\mathcal{R}_B$. ∎

Lemma 4.6.6 (cf. [111, Lem. 4.4])**.** *Assume that (HS1) and (HS2) are satisfied and pick $\overline{r}$ in such a way that (HS3$_{\overline{r}}$), (HW1$_{\overline{r}}$) and (HW2$_{\overline{r}}$) are satisfied. Fix a pair of integers $1 \le k \le 6$ and $q \ge 1$. For all $M = \frac{p}{q} \in \mathcal{M}_q$, the function*

$$\tilde{\mathcal{N}_0} : \mathcal{Y}^1_{k,M} \times \mathbb{R} \times (0,1) \quad \to \quad \mathcal{Y}_M \tag{4.6.47}$$

given by

$$[\tilde{\mathcal{N}_0}(\Phi; \theta, r)](\xi) \quad = \quad \mathcal{N}_0(\Phi(\xi); \xi, \theta, r), \qquad \xi \in p^{-1}\mathbb{Z} \tag{4.6.48}$$

is continuous.

Proof. Fix $(\Phi, \theta, r) \in \mathcal{Y}^1_{k,M} \times \mathbb{R} \times (0,1)$ and let $\varepsilon > 0$ be a small constant. Pick any triplet $(\Psi, \tilde{\theta}, \tilde{r}) \in \mathcal{Y}^1_{k,M} \times \mathbb{R} \times (0,1)$ with $\|\Phi - \Psi\|_{\mathcal{Y}^1_{k,M}} < 1$. Lemma 4.6.2 yields that $\|\Phi - \Psi\|_\infty \le C_1$ for some $C_1 > 0$. Since $\mathcal{G}$ is C^2-smooth, we can pick a constant $C_2 > 0$ in such a way that for any $V, W \in \mathbb{R}^d$ with $|V|, |W| \le \|\overline{U}_0\|_\infty + 2C_1$ and any $0 < r_1, r_2 < 1$ we have the bound

$$\begin{aligned} \Big|\mathcal{G}(V; r_1) - \mathcal{G}(W; r_1)\Big| &\le C_2\Big|V - W\Big|, \\ \Big|D\mathcal{G}(V; r_1) - D\mathcal{G}(W, r_2)\Big| &\le C_2\Big|(V, r_1) - (W, r_2)\Big|. \end{aligned} \tag{4.6.49}$$

Moreover, using two Taylor approximations we write

$$\begin{aligned} dG_1 &:= \mathcal{G}\Big(\overline{U}_\theta(\xi) + \Phi(\xi); r\Big) - \mathcal{G}\Big(\overline{U}_{\tilde{\theta}}(\xi) + \Psi(\xi); \tilde{r}\Big) \\ &= \mathcal{G}\Big(\overline{U}_\theta(\xi) + \Phi(\xi); r\Big) - \mathcal{G}\Big(\overline{U}_{\tilde{\theta}}(\xi) + \Psi(\xi); r\Big) \\ &\qquad -D_2\mathcal{G}\Big(\overline{U}_{\tilde{\theta}}(\xi) + \Psi(\xi); \zeta_2(\Psi(\xi), \xi)\Big)(r - \tilde{r}), \\ dG_2 &:= \mathcal{G}\Big(\overline{U}_\theta(\xi); r\Big) - \mathcal{G}\Big(\overline{U}_{\tilde{\theta}}(\xi); \tilde{r}\Big) \\ &= \mathcal{G}\Big(\overline{U}_\theta(\xi); r\Big) - \mathcal{G}\Big(\overline{U}_{\tilde{\theta}}(\xi); r\Big) \\ &\qquad -D_2\mathcal{G}\Big(\overline{U}_{\tilde{\theta}}(\xi)(r - \tilde{r}); \zeta_1(\xi)\Big)(r - \tilde{r}), \end{aligned} \tag{4.6.50}$$

where $\zeta_1(\xi)$ and $\zeta_2(\Psi(\xi),\xi)$ are both in between r and $\tilde{r}$. Similarly to (4.6.34) we can pick a constant $C_3 > 0$ in such a way that

$$|D_2\mathcal{G}(V,\rho)| \quad \leq \quad C_3 \min\{|V - P^-|, |V - P^+|\} \tag{4.6.51}$$

for any $0 < \rho < 1$ and $|V| \leq \|\overline{U}_0\|_\infty + 2C_1$. This allows us to obtain the pointise estimate

$$\begin{array}{lcl} d\tilde{\mathcal{N}} & := & \Big|[\tilde{\mathcal{N}_0}(\Phi;\theta,r)](\xi) - [\tilde{\mathcal{N}_0}(\Psi;\tilde{\theta},\tilde{r})](\xi)\Big| \\ & \leq & dG_1 + dG_2 + |\Phi(\xi)|\Big|D\mathcal{G}\Big(\overline{U}_\theta(\xi);r\Big) - D\mathcal{G}\Big(\overline{U}_{\tilde{\theta}}(\xi);\tilde{r}\Big)\Big| \\ & & \qquad +\Big|\Phi(\xi) - \Psi(\xi)\Big|\Big|D\mathcal{G}\Big(\overline{U}_{\tilde{\theta}}(\xi);\tilde{r}\Big)\Big| \\ & \leq & C_2\Big|\overline{U}_\theta(\xi) + \Phi(\xi) - \overline{U}_{\tilde{\theta}}(\xi) - \Psi(\xi)\Big| \\ & & \qquad +C_3(r-\tilde{r})\min\Big\{|\overline{U}_{\tilde{\theta}}(\xi) + \Psi(\xi) - P^-|, |\overline{U}_{\tilde{\theta}} + \Psi(\xi) - P^+|\Big\} \\ & & \qquad +C_2\Big|\overline{U}_\theta(\xi) - \overline{U}_{\tilde{\theta}}(\xi)\Big| + C_3(r-\tilde{r})\min\Big\{|\overline{U}_{\tilde{\theta}}(\xi) - P^-|, |\overline{U}_{\tilde{\theta}} - P^+|\Big\} \\ & & \qquad +|\Phi(\xi)|\Big|(\overline{U}_\theta(\xi),r) - (\overline{U}_{\tilde{\theta}}(\xi),\tilde{r})\Big| + \Big|\Phi(\xi) - \Psi(\xi)\Big| \\ & \leq & 2C_2\Big|\overline{U}_\theta(\xi) - \overline{U}_{\tilde{\theta}}(\xi)\Big| + (1+C_2)\big|\Phi(\xi) - \Psi(\xi)\big| \\ & & \qquad +\big(1 + |\Psi(\xi)|\big)C_3(r-\tilde{r})\min\Big\{|\overline{U}_{\tilde{\theta}}(\xi) - P^-|, |\overline{U}_{\tilde{\theta}} - P^+|\Big\} \\ & & \qquad +|\Phi(\xi)|\Big[\Big|\overline{U}_\theta(\xi) - \overline{U}_{\tilde{\theta}}(\xi)\Big| + \Big|r - \tilde{r}\Big|\Big]. \end{array} \tag{4.6.52}$$

Since $\overline{U}_\theta$ decays exponentially to its limits, we can pick $0 < \delta < 1$ in such a way that for each $\tilde{\theta} \in \mathbb{R}$ with $|\theta - \tilde{\theta}| < \delta$ and each $\xi \; ^{-1}\mathbb{Z}$ we have the bound

$$|\overline{U}_\theta(\xi) - \overline{U}_{\tilde{\theta}}(\xi)| \quad \leq \quad \min\{\tfrac{p\varepsilon}{30C_2 2^{|n|}}, \tfrac{p\varepsilon}{30(\|\Phi\|_{\mathcal{Y}_M}+1)2^{|n|}}\}. \tag{4.6.53}$$

This yields the estimates

$$\begin{array}{lcl} 2C_2 p^{-1}\sum_{\xi \in p^{-1}\mathbb{Z}}\Big|\overline{U}_\theta(\xi) - \overline{U}_{\tilde{\theta}}(\xi)\Big| & \leq & \frac{\varepsilon}{5}, \\ p^{-1}\sum_{\xi\in p^{-1}\mathbb{Z}}|\Phi(\xi)|\Big|\overline{U}_\theta(\xi) - \overline{U}_{\tilde{\theta}}(\xi)\Big| & \leq & \frac{\varepsilon}{5}. \end{array} \tag{4.6.54}$$

Moreover, similarly to (4.6.33) we pick $C_4 > 0$ in such a way that the pointwise estimate

$$\min\Big\{|\overline{U}_{\tilde{\theta}}(\xi) - P^-|, |\overline{U}_{\tilde{\theta}}(\xi) - P^+|\Big\} \quad \leq \quad \tfrac{1}{\alpha}C_4 e^{-\alpha|\xi|} \tag{4.6.55}$$

holds for any $\xi \in \mathbb{R}$ and any $\tilde{\theta} \in \mathbb{R}$ with $|\tilde{\theta} - \theta| < \delta$. As such we can pick $C_5 > 0$ in such a way that the bound

$$\|\min\Big\{|\overline{U}_{\tilde{\theta}} - P^-|, |\overline{U}_{\tilde{\theta}} - P^+|\Big\}\|_{\mathcal{Y}_M} \quad \leq \quad C_5 \tag{4.6.56}$$

holds for any $\tilde{\theta} \in \mathbb{R}$ with $|\tilde{\theta} - \theta| < \delta$.

Then we obtain for each triplet $(\Psi, \tilde{\theta}, \tilde{r}) \in \mathcal{Y}^1_{k,M} \times \mathbb{R} \times (0,1)$ with $\|\Phi - \Psi\|_{\mathcal{Y}^1_{k,M}} < \min\{1, \frac{\varepsilon}{5(1+C_2)}\}$, $|\theta - \tilde{\theta}| < \delta$ and $|r - \tilde{r}| < \min\Big\{\frac{\varepsilon}{5(1+C_1)C_3C_5}, \frac{\varepsilon}{5(\|\Phi\|_{\mathcal{Y}_M}+1)}\Big\}$ that

$$\begin{array}{lll} \|d\tilde{\mathcal{N}}\|_{\mathcal{Y}_M} & = & \|[\tilde{\mathcal{N}_0}(\Phi;\theta,r)] - [\tilde{\mathcal{N}_0}(\Psi;\tilde{\theta},\tilde{r})]\|_{\mathcal{Y}_M} \\ & \leq & \frac{\varepsilon}{5} + (1+C_2)\frac{\varepsilon}{5(1+C_2)} + (1+C_1)C_3\frac{\varepsilon}{5(1+C_1)C_3C_5}C_5 \\ & & \quad + \frac{\varepsilon}{5} + \|\Phi\|_{\mathcal{Y}_M}\frac{\varepsilon}{5(\|\Phi\|_{\mathcal{Y}_M}+1)} \\ & < & \varepsilon. \end{array} \tag{4.6.57}$$

Therefore, the function $\tilde{\mathcal{N}_0}$ is continuous in the point (Φ, θ, r), which yields the desired result. ■

Proof of Proposition 4.6.1. Recall the constants $C_{\text{unif}} > 1$ and $M_{\text{unif}} \in \mathcal{M}_q$, together with the bounds (4.6.11) and (4.6.12). We let $C > 1$ be the constant from Lemmas 4.6.3-4.6.5. For any $0 < \delta_\phi < 1$ and $0 < \delta_c < 1$ we introduce the space

$$\mathcal{Z}_{\delta_c,\delta_\phi} \quad = \quad \{(c,\Phi) \in \mathbb{R} \times \mathcal{Y}^1_{k,M} : |c - \overline{c}_0| \leq \tfrac{1}{2}\delta_c \text{ and } \|\Phi\|_{\mathcal{Y}^1_{k,M}} \leq \tfrac{1}{2}\delta_\phi\}, \tag{4.6.58}$$

together with the map

$$\begin{array}{rcl} T_{\delta_c,\delta_\phi} : \mathcal{Z}_{\delta_c,\delta_\phi} & \to & \mathbb{R} \times \mathcal{Y}^1_{k,M}, \\ (c,\Phi) & \mapsto & \left(\begin{array}{l} \gamma^*_{k,M;\theta}\Big[\mathcal{R}_A(\overline{c}_0 - c, \Phi) + \mathcal{R}_B(\Phi;\theta,r) + \mathcal{R}_C(\theta,M)\Big] \\ \mathcal{V}^*_{k,M;\theta}\Big[\mathcal{R}_A(c,\Phi) + \mathcal{R}_B(\Phi;\theta,r) + \mathcal{R}_C(\theta,M)\Big] \end{array}\right). \end{array} \tag{4.6.59}$$

Upon setting

$$\delta_\phi \quad = \quad \delta_c \quad = \quad \min\{\tfrac{1}{64C_{\text{unif}}}, \tfrac{1}{32C_{\text{unif}}C}\}, \tag{4.6.60}$$

fixing $M_* \in \mathcal{M}_q$ with $M_* \geq M_{\text{unif}}$ in such a way that the bound

$$M^{-1} \quad \leq \quad \tfrac{1}{16C_{\text{unif}}C} \tag{4.6.61}$$

holds for all $M \in \mathcal{M}_q$ with $M \geq M_*$, together with the constant

$$\delta_r \quad = \quad \tfrac{1}{32C_{\text{unif}}C}\delta_\phi, \tag{4.6.62}$$

we use Lemmas 4.6.3-4.6.4 to compute

$$\begin{array}{lll} \|T_{\delta_c,\delta_\phi}(c,\Phi)\|_{\mathbb{R}\times\mathcal{Y}^1_{k,M}} & \leq & C_{\text{unif}}\Big[|c-\overline{c}_0|\|\mathcal{D}_{k,M}\Phi\|_{\mathcal{Y}_M} \\ & & \quad + C|r-\overline{r}| + C\|\Phi\|_{\mathcal{Y}_M}\|\Phi\|_{\mathcal{Y}^1_{k,M}} + CM^{-1}\Big] \\ & \leq & C_{\text{unif}}\Big[|c-\overline{c}_0|\|\Phi\|_{\mathcal{Y}^1_{k,M}} + C|r-\overline{r}| + C\|\Phi\|^2_{\mathcal{Y}^1_{k,M}} + CM^{-1}\Big] \\ & \leq & C_{\text{unif}}\Big[\frac{1}{16C_{\text{unif}}}\delta_\phi + C\frac{1}{16C_{\text{unif}}C}\delta_\phi + C\frac{1}{16C_{\text{unif}}C}\delta_\phi + C\frac{1}{16C_{\text{unif}}C}\Big] \\ & = & \frac{1}{4}\delta_\phi \end{array} \tag{4.6.63}$$

for any $(c, \Phi) \in \mathcal{Z}_{\delta_c,\delta_\phi}$, any $\theta \in \mathbb{R}$, any $r \in [\overline{r} - \delta_r, \overline{r} + \delta_r]$ and any $M \in \mathcal{M}_q$ with $M \geq M_*$. As such, we have $T_{\delta_c,\delta_\phi}(c, \Phi) \in \mathcal{Z}_{\delta_c,\delta_\phi}$. Moreover, using Lemma 4.6.5 we obtain the estimate

$$\begin{array}{rcl} dT & := & \|T_{\delta_c,\delta_\phi}(c_1, \Phi_1) - T_{\delta_c,\delta_\phi}(c_2, \Phi_2)\|_{\mathbb{R}\times\mathcal{Y}^1_{k,M}} \\ & \leq & 2C_{\mathrm{unif}}\Big[\delta_\phi|c_1 - c_2| + 2\delta_c\|\Phi_1 - \Phi_2\|_{\mathcal{Y}^1_{k,M}} \\ & & \qquad +C|r - \overline{r}|\|\Phi_1 - \Phi_2\|_{\mathcal{Y}^1_{k,M}} + C\delta_\phi\|\Phi_1 - \Phi_2\|_{\mathcal{Y}_M}\Big] \\ & \leq & 2C_{\mathrm{unif}}\Big[\frac{1}{32C_{\mathrm{unif}}}|c_1 - c_2| + 2\frac{1}{64C_{\mathrm{unif}}}\|\Phi_1 - \Phi_2\|_{\mathcal{Y}^1_{k,M}} \\ & & \qquad +C\frac{1}{32C_{\mathrm{unif}}C}\|\Phi_1 - \Phi_2\|_{\mathcal{Y}^1_{k,M}} + C\frac{1}{32C_{\mathrm{unif}}C}\|\Phi_1 - \Phi_2\|_{\mathcal{Y}^1_{k,M}}\Big] \\ & = & \frac{1}{16}|c_1 - c_2| + \frac{3}{16}\|\Phi_1 - \Phi_2\|_{\mathcal{Y}^1_{k,M}} \\ & \leq & \frac{1}{2}\|(c_1, \Phi_1) - (c_2, \Phi_2)\|_{\mathbb{R}\times\mathcal{Y}^1_{k,M}}, \end{array} \tag{4.6.64}$$

for any $(c_1, \Phi_1), (c_2, \Phi_2) \in \mathcal{Z}_{\delta_c,\delta_\phi}$, any $\theta \in \mathbb{R}$, any $r \in [\overline{r} - \delta_r, \overline{r} + \delta_r]$ and any $M \in \mathcal{M}_q$ with $M \geq M_*$, which shows that T_{δ_c,δ_ϕ} is a contraction. The fixed point theorem now implies that the map T_{δ_c,δ_ϕ}, and therefore the fixed point problem (4.6.22), has a unique fixed point $\big(c^*_M(\theta, r), \Phi^*_M(\theta, r)\big)$. By construction the pair $\big(c_M(\theta, r), \overline{U}_M(\theta, r)\big) = \big(c^*_M(\theta, r), \overline{U}_\theta + \Phi^*_M(\theta, r)\big)$ satisfies (4.2.34) with the boundary conditions (4.2.35).

The solution to this fixed point problem depends continuously on the parameters (θ, r) on account of Lemma 4.6.6 and our observations concerning the continuity of the functions $\theta \mapsto \gamma^*_{k,M;\theta}$ and $\theta \mapsto \mathcal{V}^*_{k,M;\theta}$. In addition, the normalisation (4.2.36) follows from the normalisation of the function $\mathcal{V}^*_{k,M;\theta}$ in Proposition 4.5.1. Finally, it is clear that the pair $\big(c^*_M(\theta + p^{-1}, r), \Phi^*_M(\theta + p^{-1}, r)(\cdot - p^{-1})\big)$ is also a solution to the fixed point problem (4.6.22), which by the uniqueness of solutions yields the shift-periodicity (4.2.37). ∎

4.6.2 Local uniqueness of solutions

We now turn to the uniqueness claim in the statement of the main theorem. The main issue is to obtain the decomposition (4.6.66) below. Indeed, this implies that (4.2.39) ensures that $(c, \overline{U}, \theta)$ is captured by the fixed point argument associated to the phase ϑ.

Proposition 4.6.7. *Assume that (HS1) and (HS2) are satisfied and pick $\overline{r}$ in such a way that (HS3$_{\overline{r}}$), (HW1$_{\overline{r}}$) and (HW2$_{\overline{r}}$) are satisfied. Fix a pair of integers $q \geq 1$ and $1 \leq k \leq 6$. Then there exists a small constant $\delta > 0$ so that for each $M = \frac{p}{q} \in \mathcal{M}_q$ with $M \geq M_*$ and any $(c, \overline{U}, \theta) \in \mathbb{R} \times \ell^\infty(p^{-1}\mathbb{Z}; \mathbb{R}) \times \mathbb{R}$ that satisfies*

$$\|\overline{U} - \overline{U}_\theta\|_{\mathcal{Y}^1_{k,M}} < \delta, \tag{4.6.65}$$

the function $\overline{U}$ can be decomposed as

$$\overline{U} = \pi_{\mathcal{Y}_M}\overline{U}_{\tilde{\theta}} + \Phi \tag{4.6.66}$$

for some $\Phi \in \mathcal{Y}^1_{k,M}$ *with* $\langle \pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta}}, \Phi \rangle_{\mathcal{Y}_M} = 0$ *and some* $\tilde{\theta}$ *close to* θ.

Using a Taylor approximation we can pick for each $\tilde{\theta} \in \mathbb{R}$ a sequence $\{\zeta_{\tilde{\theta}}(p^{-1}n)\}_{n \in \mathbb{Z}}$, with $\zeta_{\tilde{\theta}}(p^{-1}n)$ in between $p^{-1}n + \theta$ and $p^{-1}n + \tilde{\theta}$ for each $n \in \mathbb{Z}$ in such a way that

$$\overline{U}_{\tilde{\theta}}(\xi) - \overline{U}_{\theta} \quad = \quad (\tilde{\theta} - \theta)\overline{U}'_{\theta}(\xi) + (\tilde{\theta} - \theta)^2 \overline{U}''_{\theta}\big(\zeta_{\tilde{\theta}}(\xi)\big) \qquad (4.6.67)$$

holds for all $\xi \in p^{-1}\mathbb{Z}$. For $\tilde{\theta} \in \mathbb{R}$ we denote $\tilde{\theta}_0$ for the unique element of $[0, 1)$ which has $\tilde{\theta} - \tilde{\theta}_0 \in \mathbb{Z}$ and pick $n \in \mathbb{Z}$ in such a way that $\theta_0 = \theta - n$. For any $\tilde{\theta} \in \mathbb{R}$ with $|\tilde{\theta} - \theta| < 1$ we can compute

$$\langle \pi_{\mathcal{Y}_M} \overline{U}'_{\theta}, \pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta}} \rangle_{\mathcal{Y}_M} \quad = \quad \langle \pi_{\mathcal{Y}_M} \overline{U}'_{\theta_0}, \pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta} - nh} \rangle_{\mathcal{Y}_M}. \qquad (4.6.68)$$

Lemma 4.6.8. *Assume that (HS1) and (HS2) are satisfied and pick* $\overline{r}$ *in such a way that (HS3$_{\overline{r}}$), (HW1$_{\overline{r}}$) and (HW2$_{\overline{r}}$) are satisfied. Fix a pair of integers* $q \geq 1$ *and* $1 \leq k \leq 6$. *Then there exists a constant* $\kappa > 0$ *in so that for any* $M \in \mathcal{M}_q$ *with* $M \geq M_*$ *and any pair* $(\theta, \tilde{\theta}) \in \mathbb{R} \times \mathbb{R}$ *with* $|\tilde{\theta} - \theta| < 1$ *we have the lower bound*

$$\langle \pi_{\mathcal{Y}_M} \overline{U}'_{\theta}, \pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta}} \rangle_{\mathcal{Y}_M} \quad > \quad \tfrac{1}{\kappa}. \qquad (4.6.69)$$

Proof. On account of Lemma 4.5.10 we can pick constants $C_1 > 0$ and $\alpha > 0$ in such a way that the bounds

$$\begin{array}{lcl} |\overline{U}'_0| & \leq & C_1 e^{-\alpha|\xi|}, \\ |\overline{U}''_0| & \leq & C_1 e^{-\alpha|\xi|}, \\ |\Phi^-_0(\xi)| & \leq & C_1 e^{-\alpha|\xi|}, \\ |(\Phi^-_0)'(\xi)| & \leq & C_1 e^{-\alpha|\xi|} \end{array} \qquad (4.6.70)$$

hold for all $\xi \in \mathbb{R}$. For any $\tilde{\theta} \in \mathbb{R}$ with $|\tilde{\theta} - \theta| < 1$ we hence obtain

$$|\Phi^-_{\tilde{\theta}-n}(\xi)| \;\leq\; C_1 e^{-\alpha|\xi + \tilde{\theta} - n|} \;\leq\; C_1 e^{\alpha|\tilde{\theta} - n|} e^{-\alpha|\xi|} \;\leq\; C_1 e^{2\alpha} e^{-\alpha|\xi|}, \qquad (4.6.71)$$

which yields

$$\|\Phi^-_{\tilde{\theta}-n}\|_{BC_{-\alpha}(\mathbb{R};\mathbb{R})} \quad \leq \quad C_1 e^{2\alpha}. \qquad (4.6.72)$$

A similar computation provides the bounds

$$\begin{array}{lcl} \|\overline{U}'_{\theta_0}\|_{BC_{-\alpha}(\mathbb{R};\mathbb{R})} & \leq & C_1 e^{2\alpha}, \\ \|\overline{U}''_{\theta_0}\|_{BC_{-\alpha}(\mathbb{R};\mathbb{R})} & \leq & C_1 e^{2\alpha}, \\ \|(\Phi^-_{\tilde{\theta}-n})'\|_{BC_{-\alpha}(\mathbb{R};\mathbb{R})} & \leq & C_1 e^{2\alpha}. \end{array} \qquad (4.6.73)$$

On account of Lemma 4.A.2 and the fact that $\langle \overline{U}'_0, \Phi^-_0 \rangle_{L^2(\mathbb{R};\mathbb{R}^d)} > 0$, we can assume without loss of generality that M_* is large enough for the bound

$$\langle \pi_{\mathcal{Y}_M} \overline{U}'_{\theta}, \pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta}} \rangle_{\mathcal{Y}_M} \quad > \quad \tfrac{1}{\kappa} \qquad (4.6.74)$$

to hold for all $M \in \mathcal{M}_q$ with $M \geq M_*$, all $\theta \in \mathbb{R}$, all $\tilde{\theta} \in \mathbb{R}$ with $|\theta - \tilde{\theta}| < 1$ and for some constant $\kappa > 0$, as desired. ■

Fix a small constant $0 < \delta_\theta < 1$. In order to find a $\tilde{\theta}$ close to θ in such a way that

$$\langle \Phi - \pi_{\mathcal{Y}_M} \overline{U}_{\tilde{\theta}}, \pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta}} \rangle_{\mathcal{Y}_M} \quad = \quad 0, \tag{4.6.75}$$

we aim to solve the fixed point problem

$$\begin{aligned} \tilde{\theta} - \theta &= F_{\theta,\delta_\theta}(\tilde{\theta}) \\ &:= -\langle \pi_{\mathcal{Y}_M} \overline{U}'_\theta, \pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta}} \rangle^{-1}_{\mathcal{Y}_M} \Big[\langle \Phi - \pi_{\mathcal{Y}_M} \overline{U}_\theta, \pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta}} \rangle_{\mathcal{Y}_M} \\ &\qquad + (\theta - \tilde{\theta})^2 \langle \pi_{\mathcal{Y}_M} \overline{U}''_\theta(\zeta_{\tilde{\theta}}(\cdot)), \pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta}} \rangle_{\mathcal{Y}_M} \Big] \end{aligned} \tag{4.6.76}$$

on the space $[\theta - \delta_\theta, \theta + \delta_\theta]$.

Lemma 4.6.9. *Assume that (HS1) and (HS2) are satisfied and pick $\overline{r}$ in such a way that (HS3$_{\overline{r}}$), (HW1$_{\overline{r}}$) and (HW2$_{\overline{r}}$) are satisfied. Consider the setting of Proposition 4.6.7. Then there exist constants $C_2 > 0$ and $C_3 > 0$ so that the bound*

$$|F_{\theta,\delta_\theta}(\tilde{\theta})| \quad \leq \quad \kappa \Big[\delta C_3 + \delta^2_\theta C_2 \Big] \tag{4.6.77}$$

holds for all $\tilde{\theta} \in [\theta - \delta_\theta, \theta + \delta_\theta]$.

Proof. For $\tilde{\theta} \in \mathbb{R}$ and $M \in \mathcal{M}_q$ with $M \geq M_*$ we can estimate

$$\begin{aligned} |\langle \pi_{\mathcal{Y}_M} \overline{U}''_\theta(\zeta_{\tilde{\theta}}(\cdot)), \pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta}} \rangle_{\mathcal{Y}_M}| &\leq p^{-1} \sum_{\xi \in p^{-1}\mathbb{Z}} |\overline{U}''_\theta(\zeta_{\tilde{\theta}}(\xi))| |\Phi^-_{\tilde{\theta}}(\xi)| \\ &\leq \|\overline{U}''_\theta\|_{L^\infty(\mathbb{R};\mathbb{R})} p^{-1} \sum_{\xi \in p^{-1}\mathbb{Z}} |\Phi^-_{\tilde{\theta}}(\xi)| \\ &= \|\overline{U}''_0\|_{L^\infty(\mathbb{R};\mathbb{R})} p^{-1} \sum_{\xi \in p^{-1}\mathbb{Z}} |\Phi^-_{\tilde{\theta}_0}(\xi)| \\ &\leq \|\overline{U}''_0\|_{L^\infty(\mathbb{R};\mathbb{R})} p^{-1} \sum_{\xi \in p^{-1}\mathbb{Z}} C_1 e^{-\alpha|\xi + \tilde{\theta}_0|} \\ &\leq \|\overline{U}''_0\|_{L^\infty(\mathbb{R};\mathbb{R})} p^{-1} \sum_{\xi \in p^{-1}\mathbb{Z}} C_1 e^{\alpha\tilde{\theta}_0} e^{-\alpha|\xi|} \\ &\leq C_1 e^{\alpha h} \|\overline{U}''_0\|_{L^\infty(\mathbb{R};\mathbb{R})} p^{-1} \sum_{\xi \in p^{-1}\mathbb{Z}} e^{-\alpha|\xi|} \\ &\leq C_2 \end{aligned} \tag{4.6.78}$$

for some constant $C_2 > 0$, since $p^{-1} \sum_{\xi \in p^{-1}\mathbb{Z}} e^{-\alpha|\xi|}$ is bounded as $p \to \infty$. A similar calculation yields the existence of a constant $C_3 > 0$ for which the bounds

$$\begin{aligned} \|\pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta}}\|_{\mathcal{Y}_M} &\leq C_3, \\ \|\pi_{\mathcal{Y}_M} \overline{U}'_{\tilde{\theta}}\|_{\mathcal{Y}_M} &\leq C_3 \end{aligned} \tag{4.6.79}$$

hold for all $\tilde{\theta} \in \mathbb{R}$ and $M \in \mathcal{M}_q$ with $M \geq M_*$. The Cauchy-Schwarz inequality now yields the bound

$$|F_{\theta,\delta_\theta}(\tilde{\theta})| \leq \kappa\Big[\delta C_3 + (\theta - \tilde{\theta})^2 C_2\Big] \leq \kappa\Big[\delta C_3 + \delta_\theta^2 C_2\Big] \tag{4.6.80}$$

for all $\tilde{\theta} \in [\theta - \delta_\theta, \theta + \delta_\theta]$. ∎

Lemma 4.6.10. *Assume that (HS1) and (HS2) are satisfied and pick $\overline{r}$ in such a way that (HS3$_{\overline{r}}$), (HW1$_{\overline{r}}$) and (HW2$_{\overline{r}}$) are satisfied. Consider the setting of Proposition 4.6.7. Then there exist constants $C_4 > 0$ and $C_7 > 0$ so that the bound*

$$|F_{\theta,\delta_\theta}(\tilde{\theta}_1) - F_{\theta,\delta_\theta}(\tilde{\theta}_1)| \leq \kappa|\tilde{\theta}_1 - \tilde{\theta}_2|\Big[\delta C_4 + \delta_\theta C_7\Big] + \kappa^2 C_3 C_4 |\tilde{\theta}_1 - \tilde{\theta}_2|\Big[\delta C_3 + \delta_\theta^2 C_2\Big] \tag{4.6.81}$$

holds for all $\tilde{\theta}_1, \tilde{\theta}_2 \in [\theta - \delta_\theta, \theta + \delta_\theta]$.

Proof. Fix $\tilde{\theta}_1, \tilde{\theta}_2 \in [\theta - \delta_\theta, \theta + \delta_\theta]$ and write

$$\tilde{\theta}_1 = (\tilde{\theta}_1)_0 + n \tag{4.6.82}$$

with $(\tilde{\theta}_1)_0 \in [0, 1)$ and $n \in \mathbb{Z}$. Using a Taylor approximation we pick a sequence $\{\tilde{\zeta}(\xi) : \xi \in p^{-1}\mathbb{Z}\}$ in such a way that $\tilde{\zeta}(\xi)$ is in between $\xi + (\tilde{\theta}_1)_0$ and $\xi + \tilde{\theta}_2 - n$ and we have the identity

$$\begin{aligned} (\Phi^-_{(\tilde{\theta}_1)_0} - \Phi^-_{\tilde{\theta}_2 - n})(\xi) &= \big((\tilde{\theta}_1)_0 - (\tilde{\theta}_2 - n)\big)(\Phi^-_{\tilde{\theta}_2 - n})'\big(\tilde{\zeta}(\xi)\big) \\ &= (\tilde{\theta}_1 - \tilde{\theta}_2)(\Phi^-_{\tilde{\theta}_2 - n})'\big(\tilde{\zeta}(\xi)\big). \end{aligned} \tag{4.6.83}$$

The Cauchy-Schwarz inequality now yields the estimate

$$\begin{aligned} d_1^2 &:= |\langle \Phi - \pi_{\mathcal{Y}_M} \overline{U}_\theta, \pi_{\mathcal{Y}_M}(\Phi^-_{\tilde{\theta}_1} - \Phi^-_{\tilde{\theta}_2})\rangle_{\mathcal{Y}_M}|^2 \\ &\leq \delta^2\Big[p^{-1} \sum_{\xi \in p^{-1}\mathbb{Z}} |(\Phi^-_{\tilde{\theta}_1} - \Phi^-_{\tilde{\theta}_2})(\xi)|^2\Big] \\ &= \delta^2\Big[p^{-1} \sum_{\xi \in p^{-1}\mathbb{Z}} |(\Phi^-_{(\tilde{\theta}_1)_0} - \Phi^-_{\tilde{\theta}_2 - n})(\xi)|^2\Big] \\ &= \delta^2\Big[p^{-1} \sum_{\xi \in p^{-1}\mathbb{Z}} |(\tilde{\theta}_1 - \tilde{\theta}_2)(\Phi^-_{\tilde{\theta}_2 - n})'\big(\tilde{\zeta}(\xi)\big)|^2\Big] \\ &\leq \delta^2 |\tilde{\theta}_1 - \tilde{\theta}_2|^2 \Big[p^{-1} \sum_{\xi \in p^{-1}\mathbb{Z}} C_1 e^{-2\alpha|\tilde{\zeta}(\xi)|}\Big] \\ &\leq C_1 \delta^2 |\tilde{\theta}_1 - \tilde{\theta}_2|^2 \Big[p^{-1} \sum_{\xi \in p^{-1}\mathbb{Z}} e^{-2\alpha|\xi|} e^{2\alpha(1+\delta_\theta)}\Big] \\ &\leq (C_4)^2 \delta^2 |\tilde{\theta}_1 - \tilde{\theta}_2|^2 \end{aligned} \tag{4.6.84}$$

for some constant $C_4 > 0$, since $\delta_\theta < 1$ and $p^{-1} \sum_{\xi \in p^{-1}\mathbb{Z}} e^{-2\alpha|\xi|}$ is bounded as $p \to \infty$.

Moreover, we obtain the estimate

$$
\begin{array}{lcl}
d_2 & := & \Big|(\theta - \tilde{\theta}_1)^2 \langle \pi_{\mathcal{Y}_M} \overline{U}''_{\tilde{\theta}_1}\big(\zeta_{\tilde{\theta}_1}(\cdot)\big), \pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta}_1} \rangle_{\mathcal{Y}_M} \\
& & \quad - (\theta - \tilde{\theta}_2)^2 \langle \pi_{\mathcal{Y}_M} \overline{U}''_{\theta}\big(\zeta_{\tilde{\theta}_2}(\cdot)\big), \pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta}_2} \rangle_{\mathcal{Y}_M} \Big| \\
& \le & (\theta - \tilde{\theta}_1)^2 \Big[\Big| \langle \pi_{\mathcal{Y}_M} \overline{U}''_{\tilde{\theta}_1}\big(\zeta_{\tilde{\theta}_1}(\cdot)\big) - \pi_{\mathcal{Y}_M} \overline{U}''_{\tilde{\theta}_2}\big(\zeta_{\tilde{\theta}_2}(\cdot)\big), \pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta}_1} \rangle_{\mathcal{Y}_M} \Big| \\
& & \quad + \Big| \langle \pi_{\mathcal{Y}_M} \overline{U}''_{\tilde{\theta}_1}\big(\zeta_{\tilde{\theta}_1}(\cdot)\big), \pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta}_1} - \pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta}_2} \rangle_{\mathcal{Y}_M} \Big| \Big] \\
& & \quad + \Big[|\tilde{\theta}_1 - \theta| + |\tilde{\theta}_2 - \theta| \Big] |\tilde{\theta}_1 - \tilde{\theta}_2| \Big| \langle \pi_{\mathcal{Y}_M} \overline{U}''_{\theta}\big(\zeta_{\tilde{\theta}_2}(\cdot)\big), \pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta}_2} \rangle_{\mathcal{Y}_M} \Big| \\
& \le & \delta_\theta^2 \Big[d_3 + d_4 \Big] + 2\delta_\theta |\tilde{\theta}_1 - \tilde{\theta}_2| C_2,
\end{array}
\tag{4.6.85}
$$

where we introduced

$$
\begin{array}{lcl}
d_3 & = & \Big| \langle \pi_{\mathcal{Y}_M} \overline{U}''_{\theta}\big(\zeta_{\tilde{\theta}_1}(\cdot)\big) - \pi_{\mathcal{Y}_M} \overline{U}''_{\theta}\big(\zeta_{\tilde{\theta}_2}(\cdot)\big), \pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta}_1} \rangle_{\mathcal{Y}_M} \Big|, \\
d_4 & = & \Big| \langle \pi_{\mathcal{Y}_M} \overline{U}''_{\theta}\big(\zeta_{\tilde{\theta}_1}(\cdot)\big), \pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta}_1} - \pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta}_2} \rangle_{\mathcal{Y}_M} \Big|.
\end{array}
\tag{4.6.86}
$$

Again using a Taylor approximation we pick a sequence $\{\overline{\zeta}(\xi) : \xi \in p^{-1}\mathbb{Z}\}$ in such a way that $\overline{\zeta}(\xi)$ is in between $\xi + (\tilde{\theta}_1)_0$ and $\xi + \tilde{\theta}_2 - n$ and we have the identity

$$
(\overline{U}_{(\tilde{\theta}_1)_0} - \overline{U}_{\tilde{\theta}_2 - n})(\xi) \quad = \quad (\tilde{\theta}_1 - \tilde{\theta}_2)(\overline{U}_{\tilde{\theta}_2 - n})'\big(\overline{\zeta}(\xi)\big).
\tag{4.6.87}
$$

The definition (4.6.67) of $\zeta_{\tilde{\theta}_1}$ and $\zeta_{\tilde{\theta}_2}$ and the Cauchy-Schwarz inequality allow us to estimate

$$
\begin{array}{lcl}
d_3^2 & = & \Big| \langle \pi_{\mathcal{Y}_M} (\overline{U}_{\tilde{\theta}_1} - \overline{U}_{\tilde{\theta}_2} + (\tilde{\theta}_1 - \tilde{\theta}_2)\overline{U}'_{\theta}), \pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta}_1} \rangle_{\mathcal{Y}_M} \Big|^2 \\
& \le & |\tilde{\theta}_1 - \tilde{\theta}_2|^2 (C_3)^4 + (C_3)^2 \Big[p^{-1} \sum_{\xi \in p^{-1}\mathbb{Z}} |\tilde{\theta}_1 - \tilde{\theta}_2|^2 |\overline{U}'_{\tilde{\theta}_2 - n}\big(\overline{\zeta}(\xi)\big)|^2 \Big] \\
& \le & (C_5)^2 |\tilde{\theta}_1 - \tilde{\theta}_2|^2
\end{array}
\tag{4.6.88}
$$

for some constant $C_5 > 0$ using a calculation similar to (4.6.84). Moreover, upon combining the ideas behind (4.6.78) and (4.6.84) we arrive at

$$
d_4 \quad \le \quad C_6 |\tilde{\theta}_1 - \tilde{\theta}_2|
\tag{4.6.89}
$$

for some constant $C_6 > 0$. This yields the bound

$$
\begin{array}{lcl}
d_2 & \le & \delta_\theta^2 \Big[d_3 + d_4 \Big] + 2\delta_\theta |\tilde{\theta}_1 - \tilde{\theta}_2| C_2 \\
& \le & \delta_\theta^2 |\tilde{\theta}_1 - \tilde{\theta}_2| \Big[C_5 + C_6 \Big] + 2\delta_\theta |\tilde{\theta}_1 - \tilde{\theta}_2| C_2 \\
& \le & C_7 \delta_\theta |\tilde{\theta}_1 - \tilde{\theta}_2|
\end{array}
\tag{4.6.90}
$$

for some constant $C_7 > 0$. The Cauchy-Schwarz inequality combined with the estimate (4.6.84) yields

$$
\begin{array}{lcl}
d_5 & := & |\langle \pi_{\mathcal{Y}_M} \overline{U}'_{\theta}, \pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta}_1} - \pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta}_2} \rangle_{\mathcal{Y}_M}| \\
& \le & C_3 C_4 |\tilde{\theta}_1 - \tilde{\theta}_2|.
\end{array}
\tag{4.6.91}
$$

We therefore can estimate

$$\begin{array}{rcl} d_6 & := & \big|\langle \pi_{\mathcal{Y}_M} \overline{U}'_\theta, \pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta}_1} \rangle^{-1}_{\mathcal{Y}_M} - \langle \pi_{\mathcal{Y}_M} \overline{U}'_\theta, \pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta}_2} \rangle^{-1}_{\mathcal{Y}_M} \big| \\ & = & \Big| \frac{d_5}{\langle \pi_{\mathcal{Y}_M} \overline{U}'_\theta, \pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta}_1} \rangle_{\mathcal{Y}_M} \langle \pi_{\mathcal{Y}_M} \overline{U}'_\theta, \pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta}_2} \rangle_{\mathcal{Y}_M}} \Big| \\ & \leq & \kappa^2 C_3 C_4 |\tilde{\theta}_1 - \tilde{\theta}_2|. \end{array} \tag{4.6.92}$$

Combining all these estimates yields

$$\begin{array}{rcl} |F_{\theta,\delta_\theta}(\tilde{\theta}_1) - F_{\theta,\delta_\theta}(\tilde{\theta}_1)| & \leq & \kappa \Big[d_1 + d_2 \Big] + d_6 \Big[\delta C_3 + \delta_\theta^2 C_2 \Big] \\ & \leq & \kappa |\tilde{\theta}_1 - \tilde{\theta}_2| \Big[\delta C_4 + \delta_\theta C_7 \Big] + \kappa^2 C_3 C_4 |\tilde{\theta}_1 - \tilde{\theta}_2| \Big[\delta C_3 + \delta_\theta^2 C_2 \Big]. \end{array} \tag{4.6.93}$$

■

Proof of Proposition 4.6.7. Upon fixing

$$\begin{array}{rcl} \delta_\theta & = & \min\{1, \frac{1}{2\kappa C_2}, \frac{1}{8\kappa C_7}, \frac{1}{8\kappa^2 C_2 C_3 C_4}\}, \\ \delta & = & \min\{\frac{\delta_\theta}{2\kappa C_3}, \frac{1}{8\kappa C_4}, \frac{1}{8\kappa^2 (C_3)^2 C_4}\}, \end{array} \tag{4.6.94}$$

we obtain, using Lemma 4.6.9 and Lemma 4.6.10, for any $\tilde{\theta}, \tilde{\theta}_1, \tilde{\theta}_2 \in [\theta - \delta_\theta, \theta + \delta_\theta]$ the estimate

$$|F_{\theta,\delta_\theta}(\tilde{\theta})| \quad \leq \quad \kappa \Big[\delta C_3 + \delta_\theta^2 C_2 \Big] \quad \leq \quad \delta_\theta, \tag{4.6.95}$$

together with

$$\begin{array}{rcl} |F_{\theta,\delta_\theta}(\tilde{\theta}_1) - F_{\theta,\delta_\theta}(\tilde{\theta}_2)| & \leq & \kappa |\tilde{\theta}_1 - \tilde{\theta}_2| \Big[\delta C_4 + \delta_\theta C_7 \Big] + \kappa^2 C_3 C_4 |\tilde{\theta}_1 - \tilde{\theta}_2| \Big[\delta C_3 + \delta_\theta^2 C_2 \Big] \\ & \leq & \frac{1}{2} |\tilde{\theta}_1 - \tilde{\theta}_2|. \end{array} \tag{4.6.96}$$

Therefore, the map F_{θ,δ_θ} maps $[\theta - \delta_\theta, \theta + \delta_\theta]$ into itself and is a contraction, so that the fixed point problem (4.6.76) has a unique solution $\tilde{\theta}$. By construction this $\tilde{\theta}$ satisfies the property that upon defining

$$\Phi \quad = \quad \overline{U} - \overline{U}_{\tilde{\theta}} \quad \in \quad \mathcal{Y}^1_{k,M}, \tag{4.6.97}$$

we have the identity

$$\langle \pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta}}, \Phi \rangle_{\mathcal{Y}_M} \quad = \quad 0. \tag{4.6.98}$$

■

Proof of Theorem 4.2.1. The items (i)-(iii) follow from proposition 4.6.1. Let $\delta > 0$ be the constant from Proposition 4.6.7, fix $M = \frac{p}{q} \in \mathcal{M}_q$ with $M \geq M_*$ and fix a triplet $(c, \overline{U}, \theta) \in \mathbb{R} \times \ell^\infty(p^{-1}\mathbb{Z}, \mathbb{R}) \times \mathbb{R}$ that satisfies (4.2.34) and (4.2.39). With Proposition 4.6.7 we fix a $\tilde{\theta} \in \mathbb{R}$ close to θ in such a way that $\overline{U}$ can be decomposed as

$$\overline{U} \quad = \quad \pi_{\mathcal{Y}_M} \overline{U}_{\tilde{\theta}} + \Phi \tag{4.6.99}$$

for some $\Phi \in \mathcal{Y}^1_{k,M}$ with $\langle \pi_{\mathcal{Y}_M} \Phi^-_{\tilde{\theta}}, \Phi \rangle_{\mathcal{Y}_M} = 0$.

Using a Taylor approximation as before, we note that

$$\|\overline{U}_\theta - \overline{U}_{\tilde{\theta}}\|_{BC^1_{-\alpha}(\mathbb{R};\mathbb{R})} \leq 2|\theta - \tilde{\theta}| C_1 e^{2\alpha}. \tag{4.6.100}$$

On account of Lemma 4.A.1 we pick a constant $C_1 > 0$ in such a way that

$$\begin{array}{rcl} \|\Phi\|_{\mathcal{Y}^1_{k,M}} & \leq & \|\overline{U} - \overline{U}_\theta\|_{\mathcal{Y}^1_{k,M}} + \|\overline{U}_\theta - \overline{U}_{\tilde{\theta}}\|_{\mathcal{Y}^1_{k,M}} \\ & \leq & \delta + C_1 \|\overline{U}_\theta - \overline{U}_{\tilde{\theta}}\|_{BC^1_{-\alpha}(\mathbb{R};\mathbb{R})} \\ & \leq & \delta + C_1 2|\theta - \tilde{\theta}| C_1 e^{2\alpha} \\ & \leq & \delta + C_1 2 C_1 e^{2\alpha} \delta_\theta \\ & := & \delta + C_2 \delta_\theta. \end{array} \tag{4.6.101}$$

Recall the constant C_{unif} from (4.6.11) and we let $C > 1$ be the constant from Lemmas 4.6.3-4.6.4. Now we decrease $\delta_\theta > 0$, while letting $\delta > 0$ be given by (4.6.94), in such a way that

$$\begin{array}{rcl} \delta & = & \frac{\delta_\theta}{2\kappa C_3}, \\ C_{\text{unif}}\Big[\delta + C_2 \delta_\theta\Big] & \leq & \frac{1}{2}, \\ 2 C_{\text{unif}} C \Big[\delta + C_2 \delta_\theta\Big] & \leq & \frac{1}{4}\delta_c. \end{array} \tag{4.6.102}$$

In particular, we see that

$$\begin{array}{rcl} \|\Phi\|_{\mathcal{Y}^1_{k,M}} & \leq & \delta + C_2 \delta_\theta \\ & = & \delta\Big[1 + C_2 2\kappa C_3\Big] \\ & := & C_3 \delta. \end{array} \tag{4.6.103}$$

Inspecting the first line of the fixed point problem (4.6.22), yields that we can write

$$\overline{c}_0 - c = (\overline{c}_0 - c)\gamma^*_{k,M;\tilde{\theta}}\Big(\mathcal{D}_{k,M}\Phi\Big) + \gamma^*_{k,M;\tilde{\theta}}\Big(\mathcal{R}_B(\Phi;\tilde{\theta},r) + \mathcal{R}_C(\tilde{\theta},M)\Big). \tag{4.6.104}$$

Since we assumed $C_{\text{unif}}\|\Phi\|_{\mathcal{Y}^1_{k,M}} \leq \frac{1}{2}$ we can solve this equation for $c = c(\Phi)$ as

$$\overline{c}_0 - c(\Phi) = \Big[1 - \gamma^*_{k,M;\tilde{\theta}}\Big(\mathcal{D}_{k,M}\Phi\Big)\Big]^{-1}\gamma^*_{k,M;\tilde{\theta}}\Big(\mathcal{R}_B(\Phi;\tilde{\theta},r) + \mathcal{R}_C(\tilde{\theta},M)\Big). \tag{4.6.105}$$

Finally, our earlier estimates yield

$$\begin{array}{rcl} |\overline{c}_0 - c(\Phi)| & \leq & 2C_{\text{unif}} C\Big[\delta_r + C_3\delta + M^{-1}\Big] \\ & \leq & \frac{1}{16}\delta_c + \frac{1}{4}\delta_c + \frac{1}{8}\delta_c \\ & \leq & \frac{1}{2}\delta_c. \end{array} \tag{4.6.106}$$

Therefore, we see that $(c(\Phi), \Phi) \in \mathcal{Z}_{\delta_c,\delta_\phi}$ and that $T_{\delta_c,\delta_\phi}(c(\Phi), \Phi) = (c(\Phi), \Phi)$. By the uniqueness of the fixed point of T_{δ_c,δ_ϕ}, we obtain $c(\Phi) = c^*_M(\tilde{\theta}, r), \Phi = \Phi^*_M(\tilde{\theta}, r)$, which implies

$$c \;=\; c_M(\tilde{\theta}, r), \qquad \overline{U} \;=\; \overline{U}_M(\tilde{\theta}, r) \tag{4.6.107}$$

as desired. ∎

4.A Auxiliary results

In this section we collect several useful results that we use throughout this paper. The first three results concern the sequence spaces $\mathcal{Y}_M$ and $\mathcal{Y}^1_{k,M}$ and their associated inner products (4.3.5)-(4.3.6).

Lemma 4.A.1 ([111, Lem. 3.1]). *Fix a pair of integers $1 \leq k \leq 6$ and $q \geq 1$, together with a constant $\eta > 0$. Then there exists a constant $C \geq 1$ for which the bounds*

$$\begin{aligned} \|\pi_{\mathcal{Y}_M} f\|_{\mathcal{Y}_M} &\leq C\|f\|_{BC_{-\eta}}, \\ \|\pi_{\mathcal{Y}^1_{k,M}} g\|_{\mathcal{Y}^1_{k,M}} &\leq C\|g\|_{BC^1_{-\eta}} \end{aligned} \tag{4.A.1}$$

hold for all $M \in \mathcal{M}_q$ and all functions $f \in BC_{-\eta}(\mathbb{R}; \mathbb{R})$ and $g \in BC^1_{-\eta}(\mathbb{R}; \mathbb{R})$.

Lemma 4.A.2 ([111, Lem. 3.4]). *Fix an integer $q \geq 1$. Then there exists $C > 1$ so that the bound*

$$\left|\langle f, g\rangle_{L^2(\mathbb{R};\mathbb{R}^d)} - \langle \pi_{\mathcal{Y}_M} f, \pi_{\mathcal{Y}_M} g\rangle_{\mathcal{Y}_M}\right| \;\leq\; CM^{-1}\|f\|_{BC^1_{-\eta}(\mathbb{R};\mathbb{R}^d)}\|g\|_{BC^1_{-\eta}(\mathbb{R};\mathbb{R}^d)} \tag{4.A.2}$$

holds for all $M \in \mathcal{M}_q$ and all functions $f, g \in BC^1_{-\eta}(\mathbb{R}; \mathbb{R}^d)$.

Lemma 4.A.3 ([111, Lem. 3.5]). *Fix an integer $q \geq 1$. For any $M = \frac{p}{q} \in \mathcal{M}_q$, the operators $\mathcal{J}_M$ and $\mathcal{J}^1_{k,M}$ defined in (4.3.19) are isometries between $\mathcal{Y}_M$ and $\mathcal{H}_M$ and between $\mathcal{Y}^1_{k,M}$ and $\mathcal{H}^1_{k,M}$ respectively.*

The following results can be seen as the fully discrete generalizations of the well-known facts

$$\langle u, u'\rangle \;=\; 0, \qquad \langle u'', u\rangle \;\leq\; 0 \tag{4.A.3}$$

that hold for smooth, localized functions u. When dealing solely with nearest-neighbour interactions as in [111] the inequality $\langle \Delta_M \Phi, \Phi\rangle_{\mathcal{H}_M} \leq 0$ follows immediately from the Cauchy-Schwarz inequality. However, in our setting, some of the coefficients α_k may not be positive definite, preventing us from taking them out of the inner product. This motivates the indirect approach that is taken in the proof of Lemma 4.A.5.

Lemma 4.A.4 ([111, Cor. 3.15]). *Fix a pair of integers $1 \leq k \leq 6$ and $q \geq 1$. There exists a constant $K > 1$ so that for all $M \in \mathcal{M}_q$ and all $\Phi \in \mathcal{H}^1_{k,M}$ we have the bound*

$$\left|\langle \Phi, \mathcal{D}_{k,M}\Phi\rangle_{\mathcal{H}_M}\right| \;\leq\; KM^{-1}\|\mathcal{D}_{k,M}\Phi\|^2_{\mathcal{H}_M}. \tag{4.A.4}$$

Lemma 4.A.5 (cf. [111, Lem. 3.13])**.** *Assume that (HS1) is satisfied. Fix an integer $q \geq 1$ and pick $M \in \mathcal{M}_q$. Then the bound*

$$\langle \Delta_M \Phi, \Phi \rangle_{\mathcal{H}_M} \quad \leq \quad 0 \tag{4.A.5}$$

holds for each $\Phi \in \mathcal{H}_M$.

Proof. Pick $\Phi \in \mathcal{H}_M$ and define the stepwise interpolation function $\tilde{\Phi} \in L^2(\mathbb{R}; \mathbb{R}^d)$ by setting

$$\tilde{\Phi}\big(\xi + \zeta M^{-1} + \varepsilon\big) \quad = \quad \Phi\big(\zeta, \xi\big) \tag{4.A.6}$$

for $\xi \in M^{-1}\mathbb{Z}$, $\zeta \in q^{-1}\mathbb{Z}_q^{\circ} \cup \{0\}$ and $0 \leq \varepsilon < M^{-1}q^{-1}$. Upon recalling that

$$\vartheta \quad = \quad \tfrac{p-nq}{q}, \qquad nM^{-1} \quad = \quad 1 - \vartheta M^{-1} \tag{4.A.7}$$

and observing that

$$1 \quad = \quad p\tfrac{q}{p}q^{-1} \quad = \quad \big((p-nq) + nq\big)M^{-1}q^{-1}, \tag{4.A.8}$$

we may compute

$$\begin{array}{lcl} T_0^m \tilde{\Phi}\big(\xi + \zeta M^{-1}\big) & = & \tilde{\Phi}\big(\xi + \zeta M^{-1} + m\big) \\ & = & \tilde{\Phi}\big(\xi + mnM^{-1} + (\zeta + m(p-nq)q^{-1})M^{-1}\big) \\ & = & \Phi\big(\zeta + m(p-nq)q^{-1}, \xi + mnM^{-1}\big) \\ & = & \Phi\big(\zeta + m\vartheta, \xi + m - m\vartheta M^{-1}\big) \\ & = & T_M^m \Phi(\zeta, \xi) \end{array} \tag{4.A.9}$$

for arbitrary $\xi \in M^{-1}\mathbb{Z}$, $\zeta \in q^{-1}\mathbb{Z}_q^{\circ} \cup \{0\}$ and $m \in \mathbb{Z}$. In particular, for $m \in \mathbb{Z}$ we obtain the identity

$$\begin{array}{lcl} \langle T_0^m \tilde{\Phi}, \tilde{\Phi} \rangle_{L^2(\mathbb{R};\mathbb{R}^d)} & = & q^{-1}M^{-1} \sum_{\xi \in M^{-1}\mathbb{Z}} \sum_{\zeta \in q^{-1}\mathbb{Z}_q^{\circ} \cup \{0\}} \big\langle T_0^m \tilde{\Phi}\big(\xi + \zeta M^{-1}\big), \tilde{\Phi}\big(\xi + \zeta M^{-1}\big) \big\rangle_{\mathbb{R}^d} \\ & = & q^{-1}M^{-1} \sum_{\xi \in M^{-1}\mathbb{Z}} \sum_{\zeta \in q^{-1}\mathbb{Z}_q^{\circ} \cup \{0\}} \big\langle T_M^m \Phi(\zeta, \xi), \Phi\big(\zeta, \xi\big) \big\rangle_{\mathbb{R}^d} \\ & = & \langle T_M^m \Phi, \Phi \rangle_{\mathcal{H}_M}. \end{array} \tag{4.A.10}$$

We hence obtain

$$\begin{array}{lcl} \langle \Delta_0 \tilde{\Phi}, \tilde{\Phi} \rangle_{L^2(\mathbb{R};\mathbb{R}^d)} & = & \tau \sum_{m>0} \alpha_m \big[\langle T_0^m \tilde{\Phi}, \tilde{\Phi} \rangle_{L^2(\mathbb{R};\mathbb{R}^d)} + \langle T_0^{-m} \tilde{\Phi}, \tilde{\Phi} \rangle_{L^2(\mathbb{R};\mathbb{R}^d)} \\ & & \qquad -2\langle \tilde{\Phi}, \tilde{\Phi} \rangle_{L^2(\mathbb{R};\mathbb{R}^d)} \big] \\ & = & \tau \sum_{m>0} \alpha_m \big[\langle T_M^m \Phi, \Phi \rangle_{\mathcal{H}_M} + \langle T_M^{-m} \Phi, \Phi \rangle_{\mathcal{H}_M} - 2\langle \Phi, \Phi \rangle_{\mathcal{H}_M} \big] \\ & = & \langle \Delta_M \Phi, \Phi \rangle_{\mathcal{H}_M}. \end{array} \tag{4.A.11}$$

The desired result now follows from [6, Lem. 2.1]. ■

We now show that $\mathcal{K}_{k,M}$ approaches $\overline{\mathcal{K}}_{q,\vartheta}$ in a more rigorous fashion. The infinite-range interactions cause complications here, because we need to interchange a limit and an infinite sum. For $\tilde{M} \in \mathcal{M}_q$ we introduce the notation $\vartheta(\tilde{M})$ to refer to the value of ϑ in (4.3.20) with $M = \tilde{M}$.

Lemma 4.A.6. *Assume that (HS1) is satisfied. Fix an integer $q \geq 1$ and consider any sequence $\{M_j\}_{j \in \mathbb{N}}$ in $\mathcal{M}_q$ with the property that $\lim_{j \to \infty} M_j = \infty$ and $\vartheta(M_j) = \vartheta$ for all $j \in \mathbb{N}$ and some $\vartheta \in q^{-1}\mathbb{Z}_q \setminus \{0\}$. Then for any $Z \in C_c^\infty(\mathbb{R}; \ell^2_{q,\perp;\infty}) \subset C_c^\infty(\mathbb{R}; \ell^2_{q,\perp})$ we have the limit*

$$\lim_{j \to \infty} \|\Delta_{M_j} Z - \Delta_{q,\vartheta} Z\|_{L^2(\mathbb{R}, \ell^2_{q,\perp})} = 0. \tag{4.A.12}$$

Proof. Fix any test function $Z \in C_c^\infty(\mathbb{R}; \ell^2_{q,\perp;\infty}) \subset C_c^\infty(\mathbb{R}; \ell^2_{q,\perp})$ and pick a sufficiently large $\mu \in \mathbb{N}$ for which $\mathrm{supp}(Z) \subset [-\mu, \mu]$. Without loss of generality we assume that $\|Z\|_{L^2(\mathbb{R}, \ell^2_{q,\perp})} = 1$. Pick $\varepsilon > 0$, together with $K \in \mathbb{Z}_{>\mu}$ in such a way that

$$\tau \sum_{m \geq K - \mu} 4|\alpha_m| < \tfrac{\varepsilon}{8}. \tag{4.A.13}$$

Moreover, by the strong continuity of the shift-semigroup [61, Example I.5.4], we can pick $J \in \mathbb{N}$ in such a way that for each $j \geq J$ and each $|m| \leq 4K + 4l$ we have the bound

$$\begin{aligned} \tau|\alpha_m| \|T^m_{M_j} Z - T^m_{q,\vartheta} Z\|_{L^2(\mathbb{R}, \ell^2_{q,\perp})} &= \tau|\alpha_m| \|Z(\cdot + m n_j M_j^{-1}) - Z(\cdot + m)\|_{L^2(\mathbb{R}, \ell^2_{q,\perp})} \\ &< \tfrac{\varepsilon}{16(\mu + K)}, \end{aligned} \tag{4.A.14}$$

together with

$$|n_j M_j^{-1} - 1| \leq \tfrac{1}{2}. \tag{4.A.15}$$

Here we introduced n_j for the value of n in (4.3.20) with $M = M_j$. Fix $j \geq J$. Since $\mathrm{supp}(Z) \subset [-\mu, \mu]$, we obtain

$$\Delta_{M_j} Z(\xi) - \Delta_{q,\vartheta} Z(\xi) = \tau \sum_{m \geq K - \mu} \alpha_m \Big[Z(\xi - m n_j M_j^{-1}) - Z(\xi - m) \Big] \tag{4.A.16}$$

for any $\xi > K$, which allows us to estimate

$$\|\Delta_{M_j} Z - \Delta_{q,\vartheta} Z\|_{L^2\big((K,\infty), \ell^2_{q,\perp}\big)} \leq \tau \sum_{m \geq K - \mu} 2|\alpha_m| \|Z\|_{L^2(\mathbb{R}, \ell^2_{q,\perp})} < \tfrac{\varepsilon}{4}. \tag{4.A.17}$$

A similar computation yields

$$\|\Delta_{M_j} Z - \Delta_{q,\vartheta} Z\|_{L^2\big((-\infty,-K), \ell^2_{q,\perp}\big)} < \tfrac{\varepsilon}{4}. \tag{4.A.18}$$

Finally, for $\xi \in [-K, K]$ we see that

$$\begin{aligned} \Delta_{M_j} Z(\xi) - \Delta_{q,\vartheta} Z(\xi) &= \tau \sum_{m=1}^{4l+4K} \alpha_m \Big[Z(\xi + m n_j M_j^{-1}) - Z(\xi + m) \\ &\qquad + Z(\xi - m n_j M_j^{-1}) - Z(\xi - m) \Big] \\ &= \tau \sum_{m=1}^{4l+4K} \alpha_m \Big[T^m_{M_j} Z(\xi) - T^m_{q,\vartheta} Z(\xi) + T^{-m}_{M_j} Z(\xi) - T^{-m}_{q,\vartheta} Z(\xi) \Big]. \end{aligned} \tag{4.A.19}$$

On account of (4.A.14), we can, hence, estimate

$$\begin{aligned}\|\Delta_{M_j} Z - \Delta_{q,\vartheta} Z\|_{L^2\left([-K,K],\ell^2_{q,\perp}\right)} &\leq \tau \sum_{m=1}^{4l+4K} |\alpha_m| \Big[\|T^m_{M_j} Z - T^m_{q,\vartheta} Z\|_{L^2(\mathbb{R},\ell^2_{q,\perp})} \\ &\qquad + \|T^{-m}_{M_j} Z - T^{-m}_{q,\vartheta} Z\|_{L^2(\mathbb{R},\ell^2_{q,\perp})} \Big] \\ &< 2(4l+4K) \tfrac{\varepsilon}{16(\mu+K)} \\ &= \tfrac{\varepsilon}{2}.\end{aligned} \tag{4.A.20}$$

Combining these estimates yields the bound

$$\|\Delta_{M_j} Z - \Delta_{q,\vartheta} Z\|_{L^2(\mathbb{R},\ell^2_{q,\perp})} < \varepsilon, \tag{4.A.21}$$

from which the desired limit follows. ■

Chapter 5

Exponential dichotomies for nonlocal differential operators with infinite-range interactions

This chapter has been submitted as W.M. Schouten-Straatman and H.J. Hupkes "Exponential Dichotomies for Nonlocal Differential Operators with Infinite Range Interactions" [149].

Abstract. We show that MFDEs with infinite range discrete and/or continuous interactions admit exponential dichotomies, building on the Fredholm theory developed by Faye and Scheel for such systems. For the half line, we refine the earlier approach by Hupkes and Verduyn Lunel. For the full line, we construct these splittings by generalizing the finite-range results obtained by Mallet-Paret and Verduyn Lunel. The finite dimensional space that is 'missed' by these splittings can be characterized using the Hale inner product, but the resulting degeneracy issues raise subtle questions that are much harder to resolve than in the finite-range case. Indeed, there is no direct analogue for the standard 'atomicity' condition that is typically used to rule out degeneracies, since it explicitly references the smallest and largest shifts.

We construct alternative criteria that exploit finer information on the structure of the MFDE. Our results are optimal when the coefficients are cyclic with respect to appropriate shift semigroups or when the standard positivity conditions typically associated to comparison principles are satisfied. We illustrate these results with explicit examples and counter-examples that involve the Nagumo equation.

Key words: Exponential dichotomies, functional differential equations of mixed type, nonlocal interactions, infinite-range interactions, Hale inner product, cyclic coefficients.

5.1 Introduction

Many physical, chemical and biological systems feature nonlocal interactions that can have a fundamental impact on the underlying dynamical behaviour. A typical mechanism to generate such nonlocality is to include dependencies on spatial averages of model components, often as part of a multi-scale approach. For example, plants take up water from the surrounding soil through their spatially-extended root network, which can be modelled by nonlocal logistic growth terms [84, 85]. The propagation of cancer cells depends on the orientation of the surrounding extracellular matrix fibres, which leads naturally to nonlocal flux terms [155]. Additional examples can be found in the fields of population dynamics [25, 86, 153, 154, 157], material science [5, 8, 71, 164] and many others.

A second fundamental route that leads to nonlocality is the consideration of spatial domains that feature some type of *discreteness*. The broken translational and rotational symmetries often lead to highly complex and surprising behaviour that disappears in the continuum limit. For example, recent experiments have established that light waves can be trapped in well-designed photonic lattices [136, 163]. Other settings where discrete topological effects play an essential role include the movement of domain walls [53], the propagation of dislocations through crystals [35] and the development of fractures in elastic bodies [156]. In fact, even the simplest discretizations of standard scalar reaction-diffusion systems are known to have far richer properties than their continuous local counterparts [40, 42, 105].

Myelinated nerve fibres A commonly used modelling prototype to illustrate these issues concerns the propagation of electrical signals through nerve fibres. These nerve fibres are insulated by segments of myelin coating that are separated by periodic gaps at the so-called nodes of Ranvier [143]. Signals travel quickly through the coated regions, but lose strength rapidly. The movement through the gaps is much slower, but the signal is chemically reinforced in preparation for the next segment [127].

One of the first mathematical models proposed to capture this propagation was the FitzHugh-Nagumo partial differential equation (PDE) [76]. This model is able to reproduce the travelling pulses observed in nature [75] and has been studied extensively as a consequence. These studies have led to the development of many important mathematical techniques in areas such as singular perturbation theory [31–33, 97, 117, 119] variational calculus [36], Maslov index theory [10, 37, 46, 47, 101] and stochastic dynamics [92–94]. However, as a fully local equation it is unable to incorporate the discrete structure in a direct fashion.

In order to repair this, Keener and Sneyd [123] proposed to replace the FitzHugh-Nagumo PDE by its discretized counterpart

$$\begin{aligned} \dot{u}_j &= u_{j+1} + u_{j-1} - 2u_j + g(u_j; a) - w_j, \\ \dot{w}_j &= \rho[u_j - w_j], \end{aligned} \tag{5.1.1}$$

indexed on the spatial lattice $j \in \mathbb{Z}$. Here the variable u_j describes the potential on the j^{th} node of Ranvier, while w_j describes a recovery component. The nonlinearity can be taken as the bistable cubic $g(u; a) = u(1 - u)(u - a)$ for some $a \in (0, 1)$ and $0 < \rho \ll 1$ is a small parameter. Such an infinite system of coupled ODEs is referred to as a lattice differential equation (LDE)—a class of equations that arises naturally when discretizing the spatial derivatives in PDEs.

Since we are mainly interested in the propagation of electrical pulses, we introduce the travelling wave Ansatz

$$(u_j, w_j)(t) = (\overline{u}, \overline{w})(j + ct), \qquad (\overline{u}, \overline{w})(\pm\infty) = 0. \tag{5.1.2}$$

Here c is the speed of the wave and the smooth functions $(\overline{u}, \overline{w}) : \mathbb{R} \to \mathbb{R}^2$ represent the two waveprofiles. Plugging (5.1.2) into the LDE (5.1.1) yields the differential equation

$$\begin{aligned} c\overline{u}'(\sigma) &= \overline{u}(\sigma + 1) + \overline{u}(\sigma - 1) - 2\overline{u}(\sigma) + g(\overline{u}(\sigma); a) - \overline{w}(\sigma), \\ c\overline{w}'(\sigma) &= \rho[\overline{u}(\sigma) - \overline{w}(\sigma)] \end{aligned} \tag{5.1.3}$$

in which $\sigma = j + ct$. Since this system contains both advanced (positive) and retarded (negative) shifts, such an equation is called a functional differential equation of mixed type (MFDE).

In [108, 109] Hupkes and Sandstede established the existence and nonlinear stability of such pulses, under a 'nonpinning' condition for the associated Nagumo LDE

$$\dot{u}_j = u_{j+1} + u_{j-1} - 2u_j + g(u_j; a). \tag{5.1.4}$$

This LDE arises when considering the first component of (5.1.1) with $w = 0$. It admits travelling front solutions

$$u_j(t) = \overline{u}_*(j + c_* t), \qquad \overline{u}_*(-\infty) = 0, \qquad \overline{u}_*(+\infty) = 1 \tag{5.1.5}$$

that necessarily satisfy the MFDE

$$c_* \overline{u}_*'(\sigma) = \overline{u}_*(\sigma + 1) + \overline{u}_*(\sigma - 1) - 2\overline{u}_*(\sigma) + g(\overline{u}_*(\sigma); a). \tag{5.1.6}$$

The 'nonpinning' condition mentioned above demands that the wavespeed c_*—which depends uniquely on a [131]—does not vanish. In the PDE case this is automatic for $a \neq \frac{1}{2}$, but in the discrete setting this is a nontrivial demand due to the energy barriers caused by the lattice [16, 56, 62, 99, 122, 132].

The main idea behind the approach developed in [108, 109] is to use Lin's method [104, 128] to combine the fronts (5.1.5) and their reflections to form so-called quasi-front and quasi-back solutions to (5.1.3). Such solutions admit gaps in predetermined finite-dimensional subspaces that can be closed by choosing the correct wavespeed. The existence of these subspaces is directly related to the construction of exponential dichotomies for the linear MFDE

$$cu'(\sigma) \quad = \quad u(\sigma+1) + u(\sigma-1) - 2u(\sigma) + g_u(\overline{u}_*(\sigma);a)u(\sigma), \tag{5.1.7}$$

which arises as the linearization of (5.1.6) around the front solutions (5.1.5).

Exponential dichotomies for ODEs Roughly speaking, a linear differential equation is said to admit an *exponential dichotomy* if the space of initial conditions can be written as a direct sum of a stable and an unstable subspace. Initial conditions in the former can be continued as solutions that decay exponentially in forward time, while initial conditions in the latter admit this property in backward time. In order to be more specific, we first restrict our attention to the ODE

$$\tfrac{d}{d\sigma}u \quad = \quad A(\sigma)u, \tag{5.1.8}$$

referring to the review paper by Sandstede [147] for further details. Here $u(\sigma) \in \mathbb{C}^M$ and $A(\sigma)$ is an $M \times M$ matrix for any $\sigma \in \mathbb{R}$. Let us write $\Phi(\sigma, \tau)$ for the evolution operator associated to (5.1.8), which maps $u(\tau)$ to $u(\sigma)$.

Suppose first that the system (5.1.8) is autonomous and hyperbolic, i.e. $A(\sigma) = A$ for some matrix A that has no spectrum on the imaginary axis. Writing E_0^s and E_0^u for the generalized stable respectively unstable eigenspaces of A, we subsequently obtain the decomposition

$$\mathbb{C}^M \quad = \quad E_0^s \oplus E_0^u. \tag{5.1.9}$$

In addition, each of these subspaces is invariant under the action of $\Phi(\sigma, \tau) = \exp[A(\sigma - \tau)]$, which decays exponentially on E_0^s for $\sigma > \tau$ and on E_0^u for $\sigma < \tau$.

In order to generalize such decompositions to non-autonomous settings, the splitting (5.1.9) will need to vary with the base time $\tau \in I$. Here we pick I to be one of the three intervals $\mathbb{R}^-$, $\mathbb{R}^+$ or $\mathbb{R}$. In particular, (5.1.8) is said to be *exponentially dichotomous* on I if the following properties hold.

- There exists a family of projection operators $\{P(\tau)\}_{\tau \in I}$ on $\mathbb{C}^M$ that commute with the evolution $\Phi(\sigma, \tau)$.
- The restricted operators $\Phi^s(\sigma, \tau) := \Phi(\sigma, \tau)P(\tau)$ and $\Phi^u(\sigma, \tau) := \Phi(\sigma, \tau)\big(\mathrm{id} - P(\tau)\big)$ decay exponentially for $\sigma \geq \tau$ respectively $\sigma \leq \tau$.

Many important features concerning these dichotomies were first described by Palmer in [139, 140]. For example, the well-known roughness theorem states that exponential dichotomies persist under small perturbations of the matrices $A(\sigma)$. In addition, there is a close connection with the Fredholm properties of the associated linear operators. Consider for example the family of linear operators

$$\Lambda(\lambda) : H^1(\mathbb{R};\mathbb{C}^M) \quad \to \quad L^2(\mathbb{R};\mathbb{C}^M), \qquad u \quad \mapsto \quad \tfrac{d}{d\sigma}u - A(\sigma)u - \lambda u, \tag{5.1.10}$$

defined for $\lambda \in \mathbb{C}$. Then $\Lambda(\lambda)$ is a Fredholm operator if and only if the system

$$\tfrac{d}{d\sigma}u \quad = \quad A(\sigma)u + \lambda u \tag{5.1.11}$$

admits exponential dichotomies on both $\mathbb{R}^+$ and $\mathbb{R}^-$. In addition, $\Lambda(\lambda)$ is invertible if and only if (5.1.11) admits exponential dichotomies on $\mathbb{R}$. Since systems of the form (5.1.11) arise frequently when considering the spectral properties of wave solutions to nonlinear PDEs, exponential dichotomies have a key role to play in this area. In fact, the well-known Evans function [63, 139–141] detects precisely when the dichotomies on $\mathbb{R}^-$ and $\mathbb{R}^+$ can be patched together to form a dichotomy on $\mathbb{R}$.

Exponential dichotomies for MFDEs Several important points need to be addressed before the concepts above can be extended to linear MFDEs such as (5.1.7). The first issue is that MFDEs are typically ill-posed [144], preventing a natural analogue of the evolution operator Φ to be defined. The second issue is that $\mathbb{C}^M$ is no longer an appropriate state space. For example, computing $u'(0)$ in (5.1.7) requires knowledge of u on the interval $[-1, 1]$. These issues were resolved independently and simultaneously by Mallet-Paret and Verduyn Lunel in [133] and by Härterich, Scheel and Sandstede in [96] by decomposing suitable function spaces into separate parts that individually do admit (exponentially decaying) semiflows.

Applying the results in [133] to (5.1.7), we obtain the decomposition

$$C([-1,1];\mathbb{R}) \quad = \quad P(\tau) + Q(\tau) + \Gamma(\tau) \tag{5.1.12}$$

for each $\tau \in \mathbb{R}$. Here $\Gamma(\tau)$ is finite dimensional, while functions in $P(\tau)$ and $Q(\tau)$ can be extended to exponentially decaying solutions of the MFDE (5.1.7) on the intervals $(-\infty, \tau]$ respectively $[\tau, \infty)$. In particular, the intersection $P(\tau) \cap Q(\tau)$ contains segments of functions that belong to the kernel of the associated linear operator

$$[\mathcal{L}v](\sigma) \quad = \quad -cv'(\sigma) + v(\sigma+1) + v(\sigma-1) - 2v(\sigma) + g_u(\overline{u}_*(\sigma); a)v(\sigma). \tag{5.1.13}$$

After dividing these segments out from either P or Q, the decomposition (5.1.12) becomes a direct sum. Similar results were obtained in [96], but here the authors use the augmented statespace $\mathbb{C}^M \times L^2([-1,1];\mathbb{R})$.

In many applications, it is crucial to understand the dimension of $\Gamma(\tau)$. A key tool to achieve this is the so-called Hale inner product [91], which in the present context is given by

$$\langle \psi, \phi \rangle_\tau \quad = \quad \tfrac{1}{c}\big[\psi(0)\phi(0) + \int_{-1}^{0} \psi(s+1)\phi(s)ds - \int_{0}^{1} \psi(s-1)\phi(s)ds\big] \tag{5.1.14}$$

for two functions $\phi, \psi \in C([-1,1];\mathbb{R})$. Indeed, one of the main results achieved in [133] is the identification

$$P(\tau) + Q(\tau) = \big\{\phi \in C([-1,1];\mathbb{R}) : \langle b(\tau + \cdot), \phi\rangle_\tau = 0 \text{ for every } b \in \ker \mathcal{L}^*\big\}. \tag{5.1.15}$$

Here $\mathcal{L}^*$ stands for the formal adjoint of $\mathcal{L}$, which arises by switching the sign of c in (5.1.13).

There are two potential issues that can impact the usefulness of this result. The first is that the Hale inner product could be degenerate, the second is that kernel elements of $\mathcal{L}^*$ could vanish on large intervals. For instance, [52, Ex. V.4.8] features an example system that admits compactly supported kernel elements, which are often referred to as *small solutions*. Fortunately, both types of degeneracies can be ruled out by imposing an invertibility condition on the coefficients related to the smallest and largest shifts in the MFDE. This is easy to check and obviously satisfied for (5.1.7).

These results from [96, 133] have been used in a variety of settings by now. These include the construction of travelling waves [108, 115], the stability analysis of such waves [11, 109], the study of homoclinic bifurcations [83, 104], the analysis of pseudospectral approximations [22] and the detection of indeterminacy in economic models [48]. Partial extensions of these results for MFDEs taking values in Banach spaces can be found in [102], but only for autonomous systems at present.

Infinite-range interactions In recent years, an active interest has arisen in systems that feature interactions that can take place over arbitrarily large distances. For example, diffusion models based on Lévy processes lead naturally to fractional Laplacians in the underlying PDE [2, 14]. These operators are inherently nonlocal and often feature infinitely many terms in their discretization schemes [43]. Systems of this type have been used for example to describe amorphous semiconductors [87], liquid crystals [44], porous media [19] and game theory [18]; see [27] for an accessible introduction. Examples featuring other types of infinite-range interactions include Ising models to describe the behaviour of magnetic spins on a grid [6] and SIR models to capture the spread of infectious diseases [126].

Returning to the study of nerve axons, let us now consider large networks of neurons. These neurons interact with each other over large distances through their connecting fibres [15, 23, 24, 142]. Such systems generally have a very complex structure and finding effective equations to describe their behaviour is highly challenging. One candidate that has been proposed [24] involves FitzHugh-Nagumo type models such as

$$\begin{aligned} \dot{u}_j &= h^{-2} \sum_{k \in \mathbb{Z}_{>0}} e^{-k^2}[u_{j+k} + u_{j-k} - 2u_j] + g(u_j;a) - w_j, \\ \dot{w}_j &= \rho[u_j - w_j]. \end{aligned} \tag{5.1.16}$$

Here the constant $h > 0$ represents the (scaled) discretization distance. Alternatively, one can replace or supplement the sum in (5.1.16) by including a convolution with a smooth kernel.

The travelling wave Ansatz

$$(u_j, w_j)(t) = (\overline{u}_h, \overline{w}_h)(hj + c_h t), \qquad (\overline{u}_h, \overline{w}_h)(\pm\infty) = 0 \tag{5.1.17}$$

now yields the MFDE

$$\begin{aligned} c_h \overline{u}_h'(\sigma) &= h^{-2} \sum_{k \in \mathbb{Z}_{>0}} e^{-k^2} [\overline{u}_h(\sigma + hk) + \overline{u}_h(\sigma - hk) - 2\overline{u}_h(\sigma)] \\ &\quad + g(\overline{u}_h(\sigma); a) - \overline{w}_h(\sigma) \\ c_h \overline{w}_h'(\sigma) &= \rho[\overline{u}_h(\sigma) - \overline{w}_h(\sigma)], \end{aligned} \tag{5.1.18}$$

which includes infinite-range interactions. In particular, it is no longer possible to apply the exponential splitting results from [96, 133]. Nevertheless, Faye and Scheel obtained an existence result for such waves in [69], pioneering a new approach to analyze spatial dynamics that circumvents the use of a state space. Extending the spectral convergence technique developed by Bates, Chen and Chmaj [6], we were able to show that such waves are nonlinearly stable [150], but only for small $h > 0$. In any case, at present there is no clear mechanism that allows finite-range results to be easily extended to settings with infinite-range interactions.

Infinite-range MFDEs In this paper we take a step towards building such a bridge by constructing exponential dichotomies for the non-autonomous, integro-differential MFDE

$$\dot{x}(\sigma) = \sum_{j=-\infty}^{\infty} A_j(\sigma) x(\sigma + r_j) + \int_{\mathbb{R}} \mathcal{K}(\xi; \sigma) x(\sigma + \xi) d\xi, \tag{5.1.19}$$

which is allowed to have infinite-range interactions. Here, we have $x(\sigma) \in \mathbb{C}^M$ for $t \in \mathbb{R}$ and the scalars r_j for $j \in \mathbb{Z}$ are called the *shifts*. Typically, we use $C_b(\mathbb{R})$ as our state space, but whenever this is possible we use smaller spaces to formulate sharper results. This allows us to consider settings where the shifts are unbounded in one direction only. This occurs for example when considering delay equations.

The Fredholm properties of the linear operator associated to (5.1.19) have been described by Faye and Scheel in [68]. We make heavy use of these properties here, continuing the program initiated in the bachelor thesis of Jin [116], who considered autonomous versions of (5.1.19). In such settings, it is possible to extend the techniques developed by Hupkes and Augeraud-Véron in [102] for MFDEs posed on Banach spaces. However, it is unclear at present how to generalize these methods to non-autonomous systems.

Splittings on the full line In §5.3-5.4 we construct exponential splittings for (5.1.19) on the full line. Our main result essentially states that the decomposition (5.1.12) and the characterization (5.1.15) remain valid for the state space $C_b(\mathbb{R})$. In addition, we explore the Fredholm and continuity properties of the projection operators associated to the splitting (5.1.12). Our arguments in these sections are heavily based on the framework developed by Mallet-Paret and Verduyn Lunel in [133]. However, the unbounded shifts raise some major technical challenges.

The primary complication is that the iteration scheme used in [133] to establish the exponential decay of functions in $P(\tau)$ and $Q(\tau)$ breaks down. Indeed, the authors show that there exist $L > 0$ so that supremum of the former solutions on half-lines $(-\infty, \tau_*]$ is halved each time one makes the replacement $\tau_* \mapsto \tau_* - L$. To achieve this, they exploit the fact that the behaviour of solutions on the latter interval does not 'see' the behaviour at τ_*. This is no longer true for unbounded shifts and required us to develop a novel iteration scheme that is able to separate short-range from long-range effects.

A second major complication arises whenever continuous functions are approximated by C^1-functions. Indeed, in [133] these approximations automatically have bounded derivatives, but in our case we can no longer assume that these functions live in $W^{1,\infty}(\mathbb{R})$. This prevents a direct application of the Fredholm theory in [68], forcing us to take a more involved approach to carefully isolate the regions where the unbounded derivatives occur.

The final obstacle is caused by the frequent use of the Ascoli-Arzela theorem in [133]. Indeed, in our setting we only obtain convergence on compacta instead of full uniform convergence. Fortunately, this can be circumvented relatively easily by using the exponential decay to provide the missing compactness at infinity.

Splittings on the half line We proceed in §5.5 by constructing exponential dichotomies for (5.1.19) on the half-line $\mathbb{R}^+$. In particular, for any $\tau \geq 0$ we establish the decomposition

$$C_b(\mathbb{R}) \quad = \quad Q(\tau) \oplus R(\tau). \tag{5.1.20}$$

Here $Q(\tau)$ contains (shifted) exponentially decaying functions that satisfy (5.1.19) on $[\tau, \infty)$, while (shifts of) functions in $R(\tau)$ satisfy (5.1.19) on $[0, \tau]$. This generalizes the finite-range results obtained by Hupkes and Verduyn Lunel in [104], which we achieve by following a very similar strategy.

Besides the general complications discussed above, the main technical obstruction here is that the construction of half-line solutions to inhomogeneous versions of (5.1.19) becomes rather delicate. Indeed, the approach taken in [104] modifies the inhomogeneous terms *outside* the 'influence region' of the half-line of interest. However, in our setting here this region encompasses the whole line, forcing us to revisit the problem in a more elaborate—and technical—fashion.

Degeneracies In order to successfully exploit the characterization (5.1.15) in applications, it is essential to revisit the degeneracy issues related to the Hale inner product and the kernel elements of $\mathcal{L}^*$. Unfortunately, the absence of a 'smallest' and 'largest' shift in the infinite-range setting prevents an easy generalization of the invertibility criterion discussed above. We explore this crucial issue at length in §5.6.

In order to sketch some of the issues involved, we discuss the MFDE

$$cu'(\sigma) = \sum_{k=1}^{\infty} \gamma_k[u(\sigma+k) + u(\sigma-k) - 2u(\sigma)] + g_u(\overline{u}_*(\sigma); a)u(\sigma), \tag{5.1.21}$$

which can be interpreted as an infinite-range version of the MFDE (5.1.7) that arises by linearizing the Nagumo LDE around a travelling wave $\overline{u}_*$. In particular, we again assume the limits (5.1.5). This MFDE fits into our framework provided that the coefficients γ_k decay exponentially.

For the case $\gamma_k = e^{-k}$, we construct an explicit nontrivial function ψ that satisfies $\langle \psi, \phi \rangle_\tau = 0$ for each $\phi \in C_b(\mathbb{R})$, where $\langle \cdot, \cdot \rangle_\tau$ denotes the appropriate Hale inner product for our setting. In particular, even for strictly positive coefficients there is no guarantee that the Hale inner product is nondegenerate. We also provide such examples for systems featuring convolution kernels.

One way to circumvent this problem is to focus specifically on the kernel elements in (5.1.15). If these can be chosen to be nonnegative along with the coefficients γ_k, then we are able to recover the relation between the dimension of $\Gamma(\tau)$ in (5.1.12) and the dimension of the kernel of the operator $\mathcal{L}^*$ associated to the adjoint of (5.1.19). Fortunately, such positivity conditions follow naturally for systems that admit a comparison principle.

We also explore a second avenue that can be used without sign restrictions on the coefficients γ_k. This requires us to borrow some abstract functional analytic results. In particular, whenever the collection of sequences $\{\gamma_k\}_{k \geq N}$ obtained by taking $N \in \mathbb{N}$ spans an infinite dimensional subset of $\ell^2(\mathbb{N}; \mathbb{C})$, we show that the Hale inner product is nondegenerate in a suitable sense. Fortunately, this rather abstract condition can often be made concrete. For example, we show that it can be enforced by imposing the Gaussian decay rate $\gamma_k \sim \exp[-k^2]$.

5.2 Main results

Our main results consider the integro-differential MFDE[1]

$$\dot{x}(t) = \sum_{j=-\infty}^{\infty} A_j(t)x(t+r_j) + \int_{\mathbb{R}} \mathcal{K}(\xi; t)x(t+\xi)d\xi, \tag{5.2.1}$$

[1]In the interest of readability we use t as our main variable throughout the remainder of this paper, departing from the notation σ that we used in §5.1. However, the reader should keep in mind that this variable is related to a spatial quantity for most applications.

where we take $x \in \mathbb{C}^M$ for some integer $M \geq 1$. The set of scalars $\mathcal{R} := \{r_j : j \in \mathbb{Z}\} \subset \mathbb{R}$ and the support of $\mathcal{K}(\cdot; t)$ need not be bounded. In fact, we pick two constants

$$-\infty \leq r_{\min} \leq 0 \leq r_{\max} \leq \infty, \qquad r_{\min} < r_{\max} \tag{5.2.2}$$

in such a way that

$$\begin{array}{rcll} r_j & \in & \overline{(r_{\min}, r_{\max})}, & \text{for all } j \in \mathbb{Z}, \\ \operatorname{supp}\big(\mathcal{K}(\cdot; t)\big) & \subset & \overline{(r_{\min}, r_{\max})}, & \text{for all } t \in \mathbb{R}, \end{array} \tag{5.2.3}$$

while $|r_{\min}|$ and $|r_{\max}|$ are as small as possible. One readily sees that potential solutions to (5.2.1) must be defined on intervals that have a minimal length of $r_{\max} - r_{\min}$.

Naturally, one can always artificially increase the quantities $|r_{\min}|$ and $|r_{\max}|$ by adding matrices $A_j = 0$ to (5.2.1) with large associated shifts $|r_j| \gg 1$. However, we will see that this only weakens the predictive power of our results by needlessly enlarging the relevant state spaces.

A more general version of (5.2.1) might take the form

$$\dot{x}(t) \quad = \quad \int_{r_{\min}}^{r_{\max}} d_\theta(t, \theta) x(t + \theta), \tag{5.2.4}$$

where $d_\theta(t, \theta)$ is an $M \times M$ matrix of finite Lebesgue-Stieltjes measures on $\overline{(r_{\min}, r_{\max})}$ for each $t \in \mathbb{R}$. However, the adjoint of the system (5.2.4) is not always a system of similar type, so to avoid technical complications we will restrict ourselves to the system (5.2.1).

We now formulate our two main conditions on the coefficients in (5.2.1), which match those used in [68]. As a preparation, we define the exponentially weighted space

$$L^1_\eta(\mathbb{R}; \mathbb{C}^{M \times M}) \quad := \quad \Big\{ \mathcal{V} \in L^1(\mathbb{R}; \mathbb{C}^{M \times M}) \Big| \| e^{\eta |\cdot|} \mathcal{V}(\cdot) \|_{L^1(\mathbb{R}; \mathbb{C}^{M \times M})} < \infty \Big\} \tag{5.2.5}$$

for any $\eta > 0$, with its natural norm

$$\|\mathcal{V}\|_\eta \quad := \quad \| e^{\eta |\cdot|} \mathcal{V}(\cdot) \|_{L^1(\mathbb{R}; \mathbb{C}^{M \times M})}. \tag{5.2.6}$$

We note that the conditions on $\mathcal{R}$ below are not actual restrictions as long as the closure $\overline{\mathcal{R}}$ is countable. Indeed, one can simply add the missing shifts to $\mathcal{R}$ and write $A_j = 0$ for the associated matrix.

Assumption (HA). For each $j \in \mathbb{Z}$ the map $t \mapsto A_j(t)$ is bounded and belongs to $C^1(\mathbb{R}; \mathbb{C}^{M \times M})$. Moreover, there exists a constant $\tilde{\eta} > 0$ for which the bound

$$\sum_{j=-\infty}^{\infty} \|A_j(\cdot)\|_\infty e^{\tilde{\eta} |r_j|} \quad < \quad \infty \tag{5.2.7}$$

holds. In addition, the set $\mathcal{R}$ is closed with $0 \in \mathcal{R}$.

Assumption (HK). There exists a constant $\tilde{\eta} > 0$ so that the following properties hold.

- The map $t \mapsto \mathcal{K}(\cdot; t)$ belongs to $C^1\big(\mathbb{R}; L^1_{\tilde{\eta}}(\mathbb{R}; \mathbb{C}^{M\times M})\big)$.
- The kernel $\mathcal{K}$ is localized in the sense that

$$\begin{array}{lcl} \sup_{t\in\mathbb{R}} \|\mathcal{K}(\cdot;t)\|_{\tilde{\eta}} + \sup_{t\in\mathbb{R}} \|\frac{d}{dt}\mathcal{K}(\cdot;t)\|_{\tilde{\eta}} & < & \infty, \\ \sup_{t\in\mathbb{R}} \|\mathcal{K}(\cdot;t-\cdot)\|_{\tilde{\eta}} + \sup_{t\in\mathbb{R}} \|\frac{d}{dt}\mathcal{K}(\cdot;t-\cdot)\|_{\tilde{\eta}} & < & \infty. \end{array} \tag{5.2.8}$$

Our third structural condition involves the behaviour of the coefficients in (5.2.1) as $t \to \pm\infty$. Following [68, 130], we say that the system (5.2.1) is asymptotically hyperbolic if the limits

$$A_j(\pm\infty) := \lim_{t\to\pm\infty} A_j(t), \qquad \mathcal{K}(\xi;\pm\infty) := \lim_{t\to\pm\infty} \mathcal{K}(\xi;t) \tag{5.2.9}$$

exist for each $j \in \mathbb{Z}$ and $\xi \in \mathbb{R}$, while the characteristic functions

$$\Delta^{\pm}(z) = zI - \int_{\mathbb{R}} \mathcal{K}(\xi;\pm\infty)e^{z\xi}d\xi - \sum_{j=-\infty}^{\infty} A_j(\pm\infty)e^{zr_j} \tag{5.2.10}$$

associated to the limiting systems

$$\dot{x}(t) = \sum_{j=-\infty}^{\infty} A_j(\pm\infty)x(t+r_j) + \int_{\mathbb{R}} \mathcal{K}(\xi;\pm\infty)x(t+\xi)d\xi \tag{5.2.11}$$

satisfy

$$\det \Delta^{\pm}(iy) \neq 0 \tag{5.2.12}$$

for all $y \in \mathbb{R}$. In fact, we require that these limiting systems are approached in a summable fashion.

Assumption (HH). The system (5.2.1) is asymptotically hyperbolic and satisfies the limits

$$\lim_{t\to\pm\infty} \sum_{j=-\infty}^{\infty} |A_j(t) - A_j(\pm\infty)|e^{\tilde{\eta}|r_j|} = 0, \tag{5.2.13}$$

together with

$$\lim_{t\to\pm\infty} \|\mathcal{K}(\cdot;t) - \mathcal{K}(\cdot;\pm\infty)\|_{\tilde{\eta}} = 0, \qquad \lim_{t\to\pm\infty} \|\mathcal{K}(\cdot;t-\cdot) - \mathcal{K}(\cdot;\pm\infty)\|_{\tilde{\eta}} = 0. \tag{5.2.14}$$

Bounded solutions to the system (5.2.1) can be interpreted as kernel elements of the linear operator $\Lambda : W^{1,\infty}(\mathbb{R}) \to L^{\infty}(\mathbb{R})$ that acts as

$$(\Lambda x)(t) = \dot{x}(t) - \sum_{j=-\infty}^{\infty} A_j(t)x(t+r_j) - \int_{\mathbb{R}} \mathcal{K}(\xi;t)x(t+\xi)d\xi. \tag{5.2.15}$$

We will write $\Lambda^* : W^{1,\infty}(\mathbb{R}) \to L^\infty(\mathbb{R})$ for the formal adjoint of this operator, which is given by

$$(\Lambda^* y)(t) \quad = \quad -\dot{y}(t) - \sum_{j=-\infty}^{\infty} A_j(t - r_j)^\dagger y(t - r_j) - \int_{\mathbb{R}} \mathcal{K}(\xi; t - \xi)^\dagger y(t - \xi) d\xi, \tag{5.2.16}$$

using $\dagger$ to denote the conjugate transpose of a matrix. Indeed, one may readily verify the identity

$$\langle y, \Lambda x \rangle_{L^2(\mathbb{R})} \quad = \quad \langle \Lambda^* y, x \rangle_{L^2(\mathbb{R})} \tag{5.2.17}$$

whenever $x, y \in H^1(\mathbb{R})$.

For convenience, we borrow the notation from [104, 133] and write

$$\mathcal{B} \quad = \quad \ker(\Lambda), \qquad \mathcal{B}^* \quad = \quad \ker(\Lambda^*). \tag{5.2.18}$$

The following result obtained by Faye and Scheel describes several useful Fredholm properties that link these kernels to the ranges of the operators Λ and Λ^*.

Proposition 5.2.1 ([68, Thm. 2])**.** *Assume that (HA), (HK) and (HH) are satisfied. Then both the operators Λ and Λ^* are Fredholm operators. Moreover, the kernels and ranges satisfy the identities*

$$\begin{array}{lcl} \mathrm{Range}(\Lambda) & = & \{h \in L^\infty(\mathbb{R}) \mid \int_{-\infty}^{\infty} y(t)^\dagger h(t) dt = 0 \textit{ for every } y \in \mathcal{B}^*\}, \\ \mathrm{Range}(\Lambda^*) & = & \{h \in L^\infty(\mathbb{R}) \mid \int_{-\infty}^{\infty} x(t)^\dagger h(t) dt = 0 \textit{ for every } x \in \mathcal{B}\} \end{array} \tag{5.2.19}$$

and the Fredholm indices can be computed by

$$\mathrm{ind}(\Lambda) \quad = \quad -\mathrm{ind}(\Lambda^*) \quad = \quad \dim \mathcal{B} - \dim \mathcal{B}^*. \tag{5.2.20}$$

Finally, there exist constants $C > 0$ and $0 < \alpha \le \tilde{\eta}$ so that the estimate

$$|b(t)| \quad \le \quad C e^{-\alpha |t|} \|b\|_\infty \tag{5.2.21}$$

holds for any $b \in \mathcal{B} \cup \mathcal{B}^*$ and any $t \in \mathbb{R}$.

5.2.1 State spaces

Let us introduce the intervals

$$D_X \quad = \quad \overline{(r_{\min}, r_{\max})}, \qquad D_Y \quad = \quad \overline{(-r_{\max}, -r_{\min})}, \tag{5.2.22}$$

together with the state spaces

$$X \quad = \quad C_b(D_X), \qquad Y \quad = \quad C_b(D_Y), \tag{5.2.23}$$

which contain bounded continuous functions that we measure with the supremum norm. Suppose now that x and y are two bounded continuous functions that are defined on (at least) the interval $t + D_X$ respectively $t + D_Y$. We then write $x_t \in X$ and $y^t \in Y$ for the segments

$$x_t(\theta) = x(t+\theta), \qquad y^t(\theta) = y(t+\theta), \tag{5.2.24}$$

in which $\theta \in D_X$ respectively $\theta \in D_Y$. This allows us to introduce the kernel segment spaces

$$\begin{aligned} B(\tau) &= \{\phi \in X \mid \phi = x_\tau \text{ for some } x \in \mathcal{B}\}, \\ B^*(\tau) &= \{\psi \in Y \mid \psi = y^\tau \text{ for some } y \in \mathcal{B}^*\} \end{aligned} \tag{5.2.25}$$

for every $\tau \in \mathbb{R}$. Observe that $B(\tau)$ and $B^*(\tau)$ are just shifted versions of $\mathcal{B}$ and $\mathcal{B}^*$ if $r_{\min} = -\infty$ and $r_{\max} = \infty$ both hold.

The Hale inner product [91] provides a useful coupling between X and Y. The natural definition in the current setting is given by

$$\begin{aligned} \langle \psi, \phi \rangle_t = {} & \psi(0)^\dagger \phi(0) - \sum_{j=-\infty}^{\infty} \int_0^{r_j} \psi(s - r_j)^\dagger A_j(t + s - r_j)\phi(s) ds \\ & - \int_{\mathbb{R}} \int_0^{r} \psi(s-r)^\dagger \mathcal{K}(r; t+s-r)\phi(s) ds dr \end{aligned} \tag{5.2.26}$$

for any pair $(\phi, \psi) \in X \times Y$. Note that, by decreasing $\tilde{\eta}$ if necessary, we can strengthen (5.2.7) to obtain

$$\sum_{j=-\infty}^{\infty} \|A_j(\cdot)\|_\infty |r_j| e^{\tilde{\eta}|r_j|} < \infty. \tag{5.2.27}$$

Together with (5.2.8), this ensures that the Hale inner product is well-defined. In Lemma 5.3.12 below we verify the identity

$$\tfrac{d}{dt}\langle y^t, x_t \rangle_t = y^\dagger(t)[\Lambda x](t) + [\Lambda^* y](t)^\dagger x(t) \tag{5.2.28}$$

for $x, y \in W^{1,\infty}(\mathbb{R})$, which indicates that the Hale inner product can be seen as the duality pairing between Λ and Λ^*.

An important role in the sequel is reserved for the subspaces

$$X^\perp(\tau) = \{\phi \in X \mid \langle \psi, \phi \rangle_\tau = 0 \text{ for every } \psi \in B^*(\tau)\}, \tag{5.2.29}$$

which have finite codimension

$$\beta(\tau) := \operatorname{codim}_X X^\perp(\tau) \leq \dim B^*(\tau) \leq \dim \mathcal{B}^*. \tag{5.2.30}$$

In the ODE case $r_{\min} = r_{\max} = 0$, so one readily concludes that $\beta(\tau) = \dim \mathcal{B}^*$. However, in the present setting it is possible for the Hale inner product to be degenerate or for kernel elements to vanish on large intervals. In these cases, the first respectively second inequality in (5.2.30) could become strict.

In the finite range setting of [133], the authors ruled out these degeneracies by imposing an atomic condition on the matrices $\{A_j\}$ corresponding to the shifts $r_{\min}$ and $r_{\max}$. However, there is no obvious way to generalize this condition when $|r_{\min}|$ or $r_{\max}$ are infinite. As an alternative, some of our results require the following technical assumption.

Assumption (HKer). Consider any nonzero $d \in \mathcal{B} \cup \mathcal{B}^*$ and $\tau \in \mathbb{R}$. Then d does not vanish on $(-\infty, \tau]$ and also does not vanish on $[\tau, \infty)$.

A similar assumption was used in [11, Assumption H3(iii)], where the authors remove the $|r_{\min}| = r_{\max}$ restriction from the exponential dichotomy constructions in [96]. However, this condition is naturally much harder to verify than the previous atomicity condition. We explore this issue at length in §5.6, where we present several scenarios under which (HKer) can be verified.

We highlight one of these scenarios in the result below, which requires sign conditions on elements of $\mathcal{B}$ and $\mathcal{B}^*$. Fortunately, for a large class of systems—including the linearization (5.1.21) of the Nagumo LDE—these are known consequences of the comparison principle.

Proposition 5.2.2 (see Prop. 5.6.10)**.** *Assume that (HA), (HK) and (HH) are satisfied. Assume furthermore that there exists $K_{\mathrm{const}} \in \mathbb{Z}_{\geq 1}$ for which the following structural conditions are satisfied.*

(a) We have $r_j = j$ for $j \in \mathbb{Z}$, which implies $r_{\min} = -\infty$ and $r_{\max} = \infty$.

(b) The function $A_j(\cdot)$ is constant and positive definite whenever $|j| \geq K_{\mathrm{const}}$.

(c) For any $|\xi| \geq K_{\mathrm{const}}$ the function $\mathcal{K}(\xi; \cdot)$ is constant and positive definite.

(d) We either have $\mathcal{B} = \{0\}$ or $\mathcal{B} = \mathrm{span}\{b\}$ for some nonnegative function b. The same holds for $\mathcal{B}^*$.

Then the nontriviality condition (HKer) is satisfied.

In §5.5-5.6 we explore some of the consequences of (HKer). In addition, we propose weaker conditions under which equality holds for one or both of the inequalities in (5.2.30). However, for now we simply state the following result.

Corollary 5.2.3 (cf. [133, Cor. 4.7], see §5.6)**.** *Assume that (HA), (HK), (HH) and (HKer) are all satisfied. Then the identities*

$$\dim B(\tau) \quad = \quad \dim \mathcal{B}, \qquad\qquad \beta(\tau) \quad = \quad \dim B^*(\tau) \quad = \quad \dim \mathcal{B}^* \tag{5.2.31}$$

hold for every $\tau \in \mathbb{R}$.

5.2.2 Exponential dichotomies on $\mathbb{R}$

We now set out to describe our exponential splittings for (5.2.1) on the full line $\mathbb{R}$. To this end, we introduce the intervals

$$D_\tau^\ominus \quad = \quad \overline{(-\infty, \tau + r_{\max})}, \qquad\qquad D_\tau^\oplus \quad = \quad \overline{(\tau + r_{\min}, \infty)} \tag{5.2.32}$$

for each $\tau \in \mathbb{R}$. Following the notation in [104, 133], this allows us to define the solution spaces

$$\begin{aligned} \mathcal{P}(\tau) &= \{x \in C_b(D_\tau^\ominus) \mid x \text{ is a bounded solution of (5.2.1) on } (-\infty, \tau]\}, \\ \mathcal{Q}(\tau) &= \{x \in C_b(D_\tau^\oplus) \mid x \text{ is a bounded solution of (5.2.1) on } [\tau, \infty)\}, \end{aligned} \tag{5.2.33}$$

together with the associated initial segments

$$\begin{array}{rcl} P(\tau) & = & \{\phi \in X \mid \phi = x_\tau \text{ for some } x \in \mathcal{P}(\tau)\}, \\ Q(\tau) & = & \{\phi \in X \mid \phi = x_\tau \text{ for some } x \in \mathcal{Q}(\tau)\}. \end{array} \tag{5.2.34}$$

For $\tau \in \mathbb{R}$ we call $x \in \mathcal{P}(\tau)$ a left prolongation of an element $\phi = x_\tau \in P(\tau)$, with a similar definition for right prolongations. Note that, if $r_{\min} = -\infty$, each $\phi \in P(\tau)$ is simply a translation of a function in $\mathcal{P}(\tau)$. The corresponding result holds for $Q(\tau)$ and $\mathcal{Q}(\tau)$ if $r_{\max} = \infty$.

Again following [133], we also work with the spaces

$$\begin{array}{rcl} \widehat{\mathcal{P}}(\tau) & = & \{x \in \mathcal{P}(\tau) \mid \int\limits_{-\infty}^{\tau + r_{\max}} y(t)^\dagger x(t) dt = 0 \text{ for every } y \in \mathcal{B}\}, \\ \widehat{\mathcal{Q}}(\tau) & = & \{x \in \mathcal{Q}(\tau) \mid \int\limits_{\tau + r_{\min}}^{\infty} y(t)^\dagger x(t) dt = 0 \text{ for every } y \in \mathcal{B}\}, \end{array} \tag{5.2.35}$$

together with

$$\begin{array}{rcl} \widehat{P}(\tau) & = & \{\phi \in X \mid \phi = x_\tau \text{ for some } x \in \widehat{\mathcal{P}}(\tau)\}, \\ \widehat{Q}(\tau) & = & \{\phi \in X \mid \phi = x_\tau \text{ for some } x \in \widehat{\mathcal{Q}}(\tau)\}. \end{array} \tag{5.2.36}$$

The integrals in (5.2.35) convergence since functions in $\mathcal{B}$ decay exponentially. Finally, we write

$$S(\tau) \;=\; P(\tau) + Q(\tau), \qquad \widehat{S}(\tau) = \widehat{P}(\tau) + \widehat{Q}(\tau). \tag{5.2.37}$$

Our first two results here provide exponential decay estimates for functions in $\widehat{\mathcal{P}}(\tau)$ and $\widehat{\mathcal{Q}}(\tau)$, together with a direct sum decomposition for $S(\tau)$. In addition, we show that the latter space can be identified with $X^\perp(\tau)$ from (5.2.29). We remark that the structure of these results matches their counterparts from [91] almost verbatim.

Theorem 5.2.4 (cf. [133, Thm. 4.2], see §5.3)**.** *Assume that (HA), (HK) and (HH) are satisfied and choose a sufficiently large $\tau_* > 0$. Then there exist constants $K_{\mathrm{dec}} > 0$ and $\alpha > 0$ so that for any $\tau \le -\tau_*$ and $p \in \mathcal{P}(\tau)$ we have the bound*

$$|p(t)| + |\dot{p}(t)| \;\le\; K_{\mathrm{dec}} e^{\alpha(t-\tau)} \|p_\tau\|_\infty, \qquad t \le \tau, \tag{5.2.38}$$

while for any $\tau \ge \tau_*$ and $q \in \mathcal{Q}(\tau)$ we have the corresponding estimate

$$|q(t)| + |\dot{q}(t)| \;\le\; K_{\mathrm{dec}} e^{-\alpha(t-\tau)} \|q_\tau\|_\infty, \qquad t \ge \tau. \tag{5.2.39}$$

In addition, the bounds (5.2.38)-(5.2.39) also hold for any $p \in \widehat{\mathcal{P}}(\tau)$ and $q \in \widehat{\mathcal{Q}}(\tau)$, now without any restriction on the value of $\tau \in \mathbb{R}$, but with possibly different values of K_{dec} and α.

Theorem 5.2.5 (cf. [133, Thm. 4.3], see §5.3)**.** *Assume that (HA), (HK) and (HH) are satisfied. For each $\tau \in \mathbb{R}$ the spaces $P(\tau)$, $Q(\tau)$, $S(\tau)$ and their counterparts $\widehat{P}(\tau)$, $\widehat{Q}(\tau)$, $\widehat{S}(\tau)$ are all closed subspaces of X. Moreover, we have the identities*

$$\begin{array}{rclcrcl} P(\tau) & = & \widehat{P}(\tau) \oplus B(\tau), & & Q(\tau) & = & \widehat{Q}(\tau) \oplus B(\tau), \\ \widehat{S}(\tau) & = & \widehat{P}(\tau) \oplus \widehat{Q}(\tau), & & S(\tau) & = & \widehat{S}(\tau) \oplus B(\tau) \\ & & & & & = & \widehat{P}(\tau) \oplus \widehat{Q}(\tau) \oplus B(\tau). \end{array} \tag{5.2.40}$$

Finally, we have the identification

$$S(\tau) \quad = \quad X^{\perp}(\tau), \tag{5.2.41}$$

where $X^{\perp}(\tau)$ is defined in (5.2.29).

However, these theorems provide no information on how the spaces $P(\tau)$ and $Q(\tau)$ depend on τ. In order to address this issue, we need to study the projections from the state space X onto the factors $\widehat{P}(\tau)$ and $\widehat{Q}(\tau)$ using the decomposition in (5.2.40). To be more precise, for a fixed $\tau_0 \in \mathbb{R}$ we write

$$X \quad = \quad \widehat{P}(\tau_0) \oplus \widehat{Q}(\tau_0) \oplus \Gamma \tag{5.2.42}$$

for a suitable finite dimensional subspace $\Gamma \subset X$. This allows us define projections $\Pi_{\widehat{P}}$ and $\Pi_{\widehat{Q}}$ onto the factors $\widehat{P}(\tau_0)$ respectively $\widehat{Q}(\tau_0)$.

In addition, we are interested in the limiting behaviour as $\tau \to \pm\infty$. To this end, we apply Theorem 5.2.5 to the two limiting systems (5.2.11), which leads to the decompositions

$$X \quad = \quad P(-\infty) \oplus Q(-\infty) \quad = \quad P(\infty) \oplus Q(\infty). \tag{5.2.43}$$

We write $\overleftarrow{\Pi}_P$ and $\overleftarrow{\Pi}_Q$ for the projections onto the factors $P(-\infty)$ and $Q(-\infty)$ respectively, together with $\overrightarrow{\Pi}_P$ and $\overrightarrow{\Pi}_Q$ for the projections onto the factors $P(\infty)$ and $Q(\infty)$.

Theorem 5.2.6 (cf. [133, Thm. 4.6], see §5.4)**.** *Assume that (HA), (HK) and (HH) are satisfied. Then the spaces $\widehat{P}(\tau)$, $\widehat{Q}(\tau)$ and $\widehat{S}(\tau)$ vary upper semicontinuously with τ, while the quantities $\dim B(\tau)$ and $\beta(\tau)$ vary lower semicontinuously with τ.*

In particular, fix $\tau_0 \in \mathbb{R}$ and consider any τ sufficiently close to τ_0. Then the restrictions

$$\begin{array}{rcl} \Pi_{\widehat{P}} : \widehat{P}(\tau) & \to & \Pi_{\widehat{P}}\big(\widehat{P}(\tau)\big) \subset \widehat{P}(\tau_0), \\ \Pi_{\widehat{Q}} : \widehat{Q}(\tau) & \to & \Pi_{\widehat{Q}}\big(\widehat{Q}(\tau)\big) \subset \widehat{Q}(\tau_0) \end{array} \tag{5.2.44}$$

of the projections associated to the decomposition (5.2.42) *are isomorphisms onto their ranges, which are closed. Moreover, the norms satisfy*

$$\lim_{\tau \to \tau_0} \|I - \Pi_{\widehat{P}}|_{\widehat{P}(\tau)}\| \quad = \quad 0, \qquad\qquad \lim_{\tau \to \tau_0} \|I - \Pi_{\widehat{Q}}|_{\widehat{Q}(\tau)}\| \quad = \quad 0, \tag{5.2.45}$$

in which I denotes the inclusion of $\widehat{P}(\tau)$ or $\widehat{Q}(\tau)$ into X.

In addition, we have the identities

$$\begin{array}{rcl} \overleftarrow{\Pi}_P\big(P(\tau)\big) & = & P(-\infty), \\ \overrightarrow{\Pi}_Q\big(Q(\tau)\big) & = & Q(\infty), \end{array} \tag{5.2.46}$$

for sufficiently negative values of τ in the first line of (5.2.46) and for sufficiently positive values of τ in the second line of (5.2.46). The associated norms satisfy the limits

$$\lim_{\tau \to -\infty} \|I - \overleftarrow{\Pi}_P|_{P(\tau)}\| \;=\; 0, \qquad \lim_{\tau \to \infty} \|I - \overrightarrow{\Pi}_Q|_{Q(\tau)}\| \;=\; 0. \tag{5.2.47}$$

These results can be strengthened if we also assume that (HKer) holds. Indeed, Corollary 5.2.3 implies that the codimension of $S(\tau)$ remains constant. This can be leveraged to obtain the following continuity properties.

Corollary 5.2.7 (cf. [133, Cor. 4.7], see §5.6)**.** *Assume that (HA), (HK), (HH) and (HKer) are all satisfied. Then the spaces $\widehat{P}(\tau)$ and $\widehat{Q}(\tau)$ vary continuously with τ, i.e. the projections $\Pi_{\widehat{P}}$ and $\Pi_{\widehat{Q}}$ from (5.2.44) are isomorphisms onto $\widehat{P}(\tau_0)$ and $\widehat{Q}(\tau_0)$ respectively. The same conclusion holds for their counterparts $P(\tau)$ and $Q(\tau)$.*

5.2.3 Exponential dichotomies on half-lines

In many applications it is useful to consider exponential dichotomies on half-lines such as $[0, \infty)$, instead of the full line. Our main goal here is to show to prove the natural generalisation of Theorem 5.2.5 to this half-line setting, along the lines of the results in [104].

In particular, we set out to obtain decompositions of the form

$$X \;=\; Q(\tau) \oplus R(\tau), \tag{5.2.48}$$

where $Q(\tau)$ is defined in (5.2.34) and segments in $R(\tau)$ should be 'extendable' to solve (5.2.1) on $[0, \tau]$. Since this is a finite interval however there is no longer a 'canonical' definition for $R(\tau)$. In fact, we define these spaces in a indirect fashion, by constructing appropriate subsets

$$\mathcal{R}(\tau) \subset \{r \in C_b(D_\tau^{\ominus}) \mid r \text{ is a bounded solution of (5.2.1) on } [0, \tau]\} \tag{5.2.49}$$

and writing

$$R(\tau) \;=\; \{\phi \in X \mid \phi = x_\tau \text{ for some } x \in \mathcal{R}(\tau)\}. \tag{5.2.50}$$

In order to achieve this, we exploit continuity properties for the projection operators that are stronger than those obtained in Theorem 5.2.6. In particular, we again impose the nontriviality condition (HKer). However, we explain in §5.5 how this condition can be weakened slightly. For example, we need less information concerning the kernel space $\mathcal{B}$ to apply our construction.

Theorem 5.2.8 (cf. [104, Thm. 4.1], see §5.5)**.** *Assume that (HA), (HK), (HH) and (HKer) are satisfied. Then for every $\tau \geq 0$ there exists a closed subspace $\mathcal{R}(\tau) \subset C_b(D_\tau^\ominus)$ that satisfies the inclusion* (5.2.49) *together with the following properties.*

(i) Recalling the spaces (5.2.34) and (5.2.50)*, the splitting* (5.2.48) *holds for every $\tau \geq 0$.*

(ii) There exist constants $K_{\mathrm{dec}} > 0$ and $\alpha > 0$ so that the exponential estimate

$$|x(t)| \quad \leq \quad K_{\mathrm{dec}} e^{-\alpha|t-\tau|} \|x_\tau\|_\infty \tag{5.2.51}$$

holds for every $x \in \mathcal{R}(\tau)$ and every pair $0 \leq t \leq \tau$.

(iii) The spaces $R(\tau)$ are invariant, in the sense that $x_t \in R(t)$ holds whenever $x \in \mathcal{R}(\tau)$ and $0 \leq t \leq \tau$. The corresponding statement holds for the spaces $Q(\tau)$.

(iv) The projections $\Pi_{Q(\tau)}$ and $\Pi_{R(\tau)}$ associated to the splitting (5.2.48) *depend continuously on $\tau \geq 0$. In addition, there exists a constant $C \geq 0$ so that the uniform bounds $\|\Pi_{Q(\tau)}\| \leq C$ and $\|\Pi_{R(\tau)}\| \leq C$ hold for all $\tau \geq 0$.*

5.3 The existence of exponential dichotomies

Our goal in this section is to establish Theorems 5.2.4-5.2.5. The strategy that we follow is heavily based on [133], allowing us to simply refer to the results there from time to time. However, the unbounded shifts force us to develop an alternative approach at several key points in the analysis. We have therefore structured this section in such a way that these modifications are highlighted.

The first main task is to show that functions in the spaces $\mathcal{P}(\tau)$ and $\mathcal{Q}(\tau)$, together with their derivatives, decay exponentially in a uniform fashion. When the shifts are unbounded, the methods developed in [133] can no longer be used to establish this exponential decay. In particular, the bound (5.3.4) below was obtained in [133], but one cannot simply make the replacement $r_{\max} \to \infty$ and still recover the desired exponential decay of solutions. Indeed, the iterative scheme in [133] breaks down, forcing us to use a different approach.

The key ingredient is to show that the cumulative influence of the large shifts decays exponentially. The following preliminary estimate will help us to quantify this.

Lemma 5.3.1. *Assume that (HA), (HK) and (HH) are satisfied. Then there exist three constants $(p, K_{\exp}, \alpha) \in \mathbb{R}^3_{>0}$ for which the bound*

$$\sum_{r_j \geq |t|} |A_j(s)| e^{\alpha |r_j|} + \int_{|t|}^{\infty} |\mathcal{K}(\xi; s)| e^{\alpha|\xi|} d\xi \quad \leq \quad K_{\exp} e^{-2\alpha|t|} \tag{5.3.1}$$

holds for all $t < -p$ and all $s \in \mathbb{R}$. In addition, if $r_{\max} < \infty$, then we can pick $p = r_{\max}$.

Proof. Suppose first that $r_{\max} = \infty$. Setting $\alpha = \frac{\tilde{\eta}}{3}$, we can derive from (5.2.7) that

$$\begin{aligned} \sum_{r_j \geq |t|} \|A_j(\cdot)\|_\infty e^{\alpha |r_j|} &\leq e^{-2\alpha t} \sum_{r_j \geq |t|} \|A_j(\cdot)\|_\infty e^{\tilde{\eta}|r_j|} \\ &\leq e^{-2\alpha t} \sum_{j=-\infty}^{\infty} \|A_j(\cdot)\|_\infty e^{\tilde{\eta}|r_j|} \end{aligned} \tag{5.3.2}$$

for $|t|$ sufficiently large. The second term in (5.3.1) can be bounded in the same fashion using (5.2.8). If $r_{\max} < \infty$ then (5.3.1) follows trivially for $p = r_{\max}$, since the left-hand side is always zero for $t < -p$ and $s \in \mathbb{R}$. ∎

Our first main result generalizes the bound (5.3.4) to the setting where $r_{\max} = \infty$. This is achieved by splitting the relevant interval $[\tau, \infty)$ into two parts $[\tau, \tau + p]$ and $[\tau + p, \infty)$ that we analyze separately. We use the ideas from [133] to study the first part, while careful estimates involving (5.3.1) allow us to control the contributions from the unbounded second interval.

Proposition 5.3.2. *Assume that (HA), (HK) and (HH) are satisfied, recall the constants $(p, K_{\exp}, \alpha) \in \mathbb{R}^3_{>0}$ from Lemma 5.3.1 and pick a sufficiently negative $\tau_- \ll -1$. Then there exists a constant $\sigma > 0$ so that for each $\tau \leq \tau^-$ and each $x \in \mathcal{P}(\tau)$ we have the bound*

$$|x(t)| \leq \max\Big\{ \tfrac{1}{2} \sup_{s \in (-\infty, \tau + p]} |x(s)|, \ K_{\exp} \sup_{s \in [p + \tau, \infty)} e^{-\alpha(s-t)} |x(s)| \Big\}, \qquad t \leq -\sigma + \tau \tag{5.3.3}$$

when $r_{\max} = \infty$, or alternatively

$$|x(t)| \leq \tfrac{1}{2} \sup_{s \in (-\infty, \tau + r_{\max}]} |x(s)|, \qquad t \leq -\sigma + \tau \tag{5.3.4}$$

when $r_{\max} < \infty$. The same[2] *bounds hold for $x \in \widehat{\mathcal{P}}(\tau)$, but now any $\tau \in \mathbb{R}$ is permitted.*

The second main complication occurs when one tries to mimic the approach in [133] to study the properties of $S(\tau)$. Although it is relatively straightforward to show that this space is closed and has finite codimension in X, the explicit description (5.2.41) for $S(\tau)$ is much harder to obtain. The arguments in [133] approximate elements of $X^\perp(\tau)$ by C^1-smooth functions and apply the Fredholm operator Λ to (extensions of) these approximants. However, when D_X is unbounded this approach breaks down, because C^1-smooth functions in X need not have a bounded derivative. One can hence no longer directly appeal to the useful Fredholm properties of Λ.

Our second main result provides an alternative approach that circumvents these difficulties. The novel idea is that we split such problematic functions into two parts that both confine the regions where the derivatives are unbounded to a half-line. This turns out to be sufficient to allow the main spirit of the analysis in [133] to proceed.

Proposition 5.3.3. *Assume that (HA), (HK) and (HH) are satisfied. Fix $\tau \in \mathbb{R}$ and let $X^\perp(\tau)$ be given by (5.2.29). Then there exists a dense subset $D \subset X^\perp(\tau)$ with $D \subset S(\tau)$.*

[2] Naturally, one may need to change the value of the constant $\sigma > 0$.

Besides these two main obstacles, we encounter smaller technical issues at many points during our analysis. For example, the lack of full uniform convergence on unbounded intervals from the Ascoli-Arzela theorem requires significant attention. In addition, manipulations involving the Hale inner product on unbounded domains raise subtle convergence issues that must be addressed.

5.3.1 Preliminaries

In this subsection, we collect several preliminary properties satisfied by the spaces introduced in (5.2.25), (5.2.33) and (5.2.34). In particular, we discuss whether functions in $P(\tau)$ or $Q(\tau)$ have unique extensions in $\mathcal{P}(\tau)$ and $\mathcal{Q}(\tau)$ and study the intersection $P(\tau) \cap Q(\tau)$.

Lemma 5.3.4. *Assume that (HA), (HK) and (HH) are satisfied and fix $\tau \in \mathbb{R}$. Then the spaces defined in §5.2 have the following properties.*

(i) We have the inequalities $\dim B(\tau) \leq \dim \mathcal{B} < \infty$ *and* $\dim B^*(\tau) \leq \dim \mathcal{B}^* < \infty$. *In addition, if* $|r_{\min}| = r_{\max} = \infty$*, then* $\dim B(\tau) = \dim \mathcal{B}$ *and* $\dim B^*(\tau) = \dim \mathcal{B}^*$.

(ii) The inclusions $\widehat{\mathcal{P}}(\tau) \subset \mathcal{P}(\tau)$, $\widehat{\mathcal{Q}}(\tau) \subset \mathcal{Q}(\tau)$, $\widehat{P}(\tau) \subset P(\tau)$ and $\widehat{Q}(\tau) \subset Q(\tau)$ have finite codimension of at most $\dim \mathcal{B}$.

(iii) We have $B(\tau) = P(\tau) \cap Q(\tau)$.

Proof. Items (i) and (ii) are clear from their definition and Proposition 5.2.1. For item (iii) we note that the inclusion $B(\tau) \subset P(\tau) \cap Q(\tau)$ is trivial. Conversely, for $\phi \in P(\tau) \cap Q(\tau)$ we pick $x \in \mathcal{P}(\tau)$ and $y \in \mathcal{Q}(\tau)$ with $\phi = x_\tau = y_\tau$, so that $x = y$ on $D_X + \tau$. This allows us to consider the function z that is defined on the real line by

$$z(t) \quad = \quad \begin{cases} x(t), & t \leq r_{\max} + \tau \\ y(t), & t \geq r_{\min} + \tau. \end{cases} \tag{5.3.5}$$

It is now easy to see that $z \in \mathcal{B}$, which implies $\phi \in B(\tau)$. ∎

Lemma 5.3.5. *Assume that (HA), (HK) and (HH) are satisfied. Then there exists $\mu^- \in (-\infty, \infty]$ such that every $\phi \in P(\tau)$ with $\tau < \mu^-$ has a unique left prolongation in $\mathcal{P}(\tau)$. Similarly, there exists $\mu^+ \in [-\infty, \infty)$ such that every $\phi \in Q(\tau)$ with $\tau > \mu^+$ has a unique right prolongation in $\mathcal{Q}(\tau)$. On the other hand, any element of $\widehat{P}(\tau)$ and $\widehat{Q}(\tau)$ has a unique left respectively right prolongation, this time for any $\tau \in \mathbb{R}$.*

Proof. We only consider the left prolongations. If $r_{\min} = -\infty$, then both results are trivial with $\mu^- = \infty$. If, on the other hand, $r_{\min} > -\infty$, then we can follow the proof of [133, Props. 4.8 and 4.10] to arrive at the desired conclusion. ∎

5.3.2 Exponential decay

Our task here is to furnish a proof for Proposition 5.3.2 and to use this result to establish Theorem 5.2.4. Our approach consists of three main steps: constructing a uniform limit for a sequence that contradicts (5.3.3), showing that this limit satisfies one of the asymptotic systems (5.2.11) and subsequently concluding that this violates the hyperbolicity assumption (HH). The main technical novelties with respect to [133] are contained in the first two steps, where we need to take special care to handle the tail contributions arising from the unbounded shifts.

Lemma 5.3.6. *Consider the setting of Proposition 5.3.2 and let $\{\sigma_n\}_{n\ge 1}$, $\{x_n\}_{n\ge 1}$ and $\{\tau_n\}_{n\ge 1}$ be sequences with the following properties.*

(a) We have $\sigma_n > 0$ for each n, together with $\sigma_n \uparrow \infty$.

(b) We either have $x_n \in \mathcal{P}(\tau_n)$ and $\tau_n \le \tau_-$ for each n or $x_n \in \widehat{\mathcal{P}}(\tau_n)$ and $\tau_n \in \mathbb{R}$ for each n.

(c) For each $n \ge 1$ we have the bound

$$|x_n(-\sigma_n + \tau_n)| \quad \ge \quad \tfrac{1}{2}, \tag{5.3.6}$$

together with the normalization

$$\sup_{s\in(-\infty,\tau_n+p]} |x_n(s)| \quad = \quad 1. \tag{5.3.7}$$

(d) If $r_{\max} = \infty$, then we have the additional bound

$$|x_n(-\sigma_n + \tau_n)| \quad \ge \quad K_{\exp} e^{\alpha(-\sigma_n+\tau_n)} \sup_{s\in[p+\tau_n,\infty)} e^{-\alpha s}|x_n(s)|. \tag{5.3.8}$$

Then upon defining $z_n(t) = x_n(t - \sigma_n + \tau_n)$ and passing to a subsequence, we have $z_n \to z$ uniformly on compact subsets of $\mathbb{R}$. Moreover, we have $z \ne 0$ and $|z| \le 1$ on $\mathbb{R}$.

Proof. We first consider the case $r_{\max} = \infty$ and treat the two possibilities $x_n \in \mathcal{P}(\tau_n)$ and $x_n \in \widehat{\mathcal{P}}(\tau_n)$ simultaneously. In particular, we establish the desired uniform convergence on the compact interval $I_L = [-L, L]$ for some arbitrary $L \ge 1$, which is contained in $(-\sigma_N, \sigma_N)$ for some sufficiently large N.

For $n \ge N$ and $t \in I_L$ we have $|z_n(t)| \le 1$. In addition, upon writing

$$\begin{array}{lcl} \overline{A}_{j,n}(t) & = & A_j(t - \sigma_n + \tau_n)x_n(t - \sigma_n + \tau_n + r_j), \\ \overline{\mathcal{K}}_n(\xi;t) & = & \mathcal{K}(\xi; t - \sigma_n + \tau_n)x_n(t - \sigma_n + \tau_n + \xi), \end{array} \tag{5.3.9}$$

we obtain

$$|\dot{z}_n(t)| \quad = \quad |\dot{x}_n(t - \sigma_n + \tau_n)| \quad \le \quad \sum_{j=-\infty}^{\infty} \left|\overline{A}_{j,n}(t)\right| + \int_{\mathbb{R}} \left|\overline{\mathcal{K}}_n(\xi;t)\right| d\xi. \tag{5.3.10}$$

We now split the sum above over the two sets

$$\begin{array}{rclcl} J_n^-(t) & = & \{j \in \mathbb{Z} \mid r_j \le p + \sigma_n - t\} & \subset & \{j \in \mathbb{Z} \mid r_j \le p\}, \\ J_n^+(t) & = & \{j \in \mathbb{Z} \mid r_j > p + \sigma_n - t\} & \subset & \{j \in \mathbb{Z} \mid r_j \ge -L + \sigma_n + p\}. \end{array} \tag{5.3.11}$$

For $j \in J_n^-(t)$ we have $t - \sigma_n + \tau_n + r_j \le \tau_n + p$, which in view of the normalization (5.3.7) allows us to write

$$\left|\overline{A}_{j,n}(t)\right| \quad \le \quad \|A_j(\cdot)\|_\infty. \tag{5.3.12}$$

On the other hand, for $j \in J_n^+(t)$ we may use (5.3.7)-(5.3.8) to obtain

$$\begin{array}{rcl} \left|\overline{A}_{j,n}(t)\right| & \le & \|A_j(\cdot)\|_\infty K_{\exp}^{-1} e^{\alpha(\sigma_n - \tau_n)} e^{\alpha(t - \sigma_n + \tau_n + r_j)} |x_n(-\sigma_n + \tau_n)| \\ & \le & \|A_j(\cdot)\|_\infty K_{\exp}^{-1} e^{\alpha t} e^{\alpha r_j} \\ & \le & \|A_j(\cdot)\|_\infty K_{\exp}^{-1} e^{\alpha L} e^{\alpha r_j}. \end{array} \tag{5.3.13}$$

In particular, we may use (5.3.1) to estimate

$$\begin{array}{rcl} \sum_{j=-\infty}^{\infty} \left|\overline{A}_{j,n}(t)\right| & \le & \sum_{j \in J_n^-(t)} \|A_j(\cdot)\|_\infty + \sum_{j \in J_n^+(t)} \|A_j(\cdot)\|_\infty K_{\exp}^{-1} e^{\alpha L} e^{\alpha r_j} \\ & \le & \sum_{r_j \le p} \|A_j(\cdot)\|_\infty + e^{-2\alpha|L - \sigma_n - p| + \alpha L} \\ & = & \sum_{r_j \le p} \|A_j(\cdot)\|_\infty + e^{\alpha(3L - 2p - 2\sigma_n)}. \end{array} \tag{5.3.14}$$

In a similar fashion, we obtain the corresponding bound

$$\int_{\mathbb{R}} \left|\overline{\mathcal{K}}_n(\xi; t)\right| d\xi \quad \le \quad \sup_{s \in \mathbb{R}} \int_{-\infty}^{p} |\mathcal{K}(\xi; s)| d\xi + e^{\alpha(3L - 2p - 2\sigma_n)}. \tag{5.3.15}$$

We hence see that both $\{z_n\}_{n \ge N}$ and $\{\dot{z}_n\}_{n \ge N}$ are uniformly bounded on I_L.

Using the Ascoli-Arzela theorem, we can now pass over to some subsequence to obtain the convergence $z_n \to z$ uniformly on compact subsets of $\mathbb{R}$. Moreover, since $z_n(0) \ge \frac{1}{2}$ for each n, we obtain $z(0) \ge \frac{1}{2}$ and thus $z \ne 0$. The bound on $z_n(t)$ obtained above implies that also $|z| \le 1$ on $\mathbb{R}$.

If $r_{\max} < \infty$ then this procedure can be repeated, but now one does not need the second terms in (5.3.14) and (5.3.15). In particular, the argument reduces to the one in [133]. ∎

Lemma 5.3.7. *Consider the setting of Proposition 5.3.2 and Lemma 5.3.6. If the sequence $\{-\sigma_n + \tau_n\}_{n \ge 1}$ is unbounded, then the limiting function z satisfies one of the limiting equations (5.2.11). If, on the other hand, the sequence $\{-\sigma_n + \tau_n\}_{n \ge 1}$ is bounded, then there exists $\beta \in \mathbb{R}$ in such a way that the function $x(t) = z(t - \beta)$ satisfies $x \in \mathcal{B}$.*

Proof. Without loss of generality we assume that $-\sigma_n + \tau_n \to \infty$ if the sequence $\{-\sigma_n + \tau_n\}_{n\geq 1}$ is unbounded or $-\sigma_n + \tau_n \to \beta$ if the sequence $\{-\sigma_n + \tau_n\}_{n\geq 1}$ is bounded. For convenience, we (re)-introduce the expressions

$$\begin{array}{rcl} \overline{A}_{j,n}(s) & = & A_j(s - \sigma_n + \tau_n) z_n(s + r_j), \\ \overline{\mathcal{K}}_n(\xi; s) & = & \mathcal{K}(\xi; s - \sigma_n + \tau_n) z_n(s + \xi) \end{array} \tag{5.3.16}$$

and use the integrated form of (5.2.1) to write

$$\begin{array}{rcl} z(t_2) - z(t_1) & = & \lim_{n\to\infty} z_n(t_2) - z_n(t_1) \\ & = & \lim_{n\to\infty} \int_{t_1}^{t_2} \sum_{j=-\infty}^{\infty} \overline{A}_{j,n}(s) ds + \lim_{n\to\infty} \int_{t_1}^{t_2} \int_{\mathbb{R}} \overline{\mathcal{K}}_n(\xi; s) d\xi ds \\ & := & \mathcal{J}_A + \mathcal{J}_{\mathcal{K}} \end{array} \tag{5.3.17}$$

for an arbitrary pair $t_1 < t_2$ that we fix. Upon introducing the tail expression

$$\mathcal{E}_{A;N} \quad = \quad \lim_{n\to\infty} \int_{t_1}^{t_2} \sum_{|j|=N+1}^{\infty} \overline{A}_{j,n}(s)\, ds \tag{5.3.18}$$

for any $N \geq 0$, we readily observe that

$$\begin{array}{rcl} \mathcal{J}_A & = & \lim_{n\to\infty} \int_{t_1}^{t_2} \sum_{j=-N}^{N} \overline{A}_{j,n}(s) ds + \mathcal{E}_{A;N} \\ & = & \int_{t_1}^{t_2} \sum_{j=-N}^{N} A_j(\infty) z(s + r_j) ds + \mathcal{E}_{A;N} \end{array} \tag{5.3.19}$$

if the sequence $\{-\sigma_n + \tau_n\}_{n\geq 1}$ is unbounded, while

$$\mathcal{J}_A \quad = \quad \int_{t_1}^{t_2} \sum_{j=-N}^{N} A_j(s + \beta) z(s + r_j) ds + \mathcal{E}_{A;N} \tag{5.3.20}$$

if the sequence $\{-\sigma_n + \tau_n\}_{n\geq 1}$ is bounded. Here we evaluated the limit using the convergence $-\sigma_n + \tau_n \to \infty$ or $-\sigma_n + \tau_n \to \beta$. Slightly adapting the estimate (5.3.14) with $L = \max\{|t_1|, |t_2|\}$, we find

$$\begin{array}{rcl} |\mathcal{E}_{A;N}| & \leq & (t_2 - t_1) \sum_{|j|>N} \|A_j(\cdot)\|_\infty + \lim_{n\to\infty} (t_2 - t_1) e^{-2\alpha\sigma_n} e^{\alpha(3L-2p)} \\ & = & (t_2 - t_1) \sum_{|j|>N} \|A_j(\cdot)\|_\infty, \end{array} \tag{5.3.21}$$

which yields $\mathcal{E}_{A;N} \to 0$ as $N \to \infty$. Since $|z| \leq 1$ on $\mathbb{R}$, we can now use the dominated convergence theorem to conclude that

$$\mathcal{J}_A \quad = \quad \int_{t_1}^{t_2} \sum_{j=-\infty}^{\infty} A_j(\infty) z(s + r_j) ds \tag{5.3.22}$$

if the sequence $\{-\sigma_n + \tau_n\}_{n\geq 1}$ is unbounded, while

$$\mathcal{J}_A \quad = \quad \int_{t_1}^{t_2} \sum_{j=-\infty}^{\infty} A_j(s + \beta) z(s + r_j) ds \tag{5.3.23}$$

if the sequence $\{-\sigma_n + \tau_n\}_{n\geq 1}$ is bounded. A similar argument for $\mathcal{J}_{\mathcal{K}}$ hence shows that z is a solution of the limiting system (5.2.11) at $+\infty$. ■

Proof of Proposition 5.3.2. Arguing by contradiction, we assume that (5.3.3) or (5.3.4) fails. We can then construct sequences $\{\sigma_n\}_{n\geq 1}$, $\{x_n\}_{n\geq 1}$ and $\{\tau_n\}_{n\geq 1}$ that satisfy properties (i)-(iv) of Lemma 5.3.6. If the sequence $\{-\sigma_n + \tau_n\}_{n\geq 1}$ is also unbounded, then Lemma 5.3.7 yields that z is a nontrivial, bounded solution of one of the limiting equations (5.2.11), contradicting the hyperbolicity of these systems.

If on the other hand the sequence $\{-\sigma_n + \tau_n\}_{n\geq 1}$ is bounded, we can assume that $-\sigma_n + \tau_n \to \beta$ for some $\beta \in \mathbb{R}$. Since necessarily $\tau_n \to \infty$, this can only happen if $x_n \in \widehat{\mathcal{P}}(\tau_n)$ for each n. Lemma 5.3.7 yields that $x_n \to x$ uniformly on compact subsets of $\mathbb{R}$ and that $0 \neq x \in \mathcal{B}$. On account of Proposition 5.2.1 we find that x decays exponentially. By definition of $\widehat{\mathcal{P}}$ we, therefore, obtain

$$0 \quad = \quad \int_{-\infty}^{\infty} x(t)^{\dagger} x_n(t)dt \quad \to \quad \int_{-\infty}^{\infty} |x(t)|^2 dt, \tag{5.3.24}$$

which yields a contradiction since $x \neq 0$. ■

We now shift our attention to the proof of Theorem 5.2.4. In particular, we set up an iteration scheme to leverage the bound (5.3.3) and show that solutions in $\mathcal{P}(\tau)$ decay exponentially. As a preparation, we provide a uniform bound on the supremum of such solutions.

Lemma 5.3.8. *Assume that (HA), (HK) and (HH) are satisfied. Recall the constant μ^- from Lemma 5.3.5 and fix $\tau^- < \mu^-$. Then there exists $C > 0$ in such a way for each $\tau \leq \tau^-$ and each $x \in \mathcal{P}(\tau)$ we have the bound*

$$\|x\|_{C_b(D_{\tau}^{\ominus})} \quad \leq \quad C\|x_{\tau}\|_{\infty}. \tag{5.3.25}$$

The same bound holds for any $x \in \widehat{\mathcal{P}}(\tau)$, with a possibly different value of C, where now any $\tau \in \mathbb{R}$ is permitted.

Proof. The bound (5.3.25) is in fact an equality with $C = 1$ if $r_{\min} = -\infty$, so we assume that $r_{\min} > -\infty$. If $r_{\max} < \infty$ the final part of the proof of [133, Thm. 4.2] can be repeated, hence we also assume that $r_{\max} = \infty$.

Arguing by contradiction, we consider sequences $\{x_n\}_{n\geq 1}, \{\tau_n\}_{n\geq 1}$ and $\{C_n\}_{n\geq 1}$ with $C_n \to \infty$ and

$$\|x_n\|_{C_b(D_{\tau_n}^{\ominus})} \quad = \quad C_n\|(x_n)_{\tau_n}\|_{\infty} \quad = \quad 1, \tag{5.3.26}$$

with either $x_n \in \mathcal{P}(\tau_n)$ and $\tau_n \leq \tau^-$ for each n or $x_n \in \widehat{\mathcal{P}}(\tau_n)$ and $\tau_n \in \mathbb{R}$.

We want to emphasize that due to the lack of a natural choice for the sequence $\{\sigma_n\}_{n\geq 1}$ which satisfies (a) of Lemma 5.3.6, we cannot immediately apply this result. However, we will follow more or less the same procedure to arrive at a slightly weaker conclusion. Note that the function $z_n(t) = x_n(t + \tau_n)$ is a solution of (5.2.1) on the interval $(-\infty, 0]$ for each value of n. In addition, we note that

$$\sup_{t \in [r_{\min}, \infty)} |z_n(t)| \quad \leq \quad \|(x_n)_{\tau_n}\|_\infty \quad = \quad C_n^{-1}. \tag{5.3.27}$$

We can now follow the proof of Lemma 5.3.6, using (5.3.27) to control the behaviour of z_n on $[0, \infty)$, and pass to a subsequence to obtain $z_n \to z$ uniformly on compact subsets of $(-\infty, 0]$. In addition, (5.3.27) allows us to extend this convergence to all compact subsets of $\mathbb{R}$, with $z_0 = 0$. For each $n \geq 1$ we pick s_n in such a way that $|x_n(-s_n + \tau_n)| = 1$. On account of Proposition 5.3.2 the set $\{s_n\}_{n \geq 1}$ is bounded, which means that z is not identically zero.

Suppose first that the sequence $\{\tau_n\}_{n \geq 1}$ is unbounded. Since each function z_n is a solution of (5.2.1) on $(-\infty, 0]$, we can follow the proof of Lemma 5.3.7 to conclude that z is a bounded solution of one of the limiting equations (5.2.11) on $(-\infty, 0]$. Moreover, since $z_0 = 0$ it follows that z is also a solution of the limiting equation (5.2.11) on $[0, \infty)$. Hence z is a nontrivial, bounded solution on $\mathbb{R}$ of one of the limiting equations (5.2.11), which yields a contradiction.

Suppose now that $\{\tau_n\}_{n \geq 1}$ is in fact a bounded sequence. Then after passing to a subsequence we obtain $\tau_n \to \tau_0$. Following the proof of Lemma 5.3.7, we see that the function $x(t) = z(t - \tau_0)$ is a nontrivial, bounded solution of (5.2.1) on $(-\infty, \tau_0]$. Since $z_0 = 0$, we get that $x_{\tau_0} = 0$ and therefore x is a nontrivial, bounded left prolongation of the zero solution from the starting point τ_0. If $\tau_0 < \mu^-$, this gives an immediate contradiction to Lemma 5.3.5. If on the other hand $\tau_0 \geq \mu^- > \tau^-$, then our assumptions allow us to conclude that $x_n \in \widehat{\mathcal{P}}(\tau_n)$ for all n. A computation similar to (5.3.24) shows that $x \in \widehat{\mathcal{P}}(\tau_0)$, which contradicts Lemma 5.3.5. This establishes (5.3.25). ∎

Lemma 5.3.9. *Assume that (HA), (HK) and (HH) are satisfied. Recall the constant μ^- from Lemma 5.3.5 and fix $\tau^- < \mu^-$. Then there exist constants $\tilde{K} > 0$ and $\tilde{\alpha} > 0$ so that the bound*

$$|x(t)| \quad \leq \quad \tilde{K} e^{\alpha(t-\tau)} \|x\|_{C_b(D_\tau^\ominus)} \tag{5.3.28}$$

holds for all $\tau \leq \tau^-$, all $x \in \mathcal{P}(\tau)$ and all $t \leq \tau + p$.

Proof. The proof of [133, Thm. 4.2] can be used to handle the case $r_{\max} < \infty$, so we assume here that $r_{\max} = \infty$. Pick any $x \in \mathcal{P}(\tau)$, which we normalize to have $\|x\|_{C_b(D_\tau^\ominus)} = 1$. Recalling the constants from Proposition 5.3.2, we assume without loss of generality that

$$K_{\exp} \quad \geq \quad 1, \qquad\qquad K_{\exp} e^{-\alpha(\sigma+p)} \quad \leq \quad \tfrac{1}{4}. \tag{5.3.29}$$

For $t \leq -\sigma + \tau$, this allows us to estimate

$$\begin{array}{lcl} |x(t)| & \leq & \max\Big\{ \frac{1}{2} \sup_{s \in (-\infty, \tau+p]} |x(s)|, K_{\exp} \sup_{s \in [p+\tau, \infty)} e^{-\alpha(s-t)} |x(s)| \Big\} \\ & \leq & \max\big\{ \frac{1}{2}, K_{\exp} e^{-\alpha(p+\tau+\sigma-\tau)} \big\} \\ & = & \frac{1}{2}. \end{array} \tag{5.3.30}$$

We aim to show, by induction, that for each integer $m \geq 0$ we have the bound

$$|x(t)| \leq 2^{-(m+1)}, \qquad t \leq t_m, \tag{5.3.31}$$

where we have introduced

$$t_m := -m(\sigma + p) - \sigma + \tau. \tag{5.3.32}$$

Indeed, if (5.3.31) holds for each $m \in \mathbb{Z}_{\geq 0}$, then we obtain the desired estimate

$$|x(t)| \leq \tilde{K} e^{\tilde{\alpha}(t-\tau)} \tag{5.3.33}$$

for any $t \leq \tau$ with $\tilde{\alpha} = \frac{\ln(2)}{\sigma+p}$ and $\tilde{K} = e^{\tilde{\alpha}(\sigma+p)}$, which concludes the proof.

The case $m = 0$ follows from (5.3.30), so we pick $M \geq 1$ and assume that (5.3.31) holds for each value of $0 \leq m \leq M-1$. Since $x \in \mathcal{P}(\tau)$ and since $\sigma > 0$ and $p > 0$, we must have $x \in \mathcal{P}(t_M + \sigma)$ as well. Fix $t \leq t_M$. Then Proposition 5.3.2 yields the bound

$$|x(t)| \leq \max\Big\{ \tfrac{1}{2} \sup_{s \in (-\infty, t_M + \sigma + p]} |x(s)|, K_{\mathrm{exp}} \sup_{s \in [t_M + \sigma + p, \infty)} e^{-\alpha(s-t)} |x(s)| \Big\}. \tag{5.3.34}$$

Since $t_M + \sigma + p = t_{M-1}$, we may apply (5.3.31) with $m = M-1$ to obtain

$$\tfrac{1}{2} \sup_{s \in (-\infty, t_M + \sigma + p]} |x(s)| \leq \tfrac{1}{2} 2^{-M} = 2^{-(M+1)}. \tag{5.3.35}$$

In addition, we may use (5.3.29) and (5.3.31) to estimate

$$\begin{aligned} K_{\mathrm{exp}} \sup_{s \in [t_m, t_{m-1}]} e^{-\alpha(s-t)} |x(s)| &\leq K_{\mathrm{exp}} e^{-\alpha\left(t_m - t_M\right)} 2^{-m} \\ &= K_{\mathrm{exp}} e^{-\alpha(M-m)(p+\sigma)} 2^{-m} \\ &\leq \left(\tfrac{1}{4}\right)^{M-m} 2^{-m} \\ &\leq 2^{-(M+1)}, \end{aligned} \tag{5.3.36}$$

for $0 \leq m \leq M-1$. Finally, we can estimate

$$\begin{aligned} K_{\mathrm{exp}} \sup_{s \in [\tau - \sigma, \infty)} e^{-\alpha(s-t)} |x(s)| &\leq K_{\mathrm{exp}} e^{-\alpha(\tau - \sigma - t_M)} \\ &= K_{\mathrm{exp}} e^{-\alpha M (p+\sigma)} \\ &\leq 2^{-(M+1)}. \end{aligned} \tag{5.3.37}$$

Combining (5.3.34) with (5.3.35)-(5.3.37) now yields the bound

$$|x(t)| \leq 2^{-(M+1)}, \tag{5.3.38}$$

as desired. ■

Proof of Theorem 5.2.4. We only show the result for the $\mathcal{P}$-spaces; the result for the $\mathcal{Q}$-spaces follows analogously. If $r_{\max} < \infty$, the proof of [133, Thm. 4.2] can be repeated, so we assume that $r_{\max} = \infty$. Pick any $x \in \mathcal{P}(\tau)$. From Lemma 5.3.9 we obtain the bound

$$|x(t)| \quad \leq \quad \tilde{K}e^{\alpha(t-\tau)}\|x\|_{C_b(D_\tau^{\ominus})} \tag{5.3.39}$$

for all $t \leq \tau + p$. Since $x \in \mathcal{P}(\tau)$, we can write

$$\dot{x}(t) \quad = \quad \sum_{j=-\infty}^{\infty} A_j(t)x(t+r_j) + \int_{\mathbb{R}} \mathcal{K}(\xi;t)x(t+\xi)d\xi \tag{5.3.40}$$

for $t \leq \tau$. Lemma 5.3.1 allows us to estimate

$$\begin{aligned} \Big|\sum_{j=-\infty}^{\infty} A_j(t)x(t+r_j)d\xi\Big| \quad &\leq \quad \sum_{t+r_j \leq \tau+p} \|A_j(\cdot)\|_\infty \tilde{K}e^{\alpha(t+r_j-\tau)}\|x\|_\infty \\ &\qquad + \sum_{t+r_j > \tau+p} \|A_j(\cdot)\|_\infty \|x\|_\infty \\ &\leq \quad \sum_{t+r_j \leq \tau} \|A_j(\cdot)\|_\infty e^{\alpha|r_j|}\tilde{K}e^{\alpha(t-\tau)}\|x\|_\infty + K_{\exp}e^{2\alpha(t-\tau)}\|x\|_\infty \\ &\leq \quad \sum_{j=-\infty}^{\infty} \|A_j(\cdot)\|_\infty e^{\alpha|r_j|}\tilde{K}e^{\alpha(t-\tau)}\|x\|_\infty + K_{\exp}e^{2\alpha(t-\tau)}\|x\|_\infty \end{aligned} \tag{5.3.41}$$

for any $t \leq \tau$. Using a similar estimate for the convolution kernel, we obtain the bound

$$\begin{aligned} |\dot{x}(t)| \quad \leq \quad & \tilde{K}e^{\alpha(t-\tau)}\|x\|_\infty \sum_{j=-\infty}^{\infty} \|A_j(\cdot)\|_\infty e^{\alpha|r_j|} + K_{\exp}e^{2\alpha(t-\tau)}\|x\|_\infty \\ & + \tilde{K}e^{\alpha(t-\tau)}\|x\|_\infty \sup_{s\in\mathbb{R}} \|\mathcal{K}(\cdot;s)\|_{\tilde{\eta}} + K_{\exp}e^{2\alpha(t-\tau)}\|x\|_\infty \end{aligned} \tag{5.3.42}$$

for any $t \leq \tau$. Since $r_{\max} = \infty$, we can derive from Lemma 5.3.8 that

$$\|x\|_\infty \quad = \quad \|x\|_{C_b(\mathcal{D}_\tau^{\ominus})} \quad \leq \quad C\|x_\tau\|_\infty. \tag{5.3.43}$$

The bounds (5.3.42)-(5.3.43) together establish the desired result. ■

5.3.3 The restriction operators π^+ and π^-

It is often convenient to split the domain D_X into the two parts

$$D_X^- \quad = \quad \overline{(r_{\min}, 0)}, \qquad\qquad D_X^+ \quad = \quad \overline{(0, r_{\max})} \tag{5.3.44}$$

and study the restriction of functions in X to the spaces

$$X^- \quad = \quad C_b(D_X^-), \qquad\qquad X^+ \quad = \quad C_b(D_X^+). \tag{5.3.45}$$

In particular, we introduce the operators $\pi^+ : X \to X^+$ and $\pi^- : X \to X^-$ that act as

$$(\pi^\pm f)(t) \quad = \quad f(t), \qquad\qquad t \in D_X^\pm. \tag{5.3.46}$$

Moreover, for a subspace $E \subset X$ we let π_E^+ and π_E^- denote the restrictions of π^+ and π^- to E. We obtain some preliminary compactness results below, leaving a more detailed analysis of these operators to §5.4.

Proposition 5.3.10 (cf. [133, Thm. 4.4]). *Assume that (HA), (HK) and (HH) are satisfied. Then for every $\tau \in \mathbb{R}$, the operators $\pi^-_{P(\tau)}$, $\pi^+_{Q(\tau)}$, $\pi^-_{\widehat{P}(\tau)}$ and $\pi^+_{\widehat{Q}(\tau)}$ are all compact.*

Proof. Suppose first that $r_{\min} = -\infty$ and fix $\tau \in \mathbb{R}$. Let $\{\phi_n\}_{n\geq 1}$ be a bounded sequence in $\widehat{P}(\tau)$ and write $\{x_n\}_{n\geq 1}$ for the corresponding sequence in $\widehat{\mathcal{P}}(\tau)$ that has $(x_n)_\tau = \phi_n$ for each $n \geq 1$. After passing to a subsequence, the exponential bound (5.2.38) allows us to obtain the convergence $x_n \to x$ uniformly on compact subsets of $(-\infty, 0]$. For any $\varepsilon > 0$, we can use (5.2.38) to pick $L \gg 1$ in such a way that $|x_n(t)| < \frac{\varepsilon}{2}$ and hence $|x(t)| \leq \frac{\varepsilon}{2}$ holds for all $t \leq -L$. The uniform convergence on $[-L, 0]$ now allows us to pick $N \gg 1$ so that $|x_n(t) - x(t)| \leq \varepsilon$ for all $t \leq 0$ and $n \geq N$. In particular, $\{x_n\}_{n\geq 1}$ converges in X^-, which shows that $\pi^-_{\widehat{P}(\tau)}$ is compact.

The case where $r_{\min} > -\infty$ can be treated as in the proof of [133, Thm. 4.4] and will be omitted. The compactness of $\pi^+_{\widehat{Q}(\tau)}$ follows by symmetry. Finally, the operators $\pi^-_{P(\tau)}$ and $\pi^+_{Q(\tau)}$ are compact since they are finite-dimensional extensions of $\pi^-_{\widehat{P}(\tau)}$ and $\pi^+_{\widehat{Q}(\tau)}$ respectively. ∎

The second part of Corollary 5.3.11 below references the subspaces $P(\pm\infty) \subset X$ and $Q(\pm\infty) \subset X$, being the spaces corresponding the limiting equations (5.2.11) with the decomposition given in (5.2.40). Since the systems (5.2.11) also satisfy the conditions (HA), (HK) and (HH), we can apply the results from the previous sections to the subspaces $P(\pm\infty)$ and $Q(\pm\infty)$.

Corollary 5.3.11 (cf. [133, Cor. 4.11]). *Assume that (HA), (HK) and (HH) are satisfied and let $\{\phi_n\}_{n\geq 1}$ and $\{\psi_n\}_{n\geq 1}$ be bounded sequences, with $\phi_n \in \widehat{P}(\tau_n)$ and $\psi_n \in \widehat{P}(\tau_0)$ for each $n \geq 1$. Suppose furthermore that $\tau_n \to \tau_0$ and that the sequence $\{\pi^+(\phi_n - \psi_n)\}_{n\geq 1}$ converges in X^+. Then after passing to a subsequence, the differences $\{\phi_n - \psi_n\}_{n\geq 1}$ converge to some $\phi \in \widehat{P}(\tau_0)$, uniformly on compact subsets of D_X.*

The conclusion above remains valid after making the replacements

$$\{\widehat{P}(\tau_n), \widehat{P}(\tau_0), \tau_0\} \mapsto \{P(\tau_n), P(-\infty), -\infty\}. \tag{5.3.47}$$

In addition, the analogous results hold for the spaces $\widehat{Q}$ and Q after replacing π^+ by π^- and $-\infty$ by $+\infty$.

Proof. For each $n \geq 1$ we let $y_n \in \widehat{\mathcal{P}}(\tau_n)$ and $z_n \in \widehat{\mathcal{P}}(\tau_0)$ denote the left prolongations of ϕ_n and ψ_n respectively. Moreover, we write $x_n(t) = y_n(t + \tau_n - \tau_0) - z_n(t)$ for $t \leq \tau_0 + r_{\max}$. Then x_n satisfies the inhomogeneous version of (5.2.1) given by

$$\dot{x}_n(t) \quad = \quad \sum_{j=-\infty}^{\infty} A_j(t)x_n(t + r_j) + \int_{\mathbb{R}} \mathcal{K}(\xi; t)x_n(t + \xi)d\xi + h_n(t), \tag{5.3.48}$$

in which h_n is defined by

$$\begin{aligned} h_n(t) \quad = \quad & \sum_{j=-\infty}^{\infty} \big(A_j(t + \tau_n - \tau_0) - A_j(t)\big)y_n(t + r_j + \tau_n - \tau_0) \\ & + \int_{\mathbb{R}} \big(\mathcal{K}(\xi; t + \tau_n - \tau_0) - \mathcal{K}(\xi; t)\big)y_n(t + \xi + \tau_n - \tau_0)d\xi. \end{aligned} \tag{5.3.49}$$

Because x_n satisfies the inhomogeneous equation (5.3.48), since $\sum_{|j|=N}^{\infty} \|A_j\|_\infty \to 0$ as $N \to \infty$, since $\sup_{t\in\mathbb{R}} \|\mathcal{K}(\cdot;t)\|_{\tilde{\eta}} < \infty$, and since both y_n and z_n enjoy the uniform exponential estimates in Theorem 5.2.4, we see that the sequence $\{x_n\}_{n\geq 1}$ is uniformly bounded and equicontinuous. Hence we can apply the Ascoli-Arzela theorem to pass over to a subsequence for which $x_n \to x$ uniformly on compact subsets of $(-\infty, \tau_0]$. Moreover, x is bounded and the convergence $x_n \to x$ is uniform on $D_X^+ + \tau_0$ since $\{\pi^+(\phi_n - \psi_n)\}_{n\geq 1}$ converges in X^+. However, in contrast to [133] we cannot conclude that this convergence is uniform on D_X, since this interval is not necessarily compact.

We see that $h_n \to 0$ in $L^1(I)$ for any bounded interval $I \subset (-\infty, \tau_0]$, again using the limit $\sum_{|j|=N}^{\infty} \|A_j\|_\infty \to 0$ as $N \to \infty$, the bound $\sup_{t\in\mathbb{R}} \|\mathcal{K}(\cdot;t)\|_{\tilde{\eta}} < \infty$ and the fact that the sequence $\{y_n\}_{n\geq 1}$ is bounded uniformly on $D_0^{\ominus}$. Similarly to the proof of Lemma 5.3.7, we obtain that $x : D_{\tau_0}^{\ominus} \to \mathbb{C}^M$ is a bounded solution of (5.2.1) on $(-\infty, \tau_0]$, which yields $x \in \mathcal{P}(\tau_0)$. Finally, for every $w \in \mathcal{B}$ we obtain

$$\begin{array}{rcl} 0 & = & \int_{-\infty}^{\tau_n + r_{\max}} w(t)^\dagger y_n(t)dt - \int_{-\infty}^{\tau_0 + r_{\max}} w(t)^\dagger z_n(t)dt \\ & = & \int_{-\infty}^{\tau_0 + r_{\max}} w(t)^\dagger x_n(t)dt - \int_{-\infty}^{\tau_n + r_{\max}} \big(w(t) - w(t - \tau_n - \tau_0)\big)^\dagger y_n(t)dt \\ & \to & \int_{-\infty}^{\tau_0 + r_{\max}} w(t)^\dagger x(t)dt, \end{array} \tag{5.3.50}$$

since w decays exponentially on account of Proposition 5.2.1. Therefore we must have $x \in \widehat{\mathcal{P}}(\tau_0)$ and thus $\phi := x_{\tau_0} \in \widehat{P}(\tau_0)$.

The result for $P(\tau_n)$ where $\tau_n \to -\infty$ follows a similar proof. We now use the estimate (5.2.38), which is valid for sufficiently small τ. Naturally, the integral computation (5.3.50) is not needed in this proof. The remaining results follow by symmetry. ∎

5.3.4 Fundamental properties of the Hale inner product

We now shift our focus towards the Hale inner product, which plays an important role throughout the remainder of the paper. In particular, we establish the identity (5.2.28), which requires special care on account of the infinite sums. In addition, we study the limiting behaviour of the Hale inner product and establish a uniform estimate that holds for exponentially decaying functions.

Lemma 5.3.12. *Assume that (HA), (HK) and (HH) are satisfied and fix two functions $x, y \in C_b(\mathbb{R})$. Suppose furthermore that x and y are both differentiable at some time $t \in \mathbb{R}$. Then we have the identity*

$$\frac{d}{dt}\langle y^t, x_t\rangle_t = y^\dagger(t)[\Lambda x](t) + [\Lambda^* y](t)^\dagger x(t). \tag{5.3.51}$$

In particular, if $y \in \mathcal{B}^*$ and either $x \in \mathcal{P}(\tau)$ or $x \in \mathcal{Q}(\tau)$ for some $\tau \in \mathbb{R}$, then $\frac{d}{dt}\langle y^t, x_t\rangle_t = 0$ for all $t \leq \tau$ or all $t \geq \tau$ respectively.

Proof. For any $t \in \mathbb{R}$ we can rewrite the Hale inner product in the form

$$\begin{array}{rcl} \langle y^t, x_t \rangle_t & = & y^t(0)^\dagger x_t(0) - \sum\limits_{j=-\infty}^{\infty} \int\limits_0^{r_j} y^t(s - r_j)^\dagger A_j(t + s - r_j) x_t(s) ds \\ & & - \int\limits_{\mathbb{R}} \int\limits_0^{r} y^t(s - r)^\dagger \mathcal{K}(r; t + s - r) x_t(s) ds dr \\ & = & y(t)^\dagger x(t) - \sum\limits_{j=-\infty}^{\infty} \int\limits_t^{t+r_j} y(s - r_j)^\dagger A_j(s - r_j) x(s) ds \\ & & - \int\limits_{\mathbb{R}} \int\limits_t^{t+r} y(s - r)^\dagger \mathcal{K}(r; s - r) x(s) ds dr. \end{array} \tag{5.3.52}$$

We aim to compute the derivative $\frac{d}{dt}\langle y^t, x_t \rangle_t$, so the main difficulty compared to [133] is that we need to interchange a derivative and an infinite sum as well as a derivative and an integral instead of a derivative and a finite sum. Since we can estimate

$$\begin{array}{rcl} \sum\limits_{j=-\infty}^{\infty} \Big| \frac{d}{dt} \int\limits_t^{t+r_j} y(s - r_j)^\dagger A_j(s - r_j) x(s) ds \Big| & = & \sum\limits_{j=-\infty}^{\infty} \Big| y(t)^\dagger A_j(t) x(t + r_j) \\ & & \quad - y(t - r_j)^\dagger A_j(t - r_j) x(t) \Big| \\ & \leq & 2\|x\|_\infty \|y\|_\infty \sum\limits_{j=-\infty}^{\infty} \|A_j(\cdot)\|_\infty, \end{array} \tag{5.3.53}$$

we see that this series converges uniformly. In a similar fashion we can estimate

$$\begin{array}{rcl} \int\limits_{\mathbb{R}} \Big| \frac{d}{dt} \int\limits_t^{t+r} y(s - r)^\dagger \mathcal{K}(r; s - r) x(s) ds \Big| dr & = & \int\limits_{\mathbb{R}} \big| y(t)^\dagger \mathcal{K}(r; t) x(t + r) \\ & & \quad - y(t - r)^\dagger \mathcal{K}(r; t - r) x(t) \big| dr \\ & \leq & \|x\|_\infty \|y\|_\infty \big[\sup\limits_{t \in \mathbb{R}} \|\mathcal{K}(\cdot; t)\|_{\tilde{\eta}} \\ & & \quad + \sup\limits_{t \in \mathbb{R}} \|\mathcal{K}(\cdot; t - \cdot)\|_{\tilde{\eta}} \big]. \end{array} \tag{5.3.54}$$

We can hence freely exchange a time derivative with the integral and sum in (5.3.53) to obtain

$$\begin{array}{rcl} \frac{d}{dt}\langle y^t, x_t \rangle_t & = & \dot{y}(t)^\dagger x(t) + y(t)^\dagger \dot{x}(t) \\ & & - \sum\limits_{j=-\infty}^{\infty} \big[y(t)^\dagger A_j(t) x(t + r_j) - y(t - r_j)^\dagger A_j(t - r_j) x(t) \big] \\ & & - \big[\int\limits_{\mathbb{R}} y(t)^\dagger \mathcal{K}(r; t) x(t + r) dr - \int\limits_{\mathbb{R}} y(t - r)^\dagger \mathcal{K}(r; t - r) x(t) dr \big] \\ & = & y^\dagger(t) [\Lambda x](t) + [\Lambda^* y](t)^\dagger x(t). \end{array} \tag{5.3.55}$$

The final statement follows trivially from (5.3.51). ∎

Lemma 5.3.13. *Assume that (HA), (HK) and (HH) are satisfied and fix two functions $x, y \in C_b(\mathbb{R})$. Suppose furthermore that $y(t)$ decays exponentially as $t \to \infty$. Then we have the limit*

$$\lim_{t \to \infty} \langle y^t, x_t \rangle_t \quad = \quad 0. \tag{5.3.56}$$

The corresponding estimate holds for $t \to -\infty$ if $y(t)$ decays exponentially as $t \to -\infty$.

Proof. Pick $0 < \beta < \tilde{\eta}$ and $K > 0$ in such a way that $|y(t)| \le Ke^{-\beta t}$ for $t \ge 0$. Upon choosing a small $\varepsilon > 0$, we first pick $N \in \mathbb{Z}_{\ge 1}$ in such a way that the bound

$$\sum_{|j|=N+1}^{\infty} |r_j A_j(s-r_j)| \|x\|_\infty \|y\|_\infty + \int_{(-\infty,-N]\cup[N,\infty)} |r\mathcal{K}(r;s-r)| \|x\|_\infty \|y\|_\infty dr \quad \le \quad \tfrac{\varepsilon}{6} \tag{5.3.57}$$

holds for all $s \in \mathbb{R}$. We pick $T > N$ in such a way that also $T > \max\{|r_j| : -N \le j \le N\}$ and that we have the estimates

$$\begin{array}{lcl} |y(t)| \|x\|_\infty & \le & \frac{\varepsilon}{3}, \\ Ke^{-\beta t} \sum_{j=-N}^{N} e^{\beta |r_j|} |r_j A_j(s)| \|x\|_\infty & \le & \frac{\varepsilon}{6}, \\ Ke^{-\beta t} \int_{-N}^{N} e^{\beta|r|} |r\mathcal{K}(r;s-r)| \|x\|_\infty dr & \le & \frac{\varepsilon}{6} \end{array} \tag{5.3.58}$$

for all $t \ge T$ and all $s \in \mathbb{R}$. In particular, we can estimate

$$\begin{array}{rcl} I_1 & := & \sum_{j=-\infty}^{\infty} \Big| \int_t^{t+r_j} y(s-r_j)^\dagger A_j(s-r_j) x(s) ds \Big| \\ & = & \sum_{j=-N}^{N} \Big| \int_t^{t+r_j} y(s-r_j)^\dagger A_j(s-r_j) x(s) ds \Big| \\ & & \quad + \sum_{|j|=N+1}^{\infty} \Big| \int_t^{t+r_j} y(s-r_j)^\dagger A_j(s-r_j) x(s) ds \Big| \\ & \le & \sup_{s\in\mathbb{R}} \sum_{j=-N}^{N} K \max\{e^{-\beta t}, e^{-\beta(t-r_j)}\} |r_j A_j(s-r_j)| \|x\|_\infty \\ & & \quad + \sup_{s\in\mathbb{R}} \sum_{|j|=N+1}^{\infty} |r_j A_j(s-r_j)| \|x\|_\infty \|y\|_\infty \\ & \le & Ke^{-\beta t} \sup_{s\in\mathbb{R}} \sum_{j=-N}^{N} e^{\beta|r_j|} |r_j A_j(s-r_j)| \|x\|_\infty + \frac{\varepsilon}{4} \\ & \le & \frac{\varepsilon}{3} \end{array} \tag{5.3.59}$$

for any $t \ge T$. In a similar fashion, we obtain the estimate

$$\begin{array}{rcl} I_2 & := & \int_{\mathbb{R}} \Big| \int_t^{t+r} y(s-r)^\dagger \mathcal{K}(r;s-r) x(s) ds \Big| dr \\ & = & \int_{-N}^{N} \Big| \int_t^{t+r} y(s-r)^\dagger \mathcal{K}(r;s-r) x(s) ds \Big| dr \\ & & \quad + \int_{(-\infty,-N]\cup[N,\infty)} \Big| \int_t^{t+r} y(s-r)^\dagger \mathcal{K}(r;s-r) x(s) ds \Big| dr \\ & \le & \frac{\varepsilon}{3} \end{array} \tag{5.3.60}$$

for $t \geq T$. The representation (5.3.52) now allows us to estimate

$$\begin{array}{rcl} |\langle y^t, x_t \rangle_t| & \leq & |y(t)||x(t)| + \sum\limits_{j=-\infty}^{\infty} \Big| \int\limits_t^{t+r_j} \big| y(s-r_j)^\dagger A_j(s-r_j)x(s) \big| ds \Big| \\ & & + \int\limits_{\mathbb{R}} \Big| \int\limits_t^{t+r} \big| y(s-r)^\dagger \mathcal{K}(r; s-r)x(s) \big| ds \Big| dr \\ & \leq & |y(t)| \|x\|_\infty + I_1 + I_2 \\ & \leq & \varepsilon, \end{array} \tag{5.3.61}$$

for any $t \geq T$, as desired. ∎

Lemma 5.3.14. *Assume that (HA), (HK) and (HH) and (HKer) are satisfied. Suppose furthermore that $r_{\min} = -\infty$ and consider a pair of constants $(K_0, \alpha_0) \in \mathbb{R}^2_{>0}$. Then there exists a positive constant $B > 0$ so that the estimate*

$$|\langle \psi, y_0 \rangle_0| \quad \leq \quad B \|\psi\|_\infty \|y_\tau\|_\infty e^{-\alpha\tau} \tag{5.3.62}$$

holds for any $\tau \geq 0$, any $\psi \in Y$ and any $y \in C_b(D^{\ominus}_\tau)$ that satisfies the exponential bound

$$|y(t)| \quad \leq \quad K_0 e^{-\alpha_0(\tau - t)} \|y_\tau\|_\infty, \qquad t \leq \tau. \tag{5.3.63}$$

Proof. Recall the constants $(K_{\exp}, \alpha, p) \in \mathbb{R}^3_{>0}$ from Lemma 5.3.1. By lowering α and increasing $K_{\exp}$ if necessary, we may assume that $\alpha \leq \alpha_0$ and $K_{\exp} \geq K_0$. A first crude estimate yields

$$\Big| \sum_{j=-\infty}^{\infty} \int\limits_0^{r_j} x(s-r_j) A_j(t+s-r_j) y(s) ds \Big| \quad \leq \quad \|x\|_\infty \sum_{j=-\infty}^{\infty} \|A_j(\cdot)\|_\infty \Big| \int\limits_0^{r_j} y(s) ds \Big|. \tag{5.3.64}$$

Splitting this sum into two parts and using the decay (5.3.63), we obtain the bound

$$\begin{array}{rcl} \sum\limits_{j=-\infty}^{\infty} \|A_j(\cdot)\|_\infty \Big| \int\limits_0^{r_j} y(s) ds \Big| & = & \sum\limits_{r_j \leq \tau} \|A_j(\cdot)\|_\infty \Big| \int\limits_0^{r_j} y(s) ds \Big| + \sum\limits_{r_j > \tau} \|A_j(\cdot)\|_\infty \int\limits_0^{r_j} |y(s)| ds \\ & \leq & \sum\limits_{r_j \leq \tau} \|A_j(\cdot)\|_\infty K_{\exp} \|y_\tau\|_\infty \Big| \int\limits_0^{r_j} e^{-\alpha(\tau - s)} ds \Big| \\ & & \quad + \sum\limits_{r_j > \tau} \|A_j(\cdot)\|_\infty r_j \|y_0\|_\infty \\ & \leq & \sum\limits_{r_j \leq \tau} \|A_j(\cdot)\|_\infty K_{\exp} \|y_\tau\|_\infty \frac{1}{\alpha} \big| e^{\alpha(r_j - \tau)} - e^{-\alpha\tau} \big| \\ & & \quad + K_{\exp} e^{-2\alpha\tau} \|y_0\|_\infty \\ & \leq & K_{\exp} \|y_\tau\|_\infty \frac{1}{\alpha} e^{-\alpha\tau} \sum\limits_{j=-\infty}^{\infty} \|A_j(\cdot)\|_\infty + K_{\exp} e^{-2\alpha\tau} \|y_\tau\|_\infty, \end{array} \tag{5.3.65}$$

where we used $r_{\min} = -\infty$ to conclude $\|y_0\|_\infty \leq \|y_\tau\|_\infty$. A similar computation for the convolution term yields the desired bound (5.3.62). ∎

5.3.5 Exponential splitting of the state space X

In the remainder of this section, we set out to establish Proposition 5.3.3 and complete the proof of Theorem 5.2.5. In particular, the main technical goal is to establish the identity (5.2.41). We start by considering the inclusion $X^{\perp}(\tau) \subset S(\tau)$, which will follow from Proposition 5.3.3 and the closedness of $S(\tau)$. In particular, we show that $C^1(D_X) \cap X^{\perp}(\tau)$ is contained in $S(\tau)$.

Again, the main complication is that the derivatives of functions x in this subset need not be bounded, which hence also holds for Λx. However, we do know that $\dot{x} - \Lambda x$ is bounded, which allows us to use a technical splitting of x to achieve the desired result. In order to establish the consequences of this splitting, we will need to exploit the fundamental properties of the Hale inner product from §5.3.4.

Lemma 5.3.15. *Assume that (HA), (HK) and (HH) are satisfied. Fix $\tau \in \mathbb{R}$ and pick a differentiable function $x \in C_b(\mathbb{R}) \cap C^1(\mathbb{R})$ with $\phi := x_\tau \in X^{\perp}(\tau)$. Recall the operator Λ from (5.2.15), write $h = \Lambda x$ and consider the functions h_- and h_+ given by*

$$h_-(t) \quad = \quad \begin{cases} h(t), & t \le \tau, \\ 0, & t > \tau, \end{cases} \qquad h_+(t) \quad = \quad \begin{cases} 0, & t \le \tau, \\ h(t), & t > \tau. \end{cases} \tag{5.3.66}$$

Then there exists a decomposition $x = x_- + x_+$ with $x_-, x_+ \in C_b(\mathbb{R}) \cap C^1(\mathbb{R})$ for which we have the inclusions

$$h_- - \Lambda x_- \quad \in \quad \mathrm{Range}(\Lambda), \qquad h_+ - \Lambda x_+ \quad \in \quad \mathrm{Range}(\Lambda). \tag{5.3.67}$$

Proof. We choose the decomposition $x_- + x_+ = x$ with $x_\pm \in C_b(\mathbb{R}) \cap C^1(\mathbb{R})$ in such a way that $x_- = 0$ on $[\tau + 1, \infty)$, while $x_+ = 0$ on $(-\infty, \tau - 1]$. Although the derivative of x_- need not be bounded on $(-\infty, \tau]$, while the derivative of x_+ need not be bounded on $[\tau, \infty)$, we do claim that

$$h_- - \Lambda x_- \quad \in \quad L^\infty(\mathbb{R}), \qquad h_+ - \Lambda x_+ \quad \in \quad L^\infty(\mathbb{R}). \tag{5.3.68}$$

To see this, we note that by construction Λx^- is bounded on $[\tau, \infty)$, while Λx^+ is bounded on $(-\infty, \tau]$. In particular, $h_+ - \Lambda x^+$ is automatically bounded on $(-\infty, \tau]$. On the other hand, for $t \ge \tau$ we may compute

$$h_+(t) - [\Lambda x_+](t) \quad = \quad h(t) - [\Lambda x](t) + [\Lambda x_-](t) \quad = \quad [\Lambda x_-](t), \tag{5.3.69}$$

which shows that $h_+ - \Lambda x_+$ is also bounded on $[\tau, \infty)$. The claim for x_- follows by symmetry.

We now set out to show that $h_+ - \Lambda x_+ \in \mathrm{Range}(\Lambda)$ by exploiting the characterization (5.2.19). In particular, pick any $u \in \mathcal{B}^*$ and consider the integral

$$\begin{aligned} \mathcal{I} \quad &:= \quad \int_{-\infty}^{\infty} u(t)^\dagger \big[h_+(t) - (\Lambda x_+)(t)\big] dt \\ &= \quad \int_{\tau}^{\infty} u(t)^\dagger \big[h_+(t) - (\Lambda x_+)(t)\big] dt + \int_{-\infty}^{\tau} u(t)^\dagger \big[h_+(t) - (\Lambda x_+)(t)\big] dt \\ &=: \quad \mathcal{I}^+ + \mathcal{I}^-. \end{aligned} \tag{5.3.70}$$

Exploiting (5.3.69) we obtain

$$\begin{array}{lcl}\mathcal{I}^+ & = & \int_\tau^\infty u(t)^\dagger\big[\dot{x}_-(t) - \sum_{j=-\infty}^{\infty} A_j(t)x_-(t+r_j) - \int_{\mathbb{R}} \mathcal{K}(\xi;t)x_-(t+\xi)d\xi\big]\,dt \\ & = & \int_\tau^\infty u(t)^\dagger[\Lambda x_-](t)dt.\end{array} \tag{5.3.71}$$

Since $\Lambda^* u = 0$ we can immediately exploit the fundamental property of the Hale inner product from Lemma 5.3.12 to obtain

$$\begin{array}{lcl}\mathcal{I}^+ & = & \int_\tau^\infty u(t)^\dagger[\Lambda x_-](t)dt \\ & = & \int_\tau^\infty \Big(u(t)^\dagger[\Lambda x_-](t) + [\Lambda^* u](t)x_-(t)\Big)\,dt \\ & = & \int_\tau^\infty \frac{d}{dt}\langle u^t, (x_-)_t\rangle_t\,dt \\ & = & \lim_{t\to\infty}\langle u^t, (x_-)_t\rangle_t - \langle u^\tau, (x_-)_\tau\rangle_\tau \\ & = & -\langle u^\tau, (x_-)_\tau\rangle_\tau.\end{array} \tag{5.3.72}$$

The final equality follows in consideration of Lemma 5.3.13, since the function $u \in \mathcal{B}^*$ decays exponentially on account of Proposition 5.2.1. In a similar fashion, we obtain

$$\mathcal{I}^- \quad = \quad -\langle u^\tau, (x_+)_\tau\rangle_\tau. \tag{5.3.73}$$

As such, we can use $\phi \in X^\perp(\tau)$ to compute

$$\mathcal{I} \;=\; \mathcal{I}^+ + \mathcal{I}^- \;=\; -\langle u^\tau, x_\tau\rangle_\tau \;=\; -\langle u^\tau, \phi\rangle_\tau \;=\; 0. \tag{5.3.74}$$

The identity (5.2.19) now yields the desired conclusion. ■

Proof of Proposition 5.3.3. Inspecting the definition of the Hale inner product (5.2.26), we readily see that the map $\phi \mapsto \langle\psi,\phi\rangle_\tau$ is continuous for any $\tau \in \mathbb{R}$ and any $\psi \in B^*(\tau)$. In particular, the space $X^\perp(\tau)$ is closed and has finite codimension in X. We now write

$$\mathcal{E} \quad = \quad C^1(D_X) \cap X^\perp(\tau) \tag{5.3.75}$$

and note that $\mathcal{E}$ is indeed dense in $X^\perp(\tau)$ by [133, Lem. 4.14]. Pick any $\phi \in \mathcal{E}$ and extend it arbitrarily to a bounded C^1 function $x : \mathbb{R} \to \mathbb{C}^M$ that has $x_\tau = \phi$. Recalling the functions $h_\pm$ and $x_\pm$ from Lemma 5.3.15, we use this result to find a function $\tilde{y} \in W^{1,\infty}$ that has $\Lambda\tilde{y} = h_+ - \Lambda x_+$. Writing $y = \tilde{y} + x_+ \in C_b(\mathbb{R})$, we see that

$$[\Lambda y](t) \quad = \quad h_+(t) \tag{5.3.76}$$

which vanishes for $t \leq \tau$. In particular, we have $y \in \mathcal{P}(\tau)$. In a similar fashion, we can find a function $z \in \mathcal{Q}(\tau)$ with $\Lambda z = h_-$.

Writing $w = x - y - z \in C_b(\mathbb{R})$, we readily compute

$$\Lambda w \quad = \quad h - h_+ - h_- \quad = \quad 0, \tag{5.3.77}$$

which implies $w \in \mathcal{B}$ and hence

$$\phi \quad = \quad x_\tau \quad = \quad y_\tau + z_\tau + w_\tau \in P(\tau) + Q(\tau) + B(\tau) \quad = \quad S(\tau), \tag{5.3.78}$$

as desired. ■

We now turn to the remaining inclusion $S(\tau) \subset X^\perp(\tau)$. As before, we exploit the fundamental identity (5.2.28). Combined with the exponential decay of functions in $P(\tau)$ and $Q(\tau)$, this will allow us to show that both spaces are contained in $X^\perp(\tau)$.

Lemma 5.3.16. *Assume that (HA), (HK) and (HH) are satisfied. Then for each $\tau \in \mathbb{R}$ we have the inclusion $S(\tau) \subset X^\perp(\tau)$.*

Proof. By symmetry, it suffices to show that $P(\tau) \subset X^\perp(\tau)$. To this end, we pick $x \in \mathcal{P}(\tau)$ and $y \in \mathcal{B}^*$ and note that

$$\tfrac{d}{dt}\langle y^t, x_t \rangle_t \quad = \quad 0 \tag{5.3.79}$$

for all $t \leq \tau$ by Lemma 5.3.12. Since $x(t)$ is bounded as $t \to -\infty$ while $y(t) \to 0$ at an exponential rate, we may use Lemma 5.3.13 to obtain

$$\langle y^\tau, x_\tau \rangle_\tau \quad = \quad \lim_{t \to -\infty} \langle y^t, x_t \rangle_t \quad = \quad 0, \tag{5.3.80}$$

as desired. ■

The remainder of the proof of Theorem 5.2.5 uses arguments that are very similar to those in [133]. The main point is that the compactness properties obtained in §5.3.3 allow us to show that $\widehat{P}(\tau)$ and $\widehat{Q}(\tau)$ are closed, which allows the computations above to be leveraged.

Lemma 5.3.17. *Assume that (HA), (HK) and (HH) are satisfied. Then for each $\tau \in \mathbb{R}$, the spaces $P(\tau)$, $Q(\tau)$, $\widehat{P}(\tau)$ and $\widehat{Q}(\tau)$ are all closed subspaces of X.*

Proof. Let $\{\phi_n\}_{n \geq 1}$ be a sequence in $\widehat{P}(\tau)$ that converges in X to some $\phi \in X$. Picking $\tau_n = \tau$ and $\psi_n = 0$ in Corollary 5.3.11 then immediately implies that $\{\phi_n\}_{n \geq 1}$ converges uniformly on compact sets to some $\hat{\phi} \in \widehat{P}(\tau)$. By necessity we hence have $\phi = \hat{\phi}$, which means that $\widehat{P}(\tau)$ and by symmetry $\widehat{Q}(\tau)$ are both closed. This subsequently must also hold for the finite dimensional extensions $P(\tau)$ and $Q(\tau)$. ■

Lemma 5.3.18. *Assume that (HA), (HK) and (HH) are satisfied. Then for each $\tau \in \mathbb{R}$ the spaces $S(\tau)$ and $\widehat{S}(\tau)$ are closed subspaces of X. Moreover, the decompositions* (5.2.40) *hold.*

Proof. In view of Proposition 5.3.10 and Lemma 5.3.17, the result can be obtained by following the proof of [133, Prop. 4.12 & Prop. 4.13], together with the first part of the proof of [133, Thm. 4.3]. ■

Proof of Theorem 5.2.5. Every statement except the identity (5.2.41) follows from Lemma 5.3.17 and Lemma 5.3.18. In addition, Lemma 5.3.16 yields the inclusion $S(\tau) \subset X^{\perp}(\tau)$, while Proposition 5.3.3 yields the inclusion $D \subset S(\tau)$ for some dense set $D \subset X^{\perp}(\tau)$. Since $S(\tau)$ is closed, we immediately obtain (5.2.41). ∎

5.4 Fredholm properties of the projections $\Pi_{\widehat{P}}$ and $\Pi_{\widehat{Q}}$

The goal of this section is to understand the projection operators $\Pi_{\widehat{P}}$ and $\Pi_{\widehat{Q}}$ associated to the decomposition (5.2.42). In contrast to the previous section, we can follow the approach from [133] relatively smoothly here. The main difficulty is that the arguments in [133] often use Corollary 5.3.11 to conclude that certain subsequences converge uniformly, while we can only conclude that this convergence takes place on compact subsets. The primary way in which we circumvent this issue is by appealing to the exponential estimates in Theorem 5.2.4.

As a bonus, we also obtain information on the Fredholm properties of the restriction operators $\pi^{\pm}$ introduced in §5.3.3. In particular, besides proving Theorem 5.2.6, we also establish the following two results.

Proposition 5.4.1 (cf. [133, Thm. 4.5]). *Assume that (HA), (HK) and (HH) are satisfied. Then the operators $\pi^+_{P(\tau)}$, $\pi^-_{Q(\tau)}$, $\pi^+_{\widehat{P}(\tau)}$ and $\pi^-_{\widehat{Q}(\tau)}$ are all Fredholm for every $\tau \in \mathbb{R}$. Recalling the function $\beta(\tau)$ defined in (5.2.30), the Fredholm indices satisfy the identities*

$$\begin{aligned} \operatorname{ind}(\pi^+_{P(\tau)}) + \operatorname{ind}(\pi^-_{Q(\tau)}) &= -M + \dim B(\tau) - \beta(\tau), \\ \operatorname{ind}(\pi^+_{\widehat{P}(\tau)}) + \operatorname{ind}(\pi^-_{\widehat{Q}(\tau)}) &= -\big(M + \dim B(\tau) + \beta(\tau)\big). \end{aligned} \tag{5.4.1}$$

Proposition 5.4.2 (cf. [133, Thm. 4.6]). *Assume that (HA), (HK) and (HH) are satisfied. Fix $\tau_0 \in \mathbb{R}$ and consider the projections $\Pi_{\widehat{P}}$ and $\Pi_{\widehat{Q}}$ associated to the decomposition* (5.2.42). *Then we have the identities*

$$\begin{aligned} \operatorname{ind}(\pi^+_{\widehat{P}(\tau)}) &= \operatorname{ind}(\pi^+_{\widehat{P}(\tau_0)}) - \operatorname{codim}_{\widehat{P}(\tau_0)} \Pi_{\widehat{P}}\big(\widehat{P}(\tau)\big), \\ \operatorname{ind}(\pi^-_{\widehat{Q}(\tau)}) &= \operatorname{ind}(\pi^-_{\widehat{Q}(\tau_0)}) - \operatorname{codim}_{\widehat{Q}(\tau_0)} \Pi_{\widehat{Q}}\big(\widehat{Q}(\tau)\big). \end{aligned} \tag{5.4.2}$$

Moreover, the quantities $\operatorname{ind}(\pi^+_{\widehat{P}(\tau)})$ and $\operatorname{ind}(\pi^-_{\widehat{Q}(\tau)})$ vary upper semicontinuously with τ. In addition, we have the identities

$$\begin{aligned} \operatorname{ind}(\pi^+_{\widehat{P}(\tau)}) + \dim B(\tau) &= \operatorname{ind}(\pi^+_{P(\tau)}) &= \operatorname{ind}(\pi^+_{P(-\infty)}), \\ \operatorname{ind}(\pi^-_{\widehat{Q}(\tau)}) + \dim B(\tau) &= \operatorname{ind}(\pi^-_{Q(\tau)}) &= \operatorname{ind}(\pi^-_{Q(\infty)}), \end{aligned} \tag{5.4.3}$$

for sufficiently negative values of τ in the first line of (5.4.3) and for sufficiently positive values of τ in the second line of (5.4.3).

We first need to study the projection operators π^+ and π^- from (5.3.46) in more detail. We proceed largely along the lines of [133], taking a small detour in order to establish that the ranges are closed.

Lemma 5.4.3. *Assume that (HA), (HK) and (HH) are satisfied. Then the operators $\pi^+_{\widehat{P}(\tau)}$ and $\pi^-_{\widehat{Q}(\tau)}$ have finite dimensional kernels for each $\tau \in \mathbb{R}$.*

Proof. This can be established by repeating the first half of the proof of [133, Lem. 3.8]. ∎

Lemma 5.4.4 (cf. [133, Lem. 3.8])**.** *Assume that (HA), (HK) and (HH) are satisfied. Then the operators $\pi^+_{\widehat{P}(\tau)}$ and $\pi^-_{\widehat{Q}(\tau)}$ have closed ranges for each $\tau \in \mathbb{R}$.*

Proof. By symmetry, we pick $\tau \in \mathbb{R}$ and restrict attention to the operator $\pi^+_{\widehat{P}(\tau)}$. We fix a closed complement $C \subset \widehat{P}(\tau)$ for the finite dimensional space $\ker(\pi^+_{\widehat{P}(\tau)})$, so that $\widehat{P}(\tau) = \ker(\pi^+_{\widehat{P}(\tau)}) \oplus C$. We now consider a sequence $\{\phi_n\}_{n\geq 1} \subset C$ and suppose that the restrictions $\psi_n = \pi^+_{\widehat{P}(\tau)}\phi_n$ satisfy the uniform convergence $\psi_n \to \psi$ on D^+_X. If the sequence $\{\phi_n\}_{n\geq 1}$ is bounded, then an application of Corollary 5.3.11 immediately yields that $\phi_n \to \phi \in \widehat{P}(\tau)$ uniformly on compacta, after passing to a subsequence. This implies that $\psi = \pi^+_{\widehat{P}(\tau)}\phi$ and thus $\psi \in \text{Range}(\pi^+_{\widehat{P}(\tau)})$, as desired.

Let us assume therefore that $\|\phi_n\|_\infty \uparrow \infty$ and consider the rescaled sequence $\tilde{\phi}_n = \|\phi_n\|_\infty^{-1}\phi_n$, which satisfies

$$\pi^+_{\widehat{P}(\tau)}\tilde{\phi}_n \quad = \quad \|\phi_n\|_\infty^{-1}\psi_n \to 0 \tag{5.4.4}$$

uniformly on D^+_X. We may again apply Corollary 5.3.11 to obtain $\tilde{\phi}_n \to \tilde{\phi} \in \widehat{P}(\tau)$ uniformly on compacta, with $\pi^+\tilde{\phi} = 0$. In contrast to the setting of [133, Lem. 3.8], this convergence is not immediately uniform on the (possibly unbounded) interval D^-_X. On account of Proposition 5.3.10, the operator $\pi^-_{\widehat{P}(\tau)}$ is compact, so we can pass to yet another subsequence to obtain the limit $\tilde{\phi}_n \to \tilde{\phi}$ uniformly on D^-_X. As such, $\tilde{\phi}_n \to \tilde{\phi}$ uniformly both on D^-_X and on D^+_X, so the convergence is uniform on D_X. Moreover, $\pi^+_{\widehat{P}(\tau)}\tilde{\phi} = 0$, so $\tilde{\phi} \in \ker(\pi^+_{\widehat{P}(\tau)})$. Since the convergence $\tilde{\phi}_n \to \tilde{\phi}$ is uniform on D_X we get $\|\tilde{\phi}\|_\infty = 1$, as $\|\tilde{\phi}_n\|_\infty = 1$ for each n. However, C is closed and $\tilde{\phi}_n \in C$ for each n, so $\tilde{\phi} \in C$. Therefore, $\tilde{\phi}$ is a nontrivial element of $\ker(\pi^+_{\widehat{P}(\tau)}) \cap C$, which yields a contradiction. ∎

Proof of Proposition 5.4.1. The proof is identical to that of [133, Prop. 4.12] and, as such, will be omitted. It uses Theorem 5.2.5, together with Lemmas 5.4.3 and 5.4.4. ∎

Lemma 5.4.5. *Assume that (HA), (HK) and (HH) are satisfied. Fix $\tau_0 \in \mathbb{R}$ and consider the projections $\Pi_{\widehat{P}}$ and $\Pi_{\widehat{Q}}$ associated to the decomposition* (5.2.42)*. Then for*

τ sufficiently close to τ_0, the restrictions

$$\begin{aligned} \Pi_{\widehat{P}} : \widehat{P}(\tau) &\to \Pi_{\widehat{P}}\big(\widehat{P}(\tau)\big) \subset \widehat{P}(\tau_0), \\ \Pi_{\widehat{Q}} : \widehat{Q}(\tau) &\to \Pi_{\widehat{Q}}\big(\widehat{Q}(\tau)\big) \subset \widehat{Q}(\tau_0) \end{aligned} \tag{5.4.5}$$

to the subspaces $\widehat{P}(\tau)$ and $\widehat{Q}(\tau)$ are isomorphisms onto their ranges, which are closed. Moreover, we have the limits

$$\lim_{\tau \to \tau_0} \|I - \Pi_{\widehat{P}}|_{\widehat{P}(\tau)}\| = 0, \qquad \lim_{\tau \to \tau_0} \|I - \Pi_{\widehat{Q}}|_{\widehat{Q}(\tau)}\| = 0, \tag{5.4.6}$$

in which I denotes the inclusion of $\widehat{P}(\tau)$ or $\widehat{Q}(\tau)$ into X.

Proof. By symmetry, we only have to consider the projection $\Pi_{\widehat{P}}$. In order to establish the limit (5.4.6), we pick an arbitrary bounded sequence $\{\phi_n\}_{n\geq 1}$ that has $\phi_n \in \widehat{P}(\tau_n)$ and $\tau_n \to \tau_0$. Using the decomposition (5.2.42), we write

$$\phi_n = \rho_n + \psi_n + \sigma_n, \tag{5.4.7}$$

with $\rho_n \in \widehat{P}(\tau_0)$, $\psi_n \in \widehat{Q}(\tau_0)$ and $\sigma_n \in \Gamma$ for each n. Then each of the sequences $\{\rho_n\}_{n\geq 1}$, $\{\psi_n\}_{n\geq 1}$ and $\{\sigma_n\}_{n\geq 1}$ is bounded. It is sufficient to show that $\phi_n - \rho_n \to 0$ for some subsequence. Note that this also establishes the claim that the restriction in (5.4.5) is an isomorphism with closed range.

By Proposition 5.3.10 and the finite dimensionality of Γ, we can pass over to a subsequence for which both $\{\pi^+_{\widehat{Q}(\tau_0)}\psi_n\}_{n\geq 1}$ and $\{\sigma_n\}_{n\geq 1}$ converge. As such, $\{\pi^+(\phi_n - \rho_n)\}_{n\geq 1}$ converges, so Corollary 5.3.11 implies that $\phi_n - \rho_n \to \phi \in \widehat{P}(\tau_0)$ uniformly on compact subsets of D_X after passing to a further subsequence. In particular, we obtain the convergence $\psi_n + \sigma_n \to \phi$, uniformly on D_X^+ and uniformly on compact subsets of D_X^-.

If $r_{\min} \neq -\infty$ then the convergence $\psi_n + \sigma_n \to \phi$ is in fact uniform on D_X, allowing us to follow the approach in [133]. In particular, we obtain $\phi \in \big(\widehat{Q}_{\tau_0} \oplus \Gamma\big) \cap \widehat{P}_{\tau_0}$ and hence $\phi = 0$ as desired. Assuming therefore that $r_{\min} = -\infty$, we use the convergence of $\{\sigma_n\}_{n\geq 1}$ to conclude that $\psi_n \to \psi$ uniformly on D_X^+ and uniformly on compact subsets of $(-\infty, 0]$. For any $n \geq 1$ we write $y_n \in \widehat{\mathcal{Q}}(\tau_0)$ for the right-extension of ψ_n, i.e., $\psi_n = (y_n)_{\tau_0}$. Using the uniform estimates in Theorem 5.2.4 for large positive t, we can use the Ascoli-Arzela theorem to pass to a subsequence that has $y_n \to y$, uniformly on compact subsets of $\mathbb{R}$. Necessarily we have

$$y_{\tau_0}(t) = \psi(t), \qquad t \in \overline{(-1, r_{\max})}. \tag{5.4.8}$$

Since $\psi_n \to \psi$ uniformly on D_X^+, it follows that $y_n \to y$ uniformly on $\tau_0 + D_X^+$. We can hence follow the proof of Lemma 5.3.7 to see that y is a solution of (5.2.1) on $[\tau_0, \infty)$ and therefore $\psi \in Q(\tau_0)$. Similarly to the proof of Corollary 5.3.11 we even get $\psi \in \widehat{Q}(\tau_0)$. This yields $\phi \in \big(\widehat{Q}(\tau_0) \oplus \Gamma\big) \cap \widehat{P}(\tau_0)$ and therefore $\phi = 0$. ■

Proof of Theorem 5.2.6. The first statement and (5.2.45) follow from Lemma 5.4.5, while the lower semicontinuity of $\dim B(\tau)$ and $\beta(\tau)$ and the limit in (5.2.47) can be established in a fashion similar to the proof of [133, Thm. 4.6].

It remains to show that (5.2.46) holds. Following [133], it suffices to find a bounded function $y : \mathbb{R} \to \mathbb{C}^M$ that satisfies the inhomogeneous system

$$\dot{y}(t) = \sum_{j=-\infty}^{\infty} A_j^{(\tau)}(t) y(t + r_j) + \int_{\mathbb{R}} \mathcal{K}^{(\tau)}(\xi; t) y(t + \xi) d\xi + h^{(\tau)}(t), \tag{5.4.9}$$

in which we have introduced the coefficients

$$A_j^{(\tau)}(t) = \begin{cases} A_j(t+\tau), & t < 0, \\ A_j(-\infty), & t \geq 0, \end{cases} \qquad \mathcal{K}^{(\tau)}(\xi; t) = \begin{cases} \mathcal{K}(\xi; t+\tau), & t < 0, \\ \mathcal{K}(\xi; -\infty), & t \geq 0, \end{cases} \tag{5.4.10}$$

together with the inhomogeneity

$$h^{(\tau)}(t) = \sum_{j=-\infty}^{\infty} \big(A_j^{\tau}(t) - A_j(-\infty)\big) x(t + r_j) + \int_{\mathbb{R}} \big(\mathcal{K}^{\tau}(\xi; t) - \mathcal{K}(\xi; -\infty)\big) x(t + \xi) d\xi. \tag{5.4.11}$$

This can be achieved by following the same steps as in [133], but now using the proof of [68, Lem. 3.1 (step 3)][3] instead of the results in [130]. ∎

Proof of Proposition 5.4.2. The proof is identical to that of [133, Thm. 4.6] and, as such, will be omitted. It uses Theorems 5.2.5 and 5.2.6. ∎

5.5 Exponential dichotomies on half-lines

In this section, we adapt the approach of [104] to obtain exponential splittings for (5.2.1) on the half-line $[0, \infty)$. The main idea is to explicitly construct suitable finite-dimensional enlargements of $\mathcal{P}(\tau)$ for $\tau \geq 0$. The extra functions $\{y_{(\tau)}\}_{\tau \geq 0}$ satisfy (5.2.1) on $[0, \tau]$, but not on $(-\infty, \tau]$. In fact, we will exploit the fundamental identity (5.2.28) to guarantee that the segments $\{(y_{(\tau)})_\tau\}$ are not contained in $S(\tau)$.

In order to achieve this, we need to construct inverses for the Fredholm operator Λ restricted to half-lines. In the ODE case one can write down explicit variation-of-constants formula's to achieve this, but such constructions are problematic at best in the current setting. Instead, we follow [104] and solve $\Lambda x = h$ by appropriately modifying h *outside* the half-line of interest in order to satisfy $\langle y, h \rangle_{L^2} = 0$ for all $y \in \mathcal{B}^*$. In order to ensure that such a modification is not precluded by degeneracy issues, we need to assume that (HKer) holds.

[3] The matrices A_j^{τ} and $\mathcal{K}^{\tau}$ need not be continuous, while in [68] the coefficients are assumed to be continuous. However, the continuity is not used in the parts of the proof that are relevant for us.

The main complication in the setting $r_{\min} = -\infty$ is that this modification of h is visible directly in the equation satisfied by x, rather than only indirectly via the Fredholm properties of Λ as in [104]. This raises issues when using a standard bootstrapping procedure to obtain estimates on $\dot{x}$. Naturally, the unbounded shifts also cause technical problems similar to those encountered in §5.3-5.4, but fortunately the same tricks also work here.

Remark 5.5.1. In fact, in this section it suffices to assume a weaker version of the nontriviality condition (HKer). In particular, we do not need the condition that each nonzero $d \in \mathcal{B}$ vanishes on the intervals $(-\infty, \tau]$ for $\tau < 0$ or $[\tau, \infty)$ for $\tau > 0$. This is because the formulation of Theorem 5.2.8 references the specific half-line $\mathbb{R}^+$, rather than arbitrary half-lines.

5.5.1 Strategy

In order to ensure that the spaces we construct are invariant with respect to τ, we need to slightly modify the τ-dependent normalization condition used in (5.2.35). Indeed, upon writing

$$\begin{array}{rcl} \tilde{\mathcal{P}}(\tau) & = & \{x \in \mathcal{P}(\tau) \mid \int_{-\infty}^{\min(\tau + r_{\max}, 0)} y(t)^\dagger x(t) dt = 0 \text{ for every } y \in \mathcal{B}\}, \\ \tilde{\mathcal{Q}}(\tau) & = & \{x \in \mathcal{Q}(\tau) \mid \int_{\infty}^{\max(\tau + r_{\min}, 0)} y(t)^\dagger x(t) dt = 0 \text{ for every } y \in \mathcal{B}\}, \\ \tilde{P}(\tau) & = & \{\phi \in X \mid \phi = x_\tau \text{ for some } x \in \tilde{\mathcal{P}}(\tau)\}, \\ \tilde{Q}(\tau) & = & \{\phi \in X \mid \phi = x_\tau \text{ for some } x \in \tilde{\mathcal{Q}}(\tau)\}, \end{array} \tag{5.5.1}$$

we see that the upper bounds for the defining integrals are now constant for $\tau \geq 0$ and $\tau \leq 0$ respectively. In view of the nontriviality assumption (HKer), all the conclusions from the previous sections remain valid for these new spaces. In particular, we have the following result.

Corollary 5.5.2 (cf. [104, Prop. 4.2])**.** *Assume that (HA), (HK), (HH) and (HKer) are satisfied. Recall the spaces $S(\tau)$ from Theorem 5.2.5. Then we have the direct sum decomposition*

$$S(\tau) \quad = \quad \tilde{P}(\tau) \oplus \tilde{Q}(\tau) \oplus B(\tau) \tag{5.5.2}$$

for any $\tau \in \mathbb{R}$.

Our first goal is to find an explicit complement for the space $S(\tau)$ in X. In view of the identification $S(\tau) = X^\perp(\tau)$, we actually build a duality basis for $B^*(\tau)$ with respect to the Hale inner product; see (5.5.3).

Proposition 5.5.3 (cf. [104, Lem. 4.3])**.** *Assume that (HA), (HK), (HH) and (HKer) are satisfied. Write $n_d = \dim(\mathcal{B}^*)$ and choose a basis $\{d^i : 1 \leq i \leq n_d\}$ for $\mathcal{B}^*$. Then there exists a constant $r_0 > 0$, together with a family of functions $y^i_{(\tau)} \in C_b(D^{\ominus}_\tau)$, defined for every $\tau \geq 0$ and every integer $1 \leq i \leq n_d$, that satisfies the following properties.*

(i) For any $\tau \geq 0$ and any integer $1 \leq i \leq n_d$ we have $\big[\Lambda y^i_{(\tau)}\big](t) = 0$ for every $t \in (\infty, -r_0] \cup [0, \tau]$.

(ii) For any $0 \le t \le \tau$ and any pair $1 \le i, j \le n_d$ we have the identity

$$\langle (d^i)^t, (y^j_{(\tau)})_t \rangle_t \quad = \quad \delta_{ij}. \tag{5.5.3}$$

(iii) For any fixed constant $t \ge 0$ and fixed integer $1 \le i \le n_d$, the map $\tau \mapsto (y^i_{(\tau)})_t$ is continuous from the interval $[t, \infty)$ into the state space X.

(iv) For any triplet $0 \le t \le \tau_1 \le \tau_2$ and any integer $1 \le i \le n_d$, we have the inclusion

$$\big[y^i_{(\tau_1)} - y^i_{(\tau_2)}\big]_t \in \tilde{P}(t). \tag{5.5.4}$$

(v) For any $\tau \ge 0$ and any integer $1 \le i \le n_d$, the integral condition

$$\textstyle\int_{-\infty}^0 b(t)^\dagger y_{(\tau)}(t) dt \quad = \quad 0 \tag{5.5.5}$$

holds for all $b \in \mathcal{B}$.

Upon using the functions in Proposition 5.5.3 to introduce the finite-dimensional spans

$$\mathcal{Y}(\tau) \quad = \quad \operatorname{span}\big\{y^i_{(\tau)}\big\}_{i=1}^{n_d}, \qquad\qquad Y(\tau) \quad = \quad \operatorname{span}\big\{(y^i_{(\tau)})_\tau\big\}_{i=1}^{n_d}, \tag{5.5.6}$$

we can now define the spaces $R(\tau)$ and $\mathcal{R}(\tau)$ that appear in Theorem 5.2.8 by writing

$$\mathcal{R}(\tau) \quad = \quad \tilde{\mathcal{P}}(\xi) \oplus \mathcal{Y}(\tau), \qquad\qquad R(\tau) \quad = \quad \tilde{P}(\xi) \oplus Y(\tau). \tag{5.5.7}$$

The identities in (5.5.3) show that the dimension of the space $Y(\tau)$ is precisely n_d. Moreover, in combination with Theorem 5.2.5 they yield

$$S(\tau) \cap Y(\tau) \quad = \quad \{0\}, \tag{5.5.8}$$

which means that we have the direct sum decomposition

$$X \quad = \quad \tilde{P}(\tau) \oplus Y(\tau) \oplus Q(\tau) \tag{5.5.9}$$

for any $\tau \ge 0$.

Our final main result here generalizes the exponential decay estimates contained in Theorem 5.2.4 to the half-line setting. The main obstacle here is that it is more involved to control the derivative of functions in $\mathcal{Y}(\tau)$, preventing a direct application of the Ascoli-Arzela theorem. Indeed, these functions have a nonzero right-hand side on the interval $[-r_0, 0]$ when substituted into (5.2.1).

Proposition 5.5.4 (cf. [104, Prop. 4.4])**.** *Assume that (HA), (HK), (HH) and (HKer) are satisfied. Then for any $\tau \ge 0$, every function $x \in \mathcal{R}(\tau)$ is C^1-smooth on $(-\infty, \tau]$. In addition, there exist constants $K_{\mathrm{dec}} > 0$ and $\alpha > 0$ in such a way that for all $\tau \ge 0$ and all $t \le \tau$ we have the pointwise estimate*

$$|x(t)| + |\dot{x}(t)| \quad \le \quad K_{\mathrm{dec}} e^{-\alpha(\tau - t)} \|x_\tau\|_\infty \tag{5.5.10}$$

for every $x \in \mathcal{R}(\tau)$.

5.5.2 Construction of $\mathcal{Y}(\tau)$

In order to construct the functions $\{y^i_{(\tau)}\}$ from Proposition 5.5.3, we will use the freedom we still have to choose complements for the range and the kernel of the operator Λ.

Lemma 5.5.5 (cf. [104, Lem. 3.4]). *Assume that (HA), (HK), (HH) and (HKer) are satisfied and fix $\tau \in \mathbb{R}$. Write $n_d = \dim(\mathcal{B}^*)$ and choose a basis $\{d^i : 1 \le i \le n_d\}$ for $\mathcal{B}^*$. Then there exists a constant $r_0^{(\tau)} > 0$, together with functions*

$$\{\phi^i_{(\tau)} : 1 \le i \le n_d\} \subset C_b[\tau, \tau + r_0^{(\tau)}], \qquad \{\psi^i_{(\tau)} : 1 \le i \le n_d\} \subset C_b[\tau - r_0^{(\tau)}, \tau] \tag{5.5.11}$$

that satisfy the identities

$$\begin{aligned} \int_{\tau}^{\tau+r_0^{(\tau)}} d^i(t)^\dagger \phi^j_{(\tau)}(t)dt &= \delta_{i,j}, \\ \int_{\tau-r_0^{(\tau)}}^{\tau} d^i(t)^\dagger \psi^j_{(\tau)}(t)dt &= \delta_{i,j} \end{aligned} \tag{5.5.12}$$

for any $1 \le i, j \le n_d$, together with

$$\begin{aligned} 0 &= \phi^j_{(\tau)}(\tau) &&= \phi^j_{(\tau)}(\tau + r_0^{(\tau)}), \\ 0 &= \psi^j_{(\tau)}(\tau - r_0^{(\tau)}) &&= \psi^j_{(\tau)}(\tau), \end{aligned} \tag{5.5.13}$$

for any $1 \le j \le n_d$.

Proof. By symmetry, we only consider the construction of the functions $\{\psi^i_{(\tau)} : 1 \le i \le n_d\}$. We first note that the restriction operator

$$\mathcal{B}^* \to C_b[-r_0^{(\tau)} + \tau, \tau], \qquad d \mapsto d|_{[-r_0^{(\tau)}+\tau,\tau]} \tag{5.5.14}$$

is injective for some $r_0^{(\tau)} > 0$. This follows trivially from (HKer) and the fact that $\mathcal{B}^*$ is finite dimensional.

Let us denote $[\cdot,\cdot]_\tau$ for the integral product

$$[\psi, \phi]_\tau = \int_{\tau-r_0^{(\tau)}}^{\tau} \psi(t)^\dagger \phi(t)dt. \tag{5.5.15}$$

Consider any set of functions $\{\tilde{\psi}^i : 1 \le i \le n_d\} \subset C_b[\tau - r_0^{(\tau)}, \tau]$ with

$$0 = \tilde{\psi}^j(\tau - r_0^{(\tau)}) = \tilde{\psi}^j(\tau), \tag{5.5.16}$$

for which the $n_d \times n_d$-matrix Z with entries $Z_{ij} = [d^i|_{[-r_0^{(\tau)}+\tau,\tau]}, \tilde{\psi}^j]_\tau$ is invertible. This is possible on account of the linear independence of the sequence $\{d^i|_{[-r_0^{(\tau)}+\tau,\tau]} : 1 \le i \le n_d\}$. For any integer $1 \le j \le n_d$ we can now choose

$$\psi^j_{(\tau)} = \sum_{k=1}^{n_d} Z^{-1}_{kj} \tilde{\psi}^k. \tag{5.5.17}$$

By construction, we have $\psi^j_{(\tau)}(\tau - r_0^{(\tau)}) = \psi^j_{(\tau)}(\tau) = 0$ and we can compute

$$\begin{array}{rcl} \int\limits_{\tau - r_0^{(\tau)}}^{\tau} d^i(t)^\dagger \psi^j_{(\tau)}(t)dt & = & \Big[d^i|_{[-r_0^{(\tau)}+\tau,\tau]}, \psi^j \Big]_\tau \\ & = & \sum\limits_{k=1}^{n_d} Z^{-1}_{kj} \Big[d^i|_{[-r_0^{(\tau)}+\tau,\tau]}, \tilde{\psi}^k \Big]_\tau \\ & = & \sum\limits_{k=1}^{n_d} Z^{-1}_{kj} Z_{ik} \\ & = & \delta_{i,j} \end{array} \tag{5.5.18}$$

for any $1 \le i, j \le n_d$, as desired. ∎

Lemma 5.5.6 (cf. [104, Pg. 13]). *Assume that (HA), (HK), (HH) and (HKer) are satisfied and fix $\tau \in \mathbb{R}$. Then there exist bounded linear operators*

$$\begin{array}{rcl} \Lambda^{-1}_{+;\tau} : L^\infty\big([\tau,\infty);\mathbb{C}^M\big) & \to & W^{1,\infty}(D^\oplus_\tau;\mathbb{C}^M), \\ \Lambda^{-1}_{-;\tau} : L^\infty\big((-\infty,\tau];\mathbb{C}^M\big) & \to & W^{1,\infty}(D^\ominus_\tau;\mathbb{C}^M) \end{array} \tag{5.5.19}$$

with the property that the identities

$$\begin{array}{rclr} [\Lambda\Lambda^{-1}_{+;\tau} f](t) & = & f(t), & t \ge \tau, \\ [\Lambda\Lambda^{-1}_{-;\tau} g](s) & = & g(s) & t \le \tau \end{array} \tag{5.5.20}$$

hold for $f \in L^\infty\big([\tau,\infty);\mathbb{C}^M\big)$ and $g \in L^\infty\big((-\infty,\tau];\mathbb{C}^M\big)$.

Proof. By symmetry, we will only construct the operator $\Lambda^{-1}_{+;\tau}$. We write $R = \mathrm{Range}(\Lambda)$ and $K = \mathcal{B}$. Let $R^\perp$ and $K^\perp$ be arbitrary complements of R and K respectively, so that we have

$$W^{1,\infty}(\mathbb{R};\mathbb{C}^M) \;=\; K \oplus K^\perp, \qquad L^\infty(\mathbb{R};\mathbb{C}^M) \;=\; R \oplus R^\perp. \tag{5.5.21}$$

Let π_R and $\pi_{R^\perp}$ denote the projections corresponding to this splitting. Then $\Lambda : K^\perp \to R$ is invertible, with a bounded inverse $\Lambda^{-1} \in \mathcal{L}(R, K^\perp)$.

We let $r_0^{(\tau)} > 0$ and $\{\psi^i_{(\tau)} : 1 \le i \le n_d\}$ be the constant and the functions from Lemma 5.5.5 for this value of τ. For $1 \le i \le n_d$ we write $g^i_{(\tau)} \in L^\infty(\mathbb{R};\mathbb{C}^M)$ for the function that has $g^i_{(\tau)} = \psi^i_{(\tau)}$ on $[-r_0^{(\tau)}+\tau, \tau]$, while $g^i_{(\tau)} = 0$ on $(-\infty, -r_0^{(\tau)}+\tau) \cup (\tau,\infty)$. Since we have $g^i_{(\tau)} \notin R$ for $1 \le i \le n_d$ by Proposition 5.2.1 and these functions are linearly independent, we can explicitly choose the projection $\pi_{R^\perp}$ to be given by

$$\pi_{R^\perp} f \;=\; \sum_{i=1}^{n_d} \big[\textstyle\int_{-\infty}^{\infty} d^i(t)^\dagger f(t)dt \big] g^i_{(\tau)}. \tag{5.5.22}$$

Upon writing $\mathbb{1}_{[\tau,\infty)}$ for the indicator function on $[\tau,\infty)$, we can define the inverse of Λ on the positive half-line $[\tau,\infty)$ by

$$\Lambda^{-1}_{+;\tau} f \;=\; \Lambda^{-1} \pi_{\mathcal{R}} \mathbb{1}_{[\tau,\infty)} f. \tag{5.5.23}$$

By construction, we have $g^i_{(\tau)}(t) = 0$ for all $t \geq \tau$ and all $1 \leq i \leq n_d$. As such, we have $[\pi_{R^\perp} \mathbb{1}_{[\tau,\infty)} f](t) = 0$ for any $t \geq \tau$ and any $f \in L^\infty([\tau,\infty);\mathbb{C}^M)$. Hence a short computation shows that we have

$$[\Lambda\Lambda^{-1}_{+;\tau} f](t) \quad = \quad [\mathbb{1}_{[\tau,\infty)} f](t) - \big[\pi_{\mathbb{R}^\perp} \mathbb{1}_{[\tau,\infty)} f\big](t) \quad = \quad f(t) \tag{5.5.24}$$

for $t \geq \tau$ and $f \in L^\infty([\tau,\infty);\mathbb{C}^M)$, as desired. ■

For notational convenience, we write

$$r_0 \quad := \quad r_0^{(0)}, \qquad\qquad \psi^i \quad := \quad \psi^i_{(0)} \tag{5.5.25}$$

for the constant and functions obtained in Lemma 5.5.5 for $\tau = 0$. As in the proof of Lemma 5.5.6, we also write $g^i \in L^\infty(\mathbb{R};\mathbb{C}^M)$ for the function

$$g^i(t) \quad = \quad \begin{cases} \psi^i(t), & t \in [-r_0, 0] \\ 0, & t \in (\infty, -r_0] \cup [0,\infty). \end{cases} \tag{5.5.26}$$

On account of the identity (5.5.13), we note that the function g^i is continuous.

Proof of Proposition 5.5.3. For any $k \in \mathbb{Z}_{\geq 1}$ we write $\Lambda^{-1}_{-;k}$ for the inverse operator constructed in Lemma 5.5.6 for the half-line $(-\infty, k]$, together with $y^i_{(k)} = \Lambda^{-1}_{-;k} g^i$. Assumption (HKer) implies that any basis of $\mathcal{B}$ remains linearly independent when restricted to the interval $(-\infty, 0]$. As such, we can add an appropriate element of $\mathcal{B}$ to $y_{(k)}$ to ensure that the integral condition (5.5.5) is satisfied. For any integer $1 \leq j \leq n_d$, Lemma 5.3.12 and the exponential decay of the function d^j allow us to compute

$$\begin{aligned} \langle (d^j)^t, (y^i_{(k)})_t \rangle_t &= \int_{-\infty}^t d^j(s)^\dagger \big[\Lambda y^i_{(k)}\big](s) dt \\ &= \int_{-r_0}^0 d^j(s)^\dagger g^i(s) ds \\ &= \delta_{ij} \end{aligned} \tag{5.5.27}$$

for any $0 \leq t \leq k$. We now pick a continuous function $\chi : [0,\infty) \to [0,1]$ that is zero near even integers and one near odd integers. Upon defining

$$y^i_{(\tau)} \quad = \quad \chi(2\tau) y^i_{(\lceil \tau \rceil)} + \big[1 - \chi(2\tau)\big] y^i_{(\lceil \tau + \frac{1}{2} \rceil)}, \tag{5.5.28}$$

in which $\lceil \tau \rceil$ denotes the closest integer larger or equal to τ, it is easy to see that properties (i) through (v) are all satisfied. ■

5.5.3 Exponential decay

We now focus on the exponential decay of functions in $\mathcal{Y}(\tau)$, noting that Theorem 5.2.4 already captures the corresponding behaviour for functions in $\widetilde{\mathcal{P}}(\tau)$. The technical issues that we encountered during the proof of Theorem 5.2.4 persist in this half-line setting. In particular, we need to control the behaviour of functions in $\mathcal{Y}(\tau)$ on a left half-line and a right half-line at the same time.

In addition, in the proof of the corresponding result in [104], the authors were explicitly able to avoid the region where $\Lambda y^i_{(\tau)}$ is nonzero when considering the states $(y^i_{(\tau)})_\tau$. This is of course no longer possible in our setting when $|r_{\min}|$ is infinite. As such, we need to control the value of $\Lambda y^i_{(\tau)}$ in a more rigorous fashion.

Our first result can be see as the analogue of Lemma 5.3.6, but now the goal is to obtain estimates on Λy_n for bounded sequences $\{y_n \in \mathcal{Y}(\tau_n)\}$. As a preparation, we recall from the proof of Proposition 5.5.3 that the identity

$$\Lambda y \quad = \quad \sum_{i=1}^{n_d} g^i \langle (d^i)^0, y_0 \rangle_0 \tag{5.5.29}$$

holds for $y \in \mathcal{Y}(\tau)$. In addition, we recall the constants $p > 0, K_{\exp} > 0$ and $\alpha > 0$ introduced in Lemma 5.3.1.

Lemma 5.5.7. *Assume that (HA), (HK), (HH) and (HKer) are satisfied and let $\{\sigma_n\}_{n\geq 1}$, $\{y_n\}_{n\geq 1}$ and $\{\tau_n\}_{n\geq 1}$ be sequences with the following properties.*

(a) We have $\sigma_n > 0$ for each n, together with $\sigma_n \uparrow \infty$.

(b) We have $y_n \in \mathcal{Y}(\tau_n)$ and $\tau_n \geq 0$ for each n.

(c) For each $n \geq 1$ we have the bound

$$|y_n(-\sigma_n + \tau_n)| \quad \geq \quad \tfrac{1}{2}, \tag{5.5.30}$$

together with the normalization

$$\sup_{s \in (-\infty, \tau_n + p]} |y_n(s)| \quad = \quad 1. \tag{5.5.31}$$

(d) If $r_{\max} = \infty$, then we have the additional bound

$$|y_n(-\sigma_n + \tau_n)| \quad \geq \quad K_{\exp} e^{\alpha(-\sigma_n + \tau_n)} \sup_{s \in [p + \tau_n, \infty)} e^{-\alpha s} |y_n(s)|. \tag{5.5.32}$$

(e) The limit $-\sigma_n + \tau_n \to \beta_0$ holds for some $\beta_0 \in \mathbb{R}$.

Then the set of scalars $\{\langle (d^i)^0, (y_n)_0 \rangle_0\}$ is bounded uniformly for $n \geq 1$ and $1 \leq i \leq n_d$.

Proof. Suppose first that $r_{\max} = \infty$. Fixing $n \in \mathbb{Z}_{\geq 1}$ and $1 \leq i \leq n_d$, we can use the bounds (5.5.30) and (5.5.32) to estimate

$$\begin{aligned} |\langle (d^i)^0, (y_n)_0 \rangle_0| \quad &\leq \quad |d^i(0)^\dagger y_n(0)| + \Big| \sum_{j=-\infty}^{\infty} \int_0^{r_j} d^i(s - r_j)^\dagger A_j(s - r_j) y_n(s) ds \Big| \\ &\qquad + \Big| \int_{\mathbb{R}} \int_0^r d^i(s - r)^\dagger \mathcal{K}(r; s - r) y(s) ds dr \Big| \\ &\leq \quad |d^i(0)| + S_1(i, n) + S_2(i, n) + S_3(i, n) \\ &\qquad + I_1(i, n) + I_2(i, n) + I_3(i, n), \end{aligned} \tag{5.5.33}$$

in which we have defined

$$\begin{array}{lcl} S_1(i,n) & = & \sum_{r_j \le p+\tau_n} \big| \int_0^{r_j} d^i(s-r_j)^\dagger A_j(s-r_j) ds \big|, \\ S_2(i,n) & = & \sum_{r_j > p+\tau_n} \big| \int_0^{p+\tau_n} d^i(s-r_j)^\dagger A_j(s-r_j) ds \big|, \\ S_3(i,n) & = & \sum_{r_j > p+\tau_n} \big| \int_{p+\tau_n}^{r_j} d^i(s-r_j)^\dagger A_j(s-r_j) K_{\exp}^{-1} e^{\alpha(\sigma_n - \tau_n)} e^{\alpha s} ds \big|, \end{array} \tag{5.5.34}$$

together with the corresponding expressions $I_1(i,n)$, $I_2(i,n)$ and $I_3(i,n)$ related to the integrals involving $\mathcal{K}$.

We easily obtain the bounds

$$\begin{array}{lcl} |S_1(i,n)| & \le & \max_{1 \le k \le n_d} \sum_{j=-\infty}^{\infty} \big| \int_0^{r_j} |d^k(s-r_j)^\dagger A_j(s-r_j)| ds \big|, \\ |S_2(i,n)| & \le & \max_{1 \le k \le n_d} \sum_{r_j > p} \int_0^{\infty} |d^k(s-r_j)^\dagger A_j(s-r_j)| ds, \\ |I_1(i,n)| & \le & \max_{1 \le k \le n_d} \int_{\mathbb{R}} \big| \int_0^{r} |d^k(s-r)^\dagger \mathcal{K}(r; s-r)| ds \big| dr, \\ |I_2(i,n)| & \le & \max_{1 \le k \le n_d} \int_p^{\infty} \int_0^{\infty} |d^k(s-r)^\dagger \mathcal{K}(r; s-r)| ds dr, \end{array} \tag{5.5.35}$$

which are uniform in i and n. Turning to the remaining expressions, we pick a small $\varepsilon > 0$ and assume that n is large enough to have $|\beta_0 + \sigma_n - \tau_n| < \varepsilon$. This allows us to estimate

$$\begin{array}{lcl} |S_3(i,n)| & \le & K_{\exp}^{-1} e^{\alpha(\sigma_n - \tau_n)} \sum_{r_j > p+\tau_n} \|A_j\|_\infty |r_j| e^{\alpha r_j} \|d^i\|_\infty \\ & \le & K_{\exp}^{-1} e^{\alpha(\sigma_n - \tau_n)} K_{\exp} e^{-2\alpha(p+\tau_n)} \|d^i\|_\infty \\ & \le & e^{-2\alpha(p+\tau_n)} e^{\alpha\beta_0 + \alpha\varepsilon} \max_{1 \le k \le n_d} \|d^k\|_\infty, \end{array} \tag{5.5.36}$$

with a corresponding bound for I_3. In particular, both $S_3(i,n)$ and $I_3(i,n)$ converge to 0 as $n \to \infty$, so they can be bounded from above uniformly in i and n.

In the case where $r_{\max} < \infty$, we can repeat this procedure with $p = r_{\max}$. The quantities $S_3(i,n)$ and $I_3(i,n)$ are identically zero in this case. ■

Lemma 5.5.8. *Assume that (HA), (HK), (HH) and (HKer) are satisfied and suppose that $r_{\max} = \infty$. Then for each $\tau \ge 0$ and each $y \in \mathcal{Y}(\tau)$ we have the bound*

$$|y(t)| \le \max\Big\{ \tfrac{1}{2} \sup_{s \in (-\infty, \tau+p]} |y(s)|, K_{\exp} \sup_{s \in [p+\tau, \infty)} e^{-\alpha(s-t)} |y(s)| \Big\}, \qquad t \le -\sigma + \tau. \tag{5.5.37}$$

If $r_{\max} < \infty$, then the same statements hold with (5.5.37) replaced by

$$|y(t)| \le \tfrac{1}{2} \sup_{s \in (-\infty, \tau + r_{\max}]} |y(s)|, \qquad t \le -\sigma + \tau. \tag{5.5.38}$$

Proof. Arguing by contradiction, let us consider sequences $\{\sigma_n\}_{n \ge 1}$, $\{\tau_n\}_{n \ge 1}$ and $\{y_n\}_{n \ge 1}$ that satisfy properties (a)-(d) in Lemma 5.5.7. If the sequence $\{-\sigma_n + \tau_n\}_{n \ge 1}$

is unbounded then we can follow the proof of Proposition 5.3.2 to arrive at a contradiction, since the interval $[-r_0, 0]$ on which Λy_n might be nonzero gets 'pushed out' towards $\pm\infty$.

Suppose therefore that $-\sigma_n + \tau_n \to \beta_0 \in \mathbb{R}$, possibly after passing to a subsequence. Combining Lemma 5.5.7 with (5.5.29) shows that $\{\Lambda y_n\}_{n\geq 1}$ is uniformly bounded, which allows us to apply the Ascoli-Arzela theorem to conclude that $y_n \to y_*$ uniformly on compact subsets of $\mathbb{R}$. A computation similar to the proof of Lemma 5.3.7 shows that $[\Lambda y_*](t) = 0$ for every $t \geq 0$, since the functions g^i vanish for these values of t. In particular, we must have $(y_*)_0 \in Q(0)$. On account of Theorem 5.2.5, we obtain $(y_*)_0 \in X^{\perp}(0)$, which yields

$$\langle (d^i)^0, (y_*)_0 \rangle_0 \quad = \quad 0 \tag{5.5.39}$$

for each $1 \leq i \leq n_d$. In view of (5.5.29) this means that $\Lambda y_n \to 0$ uniformly on every compact subset of the real line. In particular, we must have $\Lambda y_* = 0$ on the entire real line, which implies that $y_* \in \mathcal{B}$. However, this contradicts the integral condition (5.5.5). ∎

Lemma 5.5.9. *Assume that (HA), (HK), (HH) and (HKer) are satisfied. Then there exists $C > 0$ so that for all $\tau \geq 0$ and all $y \in \mathcal{Y}(\tau)$ we have the bound*

$$\|y\|_{C_b(D^{\ominus}_{\tau})} \quad \leq \quad C\|y_\tau\|_\infty. \tag{5.5.40}$$

Proof. The bound (5.5.40) is in fact an equality with $C = 1$ if $r_{\min} = -\infty$. Hence we assume that $r_{\min} > -\infty$. Arguing by contradiction, we can pick sequences $\{y_n\}_{n\geq 1}, \{\tau_n\}_{n\geq 1}$ and $\{C_n\}_{n\geq 1}$ with $C_n \to \infty$ with $\tau_n \geq 0$ and $y_n \in \mathcal{Y}(\tau_n)$ for each n in such a way that we have the identity

$$\|y_n\|_{C_b(D^{\ominus}_{\tau_n})} \quad = \quad C_n\|(y_n)_{\tau_n}\|_\infty \quad = \quad 1. \tag{5.5.41}$$

If the sequence $\{\tau_n\}_{n\geq 1}$ is unbounded we can follow the first half of the proof of Lemma 5.3.8 to arrive at a contradiction.

Hence we suppose that, after passing to a subsequence, we have $\tau_n \to \tau_* \geq 0$. Since the bounds on the functions $\{y_n\}_{n\geq 1}$ are stronger than those in (5.5.30) or (5.5.32), we can repeat the procedure from Lemma 5.5.7 to conclude that $y_n \to y_*$ uniform on compact subsets of $(-\infty, \tau_*]$. For each $n \geq 1$ we pick s_n in such a way that $|y_n(-s_n + \tau_n)| = 1$. On account of Lemma 5.5.8, the set $\{s_n\}_{n\geq 1}$ is bounded. Hence, we obtain that

$$y_*(t) \quad \neq \quad 0, \ \text{ for some } t \in \overline{(r_{\min} + \tau_*, \sigma + \tau_*)}. \tag{5.5.42}$$

In addition, we have $(y_n)_{\tau_n} \to 0$ uniformly as $n \to \infty$, so we even obtain that $y_n \to y_*$ uniformly on $D^+_X + \tau_*$. If $r_{\max} < \infty$, we set $y_* = 0$ on $(r_{\max} + \tau_*, \infty)$. In particular, we have $(y_*)_{\tau_*} = 0$ and thus $[\Lambda y_*](t) = 0$ for any $t \in [\tau_*, \infty)$. Moreover, we have $[\Lambda y_*](t) = 0$ for any $t \in [0, \tau_*]$, since Λy_n is zero for these values of t for each $n \in \mathbb{Z}_{\geq 1}$.

This means that $y_* \in \mathcal{Q}(0)$ and, as before, this yields a contradiction. ∎

Proof of Proposition 5.5.4. Using Lemmas 5.5.8 and 5.5.9, we can extend the proof of Theorem 5.2.4 to also include functions in $\mathcal{Y}(\tau)$. As such, for all $\tau \geq 0$ and $x \in \mathcal{R}(\tau)$ we have the pointwise estimate

$$|x(t)| \quad \leq \quad K_{\mathrm{dec}} e^{-\alpha(\tau - t)} \|x_\tau\|_\infty, \qquad t \leq \tau. \tag{5.5.43}$$

The exponential decay of $\dot{x}$ for $x \in \tilde{P}(\tau)$ follows directly from Theorem 5.2.4. Let us therefore consider an arbitrary $y \in \mathcal{Y}(\tau)$, which satisfies the exponential bound (5.3.63). Recalling the constant $B > 0$ from Lemma 5.3.14, we write

$$\begin{array}{rcl} C & = & \sum_{j=-\infty}^{\infty} \|A_j(\cdot)\|_\infty e^{\alpha|r_j|} + \sup_{t \in \mathbb{R}} \|\mathcal{K}(\cdot;t)\|_\alpha, \\ \tilde{B} & = & B e^{\alpha r_0} \sum_{i=1}^{n_d} \|(d^i)^0\|_\infty \|g^i\|_\infty. \end{array} \tag{5.5.44}$$

Recalling the bound (5.3.62) and the identity (5.5.29), we obtain that

$$\begin{array}{rcl} |\Lambda y|(t) & = & \big| \sum_{i=1}^{n_d} g^i \langle (d^i)^0, y_0 \rangle_0 \big| \\ & \leq & \sum_{i=1}^{n_d} \|g^i\|_\infty \big| \langle (d^i)^0, y_0 \rangle_0 \big| \\ & \leq & B \sum_{i=1}^{n_d} \|g^i\|_\infty \|(d^i)^0\|_\infty \|y_\tau\|_\infty e^{-\alpha\tau} \\ & \leq & e^{-\alpha(\tau - t)} \tilde{B} \|y_\tau\|_\infty \end{array} \tag{5.5.45}$$

for any $-r_0 \leq t \leq 0$. Since $g^i(t) = 0$ for $t \geq 0$ and $t \leq -r_0$ and since g^i is continuous, we see that (5.5.45) is, in fact, valid for any $t \leq \tau$. As such, we immediately obtain

$$|\dot{y}(t)| \quad \leq \quad K_{\mathrm{dec}} e^{-\alpha(\tau - t)} \Big(C + \tilde{B} \Big) \|y_\tau\|_\infty, \qquad t \leq \tau \tag{5.5.46}$$

for any $\tau \geq 0$ and any $y \in \mathcal{Y}(\tau)$.

For the final statement we first recall the identity (5.5.29). Since the coefficients $A_j(t)$ and $\mathcal{K}(\cdot;t)$ depend continuously on t and since the functions g^i are continuous, the identity (5.5.29) yields that $\dot{x}$ is continuous on $(-\infty, \tau]$ for any $x \in \mathcal{R}(\tau)$ and any $\tau \geq 0$. ∎

5.5.4 Projection operators

In order to complete the proof of Theorem 5.2.8, we need to consider the behaviour of several projection operators. In particular, we recall the splitting

$$X \quad = \quad P(\infty) \oplus Q(\infty) \tag{5.5.47}$$

corresponding to the hyperbolic limiting system (5.2.11) at $+\infty$, together with the notation $\overrightarrow{\Pi}_P$ and $\overrightarrow{\Pi}_Q$ for the projections onto the factors $P(\infty)$ and $Q(\infty)$. In addition, we recall the decompositions

$$X \quad = \quad R(\tau) \oplus Q(\tau) \quad = \quad \tilde{P}(\tau) \oplus Y(\tau) \oplus Q(\tau), \qquad \tau \geq 0 \tag{5.5.48}$$

obtained above in this section and write $\Pi_{\tilde{P}(\tau)}$, $\Pi_{Y(\tau)}$ and $\Pi_{Q(\tau)}$ for the corresponding projections.

Our first result can be seen as a supplement for the bound (5.2.47) in Theorem 5.2.6. Indeed, together these bounds allow the full structure of the two decompositions above to be compared with each other for $\tau \gg 1$.

Lemma 5.5.10 (cf. [104, Lem. 4.5]). *Assume that (HA), (HK), (HH) and (HKer) are satisfied. Then we have the limit*

$$\lim_{\tau \to \infty} \|I - \overrightarrow{\Pi}_P|_{R(\tau)}\| \quad = \quad 0. \tag{5.5.49}$$

Proof. If $r_{\min} > -\infty$ then we can follow the proof of [104, Lem. 4.5] to obtain the desired result, so we assume that $r_{\min} = -\infty$. Recalling the positive constants K_{dec} and α from Proposition 5.5.4, we write

$$C \quad = \quad \sum_{j=-\infty}^{\infty} \|A_j(\cdot)\|_\infty e^{\alpha |r_j|} + \sup_{t \in \mathbb{R}} \|\mathcal{K}(\cdot;t)\|_\alpha + \sum_{j=-\infty}^{\infty} |A_j(\infty)| e^{\alpha |r_j|} + \|\mathcal{K}(\cdot;\infty)\|_\alpha. \tag{5.5.50}$$

Fix an arbitrary $\varepsilon > 0$ and pick $\tau_0 \gg 1$ in such a way that the bounds

$$\begin{aligned} &4K_{\mathrm{dec}}\big(1+C\big)e^{-\alpha\tau_0} && < \tfrac{\varepsilon}{2}, \\ &\sum_{j=-\infty}^{\infty} \big|A_j(t) - A_j^+(\infty)\big| + \|\mathcal{K}(\cdot;t) - \mathcal{K}(\cdot;\infty)\|_\alpha + \|\mathcal{K}(\cdot;t-\cdot) - \mathcal{K}(\cdot;\infty)\|_\alpha && < \tfrac{\varepsilon}{2} \end{aligned} \tag{5.5.51}$$

hold for all $t \geq \tau_0$. Recall the constant r_0 from (5.5.25) and fix any $\tau \geq 2\tau_0 + p + r_0$.

First we pick any $y \in \mathcal{R}(\tau)$ and write $\phi = y_\tau \in R(\tau)$. We now set out to show that

$$\|\overrightarrow{\Pi}_Q \phi\|_\infty \quad \leq \quad \varepsilon C' \|\phi\|_\infty, \tag{5.5.52}$$

for some constant $C' > 0$. Indeed, this upper bound implies that

$$\|I - \overrightarrow{\Pi}_P|_{Y(\tau)}\| \quad = \quad \|\overrightarrow{\Pi}_Q|_{Y(\tau)}\| \quad \leq \quad C'\varepsilon, \tag{5.5.53}$$

which yields the desired result.

On account of Proposition 5.5.4 we note that y is continuously differentiable on $(-\infty, \tau]$, which yields that ϕ is continuously differentiable on $(-\infty, 0]$. In addition, Proposition 5.5.4 implies that both ϕ and $\dot{\phi}$ decay exponentially for $t \to -\infty$. which means that $\phi|_{(-\infty,0]} \in C_b^1\big((-\infty,0]\big)$. We can hence approximate ϕ by functions $\{\phi_k\}_{k\geq 1}$

in $C_b^1(D_X)$ which have $\phi_k(t) = \phi(t)$ for any $t \in (-\infty, 0]$. These functions can be extended to C^1-smooth functions $\{y_k\}_{\geq 1}$, defined on $\mathbb{R}$, which have $(y_k)_\tau = \phi_k$. As such, they have $y_k(t) = y(t)$ for any $t \leq \tau$. Due to the uniform bound on both y and $\dot{y}$ from Proposition 5.5.4 we can pick the functions $\{y_k\}_{k\geq 1}$ in such a way that the bound

$$|\dot{y}_k(t)| + |y_k(t)| \quad \leq \quad 4K_{\mathrm{dec}} e^{-\alpha(\tau - t)} \Big(1 + C\Big) \|y_\tau\|_\infty \tag{5.5.54}$$

holds for any $t \leq 0$ and any $k \in \mathbb{Z}_{\geq 1}$.

We now introduce the Heaviside function H_τ that acts as $H_\tau(t) = I$ if $t \geq \tau$ and zero otherwise, together with the operator

$$[\Lambda_\infty x](t) \quad = \quad \dot{x}(t) - \sum_{j=-\infty}^{\infty} A_j(\infty) x(t + r_j) - \int_{\mathbb{R}} \mathcal{K}(s; \infty) x(t + s) ds. \tag{5.5.55}$$

Recalling the splitting (5.5.47), we observe that for any function $x \in C_b^1(\mathbb{R})$ we have

$$\big(\Lambda_\infty^{-1} H_\tau \Lambda_\infty x\big)_\tau \quad \in \quad P(\infty), \qquad \big(\Lambda_\infty^{-1} [I - H_\tau] \Lambda_\infty x\big)_\tau \quad \in \quad Q(\infty), \tag{5.5.56}$$

together with

$$x_\tau \quad = \quad \big(\Lambda_\infty^{-1} H_\tau \Lambda_\infty x\big)_\tau + \big(\Lambda_\infty^{-1} [I - H_\tau] \Lambda_\infty x\big)_\tau. \tag{5.5.57}$$

As such, we have the representation

$$\overrightarrow{\Pi}_Q x_\tau \quad = \quad \big(\Lambda_\infty^{-1} [I - H_\tau] \Lambda_\infty x\big)_\tau \tag{5.5.58}$$

for any C^1-smooth function x. For any $t \in \mathbb{R}$ and any $k \in \mathbb{Z}_{\geq 1}$, we observe that

$$\begin{array}{rcl} [\Lambda_\infty y_k](t) & = & [\Lambda y_k](t) + \sum_{j=-\infty}^{\infty} \big[A_j(t) - A_j(\infty)\big] y_k(t + r_j) \\ & & \quad + \int_{\mathbb{R}} \big(\mathcal{K}(s; t) - \mathcal{K}(s; \infty)\big) y_k(t + s) ds. \end{array} \tag{5.5.59}$$

Since $[\Lambda y_k](t) = [\Lambda y](t) = 0$ for $\tau_0 \leq t \leq \tau$ and any $k \in \mathbb{Z}_{\geq 1}$, we may hence estimate

$$\begin{array}{rcl} \|[I - H_\tau] \Lambda_\infty y_k\|_\infty & \leq & \sup_{t \leq \tau_0} \big[|\dot{y}_k(t)| + C\|y_t\|_\infty\big] + \sup_{\tau_0 \leq t \leq \tau} \frac{\varepsilon}{2} \|(y_k)_t\|_\infty \\ & \leq & 4K_{\mathrm{dec}} \big(1 + C\big) e^{-\alpha(\tau - \tau_0)} \|\phi_k\|_\infty + \frac{\varepsilon}{2} \|\phi_k\|_\infty \\ & \leq & 4K_{\mathrm{dec}} \big(1 + C\big) e^{-\alpha\tau_0} \|\phi_k\|_\infty + \frac{\varepsilon}{2} \|\phi_k\|_\infty \\ & \leq & \varepsilon \|\phi_k\|_\infty. \end{array} \tag{5.5.60}$$

By the boundedness of the operator Λ_∞^{-1}, we find that there exists a constant $C' > 0$ that allows us to write

$$\|\overrightarrow{\Pi}_Q \phi_k\|_\infty \quad = \quad \|\big(\Lambda_\infty^{-1} [I - H_\tau] \Lambda_\infty y_k\big)_\tau\|_\infty \quad \leq \quad \varepsilon C' \|\phi_k\|_\infty. \tag{5.5.61}$$

The operator $\overrightarrow{\Pi}_Q$ is continuous, so we can take the limit $k \to \infty$ to obtain

$$\|\overrightarrow{\Pi}_Q \phi\|_\infty \quad \leq \quad \varepsilon C' \|\phi\|_\infty. \tag{5.5.62}$$

This yields the desired bound

$$\|I - \overrightarrow{\Pi}_P|_{Y(\tau)}\| \quad = \quad \|\overrightarrow{\Pi}_Q|_{Y(\tau)}\| \quad \leq \quad C'\varepsilon. \tag{5.5.63}$$

■

Lemma 5.5.11 (cf. [104, Lem. 4.6]). *Assume that (HA), (HK), (HH) and (HKer) are satisfied and fix* $\tau_0 \geq 0$. *Then we have the limits*

$$\begin{aligned} \|[I - \Pi_{\tilde{P}(\tau_0)}]|_{\tilde{P}(\tau)}\| &\to 0 \quad \text{as } \tau \to \tau_0, \\ \|[I - \Pi_{Y(\tau_0)}]|_{Y(\tau)}\| &\to 0 \quad \text{as } \tau \to \tau_0, \\ \|[I - \Pi_{Q(\tau_0)}]|_{Q(\tau)}\| &\to 0 \quad \text{as } \tau \to \tau_0. \end{aligned} \tag{5.5.64}$$

Proof. The first and the third limit follow from Theorem 5.2.6. The second limit follows from the finite dimensionality of the spaces Y and from item (iii) of Proposition 5.5.3. ■

Lemma 5.5.12 (cf. [104, Lem. 4.7]). *Assume that (HA), (HK), (HH) and (HKer) are satisfied. Then the projections* $\Pi_{Q(\tau)}$ *from Lemma 5.5.11 can be uniformly bounded for all* $\tau \geq 0$.

Proof. The proof is identical to that of [104, Lem. 4.7] and, as such, will be omitted. It uses Proposition 5.3.10, together with Lemmas 5.5.10 and 5.5.11. ■

Corollary 5.5.13 (cf. [104, Cor. 4.8]). *Assume that (HA), (HK), (HH) and (HKer) are satisfied. Then the projections* $\Pi_{R(\tau)}$ *and* $\Pi_{Q(\tau)}$ *corresponding to the first splitting in* (5.5.48) *depend continuously on* $\tau \in \mathbb{R}_{\geq 0}$. *In addition, we have the limits*

$$\lim_{\tau\to\infty} \|\Pi_{Q(\tau)} - \overrightarrow{\Pi}_Q\| \quad = \quad 0, \qquad \lim_{\tau\to\infty} \|\Pi_{R(\tau)} - \overrightarrow{\Pi}_P\| \quad = \quad 0. \tag{5.5.65}$$

Proof. The proof is identical to that of [104, Cor. 4.8] and, as such, will be omitted. It uses Lemmas 5.5.10 and 5.5.12. ■

Proof of Theorem 5.2.8. Upon defining the space $R(\tau)$ by (5.5.7), the exponential decay rates follow from Theorem 5.2.4 and Proposition 5.5.4. The continuity of the projections follows from Corollary 5.5.13, while the uniform bounds on the projections follow from Lemma 5.5.12. ■

5.6 Degeneracies and their avoidance

In this section, we set out to prove Corollaries 5.2.3 and 5.2.7. In fact, our main result below formulates alternative conditions that can be used instead of (HKer) to obtain the same conclusions. These alternatives involve the Hale inner product, which we require to be (partially) nondegenerate in the following sense.

Definition 5.6.1. *Let $F \subset Y$ be a subset with $0 \in F$ and fix $\tau \in \mathbb{R}$. We say that the Hale inner product is left-nondegenerate at τ for functions in F if $\psi = 0$ is the only function $\psi \in F$ for which $\langle \psi, \phi \rangle_\tau = 0$ holds for every $\phi \in X$.*

Definition 5.6.2. *Let $E \subset X$ be a subset with $0 \in E$ and fix $\tau \in \mathbb{R}$. We say that the Hale inner product is right-nondegenerate at τ for functions in E if $\phi = 0$ is the only function $\phi \in E$ for which $\langle \psi, \phi \rangle_\tau = 0$ holds for every $\psi \in Y$.*

Proposition 5.6.3 (cf. [133, Cor. 4.7])**.** *Assume that (HA), (HK) and (HH) are satisfied. Suppose furthermore that at least one of the following three conditions is satisfied.*

(a) The nontriviality condition (HKer) holds.

(b) We have $|r_{\min}| = r_{\max} = \infty$ and the Hale inner product is left-nondegenerate for functions in $B^*(\tau)$ at each $\tau \in \mathbb{R}$.

(c) We have $r_{\min} < 0 < r_{\max}$ and for each $\tau \in \mathbb{R}$ the Hale inner product at τ is both left-nondegenerate for functions in $B^*(\tau)$ and right-nondegenerate for functions in $B(\tau)$.

Then the identities

$$\dim B(\tau) \quad = \quad \dim \mathcal{B}, \qquad\qquad \beta(\tau) \quad = \quad \dim B^*(\tau) \quad = \quad \dim \mathcal{B}^* \tag{5.6.1}$$

hold for every $\tau \in \mathbb{R}$. Moreover, the four Fredholm indices appearing in (5.4.1) are independent of τ and given by (5.6.1). In addition, the first equation in (5.4.1) becomes

$$\mathrm{ind}(\pi^+_{P(\tau)}) + \mathrm{ind}(\pi^-_{Q(\tau)}) \quad = \quad -M + \mathrm{ind}(\Lambda) \tag{5.6.2}$$

with Λ as in (5.2.15). Finally, the spaces $P(\tau)$, $Q(\tau)$, $\widehat{P}(\tau)$ and $\widehat{Q}(\tau)$ all vary continuously with respect to τ.

In §5.6.1 we provide various structural conditions on the system (5.2.1) that allow the conditions (a)-(c) above to be verified. They turn out to be closely related, as illustrated by the examples that we provide in §5.6.2. We establish our main result in §5.6.3, where we also describe how partial results can be obtained under weaker conditions.

5.6.1 Structural conditions

In order to use Proposition 5.6.3 to compute the codimension of the space $S(\tau)$ in X, we either need to establish the nondegeneracy of the Hale inner product or show that the nontriviality condition (HKer) is satisfied. However, it is by no means clear how this can be achieved in practice for concrete systems. Our goal here is to describe several more-or-less explicit criteria that can be used to verify these nondegeneracy and nontriviality conditions.

Some of these criteria reference the adjoint of the system (5.2.1), which is closely related to the operator Λ^* defined in (5.2.16). This system is given by

$$\dot{y}(t) \quad = \quad -\sum_{j=-\infty}^{\infty} A_j(t-r_j)^\dagger y(t-r_j) - \int_{\mathbb{R}} \mathcal{K}(\xi; t-\xi)^\dagger y(t-\xi) d\xi. \tag{5.6.3}$$

Most of our conditions impose the following basic structural condition, which demands that the coefficients corresponding to large shifts are autonomous. This is valid for many common reaction-diffusion systems such as those studied in [6, 150]. Indeed, the large shifts usually arise from discretizations of the diffusion, which is typically autonomous. The nonautonomous reaction terms are typically localized in space.

Assumption (hB). There exists a constant $K_{\text{const}} \in \mathbb{Z}_{\geq 1}$ together with families of diagonal matrices

$$\{\tilde{A}_j : j \in \mathbb{Z} \text{ with } |j| \geq K_{\text{const}}\} \subset \mathbb{C}^{M\times M}, \quad \{\tilde{\mathcal{K}}(\xi) : \xi \in \mathbb{R} \text{ with } |\xi| \geq K_{\text{const}}\} \subset \mathbb{C}^{M\times M}, \tag{5.6.4}$$

so that the following structural conditions are satisfied.

(a) We have $r_j = j$ for $j \in \mathbb{Z}$, which implies $r_{\min} = -\infty$ and $r_{\max} = \infty$.

(b) We have $A_j(t) = \tilde{A}_j$ for all $t \in \mathbb{R}$ whenever $|j| \geq K_{\text{const}}$.

(c) We have $\mathcal{K}(\xi; t) = \tilde{\mathcal{K}}(\xi)$ for all $t \in \mathbb{R}$ whenever $|\xi| \geq K_{\text{const}}$.

Remark 5.6.4. The assumption (hB) can be relaxed by assuming that there exists a basis for $\mathbb{C}^M$ on which the matrices $\tilde{A}_j$ for $j \leq -K_{\text{const}}$ are diagonal, together with a separate basis on which the matrices $\tilde{A}_j$ for $j \geq K_{\text{const}}$ are diagonal. However, for notational simplicity, we do not pursue such an approach.

Remark 5.6.5. The condition (hB) can be relaxed to include shifts r_j with $|r_j| < K_{\text{const}}$ that are not equidistant. In addition, there does not need to be any limit on the number of these small shifts. However, for notational simplicity we do not pursue such a level of generality.

We divide our discussion into several scenarios for the unbounded coefficients that we each discuss in turn. Our general results are formulated at the end of this subsection.

5.6.1.1 Bounded shifts and compact support

The methods from [133] can be applied almost directly when the nonlocal terms all have finite range, except that we need to take care of accumulation points of the shifts. In any case, it is straightforward to formulate the appropriate atomic condition at a point $\tau \in \mathbb{R}$.

Assumption (hFin). We have $|r_{\min}| + r_{\max} < \infty$ and there is a small $\delta > 0$ so that the convolution kernel $\mathcal{K}(\cdot; t)$ is supported in the interval $[r_{\min} + \delta, r_{\max} - \delta]$ for each

$t \in \mathbb{R}$. In addition, neither $r_{\min}$ nor $r_{\max}$ is an accumulation point of the set of shifts $\mathcal{R}$ and there are unique integers $j_{\min}, j_{\max}$ that satisfy

$$r_{\min} = r_{j_{\min}}, \qquad r_{\max} = r_{j_{\max}}. \tag{5.6.5}$$

Finally, we have $\det\big(A_{j_{\min}}(t)\big) \neq 0$ for a dense set of $t \in [\tau + r_{\min}, \tau - r_{\min}]$, together with $\det\big(A_{j_{\max}}(t)\big) \neq 0$ for a dense set of $t \in [\tau - r_{\max}, \tau + r_{\max}]$.

5.6.1.2 Unbounded shifts and compact support

We here consider the case where the discrete shifts are unbounded, but the convolution kernels all have finite support. For convenience, we formulate this as an assumption.

Assumption (hSh1). Assumption (hB) is satisfied. In addition, $\mathcal{K}(\cdot; t)$ is supported in the interval $[-K_{\mathrm{const}}, K_{\mathrm{const}}]$ for each $t \in \mathbb{R}$.

Our approach here exploits the functional analytic framework of cyclic vectors for the backward shift operator on ℓ^2, which was first described in [54]. This framework allows us to find sufficient conditions under which the nontriviality condition (HKer) holds and the Hale inner product is nondegenerate for exponentially decaying functions. Reversely, we also provide a condition that guarantees the Hale inner product to be degenerate, even for exponentially decaying functions; see Proposition 5.6.6 below.

Let us first collect the necessary terminology. We consider the backward shift operator S on the sequence space $\ell^2(\mathbb{N}_0; \mathbb{C})$, defined by

$$S : \ell^2(\mathbb{N}_0; \mathbb{C}) \to \ell^2(\mathbb{N}_0; \mathbb{C}), \qquad (a_n)_{n \geq 0} \mapsto (a_n)_{n \geq 1}. \tag{5.6.6}$$

We call a vector $a = (a_n)_{n \geq 0} \in \ell^2(\mathbb{N}_0; \mathbb{C})$ *cyclic* if the span of the set $\{S^N a : N \geq 0\}$ is dense in $\ell^2(\mathbb{N}_0; \mathbb{C})$. Our main condition here demands that the diagonal elements of the matrices $\tilde{A}_j$ can be used to form such cyclic sequences. Our first result shows that this is in fact essential for the nondegeneracy of the Hale inner product.

Assumption (hSh2). Upon writing $j_n = K_{\mathrm{const}} + n$ together with

$$\alpha^{(k)} = \big(\tilde{A}^{(k,k)}_{-j_n}\big)_{n \geq 0} \subset \ell^2(\mathbb{N}_0; \mathbb{C}) \qquad \beta^{(k)} = \big(\tilde{A}^{(k,k)}_{j_n}\big)_{n \geq 0} \subset \ell^2(\mathbb{N}_0; \mathbb{C}), \tag{5.6.7}$$

the sequences $\alpha^{(k)}$ and $\beta^{(k)}$ are cyclic for the backwards shift operator for any $1 \leq k \leq M$.

Proposition 5.6.6 (see §5.6.5)**.** *Assume that (HA), (HK) and (HH) and (hSh1) are all satisfied. If the cyclicity condition (hSh2) is not satisfied, then there exists a nonzero function $\psi \in Y$ that decays exponentially and satisfies $\langle \psi, \phi \rangle_\tau = 0$ for every $\phi \in X$ and each $\tau \in \mathbb{R}$.*

For the backward shift operator on $\ell^2(\mathbb{N}_0; \mathbb{C})$, the criterion for an exponentially decaying sequence to be cyclic can be made explicit; see §5.6.4. This allows us to formulate two results that can be used to verify (hSh2).

Lemma 5.6.7 (see §5.6.4)**.** *Assume that (HA), (HK) and (HH) and (hSh1) are all satisfied. Consider the functions $f^{(k)}$ and $g^{(k)}$ that are defined on their natural domain by*

$$f^{(k)}(z) \quad = \quad \sum_{j=K_{\mathrm{const}}}^{\infty} \tilde{A}_{-j}^{(k,k)} z^j, \qquad g^{(k)}(z) \quad = \quad \sum_{j=K_{\mathrm{const}}}^{\infty} \tilde{A}_{j}^{(k,k)} z^j. \tag{5.6.8}$$

Then the cyclicity condition (hSh2) is satisfied if and only if the functions $f^{(k)}$ and $g^{(k)}$ are not rational functions for any $1 \le k \le M$.

Lemma 5.6.8 (see §5.6.4)**.** *Assume that (HA), (HK) and (HH) and (hSh1) are all satisfied and consider the sequences $\alpha^{(k)}$ and $\beta^{(k)}$ defined in* (5.6.7)*. Then the sets $\{S^N\alpha^{(k)} : N \ge 0\}$ and $\{S^N\beta^{(k)} : N \ge 0\}$ are both infinite dimensional for each $1 \le k \le M$ if and only if the cyclicity condition (hSh2) is satisfied.*

5.6.1.3 Bounded shifts, unbounded support

We now consider the reverse of the setting discussed in §5.6.1.2. In particular, we assume that the discrete shifts are bounded.

Assumption (hCyc1). Assumption (hB) is satisfied, with $\tilde{A}_j = 0$ whenever $|j| \ge K_{\mathrm{const}}$.

In this case, one is interested in the translation semigroup $\{S_t\}_{t\ge 0}$ on the space L^1, which acts as

$$(S_t f)(s) = f(s+t) \tag{5.6.9}$$

for $f \in L^1\big([0,\infty);\mathbb{C}\big)$. A function $f \in L^1\big([0,\infty);\mathbb{C}\big)$ is said to be *cyclic* for the translation semigroup if $\mathrm{span}\{S_t f : t \ge 0\}$ is dense in $L^1\big([0,\infty);\mathbb{C}\big)$. We impose the following counterpart to (hSh2), which will allow us to establish (HKer) together with the non-degeneracy of the Hale inner product for bounded functions.

Assumption (hCyc2). For any $1 \le k \le M$, the functions

$$f^{(k)}(s) \quad = \quad \tilde{\mathcal{K}}(K_{\mathrm{const}} + s)^{(k,k)}, \qquad\qquad g^{(k)}(s) \quad = \quad \tilde{\mathcal{K}}(-K_{\mathrm{const}} - s)^{(k,k)} \tag{5.6.10}$$

are cyclic for the translation semigroup on $L^1\big([0,\infty);\mathbb{C}\big)$.

It is well-known that there exist kernels that satisfy (hCyc2) and (HK), see Lemma 5.6.15 below. In addition, translates of such kernels remain cyclic. However, we are unaware of any criterion to explicitely characterize them. This prevents us from formulating a result analogous to Lemma 5.6.7.

5.6.1.4 Positive-definite coefficients

Our final scenario requires information on the sign of the coefficient functions (5.6.4) and the kernel elements in $\mathcal{B}^*$. Such information can typically be obtained by applying Krein-Rutman type arguments, see for example [39, 110, 131]. In each of these examples the kernels $\mathcal{B}$ and $\mathcal{B}^*$ are at most one-dimensional. Notice that our main condition here is weaker than the requirements formulated in Proposition 5.2.2. For convenience we split the conditions on the coefficients and the kernels into separate assumptions.

Assumption (hPos1). Assumption (hB) is satisfied and the matrices (5.6.4) are all positive semidefinite. Finally, at least one of the following two conditions holds.

(a) For each $m \geq K_{\text{const}}$ there exist $i \geq m$ and $j \leq -m$ for which the matrices $\tilde{A}_i$ and $\tilde{A}_j$ are positive definite.

(b) The map $s \mapsto \tilde{\mathcal{K}}(s)$ is continuous on $(-\infty, -K_{\text{const}}] \cup [K_{\text{const}}, \infty)$. In addition, for each $m \geq K_{\text{const}}$ there exists $s \geq m$ and $r \leq -m$ for which the matrices $\tilde{\mathcal{K}}(s)$ and $\tilde{\mathcal{K}}(r)$ are positive definite.

Assumption (hPos2). The adjoint kernel satisfies $\mathcal{B}^* = \{0\}$ or $\mathcal{B}^* = \text{span}\{b\}$ for some nonnegative function b.

In Proposition 5.6.10 below, we show that the nontriviality condition (HKer) is satisfied if (hPos1) holds, while (hPos2) holds both for the system (5.2.1) as well as its adjoint (5.6.3). On the other hand, the left-nondegeneracy of the Hale inner product follows from the positivity condition (hPos1) without any additional assumptions on $\mathcal{B}$ or $\mathcal{B}^*$.

5.6.1.5 Summary of results

Our main results for this subsection can now be formulated as follows.

Proposition 5.6.9 (see §5.6.5)**.** *Assume that (HA), (HK) and (HH) are satisfied. Then we have the following implications.*

(i) If the atomic condition (hFin) is satisfied at some point $\tau \in \mathbb{R}$, then the Hale inner product $\langle \cdot, \cdot \rangle_\tau$ is left-nondegenerate at τ for functions in Y and right-nondegenerate at τ for functions in X.

(ii) If the cyclicity conditions (hSh1) and (hSh2) are satisfied, then at each $\tau \in \mathbb{R}$ the Hale inner product $\langle \cdot, \cdot \rangle_\tau$ is left-nondegenerate and right-nondegenerate for exponentially decaying functions.

(iii) If the cyclicity conditions (hCyc1) and (hCyc2) are satisfied, then at each $\tau \in \mathbb{R}$ the Hale inner product $\langle \cdot, \cdot \rangle_\tau$ is left-nondegenerate for functions in Y and right-nondegenerate for functions in X.

(iv) If the positivity condition (hPos1) is satisfied, then at each $\tau \in \mathbb{R}$ the Hale inner product $\langle \cdot, \cdot \rangle_\tau$ is left-nondegenerate and right-nondegenerate for nonnegative functions.

In each of the cases (i)-(iii), the quantity in (5.2.30) satisfies $\beta(\tau) = \dim B^*(\tau)$. This also holds for case (iv) provided that the positivity condition (hPos2) is satisfied.

Proposition 5.6.10 (see §5.6.6)**.** *Assume that (HA), (HK) and (HH) are satisfied. Then we have the following implications.*

(i) If the atomic condition (hFin) is satisfied at each $\tau \in \mathbb{R}$, then the nontriviality condition (HKer) is satisfied for the system (5.2.1).

(ii) If the cyclicity conditions (hSh1) and (hSh2) are satisfied, then the nontriviality condition (HKer) is satisfied for the system (5.2.1).

(iii) If the cyclicity conditions (hCyc1) and (hCyc2) are satisfied, then the nontriviality condition (HKer) is satisfied for the system (5.2.1).

(iv) If the positivity condition (hPos1) is satisfied and (hPos2) holds both for (5.2.1) and its adjoint (5.6.3), then the nontriviality condition (HKer) is satisfied for the system (5.2.1).

Note that the nontriviality condition (HKer) does not directly imply that the the Hale inner product is nondegenerate in some form. Instead, it enables us construct an explicit complement to the space $S(\tau)$. In particular, the nondegeneracy of the Hale inner product is useful, but not necessary to compute the codimension $\beta(\tau)$.

5.6.2 Examples

In order to illustrate the results above, we consider the infinite-range nonlinear MFDE

$$\begin{array}{rcl}\dot{u}(t) & = & \sum_{k=1}^{\infty}\gamma_k[u(t+k)+u(t-k)-2u(t)]+\int_0^{\infty}\theta(\xi)[u(t+\xi)+u(t-\xi)-2u(t)]d\xi \\ & & +g\big(u(t);a\big),\end{array} \tag{5.6.11}$$

in which the nonlinearity g is given by the cubic nonlinearity

$$g(u;a)=u(1-u)(u-a), \qquad a\in(0,1), \tag{5.6.12}$$

while the sequence γ and the function θ decay exponentially. This MFDE can be interpreted as the travelling wave equation for a nonlocal version of the Nagumo PDE. One is typically interested in the front solutions, which satisfy the limits

$$\lim_{t\to-\infty}\overline{u}(t) \;=\; 0, \qquad \lim_{t\to\infty}\overline{u}(t) \;=\; 1. \tag{5.6.13}$$

Results concerning the existence of such these solutions in a variety of settings can be found in [6, 95, 122, 131]. For our purposes here, we will simply assume such a solution exists and consider the associated linearization of (5.6.11), which is given by

$$\begin{array}{rcl}\dot{u}(t) & = & \sum_{k=1}^{\infty}\gamma_k[u(t+k)+u(t-k)-2u(t)]+\int_0^{\infty}\theta(\xi)\big[u(t+\xi)+u(t-\xi)-2u(t)\big]d\xi \\ & & +g_u(\overline{u}(t);a)u(t).\end{array} \tag{5.6.14}$$

We remark that a simple differentiation automatically yields $\frac{d}{dt}\overline{u}\in\mathcal{B}$.

In this setting, the Hale inner product is given by

$$\begin{array}{rcl}\langle\psi,\phi\rangle_\tau & = & \overline{\psi(0)}\phi(0)+\sum_{k=1}^{\infty}\int_{-k}^{0}\overline{\psi(s+k)}\gamma_k\phi(s)ds-\sum_{k=1}^{\infty}\int_0^k\overline{\psi(s-k)}\gamma_{|k|}\phi(s)ds \\ & & -\int_{\mathbb{R}}\int_0^r\overline{\psi(s-r)}\theta(|r|)\phi(s)dsdr,\end{array} \tag{5.6.15}$$

which is independent of τ and the function $\overline{u}$. With the exception of (hPos2), we can hence investigate the validity of our assumptions and the nondegeneracy of the Hale inner product without any knowledge regarding the wave $\overline{u}$ besides the limits (5.6.13).

For example, we note that (hB) is automatically satisfied with $K_{\mathrm{const}} = 1$ and

$$\tilde{A}_j = \gamma_{|j|} \qquad \tilde{\mathcal{K}}(\xi) = \theta(|\xi|) \tag{5.6.16}$$

for $|j| \geq 1$ and $\xi \neq 0$. In addition, we have

$$A_0(t) = -2 \sum_{k=1}^{\infty} \gamma_k - 2 \int_0^{\infty} \theta(\xi) d\xi + g_u(\overline{u}(t); a). \tag{5.6.17}$$

In particular, it is clear that (HA) and (HK) hold. However, one needs additional information on the coefficients in order to verify the hyperbolicity assumption (HH).

We consider various choices for γ and θ in our discussion below. In each case we are able to distinguish whether or not the Hale inner product is degenerate. For each of the two degenerate cases, we construct an explicit nontrivial function $\psi \in Y$ for which $\langle \psi, \phi \rangle_\tau = 0$ for all $\phi \in X$ and all $\tau \in \mathbb{R}$. However, we emphasize again that this does not prevent us from showing that (HKer) holds.

5.6.2.1 Positive coefficients

Consider the system (5.6.14) and suppose that the coefficients $\{\gamma_k\}_{k \geq 1}$ and the convolution kernels $\theta(\xi)$ are positive. The bistablity of the nonlinearity g then allows us to conclude that the hyperbolicity condition (HH) is satisfied. In addition, (hPos1) holds and hence the Hale inner product is nondegenerate for nonnegative functions.

These positivity conditions imply that a comparison principle holds for (5.6.14). In such a setting, one can typically derive that the kernels $\mathcal{B}$ and $\mathcal{B}^*$ are both one-dimensional and spanned by a strictly positive function. For example, the wave $\overline{u}$ is typically monotonically increasing and the associated derivative $\frac{d}{dt}\overline{u}$ spans $\mathcal{B}$ and is strictly positive. Results of this type have been proven in various settings, see for example [7, 8, 38]. In each case, the system (5.6.14) together with its adjoint (5.6.3) satisfy (hPos2). In particular, the nontriviality condition (HKer) holds.

5.6.2.2 noncyclic shift coefficients

Consider the system (5.6.14) with $\theta(\xi) = 0$ for each $t \in \mathbb{R}$ and $\gamma_k = e^{-k}$ for $k \geq 1$. This system satisfies (hSh1). Since the coefficients $\{\gamma_k\}_{k \geq 1}$ are positive, the results from §5.6.2.1 show that (HH) is satisfied and that the Hale inner product for the system (5.6.14) is nondegenerate for nonnegative functions.

However, it is easy to see that

$$\sum_{k \geq 1} \gamma_k z^k = \frac{z}{e - z}, \tag{5.6.18}$$

which is a rational function. Hence, this system does not satisfy (hSh2) on account of Lemma 5.6.7. Alternatively, letting S denote the backwards shift operator on $\ell^2(\mathbb{N}_0;\mathbb{C})$, the sequence $\alpha = (\gamma_k)_{k\geq 1}$ satisfies $S^N\alpha = e^{-N}\alpha$ for any $N \geq 0$. In particular, the set $\text{span}\{S^N\alpha : N \geq 0\}$ is one-dimensional, which in view of Lemma 5.6.8 again shows that (hSh2) is not satisfied. In particular, Proposition 5.6.6 implies that the Hale inner product is not nondegenerate for all exponentially decaying functions.

To make this more explicit, we consider the continuous, bounded function $\psi : \mathbb{R} \to \mathbb{R}$ that has

$$\psi(s) = 0 \text{ for } s \leq 1, \qquad \psi\Big(\frac{3}{2}\Big) = 1, \qquad \psi\Big(\frac{5}{2}\Big) = -e, \qquad \psi(s) = 0 \text{ for } s \geq 3 \tag{5.6.19}$$

and is linear in the missing segments. This choice is motivated by the fact that

$$\beta = (1, -e, 0, 0, ...) \in \ell^2(\mathbb{N}_0;\mathbb{C}) \tag{5.6.20}$$

is perpendicular to the set $\text{span}\{S^N\alpha : N \geq 0\}$ and ensures that

$$\sum_{k=m}^{\infty} \psi\big(\tilde{s} + k + 1 - m\big)\gamma_k \;=\; \tilde{s}\big(e^{-m} - e \cdot e^{-(m+1)}\big) \;=\; 0, \tag{5.6.21}$$

for any $m \in \mathbb{Z}_{\geq 1}$ and any $\tilde{s} \in [0, 1)$. For an arbitrary $s \leq 0$ we make the decomposition

$$s \;=\; \tilde{s} + 1 - m \tag{5.6.22}$$

for some integer $m \geq 1$ and $\tilde{s} \in [0, 1)$. Applying (5.6.21), we now compute

$$\sum_{k\geq 1-s} \psi(s + k)\gamma_k \;=\; \sum_{k\geq m-\tilde{s}} \psi(\tilde{s} + k + 1 - m)\gamma_k \;=\; 0 \tag{5.6.23}$$

since the final sum in fact ranges over $k \geq m$.

Since $\psi(s) = 0$ for $s \leq 1$, the Hale inner product reduces to

$$\begin{aligned} \langle \psi, \phi\rangle_\tau \;&=\; \sum_{k=1}^{\infty} \int_{-k}^{0} \psi(s + k)\gamma_k\phi(s)ds \\ &=\; \sum_{k=1}^{\infty} \int_{-k+1}^{0} \psi(s + k)\gamma_k\phi(s)ds, \end{aligned} \tag{5.6.24}$$

for $\phi \in C_b(\mathbb{R})$ and $\tau \in \mathbb{R}$. The dominated convergence theorem allows us to interchange the sum and the infinite integral, which yields

$$\langle \psi, \phi\rangle_\tau \;=\; \int_{-\infty}^{0} \sum_{k\geq 1-s} \psi(s + k)\gamma_k\phi(s)ds \;=\; 0, \tag{5.6.25}$$

for any $\phi \in C_b(\mathbb{R})$ and $\tau \in \mathbb{R}$. Since $\gamma_k > 0$ for any $k \in \mathbb{Z}_{\geq 1}$, this example shows that a naive generalization of the atomic condition (hFin) is not sufficient to establish the nondegeneracy of the Hale inner product.

5.6.2.3 noncyclic convolution kernel

Consider the system (5.6.14) with $\theta(\xi) = \exp(-\xi)$ and $\gamma_k = 0$ for $k \geq 1$. This system satisfies (hCyc1). Since the kernel θ is positive, the results from §5.6.2.1 again show that (HH) is satisfied and that the Hale inner product for the system (5.6.14) is non-degenerate for nonnegative functions.

However, the identity

$$\theta(t+\xi) \quad = \quad \exp(-t)\theta(\xi), \qquad\qquad (\xi, t) \in R^2_{\geq 0} \tag{5.6.26}$$

directly implies that $\mathrm{span}\{\theta(\cdot + t) : t \geq 0\}$ is one dimensional in $L^1\big([0,\infty);\mathbb{C}\big)$. In particular, the cyclicity condition (hCyc2) fails to be satisfied. While we cannot appeal to a general result here, we can show by hand that the Hale inner product is degenerate for an exponentially decaying function.

To this end, we consider the bounded, continuous function $\psi : \mathbb{R} \to \mathbb{R}$ that has

$$\psi(s) = 0 \text{ for } s \leq 0, \qquad \psi(1) = -1, \qquad \psi(2) = 0, \qquad \psi(3) = e^2, \qquad \psi(s) = 0 \text{ for } s \geq 4 \tag{5.6.27}$$

and is linear in the missing segments. By construction, the identity

$$\int_0^\infty \psi(r)\theta(r-s)dr \quad = \quad \exp(s)\int_0^\infty \psi(r)\theta(r)dr \quad = \quad 0 \tag{5.6.28}$$

holds for any $s \leq 0$. For any $\phi \in C_b(\mathbb{R})$ we can again use the dominated convergence theorem to compute

$$\begin{aligned}
\langle \psi, \phi \rangle_\tau &= \psi(0)\phi(0) - \int_{\mathbb{R}}\int_0^r \psi(s-r)^\dagger \theta(|r|)\phi(s)dsdr \\
&= -\int_0^\infty \int_{-r}^0 \psi(s+r)^\dagger \theta(r)\phi(s)dsdr \\
&= -\int_{-\infty}^0 \int_0^\infty \mathbb{1}_{\{s\in[-r,0]\}}\psi(s+r)^\dagger\theta(r)\phi(s)drds \\
&= -\int_{-\infty}^0 \int_{-s}^\infty \psi(s+r)^\dagger\theta(r)\phi(s)drds \\
&= -\int_{-\infty}^0 \int_0^\infty \psi(r)^\dagger\theta(r-s)\phi(s)drds \\
&= 0
\end{aligned} \tag{5.6.29}$$

for any $\tau \in \mathbb{R}$.

5.6.2.4 Cyclic shifts with mixed coefficients

For our final example, we choose $\theta = 0$ and consider a sequence γ that admits Gaussian decay. In particular, we write

$$\gamma_k \quad = \quad \tfrac{1}{h^2}c_k \exp(-k^2), \qquad\qquad h > 0, \tag{5.6.30}$$

for some bounded sequence $\{c_k\}_{k\geq 1}$ that can have both positive and negative elements, but must be uniformly bounded away from zero. In particular, (hSh1) is satisfied, but this may not hold for the positivity condition (hPos1). In order to verify the hyperbolicity condition (HH), it suffices to impose the restriction

$$\sum_{k>0} c_k \exp(-k^2)\Big(1-\cos(kz)\Big) \quad > \quad 0, \qquad\qquad z\in(0,2\pi); \tag{5.6.31}$$

see [150, Lem. 5.6]. This can be interpreted as the statement that the sum in (5.6.14) is spectrally similar to the Laplacian.

We now set out to establish the cyclicity condition (hSh2) by appealing to Lemma 5.6.8. Recalling the backward shift operator (5.6.6), we consider the vector $e = (e_n)_{n\geq 0} \in \ell^2(\mathbb{N}_0;\mathbb{C})$ given by

$$e_{k-1} \quad = \quad c_k \exp(-k^2), \qquad\qquad k\geq 1 \tag{5.6.32}$$

and set out to show that the set

$$\mathcal{A} \quad := \quad \mathrm{span}\{S^N e : N\geq 0\} \tag{5.6.33}$$

is an infinite dimensional subspace of $\ell^2(\mathbb{N}_0;\mathbb{C})$.

Arguing by induction, we pick $\ell\geq 1$ and assume that the vectors $e, Se, ..., S^{\ell-1}e$ are linearly independent. Suppose now that we have a nonzero multiplet $(\lambda_0, ..., \lambda_\ell)\in\mathbb{C}^{\ell+1}$ for which

$$\sum_{i=0}^{\ell} \lambda_i S^i e \quad = \quad 0. \tag{5.6.34}$$

Let $0\leq i_* < \ell$ be the smallest integer with $\lambda_{i_*}\neq 0$. Our assumption on c implies that the sequence $\{|\frac{c_k}{c_{k+1}}|\}_{k\geq 1}$ is uniformly bounded away from zero, which implies that the quotient

$$|\tfrac{e_{k-1}}{e_k}| \quad = \quad |\tfrac{c_k}{c_{k+1}}|\exp(2k+1) \tag{5.6.35}$$

grows to infinity as $k\to\infty$. In particular, by picking a sufficiently large index $K\gg 1$ we obtain the bound

$$\begin{array}{rcl} |\sum_{i=i_*+1}^{\ell}\lambda_i(S^ie)_K| & \leq & \sum_{i=i_*+1}^{\ell}|\lambda_i e_{K+i}| \\ & < & |\lambda_{i_*}e_{K+i_*}| \\ & = & |\lambda_{i_*}(S^{i_*}e)_K|, \end{array} \tag{5.6.36}$$

which contradicts the K-th component of the identity (5.6.34). In particular, Proposition 5.6.9 yields that there is no exponentially decaying $\psi\in Y$ that has $\langle\psi,\phi\rangle_\tau = 0$ for each $\phi\in X$.

If $h > 0$ is sufficiently small, then the existence of a travelling front solution for (5.6.11) is guaranteed by [6, Thm. 1]. One can subsequently use Proposition 5.6.10 to conclude that the nontriviality condition (HKer) is satisfied.

5.6.3 (Co)-dimension counting

The main goal of this subsection is to establish the identities (5.6.1) concerning the dimensions of $B(\tau)$ and $B^*(\tau)$ and the codimension of $S(\tau)$. The remainder of the statements in Proposition 5.6.3 follow readily from these computations, using the main results in §5.2. We aim to use as little information as possible, providing partial results under weaker conditions.

Lemma 5.6.11. *Assume that (HA), (HK) and (HH) are satisfied. Fix $\tau \in \mathbb{R}$ and suppose first that the Hale inner product is left-nondegenerate at τ for functions in $B^*(\tau)$. Then the identity*

$$\beta(\tau) \quad = \quad \dim B^*(\tau) \tag{5.6.37}$$

holds. Alternatively, if the nontriviality condition (HKer) is satisfied, then the identity (5.6.37) is valid for all $\tau \in \mathbb{R}$.

Proof. In the first case, this follows directly from the characterisation of $S(\tau)$ given by (5.2.41). In the second case, the statement for $\tau \geq 0$ follows from the direct sum decomposition (5.5.9) and the identities in (5.5.3). Using symmetry arguments this can be extended to $\tau < 0$. ∎

Lemma 5.6.12. *Assume that (HA), (HK) and (HH) are satisfied. Fix $\tau \in \mathbb{R}$ and suppose first that any nonzero $d \in \mathcal{B} \cup \mathcal{B}^*$ does not vanish on $(-\infty, \tau]$ and does not vanish on $[\tau, \infty)$. Then we have the identities*

$$\dim B(\tau) \quad = \quad \dim \mathcal{B}, \qquad\qquad \dim B^*(\tau) \quad = \quad \dim \mathcal{B}^*. \tag{5.6.38}$$

In particular, if the nontriviality condition (HKer) holds then (5.6.38) is valid for each $\tau \in \mathbb{R}$.

Proof. Since the statements hold trivially if $|r_{\min}| = r_{\max} = \infty$ on account of Lemma 5.3.4, we will use symmetry to assume without loss that $r_{\max} < \infty$. Arguing by contradiction to establish the first identity, let us consider a nontrivial kernel element $x \in \mathcal{B}$ that has $x_\tau = 0$. If $r_{\min} = -\infty$, this means that x vanishes identically on $D_\tau^\ominus$ and hence $(-\infty, \tau]$, violating our assumption. On the other hand, if $r_{\min} > -\infty$ we can assume without loss that x does not vanish on $(r_{\max}, \infty)$. Upon introducing the new function

$$\tilde{x}(t) \quad = \quad \begin{cases} x(t), & t \geq \tau + r_{\min}, \\ 0, & t < \tau + r_{\min}, \end{cases} \tag{5.6.39}$$

we see that $\tilde{x}$ is a nontrivial element of $\mathcal{B}$ that vanishes on $D_\tau^\ominus$, again violating our assumption. The second identity in (5.6.38) can be obtained in a similar fashion. ∎

Lemma 5.6.13. *Assume that (HA), (HK) and (HH) are satisfied and that $r_{\min} < 0 < r_{\max}$. Suppose that for each $\tau \in \mathbb{R}$ the Hale inner product is left-nondegenerate for functions in $B^*(\tau)$. Then we have the identity*

$$\dim B^*(\tau) \quad = \quad \dim \mathcal{B}^* \tag{5.6.40}$$

for any $\tau \in \mathbb{R}$. Similarly, if for each $\tau \in \mathbb{R}$ the Hale inner product is right-nondegenerate for functions in $B(\tau)$, then the identity

$$\dim B(\tau) \quad = \quad \dim \mathcal{B} \tag{5.6.41}$$

holds for each $\tau \in \mathbb{R}$.

Proof. Both identities follow trivially from Lemma 5.3.4 if $|r_{\min}| = r_{\max} = \infty$. By symmetry we only consider the identity (5.6.40). Suppose that (5.6.40) fails, allowing us to pick a nonzero $y \in \mathcal{B}^*$ that has $y^\tau = 0$ for some $\tau \in \mathbb{R}$. Possibly after increasing τ, we may assume by symmetry that $r_{\min} > -\infty$ and that there exists a small $0 < \varepsilon < |r_{\min}|$ so that

$$y(\tau - r_{\min} + \delta) \quad \neq \quad 0 \tag{5.6.42}$$

holds for each $\delta \in (0, \varepsilon)$. In particular, $0 \neq y^{\tau+\varepsilon} \in B^*(\tau + \varepsilon)$, so by the left-nondegeneracy of the Hale inner product at $\tau + \varepsilon$, we can pick $\phi \in X$ with

$$\langle y^{\tau+\varepsilon}, \phi \rangle_{\tau+\varepsilon} \quad \neq \quad 0. \tag{5.6.43}$$

Without loss, we can assume that ϕ is differentiable, allowing us to pick a differentiable function $x \in C_b(\mathbb{R})$ that has $\phi = x_{\tau+\varepsilon}$. On account of Lemma 5.3.12 we can compute

$$\tfrac{d}{dt} \langle y^t, x_t \rangle_t \quad = \quad y^*(t)[\Lambda x](t) + [\Lambda^* y](t) x(t) \quad = \quad 0 \tag{5.6.44}$$

for any $t \in (\tau - r_{\max}, \tau - r_{\min})$, since $y^\tau = 0$ and since $y \in \mathcal{B}^*$. As such, $\langle y^t, x_t \rangle_t$ is constant on $(\tau - r_{\max}, \tau - r_{\min}]$. Since $y^\tau = 0$, it follows that $\langle y^\tau, x_\tau \rangle_\tau = 0$. However, this yields the identity

$$0 \quad = \quad \langle y^{\tau+\varepsilon}, x_{\tau+\varepsilon} \rangle_{\tau+\varepsilon} \quad = \quad \langle y^{\tau+\varepsilon}, \phi \rangle_{\tau+\varepsilon}, \tag{5.6.45}$$

which contradicts (5.6.43). ∎

Proof of Proposition 5.6.3. We first aim to establish (5.6.1). If the nontriviality condition (HKer) holds, this follows by combining Lemmas 5.6.11 and Lemma 5.6.12. Alternatively, if (b) holds, then (5.6.1) follows by combining Proposition 5.6.9 with Lemmas 5.3.4 and 5.6.11. Finally, if (c) holds, then (5.6.1) follows by combining Proposition 5.6.9 with Lemmas 5.6.11 and 5.6.13.

Turning to the Fredholm indices, we remark that the right-hand side of (5.4.1) is now constant in τ. Since both $\operatorname{ind}(\pi^+_{P(\tau)})$ and $\operatorname{ind}(\pi^-_{Q(\tau)})$ are upper semi-continuous by Proposition 5.4.2, both these factors must be constant as well. By Theorem 5.2.5 the inclusions $\widehat{P}(\tau) \subset P(\tau)$ and $\widehat{Q}(\tau) \subset Q(\tau)$ have constant codimension $\dim B(\tau) = \dim \mathcal{B}$. Hence the indices $\operatorname{ind}(\pi^+_{\widehat{P}(\tau)})$ and $\operatorname{ind}(\pi^-_{\widehat{Q}(\tau)})$ are also constant. Moreover, these four subspaces vary continuously in τ. Finally, the identity (5.6.2) follows from (5.4.1) and (5.6.1), using the value of $\operatorname{ind}(\Lambda)$ given in Proposition 5.2.1. ∎

Proof of Corollaries 5.2.3 and 5.2.7. These results follow directly from Proposition 5.6.3. ∎

5.6.4 Cyclic coefficients

In this subsection, we collect several results from the literature concerning the cyclicity of the backwards shift operator and the translation semigroup. In addition, we translate these results into our setting and explore their consequences.

Proposition 5.6.14 ([54, Thm. 2.2.4, Rem. 2.2.6])**.** *Consider a sequence $\alpha = (\alpha_n)_{n\geq 0} \in \ell^2(\mathbb{N}_0;\mathbb{C})$ that decays exponentially and write f for the associated function*

$$f(z) \quad = \quad \sum_{n=0}^{\infty} \alpha_n z^n, \tag{5.6.46}$$

defined on its natural domain in $\mathbb{C}$. Then the sequence α is cyclic for the backwards shift operator (5.6.6) *if and only if f is not a rational function. In fact, if α is not cyclic, then* $\mathrm{span}\{S^N\alpha : N \geq 0\}$ *is finite dimensional in $\ell^2(\mathbb{N}_0;\mathbb{C})$.*

Lemma 5.6.15. *For any $T > 0$ and any function $f \in L^1\big([0,\infty);\mathbb{C}\big)$ that is cyclic for the translation group $(S_t)_{t\geq 0}$ defined in* (5.6.9)*, the shifted function $s \mapsto f(s+T)$ is also cyclic for $(S_t)_{t\geq 0}$. In addition, for any $\tilde{\eta} > 0$, there exists a function $f \in L^1_{\tilde{\eta}}\big([0,\infty);\mathbb{C}\big)$ that is cyclic for the translation group $(S_t)_{t\geq 0}$. In particular, there exists a convolution kernel that satisfies both (HK) and (hCyc2).*

Proof. The first statement follows directly from [134, Lem. 1]. Turning to the existence claim, we fix $\tilde{\eta} > 0$ and let $(T_t)_{t\geq 0}$ be the translation semigroup on $L^1_{\tilde{\eta}}\big([0,\infty);\mathbb{C}\big)$. It follows from [135, Thm. 1(i)] that there exists $f \in L^1_{\tilde{\eta}}\big([0,\infty);\mathbb{C}\big)$ that is supercyclic for $(T_t)_{t\geq 0}$, which means that $\{\lambda S(t) f : t \geq 0, \lambda \in \mathbb{R}\}$ is dense in $L^1_{\tilde{\eta}}\big([0,\infty);\mathbb{C}\big)$. Such a function is clearly also cyclic for $(T_t)_{t\geq 0}$ (with respect to the norm $\|\cdot\|_{\tilde{\eta}}$). We write

$$\mathcal{D} \quad = \quad \mathrm{span}\{T(t)f : t \geq 0\} \quad = \quad \mathrm{span}\{S(t)f : t \geq 0\}. \tag{5.6.47}$$

Since $L^1_{\tilde{\eta}}\big([0,\infty);\mathbb{C}\big)$ contains all compactly supported functions, we see that $L^1_{\tilde{\eta}}\big([0,\infty);\mathbb{C}\big)$ is dense in $L^1\big([0,\infty);\mathbb{C}\big)$ with respect to the usual norm $\|\cdot\|_{L^1}$. Hence it is sufficient to show that $\mathcal{D}$ is dense in $L^1_{\tilde{\eta}}\big([0,\infty);\mathbb{C}\big)$ with respect to $\|\cdot\|_{L^1}$. Fix any $g \in L^1_{\tilde{\eta}}\big([0,\infty);\mathbb{C}\big)$ and let $\{g_n\}_{n\geq 1}$ be a sequence in $\mathcal{D}$ with

$$\lim_{n\to\infty} \|g_n - g\|_{\tilde{\eta}} \quad = \quad 0. \tag{5.6.48}$$

For $n \in \mathbb{N}$ we can compute

$$\begin{aligned} \|g_n - g\|_{\tilde{\eta}} &= \int_{\mathbb{R}} e^{\tilde{\eta}|\xi|} |g_n(\xi) - g(\xi)| d\xi \\ &\geq \int_{\mathbb{R}} |g_n(\xi) - g(\xi)| d\xi \\ &= \|g_n - g\|_{L^1}, \end{aligned} \tag{5.6.49}$$

which immediately implies that also $g_n \to g$ in $L^1\big([0,\infty);\mathbb{C}\big)$, as desired. Hence f is cyclic for the translation group $(S_t)_{t\geq 0}$. In particular, the convolution kernel

$$\mathcal{K}(\xi; t) \quad = \quad f(|\xi|) \tag{5.6.50}$$

satisfies both (HK) and (hCyc2). ∎

Lemma 5.6.16. *Let $\{D_n\}_{n\geq 0}$ be an exponentially decaying sequence of $M\times M$ diagonal matrices. Then the following statements are equivalent.*

(i) There exists a nonzero sequence $y \in \ell^2(\mathbb{N}_0;\mathbb{C}^M)$ that satisfies

$$\sum_{n=0}^{\infty} y_n^{\dagger} D_{n+N} \quad = \quad 0 \tag{5.6.51}$$

for each $N \in \mathbb{Z}_{\geq 0}$.

(ii) There exists at least one $1 \leq k \leq M$ for which the sequence $(D_n^{(k,k)})_{n\geq 0}$ is not cyclic for the backwards shift operator on $\ell^2(\mathbb{N}_0;\mathbb{C})$.

In addition, if these statements hold, then the sequence y in (i) can be chosen to decay exponentially. Finally, if these conditions do not hold, then they also do not hold for the shifted sequence $\{D_n\}_{n\geq N}$, for any $N \in \mathbb{Z}_{\geq 0}$.

Proof. As a preparation, we introduce the sequences

$$\alpha^{(k);N} \quad = \quad (\alpha_n^{(k);N})_{n\geq 0} \quad = \quad \big(D_{n+N}^{(k,k)}\big)_{n\geq 0} \in \ell^2(\mathbb{N}_0;\mathbb{C}) \tag{5.6.52}$$

for any $N \geq 0$ and any $1 \leq k \leq M$. In addition, we define the associated subspaces

$$\mathcal{D}^{(k)} \quad = \quad \operatorname{span}\{\alpha^{(k);N} \mid N \geq 0\} \tag{5.6.53}$$

for $1 \leq k \leq M$.

Let us first assume that (i) holds, but that (ii) fails. Then the subspaces $\mathcal{D}^{(k)}$ are all dense in $\ell^2(\mathbb{N}_0;\mathbb{C})$. In addition, our diagonality assumption together with (5.6.51) implies that

$$\Big\langle y^{(k)}, \alpha^{(k);N} \Big\rangle_{\ell^2(\mathbb{N}_0;\mathbb{C})} \quad = \quad 0 \tag{5.6.54}$$

for any $N \in \mathbb{Z}_{\geq 0}$ and $1 \leq k \leq M$ and thus

$$\big\langle y^{(k)}, d \big\rangle_{\ell^2(\mathbb{N}_0;\mathbb{C})} \quad = \quad 0 \tag{5.6.55}$$

for any $d \in \mathcal{D}^{(k)}$ and $1 \leq k \leq M$. Together these two properties yield the contradiction $y = 0$.

Let us now assume that (ii) holds. Then Proposition 5.6.14 implies there exists $1 \leq k_0 \leq M$ for which the subspace $\mathcal{D}^{(k_0)}$ defined in (5.6.53) is finite dimensional, with a basis that consists of exponentially decaying sequences. In particular, we can pick an exponentially decaying sequence $\psi \in \ell^2(\mathbb{N}_0;\mathbb{C})$ that satisfies $\langle \psi, d\rangle_{\ell^2(\mathbb{N}_0;\mathbb{C})} = 0$ for any $d \in \mathcal{D}^{(k_0)}$. Upon writing $y = (0, ..., 0, \psi, 0, ..., 0) \in \ell^2(\mathbb{N}_0;\mathbb{C}^M)$, where ψ takes the k_0^{th} position, we hence see that (5.6.51) is satisfied by construction.

The final statement follows from the characterization in Proposition 5.6.14, which implies that (non)-cyclicity is preserved under translation. Indeed, if the function f defined in (5.6.46) is not a rational function, then the function

$$f_N(z) \quad = \quad z^{-N}\big[f(z) - \sum_{n=0}^{N-1} \alpha_n z^n\big] \tag{5.6.56}$$

associated to the shifted sequence $S^N \alpha$ is also not rational. ■

Proof of Lemmas 5.6.7 and 5.6.8. Both results follow directly from Proposition 5.6.14 and Lemma 5.6.16. ■

5.6.5 Nondegeneracy of the Hale inner product

In this subsection we show how the nondegeneracy of the Hale inner product can be derived from the conditions formulated in §5.6.1. In particular, we establish Propositions 5.6.6 and 5.6.9.

As a convenience, we first connect the right-nondegeneracy properties for the system (5.2.1) to the left-nondegeneracy properties for the adjoint system (5.6.3). This will allows us to focus solely on the left-nondegeneracy of the Hale inner product with respect to functions in $B^*(\tau)$.

Lemma 5.6.17. *Assume that (HA), (HK) and (HH) are satisfied. Fix $\tau \in \mathbb{R}$ and $E \subset X$ with $0 \in E$. Then the Hale inner product for the system (5.2.1) at τ is right-nondegenerate for functions in E if and only if the Hale inner product for the adjoint system (5.6.3) at τ is left-nondegenerate for functions in E.*

Proof. For any $\phi \in X$, $\psi \in Y$ and $\tau \in \mathbb{R}$, the Hale inner product for the adjoint system (5.6.3) is given by

$$\begin{array}{lcl} \langle \phi, \psi \rangle_{\tau}^{\mathrm{adj}} & = & \phi(0)^{\dagger}\psi(0) + \sum_{j=-\infty}^{\infty} \int_{0}^{-r_j} \phi(s+r_j)^{\dagger} A_j(\tau + s - r_j)^{\dagger}\psi(s) ds \\ & & + \int_{\mathbb{R}} \int_{0}^{r} \phi(s-r)^{\dagger} \mathcal{K}(s-r; \tau+s-r)^{\dagger}\psi(s) ds dr. \end{array} \tag{5.6.57}$$

A short computation shows that

$$\begin{array}{lcl} \overline{\langle \phi, \psi \rangle_{\tau}^{\mathrm{adj}}} & = & \psi(0)^{\dagger}\phi(0) - \sum_{j=-\infty}^{\infty} \int_{0}^{r_j} \psi(s-r_j)^{\dagger} A_j(\tau+s-r_j)\phi(s) ds \\ & & - \int_{\mathbb{R}} \int_{0}^{r} \psi(s-r)^{\dagger} \mathcal{K}(r; \tau+s-r)\phi(s) ds dr \\ & = & \langle \psi, \phi \rangle_{\tau}, \end{array} \tag{5.6.58}$$

which directly implies the desired result. ■

We proceed by discussing the cyclicity criteria in introduced in §5.6.1.2 and §5.6.1.3. The following preparatory result will help us to link the discussion in §5.6.4 to the degeneracy properties of the Hale inner product.

Lemma 5.6.18. *Assume that (HA), (HK), (HH) and (hB) are satisfied and fix $\tau \in \mathbb{R}$. Pick any $\psi \in Y$ that does not vanish on D_Y^+ and satisfies $\langle \psi, \phi \rangle_{\tau} = 0$ for every $\phi \in X$. Writing*

$$\sigma = \inf\{s \in D_Y^+ \mid \psi(s) \neq 0\}, \tag{5.6.59}$$

there exist $\varepsilon > 0$ and $N_0 \in \mathbb{Z}_{\geq K_{\mathrm{const}}}$ so that the identity

$$\sum_{j=0}^{\infty} \psi(s+j)^{\dagger} \tilde{A}_{-j-N} + \int_{\sigma-s}^{\infty} \psi(s+r)^{\dagger} \tilde{\mathcal{K}}(-r-N) dr \quad = \quad 0 \tag{5.6.60}$$

holds for almost every $s \in (\sigma, \sigma+\varepsilon)$ and every integer $N \geq N_0$. In addition, if $\tilde{A}_j = 0$ for each $j \leq -N_0$, then we in fact have

$$\int_{0}^{\infty} \psi(\sigma+r)^{\dagger} \tilde{\mathcal{K}}(-r-\theta) dr \quad = \quad 0 \tag{5.6.61}$$

for all (reals) $\theta \geq N_0 + \varepsilon$.

Proof. We first pick an arbitrary $s < 0$ with $s \notin \mathbb{Z}$. Using a sequence of functions supported on small intervals that shrink to the singleton $\{s\}$, we can use (5.2.26) to conclude that

$$\sum_{j<s} \psi(s-j)^{\dagger} A_j(\tau+s-j) + \int_{-\infty}^{s} \psi(s-r)^{\dagger} \mathcal{K}(r; \tau+s-r) dr \quad = \quad 0. \tag{5.6.62}$$

Imposing the further restriction $s \leq -K_{\mathrm{const}}$, this can be rephrased as

$$\sum_{j<s} \psi(s-j)^{\dagger} \tilde{A}_j + \int_{-\infty}^{s} \psi(s-r)^{\dagger} \tilde{\mathcal{K}}(r) dr \quad = \quad 0. \tag{5.6.63}$$

We now choose $\varepsilon > 0$ to be so small that $(\sigma, \sigma+\varepsilon)$ contains no integers. Then for any sufficiently large integer $N \gg 1$, we can combine (5.6.63) together with the definition of σ to conclude that

$$\sum_{j<s-\sigma} \psi(s-j)^{\dagger} \tilde{A}_j + \int_{-\infty}^{s-\sigma} \psi(s-r)^{\dagger} \tilde{\mathcal{K}}(r) dr \quad = \quad 0 \tag{5.6.64}$$

for all $s \in (\sigma - N, \sigma + \varepsilon - N)$. This yields (5.6.60) upon introducing new variables

$$(s', j', r') \quad = \quad (s+N, -j-N, -r-N) \tag{5.6.65}$$

and dropping the primes, noting that $\lceil \sigma - s' \rceil = 0$. The final statement follows from the fact that we no longer need to rule out integer values of s' above, together with the replacement $r \mapsto r + \sigma - s$. ∎

Lemma 5.6.19. *Assume that (HA), (HK) and (HH) are satisfied and fix $\tau \in \mathbb{R}$. Assume moreover that the cyclicity conditions (hSh1)-(hSh2) are satisfied. Then the Hale inner product at τ is left-nondegenerate for exponentially decaying functions.*

Proof. Assume that $\psi \in Y$ decays exponentially and has $\langle \psi, \phi \rangle_{\tau} = 0$ for every $\phi \in X$. Exploiting symmetry, we assume further that ψ does not vanish on D_Y^+ and set out to find a contradiction. Recalling the setting of Lemma 5.6.18 and remembering that $\tilde{\mathcal{K}} = 0$ on $(-\infty, -K_{\mathrm{const}}]$, we obtain from (5.6.60) that the identity

$$\sum_{j=0}^{\infty} \psi(s+j)^{\dagger} \tilde{A}_{-j-N} \quad = \quad 0 \tag{5.6.66}$$

holds for almost every $s \in (\sigma, \sigma + \varepsilon)$ and every $N \geq N_0 \geq K_{\mathrm{const}}$.

By (hSh2) and the invariance of cyclicity under translations, the sequences $(\tilde{A}^{(k,k)}_{-j})_{j \geq N}$ are cyclic for each $1 \leq k \leq M$. In particular, Lemma 5.6.16 implies that that the sequence $\psi(s + \mathbb{N}_0) \in \ell^2(\mathbb{N}_0; \mathbb{C}^M)$ and hence also the first coordinate $\psi(s)$ must vanish for all $s \in (\sigma, \sigma + \varepsilon)$. This contradicts the definition of σ. ■

Proof of Proposition 5.6.6. Assume without loss of generality that the sequence $(\tilde{A}^{(k,k)}_{-j})_{j \geq K_{\mathrm{const}}}$ is not cyclic for the backwards shift operator. Lemma 5.6.16 then allows us to pick an exponentially decaying nonzero sequence

$$y \quad = \quad (y_n)_{n \geq 0} \in \ell^2(\mathbb{N}_0; \mathbb{C}^M) \tag{5.6.67}$$

for which the identity

$$\sum_{j=0}^{\infty} y_j^{\dagger} \tilde{A}_{-j-N} \quad = \quad 0 \tag{5.6.68}$$

holds for all integers $N \geq K_{\mathrm{const}}$.

We now define a continuous, bounded function $\psi : D_Y \to \mathbb{C}^M$ by writing

$$\psi(s) \quad = \quad 0, \qquad\qquad s \in (-\infty, K_{\mathrm{const}}), \tag{5.6.69}$$

together with

$$\psi(j) \quad = \quad 0, \qquad \psi(j + \tfrac{1}{2}) \quad = \quad y_{j - K_{\mathrm{const}}}, \qquad\qquad j \in \mathbb{Z}_{\geq K_{\mathrm{const}}} \tag{5.6.70}$$

and performing a linear interpolation between these prescribed values. This construction implies that

$$\sum_{j=N}^{\infty} \psi\big(\tilde{s} + j + K_{\mathrm{const}} - N\big)^{\dagger} \tilde{A}_{-j} \quad = \quad \tilde{s} \sum_{j=0}^{\infty} y_j^{\dagger} \tilde{A}_{-N-j} \quad = \quad 0, \tag{5.6.71}$$

for any integer $N \geq K_{\mathrm{const}}$ and any $\tilde{s} \in [0, 1)$.

Let us now consider an arbitrary $s \leq 0$ and make the decomposition

$$s \quad = \quad \tilde{s} + K_{\mathrm{const}} - N \tag{5.6.72}$$

for some integer $N \geq K_{\mathrm{const}}$ and $\tilde{s} \in [0, 1)$. Applying (5.6.71), we now compute

$$\sum_{j \geq K_{\mathrm{const}} - s} \psi(s + j) \tilde{A}_{-j} \quad = \quad \sum_{j \geq N - \tilde{s}} \psi(\tilde{s} + j + K_{\mathrm{const}} - N) \tilde{A}_{-j} \quad = \quad 0 \tag{5.6.73}$$

since the final sum in fact ranges over $j \geq N$.

For any $\phi \in X$, we note that (5.2.26) reduces to

$$\begin{aligned} \langle \psi, \phi \rangle_{\tau} \quad = \quad & - \sum_{j=-\infty}^{\infty} \int_0^{j} \psi(s - j)^{\dagger} A_j(\tau + s - j) \phi(s) ds \\ & - \int_{\mathbb{R}} \int_0^{r} \psi(s - r)^{\dagger} \mathcal{K}(r; t + s - r) \phi(s) ds dr \end{aligned} \tag{5.6.74}$$

since $\psi(0) = 0$. Exploiting (hB), this can be further simplified and recast as

$$\begin{array}{lcl} \langle \psi, \phi \rangle_\tau & = & -\sum\limits_{j=-\infty}^{-K_{\mathrm{const}}} \int\limits_0^j \psi(s-j)^\dagger \tilde{A}_j \phi(s) ds \\ & = & -\sum\limits_{j=K_{\mathrm{const}}}^{\infty} \int\limits_0^{K_{\mathrm{const}}-j} \psi(s+j)^\dagger \tilde{A}_{-j} \phi(s) ds. \end{array} \tag{5.6.75}$$

The dominated convergence theorem allows us to interchange the infinite sum and the integral, which yields

$$\langle \psi, \phi \rangle_\tau \quad = \quad -\int\limits_0^{-\infty} \sum_{j \geq K_{\mathrm{const}}-s} \psi(s+j)^\dagger \tilde{A}_{-j} \phi(s) ds \quad = \quad 0 \tag{5.6.76}$$

on account of (5.6.73). ∎

Lemma 5.6.20. *Assume that (HA), (HK) and (HH) are satisfied and fix $\tau \in \mathbb{R}$. Assume moreover that the cyclicity conditions (hCyc1)-(hCyc2) are satisfied. Then the Hale inner product at τ is left-nondegenerate for functions in Y.*

Proof. Assume that $\psi \in Y$ has $\langle \psi, \phi \rangle_\tau = 0$ for every $\phi \in X$. Exploiting symmetry, we assume further that ψ does not vanish on D_Y^+ and set out to find a contradiction.

We pick $1 \leq k \leq M$ for which $\psi^{(k)}$ does not vanish on D_Y^+. Recalling the setting of Lemma 5.6.18 and remembering that $\tilde{A}_j = 0$ for each $|j| \geq K_{\mathrm{const}}$, we obtain from (5.6.61) that the identity

$$\int\limits_0^\infty \psi(\sigma + r)^\dagger \tilde{\mathcal{K}}(-r-\theta) dr \quad = \quad 0 \tag{5.6.77}$$

holds for every $\theta \geq N + \varepsilon$. We introduce the subspace

$$\mathcal{D} \quad = \quad \mathrm{span}\big\{ t \mapsto \tilde{\mathcal{K}}^{(k,k)}(-t-r) \mid r \geq N + \varepsilon \big\}, \tag{5.6.78}$$

which is dense in $L^1\big([0,\infty);\mathbb{C}\big)$ by (hCyc2) and Lemma 5.6.15. We therefore have

$$\int\limits_0^\infty \psi^{(k)}(\sigma + r)^* f(r) dr \quad = \quad 0 \tag{5.6.79}$$

for every $f \in \mathcal{D}$.

We fix any $f \in L^1\big([0,\infty);\mathbb{C}\big)$ and let $\{f_n\}_{n \geq 1}$ be a sequence in $\mathcal{D}^{(k)}$ with $f_n \to f$. Using (5.6.79) we can estimate

$$\begin{array}{lcl} \Big| \int\limits_0^\infty \psi^{(k)}(\sigma + r)^* f(r) dr \Big| & = & \Big| \int\limits_0^\infty \psi^{(k)}(\sigma + r)^* \big(f(r) - f_n(r)\big) dr \Big| \\ & \leq & \|\psi\|_\infty \int\limits_0^\infty |f(r) - f_n(r)| dr, \end{array} \tag{5.6.80}$$

which converges to 0 as $n \to \infty$. Hence (5.6.79) holds for any $f \in L^1\big([0,\infty);\mathbb{C}\big)$. In particular, we pick $s \in (\sigma, \sigma+\varepsilon)$ for which $\psi^{(k)}(s) \neq 0$ and we let $f \in L^1\big([0,\infty);\mathbb{C}\big)$ be a sufficiently small peak function, centered around $s-\sigma$. This immediately yields

$$\int_0^\infty \psi^{(k)}(\sigma+r)^* f(r)dr \quad \neq \quad 0, \tag{5.6.81}$$

which contradicts (5.6.79). ∎

Lemma 5.6.21. *Assume that (HA), (HK) and (HH) are satisfied and fix $\tau \in \mathbb{R}$. Assume moreover that the positivity condition (hPos1) is satisfied. Then the Hale inner product at τ is left-nondegenerate for nonnegative functions.*

Proof. Assume that $\psi \in Y$ is nonnegative and has $\langle \psi, \phi \rangle_\tau = 0$ for every $\phi \in X$. Exploiting symmetry, we assume further that ψ does not vanish on D_Y^+ and set out to find a contradiction. Recalling the setting of Lemma 5.6.18, we obtain from (5.6.60) that the identity

$$\sum_{j=0}^{\infty} \psi(s+j)^\dagger \tilde{A}_{-j-N} + \int_{\sigma-s}^{\infty} \psi(s+r)^\dagger \tilde{\mathcal{K}}(-r-N)dr \quad = \quad 0 \tag{5.6.82}$$

holds for almost every $s \in (\sigma, \sigma+\varepsilon)$ and every $N \geq N_0 \geq K_{\mathrm{const}}$. In addition, the definition of σ allows us to conclude $\psi(s) > 0$ for $s \in (\sigma, \sigma+\varepsilon)$.

Since the matrices (5.6.4) are all positive semidefinite, we have

$$\big(\psi(s+j)^\dagger \tilde{A}_{-j-N}\big)^{(k)} \quad \geq \quad 0, \qquad\qquad s \in (\sigma, \sigma+\varepsilon) \tag{5.6.83}$$

for all $j \geq 0$, $1 \leq k \leq M$ and $N \geq N_0$, together with

$$\big(\psi(s+r)^\dagger \tilde{\mathcal{K}}(-r-N)\big)^{(k)} \quad \geq \quad 0, \qquad\qquad s \in (\sigma, \sigma+\varepsilon) \tag{5.6.84}$$

for all $r \geq \sigma - s$, $1 \leq k \leq M$ and all $N \geq N_0$. On the other hand, fixing $j = 0$ and $r = 0$, item (a) and (b) in (hPos1) allow us to find $N \geq N_0$ for which one or both of the inequalities (5.6.83)-(5.6.84) are strict. This immediately contradicts (5.6.82). ∎

Lemma 5.6.22. *Assume that (HA), (HK) and (HH) are satisfied. Assume moreover that the atomic condition (hFin) is satisfied at some point $\tau \in \mathbb{R}$. Then the Hale inner product at τ is left-nondegenerate for functions in Y.*

Proof. The proof is identical to that of [133, Prop. 4.16] and, as such, will be omitted. ∎

Proof of Proposition 5.6.9. The statements (i)-(iv) follow from Lemmas 5.6.17, 5.6.19, 5.6.20, 5.6.21 and 5.6.22. The final statement follows from the representation (5.2.30), applying Proposition 5.2.1 for (ii)-(iii) or using the nonnegative $B^*(\tau)$-basis for (iv). ∎

Remark 5.6.23. The conclusion in Lemma 5.6.19 that the Hale inner product is non-degenerate for exponentially decaying functions cannot easily be generalized to bounded functions. Indeed, the key argument is that the sequence $\psi(s+\mathbb{N}_0)^{(k)}$ is perpendicular to a dense subspace of $\ell^2(\mathbb{N}_0;\mathbb{C})$. This sequence is in ℓ^2 itself on account of the exponential decay of ψ and must therefore vanish. However, it *is* possible for nontrivial ℓ^∞ sequences to be perpendicular to a dense subspace of $\ell^2(\mathbb{N}_0;\mathbb{C})$; see the discussion at [1]. In a similar fashion, we do not expect Lemma 5.6.21 to be easily generalizable.

5.6.6 The nontriviality condition (HKer)

In this final subsection we show how the nontriviality condition (HKer) can be verified.

Lemma 5.6.24. *Assume that (HA), (HK) and (HH) are satisfied. Suppose that the atomic condition (hFin) holds for the system (5.2.1) at each $\tau \in \mathbb{R}$. Then the nontriviality condition (HKer) is also satisfied.*

Proof. By symmetry and the fact the adjoint system (5.6.3) also satisfies (hFin), it suffices to show that any nonzero $d \in \mathcal{B}$ cannot vanish on $(-\infty, 0]$. Arguing by contradiction, we assume that $d = 0$ identically on $(-\infty, 0]$. Defining $\sigma = \inf\{s \in \mathbb{R} : d(s) \neq 0\}$, we have $0 \leq \sigma < \infty$ by construction.

Recalling the constant $\delta > 0$ from (hFin), we pick $0 < \varepsilon < \delta$ sufficiently small to have $d(\sigma+\varepsilon) \neq 0$ and $r_j + \varepsilon < r_{\max}$ for any $j \in \mathbb{Z}$ with $r_j \neq r_{\max}$. Evaluating (5.2.1) at $t = \sigma + \varepsilon - r_{\max}$ now yields

$$\begin{array}{lll} 0 & = & -\dot{d}(t) + \sum\limits_{j=1}^{\infty} A_j(t) d(t+r_j) + \int\limits_{\mathbb{R}} \mathcal{K}(\xi;t) d(t+\xi) d\xi \\ & = & A_{j_{\max}}(\sigma+\varepsilon-r_{\max}) d(\sigma+\varepsilon). \end{array} \tag{5.6.85}$$

Since the matrix $A_{j_{\max}}(\sigma+\varepsilon-r_{\max})$ is nonsingular, we obtain the desired contradiction $d(\sigma+\varepsilon) = 0$. ∎

Lemma 5.6.25. *Assume that (HA), (HK) and (HH) are satisfied, together with the cyclicity conditions (hSh1)-(hSh2). Then the nontriviality condition (HKer) also holds.*

Proof. By symmetry and the fact the adjoint system (5.6.3) also satisfies (hSh1)-(hSh2), it suffices to show that any nonzero $d \in \mathcal{B}$ cannot vanish on $(-\infty, 0]$. Writing $\sigma = \inf\{s \in \mathbb{R} : d(s) \neq 0\}$, we have $0 \leq \sigma < \infty$ by construction. Recalling the constant $K_{\mathrm{const}} \in \mathbb{Z}_{\geq 0}$ from (hB), we use (5.2.1) to conclude that

$$\begin{array}{lll} 0 & = & -\dot{d}(s) + \sum\limits_{j\in\mathbb{Z}} A_j(s) d(s+j) + \int\limits_{\mathbb{R}} \mathcal{K}(\xi;s) d(s+\xi) d\xi \\ & = & \sum\limits_{j \geq \sigma - s} \tilde{A}_j d(s+j) \end{array} \tag{5.6.86}$$

for any $s \in (-\infty, -K_{\mathrm{const}}]$.

We now pick an integer N_0 and a constant $\varepsilon > 0$ in such a way that $N_0 > \sigma + K_{\mathrm{const}}$ and $d(\sigma + \varepsilon) \neq 0$ both hold. Then for any integer $N \geq N_0$ and any $s \in (\sigma, \sigma + \varepsilon)$, we can use (5.6.86) to conclude

$$\sum_{j=0}^{\infty} d(s+j)^{\dagger} \tilde{A}_{j+N} \quad = \quad 0, \tag{5.6.87}$$

which closely resembles (5.6.66). We can hence follow the proof of Lemma 5.6.19 to obtain the contradiction $d = 0$. ■

Lemma 5.6.26. *Assume that (HA), (HK) and (HH) are satisfied. Suppose that the cyclicity conditions (hCyc1)-(hCyc2) are satisfied for the system (5.2.1). Then the nontriviality condition (HKer) is satisfied for the system (5.2.1).*

Proof. By symmetry and the fact the adjoint system (5.6.3) also satisfies (hCyc1)-(hCyc2), it suffices to show that any nonzero $d \in \mathcal{B}$ cannot vanish on $(-\infty, 0]$. We can follow the proof of Lemmas 5.6.20 and 5.6.25 to arrive at a contradiction. ■

Lemma 5.6.27. *Assume that (HA), (HK) and (HH) and (hPos1) are satisfied. Suppose furthermore that the positivity condition (hPos2) holds for both the system (5.2.1) and the adjoint system (5.6.3). Then the nontriviality condition (HKer) is also satisfied.*

Proof. By symmetry, it suffices to show that any nonzero, nonnegative $d \in \mathcal{B}$ cannot vanish on $(-\infty, 0]$. Write $\sigma = \inf\{s \in \mathbb{R} : d(s) \neq 0\}$ and recall the constant $K_{\mathrm{const}} \in \mathbb{Z}_{\geq 0}$ from (hB). Using (5.2.1) we see that

$$\begin{array}{lcl} 0 & = & -\dot{d}(s) + \sum_{j \in \mathbb{Z}} A_j(s) d(s+j) + \int_{\mathbb{R}} \mathcal{K}(\xi; s) d(s+\xi) d\xi \\ & = & \sum_{j \geq \sigma - s} \tilde{A}_j d(s+j) + \int_{\sigma - s}^{\infty} \tilde{\mathcal{K}}(\xi) d(s+\xi) d\xi \end{array} \tag{5.6.88}$$

for any $s \in (-\infty, -K_{\mathrm{const}}]$.

We now pick an integer N_0 and a constant $\varepsilon > 0$ in such a way that $d(\sigma + \delta) \neq 0$ for each $0 < \delta < \varepsilon$ and $N_0 > \sigma + K_{\mathrm{const}} + \varepsilon$ both hold. If (a) holds in (hPos1), we pick $N \geq N_0$ in such a way that $\tilde{A}_N$ is positive definite. Picking $s = \sigma + \varepsilon - N \in (-\infty, -K_{\mathrm{const}}]$, we arrive at the contradiction

$$0 \quad \geq \quad \left(\tilde{A}_N d(\sigma + \varepsilon)\right)^{(k)} \quad > \quad 0 \tag{5.6.89}$$

for some $1 \leq k \leq M$. On the other hand, if (b) holds in (hPos1), we pick $\theta \geq N_0$ in such a way that $\tilde{\mathcal{K}}_{\theta+\delta}$ is positive definite whenever $|\delta| \leq \frac{\varepsilon}{4}$. Picking $s = \sigma + \frac{\varepsilon}{2} - \theta \in (-\infty, -K_{\mathrm{const}}]$, we obtain

$$0 \quad \geq \quad C \inf_{t \in [\frac{\varepsilon}{4}, \frac{3\varepsilon}{4}]} \{d(\sigma + t)^{(k)}\} \quad > \quad 0 \tag{5.6.90}$$

for some constant $C > 0$ and some $1 \leq k \leq M$, a contradiction. ■

Proof of Proposition 5.6.10. This follows directly from Lemmas 5.6.24-5.6.27. ■

Chapter 6

Parameter-dependent exponential dichotomies for nonlocal differential operators

---

6.1 Introduction and main result

In this short, final chapter, we extend parts of the theory from Chapter 5 to include MFDEs such as (5.2.1) that depend smoothly on a parameter μ. For each individual μ one can construct the corresponding exponential splitting using our previous results, but this construction contains some noncanonical choices that do not necessarily preserve the smoothness in μ. Often in applications, this smoothness is necessary in order to obtain uniform estimates and close bifurcation arguments.

For example, exponential dichotomies play a major role in the construction and stability analysis [108, 109] of travelling pulse solutions to the FitzHugh-Nagumo LDE (5.1.1). In particular, Hupkes and Sandstede consider a family of linearisations of the Nagumo MFDE of the form

$$cu'(\sigma) \quad = \quad u(\sigma+1) + u(\sigma-1) - 2u(\sigma) + g_u\big(\Theta(\vartheta, c, \rho)(\sigma), a\big)u(\sigma). \qquad (6.1.1)$$

Here, the relevant parameters are the wavespeed c, which should be close to the wavespeed of the travelling front solution (5.1.5), the parameter ρ from the corresponding FitzHugh-Nagumo system, which should be close to 0, and a phase shift ϑ. Using exponential dichotomies for (6.1.1), the authors construct quasi-front and quasi-back solutions to (5.1.1).

Since we work in more or less the same setting as in Chapter 5 and use several key results from that chapter, we will reuse the notation and assumptions introduced there. In particular, we consider the parameter-dependent system

$$\begin{array}{rcl} \dot{x}(t) & = & \sum_{j=-\infty}^{\infty} A_j(t;\mu)x(t+r_j) + \int_{\mathbb{R}} \mathcal{K}(\xi;t;\mu)x(t+\xi)d\xi \\ & := & L(t,\mu)x_t. \end{array} \tag{6.1.2}$$

Here the parameter μ takes values in an open set $U \subset \mathbb{R}^p$, for some integer $p \geq 1$ and the notation x_t was introduced in (5.2.24). The corresponding linear operators $\Lambda(\mu) : W^{1,\infty}(\mathbb{R}; C^M) \to L^\infty(\mathbb{R}; \mathbb{C}^M)$ are given by

$$(\Lambda(\mu)x)(t) \quad = \quad \dot{x}(t) - \sum_{j=-\infty}^{\infty} A_j(t;\mu)x(t+r_j) - \int_{\mathbb{R}} \mathcal{K}(\xi;t;\mu)x(t+\xi)d\xi. \tag{6.1.3}$$

We assume that the system (6.1.2) depends C^k-smoothly on μ in the following sense.

Assumption (HC). The linear operators $\Lambda(\mu)$ corresponding to the system (6.1.2) depends C^k-smoothly on the parameter $\mu \in U$ for some integer $k \geq 0$. In addition, Assumption (HKer) holds for some $\mu_0 \in U$, while Assumptions (HA), (HK) and (HH) hold uniformly for $\mu \in U$. That is, the constant $\tilde{\eta}$ and the upper bounds for the quantities in (5.2.7) and (5.2.8) can be chosen independently of $\mu \in U$. Finally, the limiting operators $\Lambda_{\pm\infty}(\mu)$ depend C^k-smoothly on $\mu \in U$.

Our main result below shows that the exponential splittings which were obtained in §5.5 can be constructed in such a way that the smoothness in the parameter μ is preserved. The concession we have to make is that the space $R(\tau;\mu)$ will be no longer invariant in the sense of Theorem 5.2.8. We view the results in this chapter as another step in the ongoing effort to close the gap between MFDEs with finite-range and with infinite-range interactions. In particular, we expect our results to play an important part in the stability analysis of the FitzHugh-Nagumo LDE with infinite-range interactions (5.1.16), which, at present, is an open problem if $h > 0$ is sufficiently far away from 0.

Theorem 6.1.1 (cf. [104, Thm. 5.1]). *Assume that (HC) is satisfied. Then there exists an open neighbourhood $\mu_0 \in U' \subset U$ in such a way that for any $\mu \in U'$ and any $\tau \geq 0$ there exist subspaces $Q(\tau,\mu), R(\tau,\mu) \subset X$ that satisfy the following properties.*

(i) We have the direct sum decomposition

$$X \quad = \quad Q(\tau;\mu) \oplus R(\tau;\mu) \tag{6.1.4}$$

(ii) Each $\phi \in Q(\tau;\mu)$ can be extended to a solution $E_{\tau,\mu}\phi$ of (6.1.2) on the interval $[\tau,\infty)$, while each $\psi \in R(\tau;\mu)$ can be extended to a solution $E_{\tau,\mu}\psi$ of (6.1.2) on the interval $(-\infty,-r_0] \cup [0,\tau]$. [1]

[1] Here the constant $r_0 > 0$ is defined Proposition 5.5.3.

(iii) *The maps $\mu \mapsto \Pi_{Q(\tau;\mu)}$ and $\mu \mapsto \Pi_{R(\tau;\mu)}$ are C^k-smooth and all derivatives can be bounded uniformly for $\tau \geq 0$.*

(iv) *There exist constants $K > 0$ and $\alpha > 0$ in such a way that we have the pointwise exponential estimates for each $\phi \in X$ and each integer $0 \leq \ell \leq k$*

$$\begin{array}{lcll} |D^\ell_\mu E_{\tau,\mu}\Pi_{Q(\tau;\mu)}\phi|(t) & \leq & Ke^{-\alpha|t-\tau|}\|\phi\|_\infty, & \text{for every } t \geq \tau, \\ |D^\ell_\mu E_{\tau,\mu}\Pi_{R(\tau;\mu)}\phi|(t) & \leq & Ke^{-\alpha|t-\tau|}\|\phi\|_\infty, & \text{for every } t \leq \tau, \\ |\Lambda(\mu)E_{\tau,\mu}\Pi_{R(\tau;\mu)}\phi|(t) & \leq & Ke^{-\alpha|t-\tau|}\|\phi\|_\infty, & \text{for every } t \leq \tau. \end{array} \tag{6.1.5}$$

Our results are primarily based on the approach from [104, §3,5], where Hupkes and Verduyn Lunel construct exponential splittings for parameter-dependent MFDEs with finite-range interactions. The main difficulty here is that in [104] these splittings are obtained by solving a linear equation on a space of functions, defined on the interval $D^\oplus_\tau$, with an exponential weight. However, several operators that are involved in this linear equation, such as the inclusion of the space $Q(\tau)$ into such an exponentially weighted space, lose their boundedness if $r_{\min} = -\infty$. As a workaround, we reconsider the problem on a space with a one-sided exponential weight. However, this change complicates several of the key technical computations.

6.2 One-sided exponential weights

We start by expanding the Fredholm theory from [68] for the system (5.2.1) to spaces with a one-sided exponential weight. For any $\eta \in \mathbb{R}$ and $f \in L^1_{\text{loc}}(\mathbb{R};\mathbb{C}^M)$ we introduce the function

$$[e^+_\eta f](x) \quad = \quad e^{\eta(x^+)}f(x), \tag{6.2.1}$$

where

$$x^+ \quad = \quad \begin{cases} x, & x \geq 0, \\ 0, & x < 0. \end{cases} \tag{6.2.2}$$

This allows us to define the spaces

$$\begin{array}{lcl} L^\infty_{\eta,+}(\mathbb{R};\mathbb{C}^M) & = & \{f \in L^1_{\text{loc}}(\mathbb{R};\mathbb{C}^M) \,|\, e^+_{-\eta}f \in L^\infty(\mathbb{R};\mathbb{C}^M)\}, \\ W^{1,\infty}_{\eta,+}(\mathbb{R};\mathbb{C}^M) & = & \{f \in L^1_{\text{loc}}(\mathbb{R};\mathbb{C}^M) \,|\, e^+_{-\eta}f \in W^{1,\infty}(\mathbb{R};\mathbb{C}^M)\}, \end{array} \tag{6.2.3}$$

with the corresponding norms

$$\begin{array}{lcl} \|f\|_{L^\infty_{\eta,+}(\mathbb{R};\mathbb{C}^M)} & := & \|e^+_{-\eta}f\|_{L^\infty(\mathbb{R};\mathbb{C}^M)}, \\ \|f\|_{W^{1,\infty}_{\eta,+}(\mathbb{R};\mathbb{C}^M)} & := & \|e^+_{-\eta}f\|_{W^{1,\infty}(\mathbb{R};\mathbb{C}^M)}. \end{array} \tag{6.2.4}$$

For sufficiently small $|\eta|$ we can consider the shifted operator $\tilde{\Lambda}_{\eta,+} : W^{1,\infty}(\mathbb{R};\mathbb{C}^M) \to L^\infty(\mathbb{R};\mathbb{C}^M)$ that acts as

$$\tilde{\Lambda}_{\eta,+}x \quad = \quad e^+_\eta \Lambda e^+_{-\eta}x. \tag{6.2.5}$$

Lemma 6.2.1. *Assume that (HA), (HK) and (HH) are satisfied. Pick any $\eta \in \mathbb{R}$ with $|\eta| < \frac{\tilde{\eta}}{4}$. Writing $\tilde{\Delta}^{\pm}_{\eta,+}$ for the characteristic equations defined in (5.2.10) for the operator (6.2.5), we have the identities*

$$\tilde{\Delta}^{+}_{\eta,+}(z) \quad = \quad \Delta^{+}(z-\eta), \qquad \tilde{\Delta}^{-}_{\eta,+}(z) \quad = \quad \Delta^{-}(z). \tag{6.2.6}$$

In addition, the adjoint operator $(\tilde{\Lambda}_{\eta,+})^{*}$ is given by

$$(\tilde{\Lambda}_{\eta,+})^{*} \quad = \quad \widetilde{\Lambda^{*}}_{-\eta,+}. \tag{6.2.7}$$

Proof. For $j \in \mathbb{Z}$ we see that

$$e^{\eta(t^{+})}e^{-\eta(t+r_j)^{+}} \quad = \quad e^{-\eta r_j} \tag{6.2.8}$$

for t sufficiently positive, while

$$e^{\eta(t^{+})}e^{-\eta(t+r_j)^{+}} \quad = \quad 1 \tag{6.2.9}$$

for t sufficiently negative. Similarly for $x \in W^{1,\infty}(\mathbb{R};\mathbb{C}^{M})$ we can compute

$$\big(e^{-\eta(t^{+})}x(t)\big)' \quad = \quad -\eta e^{-\eta(t^{+})}x(t) + e^{-\eta(t^{+})}x'(t) \tag{6.2.10}$$

for t sufficiently positive, while

$$\big(e^{-\eta(t^{+})}x(t)\big)' \quad = \quad x'(t) \tag{6.2.11}$$

for t sufficiently negative. Finally for $x \in W^{1,\infty}(\mathbb{R};\mathbb{C}^{M})$ we see that

$$\begin{aligned} e^{\eta(t^{+})}\int_{\mathbb{R}}\mathcal{K}(\xi;t)e^{-\eta(\xi+t)^{+}}x(\xi+t)d\xi \quad &= \quad e^{\eta t}\int_{-\infty}^{-t}\mathcal{K}(\xi;t)x(\xi+t)d\xi \\ &\qquad + \int_{-t}^{\infty}\mathcal{K}(\xi;t)e^{-\eta\xi}x(\xi+t)d\xi \end{aligned} \tag{6.2.12}$$

for t positive, while

$$\begin{aligned} e^{\eta(t^{+})}\int_{\mathbb{R}}\mathcal{K}(\xi;t)e^{-\eta(\xi+t)^{+}}x(\xi+t)d\xi \quad &= \quad \int_{-\infty}^{-t}\mathcal{K}(\xi;t)x(\xi+t)d\xi \\ &\qquad + e^{-\eta t}\int_{-t}^{\infty}\mathcal{K}(\xi;t)e^{-\eta\xi}x(\xi+t)d\xi \end{aligned} \tag{6.2.13}$$

for t negative. These computations directly imply the identities (6.2.6).

In addition, a short computation shows that

$$\begin{aligned} \langle y, \tilde{\Lambda}_{\eta,+}x\rangle_{L^2(\mathbb{R};\mathbb{C}^M)} \quad &= \quad \int y(t)^{\dagger}e^{\eta(t^{+})}(\Lambda e^{+}_{-\eta}x)(t)dt \\ &= \quad \int (e^{+}_{\eta}y)(t)^{\dagger}(\Lambda e^{+}_{-\eta}x)(t)dt \\ &= \quad \int (\Lambda^{*}e^{+}_{\eta}y)(t)^{\dagger}e^{-\eta(t^{+})}x(t)dt \\ &= \quad \int (e^{+}_{-\eta}\Lambda^{*}e^{+}_{\eta}y)(t)^{\dagger}x(t)dt \\ &= \quad \langle \widetilde{\Lambda^{*}}_{-\eta,+}y, x\rangle_{L^2(\mathbb{R};\mathbb{C}^M)}, \end{aligned} \tag{6.2.14}$$

which implies (6.2.7), as desired. ∎

Lemma 6.2.1 allows us to define the Fredholm operators $\Lambda_{(\eta)} : W^{1,\infty}_{\eta,+}(\mathbb{R};\mathbb{C}^M) \to L^\infty_{\eta,+}(\mathbb{R};\mathbb{C}^M)$ that act as

$$\Lambda_{(\eta),+} \quad = \quad e^+_\eta \circ \tilde{\Lambda}_{-\eta,+} \circ e^+_{-\eta}. \tag{6.2.15}$$

Our main result here shows that the natural adjoint $\Lambda^*_{(-\eta),+} : W^{1,\infty}_{-\eta,+}(\mathbb{R};\mathbb{C}^M) \to L^\infty_{-\eta,+}(\mathbb{R};\mathbb{C}^M)$ is given by

$$\Lambda^*_{(-\eta),+} \quad = \quad e^+_{-\eta} \circ \widetilde{\Lambda^*}_{\eta,+} \circ e^+_\eta. \tag{6.2.16}$$

Note that for $x \in W^{1,\infty}_{\eta,+}(\mathbb{R};\mathbb{C}^M) \cap W^{1,\infty}(\mathbb{R};\mathbb{C}^M)$ and $y \in W^{1,\infty}_{-\eta,+}(\mathbb{R};\mathbb{C}^M) \cap W^{1,\infty}(\mathbb{R};\mathbb{C}^M)$ we simply have

$$\Lambda x \quad = \quad \Lambda_{(\eta),+} x, \qquad \Lambda^* y \quad = \quad \Lambda^*_{(-\eta),+} y. \tag{6.2.17}$$

The main reasons we constructed the operators $\Lambda_{(\eta),+}$ in this fashion are that it is not a-priori clear that Λ maps $W^{1,\infty}_{\eta,+}(\mathbb{R};\mathbb{C}^M)$ into $L^\infty_{\eta,+}(\mathbb{R};\mathbb{C}^M)$ and whether these operators remain Fredholm operators. We note that $\Lambda_{(0),+} = \Lambda$, since we have the identities $W^{1,\infty}_{0,+}(\mathbb{R};\mathbb{C}^M) = W^{1,\infty}(\mathbb{R};\mathbb{C}^M)$ and $L^\infty_{0,+}(\mathbb{R};\mathbb{C}^M) = L^\infty(\mathbb{R};\mathbb{C}^M)$. The following result is the equivalent of Proposition 5.2.1 for the operator $\Lambda_{(\eta),+}$.

Proposition 6.2.2 (cf. [104, Prop. 3.2])**.** *Assume that (HA), (HK) and (HH) are satisfied. Pick any $\eta \in \mathbb{R}$ with $|\eta| < \frac{\tilde{\eta}}{4}$ for which the characteristic equation $\det \Delta^+(z) = 0$ has no roots with $\mathrm{Re}\, z = \eta$. Then both the operators $\Lambda_{(\eta),+} : W^{1,\infty}_{\eta,+}(\mathbb{R};\mathbb{C}^M) \to L^\infty_{\eta,+}(\mathbb{R};\mathbb{C}^M)$ and $\Lambda^*_{(-\eta),+} : W^{1,\infty}_{-\eta,+}(\mathbb{R};\mathbb{C}^M) \to L^\infty_{-\eta,+}(\mathbb{R};\mathbb{C}^M)$ are Fredholm operators. Moreover, the ranges admit the characterisation*

$$\begin{aligned} \mathcal{R}\big(\Lambda_{(\eta),+}\big) \quad &= \quad \big\{h \in L^\infty(\mathbb{R}) \big| \int_{-\infty}^{\infty} y(t)^* h(t) dt = 0 \textit{ for every } y \in \ker(\Lambda^*_{(-\eta),+})\big\}, \\ \mathcal{R}\big(\Lambda^*_{(-\eta),+}\big) \quad &= \quad \big\{h \in L^\infty(\mathbb{R}) \big| \int_{-\infty}^{\infty} x(t)^* h(t) dt = 0 \textit{ for every } x \in \ker(\Lambda_{(\eta),+})\big\}. \end{aligned} \tag{6.2.18}$$

The Fredholm indices can be computed by

$$\mathrm{ind}(\Lambda_{(\eta),+}) \quad = \quad -\mathrm{ind}(\Lambda^*_{(-\eta),+}) \quad = \quad \dim\ker(\Lambda_{(\eta),+}) - \dim\ker(\Lambda_{(-\eta),+}). \tag{6.2.19}$$

Finally, there exist constants $K > 0$ and $0 < \alpha \leq \tilde{\eta}$ so that

$$|e^+_{-\eta} x(t)| \quad \leq \quad K e^{-\alpha|t|} \|e^+_{-\eta} x\|_\infty \tag{6.2.20}$$

holds for any $x \in \ker(\Lambda_{(\eta),+})$ and any $t \in \mathbb{R}$, while the bound

$$|e^+_\eta x(t)| \quad \leq \quad K e^{-\alpha|t|} \|e^+_\eta x\|_\infty \tag{6.2.21}$$

holds for any $x \in \ker(\Lambda^*_{(-\eta),+})$ and any $t \in \mathbb{R}$.

Proof. These results follow from Proposition 5.2.1 and Lemma 6.2.1, together with the identities

$$\begin{array}{lllll} \ker\big(\Lambda_{(\eta),+}\big) & = & e^{+}_{\eta}\ker\big(\tilde{\Lambda}_{-\eta,+}\big), & & \\ \ker\big(\Lambda^{*}_{(-\eta),+}\big) & = & e^{+}_{-\eta}\ker\big(\widetilde{\Lambda^{*}}_{\eta,+}\big) & = & e^{+}_{-\eta}\ker\big((\tilde{\Lambda}_{-\eta,+})^{*}\big), \\ \mathrm{Range}\big(\Lambda_{(\eta),+}\big) & = & e^{+}_{\eta}\mathrm{Range}\big(\tilde{\Lambda}_{-\eta,+}\big), & & \\ \mathrm{Range}\big(\Lambda^{*}_{(-\eta),+}\big) & = & e^{+}_{-\eta}\mathrm{Range}\big(\widetilde{\Lambda^{*}}_{\eta,+}\big) & = & e^{+}_{-\eta}\mathrm{Range}\big((\tilde{\Lambda}_{-\eta,+})^{*}\big). \end{array} \quad (6.2.22)$$

∎

We now shift our attention to the parameter-dependent system (6.1.2). The following result shows that we can find a quasi-inverse for this system that depends smoothly on μ.

Proposition 6.2.3 (cf. [104, Prop. 3.3])**.** *Assume that (HC) is satisfied. Pick any $\eta \in \mathbb{R}$ with $|\eta| < \frac{\tilde{\eta}}{4}$ for which the characteristic equation $\det \Delta^{+}(z) = 0$ for $\mu = \mu_0$ has no roots with $\mathrm{Re}\, z = \eta$. Write $\mathcal{R} = \mathrm{Range}\big(\Lambda_{(\eta),+}(\mu_0)\big)$ and pick a complement $\mathcal{R}^{\perp}$ for $\mathcal{R}$ in $L^{\infty}_{\eta,+}(\mathbb{R};\mathbb{C}^M)$. Then there exists an open neighbourhood $\mu_0 \in U' \subset U$, together with a C^k-smooth function*

$$\mathcal{C}_{(\eta),+} : U' \rightarrow \mathcal{L}\big(L^{\infty}_{\eta,+}(\mathbb{R};\mathbb{C}^M), \mathcal{R}^{\perp}\big) \quad (6.2.23)$$

and a C^k-smooth quasi-inverse

$$\Lambda^{\mathrm{qinv}}_{(\eta),+} : U' \rightarrow \mathcal{L}\big(L^{\infty}_{\eta,+}(\mathbb{R};\mathbb{C}^M), W^{1,\infty}_{\eta,+}(\mathbb{R};\mathbb{C}^M)\big) \quad (6.2.24)$$

that satisfy the following properties.

(i) For any $\mu \in U'$ we have the upper bound

$$\dim\Big(\ker\big(\Lambda_{(\eta),+}(\mu)\big)\Big) \leq \dim\Big(\ker\big(\Lambda_{(\eta),+}(\mu_0)\big)\Big). \quad (6.2.25)$$

(ii) For any $\mu \in U'$ and any $f \in L^{\infty}(\mathbb{R};\mathbb{C}^M)$ we have the identity

$$\Lambda_{(\eta),+}(\mu)\Lambda^{\mathrm{qinv}}_{(\eta),+}(\mu) f = f + \mathcal{C}_{(\eta),+}(\mu) f. \quad (6.2.26)$$

Moreover, the restriction of the map $\mathcal{C}_{(\eta),+}(\mu_0)$ to $\mathcal{R}$ vanishes identically.

Proof. Upon choosing

$$\begin{array}{lll} \Lambda^{\mathrm{qinv}}_{(\eta),+}(\mu) f & = & \big[\pi_{\mathcal{R}}\Lambda_{(\eta),+}(\mu)\big]^{-1}\pi_{\mathcal{R}} f, \\ \mathcal{C}_{(\eta),+}(\mu) f & = & -\pi_{\mathcal{R}^{\perp}} f + \pi_{\mathcal{R}^{\perp}}\Lambda_{(\eta),+}(\mu)\Lambda^{\mathrm{qinv}}_{(\eta),+}(\mu) f, \end{array} \quad (6.2.27)$$

we can directly follow the proof of [104, Prop. 3.3] to arrive at the desired result. ∎

In a similar fashion, we introduce the function

$$[e_\eta^- f](x) \quad = \quad e^{\eta(x^-)} f(x), \tag{6.2.28}$$

where

$$x^- \quad = \quad \begin{cases} |x|, & x \leq 0, \\ 0, & x > 0, \end{cases} \tag{6.2.29}$$

together with the spaces

$$\begin{array}{rcl} L^\infty_{\eta,-}(\mathbb{R};\mathbb{C}^M) & = & \{f \in L^1_{\mathrm{loc}}(\mathbb{R};\mathbb{C}^M) \,|\, e^-_{-\eta} f \in L^\infty(\mathbb{R};\mathbb{C}^M)\}, \\ W^{1,\infty}_{\eta,-}(\mathbb{R};\mathbb{C}^M) & = & \{f \in L^1_{\mathrm{loc}}(\mathbb{R};\mathbb{C}^M) \,|\, e^-_{-\eta} f \in W^{1,\infty}(\mathbb{R};\mathbb{C}^M)\}, \end{array} \tag{6.2.30}$$

with the corresponding norms

$$\begin{array}{rcl} \|f\|_{L^\infty_{\eta,+}(\mathbb{R};\mathbb{C}^M)} & := & \|e^+_{-\eta} f\|_{L^\infty(\mathbb{R};\mathbb{C}^M)}, \\ \|f\|_{W^{1,\infty}_{\eta,+}(\mathbb{R};\mathbb{C}^M)} & := & \|e^+_{-\eta} f\|_{W^{1,\infty}(\mathbb{R};\mathbb{C}^M)}. \end{array} \tag{6.2.31}$$

For sufficiently small $|\eta|$ we can consider the shifted operator $\tilde{\Lambda}_{\eta,-} : W^{1,\infty}(\mathbb{R};\mathbb{C}^M) \to L^\infty(\mathbb{R};\mathbb{C}^M)$ which acts as

$$\tilde{\Lambda}_{\eta,-} x \quad = \quad e^-_\eta \Lambda e^-_{-\eta} x \tag{6.2.32}$$

and we can define the Fredholm operators $\Lambda_{(\eta),-} : W^{1,\infty}_{\eta,-}(\mathbb{R};\mathbb{C}^M) \to L^\infty_{\eta,-}(\mathbb{R};\mathbb{C}^M)$ by

$$\Lambda_{(\eta),-} \quad = \quad e^-_\eta \circ \tilde{\Lambda}_{-\eta,-} \circ e^-_{-\eta}. \tag{6.2.33}$$

Remark 6.2.4. The equivalent statements in Propositions 6.2.2-6.2.3 can be proven for the operator $\Lambda_{(\eta),-}$ under the assumption that the characteristic equation $\det \Delta^-(z) = 0$ has no roots with $\operatorname{Re} z = -\eta$, instead of the condition on Δ^+.

For notational simplicity, we use the shorthand

$$\Lambda^{\mathrm{qinv}}(\mu) \quad := \quad \Lambda^{\mathrm{qinv}}_{(0),+}(\mu) \quad = \quad \Lambda^{\mathrm{qinv}}_{(0),-}(\mu). \tag{6.2.34}$$

The half-line inverses from Lemma 5.5.6 can also be chosen to depend smoothly on the parameter μ. We recall that the intervals $D^\oplus_\tau$ and $D^\ominus_\tau$ were defined in (5.2.32), while the interval D_X was defined in (5.2.22).

Lemma 6.2.5 (cf. [104, Pg. 13])**.** *Assume that (HC) is satisfied. Recall the open neighbourhood U' of μ_0 from Proposition 6.2.3 and fix $\tau \in \mathbb{R}$. Then there exist bounded linear operators*

$$\begin{array}{rcl} \Lambda^{-1}_{+;\tau}(\mu) : L^\infty\big([\tau,\infty);\mathbb{C}^M\big) & \to & W^{1,\infty}(D^\oplus_\tau;\mathbb{C}^M), \\ \Lambda^{-1}_{-;\tau}(\mu) : L^\infty\big((-\infty,\tau];\mathbb{C}^M\big) & \to & W^{1,\infty}(D^\ominus_\tau;\mathbb{C}^M), \end{array} \tag{6.2.35}$$

defined for $\mu \in U'$, in such a way that the identities

$$\begin{array}{rcl} [\Lambda(\mu)\Lambda^{-1}_{+;\tau}(\mu) f](t) & = & f(t), \;\; t \geq \tau, \\ [\Lambda(\mu)\Lambda^{-1}_{-;\tau}(\mu) g](s) & = & g(s), \;\; s \leq \tau \end{array} \tag{6.2.36}$$

hold for $f \in L^\infty\big([\tau,\infty);\mathbb{C}^M\big)$ and $g \in L^\infty\big((-\infty,\tau];\mathbb{C}^M\big)$. The operators $\Lambda_{\pm;\tau}$ depend C^k-smoothly on the parameter μ.

In addition, if $\tau > 0$ is sufficiently large, there exists bounded linear operators

$$\Lambda^{-1}_{\diamond;\tau}(\mu) : L^\infty\big([0,\tau];\mathbb{C}^M\big) \quad \to \quad W^{1,\infty}\big(D_X + \tau;\mathbb{C}^M\big), \tag{6.2.37}$$

defined for $\mu \in U'$, in such a way that the identity

$$[\Lambda(\mu)\Lambda^{-1}_{\diamond;\tau}(\mu)f](t) \quad = \quad f(t), \;\; t \in [0,\tau] \tag{6.2.38}$$

holds for $f \in L^\infty\big([0,\tau];\mathbb{C}^M\big)$. The operators $\Lambda_{\diamond;\tau}$ depend C^k-smoothly on the parameter μ.

Proof. Using the quasi-inverse $\Lambda^{\mathrm{qinv}}(\mu)$ instead of the inverse Λ^{-1}, the proof of Lemma 5.5.6 carries over to the current setting. ■

6.3 Construction of exponential splittings

In this section, we set out to prove Theorem 6.1.1. For $\tau \geq 0$ and $\mu \in U$ we write $Q(\tau,\mu)$ for the space $Q(\tau)$ from Theorem 5.2.8 at this value of μ. In addition, we write $Q(\tau) := Q(\tau,\mu_0)$. Moreover, we introduce, for notational clarity, the evaluation operator ev_t given by

$$\mathrm{ev}_t\phi \quad = \quad \phi_t. \tag{6.3.1}$$

We will be mainly working in the spaces

$$\begin{array}{lcl} BC^{\oplus}_{\tau,\eta} & = & \big\{ f \in C_b\big(D^{\oplus}_\tau,\mathbb{C}^M\big) \,|\, e^+_{-\eta}f \in C_b\big(D^{\oplus}_\tau,\mathbb{C}^M\big)\big\}, \\ BC^{\ominus}_{\tau,\eta} & = & \big\{ f \in C_b\big(D^{\ominus}_\tau,\mathbb{C}^M\big) \,|\, e^-_{-\eta}f \in C_b\big(D^{\ominus}_\tau,\mathbb{C}^M\big)\big\} \end{array} \tag{6.3.2}$$

for $\tau \geq 0$ and $\eta \in \mathbb{R}$, with the corresponding norms

$$\|f\|_{BC^{\oplus}_{\tau,\eta}} \quad = \quad \|e^+_{-\eta}f\|_\infty, \qquad \|f\|_{BC^{\ominus}_{\tau,\eta}} \quad = \quad \|e^-_{-\eta}f\|_\infty. \tag{6.3.3}$$

This choice of spaces is in essential in our analysis and in major contrast to the finite-range setting in [104]. Indeed, there the authors consider weighted spaces, defined on the interval $D^{\oplus}_\tau$, where the weight decays exponentially in positive direction, while it grows exponentially in the direction of $r_{\min} + \tau$. An essential step in the analysis is that the inclusion of the space $Q(\tau)$ into the exponentially weighted space is a bounded linear operator. However, this is the case if and only if $r_{\min} > -\infty$. By contrast, the inclusion of $Q(\tau)$ into the space $BC^{\oplus}_{\tau,\eta}$ is bounded for $\eta < 0$ sufficiently close to 0.

The key ingredients to establish Theorem 6.1.1 are the following two results that we establish in the sequel. Basically, they state that $Q(\tau,\mu)$ and $R(\tau,\mu)$ can be constructed as a graph over $Q(\tau,\mu_0)$ and $R(\tau,\mu_0)$. For $\psi \in Q(\tau,\mu)$, we write $E_{\tau,\mu}\psi$ for the extension of the function ψ. That is, $E_{\tau,\mu}\psi$ is a solution of (6.1.2) on the interval $[\tau,\infty)$.

Proposition 6.3.1 (cf. [104, Lem. 5.2])**.** *Assume that (HC) is satisfied. Consider the splitting $X = Q(\tau) \oplus R(\tau)$ for $\tau \geq 0$ for the system (6.1.2) at $\mu = \mu_0$. Then there exists an open neighbourhood $\mu_0 \in U' \subset U$, together with C^k-smooth functions $u^*_{Q(\tau)} : U' \to \mathcal{L}(Q(\tau), X)$, defined for $\tau \geq 0$, that satisfy the following properties.*

(i) For each $\mu \in U'$ we have the identity

$$\Pi_{Q(\tau)} u^*_{Q(\tau)}(\mu) \quad = \quad I \tag{6.3.4}$$

and the limit

$$\lim_{\mu \to \mu_0} [I - \Pi_{Q(\tau)}] u^*_{Q(\tau)}(\mu) \quad = \quad 0, \tag{6.3.5}$$

holds uniformly for $\tau \geq 0$.

(ii) For $\mu \in U'$ the operator norms of the maps $u^*_{Q(\tau)}(\mu)$ are bounded uniformly for $\tau \geq 0$.

(iii) For $\mu \in U'$ we have $Q(\tau;\mu) = \mathrm{Range}\big(u^*_{Q(\tau)}(\mu)\big)$.

(iv) There exist constants $K > 0$ and $\alpha > 0$ in so that the bound

$$\big|D^\ell_\mu E_{\tau,\mu} u^*_{Q(\tau)}(\mu)\phi\big|(t) \quad \leq \quad K e^{-\alpha|t-\tau|} \|\phi\|_\infty \tag{6.3.6}$$

holds for each $\mu \in U'$, each $0 \leq \tau \leq t$, each $\phi \in Q(\tau)$ and each integer $0 \leq \ell \leq k$.

Recall that the space $R(\tau, \mu_0)$ is constructed as a finite-dimensional enlargement of the space $\tilde{P}(\tau, \mu_0)$. However, it is unclear whether this finite-dimensional space can be constructed in such a way that it depends smoothly on the parameter μ. As such, we simply construct the space $R(\tau, \mu)$ in a fashion similar to Proposition 6.3.1 and treat this as its definition. The price we have to pay is that this space is no longer invariant.

Proposition 6.3.2 (cf. [104, Lem. 5.3])**.** *Assume that (HC) is satisfied. Consider the splitting $X = Q(\tau) \oplus R(\tau)$ for $\tau \geq 0$ for the system (6.1.2) at $\mu = \mu_0$. Then there exists an open neighbourhood $\mu_0 \in U' \subset U$, together with C^k-smooth functions $u^*_{R(\tau)} : U' \to \mathcal{L}(R(\tau), X)$, defined for $\tau \geq 0$, that satisfy the following properties.*

(i) For each $\mu \in U'$ we have the identity

$$\Pi_{R(\tau)} u^*_{R(\tau)}(\mu) \quad = \quad I \tag{6.3.7}$$

and the limit

$$\lim_{\mu \to \mu_0} [I - \Pi_{R(\tau)}] u^*_{R(\tau)}(\mu) \quad = \quad 0, \tag{6.3.8}$$

holds uniformly for $\tau \geq 0$.

(ii) For $\mu \in U'$ we have that the operator norms of the maps $u^*_{R(\tau)}(\mu)$ are bounded uniformly for $\tau \geq 0$.

(iii) Writing $R(\tau;\mu) = \mathrm{Range}\big(u^*_{R(\tau)}(\mu)\big)$, each $\psi \in R(\tau;\mu)$ extends to a solution $E_{\tau,\mu}\psi$ of (6.1.2) on the interval $(-\infty, -r_0] \cup [0, \tau]$. In addition, the space $R(\tau;\mu) \subset X$ is closed.

(iv) There exist constants $K > 0$ and $\alpha > 0$ in such a way that we have the bound

$$|D^{\ell}_{\mu} E_{\tau,\mu} u^{*}_{R(\tau)}(\mu)\phi|(t) \quad \leq \quad K e^{-\alpha|t-\tau|}\|\phi\|_{\infty} \tag{6.3.9}$$

for each $\mu \in U'$, each $t \leq \tau$, each $\phi \in R(\tau)$ and each integer $0 \leq \ell \leq k$.

(v) We have the uniform bound

$$\big|\Lambda(\mu) E_{\tau,\mu} u^{*}_{R(\tau)}(\mu)\phi\big|(t) \quad \leq \quad K e^{-\alpha|t-\tau|}\|\phi\|_{\infty} \tag{6.3.10}$$

for each $\mu \in U'$, each $t \in [-r_0, 0]$ and each $\phi \in R(\tau)$.

Proof of Theorem 6.1.1. On account of Propositions 6.3.1 and 6.3.2 we can repeat the arguments used in the proof of [104, Thm. 5.1] to arrive at the desired result. ■

For any $\tau \geq 0$ and $\eta > 0$, we introduce the map $\mathcal{G}_{\tau;\eta} : U \to \mathcal{L}\big(BC^{\oplus}_{\tau,-\eta}\big)$, defined by

$$\mathcal{G}_{\tau;\eta}(\mu)u \quad = \quad \Lambda^{\mathrm{qinv}}_{(-\eta),+}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u - \iota_{\tau;\eta}\Pi_{Q(\tau)}\mathrm{ev}_0\Lambda^{\mathrm{qinv}}_{(-\eta),+}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u. \tag{6.3.11}$$

Here we introduced the notation

$$[L(\mu)u](t) \quad = \quad L(t,\mu)u_t, \tag{6.3.12}$$

together with the map $\iota_{\tau;\eta}$ which is the inclusion from $Q(\tau)$ into $BC^{\oplus}_{\tau,-\eta}$ for $\tau \geq 0$.

The proof of Proposition 6.3.1 consists of a number of steps. We start by showing that the map $\mathcal{G}_{\tau,\alpha}$ from (6.3.11) is well-defined and bounded for some specified $\alpha > 0$. Then we use this map $\mathcal{G}_{\tau,\alpha}$ to construct the functions $u^{*}_{Q(\tau)}$. Most of our focus will go to the identity $Q(\tau;\mu) = \mathrm{Range}\big(u^{*}_{Q(\tau)}(\mu)\big)$, since the other bounds and identities follow relatively quickly from the definition.

Lemma 6.3.3. *Consider the setting of Proposition 6.3.1 and suppose that $r_{\min} = -\infty$. Then there exists a constant $\alpha > 0$ so that the map*

$$\mathcal{G}_{\tau} \quad := \quad \mathcal{G}_{\tau;\alpha} \tag{6.3.13}$$

is a well-defined map $\mathcal{G}_{\tau} : U \to \mathcal{L}\big(BC^{\oplus}_{\tau,-\alpha}\big)$. In addition, there exists an open neighbourhood $\mu_0 \in U' \subset U$, together with a constant $C > 0$, so that for all $\mu \in U'$ we have the uniform bounds

$$\|\mathcal{G}_{\tau}(\mu)\| \quad \leq \quad \tfrac{1}{2}, \qquad \|D^{\ell}_{\mu}\mathcal{G}_{\tau}(\mu)\| \quad \leq \quad C \tag{6.3.14}$$

for all $\tau \geq 0$ and all integers $1 \leq \ell \leq k$.

Proof. We let $K \geq 1$ and $0 < \alpha < \tilde{\eta}$ be the constants from Theorem 5.2.8 applied to the system (6.1.2) at $\mu = \mu_0$. Without loss of generality we can assume that α is so small that the characteristic equation $\det \Delta^{+}(z)$ for $\mu = \mu_0$ has no roots with $\mathrm{Re}\, z = -\alpha$, which allows us to consider the quasi-inverse $\Lambda^{\mathrm{qinv}}_{(-\alpha),+}$ from Proposition 6.2.3. We also can assume without loss of generality that $e^{\pm}_{2\alpha} b \in W^{1,\infty}(\mathbb{R};\mathbb{C}^M)$ for any

$b \in \mathcal{B} \cup \mathcal{B}^*$.

We start by showing that the map $\mathcal{G}_\tau := \mathcal{G}_{\tau;\alpha}$ is well-defined by showing that the inclusion map $\iota_{\tau;\alpha}$ and the evaluation operator ev_0 map $Q(\tau)$ into $BC^{\oplus}_{\tau,-\alpha}$ and $BC^{\oplus}_{\tau,-\alpha}$ into X respectively.

On account of Theorem 5.2.4, the map

$$\iota_\tau \quad := \quad \iota_{\tau;\alpha} \tag{6.3.15}$$

is a well-defined and bounded map $\iota_\tau : Q(\tau) \to BC^{\oplus}_{\tau,-\alpha}$, since we assumed that $r_{\min} = -\infty$. In addition, we have the bound

$$\|\iota_\tau \phi\|_{BC^{\oplus}_{\tau,-\alpha}} \quad \leq \quad K_{\mathrm{dec}} \|\phi\|_\infty \tag{6.3.16}$$

for $\phi \in Q(\tau)$.

Let $\phi \in BC^{\oplus}_{\tau,-\alpha}$ be given. Then we obtain the pointwise estimate

$$|(\mathrm{ev}_0 \phi)(t)| \quad = \quad e^{-\alpha(t^+)} |e^{\alpha(t^+)} \phi(t)| \quad \leq \quad e^{-\alpha t} \|\phi\|_{BC^{\oplus}_{\tau,-\alpha}} \tag{6.3.17}$$

for any $t \in D^+_X$, while

$$|(\mathrm{ev}_0 \phi)(t)| \quad = \quad |e^{\alpha(t^+)} \phi(t)| \quad \leq \quad \|\phi\|_{BC^{\oplus}_{\tau,-\alpha}} \tag{6.3.18}$$

for $t \in D^-_X$.

Hence, the norms of the operators ev_0 and ι_τ are bounded by 1 and K_{dec} respectively. In addition, the projections $\Pi_{Q(\tau)}$ are uniformly bounded in norm on account of Theorem 5.2.8. Since the map $\mu \mapsto L(\mu)$ is C^k-smooth, we see that $\mathcal{G}_\tau$ is smooth as a map from U into $\mathcal{L}\big(BC^{\oplus}_{\tau,-\alpha}\big)$. The uniform bounds on the operators ι_τ, $\Pi_{Q(\tau)}$ and ev_0 now yield the uniform bound (6.3.14) for $\tau \geq 0$, integers $1 \leq \ell \leq k$ and μ sufficiently close to μ_0. ∎

In particular, we can define the bounded linear maps

$$\begin{array}{rcl} v^*_{Q(\tau)}(\mu) : Q(\tau) & \to & BC^{\oplus}_{\tau,-\alpha}, \\ \phi & \mapsto & \big[I - \mathcal{G}_\tau(\mu)\big]^{-1} \iota_\tau \phi, \end{array} \tag{6.3.19}$$

together with

$$u^*_{Q(\tau)}(\mu) \quad = \quad \mathrm{ev}_0 v^*_{Q(\tau)}(\mu). \tag{6.3.20}$$

Lemma 6.3.4. *Consider the setting of Lemma 6.3.3. Then the functions $u^*_{Q(\tau)}(\mu)$ defined in (6.3.20) satisfy items (ii) and (iv) of Proposition 6.1.4.*

Proof. The uniform bound on the operator norm of $u^*_{Q(\tau)}(\mu)$ and the exponential estimate (6.3.6) follow directly from the definition (6.3.20), together with the uniform bounds (6.3.14) and (6.3.16). ∎

Lemma 6.3.5. *Consider the setting of Lemma 6.3.3. Then we have the identity (6.3.4) and the limit (6.3.5) holds uniformly for $\tau \geq 0$.*

Proof. Pick any $\tau \geq 0$ and $u \in BC^{\oplus}_{\tau,-\alpha}$. Then we can compute

$$\begin{array}{lcl} \iota_\tau \Pi_{Q(\tau)} \mathrm{ev}_0 \iota_\tau \Pi_{Q(\tau)} \mathrm{ev}_0 u & = & \iota_\tau \Pi_{Q(\tau)} \Pi_{Q(\tau)} \mathrm{ev}_0 u \\ & = & \iota_\tau \Pi_{Q(\tau)} \mathrm{ev}_0 u. \end{array} \tag{6.3.21}$$

In particular, we see from the definition (6.3.11) that

$$\iota_\tau \Pi_{Q(\tau)} \mathrm{ev}_0 \mathcal{G}_\tau(\mu) \quad = \quad 0. \tag{6.3.22}$$

This implies

$$\Pi_{Q(\tau)} \mathrm{ev}_0 \mathcal{G}_\tau(\mu) \quad = \quad 0, \tag{6.3.23}$$

which yields

$$\pi_{Q(\tau)} u^*_{Q(\tau)}(\mu) \quad = \quad \Pi_{Q(\tau)} \mathrm{ev}_0 \big[I - \mathcal{G}_\tau(\mu)\big]^{-1} \iota_\tau \quad = \quad I, \tag{6.3.24}$$

as desired. The remainder term (6.3.5) can be bounded by considering the identity

$$\big[I - \Pi_{Q(\tau)}\big] u^*_{Q(\tau)}(\mu) \quad = \quad \mathrm{ev}_0 \Big[\big[I - \mathcal{G}_\tau(\mu)\big]^{-1} - I \Big] \iota_\tau, \tag{6.3.25}$$

which approaches 0 as $\mu \to \mu_0$, uniformly for $\tau \geq 0$. ∎

We now set out to show that $\mathrm{Range}\big(u^*_{Q(\tau)}(\mu)\big) = Q(\tau, \mu)$. The "$\subset$"-embedding can be established by a relatively direct calculation. The "$\supset$"-embedding follows from the property (6.3.14) for $\mathcal{G}_\tau$.

Lemma 6.3.6. *Consider the setting of Lemma 6.3.3. Then we have the inclusion $\mathrm{Range}\big(u^*_{Q(\tau)}(\mu)\big) \subset Q(\tau, \mu)$.*

Proof. Similarly to (5.5.26), we pick a basis for $\mathrm{Range}\big(\Lambda_{(-\alpha),+}(\mu_0)\big)^\perp$ that consists of continuous functions for which the support is contained in the interval $[-r_0, 0]$. We recall the C^k-smooth operator

$$\mathcal{C}_{(-\alpha),+} : U' \quad \to \quad \mathcal{L}\Big(L^\infty_{(-\alpha),+}(\mathbb{R}; \mathbb{C}^M), \mathrm{Range}\big(\Lambda_{(-\alpha),+}(\mu_0)\big)^\perp\Big) \tag{6.3.26}$$

from Proposition 6.2.3. Recall that α was chosen small enough to have $e^{\pm}_{2\alpha} b \in W^{1,\infty}(\mathbb{R}; \mathbb{C}^M)$ for any $b \in \mathcal{B} \cup \mathcal{B}^*$. Since $\alpha > 0$, we have $L^\infty_{(-\alpha),+}(\mathbb{R}; \mathbb{C}^M) \subset L^\infty(\mathbb{R}; \mathbb{C}^M)$. As such, we have $\Lambda(\mu)x = \Lambda_{(-\alpha),+}(\mu)x$ for any $x \in W^{1,\infty}_{(-\alpha),+}(\mathbb{R}; \mathbb{C}^M)$ and any $\mu \in U'$ on account of (6.2.17). Pick $\phi \in Q(\tau)$ and write

$$u(t) \quad = \quad [v^*_{Q(\tau)}(\mu)\phi](t - \tau), \tag{6.3.27}$$

so that

$$\mathrm{ev}_\tau u \quad = \quad u^*_{Q(\tau)}(\mu)\phi. \tag{6.3.28}$$

Writing

$$u_\tau(t) \quad = \quad u(t+\tau), \tag{6.3.29}$$

we can compute

$$u(t) \quad = \quad [\iota_\tau \phi](t-\tau) + [\mathcal{G}_\tau(\mu)u_\tau](t-\tau) \tag{6.3.30}$$

for $t \in \mathbb{R}$, so that

$$[\Lambda(\mu)u](t) \quad = \quad \big[\Lambda(\mu)\iota_\tau\phi(\cdot - \tau)\big](t) + \big[\Lambda(\mu)\mathcal{G}_\tau(\mu)u_\tau(\cdot-\tau)\big](t). \tag{6.3.31}$$

For $t \in \mathbb{R}$ we can now compute

$$[\Lambda(\mu)\iota_\tau\phi(\cdot-\tau)](t) \quad = \quad \Big[\big[L(\mu_0) - L(\mu)\big]\iota_\tau\phi(\cdot-\tau)\Big](t) + \big[\Lambda(\mu_0)\iota_\tau\phi(\cdot-\tau)\big](t), \tag{6.3.32}$$

together with

$$\begin{array}{rcl} L & := & \big[\Lambda(\mu)\mathcal{G}_\tau(\mu)u_\tau(\cdot-\tau)\big](t) \\ & = & \Big[\big[L(\mu_0) - L(\mu)\big]\mathcal{G}_\tau(\mu)u_\tau(\cdot-\tau)\Big](t) + \Big[\Lambda(\mu_0)[\mathcal{G}_\tau(\mu)u_\tau](\cdot-\tau)\Big](t) \\ & = & \Big[\big[L(\mu_0) - L(\mu)\big]\big[\mathcal{G}_\tau(\mu)[I - \mathcal{G}_\tau(\mu)]^{-1}\iota_\tau\phi(\cdot-\tau)\big]\Big](t) \\ & & \qquad +\Big[\Lambda(\mu_0)\Lambda^{\mathrm{qinv}}_{(-\alpha),+}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u_\tau(\cdot-\tau)\Big](t) \\ & & \qquad -\Big[\Lambda(\mu_0)\iota_\tau\Pi_{Q(\tau)}\mathrm{ev}_0\Lambda^{\mathrm{qinv}}_{(-\alpha),+}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u_\tau(\cdot-\tau)\Big](t) \\ & := & L_1 + L_2 + L_3. \end{array} \tag{6.3.33}$$

We can compute

$$\begin{array}{rcl} L_1 & = & \Big[\big[L(\mu_0) - L(\mu)\big]\big[\mathcal{G}_\tau(\mu)[I - \mathcal{G}_\tau(\mu)]^{-1}\iota_\tau\phi(\cdot-\tau)\big]\Big](t) \\ & = & -\Big[\big[L(\mu_0) - L(\mu)\big]\iota_\tau\phi(\cdot-\tau)\Big](t) \\ & & \qquad +\Big[\big[L(\mu_0) - L(\mu)\big]\big[[I - \mathcal{G}_\tau(\mu)]^{-1}\iota_\tau\phi(\cdot-\tau)\big]\Big](t) \\ & = & -\Big[\big[L(\mu_0) - L(\mu)\big]\iota_\tau\phi(\cdot-\tau)\Big](t) + \Big[\big[L(\mu_0) - L(\mu)\big]u_\tau(\cdot-\tau)\Big](t). \end{array} \tag{6.3.34}$$

Moreover, an application of Proposition 6.2.3 yields

$$\begin{array}{rcl} L_2 & = & \Big[\Lambda(\mu_0)\Lambda^{\mathrm{qinv}}_{(-\alpha),+}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u_\tau(\cdot-\tau)\Big](t) \\ & = & \Big[\big[L(\mu) - L(\mu_0)\big]u_\tau(\cdot-\tau)\Big](t) + \Big[\mathcal{C}_{(-\alpha),+}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u_\tau(\cdot-\tau)\Big](t). \end{array} \tag{6.3.35}$$

Combining (6.3.31), (6.3.32), (6.3.34) and (6.3.35), we obtain

$$\begin{array}{rcl} [\Lambda(\mu)u](t) & = & \Big[\big[L(\mu_0)-L(\mu)\big]\iota_\tau\phi(\cdot-\tau)\Big](t)+\big[\Lambda(\mu_0)\iota_\tau\phi(\cdot-\tau)\big](t) \\ & & -\Big[\big[L(\mu_0)-L(\mu)\big]\iota_\tau\phi(\cdot-\tau)\Big](t)+\Big[\big[L(\mu_0)-L(\mu)\big]u_\tau(\cdot-\tau)\Big](t) \\ & & +\Big[\big[L(\mu)-L(\mu_0)\big]u_\tau(\cdot-\tau)\Big](t) \\ & & +\Big[\mathcal{C}_{(-\alpha),+}(\mu_0)\big[L(\mu)-L(\mu_0)\big]u_\tau(\cdot-\tau)\Big](t) \\ & & -\Big[\Lambda(\mu_0)\iota_\tau\Pi_{Q(\tau)}\mathrm{ev}_0\Lambda^{\mathrm{qinv}}_{(-\alpha),+}(\mu_0)\big[L(\mu)-L(\mu_0)\big]u_\tau(\cdot-\tau)\Big](t) \\ & = & \big[\Lambda(\mu_0)\iota_\tau\phi(\cdot-\tau)\big](t)+\Big[\mathcal{C}_{(-\alpha),+}(\mu_0)\big[L(\mu)-L(\mu_0)\big]u_\tau(\cdot-\tau)\Big](t) \\ & & -\Big[\Lambda(\mu_0)\iota_\tau\Pi_{Q(\tau)}\mathrm{ev}_0\Lambda^{\mathrm{qinv}}_{(-\alpha),+}(\mu_0)\big[L(\mu)-L(\mu_0)\big]u_\tau(\cdot-\tau)\Big](t), \end{array} \tag{6.3.36}$$

for any $t\in\mathbb{R}$. For $t\geq\tau$ we obtain

$$\big[\Lambda(\mu_0)\iota_\tau\phi(\cdot-\tau)\big](t) \quad = \quad 0, \tag{6.3.37}$$

since $\phi\in Q(\tau)$. In addition, we recall that we chose $\mathcal{C}_{(-\alpha),+}(\mu_0)v(s)$ to be identically zero for $s\geq 0$. Finally, for $t\geq\tau$ we obtain

$$\Big[\Lambda(\mu_0)\iota_\tau\Pi_{Q(\tau)}\mathrm{ev}_0\Lambda^{\mathrm{qinv}}_{(-\alpha),+}(\mu_0)\big[L(\mu)-L(\mu_0)\big]u_\tau(\cdot-\tau)\Big](t) \quad = \quad 0 \tag{6.3.38}$$

by definition of $Q(\tau)$. Hence we must have

$$\begin{array}{rcl} [\Lambda(\mu)u](t) & = & \big[\Lambda(\mu_0)\iota_\tau\phi(\cdot-\tau)\big](t)+\Big[\mathcal{C}_{(-\alpha),+}(\mu_0)\big[L(\mu)-L(\mu_0)\big]u_\tau(\cdot-\tau)\Big](t) \\ & & -\Big[\Lambda(\mu_0)\iota_\tau\Pi_{Q(\tau)}\mathrm{ev}_0\Lambda^{\mathrm{qinv}}_{(-\alpha),+}(\mu_0)\big[L(\mu)-L(\mu_0)\big]u_\tau(\cdot-\tau)\Big](t) \\ & = & 0 \end{array} \tag{6.3.39}$$

for any $t\geq\tau$. In particular, we get $u\in\mathcal{Q}(\tau,\mu)$ and thus $u^*_{Q(\tau)}(\mu)\phi\in Q(\tau,\mu)$, as desired. ■

Lemma 6.3.7. *Consider the setting of Lemma 6.3.3. Then we have the inclusion* $\mathrm{Range}\big(u^*_{Q(\tau)}(\mu)\big)\supset Q(\tau,\mu)$.

Proof. We pick $q^1_\mu\in\mathcal{Q}(\tau,\mu)$ and write

$$\begin{array}{rcl} \phi & = & \Pi_{Q(\tau)}\mathrm{ev}_0 q^1_\mu, \\ q^2_\mu(t) & = & [v^*_{Q(\tau)}(\mu)\phi](t-\tau). \end{array} \tag{6.3.40}$$

By Lemma 6.3.6, we see that $q^2_\mu\in\mathcal{Q}(\tau,\mu)$ and therefore also $q_\mu:=q^1_\mu-q^2_\mu\in\mathcal{Q}(\tau,\mu)$. Moreover, we can compute

$$\begin{array}{rcl} \Pi_{Q(\tau)}\mathrm{ev}_0 q_\mu & = & \Pi_{Q(\tau)}\mathrm{ev}_0 q^1_\mu-\Pi_{Q(\tau)}u^*_{Q(\tau)}(\mu)\phi \\ & = & \phi-\phi \\ & = & 0 \end{array} \tag{6.3.41}$$

using (6.3.4). Upon setting

$$q_{\mu_0} = \Lambda^{\mathrm{qinv}}_{(-\alpha),+}(\mu_0)\big[L(\mu) - L(\mu_0)\big]q_\mu - q_\mu, \tag{6.3.42}$$

we note that

$$\begin{array}{lll} \Lambda(\mu_0)q_{\mu_0} & = & \big[L(\mu) - L(\mu_0)\big]q_\mu + \mathcal{C}_{(-\alpha),+}(\mu_0)\big[L(\mu) - L(\mu_0)\big]q_\mu - \Lambda(\mu_0)q_\mu \\ & = & -\Lambda(\mu)q_\mu + \mathcal{C}_{(-\alpha),+}(\mu_0)\big[L(\mu) - L(\mu_0)\big]q_\mu, \end{array} \tag{6.3.43}$$

since $L(\mu) - L(\mu_0) - \Lambda(\mu_0) = -\Lambda(\mu)$. In particular, we see that the right-hand side of (6.3.43) is zero on the halfline $[\tau, \infty)$, so we must have $q_{\mu_0} \in \mathcal{Q}(\tau)$ and hence

$$\begin{array}{lll} \mathcal{G}_\tau(\mu)q_\mu & = & q_\mu + q_{\mu_0} - \iota_\tau \Pi_{Q(\tau)}\mathrm{ev}_0[q_\mu + q_{\mu_0}] \\ & = & q_\mu + q_{\mu_0} - q_{\mu_0} \\ & = & q_{\mu_0}. \end{array} \tag{6.3.44}$$

This yields $q_\mu \in \ker(I - \mathcal{G}_\tau(\mu)) = \{0\}$, which implies $\mathrm{ev}_\tau q^1_\mu = \mathrm{ev}_\tau q^2_\mu \in \mathrm{Range}\big(u^*_{Q(\tau)}(\mu)\big)$ and completes the proof. ∎

Proof of Proposition 6.3.1. In the case where $r_{\min} > -\infty$ we can follow the proof of [104, Lem. 5.2], so we assume that $r_{\min} = -\infty$. In that case, the desired result follows directly from Lemmas 6.3.3-6.3.7. ∎

For the proof of Proposition 6.3.2, we can proceed in the same fashion as in the proof of Proposition 6.3.1, where instead of the spaces $BC^{\oplus}_{\tau,-\alpha}$, we use the space $BC^{\ominus}_{\tau,-\alpha}$. It only remains to show that $\mathrm{Range}\big(u^*_{R(\tau)}(\mu)\big) \subset X$ is closed and to establish (6.3.10).

Lemma 6.3.8. *Consider the setting of Proposition 6.3.2. Then* $\mathrm{Range}\big(u^*_{R(\tau)}(\mu)\big) \subset X$ *is closed.*

Proof. Consider a sequence $\{\phi_j\}_{j\geq 1}$ in $R(\tau)$ and, writing $\psi_j = u^*_{R(\tau)}(\mu)\phi_j$, assume that $\psi_j \to \psi_*$. By (6.3.7) we see that $\Pi_{R(\tau)}\psi_j = \phi_j$ and by the continuity of $\Pi_{R(\tau)}$ this yields $\phi_j \to \Pi_{R(\tau)}\psi_* := \phi_*$. Since the operator $u^*_{R(\tau)}(\mu)$ is bounded, we must have $u^*_{R(\tau)}(\mu)\big[\phi_j - \phi_*\big] \to 0$ and therefore $\psi_* = u^*_{R(\tau)}(\mu)\phi_*$, as desired. ∎

Lemma 6.3.9. *Consider the setting of Proposition 6.3.2. Then the uniform bound (6.3.10) holds for each $\mu \in U'$, each $t \in [-r_0, 0]$ and each $\phi \in R(\tau)$.*

Proof. We fix $\mu \in U'$, $-r_0 \leq t \leq 0$ and $\phi \in R(\tau)$ and write

$$u = E_{\tau,\mu}u^*_{R(\tau)}(\mu)\phi. \tag{6.3.45}$$

From (6.3.36) we can derive that

$$\begin{array}{lll} [\Lambda(\mu)u](t) & = & \big[\Lambda(\mu_0)\iota_\tau\phi(\cdot - \tau)\big](t) + \Big[\mathcal{C}_{(-\alpha),-}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u\Big](t) \\ & & -\Big[\Lambda(\mu_0)\iota_\tau\Pi_{R(\tau)}\mathrm{ev}_0\Lambda^{\mathrm{qinv}}_{(-\alpha),-}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u\Big](t) \\ & := & L_1 + L_2 + L_3. \end{array} \tag{6.3.46}$$

On account of Proposition 5.5.4, we immediately obtain the bound

$$|L_1| = \big|[\Lambda(\mu_0)\iota_\tau\phi(\cdot - \tau)](t)\big| \leq K_1 e^{-\alpha|t-\tau|}\|\phi\|_\infty \tag{6.3.47}$$

for some $K_1 > 0$. Recall that α was chosen small enough to have $e^{\pm}_{2\alpha} b \in W^{1,\infty}(\mathbb{R};\mathbb{C}^M)$ for any $b \in \mathcal{B} \cup \mathcal{B}^*$. Let $\{d^i\}_{i=1}^{n_d}$ denote a basis for $\ker(\Lambda(\mu_0)^*)$. In particular, we can pick a constant $K_2 > 0$ in such a way that the exponential bound

$$|d^i(\xi)| \leq K_2 e^{-2\alpha|\xi|} \tag{6.3.48}$$

holds for any $\xi \in \mathbb{R}$ and any integer $1 \leq i \leq n_d$. Using the representations from Proposition 6.2.3 and from (5.5.22) we can compute

$$\begin{array}{lcl} L_2 & = & \Big[\mathcal{C}_{(-\alpha),-}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u\Big](t) \\ & = & -\pi_{\mathcal{R}^\perp}\Big[\big[L(\mu) - L(\mu_0)\big]u\Big](t) \\ & & \quad +\pi_{\mathcal{R}^\perp}\Big[\Lambda_{(-\alpha),-}(\mu_0)\big[\Pi_{\mathcal{R}}\Lambda_{(-\alpha),-}(\mu_0)\big]^{-1}\pi_{\mathcal{R}}\big[L(\mu) - L(\mu_0)\big]u\Big](t) \\ & = & -\pi_{\mathcal{R}^\perp}\Big[\big[L(\mu) - L(\mu_0)\big]u\Big](t) \\ & = & \sum\limits_{i=1}^{n_d}\Big[\int\limits_{-\infty}^{\infty} d^i(\xi)^*\big[L(\mu_0) - L(\mu)\big]u(\xi)d\xi\Big]g^i(t). \end{array} \tag{6.3.49}$$

On account of the exponential decay (6.3.9), we can pick a constant $K_3 > 0$, independent of μ and u, for which the bound

$$\big|\big[L(\mu) - L(\mu_0)\big]u\big|(\xi) \leq K_3 e^{-\alpha(\tau-\xi)}\|\phi\|_\infty \tag{6.3.50}$$

holds for any $\xi \leq \tau$, while the bound

$$\big|\big[L(\mu) - L(\mu_0)\big]u\big|(\xi) \leq K_3\|\phi\|_\infty \tag{6.3.51}$$

holds for any $\xi > \tau$. In particular, we can estimate

$$\begin{array}{lcl} |L_2| & \leq & \sum\limits_{i=1}^{n_d}\Big[\int\limits_{-\infty}^{\tau} K_2 e^{-2\alpha|\xi|}K_3 e^{-\alpha(\tau-\xi)}\|\phi\|_\infty d\xi + \int\limits_{\tau}^{\infty} K_2 e^{-2\alpha|\xi|}K_3\|\phi\|_\infty d\xi\Big]|g^i|(t) \\ & \leq & e^{-\alpha\tau}K_2K_3\|\phi\|_\infty\Big[\int\limits_{-\infty}^{0} e^{3\alpha\xi}d\xi + \int\limits_{0}^{\tau} e^{-\alpha\xi}d\xi + (2\alpha)^{-1}\Big]\|g^i\|_\infty \\ & \leq & e^{-\alpha(\tau-t)}K_2K_3\|\phi\|_\infty\Big[\int\limits_{-\infty}^{0} e^{3\alpha\xi}d\xi + \int\limits_{0}^{\infty} e^{-\alpha\xi}d\xi + (2\alpha)^{-1}\Big]\|g^i\|_\infty e^{\alpha r_0}. \end{array} \tag{6.3.52}$$

Finally, we obtain the bound

$$\begin{array}{lcl} |L_3| & = & \Big|\Big[\Lambda(\mu_0)\iota_\tau\Pi_{R(\tau)}\mathrm{ev}_0\Lambda^{\mathrm{qinv}}_{(-\alpha),-}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u\Big](t)\Big| \\ & \leq & Ke^{-\alpha|t-\tau|}\|\Pi_{R(\tau)}\mathrm{ev}_0\Lambda^{\mathrm{qinv}}_{(-\alpha),-}(\mu_0)\big[L(\mu) - L(\mu_0)\big]u\|_\infty \\ & \leq & K_3 e^{-\alpha|t-\tau|}\|\phi\|_\infty \end{array} \tag{6.3.53}$$

for some constant $K_3 > 0$, using the uniform bounds and the exponential decay in Theorem 5.2.8 and the bound (6.3.9).

■

Proof of Proposition 6.3.2. The desired result follows from Lemmas 6.3.8 and 6.3.9. ■

Bibliography

[1] Bounded sequence perpendicular to dense subset of ell2. `https://math.stackexchange.com/questions/3078340/bounded-sequence-perpendicular-to-dense-subset-of-ell2`. Accessed: 18 January 2019.

[2] D. Applebaum. *Lévy processes and stochastic calculus*. Cambridge university press, 2009.

[3] D. G. Aronson and H. F. Weinberger. Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation. In *Partial differential equations and related topics (Program, Tulane Univ., New Orleans, La., 1974)*, pages 5–49. Lecture notes in Mathematics, Vol. 446. Springer, Berlin, 1975.

[4] D. Bambusi, E. Faou, and B. Grébert. Existence and stability of ground states for fully discrete approximations of the nonlinear Schrödinger equation. *Numerische Mathematik*, 123(3):461–492, 2013.

[5] P. W. Bates, F. Chen, and J. Wang. Global existence and uniqueness of solutions to a nonlocal phase-field system. In *US–Chinese Conference on Differential Equations and Applications, International Press, Cambridge, MA*, pages 14–21, 1997.

[6] P. W. Bates, X. Chen, and A. Chmaj. Traveling Waves of Bistable Dynamics on a Lattice. *SIAM Journal on Mathematical Analysis*, 35(2):520–546, 2003.

[7] P. W. Bates and A. Chmaj. A Discrete Convolution Model for Phase Transitions. *Archive for Rational Mechanics and Analysis*, 150(4):281–368, 1999.

[8] P. W. Bates, P. C. Fife, X. Ren, and X. Wang. Traveling waves in a convolution model for phase transitions. *Archive for Rational Mechanics and Analysis*, 138(2):105–136, 1997.

[9] P. W. Bates and C. Zhang. Traveling pulses for the Klein-Gordon equation on a lattice or continuum with long-range interaction. *Discrete & Continuous Dynamical Systems A*, 16(1):235–252, 2006.

[10] M. Beck, G. Cox, C. K. R. T. Jones, Y. Latushkin, K. McQuighan, and A. Sukhtayev. Instability of pulses in gradient reaction–diffusion systems:

a symplectic approach. *Philosophical Transactions of the Royal Society A*, 376(2117):20170187, 20pp, 2018.

[11] M. Beck, H. J. Hupkes, B. Sandstede, and K. Zumbrun. Nonlinear Stability of Semidiscrete Shocks for Two-Sided Schemes. *SIAM Journal on Mathematical Analysis*, 42(2):857–903, 2010.

[12] M. Beck, B. Sandstede, and K. Zumbrun. Nonlinear stability of time-periodic viscous shocks. *Archive for rational mechanics and analysis*, 196(3):1011–1076, 2010.

[13] S. Benzoni-Gavage, P. Huot, and F. Rousset. Nonlinear Stability of Semidiscrete Shock Waves. *SIAM Journal on Mathematical Analysis*, 35(3):639–707, 2003.

[14] J. Bertoin. *Lévy processes*, volume 121. Cambridge university press, Cambridge, 1996.

[15] R. Bertram, J. L. Greenstein, R. Hinch, E. Pate, J. Reisert, M. J. Sanderson, T. R. Shannon, J. Sneyd, and R. L. Winslow. *Tutorials in mathematical biosciences II: Mathematical modeling of calcium dynamics and signal transduction.* Springer Science & Business Media, 2005.

[16] W. J. Beyn. The Numerical Computation of Connecting Orbits in Dynamical Systems. *IMA Journal of Numerical Analysis*, 10(3):379–405, 1990.

[17] K. Bhattacharya. *Microstructure of martensite: why it forms and how it gives rise to the shape-memory effect*, volume 2. Oxford University Press, 2003.

[18] C. Bjorland, L. Caffarelli, and A. Figalli. Nonlocal tug-of-war and the infinity fractional laplacian. *Communications on Pure and Applied Mathematics*, 65(3):337–380, 2012.

[19] M. Bonforte, Y. Sire, and J. L. Vázquez. Existence, uniqueness and asymptotic behaviour for fractional porous medium equations on bounded domains. *Discrete & Continuous Dynamical Systems - A*, 35(12):5725–5767, 2015.

[20] J. M. Bos. Fredholm eigenschappen van systemen met interactie over een oneindig bereik. *Bachelor Thesis*, 2015 (in Dutch).

[21] J. P. Boyd. *Weakly nonlocal solitary waves and beyond-all-orders asymptotics: generalized solitons and hyperasymptotic perturbation theory*, volume 442. Springer Science & Business Media, 2012.

[22] D. Breda, S. Maset, and R. Vermiglio. Pseudospectral approximation of eigenvalues of derivative operators with non-local boundary conditions. *Applied Numerical Mathematics*, 56(3–4):318–331, 2006.

[23] P. C. Bressloff. Spatiotemporal dynamics of continuum neural fields. *Journal of Physics A: Mathematical and Theoretical*, 45(3):033001, 2011.

[24] P. C. Bressloff. *Waves in Neural Media: From single Neurons to Neural Fields.* Lecture Notes on Mathematical Modelling in the Life Sciences. Springer, New York, 2014.

[25] N. F. Britton. Spatial structures and periodic travelling waves in an integro-differential reaction-diffusion population model. *SIAM Journal on Applied Mathematics*, 50(6):1663–1688, 1990.

[26] M. Brucal-Hallare and E. S. Van Vleck. Traveling Wavefronts in an Antidiffusion Lattice Nagumo Model. *SIAM Journal on Applied Dynamical Systems*, 10(3):921–959, 2011.

[27] C. Bucur and E. Valdinoci. *Nonlocal diffusion and applications*, volume 20. Cham: Springer, 2016.

[28] J. W. Cahn. Theory of Crystal Growth and Interface Motion in Crystalline Materials. *Acta Metallurgica*, 8(8):554–562, 1960.

[29] J. W. Cahn and A. Novick-Cohen. Evolution Equations for Phase Separation and Ordering in Binary Alloys. *Journal of Statistical Physics*, 76(3–4):877–909, 1994.

[30] J. W. Cahn and E. S. Van Vleck. On the Co-existence and Stability of Trijunctions and Quadrijunctions in a Simple Model. *Acta Materialia*, 47(18):4627–4639, 1999.

[31] G. Carpenter. A Geometric Approach to Singular Perturbation Problems with Applications to Nerve Impulse Equations. *Journal of Differential Equations*, 23(3):335–367, 1977.

[32] P. Carter, B. de Rijk, and B. Sandstede. Stability of traveling pulses with oscillatory tails in the FitzHugh–Nagumo system. *Journal of Nonlinear Science*, 26(5):1369–1444, 2016.

[33] P. Carter and B. Sandstede. Fast pulses with oscillatory tails in the FitzHugh–Nagumo system. *SIAM Journal on Mathematical Analysis*, 47(5):3393–3441, 2015.

[34] P. Carter and B. Sandstede. Unpeeling a Homoclinic Banana in the FitzHugh–Nagumo System. *SIAM Journal on Applied Dynamical Systems*, 17(1):236–349, 2018.

[35] V. Celli and N. Flytzanis. Motion of a screw dislocation in a crystal. *Journal of Applied Physics*, 41(11):4443–4447, 1970.

[36] C.-N. Chen and Y. Choi. Traveling pulse solutions to FitzHugh–Nagumo equations. *Calculus of Variations and Partial Differential Equations*, 54(1):1–45, 2015.

[37] C.-N. Chen and X. Hu. Stability analysis for standing pulse solutions to FitzHugh–Nagumo equations. *Calculus of Variations and Partial Differential Equations*, 49(1-2):827–845, 2014.

[38] X. Chen. Existence, Uniqueness and Asymptotic Stability of Traveling Waves in Nonlocal Evolution Equations. *Advances in Diffential Equations*, 2(1):125–160, 1997.

[39] X. Chen, J. S. Guo, and C.-C. Wu. Traveling Waves in Discrete Periodic Media for Bistable Dynamics. *Archive for Rational Mechanics and Analysis*, 189(2):189–236, 2008.

[40] S.-N. Chow and J. Mallet-Paret. Pattern formation and spatial chaos in lattice dynamical systems. I. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 42(10):746–751, 1995.

[41] S.-N. Chow, J. Mallet-Paret, and W. Shen. Traveling Waves in Lattice Dynamical Systems. *Journal of Differential Equations*, 149(2):248–291, 1998.

[42] S.-N. Chow, J. Mallet-Paret, and E. S. Van Vleck. Dynamics of lattice differential equations. *International Journal of Bifurcation and Chaos*, 6(09):1605–1621, 1996.

[43] O. Ciaurri, L. Roncal, P. R. Stinga, J. L. Torrea, and J. L. Varona. Nonlocal discrete diffusion equations and the fractional discrete Laplacian, regularity and applications. *Advances in Mathematics*, 330:688–738, 2018.

[44] F. Ciuchi, A. Mazzulla, N. Scaramuzza, E. K. Lenzi, and L. R. Evangelista. Fractional diffusion equation and the electrical impedance: Experimental evidence in liquid-crystalline cells. *The Journal of Physical Chemistry C*, 116(15):8773–8777, 2012.

[45] W. A. Coppel. *Dichotomies in Stability Theory*, volume 629 of *Lecture Notes in Mathematics*. Springer Verlag, New York, 1978.

[46] P. Cornwell. Opening the Maslov Box for Traveling Waves in Skew-Gradient Systems: counting eigenvalues and proving (in)stability. *Indiana University Mathematics Journal*, 68:1801–1832, 2019.

[47] P. Cornwell and C. K. R. T. Jones. On the Existence and Stability of Fast Traveling Waves in a Doubly Diffusive FitzHugh–Nagumo System. *SIAM Journal on Applied Dynamical Systems*, 17(1):754–787, 2018.

[48] H. d'Albis, E. Augeraud-Véron, and H. J. Hupkes. Discontinuous Initial Value Problems for Functional Differential-Algebraic Equations of Mixed Type. *Journal of Differential Equations*, 253(7):1959–2024, 2012.

[49] T. Dauxois. Fermi, Pasta, Ulam and a mysterious lady. *Physics Today*, 61(1):55–57, 2008.

[50] E. D'Este, D. Kamin, F. Göttfert, A. El-Hady, and S. E. Hell. STED nanoscopy reveals the ubiquity of subcortical cytoskeleton periodicity in living neurons. *Cell Reports*, 10(8):1246–1251, 2015.

[51] E. D'Este, D. Kamin, C. Velte, F. Göttfert, M. Simons, and S. E. Hell. Subcortical cytoskeleton periodicity throughout the nervous system. *Nature: Scientific Reports*, 6(1):1–8, 2016.

[52] O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel, and H. O. Walther. *Delay Equations: functional-, complex-, and nonlinear analysis*, volume 110. Springer Science & Business Media, New York, 1995.

[53] S. V. Dmitriev, K. Abe, and T. Shigenari. Domain wall solutions for EHM model of crystal:: structures with period multiple of four. *Physica D: Nonlinear Phenomena*, 147(1–2):122–134, 2000.

[54] R. G. Douglas, H. S. Shapiro, and A. L. Shields. Cyclic vectors and invariant subspaces for the backward shift operator. *Annales de l'institut Fourier*, 20(1):37–76, 1970.

[55] J. J. Duistermaat and J. A. C. Kolk. *Multidimensional real analysis I: differentiation*, volume 86. Cambridge University Press, 2004.

[56] C. E. Elmer and E. S. Van Vleck. Dynamics of Monotone Travelling Fronts for Discretizations of Nagumo PDEs. *Nonlinearity*, 18(4):1605–1628, 2005.

[57] C. E. Elmer and E. S. Van Vleck. Spatially Discrete FitzHugh-Nagumo Equations. *SIAM Journal on Applied Mathematics*, 65(4):1153–1174, 2005.

[58] Christopher E. Elmer and Erik S. Van Vleck. Anisotropy, Propagation Failure, and Wave Speedup in Traveling Waves of Discretizations of a Nagumo PDE. *J. Comput. Phys.*, 185(2):562–582, 2003.

[59] Christopher E. Elmer and Erik S. Van Vleck. Existence of Monotone Traveling Fronts for BDF Discretizations of Bistable Reaction-Diffusion Equations. *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, 10(1-3):389–402, 2003. Second International Conference on Dynamics of Continuous, Discrete and Impulsive Systems (London, ON, 2001).

[60] Christopher E. Elmer and Erik S. Van Vleck. Dynamics of monotone travelling fronts for discretizations of Nagumo PDEs. *Nonlinearity*, 18(4):1605–1628, 2005.

[61] K. J. Engel and R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[62] T. Erneux and G. Nicolis. Propagating Waves in Discrete Bistable Reaction-Diffusion Systems. *Physica D: Nonlinear Phenomena*, 67(1–3):237–244, 1993.

[63] J. W. Evans. Nerve axon equations: III. Stability of the nerve impulse. *Indiana University Mathematics Journal*, 22(6):577–593, 1972.

[64] E. Faou and T. Jézéquel. Resonant time steps and instabilities in the numerical integration of Schrödinger equations. *Differential and integral equations*, 28(3/4):221–238, 2015.

[65] T. E. Faver. Nanopteron-stegoton traveling waves in spring dimer Fermi-Pasta-Ulam-Tsingou lattices. *Quarterly of Applied Mathematics*, 78(3):363–429, 2020.

[66] T. E. Faver and H. J. Hupkes. Micropteron traveling waves in diatomic Fermi-Pasta-Ulam-Tsingou lattices under the equal mass limit. *Physica D: Nonlinear Phenomena*, 410:132538, 2020.

[67] T. E. Faver and J. D. Wright. Exact Diatomic Fermi–Pasta–Ulam–Tsingou Solitary Waves with Optical Band Ripples at Infinity. *SIAM Journal on Mathematical Analysis*, 50(1):182–250, 2018.

[68] G. Faye and A. Scheel. Fredholm properties of nonlocal differential operators via spectral flow. *Indiana University Mathematics Journal*, 63:1311–1348, 2014.

[69] G. Faye and A. Scheel. Existence of pulses in excitable media with nonlocal coupling. *Advances in Mathematics*, 270:400–456, 2015.

[70] G. Faye and A. Scheel. Center manifolds without a phase space. *Transactions of the American Mathematical Society*, 370(8):5843–5885, 2018.

[71] E. Feireisl, F. Issard-Roch, and H. Petzeltová. A non-smooth version of the Lojasiewicz–Simon theorem with applications to non-local phase-field systems. *Journal of Differential Equations*, 199(1):1–21, 2004.

[72] E. Fermi, J. Pasta, and S. Ulam. Studies of Nonlinear Problems. Technical Report LA-1940, Los Alamos National Laboratory, 1955.

[73] P. C. Fife and J. B. McLeod. The approach of solutions of nonlinear diffusion equations to travelling front solutions. *Archive for Rational Mechanics and Analysis*, 65(4):335–361, 1977.

[74] R. FitzHugh. Impulses and physiological states in theoretical models of nerve membrane. *Biophysical Journal*, 1(6):445–466, 1961.

[75] R. FitzHugh. Motion picture of nerve impulse propagation using computer animation. *Journal of applied physiology*, 25(5):628–630, 1968.

[76] R. FitzHugh. *Mathematical models of excitation and propagation in nerve.* in Biological Engineering, McGraw Hill, New York, 1969.

[77] G. Friesecke and R. L. Pego. Solitary waves on FPU lattices: I. Qualitative properties, renormalization and continuum limit. *Nonlinearity*, 12(6):1601–1627, 1999.

[78] G. Friesecke and R. L. Pego. Solitary waves on FPU lattices: II. Linear implies nonlinear stability. *Nonlinearity*, 15(4):1343–1359, 2002.

[79] G. Friesecke and R. L. Pego. Solitary waves on Fermi–Pasta–Ulam lattices: III. Howland-type Floquet theory. *Nonlinearity*, 17(1):207–227, 2004.

[80] G. Friesecke and R. L. Pego. Solitary waves on Fermi–Pasta–Ulam lattices: IV. Proof of stability at low energy. *Nonlinearity*, 17(1):229–251, 2004.

[81] G. Friesecke and J. A. Wattis. Existence Theorem for Solitary Waves on Lattices. *Communications in Mathematical Physics*, 161(2):391–418, 1994.

[82] T. Gallay and E. Risler. A variational proof of global stability for bistable travelling waves. *Differential and integral equations*, 20(8):901–926, 2007.

[83] M. Georgi. Bifurcations from Homoclinic Orbits to Non-Hyperbolic Equilibria in Reversible Lattice Differential Equations. *Nonlinearity*, 21(4):735–763, 2008.

[84] E. Gilad, J. von Hardenberg, A. Provenzale, M. Shachak, and E. Meron. Ecosystem engineers: from pattern formation to habitat creation. *Physical Review Letters*, 93(9):098105, 2004.

[85] E. Gilad, J. von Hardenberg, A. Provenzale, M. Shachak, and E. Meron. A mathematical model of plants as ecosystem engineers. *Journal of Theoretical Biology*, 244(4):680–691, 2007.

[86] M. Grinfeld, G. Hines, V. Hutson, K. Mischaikow, G. T. Vickers, et al. Non-local dispersal. *Differential and Integral Equations*, 18(11):1299–1320, 2005.

[87] Q. Gu, E.A. Schiff, S. Grebner, F. Wang, and R. Schwarz. Non-Gaussian transport measurements and the Einstein relation in amorphous silicon. *Physical Review Letters*, 76(17):31–96, 1996.

[88] C. Gui and M. Zhao. Traveling wave solutions of Allen–Cahn equation with a fractional Laplacian. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, 32(4):785–812, 2015.

[89] J. S. Guo and C.-C. Wu. Uniqueness and stability of traveling waves for periodic monostable lattice dynamical system. *Journal of Differential Equations*, 246(10):3818—-3833, 2009.

[90] E. Hairer and G. Wanner. *Solving ordinary differential equations. II*, volume 14 of *Springer series in Computational Mathematics*. Springer-Verlag, Berlin Heidelberg, 1996.

[91] J. K. Hale and S. M. Verduyn Lunel. *Introduction to Functional Differential Equations*. Springer–Verlag, New York, 1993.

[92] C. H. S. Hamster and H. J. Hupkes. Stability of travelling waves for reaction-diffusion equations with multiplicative noise. *SIAM Journal on Applied Dynamical Systems*, 18(1):205–278, 2019.

[93] C. H. S. Hamster and H. J. Hupkes. Stability of traveling waves for systems of reaction-diffusion equations with multiplicative noise. *SIAM Journal on Mathematical Analysis*, 52(2):1386–1426, 2020.

[94] C. H. S. Hamster and H. J. Hupkes. Travelling waves for reaction-diffusion equations forced by translation invariant noise. *Physica D: Nonlinear Phenomena*, 401:132233, 2020.

[95] D. Hankerson and B. Zinner. Wavefronts for a Cooperative Tridiagonal System of Differential Equations. *Journal of Dynamics and Differential Equations*, 5(2):359–373, 1993.

[96] J. Härterich, B. Sandstede, and A. Scheel. Exponential dichotomies for linear non-autonomous functional differential equations of mixed type. *Indiana University Mathematics Journal*, 51(5):1081–1109, 2002.

[97] S. Hastings. On Travelling Wave Solutions of the Hodgkin-Huxley Equations. *Archive for Rational Mechanics and Analysis*, 60(3):229–257, 1976.

[98] A. L. Hodgkin and A. F. Huxley. A quantitative description of membrane current and its application to conduction and excitation in nerve. *The Journal of Physiology*, 117(4):500–544, 1952.

[99] A. Hoffman and J. Mallet-Paret. Universality of Crystallographic Pinning. *Journal of Dynamics and Differential Equations*, 22(2):79–119, 2010.

[100] A. Hoffman and J. D. Wright. Nanopteron solutions of diatomic Fermi–Pasta–Ulam–Tsingou lattices with small mass-ratio. *Physica D: Nonlinear Phenomena*, 358:33–59, 2017.

[101] P. Howard and A. Sukhtayev. The Maslov and Morse indices for Schrödinger operators on [0, 1]. *Journal of Differential Equations*, 260(5):4499–4549, 2016.

[102] H. J. Hupkes and E. Augeraud-Véron. Well-posed of Initial Value Problems on Hilbert Spaces. *Preprint*.

[103] H. J. Hupkes and S. M. Verduyn Lunel. Center Manifold Theory for Functional Differential Equations of Mixed Type. *Journal of Dynamics and Differential Equations*, 19(2):497–560, 2007.

[104] H. J. Hupkes and S. M. Verduyn Lunel. Lin's Method and Homoclinic Bifurcations for Functional Differential Equations of Mixed Type. *Indiana University Mathematics Journal*, 58:2433–2487, 2009.

[105] H. J. Hupkes, L. Morelli, W. M. Schouten-Straatman, and E. S. Van Vleck. Traveling waves and pattern formation for spatially discrete bistable reaction-diffusion equations (survey). In *International Conference on Difference Equations and Applications: 24th ICDEA, Dresden, Germany, May 21-25, 2018 312*, pages 55–112. Springer, 2020.

[106] H. J. Hupkes, L. Morelli, and P. Stehlík. Bichromatic travelling waves for lattice Nagumo equations. *SIAM Journal on Applied Dynamical Systems*, 18(2):973–1014, 2019.

[107] H. J. Hupkes, L. Morelli, P. Stehlík, and V. Švígler. Multichromatic travelling waves for lattice Nagumo equations. *Applied Mathematics and Computation*, 361:430–452, 2019.

[108] H. J. Hupkes and B. Sandstede. Travelling Pulse Solutions for the Discrete FitzHugh-Nagumo System. *SIAM Journal on Applied Dynamical Systems*, 9(3):827–882, 2010.

[109] H. J. Hupkes and B. Sandstede. Stability of Pulse Solutions for the Discrete FitzHugh-Nagumo System. *Transactions of the American Mathematical Society*, 365(1):251–301, 2013.

[110] H. J. Hupkes and E. S. Van Vleck. Negative diffusion and traveling waves in high dimensional lattice systems. *SIAM Journal on Mathematical Analysis*, 45(3):1068–1135, 2013.

[111] H. J. Hupkes and E. S. Van Vleck. Travelling Waves for Complete Discretizations of Reaction Diffusion Systems. *Journal of Dynamics and Differential Equations*, 28(3–4):955–1006, 2016.

[112] H. J. Hupkes and E. S. Van Vleck. Travelling Waves for Adaptive Grid Discretizations of Reaction-Diffusion Systems I: Well-posedness. *Preprint*, 2019.

[113] H. J. Hupkes and E. S. Van Vleck. Travelling Waves for Adaptive Grid Discretizations of Reaction-Diffusion Systems II: Linear Theory. *Preprint*, 2019.

[114] H. J. Hupkes and E. S. Van Vleck. Travelling Waves for Adaptive Grid Discretizations of Reaction-Diffusion Systems III: Nonlinear Theory. *Preprint*, 2019.

[115] G. Iooss and G. James. Localized Waves in Nonlinear Oscillator Chains. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 15:015113, 2005.

[116] G. Jin. Fredholm index of nonlocal differential operators via spectral flow and exponential dichotomy. *Bachelor Thesis*, 2018.

[117] C. K. R. T. Jones. Stability of the Travelling Wave Solutions of the FitzHugh-Nagumo System. *Transactions of the American Mathematical Society*, 286(2):431–469, 1984.

[118] C. K. R. T. Jones. Geometric singular perturbation theory. In R. Johnson, editor, *Dynamical Systems*, volume 1609 of *Lecture Notes in Mathematics*, pages 44–118. Springer, Berlin, Heidelberg, 1995.

[119] C. K. R. T. Jones, N. Kopell, and R. Langer. Construction of the FitzHugh-Nagumo Pulse using Differential Forms. In H. Swinney, G. Aris, and D. G. Aronson, editors, *Patterns and Dynamics in Reactive Media*, volume 37 of *IMA Volumes in Mathematics and its Applications*, pages 101–116. Springer New York, 1991.

[120] A. Kaminaga, V. K. Vanag, and I. R. Epstein. A Reaction–Diffusion Memory Device. *Angewandte Chemie International Edition*, 45(19):3087–3089, 2006.

[121] T. Kapitula and K. Promislow. *Spectral and dynamical stability of nonlinear waves*, volume 185. Springer, New York, 2013.

[122] J. P. Keener. Propagation and its Failure in Coupled Systems of Discrete Excitable Cells. *SIAM Journal on Applied Mathematics*, 47(3):556–572, 1987.

[123] J. P. Keener and J. Sneyd. *Mathematical Physiology*, volume 1. Springer, New York, 1998.

[124] M. Krupa, B. Sandstede, and P. Szmolyan. Fast and Slow Waves in the FitzHugh-Nagumo Equation. *Journal of Differential Equations*, 133(1):49–97, 1997.

[125] C. Kwok. Waves in discrete spatial domains. *Bachelor Thesis*, 2013.

[126] W.-T. Li, G. Lin, C. Ma, and F.-Y. Yang. Traveling wave solutions of a nonlocal delayed sir model without outbreak threshold. *Discrete & Continuous Dynamical Systems-B*, 19(2):467–484, 2014.

[127] R. S. Lillie. Factors Affecting Transmission and Recovery in the Passive Iron Nerve Model. *Journal of General Physiology*, 7(4):473–507, 1925.

[128] X. B. Lin. Using Melnikov's Method to Solve Shilnikov's Problems. *Proceedings of the Royal Society Edinburgh*, 116(3–4):295–325, 1990.

[129] J. Mallet-Paret. Spatial Patterns, Spatial Chaos and Traveling Waves in Lattice Differential Equations. In S.J. van Strien and S. M. Verduyn Lunel, editors, *Stochastic and Spatial Structures of Dynamical Systems*, volume 45, pages 105–129. Royal Netherlands Academy of Sciences, Amsterdam, 1996.

[130] J. Mallet-Paret. The Fredholm Alternative for Functional Differential Equations of Mixed Type. *Journal of Dynamics and Differential Equations*, 11:1–47, 1999.

[131] J. Mallet-Paret. The Global Structure of Traveling Waves in Spatially Discrete Dynamical Systems. *Journal of Dynamics and Differential Equations*, 11(1):49–127, 1999.

[132] J. Mallet-Paret. *Crystallographic pinning: direction dependent pinning in lattice differential equations*. Lefschetz Center for Dynamical Systems and Center for Control Sciences, 2001.

[133] J. Mallet-Paret and S. M. Verduyn Lunel. Exponential Dichotomies and Wiener-Hopf Factorizations for Mixed-Type Functional Differential Equations. *Journal of Differential Equations, to appear*, 2001.

[134] M. Matsui and F. Takeo. Supercyclic translation semigroups of linear operators (topics in information sciences and applied functional analysis). 数理解析研究所講究録, 1186:49–56, 2001.

[135] M. Matsui, M. Yamada, and F. Takeo. Supercyclic and chaotic translation semigroups. *Proceedings of the American Mathematical Society*, pages 3535–3546, 2003.

[136] S. Mukherjee, A. Spracklen, D. Choudhury, N. Goldman, P. Öhberg, E. Andersson, and R. R. Thomson. Observation of a Localized Flat-Band State in a Photonic Lieb Lattice. *Physical Review Letters*, 114(24):245504, 2015.

[137] J. Nagumo, S. Arimoto, and S. Yoshizawa. An active pulse transmission line simulating nerve axon. *Proceedings of the IRE*, 50(10):2061–2070, 1962.

[138] M. Or-Guil, M. Bode, C. P. Schenk, and H. G. Purwins. Spot Bifurcations in Three-Component Reaction-Diffusion Systems: The Onset of Propagation. *Physical Review E*, 57(6):6432, 1998.

[139] K. J. Palmer. Exponential dichotomies and transversal homoclinic points. *Journal of Differential Equations*, 55(2):225–256, 1984.

[140] K. J. Palmer. Exponential dichotomies and Fredholm operators. *Proceedings of the American Mathematical Society*, 104(1):149–156, 1988.

[141] R. L. Pego and M. I. Weinstein. Eigenvalues, and instabilities of solitary waves. *Philosophical Transactions of the Royal Society of London. Series A: Physical and Engineering Sciences*, 340(1656):47–94, 1992.

[142] D. J. Pinto and G. B. Ermentrout. Spatially structured activity in synaptically coupled neuronal networks: I. Traveling fronts and pulses. *SIAM Journal on Applied Mathematics*, 62(1):206–225, 2001.

[143] L. A. Ranvier. *Lećons sur l'Histologie du Système Nerveux, par M. L. Ranvier, recueillies par M. Ed. Weber.* F. Savy, Paris, 1878.

[144] A. Rustichini. Functional Differential Equations of Mixed Type: the Linear Autonomous Case. *Journal of Dynamics and Differential Equations*, 1(2):121–143, 1989.

[145] A. Rustichini. Hopf Bifurcation for Functional-Differential Equations of Mixed Type. *Journal of Dynamics and Differential Equations*, 1(2):145–177, 1989.

[146] N. Sabourova. Real and complex operator norms. *Licentiate Thesis*, 2007.

[147] B. Sandstede. Stability of travelling waves. In *Handbook of dynamical systems*, volume 2, pages 983–1055. Elsevier, 2002.

[148] C. P. Schenk, M. Or-Guil, M. Bode, and H. G. Purwins. Interacting Pulses in Three-component Reaction-Diffusion Systems on Two-Dimensional Domains. *Physical Review Letters*, 78(19):3781, 1997.

[149] W. M. Schouten-Straatman and H. J. Hupkes. Exponential dichotomies for nonlocal differential operators with infinite-range interactions. *arXiv preprint arXiv: 2001.11257*, 2019.

[150] W. M. Schouten-Straatman and H. J. Hupkes. Nonlinear Stability of Pulse Solutions for the Discrete FitzHugh-Nagumo equation with Infinite-Range Interactions. *Discrete & Continuous Dynamical Systems-A*, 39(9):5017–5083, 2019.

[151] W. M. Schouten-Straatman and H. J. Hupkes. Travelling waves for spatially discrete systems of FitzHugh-Nagumo type with periodic coefficients. *SIAM Journal on Mathematical Analysis*, 51(4):3492–3532, 2019.

[152] W. M. Schouten-Straatman and H. J. Hupkes. Travelling wave solutions for fully discrete FitzHugh-Nagumo type equations with infinite-range interactions. *arXiv preprint arXiv: 2010.11789*, 2020.

[153] W. Shen and X. Xie. Spectral theory for nonlocal dispersal operators with time periodic indefinite weight functions and applications. *Discrete & Continuous Dynamical Systems-Series B*, 22(3):1023, 2017.

[154] W. Shen and A. Zhang. Spreading speeds for monostable equations with nonlocal dispersal in space periodic habitats. *Journal of Differential Equations*, 249(4):747–795, 2010.

[155] R. Shuttleworth and D. Trucu. Multiscale modelling of fibres dynamics and cell adhesion within moving boundary cancer invasion. *Bulletin of Mathematical Biology*, 81(7):2176–2219, 2019.

[156] L. I. Slepyan. *Models and phenomena in fracture mechanics*. Springer Science & Business Media, 2012.

[157] J.-W. Sun, W.-T. Li, and Z.-C. Wang. The periodic principal eigenvalues with applications to the nonlocal dispersal logistic equation. *Journal of Differential Equations*, 263(2):934–971, 2017.

[158] A. K. Tagantsev, L. E. Cross, and J. Fousek. *Domains in ferroic crystals and thin films*, volume 13. Springer New York, 2010.

[159] A. Vainchtein and E. S. Van Vleck. Nucleation and Propagation of Phase Mixtures in a Bistable Chain. *Physical Review B*, 79(14):144123, 2009.

[160] A. Vainchtein, E. S. Van Vleck, and A. Zhang. Propagation of periodic patterns in a discrete system with competing interactions. *SIAM Journal on Applied Dynamical Systems*, 14(2):523–555, 2015.

[161] P. van Heijster and B. Sandstede. Bifurcations to Travelling Planar Spots in a Three-Component FitzHugh–Nagumo system. *Physica D: Nonlinear Phenomena*, 275:19–34, 2014.

[162] E. S. Van Vleck and A. Zhang. Competing interactions and traveling wave solutions in lattice differential equations. *Communications on Pure & Applied Analysis*, 15(2):457–475, 2016.

[163] R. A. Vicencio, C. Cantillano, L. Morales-Inostroza, B. Real, C. Mejía-Cortés, S. Weimann, A. Szameit, and M. I. Molina. Observation of Localized States in Lieb Photonic Lattices. *Physical Review Letters*, 114(24):245503, 2015.

[164] X. Wang. Metastability and stability of patterns in a convolution model for phase transitions. *Journal of Differential Equations*, 183(2):434–461, 2002.

[165] K. Xu, G. Zhong, and X. Zhuang. Actin, spectrin, and associated proteins form a periodic cytoskeletal structure in axons. *Science*, 339(6118):452–456, 2013.

[166] E. Yanagida. Stability of Fast Travelling Wave Solutions of the FitzHugh-Nagumo Equations. *Journal of Mathematical Biology*, 22(1):81–104, 1985.

[167] K. Zumbrun. Instantaneous Shock Location and One-Dimensional Nonlinear Stability of Viscous Shock Waves. *Quarterly of applied mathematics*, 69(1):177–202, 2011.

[168] K. Zumbrun and P. Howard. Pointwise Semigroup Methods and Stability of Viscous Shock Waves. *Indiana University Mathematics Journal*, 47(3):741–871, 1998.

Samenvatting

In een tijd die inmiddels lang vervlogen lijkt, waren er voetbalwedstrijden waar duizenden mensen op elkaar gepakt op de tribunes zaten. Bij zo'n wedstrijd was het gebruikelijk dat de toeschouwers af en toe spontaan een *wave*[2] vormden: door beurtelings op te staan en weer te gaan zitten lijkt er een golfbeweging door het publiek te gaan; zie Figuur 6.1(a). Met de huidige coronavirusmaatregelen kan een wave natuurlijk nog steeds plaatvinden, al zullen de toeschouwers wel op anderhalve meter afstand uit elkaar zitten. Dat de afstand groter is, betekent echter niet dat we het niet meer over een golfverschijnsel kunnen hebben. Er zit natuurlijk wel een limiet op: als bij wijze van spreken de toeschouwers allemaal honderden meters uit elkaar zitten, dan wordt het onmogelijk om zonder verdere communicatie nog een wave te kunnen doen. Dat het medium waar de golf zich door probeert te bewegen discreet is en wat de afstand tussen de componenten van het medium is, heeft blijkbaar invloed op de vraag of de golf kan bestaan.

Aan de andere kant kennen wij golfverschijnselen voornamelijk uit scenario's waar het medium een continu geheel is, zoals bij watergolven of als een elektrische stroom door een draad beweegt. Sommige golfverschijnselen lijken zich echter door een continu medium te bewegen, terwijl ze dat in feite helemaal niet doen. Het bekendste voorbeeld hiervan is de propagatie van elektrische signalen door zenuwbanen; zie Figuur 6.1(c). Deze signalen kunnen namelijk enkel propageren als de zenuwbaan is omhuld met een meyline coating. In deze coating zitten gaten op vaste afstand van elkaar. Deze gaten worden ook wel de Ranvierknopen genoemd. In de gecoate regio's beweegt het elektrische signaal snel, maar verliest wel veel kracht. Aan de andere kant beweegt het signaal veel langzamer in de Ranvierknopen, maar herstelt de signaalsterkte zich wel. Als je met een microscoop naar dit proces kijkt, lijkt het alsof het signaal springt van een Ranvierknoop naar diens buurknoop. In feite is het dus logischer om dit proces te beschouwen als een golfverschijnsel door een discreet medium, namelijk de Ranvierknopen, dan door de hele zenuwbaan.

Wiskundige modellen proberen de dynamische eigenschappen van dit soort processen in vergelijkingen te vangen. Daarbij moet altijd een belangrijke balans worden gezocht: hoe nauwkeuriger en preciezer je het model probeert te maken, hoe moeilijker het is om er nog iets zinnigs over te bewijzen. Aan de andere kant moet het model ook

[2] Dit wordt ook wel een Mexican wave genoemd.

weer niet zo simpel worden dat het elke connectie met de realiteit verliest.

Een van de eerste wiskundige modellen die de propagatie van elektrische signalen door zenuwbanen probeerde te beschrijven waren de zogenaamde Hodgkin-Huxley-vergelijkingen. Dit model is gebaseerd op experimenten op reuzeinktvissen en is voor het eerst geformuleerd in de jaren 1950; zie Figuur 6.1(b). Wiskundig gezien was het een erg ingewikkeld model[3]. Daarom hebben Richard FitzHugh en Jinichi Nagumo in de jaren 1960 een versimpeld model geïntroduceerd, wat inmideels bekend staat als het FitzHugh-Nagumo model. De eerste vraag die wiskundigen bij dit soort modellen stellen is of er golven zijn die aan de vegelijkingen in dit model voldoen. Tenslotte probeert het FitzHugh-Nagumo-model de propagatie van elektrische signalen, wat een golfverschijnsel is, te beschrijven. Richard FitzHugh heeft dat in 1968 (!) al met een computeranimatie laten zien, maar wiskundigen houden van zekerheid en willen dat dus graag bewijzen. Vanaf de jaren 1970 zijn er vele wiskundige publicaties verschenen over het bestaan van golfoplossingen in het FitzHugh-Nagumo-model.

Er is echter een groot probleem met het FitzHugh-Nagumo-model: de hele discrete structuur met de Ranvierknopen en de meyline coating komt niet direct terug in het model. In eerste instantie was dat niet zo erg: wiskundigen beginnen vaak met het begrijpen van een simpeler model voordat ze generalisaties gaan bekijken. De discrete structuur is echter wel een essentieel onderdeel van het onderliggende biologische proces. Om deze discrete structuur in te bouwen hebben James Keener en James Sneyd in 1998 een discrete versie van het FitzHugh-Nagumo-model geformuleerd. Zoals wel te verwachten was, bleek het veel lastiger te zijn om iets wiskundigs te bewijzen over dit model. Pas in 2009 is het de jonge wiskundige Hermen Jan Hupkes, samen met zijn PostDoc-begeleider Bjorn Sandstede, gelukt om te bewijzen dat er golfoplossingen bestaan in het discrete FitzHugh-Nagumo-model. Dat het zo lang duurde, kwam onder andere doordat er veel minder algemene wiskundige theorie bekend is voor discrete systemen dan voor continue systemen.

In dit proefschrift zet ik de volgende stap in het accurater maken van het FitzHugh-Nagumo-model. Om precies te zijn, analyseer ik drie verschillende generalisaties van het discrete FitzHugh-Nagumo-model: oneindig bereik, periodieke interacties en tijdsdiscretisaties; zie Figuur 6.1(d).

Oneindig bereik In het standaard discrete FitzHugh-Nagumo-model wordt aangenomen dat elke Ranvierknoop alleen zijn twee directe buurknopen 'ziet'. In hoofdstuk 2 nemen we echter aan dat elke knoop in direct contact staat met al zijn buren. In veel systemen in de wereld om ons heen is dat ook een veel logischere aanname: als er in een voetbalstadion een wave wordt gedaan, dan ga je al veel eerder klaarzitten, misschien zelfs al half opstaan, terwijl de wave zich nog aan de andere kant van het stadion bevindt. Ook in de context van zenuwbanen is het een natuurlijkere aanname: dit soort zenuwbanen

[3]Alan Hodgkin en Andrew Huxley waren dan ook geen wiskundigen maar biophysici en in dat soort vakgebieden verkiest men liever accuraatheid boven oplosbaarheid.

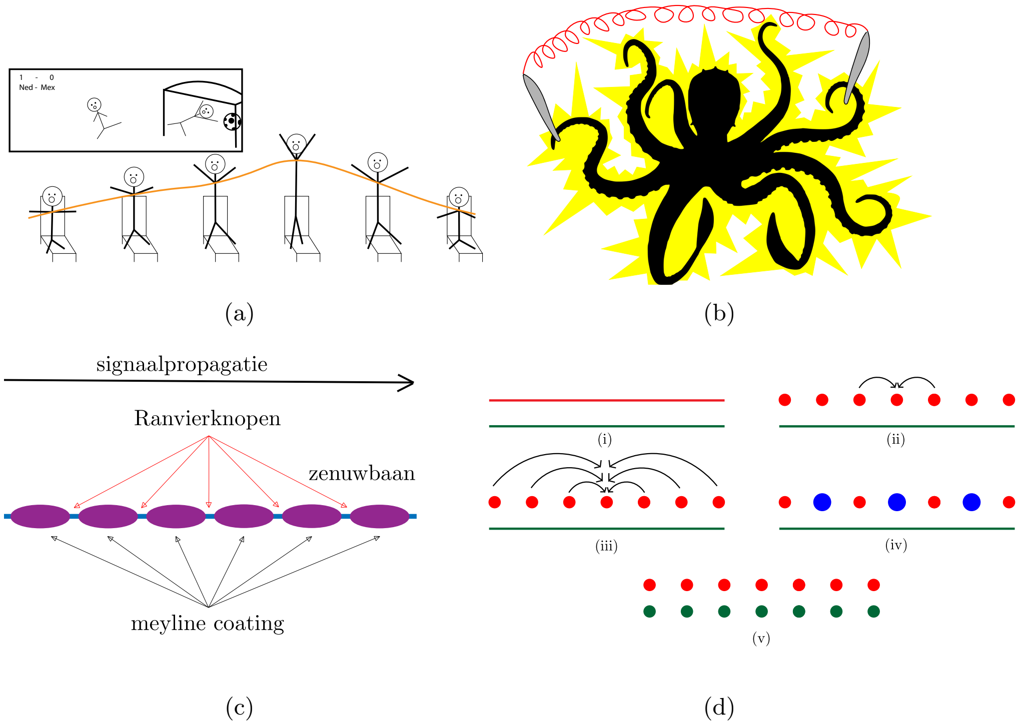


Figure 6.1: *(a) Een wave in een voetbalstadion. De oranje lijn laat het golfprofiel zien. (b) Cartoon van het experiment waar Hodgkin en Huxley een elektrisch signaal door een reuzeinktvis sturen. (c) Schematische weergave van de propagatie van elektrische signalen door zenuwbanen. (d) De verschillende soorten van het FitzHugh-Nagumo-model met in rood en blauw de ruimte en in groen de tijd: (i) het klassieke, volledig continue model; (ii) het ruimtelijk gediscretiseerde model, waar elke knoop enkel zijn directe buurknopen ziet; (iii) het ruimtelijk gediscretiseerde model met oneindig bereik, waar elke knoop alle andere knopen direct ziet; (iv) het periodieke model, waar twee verschillende typen knopen elkaar afwisselen; (v) het model waar zowel de ruimte als de tijd is gediscretiseerd.*

vormen een complex netwerk waar interacties over lange afstanden plaatsvinden. We bewijzen dat er ook in dit soort systemen golfverschijnselen op kunnen treden. We lopen daar echter wel tegen een ernstige beperking aan: we zijn genoodzaakt om aan te nemen dat de afstand tusen twee opeenvolgende Ranvierknopen 'voldoende klein' is. Voldoende klein is echter een nogal vaag begrip[4]: hoe weet ik nou of in mijn toepassing de afstand klein genoeg is? Is de afstand in de zenuwbanen in mijn lichaam klein genoeg? Daar geeft de wiskunde helaas (nog) geen antwoord op. Sterker nog: in het standaard discrete FitzHugh-Nagumo-model is deze beperking niet aanwezig. Wiskundig gezien komt dit doordat wij een totaal andere bewijstechniek gebruiken dan deze eerdere resultaten. Dit was noodzakelijk, omdat voor system waar alles direct van elkaar afhangt er nog veel minder algemene theorie beschikbaar is dan voor system die alleen van hun directe buren afhangen. In hoofdstuk 5 en 6 bouwen we een deel van deze missende theorie weer op voor systemen waar alles direct van elkaar afhangt. Wij verwachten dat deze theorie uiteindelijk voldoende, en zeker nodig, gaat zijn om de beperking van de kleine afstand weg te werken. Dit is iets om in de toekomst naar uit te kijken.

Periodieke interacties In alle eerdere FitzHugh-Nagumo-modellen wordt aangenomen dat alle Ranvierknopen identiek zijn. Echter hebben enkele recente experimenten aangetoond dat bepaalde eiwitten zich slechts om en om aan de Ranvierknopen hechten. In hoofdstuk 3 bouwen en analyseren we een model dat deze periodieke interacties meeneemt. We bewijzen dat dit systeem golfoplossingen toelaat. Echter zien deze golven er anders uit dan bij eerdere modellen. Normaal gesproken heeft een golfoplossing één vaste vorm die zich door de ruimte beweegt. Als we de Ranvierknopen nummeren, dan zien we nu dat er zich tegelijkertijd twee verschillende golven door de zenuwbaan bewegen, een door de even Ranvierknopen en een door de oneven Ranvierknopen. Of dat realistischer is, valt echter lastig te zeggen, omdat onze theoretische resultaten nog niet dicht genoeg bij de realiteit staan voor dit soort uitspraken.

Tijdsdiscretisaties De laatste generalisatie die we bekijken komt niet voort uit een poging het FitzHugh-Nagumo-model realistischer te maken, maar heeft te maken met de implementatie van dit soort modellen in computersimulaties. Er wordt wel gezegd dat mensen het lastig vinden om oneindigheid te bevatten, maar computers zijn er in elk geval nog veel slechter in. Hoewel we namelijk telkens wel aannemen dat onze ruimte discreet is, geldt dat natuurlijk niet voor de tijd: tijd is een continu geheel[5]. Een computer zal in een simulatie echter altijd tijd in kleine stukjes op moeten delen. Er zijn in de loop der jaren vele methodes ontwikkeld om dat op een nauwkeurige manier te doen. Als je een golf probeert te simuleren voor een bepaalde tijd, is typisch de vraag hoe groot de fout is aan het eind van de simulatie. In hoofdstuk 4 draaien we dit vraagstuk om. We zien het systeem met gediscretiseerde tijdsstappen als het systeem wat we gaan analyseren. Dit noemen we een volledige discretisatie. In het bijzonder bekijken we het volledig gediscretiseerde FitzHugh-Nagumo-model. Ook in dit soort systemen kun je je afvragen of er golfoplossingen kunnen bestaan. We onderzoeken

[4] Intuïtief gezien dan, wiskundig gezien heeft het gewoon een nette, precieze definitie.

[5] Als ik niet na mijn bachelor met natuurkunde was gestopt, had ik daar misschien Plancktijden en dergelijke tegen in kunnen werpen, maar dat nemen we verder toch niet mee in onze modellen.

welke simulatiemethoden hiervoor geschikt zijn. Voor zes bekende methoden bewijzen we dat er golfoplossingen kunnen bestaan.

Wiskunde is natuurlijk nooit af en dat zal vermoedelijk ook voor de analyse van het FitzHugh-Nagumo-model gelden. Naast de genoemde generalisaties hebben anderen, onder andere enkelen in Leiden, vele andere extensies onderzocht. Of we ooit de golfoplossingen van volledig gediscretiseerde, periodieke, stochastische, hoger-dimensionale FitzHugh-Nagumo-model met oneindig bereik gaan aantonen, valt nog maar te bezien. In elk geval zijn er nog genoeg interessante vraagstukken over dat wiskundigen nog wel even doorkunnen met het analyseren van FitzHugh-Nagumo-modellen.

Dankwoord

Op deze plek wil ik de gelegenheid nemen om de mensen te bedanken die mijn promotietraject tot zo'n leuke, inspirerende en zinnige ervaring hebben gemaakt.

Allereerst gaat mijn dank uit naar de man die mij eigenhandig heeft getransformeerd van een student in een wetenschapper. Hermen Jan, zoals dat wel vaker gaat, dacht ik toen ik net aan mijn promotie begon dat ik al best wat wist en al best wat kon. Jij hebt mij vriendelijk, doch beslist laten blijken dat op beide aspecten nog wel een en ander aan te merken viel. Vooral mijn schrijfstijl heeft de nodige aandacht gevergd in de afgelopen jaren. De pagina's vol rode opmerkingen hebben mij echt geholpen om mijzelf te ontwikkelen en ik merk dat ik de laatste tijd veel minder opzie tegen bepaalde aspecten van het schrijfproces dan een paar jaar geleden. Je deur stond altijd open voor mijn vragen en je was betrokken bij wat er bij mij speelde, zowel op professioneel als op privé gebied. Jouw onuitputbare bron van kennis en ideeën heeft gezorgd dat wij, naar mijn mening, echt interessante wiskunde hebben kunnen ontwikkelen.

Naast alle inhoudelijke aspecten van mijn promotie, heb ik de afgelopen jaren het Mathematisch Instituut in Leiden als een warme en gezellige omgeving ervaren. In de eerste plaats wil ik de vele kantoorgenoten die ik heb de afgelopen jaren heb gehad daarvoor bedanken, te weten Amine, Dylan, Hent, Jan Pieter, Mark en Stefan (in kamer 217) en Christian, David, Olfa en Robbin (in kamer 202). Dylan, jij bent op dezelfde dag begonnen als ik en ik heb het altijd fijn gevonden dat we op die manier tegelijk op konden trekken. Mark, we hebben veel gelachen, geklaagd en het gezellig gehad samen en jouw aanwezigheid maakte kamer 217 altijd tot een vrolijke en levendige ruimte. Christian, nadat we jaren geleden in hetzelfde mentorgroepje bij SSR zaten, kwamen we elkaar nu weer tegen bij dezelfde begeleider, waardoor we elkaars successen en worstelingen op een dieper niveau begrepen. David, ik heb onze soepele samenwerking bij het assisteren van Linear Analysis en Functional Analysis altijd erg gewaardeerd en ik vind dat we dat samen echt goed voor elkaar hebben gekregen. Olfa, ik heb jou leren kennen als een betrokken persoon en ik vond het heel fijn dat jij kamer 202 tijdens de lockdown hebt aangespoord om contact te blijven houden. Ook alle mensen met wie ik samen naar buitenlandse conferenties ben geweest, naast eerdergenoemenden ook Leonardo, Mia en Timothy, wil ik heel erg bedanken voor hun gezelschap en steun bij het geven van presentaties daar. Die conferenties zijn een waardevolle verrijking geweest voor mijn promotie. Daarnaast wil ik alle anderen op het Mathematisch Instituut bedanken met wie ik een band heb opgebouwd en met wie ik altijd veel plezier heb beleefd bij de lunch, koffiepauze en PhD-colloquia. Ik heb mij altijd thuis gevoeld in Leiden.

Naast mijn vrienden op werk, wil ik ook mijn vrienden buiten werk bedanken voor hun steun. In het bijzonder wil ik de vrienden voor het leven die ik bij het dispuut M.O.C.C.A. heb gemaakt bedanken. Zonder hen zou mijn tijd in Leiden, zowel tijdens mijn studie als tijdens mijn promotie, niet half zo veel waard zijn geweest.

Daarnaast is ook mijn familie mij tot zeer grote steun geweest en wil ik ze daar zeer hartelijk voor bedanken. In de eerste plaats pap, mam, Bernd en Iris: jullie hebben altijd gezorgd dat ik het beste uit mezelf heb gehaald en jullie hebben mij en Mayke door en door gesteund bij alle hoogtepunten, maar ook alle zorgen in de afgelopen jaren. Daarnaast wil ik ook mijn schoonfamilie, Lidwien, Peter, Rymke en Ruben bedanken. Ik heb me altijd welkom en thuis gevoeld bij jullie.

Tot slot wil ik de persoon bedanken die mijn dank het meest verdient. Mayke, ik heb alle successen, teleurstellingen, zorgen en ervaringen met jou kunnen delen en je hebt mij altijd gesteund. Op wiskundig gebied heb jij meerdere malen mij de cruciale ideeën gegeven waardoor ik weer verder kon. Ook voor jouw advies en hulp bij het maken van illustraties voor presentaties en artikelen ben ik je erg dankbaar. Ik zou het zelf nooit zo mooi hebben gekund.

Curriculum Vitae

Willem Migchel Schouten-Straatman was born on 8 June 1992 in Baarn, the Netherlands. From 2004 to 2010 he attended the Kennemer Lyceum in Overveen, where he obtained his Gymnasium diploma. Subsequently, he started his double bachelor program Mathematics and Physics at Leiden University. He completed both in 2014, the bachelor Mathematics *cum laude*, with the bachelor thesis “Elastic instabilities in monoholar and biholar patterned networks” under the supervision of dr. Vivi Rottschäfer and Prof. dr. Martin van Hecke. In that same year, Willem entered the master program Applied Mathematics, again at Leiden University. He graduated *summa cum laude* after finishing his master thesis “The Riesz-Kantorovich formula for lexicographically ordered spaces” under the supervision of dr. Onno van Gaans in 2016. The main results of his master thesis have been published in a scientific journal one year later.

In September 2016, Willem started his PhD at Leiden University under the supervision of dr. Hermen Jan Hupkes, with Prof. dr. Arjen Doelman as his promotor. Willem’s research has resulted in several published papers, which form the core of this thesis. He participated and presented his work in multiple international conferences, such as SIAM Conference on Nonlinear Waves and Coherent Structures (NWCS18), SIAM Conference on Applications of Dynamical Systems (DS19) and EquaDiff 2019. He also organised the yearly NDNS+ PhD days in 2019 and provided on-site support at EquaDiff 2019. Finally, Willem was short-listed for the KWG PhD prize in 2020.

During his studies, Willem was actively involved in the education at Leiden University. In particular, he was a tutor for 5 years and a student assistant for 4 years before starting his PhD. He continued similar educational activities during his PhD. In addition, he supervised the bachelor thesis of Guanyu Jin in 2018. Willem was a member of the student association SSR-Leiden and played the violin in the Leiden student orchestra Collegium Musicum for the major part of his time in Leiden. He married his wife Mayke in 2018.