Approach to Markov operators on spaces of measures by means of equicontinuity

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Chapter 4

Equicontinuous families of Markov operators in view of asymptotic stability

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Abstract:

The relation between equicontinuity – the so-called e–property and stability of Markov operators is studied. In particular, it is shown that any asymptotically stable Markov operator with an invariant measure such that the interior of its support is non-empty satisfies the e–property.

4.1 Introduction

This chapter is centered around two concepts of equicontinuity for Markov operators defined on probability measures on Polish spaces: the *e-property* and the *Cesàro e-property*. Both appeared as a condition (among others) in the study of ergodicity of Markov operators. In particular they are very useful in proving the existence of a unique invariant measure and its *asymptotic stability*: at whatever probability measure one starts, the iterates under the Markov operator will weakly converge to the invariant measure. The first concept appeared in [LS06, SW12] while the second was introduced in [Wor10] as a theoretical generalisation of the first. It allowed the author to extend various results by replacing the e-property condition by the apparently weaker Cesàro e-property condition.

Interest in equicontinuous families of Markov operators existed already before the introduction of the e-property. Jamison [Jam64], working on compact metric state spaces, introduced the concepts of (dual) Markov operators on the continuous functions that are 'uniformly stable' or 'uniformly stable in mean' to obtain a kind of asymptotic stability results in this setting. Meyn and Tweedie [MT09] introduced the so-called '*e-chains*' on locally compact Hausdorff topological state spaces, for similar purposes. See also [Zah14] for results in a locally compact metric setting. The above mentioned concepts were used in proving ergodicity for some Markov chains (see [Ste94, Cza12, CH14, ESvR12, GL15, KPS10]).

It is worth mentioning here that similar concepts appear in the study of mean equicontinuous dynamical systems mainly on compact spaces (see for instance [LTY15]). However it must be stressed here that our space of Borel probability measures defined on some Polish space is non-compact, typically, in the generality in which we consider the question.

While studying the e–property, the natural question arose whether any asymptotically stable Markov operator satisfies this property. Proposition 6.4.2 in [MT09] asserts this holds when the phase space is compact. In particular, the authors claimed that the stronger e–chain property is satisfied. Unfortunately, the proof contains a gap and an example can be constructed showing that some additional assumptions must be added for the claimed result to hold.

Striving to repair the gap of the Meyn-Tweedie result mentioned above, we show that any asymptotically stable Markov operator with an invariant measure such that the interior of its support is nonempty satisfies the e–property.

4.2 Some (counter) examples

Let (S,d) be a Polish space. By $B(x,r)$ we denote the open ball in (S,d) of radius r, centered at $x \in S$. Further $\overline{E}$, $\mathrm{Int}_S E$ denote the closure of $E \subset S$ and the interior of E, respectively. By $C_b(S)$ we denote the vector space of all bounded real-valued continuous functions on S and by $BM(S)$ all bounded real-valued Borel measurable functions, both equipped with the supremum norm $\|\cdot\|_\infty$. By $BL(S)$ we denote the subspace of $C_b(S)$ of all bounded Lipschitz functions (for the metric d on S). For $f \in BL(S)$, $|f|_L$ denotes the Lipschitz constant of f.

By $\mathcal{M}(S)$ we denote the family of all finite Borel measures on S and by $\mathcal{P}(S)$ the subfamily of all probability measures in $\mathcal{M}(S)$. For $\mu \in \mathcal{M}(S)$, its *support* is the set

$$\operatorname{supp}\mu := \{x \in S : \mu(B(x,r)) > 0 \text{ for all } r > 0\}.$$

Recall the concept of Markov operators on measures, see Section 1.2. A measure μ_* is called *invariant* if $P\mu_* = \mu_*$. A Markov operator P is *asymptotically stable* if there exists a unique invariant measure $\mu_* \in \mathcal{P}(S)$ such that $P^n\mu \to \mu_*$ weakly as $n \to \infty$ for every $\mu \in \mathcal{P}(S)$.

A linear operator $U : \mathrm{BM}(S) \to \mathrm{BM}(S)$ is called dual to P if

$$\langle P\mu, f\rangle = \langle \mu, Uf\rangle \text{ for all } \mu \in \mathcal{M}^+(S), f \in BM(S).$$

If such operator U exists, it is unique and we call the Markov operator P *regular*. U is positive and satisfies $U\mathbb{1} = \mathbb{1}$. The Markov operator P is a *Markov-Feller operator* if it is regular and the dual operator U maps the space of continuous bounded functions $C_b(S)$ into itself.

A Feller operator P satisfies the *e–property* at $z \in S$ if for any $f \in BL(S)$ we have

$$\lim_{x\to z} \sup_{n\geq 0, n\in\mathbb{N}} |U^n f(x) - U^n f(z)| = 0, \tag{4.1}$$

i.e. if the family of iterates $\{U^n f : n \in \mathbb{N}\}$ is equicontinuous at $z \in S$. We say that a Feller operator satisfies the *e–property* if it satisfies it at any $z \in S$.

D. Worm slightly generalized the e–property introducing the Cesàro e–property (see [Wor10]). Namely, a Feller operator P will satisfy the *Cesàro e–property* at $z \in S$ if for any $f \in BL(S)$

we have

$$\lim_{x\to z} \sup_{n\geq 0, n\in\mathbb{N}} \left| \frac{1}{n}\sum_{k=1}^{n} U^k f(x) - \frac{1}{n}\sum_{k=1}^{n} U^k f(z) \right| = 0. \tag{4.2}$$

Analogously a Feller operator satisfies the *Cesàro e–property* if it satisfies this property at any $z \in S$.

Let us recall Proposition 6.4.2 in [MT09] that contains - informally - a gap in its proof (slightly reformulated):

Proposition 4.2.1. *Suppose that the Markov chain Φ has the Feller property, and that there exists a unique probability measure π such that for every x*

$$P^n(x,\cdot) \to \pi \quad \textit{weakly as } n \to \infty$$

Then Φ is an e-chain.

The following example shows that Proposition 6.4.2 fails.

Example 4.2.2. *Let $S = \{1/n : n \geq 1\} \cup \{0\}$ and let $T : S \to S$ be given by the following formula:*

$$T(0) = T(1) = 0 \qquad \textit{and} \qquad T(1/n) = 1/(n-1) \quad \textit{for } n \geq 2.$$

The operator $P : \mathcal{M}(S) \to \mathcal{M}(S)$ given by the formula $P\mu = T_*(\mu)$ (the pushforward measure) is asymptotically stable but it does not satisfy the e–property at 0.

For a Markov operator Jamison [Jam64] introduced the property of *uniform stability in mean* when $\{U^n f : n \in \mathbb{N}\}$ is an equicontinuous family of functions in the space of real-valued continuous function $C(S)$ for every $f \in C(S)$. Here S is a compact metric space. Since the space of bounded Lipschitz functions is dense for the uniform norm in the space of bounded uniformly continuous functions, this property coincides with the Cesàro e–property for compact metric spaces. Now, if the Markov operator P on the compact metric space is asymptotically stable, with the invariant measure $\mu_* \in P(S)$, then $\frac{1}{n}\sum_{k=1}^{n} U^k f \to \langle f, \mu_*\rangle$ pointwise, for every $f \in C(S)$. According to Theorem 2.3 in [Jam64] this implies that P is uniformly stable in mean, i.e. has the Cesàro e–property.

Example 4.2.3. *Let $(k_n)_{n\geq 1}$ be an increasing sequence of prime numbers. Set*

$$S := \{(\overbrace{0,\dots,0,}^{k_n^i - 1 - times}\ i/k_n, 0, \dots) \in l^\infty : i \in \{0,\dots,k_n\}, n \in \mathbb{N}\}.$$

The set S endowed with the l^∞-norm $\|\cdot\|_\infty$ is a (noncompact) Polish space. Define $T : S \to S$

by the formula

$$T((0,\dots)) = T((\overbrace{0,\dots,0,}^{k_n^{k_n}-1-times}\ 1,0,\dots)) = (0,\dots,0,\dots) \qquad \text{for } n \in \mathbb{N}$$

and

$$T((\overbrace{0,\dots,0,}^{k_n^{i}-1-times}\ i/k_n,0,\dots)) = (\overbrace{0,\dots,0,}^{k_n^{i+1}-1-times}\ (i+1)/k_n,0,\dots) \quad \text{for } i \in \{1,\dots,k_n-1\}, n \in \mathbb{N}.$$

The operator $P : \mathcal{M}(S) \to \mathcal{M}(S)$ *given by the formula* $P\mu = T_*(\mu)$ *is asymptotically stable but it does not satisfy the Cesàro e–property at* 0*. Indeed, if we take an arbitrary continuous function* $f : S \to \mathbb{R}_+$ *such that* $f((0,\dots,0,\dots)) = 0$ *and* $f(x) = 1$ *for* $x \in S$ *such that* $\|x\|_\infty \geq 1/2$ *we have*

$$\frac{1}{k_n}\sum_{i=1}^{k_n} U^i f((\overbrace{0,\dots,0}^{k_n-1},1/k_n,0,\dots)) - \frac{1}{k_n}\sum_{i=1}^{k_n} U^i f((0,\dots)) \geq 1/2.$$

4.3 Main result

We are in a position to formulate the main result of this chapter. Recall that a metric d is called admissible for the Polish space S if d metrizes the topology on S and the metric space (S,d) is separable and complete.

Theorem 4.3.1. *Let* P *be an asymptotically stable Feller operator and let* μ_* *be its unique invariant measure. If* $\operatorname{Int}_S(\operatorname{supp}\mu_*) \neq \emptyset$*, then* P *satisfies the e–property for any admissible metric* d *on* S*.*

Its proof involves the following two lemmas:

Lemma 4.3.2. *Let* P *be an asymptotically stable Feller operator and let* μ_* *be its unique invariant measure. Let* U *be dual to* P*. If* $\operatorname{Int}_S(\operatorname{supp}\mu_*) \neq \emptyset$*, then for every admissible metric* d *on* S*,* $f \in C_b(S)$ *and any* $\varepsilon > 0$ *there exists a ball* $B \subset \operatorname{supp}\mu_*$ *such that*

$$|U^n f(x) - U^n f(y)| \leq \varepsilon \qquad \text{for any } x,y \in B,\ n \in \mathbb{N}. \tag{4.3}$$

Proof. Fix $f \in C_b(S)$ and $\varepsilon > 0$. Let W be an open set in S such that $W \subset \operatorname{supp}\mu_*$. Set $Y = \overline{W}$ and observe that the subspace Y is a complete metric space, hence a Baire space.

Set

$$Y_n := \{x \in Y : |U^m f(x) - \langle f, \mu_*\rangle| \le \varepsilon/2 \text{ for all } m \ge n\}$$

and observe that Y_n is closed and

$$Y = \bigcup_{n=1}^{\infty} Y_n.$$

By the Baire Category Theorem there exist $N \in \mathbb{N}$ such that $\mathrm{Int}_Y Y_N \neq \varnothing$. Thus there exists a set $V \subset Y_N$ open in the space Y and consequently, because of the construction of Y, an open ball $B = B(z, r_0)$ for the admissible metric d in S such that $B \subset Y_N \subset \mathrm{supp}\,\mu_*$. Since

$$|U^n f(x) - \langle f, \mu_*\rangle| \le \varepsilon/2 \qquad \text{for any } x \in B \text{ and } n \ge N,$$

condition (4.3) is satisfied for all $x, y \in B, n \ge N$. Since the $U^n f$, $n = 1, \cdots, N$ are continuous at z, there exists $r^\varepsilon \le r_0$ such that $|U^n f(z) - U^n f(x)| \le \frac{\varepsilon}{2}$ for all $x \in B(z, r^\varepsilon)$, $n = 1, \cdots, N$. Then condition (4.3) is satisfied for all $x, y \in B := B(z, r^\varepsilon)$ and $n \in \mathbb{N}$. □

Lemma 4.3.3. *Let $\alpha \ge 0$. If $\mu \in \mathcal{M}^+(S)$, $x_0 \in S$ and $r > 0$ are such that $\mu(B(x_0, r)) > \alpha$, then there exists $0 < r \le r$ such that $\mu((B(x_0, r')) > \alpha$ and $\mu(S(x_0, r)) = 0$.*

Proof. For any increasing sequence $(r_n) \subset (0, r]$ such that $r_n \uparrow r$, $\mu(B(x_0, r_n)) \to \mu(B(x_0, r)) > \alpha$. Hence there exists $n_0 \in \mathbb{N}$ such that: $\mu(B(x_0, r_{n_0})) > \alpha$.

Put $r_0 := r_{n_0}$. Then $r_0 > 0$ and $\mu(B(x_0, r')) > \alpha$ for all $r' \in [r_0, r]$. The map $\Psi : [r_0, r] \times S \mapsto \mathbb{R} : (r', x) \mapsto \frac{d(x, x_0)}{r'}$ is separately continuous in r' and x, so it is jointly Borel measurable ([Bog07a], Theorem 7.14.5, p.129).

$$\begin{aligned} \mu(B(x_0, r')) &= \int_S \mathbb{1}_{B(x_0, r')}(y) d\mu(y) \\ &= \int_S \mathbb{1}_{\left\{x : \frac{d(x, x_0)}{r'} < 1\right\}}(y) d\mu(y) \\ &= \int_S \mathbb{1}_{[0,1)}(\Psi(r', y)) d\mu(y). \end{aligned} \tag{4.4}$$

Since Ψ is jointly Borel measurable, $(r', y) \mapsto \mathbb{1}_{[0,1)}(\Psi(r', y))$ is jointly Borel measurable. By the Fubini-Tonelli Theorem (or [Bog07a], Lemma 7.6.4, p.93, or [Bog07b], Corollary 3.3.3, p.182), $\underline{\phi} : r' \mapsto \mu(B(x_0, r'))$ is Borel measurable on $[r_0, r]$. In a similar manner, one shows that $\overline{\psi} := \mu(\overline{B}(x_0, r))$ is Borel measurable, where $\overline{B}(x_0, r) := \{x \in S : d(x, x_0) \le r\}$. Put $\phi(r) := \overline{\phi}(r) - \underline{\phi}(r)$. According to Lusin's Theorem, there exists a compact subset K of $[r_0, r]$, of strictly positive Lebesgue measure, such that $\phi|_K$ is continuous. Put $S(x_0, r') := \overline{B}(x_0, r') \smallsetminus B(x_0, r') = \{x \in S : d(x, x_0) = r'\}$.

Since Lebesgue measure is non-atomic, K must have at least denumerably many distinct points. Let $(r_n)_{n\in\mathbb{N}}$ be a sequence in K that consists of distinct points. Since K is a compact space, there is a subsequence $(r_{n_k})_{k\in\mathbb{N}}$ that converges to an $r' \in K$ as $k \to \infty$.

We can construct a further subsequence from $(r_{n_k})_{k\in\mathbb{N}}$ (denoted the same for convenience), that is either strictly increasing, or strictly decreasing towards r'.

(1) $r_{n_k} \uparrow r'$:

Define $A_1 := B(x_0, r_{n_1}), A_k := B(x_0, r_{n_k}) \setminus \overline{B}(x_0, r_{n_{k-1}})$.

Then

$$B(x_0, r') = \dot{\bigcup_{k=1}^{\infty}} A_k \cup S(x_0, r_{n_k}).$$

So

$$\mu(B(x_0, r')) = \sum_{k=1}^{\infty} \mu(A_k) + \mu(S(x_0, r_{n_k})) < \infty.$$

Hence, $\lim_{k\to\infty} \mu(S(x_0, r_{n_k})) = 0$. Because $r_{n_k} \in K$ and $\phi|_K$ is continuous, we get

$$\mu(S(x_0, r)) = \lim_{k\to\infty} \mu(S(x_0, r_{n_k})) = 0.$$

(2) $r_{n_k} \downarrow r'$:

Now define $A_k := B(x_0, r_{n_k}) \setminus \overline{B(x_0, r_{n_{k+1}}}$ for $k = 1, 2, \cdots$. Then

$$B(x_0, r_{n_1}) = \bigcup_{k=1}^{\infty} [A_k \cup S(x_0, r_{n_{k+1}})] \cup \overline{B(x_0, r')}.$$

Hence, $\lim_{k\to\infty} \mu(S(x_0, r_{n_k})) = 0$, as above, yielding the conclusion that $\mu(S(x_0, r')) = 0$.

Since $\partial B(x_0, r') \subset S(x_0, r')$ we find $\mu(\partial B(x_0, r') = 0$.

□

We are now ready to prove Theorem 4.3.1.

Proof. (Theorem 4.3.1) Assume, contrary to our claim, that P does not satisfy the e–property for some admissible metric d on S. Therefore there exist a function $f \in BL(S, d) \subset C_b(S)$ and a point $x_0 \in S$ such that

$$\limsup_{x\to x_0} \sup_{n\geq 0, n\in\mathbb{N}} |U^n f(x) - U^n f(x_0)| > 0.$$

Hence, there exists $\epsilon > 0$ and $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$,

$$\sup_{x \in B(x_0,\delta)} \sup_{n \geq 0, n \in \mathbb{N}} |U^n f(x) - U^n f(x_0)| \geq 4\varepsilon.$$

Thus, one has a sequence $(x_k)_{k\in\mathbb{N}}$ such that $x_k \in (B(x_0, \frac{\delta_0}{k}))$ and

$$\sup_{n \geq 0, n\in\mathbb{N}} |U^n f(x_k) - U^n f(x_0)| \geq 3\varepsilon \quad \text{for all } k \in \mathbb{N}.$$

Let $B_f = B(z, 2r)$ be an open ball contained in $\operatorname{supp} \mu_*$ such that

$$|U^n f(x) - U^n f(y)| \leq \varepsilon \qquad \text{for all } x, y \in B_f, n \in \mathbb{N}, \tag{4.5}$$

which exists according to Lemma 4.3.2. Since $B_f \subset \operatorname{supp} \mu_x$, one has $\gamma := \mu_*(B_f) > 0$. Choose $\alpha \in (0, \gamma)$ Because P is asymptotically stable, by the Alexandrov Theorem (eg. [EK86], Theorem 3.1) one has

$$\liminf_{n\to\infty} P^n \mu(B_f) \geq \mu_*(B_f) = \gamma > \alpha \qquad \text{for all } \mu \in \mathcal{P}(S), \tag{4.6}$$

Fix $N \in \mathbb{N}$ such that $2(1-\alpha)^N \|f\|_\infty < \epsilon$. Inductively we shall define measures $\nu_i^{x_0}, \mu_i^{x_0}, \nu_i^{x_k}, \mu_i^{x_k}$ and integers $n_i, i = 1, 2, \cdots, N$ in the following way:

Equation (4.6) allows us to choose $n_1 \geq 1$ such that

$$P^{n_1} \delta_{x_0}(B(z, r)) > \alpha. \tag{4.7}$$

According to Lemma 4.3.3 it is possible to choose $0 < r_1 \leq r$ such that

$$P^{n_1} \delta_{x_0}(B(z, r_1)) > \alpha \quad \text{and} \quad P^{n_1} \delta_{x_0}(S(z, r_1)) = 0.$$

Define

$$\nu_1^x(\cdot) = \frac{P^{n_1} \delta_x(\cdot \cap B(z, r_1))}{P^{n_1} \delta_x(B(z, r_1))}. \tag{4.8}$$

Because $P^{n_1} \delta_{x_0}(S(z, r_1)) = 0$ and P is Feller, $P^{n_1} \delta_x(B(z, r_1))$ converges to $P^{n_1} \delta_{x_0}(B(z, r_1)) > \alpha > 0$ if $x \to x_0$. So ν_1^x is a well-defined probability measure, concentrated on $B(z, r_1)$, for all x sufficiently close to x_0, say if $d(x, x_0) < d_1$, and $P^{n_1} \delta_x(B(z, r_1)) > \alpha$ for such x.

Define

$$\mu_1^x(\cdot) = \frac{1}{1-\alpha} \left(P^{n_1} \delta_x(\cdot) - \alpha \nu_1^x(\cdot) \right). \tag{4.9}$$

Then $\mu_1^x \in \mathcal{P}(S)$ for all $x \in S$: $d(x, x_0) < d_1$.

Since $x_k \to x_0$, there exists $N_1 \in \mathbb{N}$ such that $d(x_k, x_0) < d_1$ for all $k \geq N_1$. If $U \subset S$ is open, then by Alexandrov's Theorem,

$$\liminf_{k\to\infty} P^{n_1}\delta_{x_k}(U \cap B(z, r_1)) \geq P^{n_1}\delta_{x_0}(U \cap B(z, r)).$$

Consequently,

$$\liminf_{k\to\infty} \nu_1^{x_k}(U) = \liminf_{k\to\infty} \frac{P^{n_1}\delta_{x_k}(U \cap B(z, r_1))}{P^{n_1}\delta_{x_k}(B(z, r_1))} \geq \frac{P^{n_1}\delta_{x_0}(U \cap B(z, r_1))}{P^{n_1}\delta_{x_0}(B(z, r_1))} = \nu_1^{x_0}(U).$$

Thus, $\nu_1^{x_k} \to \nu_1^{x_0}$ weakly as $k \to \infty$. Then also $\mu_1^{x_k} \to \mu_1^{x_0}$.

Assume that we have defined $\nu_i^{x_0}, \mu_i^{x_0}, \nu_i^{x_k}, \mu_i^{x_k}$ and n_i for $i = 1, 2, \cdots, l$, for some $l < N$ such that $\nu_i^{x_k} \to \nu_i^{x_0}, \mu_i^{x_k} \to \mu_i^{x_0}$ weakly. Then, equation (4.6) allows to pick $n_{l+1} \in \mathbb{N}$ such that

$$P^{n_{l+1}}\mu_l^{x_0}(B(z, r)) > \alpha.$$

According to Lemma 4.3.3 one can select $0 < r_{l+1} \leq r$ such that $P^{n_{l+1}}\mu_l^{x_0}(B(z, r_{l+1})) > \alpha$ and $P^{n_{l+1}}\mu_l^{x_l}(S(z, r_{l+1})) = 0$. Define

$$\nu_{l+1}^{x_k} := \frac{P^{n_{l+1}}\mu_l^{x_k}(\cdot \cap B(z, r_{l+1}))}{P^{n_{l+1}}\mu_l^{x_k}(B(z, r_{l+1}))} \tag{4.10}$$

and

$$\mu_{l+1}^{x_k} := \frac{1}{1-\alpha}(P^{n_{l+1}}\mu_l^{x_k} - \alpha\nu_{l+1}^{x_k}). \tag{4.11}$$

Because $\mu_l^{x_k} \to \mu_l^{x_0}$ weakly, and $P^{n_{l+1}}\mu_l^{x_k}(\partial B(z, r_{l+1})) = 0$.

$P^{n_{l+1}}\mu_l^{x_k}(B(z, r_{l+1})) \to P^{n_{l+1}}\mu_l^{x_0}(B(z, r_{l+1})) > \alpha > 0$ as $k \to \infty$. Thus, $\nu_{l+1}^{x_k}$ is well defined for k sufficiently large and $\nu_{l+1}^{x_k} \to \nu_{l+1}^{x_0}$, weakly, by a similar argument as for $\nu_1^{x_k} \to \nu_1^{x_0}$. We conclude from (4.11), that $\mu_{l+1}^{x_k} \to \mu_{l+1}^{x_0}$ weakly too.

Moreover, the construction is such that we have

$$P^{n_1+n_2+\cdots+n_N}\delta_{x_k} = \alpha P^{n_2+\cdots+n_N}\nu_i^{x_k} + \alpha(1-\alpha)P^{n_3+\cdots+n_N}\nu_2^{x_k} + \cdots + \alpha(1-\alpha)^{N-1}\nu_N^{x_k} + (1-\alpha)^N\mu_N^{x_k}$$

for $k = 0$ and all $k \in \mathbb{N}$ sufficiently large. By construction, $\operatorname{supp}\nu_i^{x_k} \subset \overline{B(z, r)} \subset B(z, 2r) =$

B_f. So for all $n \in \mathbb{N}$, $i = 1, 2, \cdots, N$ and k sufficiently large

$$\begin{aligned} |\langle P^n \nu_i^{x_k}, f\rangle - \langle P^n \nu_i^{x_0}, f\rangle| &= \left|\int_S U^n f(x)\nu_i^{x_k}(dx) - \int_S U^n f(y)\nu_i^{x_0}(dy)\right| \\ &\leq \int_{B_f}\int_{B_f} |U^n f(x) - U^n f(y)|\, \nu_i^{x_k}(dx)\nu_i^{x_0}(dy) \\ &\leq \epsilon. \end{aligned}$$

Moreover, there exists $N_0 \in \mathbb{N}$ such that for all $k \geq N_0$,

$$|\langle P^n \delta_{x_k} - P^n \delta_{x_0}, f\rangle| < \epsilon$$

for all $0 \leq n < n_1 + n_2 + \cdots + n_N$. For $n \geq n_1 + n_2 + \cdots + n_N$ one has for k sufficiently large,

$$\begin{aligned} P^n \delta_{x_k} &= \alpha P^{n-n_1}\nu_1^{x_k} + \alpha(1-\alpha)P^{n-n_1-n_2}\nu_2^{x_k} + \cdots + \\ &\quad + \alpha(1-\alpha)^{N-1}P^{n-n_1-\cdots-n_N}\nu_N^{x_k} + (1-\alpha)^N P^{n-n_1-\cdots-n_N}\mu_N^{x_k}. \end{aligned}$$

Therefore, for these n and k,

$$\begin{aligned} |\langle P^n \delta_{x_n}, f\rangle - \langle P^n \delta_{x_0}, f\rangle| &\leq \varepsilon(\alpha + \alpha(1-\alpha) + \cdots + \alpha(1-\alpha)^{N-1}) + 2(1-\alpha)^N \|f\|_\infty \\ &\leq \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

Thus, the construction of the $(x_k)_{k\in\mathbb{N}}$ is such that for k sufficiently large

$$3\varepsilon \leq \sup_{n\geq 0, n\in\mathbb{N}} |U^n f(x_k) - U^n f(x_0)| = \sup_{n\geq 0} |\langle P^n \delta_{x_k}, f\rangle - \langle P^n \delta_{x_0}, f\rangle| \leq 2\varepsilon$$

which is impossible. This completes the proof.

□

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