Topologies and convergence structures on vector lattices of operators
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Summary

This thesis consists of three papers that are centered around the common theme of Hausdorff uo-Lebesgue topologies and convergence structures on vector lattices and on vector lattices and vector lattice algebras of order bounded operators. Its origins lie in asking for possible analogues of the von Neumann bicommutant theorem in the context of Banach lattices and vector lattices. Apart from being interesting in their own right, such analogues are expected to be relevant for the study of vector lattice algebras and Banach lattice algebras of order bounded operators, as well as for representation theory in vector lattices and Banach lattices.

When contemplating a possible bicommutant theorem for vector lattices, the evident ingredient that is missing in that context is the weak (or strong) operator topology that figures in von Neumann's theorem. Fortunately, there are natural candidates that can take over this role. The first one is a possible Hausdorff uo-Lebesgue topology on a vector lattice algebra of operators. Such topologies on vector lattices have received considerable attention in recent years, and they appear to have a rather special position among the possible locally solid topologies. Apart from this, there are several natural convergence structures on vector lattices of order bounded operators to be considered. These come in pairs, consisting of a uniform and a strong version. For example, the general theory of vector lattices provides the definition of order convergence of a net for any vector lattice, hence also for a vector lattice of order bounded operators. For a net of operators on a vector lattice, however, one can also require that it be pointwise order convergent for every element of the underlying vector lattice. Thus there are a uniform and a strong order convergence structure on a vector lattice of order bounded operators. Likewise, there are a uniform unbounded order convergence structure and a strong unbounded order convergence structure, as well as a uniform convergence structure for a possible Hausdorff uo-Lebesgue topology (which is the topological structure as already mentioned) and a strong one with respect to such a topology on the underlying vector lattice. Thus we have six convergence structures. For each of these, one can speak of the corresponding adherence of a set of order bounded linear operators. These adherences (one of which is an actual topological closure) are all natural candidates that can take over the role of the closure in the weak (or strong) operator topology in von Neumann's theorem.

When trying to work with these convergence structures in the context of an attempted bicommutant theorem, one very quickly starts to feel the need for some 'basic facts to work with'. For example, is the adherence of a vector lattice subalgebra of the order bounded operators with respect to strong unbounded order convergence again a vector lattice subalgebra? As it turns out, this is always the case when it is contained in the orthomorphisms, but

not always when it is contained in the order continuous operators. The question is natural and the answer is easily formulated, but establishing this answer (including its 'sharpness' as an important ingredient) is a non-trivial matter. There are many more such basic, but very often non-trivial, issues to be resolved before one can get to more advanced parts of the theory such as a bicommutant theorem. Since these have not been considered to any substantial extent before, this is now undertaken in a systematic fashion in this thesis.

The first mathematical part of the thesis, Chapter 2, is still concerned with the general theory of Hausdorff uo-Lebesgue topologies on vector lattices. Starting from a Hausdorff o-Lebesgue topology on an order dense ideal, a Hausdorff uo-Lebesgue topology on the vector lattice itself is constructed. This results in a going-up-going-down procedure of supplying regular vector sublattices with such a topology that takes an earlier uniform construction of such topologies still one step further. Classical relations between convergence in measure and convergence almost everywhere are shown to be special cases of a more general result relating convergence in a Hausdorff uo-Lebesgue topology and unbounded order convergence.

In Chapter 3, Hausdorff uo-Lebesgue topologies on vector lattices of order bounded operators are constructed from Hausdoff o-Lebesgue topologies on the underlying vector lattices. The six convergence structures mentioned above are introduced in this chapter, and their relations are studied at the level of vector lattices of order bounded operators. Particular attention is paid to the orthomorphisms, where several implications between these convergences hold that are not generally valid.

In Chapter 4, after completing the investigation of the thirty-six possible implications between the six convergence structures on vector lattices of order bounded operators, these convergence structures are then considered in the context of vector lattice algebras of order bounded operators. The continuity with respect to the six convergence structures is investigated of the left and right multiplications, as well as of the simultaneous continuity of the multiplication. Results about adherences of vector lattice subalgebras being vector lattice subalgebras again are then an immediate consequence; with one exception it is also shown that these results are 'sharp'. Results are also included concerning the equality of various adherences of vector lattices (and of vector lattice algebras) of order bounded operators.

The material in Chapter 4, which builds on the earlier parts, is the most closely related to the original question regarding possible analogues of von Neumann's bicommutant theorem for vector lattices. It can, in fact, be used to obtain such analogues. These results will be published at a later date.