Topologies and convergence structures on vector lattices of operators

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Chapter 2

Vector lattices with a Hausdorff uo-Lebesgue topology

Abstract

We investigate the construction of a Hausdorff uo-Lebesgue topology on a vector lattice from a Hausdorff (o)-Lebesgue topology on an order dense ideal, and what the properties of the topologies thus obtained are. When the vector lattice has an order dense ideal with a separating order continuous dual, it is always possible to supply it with such a topology in this fashion, and the restriction of this topology to a regular sublattice is then also a Hausdorff uo-Lebesgue topology. A regular vector sublattice of $\mathrm{L}_0(X, \Sigma, \mu)$ for a semi-finite measure μ falls into this category, and the convergence of nets in its Hausdorff uo-Lebesgue topology is then the convergence in measure on subsets of finite measure. When a vector lattice not only has an order dense ideal with a separating order continuous dual, but also has the countable sup property, we show that every net in a regular vector sublattice that converges in its Hausdorff uo-Lebesgue topology always contains a sequence that is uo-convergent to the same limit. This enables us to give satisfactory answers to various topological questions about uo-convergence in this context.

2.1 Introduction and overview

In this paper, we investigate the construction of a Hausdorff uo-Lebesgue topology on a vector lattice from a Hausdorff (o)-Lebesgue topology[1] on an order dense ideal, and what the properties of the topologies thus obtained are.

After recalling the relevant notions and making the necessary preparations in Section 2.2, the key construction is carried out in Theorem 2.3.1 in Section 2.3, below. The idea of starting with a topology on an order dense ideal originates from [11] but, whereas the construction in [11] to obtain a global topology is carried out using Riesz pseudo-norms, we follow an approach using neighbourhood bases of zero that is inspired by [44]. Using such neighbourhood bases, it is possible to perform the construction under minimal hypotheses on the initial data, and thus understand how these hypotheses are reflected in the properties of the resulting global topology. The remainder of Section 2.3 is mainly concerned with showing how the general theorem relates to existing results in the literature. Our working with neighbourhood bases of zero enables us to explain certain 'pathologies' in the literature, where a topology of unbounded type is not Hausdorff, or not linear, from the general theorem.

In Section 2.4, we move to the context where the initial ideal is actually order dense and admits a Hausdorff o-Lebesgue topology. In that case, every regular vector sublattice of the global vector lattice admits a Hausdorff uo-Lebesgue topology. The resulting overview Theorem 2.4.9, below, mostly consists of a summary of results that are already in the literature, though not presented in this way. It is also recalled in that section that a regular vector sublattice admits a Hausdorff uo-Lebesgue topology when the global vector lattice admits one. Consequently, there is a going-up-going-down procedure: starting with a Hausdorff o-Lebesgue topology on an order dense ideal, one obtains a Hausdorff uo-Lebesgue topology on the global vector lattice, and then finally also one on every regular vector sublattice.

In view of the going-up-going-down construction, it is evidently desirable to have a class of vector lattices that admit Hausdorff o-Lebesgue topologies because such data can serve as 'germs' for Hausdorff uo-Lebesgue topologies. The vector lattices with separating order continuous duals form such a class, and this is exploited in Section 2.5.

Section 2.6 is concerned with regular vector sublattices of $\mathrm{L}_0(X,\Sigma,\mu)$ for a semi-finite measure μ. Via an application of the going-up-going-down procedure, every regular vector sublattice of $\mathrm{L}_0(X,\Sigma,\mu)$ admits a Hausdorff uo-Lebesgue topology. We give a rigorous proof of the fact that the convergence of nets in such a topology is the convergence in measure on subsets of finite measure. For $\mathrm{L}_p(X,\Sigma,\mu)$, we also discuss how the (in fact) unique Hausdorff uo-Lebesgue topology on these spaces can be described in various seemingly different ways that are still equivalent. The relation between these topologies and minimal and smallest Hausdorff locally solid linear topologies on these spaces is explained.

Section 2.7 is concerned with convergent sequences that can always be found 'within' nets that are convergent in a Hausdorff uo-Lebesgue topology on a vector lattice that has the countable sup property and that has an order dense ideal with a separating order continuous

[1] In the literature, what we call a o-Lebesgue topology is simply called a Lebesgue topology. Now that uo-Lebesgue topologies, with a completely analogous definition, have become objects of a more extensive study, it seems consistent to also add a prefix to the original term.

dual. The precise statement is in Theorem 2.7.6, below; this is one of the main theorems in this paper. It is in the same spirit as the fact that a sequence that converges (globally) in measure always contains a subsequence that converges almost everywhere to the same limit.

Finally, in Section 2.8, we study topological aspects of uo-convergence. The relations between uo-convergence and various order topologies are not at all well understood, but when the global vector lattice has the countable sup property, and also has an order dense ideal with a separating order continuous dual, then a reasonably satisfactory picture emerges. In Theorem 2.8.1 and Theorem 2.8.8, below, various topological closures and (sequential) adherences are then seen to be equal. It is then also possible to give a necessary and sufficient criterion for sequential uo-convergence to be topological; see Corollary 2.8.5, below.

We have tried to be as complete in the development of this part of the theory of uo-convergence as we could, and also to relate to relevant existing results in the literature whenever possible. Any omissions at this point are unintentional.

2.2 Preliminaries

In this section, we collect a number of definitions, notations, conventions and preparatory results. We refer the reader to the textbooks [2], [5], [6], [7], [36], [37], [40], [50], and [51] for general background information on vector lattices and Banach lattices.

2.2.1 Vector lattices, operators, and (unbounded) order convergence

All vector spaces are over the real numbers. Measures take their values in $[0,\infty]$ and are not supposed to satisfy any condition unless otherwise specified. All vector lattices are supposed to be Archimedean. The positive cone of a vector lattice E is denoted by E^+.

Let E be a vector lattice, and let F be a vector sublattice of E. Then F is *order dense in* E when, for every $x \in E$ with $x > 0$, there exists a $y \in F$ such that $0 < y \leq x$; F is called *super order dense in* E when, for every $x \in E^+$, there exists a sequence $(x)_{n=1}^{\infty} \subseteq F^+$ with $x_n \uparrow x$ in E. The vector sublattice F of E is order dense in E if and only if, for every $x \in E^+$, we have $x = \sup\{\, y \in F : 0 \leq y \leq x \,\}$; see [7, Theorem 1.34], for example.

A vector sublattice F of a vector lattice E is called *majorising in* E when, for every $x \in E$, there exists a $y \in F$ such that $x \leq y$. In some sources, such as [11], F is then said to be full in E.

A vector lattice E *has the countable sup property* when, for every non-empty subset S of E that has a supremum in E, there exists an at most countable subset of S that has the same supremum in E as S. In parts of the literature, such as in [36] and [51], E is then said to be order separable.

Let E be a vector lattice, and let $x \in E$. We say that a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in E is *order convergent to* $x \in E$ (denoted by $x_\alpha \xrightarrow{\text{o}} x$) when there exists a net $(y_\beta)_{\beta\in\mathcal{B}}$ in E such that $y_\beta \downarrow 0$ and with the property that, for every $\beta_0 \in \mathcal{B}$, there exists an $\alpha_0 \in \mathcal{A}$ such that $|x - x_\alpha| \leq y_{\beta_0}$ whenever α in $\mathcal{A}$ is such that $\alpha \geq \alpha_0$. Note that the index sets $\mathcal{A}$ and $\mathcal{B}$ need not be equal; for a discussion of the difference between these two possible definitions we refer to [1], for example.

Let E and F be vector lattices. The order bounded operators from E into F will be denoted by $\mathscr{L}_{\mathrm{ob}}(E,F)$, and the regular operators from E into F by $\mathscr{L}_{\mathrm{r}}(E,F)$. When F is Dedekind complete, we have $\mathscr{L}_{\mathrm{ob}}(E,F)=\mathscr{L}_{\mathrm{r}}(E,F)$, and this space is then a Dedekind complete vector lattice; see [7, Theorem 1.18], for example. We write $E^{\sim}$ for $\mathscr{L}_{\mathrm{ob}}(E,\mathbb{R})=\mathscr{L}_{\mathrm{r}}(E,\mathbb{R})$.

A linear operator $T:E\to F$ between two vector lattices E and F is *order continuous* when, for every net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in E, the fact that $x_\alpha\xrightarrow{\mathrm{o}}0$ in E implies that $Tx_\alpha\xrightarrow{\mathrm{o}}0$ in F. When T is positive one can, equivalently, require that, for every net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in E, the fact that $x_\alpha\downarrow 0$ in E imply that $Tx_\alpha\downarrow 0$ in F. An order continuous linear operator between two vector lattices is automatically order bounded; see [7, Lemma 1.54], for example. The order continuous linear operators from E into F will be denoted by $\mathscr{L}_{\mathrm{oc}}(E,F)$. In the literature, the notation $\mathscr{L}_{\mathrm{n}}(E,F)$ is often used. When F is Dedekind complete, $\mathscr{L}_{\mathrm{oc}}(E,F)$ is a band in $\mathscr{L}_{\mathrm{r}}(E,F)$; see [7, Theorem 1.57], for example. We write $E^{\sim}_{\mathrm{oc}}$ for $\mathscr{L}_{\mathrm{oc}}(E,\mathbb{R})$.

The following result is easily established using the Riesz-Kantorovich formulas and their 'dual versions'; see [7, Theorems 1.18 and 1.23], for example. We shall be interested only in the case where the lattice F in it is the real numbers and the band B is the zero band, but the general case comes at no extra cost in the routine proof.

Proposition 2.2.1. *Let E and F be vector lattices, where F is Dedekind complete, and let B be a band in F.*

(1) Let I be an ideal of E. Then the subset

$$\{\,T\in\mathscr{L}_{\mathrm{r}}(E,F):Tx\in B \textit{ for all } x\in I\,\}$$

of $\mathscr{L}_{\mathrm{r}}(E,F)$ is band in $\mathscr{L}_{\mathrm{r}}(E,F)$. For every subset S of I that generates I, it is equal to

$$\{\,T\in\mathscr{L}_{\mathrm{r}}(E,F):|T||x|\in B \textit{ for all } x\in S\,\}.$$

(2) Let $\mathscr{I}$ be an ideal of $\mathscr{L}_{\mathrm{r}}(E,F)$. Then the subset

$$\{\,x\in E:Tx\in B \textit{ for all } T\in\mathscr{I}\,\}$$

of E is an ideal of E. For every subset $\mathscr{S}$ of $\mathscr{I}$ that generates $\mathscr{I}$, it is equal to

$$\{\,x\in E:|T||x|\in B \textit{ for all } T\in\mathscr{S}\,\}.$$

It is a band in E when $\mathscr{I}\subseteq\mathscr{L}_{\mathrm{oc}}(E,F)$.

Let F be a vector sublattice of a vector lattice E. Then F is a *regular vector sublattice of E* when the inclusion map from F into E is order continuous. Equivalently, for every net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in F, the fact that $x_\alpha\downarrow 0$ in F should imply that $x_\alpha\downarrow 0$ in E. It is immediate from the latter criterion that ideals are regular vector sublattices. It is also true that order dense vector sublattices are regular vector sublattices; see [6, Theorem 1.23], for example.

Let $(x_\alpha)_{\alpha\in\mathcal{A}}$ be a net in a vector lattice E, and let $x\in E$. We say that (x_α) is *unbounded order convergent to x in E* (denoted by $x_\alpha\xrightarrow{\mathrm{uo}}x$) when $|x_\alpha-x|\wedge y\xrightarrow{\mathrm{o}}0$ in E for all $y\in$

E^+. Order convergence implies unbounded order convergence to the same limit. For order bounded nets, the two notions coincide. [2]

We shall repeatedly refer to the following collection of results; see [28, Theorem 2.8, Corollary 2.12, and Theorem 3.2].

Theorem 2.2.2. *Let E be a vector lattice, and let F be a vector sublattice of E. Take a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in F.*

(1) Suppose that F is order dense and majorising in E. Then $x_\alpha \xrightarrow{\mathrm{o}} 0$ in F if and only if $x_\alpha \xrightarrow{\mathrm{o}} 0$ in E.

(2) Suppose that F is a regular vector sublattice of E and that $(x_\alpha)_{\alpha\in\mathcal{A}}$ is order bounded in F. Then $x_\alpha \xrightarrow{\mathrm{o}} 0$ in F if and only if $x_\alpha \xrightarrow{\mathrm{o}} 0$ in E.

(3) The following are equivalent:

(a) F is a regular vector sublattice of E;

(b) for every net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in F, the fact that $x_\alpha \xrightarrow{\mathrm{uo}} 0$ in F implies that $x_\alpha \xrightarrow{\mathrm{uo}} 0$ in E;

(c) for every net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in F, $x_\alpha \xrightarrow{\mathrm{uo}} 0$ in F if and only if $x_\alpha \xrightarrow{\mathrm{uo}} 0$ in E.

In the sequel of this paper, we shall encounter restrictions of order continuous linear functionals on vector lattices to vector sublattices. For this, we include the following result. It is based on a theorem of Veksler's. It contains quite a bit more than we shall actually need, but we use the opportunity to present the results in it, and its fourth and fifth parts in particular.

Theorem 2.2.3. *Let E be a vector lattice, let F be a vector sublattice of E, and let G be a Dedekind complete vector lattice. Take $T \in \mathscr{L}_{\mathrm{oc}}(E,G)$.*

(1) Suppose that F is a regular vector sublattice of E. Then the restriction $T|_F : F \to G$ of T to F is order continuous.

(2) Suppose that F is a regular sublattice of E. When $\mathscr{L}_{\mathrm{oc}}(E,G)$ separates the points of E, then $\mathscr{L}_{\mathrm{oc}}(F,G)$ separates the points of F.

(3) Suppose that F is an order dense vector sublattice of E. Then the restriction map $T \mapsto T|_F$ is a positive linear injection from $\mathscr{L}_{\mathrm{oc}}(E,G)$ into $\mathscr{L}_{\mathrm{oc}}(F,G)$.

Suppose that F is an order dense and majorising vector sublattice of E. Then:

(4) the restriction map $T \mapsto T|_F$ is a lattice isomorphism between $\mathscr{L}_{\mathrm{oc}}(E,G)$ and $\mathscr{L}_{\mathrm{oc}}(F,G)$;

(5) $\mathscr{L}_{\mathrm{oc}}(E,G)$ separates the points of E if and only if $\mathscr{L}_{\mathrm{oc}}(F,G)$ separates the points of F.

Proof. Part (1) is clear, and then so is part (2).

It is evident from part (1) that $\mathscr{L}_{\mathrm{oc}}(F,G)$ separates the points of F when $\mathscr{L}_{\mathrm{oc}}(E,G)$ separates the points of E.

Suppose that F is an order dense (hence regular) vector sublattice of E and that $T \in \mathscr{L}_{\mathrm{oc}}(E,G)$ is such that $T|_F = 0$. Take $x \in E^+$. Then $\{\, y \in F : 0 \leq y \leq x \,\} \uparrow x$ in E. Since $T|_F = 0$, the order continuity of T on E then implies that $Tx = 0$. Hence $T = 0$, and we conclude that the restriction map $T \mapsto T|_F$ is a positive linear injection from $\mathscr{L}_{\mathrm{oc}}(E,G)$ into $\mathscr{L}_{\mathrm{oc}}(F,G)$.

[2] Although we shall not need this, it would be less than satisfactory not to mention here that the uo-continuous dual of a vector lattice (defined in the obvious way) has a very concrete description, and is often trivial. According to [27, Proposition 2.2], it is the linear span of the coordinate functionals corresponding to atoms.

Suppose that F is order dense and majorising in E.

Take $S \in \mathscr{L}_{\mathrm{oc}}(F,G)$. In that case, according to a result of Veksler's (see [7, Theorem 1.65]), each of S^+ and S^- can be extended to a positive order continuous operator from E into G. Hence S itself can be extended to an order continuous operator S^{ext} from E into G. By what we have already observed in part (3), such an order continuous extension is unique, and we conclude from this that the map $S \mapsto S^{\mathrm{ext}}$ is a positive linear injection from $\mathscr{L}_{\mathrm{oc}}(F,G)$ into $\mathscr{L}_{\mathrm{oc}}(E,G)$. It is clear that the extension and restriction maps between $\mathscr{L}_{\mathrm{oc}}(E,G)$ and $\mathscr{L}_{\mathrm{oc}}(F,G)$ are each other's inverses. We conclude that the restriction map $T \mapsto T|_F$ is a bi-positive linear bijection between $\mathscr{L}_{\mathrm{oc}}(E,G)$ and $\mathscr{L}_{\mathrm{oc}}(F,G)$. Hence it is a lattice isomorphism, as required.

One direction of the equivalence in part (5) is clear from part (2). For the converse direction, suppose that $\mathscr{L}_{\mathrm{oc}}(F,G)$ separates the points of F. Take $x \in E$ such that $Tx = 0$ for all $T \in \mathscr{L}_{\mathrm{oc}}(E,G)$. Since $\mathscr{L}_{\mathrm{oc}}(E,G)$ is an ideal of $\mathscr{L}_{\mathrm{r}}(E,F)$, Proposition 2.2.1 shows that $T|x| = 0$ for all $T \in \mathscr{L}_{\mathrm{oc}}(E,G)$. Suppose that $x \neq 0$. Then there exists a $y \in F$ such that $0 < y \leq |x|$, and we have $Ty = 0$ for all positive $T \in \mathscr{L}_{\mathrm{oc}}(E,G)$, hence for all $T \in \mathscr{L}_{\mathrm{oc}}(E,G)$. In view of part (4), this is the same as saying that $Sy = 0$ for all $S \in \mathscr{L}_{\mathrm{oc}}(F,G)$. Our assumption yields that $y = 0$; this contradiction shows that we must have $x = 0$. □

2.2.2 Topologies on vector lattices

When E is a vector space, a *linear topology on E* is a (not necessarily Hausdorff) topology that provides E with the structure of a topological vector space. When E is a vector lattice, a *locally solid linear topology on E* is a linear topology on E such that there exists a base of (not necessarily open) neighbourhoods of 0 that are solid subsets of E. For the general theory of locally solid linear topologies on vector lattices we refer to [6]. A locally solid linear topology on E that is also a locally convex linear topology is a *locally convex-solid linear topology*. In that case, there exists a base of neighbourhoods of 0 that consists of absorbing, closed, convex, and solid subsets of E; see [6, p. 59].

When E is a vector lattice, a *locally solid additive topology on E* is a topology that provides the additive group E with the structure of a (not necessarily Hausdorff) topological group, such that there exists a base of (not necessarily open) neighbourhoods of 0 that are solid subsets of E.

Let E be a vector lattice. We say that *order convergence in E is topological* when there exists a (evidently unique) topology on E such that its convergent nets are precisely the order convergent nets, with preservation of limits. It follows from the properties of order convergence that such a topology is automatically a Hausdorff linear topology. Likewise, *unbounded order convergence in E is topological* when there exists a topology on E such that its convergent nets are precisely the nets that are unbounded order convergent, with preservation of limits. Such a topology is again unique, and automatically a Hausdorff linear topology.

A topology τ on a vector lattice E is an *o-Lebesgue topology* when it is a (not necessarily Hausdorff) locally solid linear topology on E such that, for a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in E and $x \in E$, the fact that $x_\alpha \xrightarrow{\mathrm{o}} x$ in E implies that $x_\alpha \xrightarrow{\tau} x$. Equivalently, the fact that $x_\alpha \xrightarrow{\mathrm{o}} 0$ in E should imply that $x_\alpha \xrightarrow{\tau} 0$. A vector lattice need not admit a Hausdorff o-Lebesgue topology. It can be shown, see [6, Example 3.2], that C([0, 1]) does not even admit a Hausdorff locally solid linear topology such that *sequential* order convergence implies topological convergence.

A topology τ on a vector lattice E is a *uo-Lebesgue topology* when it is a (not necessarily Hausdorff) locally solid linear topology on E such that, for a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in E and $x \in E$, the fact that $x_\alpha \xrightarrow{\text{uo}} x$ in E implies that $x_\alpha \xrightarrow{\tau} x$. Equivalently, the fact that $x_\alpha \xrightarrow{\text{uo}} 0$ in E should imply that $x_\alpha \xrightarrow{\tau} 0$. Since order convergence implies unbounded order convergence, a uo-Lebesgue topology is an o-Lebesgue topology.

The following fundamental facts are from [11, Proposition 3.2, 3.4, and 6.2, and Corollary 6.3] and [44, Theorems 5.5, 5.9, and 6.4].

Theorem 2.2.4 (Conradie and Taylor)**.** *Let E be a vector lattice. Then the following are equivalent:*
(1) E admits a Hausdorff o-Lebesgue topology;
(2) E admits a Hausdorff uo-Lebesgue topology;
(3) the partially ordered set of all Hausdorff locally solid linear topologies on E has a minimal element.
When this is the case, the topologies in the parts (2) *and* (3) *are both unique, they coincide, and they are the smallest Hausdorff o-Lebesgue topology on E.*

When E admits a Hausdorff uo-Lebesgue topology, we shall denote the unique such topology by $\widehat{\tau}_E$. In [11], it is denoted by τ_m. For a given vector lattice, there may be several ways to obtain a Hausdorff uo-Lebesgue topology on it. This can then give criteria for the convergence of nets in the common resulting topology that are apparently equivalent, but not always immediately obviously so. See Remark 2.6.4 for this, for example.

Remark 2.2.5. Some caution is necessary when consulting the literature on minimal Hausdorff locally solid linear topologies because in [6, Definition 7.64] such a topology is defined as what would usually be called a *smallest* Hausdorff locally solid linear topologies. When a vector lattice E admits a complete metrisable o-Lebesgue topology, such as a Banach lattice with an order continuous norm, then it admits a smallest (in the usual sense of the word) Hausdorff locally solid linear topology; see [6, Theorem 7.65]. Combining this with Theorem 2.2.4, we see that E then admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$, and that $\widehat{\tau}_E$ is then not just the smallest Hausdorff o-Lebesgue topology, but even the smallest Hausdorff locally solid linear topology on E.

2.3 Unbounded topologies generated by topologies on ideals

We shall now describe how topologies 'of unbounded type' on vector lattices can be obtained from topologies on ideals. There are already several constructions in this vein and accompanying results in the literature; see [11, 21, 32, 44], for example. In the following result, we carry out such a construction in what appears to be the most general possible context. Starting from a locally solid (not necessarily linear or Hausdorff) additive topology on an ideal F of a vector lattice E—which need not be the restriction of a global locally solid additive topology on E—and a non-empty subset of F, we define an 'unbounded' locally solid additive topology on E. We give necessary and sufficient conditions for this new topology on E to be Hausdorff, and also for it to be a linear topology. Various known results in more special cases can then be understood from the general theorem, as will be discussed in Examples 2.3.8 to 2.3.12, below.

The subset S figuring in this construction can be replaced by the ideal that it generates without altering the result. Although it may conceptually be more natural to work with ideals than with subsets, working with arbitrary subsets has the advantage of keeping an eye on a small number of presumably relatively easily manageable 'test elements'. It is for this reason that we carry this along to later results; see also Remark 2.4.4, below. The convenience of this approach will become apparent in the proof of Theorem 2.6.1.

Theorem 2.3.1. *Let E be a vector lattice, let F be an ideal of E, and let τ_F be a (not necessarily Hausdorff) locally solid additive topology on F. Take a non-empty subset S of F.*

There exists a unique (possibly non-Hausdorff) additive topology $\mathrm{u}_S\tau_F$ on E such that, for a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in E, $x_\alpha \xrightarrow{\mathrm{u}_S\tau_F} 0$ in E if and only if $|x_\alpha| \wedge |s| \xrightarrow{\tau_F} 0$ in F for all $s \in S$.

Let $I_S \subseteq F$ be the ideal generated by S in E. For a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in E, $x_\alpha \xrightarrow{\mathrm{u}_S\tau_F} 0$ in E if and only if $|x_\alpha| \wedge |y| \xrightarrow{\tau_F} 0$ in F for all $y \in I_S$.

Furthermore:

(1) the inclusion map from F into E is τ_F–$\mathrm{u}_S\tau_F$-continuous;
(2) the topology $\mathrm{u}_S\tau_F$ on E is a locally solid additive topology;
(3) the following are equivalent:
  (a) $\mathrm{u}_S\tau_F$ is a Hausdorff topology on E;
  (b) τ_F is a Hausdorff topology on F and I_S is order dense in E;
(4) the following are equivalent:
  (i) for all $x \in E$ and $s \in S$,

$$|\varepsilon x| \wedge |s| \xrightarrow{\tau_F} 0 \tag{2.1}$$

  in F as $\varepsilon \to 0$ in $\mathbb{R}$;
  (ii) for all $x \in E$ and $y \in I_S$, $|\varepsilon x| \wedge |y| \xrightarrow{\tau_F} 0$ in F as $\varepsilon \to 0$ in $\mathbb{R}$;
  (iii) $\mathrm{u}_S\tau_F$ is a (possibly non-Hausdorff) linear topology on E.

Proof. Suppose that τ_F is a (not necessarily Hausdorff) locally solid additive topology on F.

The uniqueness of $\mathrm{u}_S\tau_F$ is clear because the nets converging to 0 and then, by translation invariance of the topology, to arbitrary points of E are prescribed.

We turn to the existence of such a topology $\mathrm{u}_S\tau_F$. Take a neighbourhood base $\{U_\lambda\}_{\lambda\in\Lambda}$ of zero in F for τ_F consisting of solid subsets of F. For $y \in I_S$ and $\lambda \in \Lambda$, set

$$V_{\lambda,y} := \{\, x \in E : |x| \wedge |y| \in U_\lambda \,\}. \tag{2.2}$$

The $V_{\lambda,y}$ are solid subsets of E since F is an ideal of E and the U_λ are solid subsets of F. Set

$$\mathscr{N}_0 := \{\, V_{\lambda,y} : \lambda \in \Lambda, y \in I_S \,\}. \tag{2.3}$$

We claim that $\mathscr{N}_0$ is a base of neighbourhoods of zero for a topology on E, which we shall already denote by $\mathrm{u}_S\tau_F$, that provides the additive group E with the structure of a topological group. Necessary and sufficient conditions on $\mathscr{N}_0$ for this can be found in [31, Theorem 3 on p.46]; we now verify these.

Take $V_{\lambda_1,y_1}, V_{\lambda_2,y_2} \in \mathscr{N}_0$. There exists a $\lambda_3 \in \Lambda$ such that $U_{\lambda_3} \subseteq U_{\lambda_1} \cap U_{\lambda_2}$. Take $x \in V_{\lambda_3,|y_1|\vee|y_2|}$. Then

$$|x| \wedge |y_1| \leq |x| \wedge (|y_1| \vee |y_2|) \in U_{\lambda_3} \subseteq U_{\lambda_1}.$$

Since F is an ideal of E and U_{λ_1} is a solid subset of F, this implies that $|x| \wedge |y_1| \in U_{\lambda_1}$, so that $x \in V_{\lambda_1,y_1}$. Likewise, $x \in V_{\lambda_2,y_2}$, and we see that $V_{\lambda_3,|y_1|\vee|y_2|} \subseteq V_{\lambda_1,y_1} \cap V_{\lambda_2,y_2}$.

It is evident that $V_{\lambda,y} = -V_{\lambda,y}$ for all $V_{\lambda,y} \in \mathscr{N}_0$.

Take $V_{\lambda,y} \in \mathscr{N}_0$. There exists a $\mu \in \Lambda$ such that $U_\mu + U_\mu \subseteq U_\lambda$. Then, for all $x_1, x_2 \in V_{\mu,y}$, we have

$$|x_1 + x_2| \wedge |y| \leq |x_1| \wedge |y| + |x_2| \wedge |y| \in U_\mu + U_\mu \subseteq U_\lambda.$$

Since F is an ideal of E and U_λ is a solid subset of F, this implies that $|x_1 + x_2| \wedge |y| \in U_\lambda$, so that $x_1 + x_2 \in V_{\lambda,y}$. Hence $V_{\mu,y} + V_{\mu,y} \subseteq V_{\lambda,y}$.

An appeal to [31, p. 46, Theorem 3] now establishes our claim.

It is clear from the definition of $\mathrm{u}_S\tau_F$ that, for a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in E, $x_\alpha \xrightarrow{\mathrm{u}_S\tau_F} 0$ in E if and only if $|x_\alpha| \wedge |y| \xrightarrow{\tau_F} 0$ in F for all $y \in I_S$.

Certainly, the fact that $|x_\alpha| \wedge |y| \xrightarrow{\tau_F} 0$ in F for all $y \in I_S$ implies that $|x_\alpha| \wedge |s| \xrightarrow{\tau_F} 0$ in F for all $s \in S$. Conversely, suppose that $(x_\alpha)_{\alpha\in\mathcal{A}}$ is a net in E such that $|x_\alpha| \wedge |s| \xrightarrow{\tau_F} 0$ in F for all $s \in S$. Take $y \in I_S$. There exist $s_1, \dots, s_n \in S$ and integers $k_1, \dots, k_n \geq 1$ such that $|y| \leq \sum_{i=1}^n k_i |s_i|$. Hence $|x_\alpha| \wedge |y| \leq \sum_{i=1}^n k_i \left(|x_\alpha| \wedge |s_i|\right)$. Since τ_F is a locally solid additive topology on F, this implies that $|x_\alpha| \wedge |y| \xrightarrow{\tau_F} 0$ in F.

We turn to the parts (1)–(4).

Since F is an ideal of E and the U_λ are solid subsets of F, we have $U_\lambda \subseteq V_{\lambda,y}$ for all $\lambda \in \Lambda$ and $y \in I_S$. This implies that the inclusion map from F into E is τ_F–$\mathrm{u}_S\tau_F$-continuous.

The topology $\mathrm{u}_S\tau_F$ is a locally solid additive topology on E by construction.

Suppose that $\mathrm{u}_S\tau_F$ is a Hausdorff topology on E. Then so is the topology it induces on F, which is weaker than τ_F. Hence τ_F is a Hausdorff topology on F. Take $x \in E$ with $x > 0$. Then there exists a $V_{\lambda,y} \in \mathscr{N}_0$ with $x \notin V_{\lambda,y}$. In particular, $x \wedge |y| \neq 0$. Hence $0 < x \wedge |y| \leq x$. Since $x \wedge |y| \in I_S$, we see that I_S is order dense in E.

Suppose, conversely, that τ_F is a Hausdorff topology on F and that I_S is order dense in E. Take $x \neq 0$ in E. There exists a $y \in I_S$ with $0 < y \leq |x|$. Pick $U_{\lambda_0} \in \{U_\lambda\}_{\lambda\in\Lambda}$ such that $y \notin U_{\lambda_0}$. Then $|x| \wedge |y| = y \notin U_{\lambda_0}$, so that $x \notin V_{\lambda_0,y}$. Hence $\bigcap_{V\in\mathscr{N}_0} V = \{0\}$. By [31, p. 48, Theorem 4], $\mathrm{u}_S\tau_F$ is a Hausdorff additive topology on the topological group E.

We shall now verify the equivalence of the parts (i)–(iii) of (4).

We prove that (i) implies (ii). Take $x \in E$ and $y \in I_S$. There exist $s_1, \dots, s_n \in S$ and integers $k_1, \dots, k_n \geq 1$ such that $|y| \leq \sum_{i=1}^n k_i |s_i|$, and it follows from this that $|\varepsilon x| \wedge |y| \leq \sum_{i=1}^n k_i \left(|\varepsilon x| \wedge |s_i|\right)$ for all $\varepsilon \in \mathbb{R}$. Since τ_F is a locally solid additive topology on F, it follows that $|\varepsilon x| \wedge |y| \xrightarrow{\tau_F} 0$ in F as $\varepsilon \to 0$ in $\mathbb{R}$.

We prove that (ii) implies (iii). Fix $\lambda \in \Lambda$ and $y \in I_S$, and take $x \in E$. Since $|\varepsilon x| \wedge |y| \xrightarrow{\tau_F} 0$ in F as $\varepsilon \to 0$ in $\mathbb{R}$, there exists a $\delta > 0$ such that $|\varepsilon x| \wedge |y| \in U_\lambda$ whenever $|\varepsilon| \leq \delta$. That is, $\varepsilon x \in V_{\lambda,y}$ whenever $|\varepsilon| \leq \delta$. This implies that $V_{\lambda,y}$ is absorbing. Furthermore, since $V_{\lambda,y}$ is a solid subset of E, it is clear that $\varepsilon x \in V_{\lambda,y}$ whenever $x \in V_{\lambda,y}$ and $\varepsilon \in [-1,1]$. Hence $V_{\lambda,y}$ is balanced. Then [5, Theorem 5.6] implies that $\mathrm{u}_S\tau_F$ is a linear topology on E.

We prove that (iii) implies (i). Take $x \in E$. Then $\varepsilon x \xrightarrow{\mathrm{u}_S\tau_F} 0$ in E as $\varepsilon \to 0$ in $\mathbb{R}$. By construction, this implies (and is, in fact, equivalent to) the fact that $|\varepsilon x| \wedge |s| \xrightarrow{\tau_F} 0$ in F for all $s \in S$.

This concludes the proof of the equivalence of the three parts of (4). The proof of the theorem is now complete. □

Definition 2.3.2. The topology $\mathrm{u}_S\tau_F$ in Theorem 2.3.1 is called the *unbounded topology on E that is generated by τ_F via S.*

Remark 2.3.3. It is clear from the two equivalent criteria in Theorem 2.3.1 for a net in E to be $\mathrm{u}_S\tau_F$-convergent to zero that $\mathrm{u}_S\tau_F = \mathrm{u}_{I_S}\tau_F$ for every non-empty subset S of F. Consequently, $\mathrm{u}_{S_1}\tau_F = \mathrm{u}_{S_2}\tau_F$ whenever S_1, S_2 are non-empty subsets of F such that $I_{S_1} = I_{S_2}$.

Remark 2.3.4. In Theorem 2.3.1, suppose that the locally solid additive topology F is the restriction $\tau_E|_F$ of a locally solid additive topology on E. It is then easy to see that $\mathrm{u}_S(\tau_E|_F) = \mathrm{u}_S\tau_E$ for every non-empty subset S of F.

Remark 2.3.5. In Theorem 2.3.1, and also in the remainder of this paper, the topologies of interest are characterised by their convergent nets. It should be noted, however, that in equations (2.2) and (2.3) the proof of Theorem 2.3.1 provides an explicit neighbourhood base of zero in E for $\mathrm{u}_S\tau_F$, in terms of a neighbourhood base of zero in F for τ_F and the ideal I_S. Suppose, for example that τ_F is a (possibly non-Hausdorff) locally convex linear topology on F that is generated by a family $\{\rho_\gamma : \gamma \in \Gamma\}$ of semi-norms on F, as will be the case in Section 2.5. Then the collection of subsets of E of the form

$$\{x \in E : \rho_i(|x| \wedge |y|) < \varepsilon \text{ for } \rho_1, \dots \rho_n \in \Gamma\},$$

where $y \in I_S$, $n \geq 1$, and $\varepsilon > 0$ are arbitrary, is a neighbourhood base of zero in E for $\mathrm{u}_S\tau_F$.

Our next result is concerned with iterating the construction in Theorem 2.3.1. It generalises what is in [44, p. 997].

Proposition 2.3.6. *Let E be a vector lattice, let F_1 be an ideal of E, and let τ_{F_1} be a (not necessarily Hausdorff) locally solid additive topology on F_1. Take a non-empty subset S_1 of F_1, and consider the unbounded topology $\mathrm{u}_{S_1}\tau_{F_1}$ on E that is generated by τ_{F_1} via S_1. Let F_2 be an ideal of E, and let $\left(\mathrm{u}_{S_1}\tau_{F_1}\right)|_{F_2}$ denote the topology on F_2 that is induced on F_2 by $\mathrm{u}_{S_1}\tau_{F_1}$. Then $\left(\mathrm{u}_{S_1}\tau_{F_1}\right)|_{F_2}$ is a locally solid additive topology on F_2. Take a non-empty subset S_2 of F_2. Then $\mathrm{u}_{S_2}\left[\left(\mathrm{u}_{S_1}\tau_{F_1}\right)|_{F_2}\right] = \mathrm{u}_{I_{S_1}\cap I_{S_2}}\tau_{F_1}$. In particular, when S is a non-empty subset of $F_1 \cap F_2$, then $\mathrm{u}_S\left[\left(\mathrm{u}_S\tau_{F_1}\right)|_{F_2}\right] = \mathrm{u}_S\tau_{F_1}$.*

Proof. It is clear from Theorem 2.3.1 that $\left(\mathrm{u}_{S_1}\tau_{F_1}\right)|_{F_2}$ is a locally solid additive topology on F_2. Let $(x_\alpha)_{\alpha\in\mathcal{A}}$ be a net in E. Then we have the following chain of equivalent statements:

$$x_\alpha \xrightarrow{\mathrm{u}_{S_2}\left[\left(\mathrm{u}_{S_1}\tau_{F_1}\right)|_{F_2}\right]} 0 \text{ in } E$$

$$
\begin{aligned}
&\iff |x_\alpha| \wedge |y_2| \xrightarrow{(\mathrm{u}_{S_1}\tau_{F_1})|_{F_2}} 0 \text{ in } F_2 \text{ for all } y_2 \in I_{S_2}\\
&\iff |x_\alpha| \wedge |y_2| \xrightarrow{\mathrm{u}_{S_1}\tau_{F_1}} 0 \text{ in } E \text{ for all } y_2 \in I_{S_2}\\
&\iff |x_\alpha| \wedge |y_2| \wedge |y_1| \xrightarrow{\tau_{F_1}} 0 \text{ in } F_1 \text{ for all } y_1 \in I_{S_1} \text{ and } y_2 \in I_{S_2}\\
&\iff |x_\alpha| \wedge |y| \xrightarrow{\tau_{F_1}} 0 \text{ in } F_1 \text{ for all } y \in I_{S_1} \cap I_{S_2}\\
&\iff x_\alpha \xrightarrow{\mathrm{u}_{I_{S_1}\cap I_{S_2}}\tau_{F_1}} 0 \text{ in } E.
\end{aligned}
$$

Hence $\mathrm{u}_{S_2}\big[\big(\mathrm{u}_{S_1}\tau_{F_1}\big)|_{F_2}\big] = \mathrm{u}_{I_{S_1}\cap I_{S_2}}\tau_{F_1}$. $\square$

Remark 2.3.7. In Proposition 2.3.6, suppose that τ_{F_1} is a (not necessarily Hausdorff) locally solid additive topology on F_1 such that, for all $x \in E$ and $s \in S_1$, $|\varepsilon x| \wedge |s| \xrightarrow{\tau_{F_1}} 0$ in F_1 as $\varepsilon \to 0$ in $\mathbb{R}$. It is then clear from Theorem 2.3.1 that $\mathrm{u}_{S_1}\tau_{F_1}$, $\big(\mathrm{u}_{S_1}\tau_{F_1}\big)|_{F_2}$, and $\mathrm{u}_{I_{S_1}\cap I_{S_2}}\tau_{F_1}$ are (possibly non-Hausdorff) locally solid linear topologies on E, F_2, and E, respectively.

We shall now explain how Theorem 2.3.1 relates to various results already in the literature.

Example 2.3.8. When $F = E$ and τ_E is a locally solid linear topology on $F = E$, the condition in equation (2.1) is automatically satisfied for any non-empty subset S of $F = E$. According to Theorem 2.3.1, $\mathrm{u}_E\tau_E$ is a locally solid linear topology on E that is Hausdorff if and only if τ_E is Hausdorff; this is [44, Theorem 2.3]. Furthermore, when A is an ideal of E, $\mathrm{u}_A\tau_E$ is a locally solid linear topology on E that is Hausdorff if and only if τ_E is Hausdorff and A is order dense in E; this is [44, Propositions 9.3 and 9.4].

Example 2.3.9. Let E be a Banach lattice. In Theorem 2.3.1, we take $F = E$, for τ_F we take the norm topology τ_E on $F = E$, and for $S \subseteq F$ we take $S = F = E$. Then the condition in equation (2.1) is satisfied. According to Theorem 2.3.1, $\mathrm{u}_E\tau_E$ is a Hausdorff locally solid linear topology on E and, for a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in E, $x_\alpha \xrightarrow{\mathrm{u}_E\tau_E} 0$ if and only if $\||x_\alpha| \wedge |y|\| \to 0$ for all $y \in E$. In [21], this type of convergence is called *unbounded norm convergence*, or *un-convergence* for short. It was already observed in [21, Section 7] that it is topological; in [32, p. 746], $\mathrm{u}_F\tau_F$ is then called the *un-topology*.

Example 2.3.10. Let E be a vector lattice, and let F be an ideal of E that is a normed vector lattice. In Theorem 2.3.1, we take for τ_F the norm topology on F, and for $S \subseteq F$ we take $S = F$. According to Theorem 2.3.1, $\mathrm{u}_F\tau_F$ is a (possibly non-Hausdorff) additive topology on E and, for a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in E, $x_\alpha \xrightarrow{\mathrm{u}_F\tau_F} 0$ if and only if $\||x_\alpha| \wedge |y|\| \to 0$ for all $y \in F$. This type of convergence is called *un-convergence with respect to X* in [32]. It was already observed that it is topological in [32, p. 747], where $\mathrm{u}_F\tau_F$ is called the *un-topology on E induced by F*.

In [32, Example 1.3], it is shown that $\mathrm{u}_F\tau_F$ can fail to be a Hausdorff topology on E. Since τ_F is a Hausdorff topology on F, Theorem 2.3.1 shows that the pertinent ideal F in [32, Example 1.3] must fail to be order dense in E; this is indeed easily seen to be the case.

Theorem 2.3.1 implies that $\mathrm{u}_F\tau_F$ is Hausdorff if and only if F is order dense in F; this is [32, Proposition 1.4].

In [32, Example 1.5], it is shown that $\mathrm{u}_F\tau_F$ can fail to be a linear topology on E. According to Theorem 2.3.1, the condition in equation (2.1) must fail to be satisfied in the context of [32, Example 1.5]; this is indeed easily seen to be the case. Theorem 2.3.1 shows that $\mathrm{u}_F\tau_F$ always provides E with an additive topology; this was also noted in [32, p. 748] in that particular context.

In [32, p. 748], the authors observe that $\mathrm{u}_F\tau_F$ is a locally solid linear topology on the vector lattice E whenever E is a normed lattice and the norm on E extends that on F, and also whenever the norm on F is order continuous. Both facts follow from Theorem 2.3.1 because equation (2.1) is then satisfied. This is clear when E is a normed lattice and the norm on E extends that on F. Suppose that the norm on F is order continuous. Take $x \in E$ and $y \in F$. Then $|\varepsilon x| \wedge |y| \xrightarrow{\mathrm{o}} 0$ in E as $\varepsilon \to 0$. Since the net $|\varepsilon x| \wedge |y|$ is order bounded in the ideal F of E, which is a regular vector sublattice of E, Theorem 2.2.2 implies that $|\varepsilon x| \wedge |y| \xrightarrow{\mathrm{o}} 0$ in F, and then $|\varepsilon x| \wedge |y| \xrightarrow{\tau_F} 0$ as $\varepsilon \to 0$ by the order continuity of the norm on F.

Example 2.3.11. For a vector lattice E, we let $|\sigma|(E, E^\sim)$ denote its absolute weak topology; the definition of this locally solid linear topology will be recalled in Section 2.5. Taking $E = F = S$ in Theorem 2.3.1 yields the so-called *unbounded absolute weak topology* $\mathrm{u}_E|\sigma|(E, E^\sim)$ on E. It is a locally solid additive topology on E that is Hausdorff if and only if $E^\sim$ separates the points of E. When E is a Banach lattice, $\mathrm{u}_E|\sigma|(E, E^\sim)$ is a Hausdorff locally solid linear topology on E. It is studied in [52].

Example 2.3.12. In [11, p. 290], a construction is given to obtain a locally solid linear topology on a vector lattice E from a locally solid linear topology on an ideal F of E. This is done using Riesz pseudo-norms, rather than by working with neighbourhood bases of zero as we have done. The key ingredient is to start with a Riesz pseudo-norm p on F, take an element u of F^+, and introduce a map $p_u : E \to \mathbb{R}$ by setting $p_u(x) := p(|x| \wedge u)$ for $x \in E$. It is then remarked that p_u is a Riesz pseudo-norm on E. This need not always be the case, however. By way of counter-example, take for E the vector lattice of all real-valued functions on $\mathbb{R}$, and for F the ideal of E consisting of all bounded functions on $\mathbb{R}$. For p, we take the supremum norm on F. For $u \in F^+$, we choose the constant function 1. We define $x \in E$ by setting $x(t) := t$ for $t \in \mathbb{R}$. Then $p_u(\lambda x) = \||\lambda x| \wedge u\| = 1$ for all non-zero $\lambda \in \mathbb{R}$, whereas we should have that $\lim_{\lambda \to 0} p_u(\lambda x) = 0$. This implies that the topologies on E that are thus constructed, although locally solid additive topologies, need not be linear topologies. This 'pathology' is similar to that in [32, Example 1.5] that was mentioned above; our example here is also quite similar to that in [32, Example 1.5]. Fortunately, in the continuation of the argument in [11], p is taken to be a Riesz pseudo-norm on F that is continuous with respect to a Hausdorff o-Lebesgue topology τ_F on F. In this context, p_u *is* a Riesz pseudo-norm on E. Indeed, since F, being an ideal of E, is a regular vector sublattice of E, Theorem 2.2.2 easily yields that $|\lambda x| \wedge u \xrightarrow{\mathrm{o}} 0$ in F as $\lambda \to 0$. Since τ_F is an o-Lebesgue topology on E, we have $|\lambda x| \wedge u \xrightarrow{\tau_F} 0$ in F as $\lambda \to 0$, and then the continuity of p on F yields that $p_u(\lambda x) \to 0$ as $\lambda \to 0$. Thus the construction in [11] proceeds correctly after all. The results of our systematic investigation with minimal hypotheses in Theorem 2.3.1, however, are more comprehensive than those in [11].

2.4 Hausdorff uo-Lebesgue topologies: going up and going down

In this section, we investigate how, via a going-up-going-down construction, the existence of a Hausdorff o-Lebesgue topology on an order dense ideal of a vector lattice E implies that every regular vector sublattice of E admits a (necessarily unique) Hausdorff uo-Lebesgue topology.

We start by going up.

Proposition 2.4.1. *Let E be a vector lattice, and let F be an ideal of E. Suppose that F admits an o-Lebesgue topology τ_F. Choose a non-empty subset S of F. Then $\mathrm{u}_S\tau_F$ is a uo-Lebesgue topology on E. It is a (necessarily unique) Hausdorff uo-Lebesgue topology on E if and only if τ_F is a Hausdorff topology on F and the ideal I_S that is generated by S is order dense in E.*

Proof. We know from Theorem 2.3.1 that $\mathrm{u}_S\tau_F$ is a locally solid additive topology on E. In order to see that it is a linear topology on E, we verify the condition in equation (2.1). Take x in E and s in S. Then $|\varepsilon x| \wedge |s| \xrightarrow{\mathrm{o}} 0$ in E as $\varepsilon \to 0$ in $\mathbb{R}$. Since F, being in ideal of E, is a regular vector sublattice of E, Theorem 2.2.2 shows that $|\varepsilon x| \wedge |s| \xrightarrow{\mathrm{o}} 0$ in F. Since τ_F is an o-Lebesgue topology on F, this implies that $|\varepsilon x| \wedge |s| \xrightarrow{\tau_F} 0$ in F as $\varepsilon \to 0$ in $\mathbb{R}$, as required.

To conclude the proof, suppose that $(x_\alpha)_{\alpha\in\mathcal{A}}$ is a net in E such that $x_\alpha \xrightarrow{\mathrm{uo}} 0$ in E. Take $s \in S$. Then $|x_\alpha| \wedge |s| \xrightarrow{\mathrm{o}} 0$ in E. Again, since F is a regular vector sublattice of E, Theorem 2.2.2 shows that $|x_\alpha| \wedge |s| \xrightarrow{\mathrm{o}} 0$ in F. Since τ_F is an o-Lebesgue topology on F, this implies that $|x_\alpha| \wedge |s| \xrightarrow{\tau_F} 0$ in F. It now follows from Theorem 2.3.1 that $x_\alpha \xrightarrow{\mathrm{u}_S\tau_F} 0$ in E, as required.

The uniqueness statement is clear from Theorem 2.2.4. □

The combination of Theorem 2.3.1 and Proposition 2.4.1 immediately yields the following.

Theorem 2.4.2. *Let E be a vector lattice. Suppose that E has an order dense ideal F that admits a Hausdorff o-Lebesgue topology. Then E admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$. This topology $\widehat{\tau}_E$ is equal to $\mathrm{u}_S\tau_F$ for every subset S of F such that the ideal $I_S \subseteq F$ that is generated by S is order dense in E.*

For a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in E, the following are equivalent:

(1) $x_\alpha \xrightarrow{\widehat{\tau}_E} 0$ *in E;*

(2) $|x_\alpha| \wedge |s| \xrightarrow{\tau_F} 0$ *in F for all $s \in S$;*

(3) $|x_\alpha| \wedge |y| \xrightarrow{\tau_F} 0$ *in F for all $y \in F$.*

Remark 2.4.3. For the case in Theorem 2.4.2 where $S = F$ and τ_F is the restriction of a Hausdorff o-Lebesgue topology on E, it was already established in [44, Theorem 9.6] that $\mathrm{u}_F\tau_F$ is a Hausdorff uo-Lebesgue topology on E. It is, therefore, of some importance to point out that not every Hausdorff o-Lebesgue topology on an order dense ideal is the restriction of a Hausdorff o-Lebesgue topology on the enveloping vector lattice. By way of example, consider the order dense ideal c_0 of ℓ^∞. Since the supremum norm on c_0 is order continuous, the usual norm topology τ_{c_0} on c_0 is a Hausdorff o-Lebesgue topology. However, there does not even exist a possibly non-Hausdorff o-Lebesgue topology τ_{ℓ^∞} on ℓ^∞ that extends τ_{c_0}. In order to see this, consider the sequence of standard unit vectors $(e)_{n=1}^\infty$ in ℓ^∞. We have $e_n \xrightarrow{\mathrm{o}} 0$ in ℓ^∞ , which would imply that $e_n \xrightarrow{\tau_{\ell^\infty}} 0$ in ℓ^∞. Since τ_{ℓ^∞} extends τ_{c_0}, we would have that $e_n \to 0$ in norm. This contradiction shows that such an extension does not exist.

Although the terminology is not used as such, the fact that $\mathrm{u}_F\tau_F$ is a Hausdorff uo-Lebesgue topology on E is implicit in the construction in [11, p. 290].

Remark 2.4.4. We are not aware of a reference where it is noted, as in part (2), that convergence of a net in the Hausdorff uo-Lebesgue topology on E can be established by using a (presumably small and manageable) subset S of F instead of the full ideal F. This nontrivial fact, which relies on the uniqueness of a Hausdorff uo-Lebesgue topology, appears to be of some practical value.

In view of the uniqueness of a Hausdorff uo-Lebesgue topology (see Theorem 2.2.4), the following is now clear from Theorem 2.4.2.

Corollary 2.4.5. *Let E be a vector lattice, and suppose that E has order dense ideals F_1 and F_2, each of which admits a Hausdorff o-Lebesgue topology. For $i = 1, 2$, choose a Hausdorff o-Lebesgue topology τ_{F_i} on F_i, and choose a non-empty subset S_i of F_i such that the ideal $I_{S_i} \subseteq F_i$ that is generated by S_i in E is order dense in E. Then $\mathrm{u}_{S_1}\tau_{F_1}$ and $\mathrm{u}_{S_2}\tau_{F_2}$ are both equal to the (necessarily unique) uo-Lebesgue topology topology $\widehat{\tau}_E$ on E.*

Remark 2.4.6. The case in Corollary 2.4.5 where $S_1 = F_1$ and $S_2 = F_2$ is [11, Proposition 3.2].

The case where, for $i = 1, 2$, $S_i = F_i$ and τ_{F_i} is the restriction to F_i of a Hausdorff o-Lebesgue topology τ_i on E, is a part of [44, Theorem 9.6]. Note, however, that our underlying proof in Proposition 2.4.1 that $\mathrm{u}_S\tau_F$ is a uo-Lebesgue topology is direct, whereas in the proof of [44, Theorem 9.6] the identification of a Hausdorff uo-Lebesgue topology as a minimal Hausdorff locally solid topology as in Theorem 2.2.4 is used.

Complementing the preceding going-up results, we cite the following going-down result; see [44, Proposition 5.12].

Proposition 2.4.7 (Taylor)**.** *Suppose that the vector lattice E admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$. Take a vector sublattice F of E. Then F is a regular vector sublattice of E if and only if the restriction $\widehat{\tau}_E|_F$ of $\widehat{\tau}_E$ to F is a (necessarily unique) Hausdorff uo-Lebesgue topology on F.*

A variation on this theme, with a wider range of topologies to use for testing the regularity of a vector sublattice, is the following.

Proposition 2.4.8. *Suppose that the vector lattice E admits a Hausdorff o-Lebesgue topology τ_E. Take a vector sublattice F of E. Then F is a regular vector sublattice of E if and only if the restriction $\tau_E|_F$ of τ_E to F is a Hausdorff o-Lebesgue topology on F.*

Proof. Once one recalls that, by definition, order convergence of a net to 0 in the regular vector sublattice F of E implies order convergence of the net to 0 in E, the proof is a straightforward minor adaptation of that of [44, Proposition 5.12]. $\square$

We now have the following overview theorem concerning Hausdorff o-Lebesgue topologies and Hausdorff uo-Lebesgue topologies on a vector lattice and on its order dense ideals. It is easily established by recalling that a uo-Lebesgue topology is an o-Lebesgue topology, that an ideal is a regular vector sublattice, and by using Theorem 2.4.2, Proposition 2.4.7, and Proposition 2.4.8.

Theorem 2.4.9. *Let E be a vector lattice, and let F be an order dense ideal of E.*
(1) Suppose that E admits a Hausdorff o-Lebesgue topology τ_E. Then the restricted topology $\tau_E|_F$ is a Hausdorff o-Lebesgue topology on E.
(2) Suppose that E admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$. Then the restricted topology $\widehat{\tau}_E|_F$ is a (necessarily unique) Hausdorff uo-Lebesgue topology on F.
(3) The following are equivalent:
  (a) F admits a Hausdorff o-Lebesgue topology;
  (b) F admits a Hausdorff uo-Lebesgue topology;
  (c) E admits a Hausdorff o-Lebesgue topology;
  (d) E admits a Hausdorff uo-Lebesgue topology.

In that case, the unique uo-Lebesgue topology $\widehat{\tau}_E$ on E equals $\mathrm{u}_S\tau_F$ for every Hausdorff o-Lebesgue topology on F and every subset S of F such that the ideal $I_S \subseteq F$ is order dense in E, and the following are equivalent:
  (i) $x_\alpha \xrightarrow{\widehat{\tau}_E} 0$ in E;
  (ii) $|x_\alpha| \wedge |s| \xrightarrow{\tau_F} 0$ in F for all $s \in S$;
  (iii) $|x_\alpha| \wedge |y| \xrightarrow{\tau_F} 0$ in F for all $y \in F$.

We conclude this section with a short discussion of Banach lattices with order continuous norms. Evidently, the norm topologies on such Banach lattices are Hausdorff o-Lebesgue topologies. As already noted in [44, p. 993], Theorem 2.4.2 allows one to identify the so-called un-topologies (see [21, Section 7] and [32, p. 746]) on such lattices as the Hausdorff uo-Lebesgue topologies that these spaces apparently admit. Consequently, we have the following result. The case where $S = E$ can be found in [44, p. 993].

Proposition 2.4.10. *Let E be a Banach lattice with an order continuous norm and norm topology τ_E. Then E admits a (necessarily unique) uo-Lebesgue topology.*

Choose a subset S of E such that the ideal I_S that is generated by S in E is order dense in E. Then:
(1) $\mathrm{u}_S\tau_E$ is the uo-Lebesgue topology $\widehat{\tau}_E$ of E;
(2) when $(x_\alpha)_{\alpha\in\mathcal{A}}$ is a net in E, then $x_\alpha \xrightarrow{\widehat{\tau}_E} 0$ in E if and only if $\| |x_\alpha| \wedge |s| \| \to 0$ for all $s \in S$; equivalently, if and only if $\| |x_\alpha| \wedge |y| \| \to 0$ for all $y \in E$.

There is an alternative reason why Banach lattices with an order continuous norms admit Hausdorff uo-Lebesgue topologies, and this results in an alternative description of these topologies; see Corollary 2.5.4, below.

Finally, suppose that E is a vector lattice that has order dense ideals F_1 and F_2 that are Banach lattices with order continuous norm topologies τ_{F_1} and τ_{F_2}, respectively. Then it is immediate from Corollary 2.4.5 that E admits a Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$, and that $\mathrm{u}_{F_1}\tau_{F_1}$ and $\mathrm{u}_{F_2}\tau_{F_2}$ are both equal to $\widehat{\tau}_E$. As discussed in Example 2.3.10, this can, using the terminology in [32], be rephrased as stating that F_1 and F_2 induce the same un-topology on E. We have thus retrieved [32, Theorem 2.6].

2.5 uo-Lebesgue topologies generated by absolute weak topologies on order dense ideals

In this section, we shall be concerned with vector lattices having order dense ideals with separating order continuous duals as a source for Hausdorff uo-Lebesgue topologies on the vector lattices themselves.

We start by recapitulating some facts from [6, p. 63–64]. Let E be a vector lattice, and let A be a non-empty subset of the order dual $E^\sim$ of E. For $\varphi \in A$, define the lattice seminorm $\rho_\varphi : E \to [0,\infty)$ by setting $\rho_\varphi(x) := |\varphi|(|x|)$ for $x \in E$. Then the locally convex-solid linear topology on E that is generated by the family $\{\rho_\varphi : \varphi \in A\}$ is called the *absolute weak topology generated by A on E*; it is denoted by $|\sigma|(E,A)$. With I_A denoting the ideal generated by A in $E^\sim$, we have $|\sigma|(E,A) = |\sigma|(E,I_A)$. Using Proposition 2.2.1, one easily concludes that $|\sigma|(E,A)$ is Hausdorff if and only if I_A separates the points of E. Although we shall not use it, let us still remark that it is not difficult to see that a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in E is $|\sigma|(E,A)$-convergent to zero if and only if $\varphi(x_\alpha) \to 0$ uniformly for φ in each fixed order interval of I_A. Thus absolute weak topologies are more natural than is perhaps apparent from their definition.

The following is now clear.

Lemma 2.5.1. *Let E be a vector lattice, and let A be a non-empty subset of $E_{\mathrm{oc}}^\sim$. Let I_A denote the ideal that is generated by A in $E_{\mathrm{oc}}^\sim$. Then $|\sigma|(E,A) = |\sigma|(E,I_A)$ is an o-Lebesgue topology on E that is even locally convex-solid. It is a Hausdorff topology if and only if I_A separates the points of E. When $(x_\alpha)_{\alpha\in\mathcal{A}}$ is a net in E, then $x_\alpha \xrightarrow{|\sigma|(E,A)} 0$ in E if and only if $|\varphi|(|x_\alpha|) \to 0$ for all $\varphi \in A$; equivalently, if and only if $|\varphi|(|x_\alpha|) \to 0$ for all $\varphi \in I_A$.*

Now that Lemma 2.5.1 provides a whole class of vector lattices admitting Hausdorff o-Lebesgue topologies, we can use these as input for Theorem 2.4.2. Taking the convergence statements in Lemma 2.5.1 into account, we arrive at the following.

Theorem 2.5.2. *Let E be a vector lattice. Suppose that E has an order dense ideal F such that $F_{\mathrm{oc}}^\sim$ separates the points of F. Then E admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$.*

Choose a subset A of $F_{\mathrm{oc}}^\sim$ such that the ideal I_A that is generated by A in $F_{\mathrm{oc}}^\sim$ separates the points of F, and choose a subset S of F such that the ideal $I_S \subseteq F$ that is generated by S is order dense in E. Then:

(1) $\mathrm{u}_S|\sigma|(F,A)$ and $\mathrm{u}_F|\sigma|(F,I_A)$ are both equal to $\widehat{\tau}_E$;

(2) *for a net* $(x_\alpha)_{\alpha\in\mathcal{A}}$ *in* E, $x_\alpha \xrightarrow{\widehat{\tau}_E} 0$ *in* E *if and only if* $|\varphi|(|x_\alpha|\wedge|s|)\to 0$ *for all* $\varphi\in A$ *and* $s\in S$; *equivalently, if and only if* $|\varphi|(|x_\alpha|\wedge|y|)\to 0$ *for all* $\varphi\in F_{\mathrm{oc}}^{\sim}$ *and* $y\in F$.

For the sake of completeness, we recall that a regular vector sublattice of a vector lattice E as in the theorem also has a (necessarily unique) Hausdorff uo-Lebesgue topology, and that this topology is the restriction of $\widehat{\tau}_E$ to the vector sublattice.

Remark 2.5.3. As noted in Remark 2.3.5, one can give an explicit neighbourhood base at zero for the topology $\widehat{\tau}_E$ in Theorem 2.5.2.

For Banach lattices with order continuous norms, the order/norm dual consists of order continuous linear functionals only. Hence we have the following result, which should be compared to Proposition 2.4.10 where the same Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$ is also identified as the un-topology.

Corollary 2.5.4. *A Banach lattice* E *with an order continuous norm admits a (necessarily unique) Hausdorff uo-Lebesgue topology* $\widehat{\tau}_E$*, namely* $\mathrm{u}_E|\sigma|(E,E^*)$.

The following gives a necessary condition for convergence in a Hausdorff uo-Lebesgue topology. It is essential in the proof of Theorem 2.7.6, below.

Proposition 2.5.5. *Let* E *be a vector lattice that admits a (necessarily unique) Hausdorff uo-Lebesgue topology* $\widehat{\tau}_E$*, and let* $(x_\alpha)_{\alpha\in\mathcal{A}}$ *be a net in* E *such that* $x_\alpha \xrightarrow{\widehat{\tau}_E} 0$ *in* E. *Take an ideal* F *of* E *such that* $F_{\mathrm{oc}}^{\sim}$ *separates the points of* F. *Then* $|\varphi|(|x_\alpha|\wedge|y|)\longrightarrow 0$ *for all* $\varphi\in F_{\mathrm{oc}}^{\sim}$ *and* $y\in F$.

Proof. Take $\varphi\in F_{\mathrm{oc}}^{\sim}$ and $y\in F$. Since $\widehat{\tau}_E$ is a locally solid topology, we have $|x|_\alpha\wedge|y| \xrightarrow{\widehat{\tau}_E} 0$ in E. It follows from Proposition 2.4.7 that F has a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_F$ and that $|x|_\alpha\wedge|y| \xrightarrow{\widehat{\tau}_F} 0$. Now we apply Theorem 2.5.2 with $E=F$ to see that $|\varphi|((|x_\alpha|\wedge|y|)\wedge|y|)\longrightarrow 0$. □

We shall now consider the order dual $E^{\sim}$ of a vector lattice E. For $x\in E$, we set

$$\varphi_x(\varphi) := \varphi(x)$$

for $\varphi\in E^{\sim}$. Then $\varphi_x\in(E^{\sim})_{\mathrm{oc}}^{\sim}$, and the map $\varphi: E\to E^{\sim}$ is a lattice homomorphism; see [6, p. 43]. Since $\varphi(E)$ already separates the points of $E^{\sim}$, we see that $(E^{\sim})_{\mathrm{oc}}^{\sim}$ separates the points of $E^{\sim}$.

We can now apply Theorem 2.5.2 twice. In both cases, we replace E with $E^{\sim}$, and we choose $E^{\sim}$ for both F and S. In the first application, we choose $(E^{\sim})_{\mathrm{oc}}^{\sim}$ for A; in the second, we choose $\varphi(E)$. The result is as follows.

Corollary 2.5.6. *Let* E *be a vector lattice. Then the order dual* $E^{\sim}$ *of* E *admits a (necessarily unique) Hausdorff uo-Lebesgue topology* $\widehat{\tau}_{E^{\sim}}$.

Moreover:

(1) $\mathrm{u}_{E^{\sim}}|\sigma|(E^{\sim},(E^{\sim})_{\mathrm{oc}}^{\sim})$ *and* $\mathrm{u}_{E^{\sim}}|\sigma|(E^{\sim},E)$ *are both equal to* $\widehat{\tau}_{E^{\sim}}$;

(2) *when* $(\varphi_\alpha)_{\alpha\in\mathcal{A}}$ *is a net in* $E^{\sim}$, *then* $\varphi_\alpha \xrightarrow{\widehat{\tau}_{E^{\sim}}} 0$ *in* E *if and only if* $|\xi|(|\varphi_\alpha|\wedge|\varphi|)\to 0$ *for all* $\xi\in(E^{\sim})_{\mathrm{oc}}^{\sim}$ *and* $\varphi\in E^{\sim}$; *equivalently, if and only if* $(|\varphi_\alpha|\wedge|\varphi|)(|x|)\to 0$ *for all* $x\in E$ *and* $\varphi\in E^{\sim}$.

Remark 2.5.7.

(1) As in the case of Theorem 2.5.2, Remark 2.3.5 shows how to give an explicit neighbourhood base at zero for the topology $\widehat{\tau}_{E^\sim}$ in Corollary 2.5.6.

(2) By Proposition 2.4.7, every regular sublattice of the order dual of a vector lattice also admits a (necessarily unique) Hausdorff Lebesgue topology that can be described in two ways. For an ideal, one of these descriptions is already in [44, Example 5.8].

(3) Corollary 2.5.6 shows that, in particular, the norm/order dual E^* of a Banach lattice admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_{E^*}$, namely the so-called unbounded absolute weak *-topology $\mathrm{u}_{E^*}|\sigma|(E^*, E)$. This was already observed in [44, Lemma 6.6].

2.6 Regular vector sublattices of $\mathrm{L}_0(X, \Sigma, \mu)$ for semi-finite measures

Let (X, Σ, μ) be a measure space, and write $\mathrm{L}_0(X, \Sigma, \mu)$ for the vector lattice of all real-valued Σ-measurable functions on X, with identification of two functions when they agree μ-almost everywhere. In this section we show that, for semi-finite μ, every regular sublattice of $\mathrm{L}_0(X, \Sigma, \mu)$ admits a (necessarily unique) Hausdorff uo-Lebesgue topology, and that a net converges in this topology if and only if it converges in measure on subsets of finite measure; see Theorem 2.6.3, below.

For some regular sublattices of $\mathrm{L}_0(X, \Sigma, \mu)$, it is quite obvious that they admit a Hausdorff uo-Lebesgue topology. Recall that the spaces $\mathrm{L}_p(X, \Sigma, \mu)$ for p such that $1 \leq p < \infty$ have order continuous norms for all measures μ; see [5, Theorem 13.7], for example. Hence their norm topologies are Hausdorff o-Lebesgue topologies, and then their un-topologies are the Hausdorff uo-Lebesgue topologies on these spaces. Alternatively, one can observe that their order continuous duals separate their points, and then also identify the Hausdorff uo-Lebesgue topologies on these spaces as the unbounded absolute weak topologies. In a similar vein, when μ is σ-finite, every ideal of $L_0(X, \Sigma, \mu)$ that can be supplied with a lattice norm has a separating order continuous dual. This result of Lozanovsky's (see [2, Theorem 5.25], for example) then implies that such a normed function space admits a Hausdorff uo-Lebesgue topology.

How about the spaces $\mathrm{L}_p(X, \Sigma, \mu)$ for $0 \leq p < 1$? There is no norm to work with, and it may well be the case that their order continuous duals are even trivial. Indeed, when μ is atomless, then, according to a results of Day's, the order continuous dual of $\mathrm{L}_p(X, \Sigma, \mu)$ is trivial for $0 < p < 1$; see [5, Theorem 13.31], for example. According to [51, Exercise 25.2], the order continuous dual of $\mathrm{L}_0(X, \Sigma, \mu)$ is trivial for every σ-finite measure with the property that, for any measurable subset A such that $0 < \mu(A) < \infty$ and for any α such that $0 < \alpha < \mu(A)$, there exists a measurable subset A' of A such that $\mu(A') = \alpha$. Taking [49, Exercise 10.12 on p. 67] into account, we see that, in particular, the order continuous dual of $\mathrm{L}_0(X, \Sigma, \mu)$ is trivial for all atomless σ-finite measures.

In spite of the failure of the two obvious approaches, it is still possible to show that all spaces $\mathrm{L}_p(X, \Sigma, \mu)$ for $0 \leq p < 1$ admit Hausdorff uo-Lebesgue topologies, provided that the measure is semi-finite. For such μ, this is even true for all regular vector sublattices of

$\mathrm{L}_0(X,\Sigma,\mu)$. This can be seen via the going-up-going-down approach from Section 2.4, and we shall now elaborate on this. We start with a few preliminary remarks.

Recall that a measure space (X,Σ,μ) is said to be *semi-finite* if, for any $A\in\Sigma$ with $\mu(A)=\infty$, there exists a measurable subset A' of A such that $0<\mu(A')<\infty$. Every σ-finite measure is semi-finite. For an arbitrary measure μ and an arbitrary p such that $1\leq p<\infty$, it is easy to see that the ideal $\mathrm{L}_p(X,\Sigma,\mu)$ of $\mathrm{L}_0(X,\Sigma,\mu)$ is order dense in $\mathrm{L}_0(X,\Sigma,\mu)$ if and only if μ is semi-finite. In that case, the ideal that is generated in $\mathrm{L}_0(X,\Sigma,\mu)$ by the subset $S:=\{\,1_A : A\in\Sigma \text{ has finite measure}\,\}$ of $\mathrm{L}_p(X,\Sigma,\mu)$ is obviously also order dense in $\mathrm{L}_0(X,\Sigma,\mu)$.

Let (X,Σ,μ) be a measure space. Take $f\in\mathrm{L}_0(X,\Sigma,\mu)$. Then a net $(f_\alpha)_{\alpha\in\mathcal{A}}$ in $\mathrm{L}_0(X,\Sigma,\mu)$ *converges to f in measure on subsets of finite measure* when, for all $A\in\Sigma$ such that $\mu(A)<\infty$ and for all $\varepsilon>0$, $\mu(\{\,x\in A : |f_\alpha(x)-f(x)|\geq\varepsilon\,\})\to 0$. In that case, we write $f_\alpha\xrightarrow{\mu^*}f$, using as asterisk to distinguish this convergence from the perhaps more usual global convergence in measure.

The following is the core result of this section. We recall that, as already mentioned, the spaces $\mathrm{L}_p(X,\Sigma,\mu)$ have order continuous norms for all measures μ and for all p such that $1\leq p<\infty$, so that their norm topologies are Hausdorff o-Lebesgue topologies.

Theorem 2.6.1. *Let $E=\mathrm{L}_0(X,\Sigma,\mu)$, where μ is a semi-finite measure. Then G admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$.*

Take a net $(f_\alpha)_{\alpha\in\mathcal{A}}$ in E. Then the following are equivalent for every p such that $1\leq p<\infty$:

(1) $f_\alpha\xrightarrow{\widehat{\tau}_E}0$;

(2)

$$\int_X |f_\alpha|^p\wedge 1_A\,d\mu=\|\,|f_\alpha|\wedge|1_A|\,\|_p^p\to 0$$

for every measurable subset A of X with finite measure;

(3)

$$\int_X |f_\alpha|^p\wedge|f|^p\,d\mu=\|\,|f_\alpha|\wedge|f|\,\|_p^p\to 0$$

for every $f\in\mathrm{L}_p(X,\Sigma,\mu)$;

(4) $f_\alpha\xrightarrow{\mu^*}f$.

Proof. We know from the semi-finiteness of μ that, for p such that $1\leq p\leq\infty$, $L_p(X,\Sigma,\mu)$ is an order dense ideal of $\mathrm{L}_0(X,\Sigma,\mu)$. Since $L_p(X,\Sigma,\mu)$ admits a Hausdorff o-Lebesgue topology when $1\leq p<\infty$, Theorem 2.4.2 shows that $\mathrm{L}_0(X,\Sigma,\mu)$ admits a (necessarily unique) Hausdorff uo-Lebesgue topology, and also that the statements in the parts (1), (2), and (3) of the present theorem are equivalent for all such p.

We show that part (3) implies part (4). Take a measurable subset A of X with finite measure, and let $\varepsilon>0$. Since $\varepsilon 1_A\in\mathrm{L}_p(X,\Sigma,\mu)$, we have, by assumption,

$$\int_X |f_\alpha|^p\wedge(\varepsilon^p 1_A)\,\mathrm{d}\mu\to 0.$$

Because

$$\int_X |f_\alpha|^p \wedge (\varepsilon^p 1_A)\,\mathrm{d}\mu \geq \int_{\{x \in A : |f_\alpha(x)| \geq \varepsilon\}} \varepsilon^p\,\mathrm{d}\mu = \varepsilon^p \mu(\{x \in A : |f_\alpha(x)| \geq \varepsilon\})$$

we conclude that $\mu(\{x \in A : |f_\alpha(x)| \geq \varepsilon\}) \to 0$. Hence $f_\alpha \xrightarrow{\mu^*} 0$.

We show that part (4) implies part (2). Take a measurable subset A of X with finite measure, and take $\varepsilon > 0$. Choose a $\delta > 0$ such that $\delta^p \mu(A) < \varepsilon/2$. Then

$$\begin{aligned}\int_X |f_\alpha|^p \wedge 1_A\,\mathrm{d}\mu &= \int_{\{x \in A : |f_\alpha(x)|^p \geq \delta^p\}} |f_\alpha|^p \wedge 1_A\,\mathrm{d}\mu + \int_{\{x \in A : |f_\alpha(x)|^p < \delta^p\}} |f_\alpha|^p \wedge 1_A\,\mathrm{d}\mu \\ &\leq \int_{\{x \in A : |f_\alpha(x)|^p \geq \delta^p\}} 1\,\mathrm{d}\mu + \int_A \delta^p\,\mathrm{d}\mu \\ &\leq \mu(\{x \in A : |f_\alpha(x)| \geq \delta\}) + \varepsilon/2.\end{aligned}$$

By our assumption, there exists an $\alpha_0 \in \mathcal{A}$ such that $\mu(\{x \in A : |f_\alpha(x)| \geq \delta\}) < \varepsilon/2$ for all $\alpha \geq \alpha_0$. Then $\int_X |f_\alpha|^p \wedge 1_A\,\mathrm{d}\mu < \varepsilon$ for all $\alpha \geq \alpha_0$. Hence $\int_X |f_\alpha| \wedge 1_A\,\mathrm{d}\mu \to 0$. □

Remark 2.6.2.

(1) We are not aware of a proof of Theorem 2.6.1 in the literature. It is stated in [11, p. 292] that the parts (1) and (4) are equivalent, but there only a reference is given to [24, 65K and 63L]. Since [24, 63L] relies on the solution of the non-trivial exercise [24, Exercise 63M(j)] for which a solution is not provided, we thought it appropriate to give an independent proof in the present paper.

(2) The equivalence of the parts (3) and (4) for finite measures and sequences was also established by different methods in [45, Example 23]. Still earlier, this case was covered in [21, Corollary 4.2], with a proof in the same spirit as our proof.

As an immediate consequence of Proposition 2.4.7 and Theorem 2.6.1, we obtain the following result via our going-up-going-down approach.

Theorem 2.6.3. *Let (X, Σ, μ) be a measure space, where μ is a semi-finite measure. Take a regular vector sublattice E of $\mathrm{L}_0(X, \Sigma, \mu)$. Then E admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$. This topology $\widehat{\tau}_E$ on E is the restriction of the Hausdorff uo-Lebesgue topology on $\mathrm{L}_0(X, \Sigma, \mu)$. A net $(f_\alpha)_{\alpha \in \mathcal{A}}$ in E converges to zero in $\widehat{\tau}_E$ if and only if it satisfies one of the three equivalent criteria in the parts (2), (3), and (4) of Theorem 2.6.1. In particular, it is $\widehat{\tau}_E$-convergent to zero if and only if it converges to zero in measure on subsets of finite measure.*

Remark 2.6.4. Let p be such that $1 \leq p < \infty$. For arbitrary measures, Proposition 2.4.10 and Corollary 2.5.4 both give a description of the convergent nets in the Hausdorff uo-Lebesgue topology on $\mathrm{L}_p(X, \Sigma, \mu)$. The former as the convergent nets in the un-topology, and the latter as the convergent nets in the unbounded absolute weak topology, respectively. When μ is semi-finite, Theorem 2.6.3 gives a third description as the convergence in measure on subsets of finite measure.

Also for $p = \infty$, Theorem 2.6.3 shows that $\mathrm{L}_\infty(X, \Sigma, \mu)$ admits a (necessarily unique) Hausdorff uo-Lebesgue topology whenever μ is semi-finite, and gives a description of its convergent nets. When μ is a localisable measure, two more descriptions are possible. We refer to [25, 211G] for the definition of localisable measures, and note that σ-finite measures are localisable, and that localisable measures are semi-finite. Indeed, for localisable measures, $\mathrm{L}_\infty(X, \Sigma, \mu)$ is the order dual of $L_1(X, \Sigma, \mu)$; see [25, 243G(b)]. Hence Corollary 2.5.6 shows once more that $\mathrm{L}_\infty(X, \Sigma, \mu)$ admits a Hausdorff uo-Lebesgue topology when μ is localisable, and gives a second and third description of its convergent nets.

Remark 2.6.5. Let (X, Σ, μ) be a measure space, where μ is a semi-finite measure.

Let p be such that $0 < p < \infty$. The combination of Theorem 2.6.3 and Remark 2.2.5 shows that the topology of convergence in measure on subsets of finite measure is the *smallest* Hausdorff locally solid linear topology on $\mathrm{L}_p(X, \Sigma, \mu)$.[3] For σ-finite measures, this can already be found in [6, Theorem 7.74], where it is also established that the usual metric topology is then the largest Hausdorff locally solid linear topology.

For $p = \infty$, the combination of Theorem 2.6.3 and Theorem 2.2.4 shows that the topology of convergence in measure on subsets of finite measure is the unique *minimal* Hausdorff locally solid linear topology on $\mathrm{L}_\infty(X, \Sigma, \mu)$. It seems worthwhile to note that, when μ is, in fact, σ-finite, and also non-atomic, [6, Theorem 7.75] shows that there is now no *smallest* Hausdorff locally solid linear topology on $\mathrm{L}_\infty(X, \Sigma, \mu)$.

Remark 2.6.6. Let $(x_n)_{n=1}^\infty$ be a sequence in $\mathrm{L}_0(X, \Sigma, \mu)$, where μ is a semi-finite measure. Suppose that $f_n \longrightarrow 0$ μ-almost everywhere. Then $f_n \xrightarrow{\mu^*} 0$. This is immediate from Egoroff's theorem (see [23, Theorem 2.33], for example), but it can also be obtained (with a long detour) in the context of uo-convergence and uo-Lebesgue topologies. Indeed, by [28, Proposition 3.1], almost everywhere convergence of a sequence in $\mathrm{L}_0(X, \Sigma, \mu)$ is, for arbitrary measures, equivalent to uo-convergence in $\mathrm{L}_0(X, \Sigma, \mu)$. Since, by definition, uo-convergence implies convergence in a uo-Lebesgue topology (when this exists), an appeal to Theorem 2.6.1 also yields the desired result.

2.7 uo-convergent sequences within $\widehat{\tau}_E$-convergent nets

Let E be a vector lattice that admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$. When $(x_\alpha)_{\alpha \in \mathcal{A}}$ is a net in E such that $x_\alpha \xrightarrow{\mathrm{uo}} 0$, then, by definition, $x_\alpha \xrightarrow{\widehat{\tau}_E} 0$. The present section is concerned with results that go in the opposite direction. The main result

[3]For this conclusion, we should note here that the usual metric topology on $\mathrm{L}_p(X, \Sigma, \mu)$ is a complete o-Lebesgue topology for every measure μ and for every p such that $0 < p < \infty$. This is commonly known when $1 \leq p < \infty$. When $0 < p < 1$, then the completeness is asserted in [39, 1.47]. The fact that the metric topology is an o-Lebesgue topology for such p follows from what is stated on [6, p. 211] in the context of σ-finite measures. This implies the result for general measures. Indeed, suppose that $(f_\alpha)_{\alpha \in \mathcal{A}}$ is a net in $\mathrm{L}_p(X, \Sigma, \mu)$ such that $f_\alpha \downarrow 0$. Passing to a tail, we may suppose that the net is bounded above by an $f_{\alpha_0} \in \mathrm{L}_p(X, \Sigma, \mu)$. The support of this f_{α_0} is σ-finite. Using the fact that the elements of $\mathrm{L}_p(X, \Sigma, \mu)$ that vanish off this support form an ideal of $\mathrm{L}_p(X, \Sigma, \mu)$, it is then easily seen from the σ-finite case that the chosen tail of the net converges to zero in the metric topology of $\mathrm{L}_p(X, \Sigma, \mu)$.

is Theorem 2.7.6, below, which lies at the basis of topological considerations in Section 2.8, but we start with a few more elementary results.

For an atomic vector lattice E, the situation is as easy as can be. Recall that, by [6, Theorem 1.78], the atomic vector lattices are precisely the order dense vector sublattices of $\mathbb{R}^X$ for some set X. Combining [13, Proposition 1] and [44, Lemma 7.4], we have the following.

Proposition 2.7.1 (Taylor). *Let E be an atomic vector lattice. Then E admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$, and this topology is locally convex-solid. For a net in E, uo-convergence and $\widehat{\tau}_E$-convergence coincide, so that uo-convergence is topological. When E is an order dense vector sublattice of $\mathbb{R}^X$ for some set X, then a net in E is uo- and $\widehat{\tau}_E$-convergent if and only if it is pointwise convergent.*

For monotone nets, uo-convergence and $\widehat{\tau}_E$ convergence still always coincide, according to the following elementary lemma.

Lemma 2.7.2. *Let E be a vector lattice, and suppose that τ is a Hausdorff locally solid linear topology on E. Let $(x_\alpha)_{\alpha\in\mathcal{A}}$ be a monotone net in E and let $x \in E$. When $x_\alpha \xrightarrow{\tau} x$ in E, then $x_\alpha \xrightarrow{\mathrm{uo}} x$ in E. When $\widehat{\tau}_E$ is a (necessarily unique) Hausdorff uo-Lebesgue topology on E, then $x_\alpha \xrightarrow{\widehat{\tau}_E} x$ in E if and only if $x_\alpha \xrightarrow{\mathrm{uo}} x$ in E.*

Proof. We may suppose that $x_\alpha \downarrow$. Take $y \in E$. Then $|x_\alpha - x| \wedge |y| \downarrow$ and $|x - x_\alpha| \wedge |y| \xrightarrow{\tau} 0$. By [6, Theorem 2.21], we have $|x_\alpha - x| \wedge |y| \downarrow 0$. Hence $x_\alpha \xrightarrow{\mathrm{uo}} x$. The final statement is clear. □

For non-monotone nets in general vector lattices, it is not generally true that $\widehat{\tau}_E$-convergence implies uo-convergence. This can already fail for sequences in Banach lattices with order continuous norms. As an example, consider $E = \mathrm{L}_1([0,1])$. For $n = 1,2,\ldots$ and $k = 1,2,\ldots,n$, let f_{nk} be the characteristic function of $[\frac{k-1}{n}, \frac{k}{n}]$, and consider the sequence $f_{11}, f_{21}, f_{22}, f_{31}, f_{32}, f_{33}, f_{41}, \ldots$. It converges to zero in measure, so Theorem 2.6.1 shows that it is $\widehat{\tau}_E$-convergent to zero. On the other hand, [28, Proposition 3.1] shows that uo-convergence of a sequence in $\mathrm{L}_1([0,1])$ is the same as almost everywhere convergence. Hence the sequence is not uo-convergent to zero.

Still, something can be salvaged in the general case. As a motivating example, suppose that (X, Σ, μ) is a measure space. It is well known that a sequence in $\mathrm{L}_0(X, \Sigma, \mu)$ that converges (globally) in measure has a subsequence that converges to the same limit almost everywhere; see [23, Theorem 2.30], for example. When μ is finite, then, in view of Theorem 2.6.1 and [28, Proposition 3.1], this can be restated as saying that a $\widehat{\tau}_E$ convergent sequence in $\mathrm{L}_0(X, \Sigma, \mu)$ has a subsequence that is uo-convergent to the same limit. We shall now extend this formulation of the result to a more general context of nets and Hausdorff uo-Lebesgue topologies on vector lattices; see Theorem 2.7.6, below. In Corollary 2.7.8, below, we shall then obtain a stronger version of the motivating result for convergence in measure and convergence almost everywhere, as a specialisation of the general result.

We start with three preparatory results. The first two appear to have some independent interest.

Proposition 2.7.3. *Let E be a vector lattice with the countable sup property such that $E_{\mathrm{oc}}^{\sim}$ separates the points of E. Take $e \in E^{+}$, and let I_e denote the ideal that is generated in E by e. Then $(I_e)_{\mathrm{oc}}^{\sim}$ separates the points of I_e. In fact, there even exists a $\varphi \in (I_e)_{\mathrm{oc}}^{\sim}$ that is strictly positive on I_e.*

Proof. It is immediate from Theorem 2.2.3 that $(I_e)_{\mathrm{oc}}^{\sim}$ separates the points of I_e. It follows from Proposition 2.2.1 that the ideal of $(I_e)_{\mathrm{oc}}^{\sim}$ that is generated by a strictly positive φ in $(I_e)_{\mathrm{oc}}^{\sim}$ would already separate the points of E. We turn to the existence of such a strictly positive $\varphi \in (I_e)_{\mathrm{oc}}^{\sim}$,

Suppose first that E is Dedekind complete. For $\psi \in \left(E_{\mathrm{oc}}^{\sim}\right)^{+}$, we let

$$N_{\psi} := \{\, x \in E : \psi(|x|) = 0 \,\}$$

denote its null ideal, and we let

$$C_{\psi} := \mathrm{N}_{\psi}^{\mathrm{d}}$$

denote its carrier. Since ψ is order continuous, N_{ψ} is a band in E.

Let B_0 be the band that is generated by the subset $\{\, C_{\psi} : \psi \in \left(E_{\mathrm{oc}}^{\sim}\right)^{+} \,\}$ of E. Then

$$B_0^{\mathrm{d}} = \bigcap_{\psi \in \left(E_{\mathrm{oc}}^{\sim}\right)^{+}} C_{\psi}^{\mathrm{d}} = \bigcap_{\psi \in \left(E_{\mathrm{oc}}^{\sim}\right)^{+}} N_{\psi}^{\mathrm{dd}} = \bigcap_{\psi \in \left(E_{\mathrm{oc}}^{\sim}\right)^{+}} N_{\psi} = \{0\},$$

where in the final step we have used Proposition 2.2.1 and the fact that $E_{\mathrm{oc}}^{\sim}$ separates the points of E. We thus see that $B_0 = E$.

For $\psi \in \left(E_{\mathrm{oc}}^{\sim}\right)^{+}$, let $P_{C_{\psi}}$ denote the band projection from E onto C_{ψ}. When $\psi_1, \psi_2 \in \left(E_{\mathrm{oc}}^{\sim}\right)^{+}$ and $\psi_1 \leq \psi_2$, then $C_{\psi_1} \subseteq C_{\psi_2}$ which, by [7, Theorem 1.46], is equivalent to $P_{C_{\psi_1}} \leq P_{C_{\psi_2}}$. Therefore, the net $\{\, P_{C_{\psi}} : \psi \in \left(E_{\mathrm{oc}}^{\sim}\right)^{+} \,\}$ in $\mathscr{L}_{\mathrm{r}}(E)$ is increasing. Set

$$P := \sup \{\, P_{C_{\psi}} : \psi \in \left(E_{\mathrm{oc}}^{\sim}\right)^{+} \,\},$$

where the supremum is in $\mathscr{L}_{\mathrm{r}}(E)$. From [36, Theorem 30.5] we know that P is a band projection with B_0 as its range space. Since $B_0 = E$, it follows that $P = I$. This implies that $\{\, P_{C_{\psi}} e : \psi \in \left(E_{\mathrm{oc}}^{\sim}\right)^{+} \,\} \uparrow e$, and it follows from the fact that E has the countable sup property that there exists a sequence $(\psi_n)_{n=1}^{\infty}$ in $\left(E_{\mathrm{oc}}^{\sim}\right)^{+}$ such that $P_{C_{\psi_n}} e \uparrow e$ in E.

Consider the ideal I_e of E. Since E is Dedekind complete it is uniformly complete, so that I_e is a Banach lattice when supplied with its order unit norm $\|\cdot\|_e$. Its order dual $I_e^{\sim}$ coincides with its norm dual E^{*} and is then a Banach lattice. Choose strictly positive real numbers $\alpha_1, \alpha_2, \ldots$ such that $\sum_{n=1}^{\infty} \alpha_n \|\psi_n|_{I_e}\| < \infty$, and define $\varphi \in I_e^{\sim}$ by setting

$$\varphi := \sum_{n=1}^{\infty} \alpha_n \psi_n|_{I_e}.$$

Since I_e, being an ideal of E, is a regular vector sublattice of E, each $\psi_n|_{I_e}$ is order continuous. On observing that, being a band, $(I_e)_{\mathrm{oc}}^{\sim}$ is an order closed and, therefore, norm closed subset of the Banach lattice E^*, we see that φ is order continuous on I_e. Obviously, φ is positive.

Suppose that $x \in I_e$ is positive and that $\varphi(x) = 0$. Then $\psi_n(x) = 0$ for all $n \geq 1$. That is, $x \in N_{\psi_n}$ for all $n \geq 1$, so that $P_{C_{\psi_n}} x = 0$ for all $n \geq 1$.

Take $\lambda \geq 0$ such that $0 \leq x \leq \lambda e$. Using [7, Theorem 2.49, Theorem 2.44, and Definition 2.41], we see that there exists an order continuous operator T on E that commutes with all band projections on E and is such that $T(\lambda e) = x$. Since $P_{C_{\Psi_n}}(\lambda e) \uparrow \lambda e$ in E, we have $TP_{C_{\Psi_n}}(\lambda e) \uparrow T(\lambda e) = x$ in E. On the other hand, we know that $TP_{C_{\Psi_n}}(\lambda e) = P_{C_{\Psi_n}} T(\lambda e) = P_{C_{\Psi_n}} x = 0$ for all n. We conclude that $x = 0$. Hence φ is strictly positive on I_e. This completes the proof when E is Dedekind complete.

For general E, we note that its Dedekind completion E^δ also has the countable sup property; see [36, Theorem 32.9]. Furthermore, Theorem 2.2.3 shows that $\left(E^\delta\right)^\sim_{\mathrm{oc}}$ separates the points of E^δ. Let $I_{e,\delta}$ denote the ideal that is generated by e in E^δ. By what has been established above, there exists a $\varphi_\delta \in \left(I_{e,\delta}\right)^\sim_{\mathrm{oc}}$ that is strictly positive on $I_{e,\delta}$. Hence its restriction $\varphi_\delta|_{I_e}$ to I_e is strictly positive on I_e. This restriction is also order continuous on I_e. To see this, suppose that $(x_\alpha)_{\alpha \in \mathcal{A}}$ is a net in I_e and that $x_\alpha \xrightarrow{\mathrm{o}} 0$ in I_e. Since I_e, being an ideal of E, is a regular vector sublattice of E, and since E, being order dense in E^δ, is a regular vector sublattice of E^δ, I_e is a regular vector sublattice of E^δ. Thus $x_\alpha \xrightarrow{\mathrm{o}} 0$ in E^δ. There exists an $\alpha_0 \in \mathcal{A}$ such that the tail $(x_\alpha)_{\alpha \in \mathcal{A}, \alpha \geq \alpha_0}$ is order bounded in I_e. Since this tail is then evidently also order bounded in $I_{e,\delta}$, Theorem 2.2.2 shows that $x_\alpha \xrightarrow{\mathrm{o}} 0$ in $I_{e,\delta}$ for $\alpha \geq \alpha_0$. Then $\varphi_\delta|_{I_e}(x_\alpha) \longrightarrow 0$ for $\alpha \geq \alpha_0$ by the order continuity of φ on $I_{e,\delta}$. Consequently, $\varphi_\delta|_{I_e}(x_\alpha) \longrightarrow 0$, as required. ☐

Suppose that a vector lattice E has an order unit e and that $(x_\alpha)_{\alpha \in \mathcal{A}}$ is a net in E. According to [28, Corollary 3.5], the fact that $|x_\alpha| \wedge e \xrightarrow{\mathrm{o}} 0$ is already enough to imply that $x_\alpha \xrightarrow{\mathrm{uo}} 0$. This is a special case of the following.

Proposition 2.7.4. *Let E be a vector lattice, let S be a non-empty subset of E, and let B_S denote the band that is generated by S in E. Suppose that $(x_\alpha)_{\alpha \in \mathcal{A}}$ is a net in B_S such that $|x_\alpha| \wedge |y| \xrightarrow{\mathrm{o}} 0$ in E for all $y \in S$. Then $x_\alpha \xrightarrow{\mathrm{uo}} 0$ in E.*

Proof. Suppose first that E is Dedekind complete.

Let I_S denote the ideal that is generated by S in E. Take $y \in I_S$. Then there exist $y_1, \ldots, y_n \in S$ and $r_1, \ldots, r_n \geq 1$ such that $|y| \leq \sum_{i=1}^n r_i |y_i|$. This implies that $|x_\alpha| \wedge |y| \leq \sum_{i=1}^n r_i (|x_\alpha| \wedge |y_i|)$, so that $|x_\alpha| \wedge |y| \xrightarrow{\mathrm{o}} 0$ in E.

Take $y \in B_S$. Then there exists a net $(y_\beta)_{\beta \in \mathcal{B}}$ in I_S such that $0 \leq y_\beta \uparrow |y|$ in E. For $\alpha \in \mathcal{A}$, set $s_\alpha := \sup_{i \geq \alpha} (|x_i| \wedge |y|)$, where the supremum is in E. Clearly, $s_\alpha \downarrow$ in E. We claim that $\inf_\alpha s_\alpha = 0$ in E. To see this, take any $\beta \in \mathcal{B}$. Then

$$
\begin{aligned}
\inf_\alpha s_\alpha &= \inf_\alpha \sup_{i \geq \alpha} (|x_i| \wedge |y|) = \inf_\alpha \sup_{i \geq \alpha} \left(|x_i| \wedge |y_\beta + |y| - y_\beta|\right) \\
&\leq \inf_\alpha \sup_{i \geq \alpha} \left(|x_i| \wedge |y_\beta| + |x_i| \wedge ||y| - y_\beta|\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \inf_{\alpha}\sup_{i\geq\alpha}\big(|x_i|\wedge|y_\beta|+||y|-y_\beta|\big)\\
&=\inf_{\alpha}\sup_{i\geq\alpha}\big(|x_i|\wedge|y_\beta|\big)+||y|-y_\beta|.
\end{aligned}
$$

Since we have already established that $|x_\alpha|\wedge|y_\beta|\xrightarrow{\mathrm{o}}0$ in E, [28, Remark 2.2] shows that $\inf_\alpha\sup_{i\geq\alpha}\big(|x_i|\wedge|y_\beta|\big)=0$. Hence $\inf_\alpha s_\alpha\leq||y|-y_\beta|$ for all $\beta\in\mathcal{B}$. Since $y_\beta\uparrow|y|$ in E, we see that $\inf_\alpha s_\alpha=0$ in E, as claimed. Since obviously $|x_\alpha|\wedge|y|\leq s_\alpha$ for all $\alpha\in\mathcal{A}$, we conclude that $|x_\alpha|\wedge|y|\xrightarrow{\mathrm{o}}0$ in E.

Because $(x_\alpha)_{\alpha\in\mathcal{A}}\subseteq B_S$, it is immediate that $|x_\alpha|\wedge|y|\xrightarrow{\mathrm{o}}0$ in E for all $y\in B_S^{\mathrm{d}}$. Since $E=B_S+B_S^{\mathrm{d}}$, we conclude that $|x_\alpha|\wedge|y|\xrightarrow{\mathrm{o}}0$ in E for all $y\in E$. This completes the proof when E is Dedekind complete.

For a general vector lattice E, we let $B_{S,\delta}$ be the band that is generated by S in E^δ. Then $B_S\subseteq B_{S,\delta}$. By what we have just established, $x_\alpha\xrightarrow{\mathrm{uo}}0$ in E^δ, and then Theorem 2.2.2 shows that $x_\alpha\xrightarrow{\mathrm{uo}}0$ in E. □

Proposition 2.7.5. *Let E be a vector lattice, and let F be an order dense ideal of E. The following are equivalent:*
(1) E has the countable sup property;
(2) F has the countable sup property and F is super order dense in E.

Proof. Suppose that E has the countable sup property. Then F has the countable sup property, as is then true for any ideal of E; see [51, Theorem 17.6]. Since F is order dense in E, the fact that E has the countable sup property then implies that F is even super order dense in E; see [36, Theorem 29.3].

Suppose that F has the countable sup property and that F is super order dense in E. Then E has the countable sup property by [36, Theorem 29.4]. □

All preparations have now been made for the proof of the core result of this section.

Theorem 2.7.6. *Let E be a vector lattice with the countable sup property, and suppose that E has an order dense ideal F such that $F_{\mathrm{oc}}^{\sim}$ separates the points of F. Let G be a regular vector sublattice of E. Then G admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_G$.*

Let $(x_\alpha)_{\alpha\in\mathcal{A}}$ be a net in G and suppose that $x_\alpha\xrightarrow{\widehat{\tau}_G}x$ for some $x\in G$. Take a sequence $(\alpha_n')_{n=1}^\infty$ of indices in $\mathcal{A}$. Then there exists an increasing sequence $\alpha_1'=\alpha_1\leq\alpha_2\leq\cdots$ of indices in $\mathcal{A}$ such that $\alpha_n\geq\alpha_n'$ for all $n\geq1$ and $x_{\alpha_n}\xrightarrow{\mathrm{uo}}x$ in G. In particular, when a sequence $(x_n)_{n=1}^\infty$ in G and $x\in G$ are such that $x_n\xrightarrow{\widehat{\tau}_G}x$ in G, then there exists a subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that $x_{n_k}\xrightarrow{\mathrm{uo}}x$ in G.

Proof. In view of Proposition 2.4.7 and Theorem 2.2.2, we may (and shall) suppose that $G=E$.

We know from Theorem 2.5.2 that E admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$, The statement on subsequences is clear from the statement on nets, so we need only establish the existence of the α_n for $n\geq1$. We may suppose that $x=0$.

Suppose first that E is Dedekind complete.

For $y \in F^+$, we let $I_y \subseteq F$ denote the ideal that is generated by y in E. By Proposition 2.7.5, F inherits the countable sup property from E. Hence Proposition 2.7.3 applies to the vector lattice F. We then see that $(I_y)^\sim_{\mathrm{oc}}$ separates the points of I_y and that there even exists a strictly positive order continuous linear functional on I_y. We choose and fix such a strictly positive $\varphi_y \in (I_y)^\sim_{\mathrm{oc}}$ for each $y \in F^+$. From Proposition 2.5.5 we know that

$$\varphi_y(|x_\alpha| \wedge y) \to 0 \tag{2.4}$$

for all $y \in F^+$.

Set $\alpha_1 := \alpha'_1$. Since F is super order dense in E by Proposition 2.7.5, we can choose a sequence $\{y^1_m\}_{m=1}^\infty$ in F^+ such that $0 \le y^1_m \uparrow_m |x_{\alpha_1}|$.

For $n \ge 2$, we shall now inductively construct an indice $\alpha_n \in \mathcal{A}$ and a sequence $\{y^n_m\}_{m=1}^\infty$ in F^+ such that, for all $n \ge 2$:

(a) $\alpha_n \ge \alpha'_n$;
(b) $\alpha_n \ge \alpha_{n-1}$;
(c) $\varphi_{y^i_m}\big(|x_{\alpha_n}| \wedge y^i_m\big) < 2^{-n}$ for $i = 1, 2, \dots, n-1$ and $m = 1, 2, \dots, n$;
(d) $0 \le y^n_m \uparrow_m |x_{\alpha_n}|$ in E.

We start with $n = 2$. The elements y^1_m of F^+ are already known for all $m \ge 1$, and $\varphi_{y^1_m}\big(|x_\alpha| \wedge y^1_m\big) \to 0$ for all $m \ge 1$ by equation (2.4). Therefore, we can choose an $\alpha_2 \in \mathcal{A}$ such that $\varphi_{y^1_m}\big(|x_{\alpha_2}| \wedge y^1_m\big) < 2^{-2}$ for $m = 1, 2$. We can arrange that also $\alpha_2 \ge \alpha'_2$ and $\alpha_2 \ge \alpha_1$. Finally, we choose a sequence $(y^2_m)_{m=1}^\infty$ in F such that $0 \le y^2_m \uparrow_m |x_{\alpha_2}|$. This completes the construction for $n = 2$.

Suppose that $n \ge 2$ and that we have already constructed $\alpha_2, \dots, \alpha_n \in \mathcal{A}$ and sequences $(y^1_m)_{m=1}^\infty, \dots, (y^n_m)_{m=1}^\infty$ in F^+ satisfying the four requirements above. The elements y^i_m of F^+ are already known for all $i = 1, 2, \dots, n$ and $m \ge 1$, and $\varphi_{y^i_m}\big(|x_\alpha| \wedge y^i_m\big) \to 0$ for all such i and m by equation (2.4). Therefore, we can choose $\alpha_{n+1} \in \mathcal{A}$ such that $\varphi_{y^i_m}\big(|x_{\alpha_{n+1}}| \wedge y^i_m\big) < 2^{-(n+1)}$ for all $i = 1, 2, \dots, n$ and $m = 1, 2, \dots, n+1$. We can arrange that also $\alpha_{n+1} \ge \alpha'_{n+1}$ and $\alpha_{n+1} \ge \alpha_n$. Finally, we choose a sequence $(y^{n+1}_m)_{m=1}^\infty$ in F^+ such that $0 \le y^{n+1}_m \uparrow_m |x_{\alpha_{n+1}}|$ in E. This completes the construction for $n+1$.

Fix $i, m \ge 1$. Since $0 \le |x_{\alpha_j}| \wedge y^i_m \le y^i_m$ for all $j \ge 1$, we can define elements $z^{j,m}_n$ of $I_{y^i_m}$ for $n \ge 1$ by setting $z^{i,m}_n := \bigvee_{j=n}^\infty \big(|x_{\alpha_j}| \wedge y^i_m\big)$. Here the supremum is in the ideal $I_{y^i_m}$ in E (which, although this is immaterial, happens to coincide with the supremum in E). It is clear that $z_n \ge 0$ for $n \ge 1$ and that $z^{i,m}_n \downarrow_n$; we shall show that $z^{i,m}_n \downarrow_n 0$ in $I_{y^i_m}$. For this, we start by noting that the inequality in (c) shows that $\varphi_{y^i_m}\big(|x_{\alpha_j}| \wedge y^i_m\big) < 2^{-j}$ for all $j \ge \max(i+1, m)$. Therefore, for all $n \ge \max(i+1, m)$, we can use the order continuity of $\varphi_{y^i_m}$ on $I_{y^i_m}$ to see that

$$\begin{aligned} 0 &\le \varphi_{y^i_m}(z^{i,m}_n) \\ &= \varphi_{y^i_m}\left(\bigvee_{j=n}^\infty \big(|x_{\alpha_j}| \wedge y^m_i\big)\right) \end{aligned}$$

$$
\begin{aligned}
&= \varphi_{y_m^i}\left(\sup_{k\geq n}\left(\bigvee_{j=n}^{k}\left(|x_{\alpha_j}|\wedge y_i^m\right)\right)\right)\\
&= \lim_{\substack{k\to\infty\\k\geq n}} \varphi_{y_m^i}\left(\bigvee_{j=n}^{k}\left(|x_{\alpha_j}|\wedge y_i^m\right)\right)\\
&\leq \limsup_{\substack{k\to\infty\\k\geq n}} \varphi_{y_m^i}\left(\sum_{j=n}^{k}\left(|x_{\alpha_j}|\wedge y_i^m\right)\right)\\
&\leq \limsup_{\substack{k\to\infty\\k\geq n}} \sum_{j=n}^{k} 2^{-j}\\
&\leq 2^{-n+1}.
\end{aligned}
$$

We see from this that for the infimum $\inf_{n\geq 1} z_n^{i,m}$ in $I_{y_m^i}$ (which, although again immaterial, happens to coincide with the infimum in E) we have

$$
0 \leq \varphi_{y_m^i}\left(\inf_{n\geq 1} z_n^{i,m}\right) \leq 2^{-n+1}
$$

for all $n \geq \max(i+1, m)$. Hence $\varphi_{y_m^i}\left(\inf_{n\geq 1} z_n^{i,m}\right) = 0$. Since $\varphi_{y_m^i}$ is strictly positive on $I_{y_m^i}$, this implies that $\inf_{n\geq 1} z_n^{i,m} = 0$ in $I_{y_m^i}$, as we wanted to show.

The inequalities $0 \leq |x_{\alpha_n}| \wedge y_m^i \leq z_n^{i,m}$ for all $n \geq 1$ now show that $|x_{\alpha_n}| \wedge y_m^i \xrightarrow{\text{o}} 0$ in $I_{y_m^i}$ as $n \to \infty$, and then also $|x_{\alpha_n}| \wedge y_m^i \xrightarrow{\text{o}} 0$ in E as $n \to \infty$.

We have now shown that, for all $i, m \geq 1$, $|x_{\alpha_n}| \wedge y_m^i \xrightarrow{\text{o}} 0$ in E as $n \to \infty$.

Let B denote the band that is generated by $\{\, y_m^i : i, m \geq 1 \,\}$ in E. In view of (d) above, it is clear that the sequence $(x_{\alpha_n})_{n=1}^{\infty}$ is a sequence in B. We can now conclude from Proposition 2.7.4 that $x_{\alpha_n} \xrightarrow{\text{uo}} 0$ in E. This concludes the proof when E is Dedekind complete.

For a general vector lattice E, we pass to the Dedekind completion E^δ of E. By [36, Theorem 32.9], E^δ also has the countable sup property. We let F^δ denote the ideal that is generated in E^δ by F. Then F is obviously majorising in F^δ. Since F is order dense in E and E is order dense in E^δ, F is order dense in E^δ and then also in F^δ. We see from this that, as the notation already suggests, F^δ is the Dedekind completion of F, but what we actually need is that, by Theorem 2.2.3, $\left(F^\delta\right)_{\text{oc}}^{\sim}$ separates the points of F^δ. The fact that F is order dense in E^δ implies that $F^\delta \supseteq F$ is order dense in E^δ. Hence E^δ also admits a (necessarily) unique Hausdorff o-Lebesgue topology $\widehat{\tau}_{E^\delta}$. Moreover, Proposition 2.4.7 shows that $x_\alpha \xrightarrow{\widehat{\tau}_{E^\delta}} 0$ in E^δ. By what has been established for the Dedekind complete case, there exist indices α_n as specified such that $x_{\alpha_n} \xrightarrow{\text{uo}} 0$ in E^δ. By Theorem 2.2.2, $x_{\alpha_n} \xrightarrow{\text{uo}} 0$ in E. $\square$

For comparison, we include the following; see [6, Theorem 4.19]. We recall that a topology on a vector lattice E is a *Fatou topology* when it is a (not necessarily Hausdorff) locally solid linear topology on E that has a base of neighbourhoods of zero consisting of solid and order closed sets. A Lebesgue topology is a Fatou topology; see [6, Lemma 4.1], for example.

Theorem 2.7.7. *Let E be a vector lattice with the countable sup property that is supplied with a Hausdorff locally solid linear topology τ with the Fatou property. Suppose that $(x_\alpha)_{\alpha\in\mathcal{A}}$ is an order bounded net in E and that $x_\alpha \xrightarrow{\tau} x$ for some $x \in E$. Then there exist indices $\alpha_1 \leq \alpha_2 \leq \cdots$ in $\mathcal{A}$ such that $x_{\alpha_n} \xrightarrow{\mathrm{o}} x$.*

The hypotheses in Theorem 2.7.7 on the topology on the vector lattice are weaker than those in Theorem 2.7.6, and its conclusion is stronger. The big difference is, however, that the net in Theorem 2.7.7 is supposed to be order bounded, whereas there is no such restriction in Theorem 2.7.6.

Theorem 2.7.7 also holds when, instead of requiring E to have the countable sup property, it is required that there exist an at most countably infinite subset of E such that the band that it generates equals the carrier of τ; see [33, Theorem 6.7]. We refer to [6, Definition 4.15] for the definition of the carrier of a (not necessarily Hausdorff) locally solid topology on a vector lattice.

For a fourth result with a similar flavour, in the context of metrisable Hausdorff locally solid linear topologies on vector lattices that need not have the countable sup property, we refer to [44, Corollary 9.9]. This generalises a similar result (see [32, Corollary 3.2]) for Banach lattices.

We have the following consequence of Theorem 2.6.1 and Theorem 2.7.6.

Corollary 2.7.8. *Let (X, Σ, μ) be a measure space where μ is σ-finite. Suppose that $(f_\alpha)_{\alpha\in\mathcal{A}}$ is a net in $\mathrm{L}_0(X, \Lambda, \mu)$ such that $f_\alpha \xrightarrow{\mu^*} 0$. Take a sequence $(\alpha'_n)_{n=1}^\infty$ of indices in $\mathcal{A}$. Then there exists an increasing sequence $\alpha'_1 = \alpha_1 \leq \alpha_2 \leq \cdots$ of indices in $\mathcal{A}$ such that $\alpha_n \geq \alpha'_n$ for all $n \geq 1$ and $f_{\alpha_n} \to 0$ almost everywhere. In particular, when a sequence $(f_n)_{n=1}^\infty$ is a sequence in $\mathrm{L}_0(X, \Lambda, \mu)$ and $f_n \xrightarrow{\mu^*} 0$, then there exists a subsequence $(f_{n_k})_{k=1}^\infty$ of $(f_n)_{n=1}^\infty$ such that $f_{n_k} \to 0$ almost everywhere.*

Proof. It is known that $\mathrm{L}_0(X, \Sigma, \mu)$ has the countable sup property for every σ-finite measure μ; see [6, Theorem 7.73] or [37, Lemma 2.6.1], for example.

The combination of Theorem 2.6.1 and Theorem 2.7.6 yields a sequence of indices α_n as specified such that $f_{\alpha_n} \xrightarrow{\mathrm{uo}} 0$. Since, for a general measure μ, uo-convergence of a sequence in $\mathrm{L}_0(X, \Sigma, \mu)$ is equivalent to its convergence almost everywhere (see [28, Proposition 3.1]), the proof is complete. $\square$

Remark 2.7.9.

(1) In view of its proof, the natural condition on μ in Corollary 2.7.8 is that μ be semi-finite and have the countable sup property. It is known, however, that this is equivalent to requiring that μ be σ-finite; see [33, Proposition 6.5].

(2) For every measure μ, a sequence in $\mathrm{L}_0(X, \Lambda, \mu)$ that converges (globally) in measure has a subsequence that converges almost everywhere to the same limit; see [23, Theorem 2.30], for example. Corollary 2.7.8 does not imply this result for arbitrary measures, but once the measure is known to be σ-finite, it *does* produce the desired subsequence,

and it even does so under the weaker hypothesis of convergence in measure on subsets of finite measure.

(3) Even for finite measures, we are not aware of an existing result that, as in Corollary 2.7.8, is concerned with *nets* that converge in measure.

Remark 2.7.10. The hypothesis in Theorem 2.7.6 that E have the countable sup property cannot be relaxed to merely requiring that F have this property. As a counter-example, consider the situation where F is a Banach lattice with an order continuous norm that is an order dense ideal of a vector lattice E. Then $F_{\mathrm{oc}}^{\sim} = F^*$ separates the points of F, and it is easy to see that F has the countable sup property; the latter also follows from a more general result in [6, Theorem 4.26]. Since the norm topology on F is a Hausdorff o-Lebesgue topology on F, E has a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$. It is the topology of un-convergence with respect to F. It is possible to find such F and E, and a sequence in E that is $\widehat{\tau}_E$ convergent to zero in E, yet has no subsequence that is uo-convergent to zero in E; see [32, Example 9.6].

Theorem 2.7.6 can be specified to various situations. Here is one involving an unbounded absolute weak topology.

Corollary 2.7.11. *Let E be a vector lattice with the countable sup property. Suppose that $E_{\mathrm{oc}}^{\sim}$ separates the points of E. Let $(x_\alpha)_{\alpha\in\mathcal{A}}$ be a net in E, and suppose that $x_\alpha \xrightarrow{\mathrm{u}|\sigma|(E,E_{\mathrm{oc}}^{\sim})} x$ for some $x \in E$. Then there exist indices $\alpha_1 \leq \alpha_2 \cdots$ such that $x_{\alpha_n} \xrightarrow{\mathrm{uo}} x$.*

We conclude this section by extending another classical result from measure theory to the context of Hausdorff uo-Lebesgue topologies and uo-convergence. Suppose that $(X, \Sigma.\mu)$ is a measure space, where μ is σ-finite. Then a sequence in $\mathrm{L}_0(X,\Sigma,\mu)$ is convergent in measure on subsets of finite measure if and only if every subsequence has a further subsequence that converges to the same limit almost everywhere; see [49, Exercise 18.14 on p. 132]. This is a special case of the following.

Theorem 2.7.12. *Let E be a vector lattice with the countable sup property, and suppose that E has an order dense ideal F such that $F_{\mathrm{oc}}^{\sim}$ separates the points of F. Let G be a regular sublattice of E. Then G admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_G$. For a sequence $(x_n)_{n=1}^{\infty} \subseteq G$, $x_n \xrightarrow{\widehat{\tau}_G} 0$ in G if and only if every subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ has a further subsequence $(x_{n_{k_i}})_{i=1}^{\infty}$ such that $x_{n_{k_i}} \xrightarrow{\mathrm{uo}} 0$ in G.*

Proof. In view of Proposition 2.4.7 and Theorem 2.2.2, we may (and shall) suppose that $G = E$.

The forward implication is clear from Theorem 2.7.6. We now show the converse. When it fails that $x_n \xrightarrow{\widehat{\tau}} 0$ in E, then Theorem 2.5.2 shows that there exists an $\varphi \in F_{\mathrm{oc}}^{\sim}$, an $y \in F$, a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ and an $\varepsilon > 0$ such that $|\varphi|(|x_{n_k}| \wedge |y|) > \varepsilon$ for all k. It is then clear from the order continuity of φ that it is impossible to find a further subsequence $(x_{n_{k_i}})_{i=1}^{\infty}$ of $(x_{n_k})_{k=1}^{\infty}$ such that $x_{n_{k_i}} \xrightarrow{\mathrm{uo}} 0$ in E. $\square$

As another special case of Theorem 2.7.12, we see that a sequence in a Banach lattice with an order continuous norm is un-convergent to zero if and only if every subsequence has a further subsequence that is uo-convergent to zero. We have thus retrieved [21, Theorem 4.4].

2.8 Topological aspects of (unbounded) order convergence

In this section, we consider topological issues that are related to (sequential) order convergence and to (sequential) unbounded order convergence, with an emphasis on the latter. Theorem 2.7.6 will be seen to be an important tool.

Let E be a vector lattice, and let $A \subseteq E$. We define the *o-adherence of* A as the set of all order limits of nets in A, and denote it by $a_{\mathrm{o}}(A)$. The *σ-o-adherence of* A is the set of all order limits of sequences in A; it is denoted by $a_{\sigma\mathrm{o}}(A)$. [4] The *uo-adherence* $a_{\mathrm{uo}}(A)$ and the *σ-uo-adherence* $a_{\sigma\mathrm{uo}}(A)$ of A are similarly defined. The subset A is *o-closed* when $a_{\mathrm{o}}(A) = A$.[5] The collection of all o-closed subsets of E is easily seen to be the collection of closed sets of a topology that is called the *o-topology on* E. The closure of a subset A in the o-topology is denoted by $\overline{A}^{\mathrm{o}}$.[6] We have $a_{\mathrm{o}}(A) \subseteq \overline{A}^{\mathrm{o}}$, with equality if and only if $a_{\mathrm{o}}(A)$ is o-closed. Likewise, there are σ-o-closed subsets and a σ-o-topology, uo-closed subsets and a uo-topology, and σ-uo-closed subsets and a σ-uo-topology, with similar notations and statements about inclusions and equalities of sets. Evidently, a uo-closed subset is o-closed, and a σ-uo-closed subset is σ-o-closed.

Order convergence in a vector lattice E is hardly ever topological; according to [13, Theorem 1] or [43, Theorem 18.36], this is the case if and only if E is finite-dimensional. It is not even true that the set map $A \mapsto a_{\mathrm{o}}(A)$ is always idempotent, i.e., that the o-adherence of a set is always o-closed. It is known, for example, that in every σ-order complete Banach lattice that does *not* have an order continuous norm, there even exists a vector sublattice such that its o-adherence is not order closed; see [26, Theorem 2.7].

We know from Proposition 2.7.1 that uo-convergence in atomic vector lattices is topological. According to [43, Theorem 6.54], atomic vector lattices are, in fact, the only ones for which this is the case.

It appears to be open whether the uo-adherence of a subset of a vector lattice is always uo-closed. In [26, Problem 2.5], it is even asked whether the uo-adherence of a vector sublattice is always o-closed, which is asking for a weaker conclusion for a much more restrictive class of subsets.

Even though the topological aspects of uo-convergence are still not well understood in general, there is a class of vector lattices where we have a reasonably complete picture. In order to formulate this, we need some more notation. For a set X with a topology τ and a subset $A \subseteq X$ of X, we let $a_{\sigma\tau}(A)$ denote the σ-τ-adherence of A, i.e., $a_{\sigma\tau}(A)$ is the set consisting of all τ-limits of sequences in A. When $a_{\sigma\tau}(A) = A$, A is said to be σ-τ-closed.

[4] In [36, p. 82], our σ-o-adherence is called the pseudo order closure. In [26], our o-adherence of a subset A is called the order closure of A, and it is denoted by $\overline{A}^{\mathrm{o}}$. These two terminologies, as well as the notation $\overline{A}^{\mathrm{o}}$, could suggest that taking the (pseudo) order closure is a (sequential) closure operation for a topology. Since this is hardly ever the case, we prefer a terminology and notation that avoid this possible confusion. It is inspired by [8, Definition 1.3.1].

[5] This definition is consistent with that in [26].

[6] There is no notation for the closure operation in the o-topology in [26].

The σ-τ-closed subsets of X are the closed subsets of a topology on X that is called the σ-τ-topology on X. We let $\overline{A}^{\tau}$ and $\overline{A}^{\sigma\text{-}\tau}$ denote the τ-closure and the σ-τ-closure of a subset A of X, respectively. Then $a_{\sigma\tau}(A) \subseteq \overline{A}^{\sigma\text{-}\tau}$, with equality if and only if $a_{\sigma\tau}(A)$ is σ-τ-closed.

Theorem 2.8.1. *Let E be a vector lattice with the countable sup property, and suppose that E has an order dense ideal F such that $F_{\mathrm{oc}}^{\sim}$ separates the points of F. Let G be a regular vector sublattice of E. Then G admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_G$. For a subset A of G, the following seven subsets of G are all equal:*

(1) $a_{\sigma\widehat{\tau}_G}(A)$ and $\overline{A}^{\sigma\text{-}\widehat{\tau}_G}$;
(2) $a_{\sigma\mathrm{uo}}(A)$ and $\overline{A}^{\sigma\text{-uo}}$;
(3) $a_{\mathrm{uo}}(A)$ and $\overline{A}^{\mathrm{uo}}$;
(4) $\overline{A}^{\widehat{\tau}_G}$.

In particular, the σ-$\widehat{\tau}_G$-topology, the σ-uo-topology, and the uo-topology on G all coincide with $\widehat{\tau}_G$.

In Theorem 2.8.1, the topological closures and (σ-)adherences are to be taken with respect to the topologies and convergences in G.

Proof. The existence and uniqueness of $\widehat{\tau}_G$ are clear from Theorem 2.7.6. Using Theorem 2.7.6 for the first inclusions, we have, for an arbitrary subset A of G,

$$\overline{A}^{\widehat{\tau}_G} \subseteq a_{\sigma\mathrm{uo}}(A) \subseteq a_{\mathrm{uo}}(A) \subseteq \overline{A}^{\widehat{\tau}_G}$$

and

$$a_{\sigma\widehat{\tau}_G}(A) \subseteq a_{\sigma\mathrm{uo}}(A) \subseteq a_{\sigma\widehat{\tau}_G}(A).$$

This gives equality of $a_{\sigma\widehat{\tau}_G}(A)$, $a_{\sigma\mathrm{uo}}(A)$, $a_{\mathrm{uo}}(A)$, and $\overline{A}^{\widehat{\tau}_G}$. Since the set map $A \mapsto \overline{A}^{\widehat{\tau}_G}$ is idempotent, so is $A \mapsto a_{\sigma\widehat{\tau}_G}(A)$. Hence $a_{\sigma\widehat{\tau}_G}(A)$ is σ-$\widehat{\tau}_G$-closed, so that it coincides with the σ-τ-closure $\overline{A}^{\sigma\text{-}\widehat{\tau}_G}$ of A. A similar argument works for $\overline{A}^{\sigma\text{-uo}}$ and $\overline{A}^{\mathrm{uo}}$. □

Remark 2.8.2. Taking $G = E$ in Theorem 2.8.1, the equality of $\overline{A}^{\widehat{\tau}_G}$ and $a_{\sigma\mathrm{uo}}(A)$ implies that, for a σ-finite measure μ, a subset of $\mathrm{L}_0(X, \Sigma, \mu)$ is closed in the topology of convergence in measure on subsets of finite measure if and only if it contains the almost every limits of sequences in it. This is [25, 245L(b)].

In the context of Theorem 2.8.1, it is also possible to give a necessary and sufficient condition for sequential uo-convergence to be topological; see Corollary 2.8.5, below. The proof of the following preparatory lemma is an abstraction of the argument in [38].

Lemma 2.8.3. *Let E be a vector lattice that is supplied with a topology τ. Suppose that τ has the following properties:*

(1) for every sequence $(x_n)_{n=1}^{\infty}$ in E and for every $x \in E$, the fact that $x_n \xrightarrow{\tau} x$ implies that there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $x_{n_k} \xrightarrow{\mathrm{uo}} x$ as $k \to \infty$.
(2) there exists a sequence $(x_n)_{n=1}^{\infty}$ in E and an $x \in E$ such that $x_n \xrightarrow{\tau} x$ but $x_n \not\xrightarrow{\mathrm{uo}} x$;

Then there does not exist a topology τ' on E such that, for every sequence $(x_n)_{n=1}^{\infty}$ in E and for every $x \in E$, $x_n \xrightarrow{\text{uo}} x$ if and only if $x_n \xrightarrow{\tau'} x$.

Proof. Suppose that there were such a topology τ'. Take a sequence $(x_n)_{n=1}^{\infty}$ in E and an $x \in E$ such that $x_n \xrightarrow{\tau} x$ but $x_n \overset{\text{uo}}{\nrightarrow} x$. Then also $x_n \overset{\tau'}{\nrightarrow} x$, so that there exists a τ'-neighbourhood V of x and a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $x_{n_k} \notin V$ for all $k \geq 1$. Since also $x_{n_k} \xrightarrow{\tau} x$ as $k \to \infty$, there exists a subsequence $(x_{n_{k_i}})_{i=1}^{\infty}$ of $(x_{n_k})_{k=1}^{\infty}$ such that $x_{n_{k_i}} \xrightarrow{\text{uo}} x$ as $i \to \infty$. Hence also $x_{n_{k_i}} \xrightarrow{\tau'} x$ as $i \to \infty$. But this is impossible, since the entire sequence $(x_{n_{k_i}})_{i=1}^{\infty}$ stays outside V. $\square$

The following is a direct consequence of Lemma 2.8.3. The topology τ in it could be a uo-Lebesgue topology, but for the result to hold it need not even be a linear topology, nor need the topology τ' be.

Proposition 2.8.4. *Let E be a vector lattice that is supplied with a topology τ. Suppose that τ has the following properties:*

(1) for every sequence $(x_n)_{n=1}^{\infty}$ in E and for every $x \in E$, the fact that $x_n \xrightarrow{\text{uo}} x$ implies that $x_n \xrightarrow{\tau} x$;

(2) for every sequence $(x_n)_{n=1}^{\infty}$ in E and for every $x \in E$, the fact that $x_n \xrightarrow{\tau} x$ implies that there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $x_{n_k} \xrightarrow{\text{uo}} x$ as $k \to \infty$.

Then the following are equivalent;

(1) there exists a topology τ' on E such that, for every sequence $(x_n)_{n=1}^{\infty}$ in E and for every $x \in E$, $x_n \xrightarrow{\text{uo}} x$ if and only if $x_n \xrightarrow{\tau'} x$;

(2) for every sequence $(x_n)_{n=1}^{\infty}$ in E and for every $x \in E$, the fact that $x_n \xrightarrow{\tau} x$ implies that $x_n \xrightarrow{\text{uo}} x$.

In that case, one can take τ for τ'.

In the appropriate context, the combination of Theorem 2.7.6 and Proposition 2.8.4 yields the following necessary and sufficient condition for sequential uo-convergence to be topological. Note that there are no assumptions at all on the topology τ in its first part.

Corollary 2.8.5. *Let E be a vector lattice with the countable sup property, and suppose that E has an order dense ideal F such that $F_{\text{oc}}^{\sim}$ separates the points of F. Let G be a regular vector sublattice of E. Then G admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_G$, and the following are equivalent:*

(1) there exists a topology τ on G such that, for every sequence $(x_n)_{n=1}^{\infty}$ in G and for every $x \in G$, $x_n \xrightarrow{\text{uo}} x$ in G if and only if $x_n \xrightarrow{\tau} x$;

(2) for every sequence $(x_n)_{n=1}^{\infty}$ in G and for every $x \in G$, the fact that $x_n \xrightarrow{\widehat{\tau}_G} x$ in G implies that $x_n \xrightarrow{\text{uo}} x$ in G.

In that case, one can take $\widehat{\tau}_G$ for τ.

The proof of the following result closely follows the one in [38], where it is shown that sequential almost everywhere convergence in $L_{\infty}([0,1])$ is not topological.

Corollary 2.8.6. *Let (X,Σ,μ) be a measure space, where μ is σ-finite. Suppose that there exists an $A\in\Sigma$ with the property that, for every $k\geq 1$, there exist finitely many mutually disjoint $A_{k,1},\dots,A_{k,N_k}\in\Sigma$ such that $0<\mu(A_{k,1}),\dots,\mu(A_{k,N_k})<1/k$ and $A=\bigcup_{l=1}^{N_k}A_{k,l}$.*

Take a regular vector sublattice G of $\mathrm{L}_0(X,\Sigma,\mu)$ that contains the characteristic functions $1_{A_{k,l}}$ of all sets $A_{k,l}$ for $k=1,2,\dots$ and $l=1,\dots,N_k$. Then there does not exist a topology τ on G such that, for every sequence $(x_n)_{n=1}^{\infty}$ in G and for every $x\in G$, $x_n\xrightarrow{\mathrm{uo}}x$ in G if and only if $x_n\xrightarrow{\tau}x$.

Proof. We are in the situation of Corollary 2.8.5, where $\widehat{\tau}_G$-convergence is convergence in measure on subsets of finite measure by Theorem 2.6.1, and sequential uo-convergence is almost everywhere convergence by [28, Proposition 3.1]. Consider the following sequence in G:

$$A_{1,1},\dots,A_{1,N_1},A_{2,1},\dots,A_{2,N_2},A_{3,1},\dots,A_{3,N_3},\dots\,.$$

This sequence clearly converges to zero on subsets of finite measure, but it converges nowhere to zero on the subset A of strictly positive measure. Hence the property in part (2) of Corollary 2.8.5 does not hold, and then neither does the property in its part (1). □

Remark 2.8.7. Corollary 2.8.6 provides us with a large class of examples of vector lattices where sequential uo-convergence is not topological—so that uo-convergence is certainly not topological—but where, according to Theorem 2.8.1, the set maps $A\mapsto a_{\sigma\mathrm{uo}}(A)$ and $A\mapsto a_{\mathrm{uo}}(A)$ are both still idempotent, so that $a_{\sigma\mathrm{uo}}(A)$ is σ-uo-closed and $a_{\mathrm{uo}}(A)$ is uo-closed for every subset A of G. For all p such that $0\leq p\leq\infty$, the space $L_p([0,1])$ is such an example.

We conclude with a strengthened version of [26, Theorem 2.2]. The improvement lies in the removal of the hypothesis that E be Banach lattice, and by adding eight more equal, but not obviously equal, sets to the three equal sets in the original result.

Theorem 2.8.8. *Let E be a vector lattice with the countable sup property, and suppose that $E_{\mathrm{oc}}^{\sim}$ separates the points of E. Then E admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$. Take an ideal I of $E_{\mathrm{oc}}^{\sim}$ that separates the points of E, and take a vector sublattice F of E. Then the following eleven vector sublattices of E are all equal:*

(1) $a_{\sigma\widehat{\tau}_E}(F)$ and $\overline{F}^{\sigma\text{-}\widehat{\tau}_E}$;
(2) $a_{\sigma\mathrm{uo}}(F)$ and $\overline{F}^{\sigma\text{-uo}}$;
(3) $a_{\mathrm{uo}}(F)$ and $\overline{F}^{\mathrm{uo}}$;
(4) $\overline{F}^{\widehat{\tau}_E}$, $\overline{F}^{|\sigma|(E,I)}$, and $\overline{F}^{\sigma(E,I)}$;
(5) $(a_{\mathrm{o}}(a_{\mathrm{o}}(F)))$ and $\overline{F}^{\mathrm{o}}$.

The equality of $a_{\mathrm{uo}}(F)$, $a_{\mathrm{o}}(a_{\mathrm{o}}(F))$, and $\overline{F}^{\sigma(E,I)}$ can already be found in [26, Theorem 2.2], where it also noted that these sets coincide with the smallest order closed vector sublattice of E containing F.

Proof. The equality of the first seven subsets is clear from Theorem 2.8.1. Since we know from Theorem 2.5.2 that $\widehat{\tau}_E=\mathrm{u}_E|\sigma|(E,I)$, it follows from [44, Proposition 2.12] that $\overline{F}^{\widehat{\tau}_E}=$

$\overline{F}^{|\sigma|(E,I)}$. Furthermore, from Kaplan's theorem (see [6, Theorem 2.33], for example) we know that E, when supplied with the Hausdorff locally convex $|\sigma|(E,I)$-topology, has the same topological dual as when it is supplied with the Hausdorff locally convex $\sigma(E,I)$-topology. By the convexity of F, we have $\overline{F}^{|\sigma|(E,I)} = \overline{F}^{\sigma(E,I)}$. This argument was already used in [26, Proof of Lemma 2.1].

We turn to the two sets in part (4). It was established in [26, Lemma 2.1] that $a_{\mathrm{uo}}(F) \subseteq a_{\mathrm{o}}(a_{\mathrm{o}}(F))$; this is, in fact, valid for vector sublattices of general vector lattices. It was also observed there that, obviously, the fact that $I \subseteq E_{\mathrm{oc}}^{\sim}$ implies that $\overline{F}^{\sigma(E,I)}$ is o-closed. Using also that we already know that $a_{\mathrm{uo}}(F) = \overline{F}^{\mathrm{uo}}$, we therefore have the following chain of inclusions:

$$\overline{F}^{\mathrm{uo}} = a_{\mathrm{uo}}(F) \subseteq a_{\mathrm{o}}(a_{\mathrm{o}}(F)) \subseteq \overline{F}^{\mathrm{o}} \subseteq \overline{F}^{\sigma(E,I)}.$$

Since we also already know that $\overline{F}^{\mathrm{uo}} = \overline{F}^{\sigma(E,I)}$, the proof is complete. $\square$