Topologies and convergence structures on vector lattices of operators

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Title: Topologies and convergence structures on vector lattices of operators
Issue Date: 2021-02-02

Chapter 1

Introduction

The classical von Neumann bicommutant theorem, which was first established by von Neumann in [46], is fundamental to the theories of von Neumann algebras and of C^*-algebras and their representations. It states that if $\mathscr{A}$ is a unital C^*-subalgebra of the bounded operators $\mathscr{B}(H)$ on a Hilbert space H, then the bicommutant of $\mathscr{A}$ in $\mathscr{B}(H)$ is equal to closure of $\mathscr{A}$ in the weak (or strong) operator topology. It is natural to ask whether there are analogues of the bicommutant theorem for other spaces than Hilbert spaces.

A result for a general class of spaces does not appear to be known, but several interesting cases have been considered. For example, in [16] de Pagter and Ricker were able to show that, for a large class of measure spaces, the bicommutant theorem holds for closed unital subalgebras of the algebra of multiplications by bounded measurable functions on their L_p-spaces for finite p. In [17], the same authors extended their results to a large class of Banach function spaces, namely the fully symmetric Banach function spaces with an order continuous norm.

The existing literature on analogues of von Neumann's theorem appears to be focused on Banach spaces and their bounded operators, also when the Banach spaces under consideration have the additional structure of Banach lattices. What happens for Banach lattices when one takes the ordering into account and adapts the notion of the bicommutant accordingly? For example, suppose that E is a Banach lattice with an order continuous norm, and that $\mathscr{A}$ is a unital subalgebra of the order bounded operators on E that is closed in the regular norm. If $\mathscr{A}$ satisfies an appropriate and sufficiently lenient condition, can one then describe the bicommutant of $\mathscr{A}$ in the order bounded operators on E in a manner that is reminiscent of von Neumann's theorem? One can ask a similar question in a more algebraic context, where E is merely supposed to be a Dedekind complete vector lattice. Apart from their intrinsic interest, such results can—as the representation theory of C^*-algebras shows—expected to be relevant for representation theory in Banach lattices and in vector lattices.

The research in this thesis originates from this perspective, with an emphasis on the algebraic context.

The first things that catches the eye when considering a possible bicommutant theorem for vector lattices is that there is no obvious analogue of the weak operator topology. On the other hand, it has become increasingly clear in recent years that so-called Hausdorff uo-Lebesgue topologies exist on many vector lattices, and that these appear to be of special relevance. Perhaps such topologies (or related ones) on vector lattices of order bounded operators can take over the role of the weak operator topologies. Furthermore, a topological closure is a special case of an adherence with respect to a convergence structure. The latter abound for vector lattices of order bounded operators and they, too, are natural candidates to be needed in the picture.

Hence there certainly appear to be possibilities to find alternatives for the weak operator topology. Quite unfortunately, there is no theory of Hausdorff uo-Lebesgue topologies on vector lattices of order bounded operators at all that goes beyond that for general vector lattices, nor is there of the natural convergence structures that exist on them. As soon as one starts contemplating more advanced issues for these vector lattices of order bounded operators, such as a possible bicommutant theorem, one runs aground because of the lack of answers to basic questions. There are no 'tools to work with'. When trying to answer these questions, then, more often than not, it turns out that, even though basic in nature, such questions need by no means be easily answered. Thus attempts at a sufficiently general bicommutant theorem for vector lattices quickly come to a standstill. It is also this lack of tools that is one of the reasons that the development of a theory of Banach lattice algebras (of operators) that does even only remotely resemble that of C^*-algebras (of operators) is currently out of reach.

This thesis aims at at least partially remedying this by providing basic, but non-trivial, results that are necessary for the development of a more advanced theory of vector lattice algebras of order bounded operators on vector lattices and on Banach lattices, and possibly of vector lattice algebras and Banach lattice algebras in general. It is worth mentioning that, building on the results in this thesis, analogues of von Neumann's theorem in the context of vector lattices and Banach lattices have already been obtained that go beyond the first explorations in [18]. These will be published at a later date.

We shall now briefly outline the contents of this thesis.

In Chapter 2, the construction of a Hausdorff uo-Lebesgue topology on a vector lattice is investigated, starting from a Hausdorff o-Lebesgue topology on an order dense ideal. The approach in [44] already unifies many results on Hausdorff uo-Lebesgue topologies in the literature and the material in this chapter takes the general theory still one step further. This chapter also contains a generalisation of the classical relations between convergence in measure and convergence almost everywhere to the context of Hausdorff uo-Lebesgue topologies and unbounded order convergence.

In Chapter 3, it is shown how, given a vector lattice E and a Dedekind complete vector lattice F that is supplied with a locally solid topology, a corresponding absolute strong topology on the order bounded operators $\mathscr{L}_{\mathrm{ob}}(E,F)$ from E into F can be introduced. It is seen from this that $\mathscr{L}_{\mathrm{ob}}(E,F)$ admits a Hausdorff uo-Lebesgue topology whenever F does. In a Dedekind complete vector lattice E, for each of order convergence, unbounded order convergence, and—when applicable—convergence in a Hausdorff uo-Lebesgue topology, the relationship is investigated between the uniform convergence structure and the corresponding strong convergence structure on the order bounded operators $\mathscr{L}_{\mathrm{ob}}(E)$ on E. Particular attention is paid to the orthomorphisms on E, where the relations between these six convergence structures are especially convenient, and which are continuous with respect to the three convergence structures on E under consideration.

Whereas Chapter 3 is concerned with vector *lattices* of order bounded operators, the emphasis in Chapter 4 is on vector lattice *algebras* of order bounded operators on a Dedekind complete vector lattice. Building on the results in Chapter 3, the continuity of the left and right multiplications of such vector lattice algebras with respect to the six convergence structures is investigated. This is then used to study the simultaneous continuity of the multiplication, the results of which then enable one to give sufficient conditions for the adherences of vector lattice algebras to be vector lattice algebras again. Results are included to show that the conditions for the results are sharp in the sense that, for example, a result is no longer true for a vector lattice subalgebra of the order continuous operators when it is stated for a vector lattice subalgebra of the orthomorphisms. The chapter concludes with a section—with special attention for the orthomophisms—on the equality of various adherences of vector sublattices with respect to the convergence structures considered in this chapter.

The Chapters 2-4 can be read independently. They are based on the following three submitted papers:

- Chapter 2: Y. Deng and M. de Jeu. Vector lattices with a Hausdorff uo-Lebesgue topology. Online at http://arxiv.org/pdf/2005.14636.pdf.
- Chapter 3: Y. Deng and M. de Jeu. Convergence structures and locally solid topologies on vector lattices of operators. Online at http://arxiv.org/pdf/2008.05379.pdf.
- Chapter 4: Y. Deng and M. de Jeu. Convergence structures and Hausdorff uo-Lebesgue topologies on vector lattice algebras of operators. Online at http://arxiv.org/pdf/2011.03768.pdf.