Topologies and convergence structures on vector lattices of operators

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Contents

Chapter 1

Introduction

The classical von Neumann bicommutant theorem, which was first established by von Neumann in [46], is fundamental to the theories of von Neumann algebras and of C*-algebras and their representations. It states that if $\mathscr{A}$ is a unital C*-subalgebra of the bounded operators $\mathscr{B}(H)$ on a Hilbert space H, then the bicommutant of $\mathscr{A}$ in $\mathscr{B}(H)$ is equal to closure of $\mathscr{A}$ in the weak (or strong) operator topology. It is natural to ask whether there are analogues of the bicommutant theorem for other spaces than Hilbert spaces.

A result for a general class of spaces does not appear to be known, but several interesting cases have been considered. For example, in [16] de Pagter and Ricker were able to show that, for a large class of measure spaces, the bicommutant theorem holds for closed unital subalgebras of the algebra of multiplications by bounded measurable functions on their L_p-spaces for finite p. In [17], the same authors extended their results to a large class of Banach function spaces, namely the fully symmetric Banach function spaces with an order continuous norm.

The existing literature on analogues of von Neumann's theorem appears to be focused on Banach spaces and their bounded operators, also when the Banach spaces under consideration have the additional structure of Banach lattices. What happens for Banach lattices when one takes the ordering into account and adapts the notion of the bicommutant accordingly? For example, suppose that E is a Banach lattice with an order continuous norm, and that $\mathscr{A}$ is a unital subalgebra of the order bounded operators on E that is closed in the regular norm. If $\mathscr{A}$ satisfies an appropriate and sufficiently lenient condition, can one then describe the bicommutant of $\mathscr{A}$ in the order bounded operators on E in a manner that is reminiscent of von Neumann's theorem? One can ask a similar question in a more algebraic context, where E is merely supposed to be a Dedekind complete vector lattice. Apart from their intrinsic interest, such results can—as the representation theory of C*-algebras shows—expected to be relevant for representation theory in Banach lattices and in vector lattices.

The research in this thesis originates from this perspective, with an emphasis on the algebraic context.

The first things that catches the eye when considering a possible bicommutant theorem for vector lattices is that there is no obvious analogue of the weak operator topology. On the other hand, it has become increasingly clear in recent years that so-called Hausdorff uo-Lebesgue topologies exist on many vector lattices, and that these appear to be of special relevance. Perhaps such topologies (or related ones) on vector lattices of order bounded operators can take over the role of the weak operator topologies. Furthermore, a topological closure is a special case of an adherence with respect to a convergence structure. The latter abound for vector lattices of order bounded operators and they, too, are natural candidates to be needed in the picture.

Hence there certainly appear to be possibilities to find alternatives for the weak operator topology. Quite unfortunately, there is no theory of Hausdorff uo-Lebesgue topologies on vector lattices of order bounded operators at all that goes beyond that for general vector lattices, nor is there of the natural convergence structures that exist on them. As soon as one starts contemplating more advanced issues for these vector lattices of order bounded operators, such as a possible bicommutant theorem, one runs aground because of the lack of answers to basic questions. There are no 'tools to work with'. When trying to answer these questions, then, more often than not, it turns out that, even though basic in nature, such questions need by no means be easily answered. Thus attempts at a sufficiently general bicommutant theorem for vector lattices quickly come to a standstill. It is also this lack of tools that is one of the reasons that the development of a theory of Banach lattice algebras (of operators) that does even only remotely resemble that of C^*-algebras (of operators) is currently out of reach.

This thesis aims at at least partially remedying this by providing basic, but non-trivial, results that are necessary for the development of a more advanced theory of vector lattice algebras of order bounded operators on vector lattices and on Banach lattices, and possibly of vector lattice algebras and Banach lattice algebras in general. It is worth mentioning that, building on the results in this thesis, analogues of von Neumann's theorem in the context of vector lattices and Banach lattices have already been obtained that go beyond the first explorations in [18]. These will be published at a later date.

We shall now briefly outline the contents of this thesis.

In Chapter 2, the construction of a Hausdorff uo-Lebesgue topology on a vector lattice is investigated, starting from a Hausdorff o-Lebesgue topology on an order dense ideal. The approach in [44] already unifies many results on Hausdorff uo-Lebesgue topologies in the literature and the material in this chapter takes the general theory still one step further. This chapter also contains a generalisation of the classical relations between convergence in measure and convergence almost everywhere to the context of Hausdorff uo-Lebesgue topologies and unbounded order convergence.

In Chapter 3, it is shown how, given a vector lattice E and a Dedekind complete vector lattice F that is supplied with a locally solid topology, a corresponding absolute strong topology on the order bounded operators $\mathscr{L}_{\mathrm{ob}}(E,F)$ from E into F can be introduced. It is seen from this that $\mathscr{L}_{\mathrm{ob}}(E,F)$ admits a Hausdorff uo-Lebesgue topology whenever F does. In a Dedekind complete vector lattice E, for each of order convergence, unbounded order convergence, and—when applicable—convergence in a Hausdorff uo-Lebesgue topology, the relationship is investigated between the uniform convergence structure and the corresponding strong convergence structure on the order bounded operators $\mathscr{L}_{\mathrm{ob}}(E)$ on E. Particular attention is paid to the orthomorphisms on E, where the relations between these six convergence structures are especially convenient, and which are continuous with respect to the three convergence structures on E under consideration.

Whereas Chapter 3 is concerned with vector *lattices* of order bounded operators, the emphasis in Chapter 4 is on vector lattice *algebras* of order bounded operators on a Dedekind complete vector lattice. Building on the results in Chapter 3, the continuity of the left and right multiplications of such vector lattice algebras with respect to the six convergence structures is investigated. This is then used to study the simultaneous continuity of the multiplication, the results of which then enable one to give sufficient conditions for the adherences of vector lattice algebras to be vector lattice algebras again. Results are included to show that the conditions for the results are sharp in the sense that, for example, a result is no longer true for a vector lattice subalgebra of the order continuous operators when it is stated for a vector lattice subalgebra of the orthomorphisms. The chapter concludes with a section—with special attention for the orthomophisms—on the equality of various adherences of vector sublattices with respect to the convergence structures considered in this chapter.

The Chapters 2-4 can be read independently. They are based on the following three submitted papers:

- Chapter 2: Y. Deng and M. de Jeu. Vector lattices with a Hausdorff uo-Lebesgue topology. Online at http://arxiv.org/pdf/2005.14636.pdf.
- Chapter 3: Y. Deng and M. de Jeu. Convergence structures and locally solid topologies on vector lattices of operators. Online at http://arxiv.org/pdf/2008.05379.pdf.
- Chapter 4: Y. Deng and M. de Jeu. Convergence structures and Hausdorff uo-Lebesgue topologies on vector lattice algebras of operators. Online at http://arxiv.org/pdf/2011.03768.pdf.

Chapter 2

Vector lattices with a Hausdorff uo-Lebesgue topology

Abstract

We investigate the construction of a Hausdorff uo-Lebesgue topology on a vector lattice from a Hausdorff (o)-Lebesgue topology on an order dense ideal, and what the properties of the topologies thus obtained are. When the vector lattice has an order dense ideal with a separating order continuous dual, it is always possible to supply it with such a topology in this fashion, and the restriction of this topology to a regular sublattice is then also a Hausdorff uo-Lebesgue topology. A regular vector sublattice of $\mathrm{L}_0(X, \Sigma, \mu)$ for a semi-finite measure μ falls into this category, and the convergence of nets in its Hausdorff uo-Lebesgue topology is then the convergence in measure on subsets of finite measure. When a vector lattice not only has an order dense ideal with a separating order continuous dual, but also has the countable sup property, we show that every net in a regular vector sublattice that converges in its Hausdorff uo-Lebesgue topology always contains a sequence that is uo-convergent to the same limit. This enables us to give satisfactory answers to various topological questions about uo-convergence in this context.

2.1 Introduction and overview

In this paper, we investigate the construction of a Hausdorff uo-Lebesgue topology on a vector lattice from a Hausdorff (o)-Lebesgue topology[1] on an order dense ideal, and what the properties of the topologies thus obtained are.

After recalling the relevant notions and making the necessary preparations in Section 2.2, the key construction is carried out in Theorem 2.3.1 in Section 2.3, below. The idea of starting with a topology on an order dense ideal originates from [11] but, whereas the construction in [11] to obtain a global topology is carried out using Riesz pseudo-norms, we follow an approach using neighbourhood bases of zero that is inspired by [44]. Using such neighbourhood bases, it is possible to perform the construction under minimal hypotheses on the initial data, and thus understand how these hypotheses are reflected in the properties of the resulting global topology. The remainder of Section 2.3 is mainly concerned with showing how the general theorem relates to existing results in the literature. Our working with neighbourhood bases of zero enables us to explain certain 'pathologies' in the literature, where a topology of unbounded type is not Hausdorff, or not linear, from the general theorem.

In Section 2.4, we move to the context where the initial ideal is actually order dense and admits a Hausdorff o-Lebesgue topology. In that case, every regular vector sublattice of the global vector lattice admits a Hausdorff uo-Lebesgue topology. The resulting overview Theorem 2.4.9, below, mostly consists of a summary of results that are already in the literature, though not presented in this way. It is also recalled in that section that a regular vector sublattice admits a Hausdorff uo-Lebesgue topology when the global vector lattice admits one. Consequently, there is a going-up-going-down procedure: starting with a Hausdorff o-Lebesgue topology on an order dense ideal, one obtains a Hausdorff uo-Lebesgue topology on the global vector lattice, and then finally also one on every regular vector sublattice.

In view of the going-up-going-down construction, it is evidently desirable to have a class of vector lattices that admit Hausdorff o-Lebesgue topologies because such data can serve as 'germs' for Hausdorff uo-Lebesgue topologies. The vector lattices with separating order continuous duals form such a class, and this is exploited in Section 2.5.

Section 2.6 is concerned with regular vector sublattices of $\mathrm{L}_0(X, \Sigma, \mu)$ for a semi-finite measure μ. Via an application of the going-up-going-down procedure, every regular vector sublattice of $\mathrm{L}_0(X, \Sigma, \mu)$ admits a Hausdorff uo-Lebesgue topology. We give a rigorous proof of the fact that the convergence of nets in such a topology is the convergence in measure on subsets of finite measure. For $\mathrm{L}_p(X, \Sigma, \mu)$, we also discuss how the (in fact) unique Hausdorff uo-Lebesgue topology on these spaces can be described in various seemingly different ways that are still equivalent. The relation between these topologies and minimal and smallest Hausdorff locally solid linear topologies on these spaces is explained.

Section 2.7 is concerned with convergent sequences that can always be found 'within' nets that are convergent in a Hausdorff uo-Lebesgue topology on a vector lattice that has the countable sup property and that has an order dense ideal with a separating order continuous

[1] In the literature, what we call a o-Lebesgue topology is simply called a Lebesgue topology. Now that uo-Lebesgue topologies, with a completely analogous definition, have become objects of a more extensive study, it seems consistent to also add a prefix to the original term.

dual. The precise statement is in Theorem 2.7.6, below; this is one of the main theorems in this paper. It is in the same spirit as the fact that a sequence that converges (globally) in measure always contains a subsequence that converges almost everywhere to the same limit.

Finally, in Section 2.8, we study topological aspects of uo-convergence. The relations between uo-convergence and various order topologies are not at all well understood, but when the global vector lattice has the countable sup property, and also has an order dense ideal with a separating order continuous dual, then a reasonably satisfactory picture emerges. In Theorem 2.8.1 and Theorem 2.8.8, below, various topological closures and (sequential) adherences are then seen to be equal. It is then also possible to give a necessary and sufficient criterion for sequential uo-convergence to be topological; see Corollary 2.8.5, below.

We have tried to be as complete in the development of this part of the theory of uo-convergence as we could, and also to relate to relevant existing results in the literature whenever possible. Any omissions at this point are unintentional.

2.2 Preliminaries

In this section, we collect a number of definitions, notations, conventions and preparatory results. We refer the reader to the textbooks [2], [5], [6], [7], [36], [37], [40], [50], and [51] for general background information on vector lattices and Banach lattices.

2.2.1 Vector lattices, operators, and (unbounded) order convergence

All vector spaces are over the real numbers. Measures take their values in $[0,\infty]$ and are not supposed to satisfy any condition unless otherwise specified. All vector lattices are supposed to be Archimedean. The positive cone of a vector lattice E is denoted by E^+.

Let E be a vector lattice, and let F be a vector sublattice of E. Then F is *order dense in* E when, for every $x\in E$ with $x>0$, there exists a $y\in F$ such that $0<y\leq x$; F is called *super order dense in* E when, for every $x\in E^+$, there exists a sequence $(x)_{n=1}^{\infty}\subseteq F^+$ with $x_n\uparrow x$ in E. The vector sublattice F of E is order dense in E if and only if, for every $x\in E^+$, we have $x=\sup\{\, y\in F : 0\leq y\leq x\,\}$; see [7, Theorem 1.34], for example.

A vector sublattice F of a vector lattice E is called *majorising in* E when, for every $x\in E$, there exists a $y\in F$ such that $x\leq y$. In some sources, such as [11], F is then said to be full in E.

A vector lattice E *has the countable sup property* when, for every non-empty subset S of E that has a supremum in E, there exists an at most countable subset of S that has the same supremum in E as S. In parts of the literature, such as in [36] and [51], E is then said to be order separable.

Let E be a vector lattice, and let $x\in E$. We say that a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in E is *order convergent to* $x\in E$ (denoted by $x_\alpha\xrightarrow{\mathrm{o}}x$) when there exists a net $(y_\beta)_{\beta\in\mathcal{B}}$ in E such that $y_\beta\downarrow 0$ and with the property that, for every $\beta_0\in\mathcal{B}$, there exists an $\alpha_0\in\mathcal{A}$ such that $|x-x_\alpha|\leq y_{\beta_0}$ whenever α in $\mathcal{A}$ is such that $\alpha\geq\alpha_0$. Note that the index sets $\mathcal{A}$ and $\mathcal{B}$ need not be equal; for a discussion of the difference between these two possible definitions we refer to [1], for example.

Let E and F be vector lattices. The order bounded operators from E into F will be denoted by $\mathscr{L}_{\mathrm{ob}}(E,F)$, and the regular operators from E into F by $\mathscr{L}_{\mathrm{r}}(E,F)$. When F is Dedekind complete, we have $\mathscr{L}_{\mathrm{ob}}(E,F)=\mathscr{L}_{\mathrm{r}}(E,F)$, and this space is then a Dedekind complete vector lattice; see [7, Theorem 1.18], for example. We write $E^\sim$ for $\mathscr{L}_{\mathrm{ob}}(E,\mathbb{R})=\mathscr{L}_{\mathrm{r}}(E,\mathbb{R})$.

A linear operator $T: E\to F$ between two vector lattices E and F is *order continuous* when, for every net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in E, the fact that $x_\alpha \xrightarrow{\mathrm{o}} 0$ in E implies that $Tx_\alpha \xrightarrow{\mathrm{o}} 0$ in F. When T is positive one can, equivalently, require that, for every net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in E, the fact that $x_\alpha\downarrow 0$ in E imply that $Tx_\alpha\downarrow 0$ in F. An order continuous linear operator between two vector lattices is automatically order bounded; see [7, Lemma 1.54], for example. The order continuous linear operators from E into F will be denoted by $\mathscr{L}_{\mathrm{oc}}(E,F)$. In the literature, the notation $\mathscr{L}_{\mathrm{n}}(E,F)$ is often used. When F is Dedekind complete, $\mathscr{L}_{\mathrm{oc}}(E,F)$ is a band in $\mathscr{L}_{\mathrm{r}}(E,F)$; see [7, Theorem 1.57], for example. We write $E_{\mathrm{oc}}^\sim$ for $\mathscr{L}_{\mathrm{oc}}(E,\mathbb{R})$.

The following result is easily established using the Riesz-Kantorovich formulas and their ‘dual versions’; see [7, Theorems 1.18 and 1.23], for example. We shall be interested only in the case where the lattice F in it is the real numbers and the band B is the zero band, but the general case comes at no extra cost in the routine proof.

Proposition 2.2.1. *Let E and F be vector lattices, where F is Dedekind complete, and let B be a band in F.*

(1) Let I be an ideal of E. Then the subset

$$\{\, T\in\mathscr{L}_{\mathrm{r}}(E,F) : Tx\in B \textit{ for all } x\in I\,\}$$

of $\mathscr{L}_{\mathrm{r}}(E,F)$ is band in $\mathscr{L}_{\mathrm{r}}(E,F)$. For every subset S of I that generates I, it is equal to

$$\{\, T\in\mathscr{L}_{\mathrm{r}}(E,F) : |T||x|\in B \textit{ for all } x\in S\,\}.$$

(2) Let $\mathscr{I}$ be an ideal of $\mathscr{L}_{\mathrm{r}}(E,F)$. Then the subset

$$\{\, x\in E : Tx\in B \textit{ for all } T\in\mathscr{I}\,\}$$

of E is an ideal of E. For every subset $\mathscr{S}$ of $\mathscr{I}$ that generates $\mathscr{I}$, it is equal to

$$\{\, x\in E : |T||x|\in B \textit{ for all } T\in\mathscr{S}\,\}.$$

It is a band in E when $\mathscr{I}\subseteq\mathscr{L}_{\mathrm{oc}}(E,F)$.

Let F be a vector sublattice of a vector lattice E. Then F is a *regular vector sublattice of E* when the inclusion map from F into E is order continuous. Equivalently, for every net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in F, the fact that $x_\alpha\downarrow 0$ in F should imply that $x_\alpha\downarrow 0$ in E. It is immediate from the latter criterion that ideals are regular vector sublattices. It is also true that order dense vector sublattices are regular vector sublattices; see [6, Theorem 1.23], for example.

Let $(x_\alpha)_{\alpha\in\mathcal{A}}$ be a net in a vector lattice E, and let $x\in E$. We say that (x_α) is *unbounded order convergent to x in E* (denoted by $x_\alpha\xrightarrow{\mathrm{uo}} x$) when $|x_\alpha - x|\wedge y\xrightarrow{\mathrm{o}} 0$ in E for all $y\in$

E^+. Order convergence implies unbounded order convergence to the same limit. For order bounded nets, the two notions coincide. [2]

We shall repeatedly refer to the following collection of results; see [28, Theorem 2.8, Corollary 2.12, and Theorem 3.2].

Theorem 2.2.2. *Let E be a vector lattice, and let F be a vector sublattice of E. Take a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in F.*

(1) Suppose that F is order dense and majorising in E. Then $x_\alpha \xrightarrow{\text{o}} 0$ in F if and only if $x_\alpha \xrightarrow{\text{o}} 0$ in E.

(2) Suppose that F is a regular vector sublattice of E and that $(x_\alpha)_{\alpha\in\mathcal{A}}$ is order bounded in F. Then $x_\alpha \xrightarrow{\text{o}} 0$ in F if and only if $x_\alpha \xrightarrow{\text{o}} 0$ in E.

(3) The following are equivalent:

(a) F is a regular vector sublattice of E;

(b) for every net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in F, the fact that $x_\alpha \xrightarrow{\text{uo}} 0$ in F implies that $x_\alpha \xrightarrow{\text{uo}} 0$ in E;

(c) for every net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in F, $x_\alpha \xrightarrow{\text{uo}} 0$ in F if and only if $x_\alpha \xrightarrow{\text{uo}} 0$ in E.

In the sequel of this paper, we shall encounter restrictions of order continuous linear functionals on vector lattices to vector sublattices. For this, we include the following result. It is based on a theorem of Veksler's. It contains quite a bit more than we shall actually need, but we use the opportunity to present the results in it, and its fourth and fifth parts in particular.

Theorem 2.2.3. *Let E be a vector lattice, let F be a vector sublattice of E, and let G be a Dedekind complete vector lattice. Take $T \in \mathscr{L}_{\text{oc}}(E,G)$.*

(1) Suppose that F is a regular vector sublattice of E. Then the restriction $T|_F : F \to G$ of T to F is order continuous.

(2) Suppose that F is a regular sublattice of E. When $\mathscr{L}_{\text{oc}}(E,G)$ separates the points of E, then $\mathscr{L}_{\text{oc}}(F,G)$ separates the points of F.

(3) Suppose that F is an order dense vector sublattice of E. Then the restriction map $T \mapsto T|_F$ is a positive linear injection from $\mathscr{L}_{\text{oc}}(E,G)$ into $\mathscr{L}_{\text{oc}}(F,G)$.

Suppose that F is an order dense and majorising vector sublattice of E. Then:

(4) the restriction map $T \mapsto T|_F$ is a lattice isomorphism between $\mathscr{L}_{\text{oc}}(E,G)$ and $\mathscr{L}_{\text{oc}}(F,G)$;

(5) $\mathscr{L}_{\text{oc}}(E,G)$ separates the points of E if and only if $\mathscr{L}_{\text{oc}}(F,G)$ separates the points of F.

Proof. Part (1) is clear, and then so is part (2).

It is evident from part (1) that $\mathscr{L}_{\text{oc}}(F,G)$ separates the points of F when $\mathscr{L}_{\text{oc}}(E,G)$ separates the points of E.

Suppose that F is an order dense (hence regular) vector sublattice of E and that $T \in \mathscr{L}_{\text{oc}}(E,G)$ is such that $T|_F = 0$. Take $x \in E^+$. Then $\{\, y \in F : 0 \leq y \leq x \,\} \uparrow x$ in E. Since $T|_F = 0$, the order continuity of T on E then implies that $Tx = 0$. Hence $T = 0$, and we conclude that the restriction map $T \mapsto T|_F$ is a positive linear injection from $\mathscr{L}_{\text{oc}}(E,G)$ into $\mathscr{L}_{\text{oc}}(F,G)$.

[2] Although we shall not need this, it would be less than satisfactory not to mention here that the uo-continuous dual of a vector lattice (defined in the obvious way) has a very concrete description, and is often trivial. According to [27, Proposition 2.2], it is the linear span of the coordinate functionals corresponding to atoms.

Suppose that F is order dense and majorising in E.

Take $S \in \mathscr{L}_{\mathrm{oc}}(F,G)$. In that case, according to a result of Veksler's (see [7, Theorem 1.65]), each of S^+ and S^- can be extended to a positive order continuous operator from E into G. Hence S itself can be extended to an order continuous operator S^{ext} from E into G. By what we have already observed in part (3), such an order continuous extension is unique, and we conclude from this that the map $S \mapsto S^{\mathrm{ext}}$ is a positive linear injection from $\mathscr{L}_{\mathrm{oc}}(F,G)$ into $\mathscr{L}_{\mathrm{oc}}(E,G)$. It is clear that the extension and restriction maps between $\mathscr{L}_{\mathrm{oc}}(E,G)$ and $\mathscr{L}_{\mathrm{oc}}(F,G)$ are each other's inverses. We conclude that the restriction map $T \mapsto T|_F$ is a bi-positive linear bijection between $\mathscr{L}_{\mathrm{oc}}(E,G)$ and $\mathscr{L}_{\mathrm{oc}}(F,G)$. Hence it is a lattice isomorphism, as required.

One direction of the equivalence in part (5) is clear from part (2). For the converse direction, suppose that $\mathscr{L}_{\mathrm{oc}}(F,G)$ separates the points of F. Take $x \in E$ such that $Tx = 0$ for all $T \in \mathscr{L}_{\mathrm{oc}}(E,G)$. Since $\mathscr{L}_{\mathrm{oc}}(E,G)$ is an ideal of $\mathscr{L}_{\mathrm{r}}(E,F)$, Proposition 2.2.1 shows that $T|x| = 0$ for all $T \in \mathscr{L}_{\mathrm{oc}}(E,G)$. Suppose that $x \neq 0$. Then there exists a $y \in F$ such that $0 < y \leq |x|$, and we have $Ty = 0$ for all positive $T \in \mathscr{L}_{\mathrm{oc}}(E,G)$, hence for all $T \in \mathscr{L}_{\mathrm{oc}}(E,G)$. In view of part (4), this is the same as saying that $Sy = 0$ for all $S \in \mathscr{L}_{\mathrm{oc}}(F,G)$. Our assumption yields that $y = 0$; this contradiction shows that we must have $x = 0$. $\square$

2.2.2 Topologies on vector lattices

When E is a vector space, a *linear topology on E* is a (not necessarily Hausdorff) topology that provides E with the structure of a topological vector space. When E is a vector lattice, a *locally solid linear topology on E* is a linear topology on E such that there exists a base of (not necessarily open) neighbourhoods of 0 that are solid subsets of E. For the general theory of locally solid linear topologies on vector lattices we refer to [6]. A locally solid linear topology on E that is also a locally convex linear topology is a *locally convex-solid linear topology*. In that case, there exists a base of neighbourhoods of 0 that consists of absorbing, closed, convex, and solid subsets of E; see [6, p. 59].

When E is a vector lattice, a *locally solid additive topology on E* is a topology that provides the additive group E with the structure of a (not necessarily Hausdorff) topological group, such that there exists a base of (not necessarily open) neighbourhoods of 0 that are solid subsets of E.

Let E be a vector lattice. We say that *order convergence in E is topological* when there exists a (evidently unique) topology on E such that its convergent nets are precisely the order convergent nets, with preservation of limits. It follows from the properties of order convergence that such a topology is automatically a Hausdorff linear topology. Likewise, *unbounded order convergence in E is topological* when there exists a topology on E such that its convergent nets are precisely the nets that are unbounded order convergent, with preservation of limits. Such a topology is again unique, and automatically a Hausdorff linear topology.

A topology τ on a vector lattice E is an *o-Lebesgue topology* when it is a (not necessarily Hausdorff) locally solid linear topology on E such that, for a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in E and $x \in E$, the fact that $x_\alpha \xrightarrow{\mathrm{o}} x$ in E implies that $x_\alpha \xrightarrow{\tau} x$. Equivalently, the fact that $x_\alpha \xrightarrow{\mathrm{o}} 0$ in E should imply that $x_\alpha \xrightarrow{\tau} 0$. A vector lattice need not admit a Hausdorff o-Lebesgue topology. It can be shown, see [6, Example 3.2], that C([0, 1]) does not even admit a Hausdorff locally solid linear topology such that *sequential* order convergence implies topological convergence.

A topology τ on a vector lattice E is a *uo-Lebesgue topology* when it is a (not necessarily Hausdorff) locally solid linear topology on E such that, for a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in E and $x \in E$, the fact that $x_\alpha \xrightarrow{\text{uo}} x$ in E implies that $x_\alpha \xrightarrow{\tau} x$. Equivalently, the fact that $x_\alpha \xrightarrow{\text{uo}} 0$ in E should imply that $x_\alpha \xrightarrow{\tau} 0$. Since order convergence implies unbounded order convergence, a uo-Lebesgue topology is an o-Lebesgue topology.

The following fundamental facts are from [11, Proposition 3.2, 3.4, and 6.2, and Corollary 6.3] and [44, Theorems 5.5, 5.9, and 6.4].

Theorem 2.2.4 (Conradie and Taylor)**.** *Let E be a vector lattice. Then the following are equivalent:*

(1) E admits a Hausdorff o-Lebesgue topology;
(2) E admits a Hausdorff uo-Lebesgue topology;
(3) the partially ordered set of all Hausdorff locally solid linear topologies on E has a minimal element.

When this is the case, the topologies in the parts (2) *and* (3) *are both unique, they coincide, and they are the smallest Hausdorff o-Lebesgue topology on E.*

When E admits a Hausdorff uo-Lebesgue topology, we shall denote the unique such topology by $\widehat{\tau}_E$. In [11], it is denoted by τ_m. For a given vector lattice, there may be several ways to obtain a Hausdorff uo-Lebesgue topology on it. This can then give criteria for the convergence of nets in the common resulting topology that are apparently equivalent, but not always immediately obviously so. See Remark 2.6.4 for this, for example.

Remark 2.2.5. Some caution is necessary when consulting the literature on minimal Hausdorff locally solid linear topologies because in [6, Definition 7.64] such a topology is defined as what would usually be called a *smallest* Hausdorff locally solid linear topologies. When a vector lattice E admits a complete metrisable o-Lebesgue topology, such as a Banach lattice with an order continuous norm, then it admits a smallest (in the usual sense of the word) Hausdorff locally solid linear topology; see [6, Theorem 7.65]. Combining this with Theorem 2.2.4, we see that E then admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$, and that $\widehat{\tau}_E$ is then not just the smallest Hausdorff o-Lebesgue topology, but even the smallest Hausdorff locally solid linear topology on E.

2.3 Unbounded topologies generated by topologies on ideals

We shall now describe how topologies 'of unbounded type' on vector lattices can be obtained from topologies on ideals. There are already several constructions in this vein and accompanying results in the literature; see [11, 21, 32, 44], for example. In the following result, we carry out such a construction in what appears to be the most general possible context. Starting from a locally solid (not necessarily linear or Hausdorff) additive topology on an ideal F of a vector lattice E—which need not be the restriction of a global locally solid additive topology on E—and a non-empty subset of F, we define an 'unbounded' locally solid additive topology on E. We give necessary and sufficient conditions for this new topology on E to be Hausdorff, and also for it to be a linear topology. Various known results in more special cases can then be understood from the general theorem, as will be discussed in Examples 2.3.8 to 2.3.12, below.

The subset S figuring in this construction can be replaced by the ideal that it generates without altering the result. Although it may conceptually be more natural to work with ideals than with subsets, working with arbitrary subsets has the advantage of keeping an eye on a small number of presumably relatively easily manageable 'test elements'. It is for this reason that we carry this along to later results; see also Remark 2.4.4, below. The convenience of this approach will become apparent in the proof of Theorem 2.6.1.

Theorem 2.3.1. *Let E be a vector lattice, let F be an ideal of E, and let τ_F be a (not necessarily Hausdorff) locally solid additive topology on F. Take a non-empty subset S of F.*

There exists a unique (possibly non-Hausdorff) additive topology $\mathrm{u}_S\tau_F$ on E such that, for a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in E, $x_\alpha \xrightarrow{\mathrm{u}_S\tau_F} 0$ in E if and only if $|x_\alpha| \wedge |s| \xrightarrow{\tau_F} 0$ in F for all $s \in S$.

Let $I_S \subseteq F$ be the ideal generated by S in E. For a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in E, $x_\alpha \xrightarrow{\mathrm{u}_S\tau_F} 0$ in E if and only if $|x_\alpha| \wedge |y| \xrightarrow{\tau_F} 0$ in F for all $y \in I_S$.

Furthermore:

(1) the inclusion map from F into E is τ_F–$\mathrm{u}_S\tau_F$-continuous;

(2) the topology $\mathrm{u}_S\tau_F$ on E is a locally solid additive topology;

(3) the following are equivalent:

  (a) $\mathrm{u}_S\tau_F$ is a Hausdorff topology on E;

  (b) τ_F is a Hausdorff topology on F and I_S is order dense in E;

(4) the following are equivalent:

  (i) for all $x \in E$ and $s \in S$,

$$|\varepsilon x| \wedge |s| \xrightarrow{\tau_F} 0 \tag{2.1}$$

  in F as $\varepsilon \to 0$ in $\mathbb{R}$;

  (ii) for all $x \in E$ and $y \in I_S$, $|\varepsilon x| \wedge |y| \xrightarrow{\tau_F} 0$ in F as $\varepsilon \to 0$ in $\mathbb{R}$;

  (iii) $\mathrm{u}_S\tau_F$ is a (possibly non-Hausdorff) linear topology on E.

Proof. Suppose that τ_F is a (not necessarily Hausdorff) locally solid additive topology on F.

The uniqueness of $\mathrm{u}_S\tau_F$ is clear because the nets converging to 0 and then, by translation invariance of the topology, to arbitrary points of E are prescribed.

We turn to the existence of such a topology $\mathrm{u}_S\tau_F$. Take a neighbourhood base $\{U_\lambda\}_{\lambda\in\Lambda}$ of zero in F for τ_F consisting of solid subsets of F. For $y \in I_S$ and $\lambda \in \Lambda$, set

$$V_{\lambda,y} := \{\, x \in E : |x| \wedge |y| \in U_\lambda \,\}. \tag{2.2}$$

The $V_{\lambda,y}$ are solid subsets of E since F is an ideal of E and the U_λ are solid subsets of F. Set

$$\mathscr{N}_0 := \{\, V_{\lambda,y} : \lambda \in \Lambda, y \in I_S \,\}. \tag{2.3}$$

We claim that $\mathscr{N}_0$ is a base of neighbourhoods of zero for a topology on E, which we shall already denote by $\mathrm{u}_S\tau_F$, that provides the additive group E with the structure of a topological group. Necessary and sufficient conditions on $\mathcal{N}_0$ for this can be found in [31, Theorem 3 on p.46]; we now verify these.

Take $V_{\lambda_1,y_1}, V_{\lambda_2,y_2} \in \mathcal{N}_0$. There exists a $\lambda_3 \in \Lambda$ such that $U_{\lambda_3} \subseteq U_{\lambda_1} \cap U_{\lambda_2}$. Take $x \in V_{\lambda_3,|y_1|\vee|y_2|}$. Then

$$|x| \wedge |y_1| \leq |x| \wedge (|y_1| \vee |y_2|) \in U_{\lambda_3} \subseteq U_{\lambda_1}.$$

Since F is an ideal of E and U_{λ_1} is a solid subset of F, this implies that $|x| \wedge |y_1| \in U_{\lambda_1}$, so that $x \in V_{\lambda_1,y_1}$. Likewise, $x \in V_{\lambda_2,y_2}$, and we see that $V_{\lambda_3,|y_1|\vee|y_2|} \subseteq V_{\lambda_1,y_1} \cap V_{\lambda_2,y_2}$.

It is evident that $V_{\lambda,y} = -V_{\lambda,y}$ for all $V_{\lambda,y} \in \mathcal{N}_0$.

Take $V_{\lambda,y} \in \mathcal{N}_0$. There exists a $\mu \in \Lambda$ such that $U_\mu + U_\mu \subseteq U_\lambda$. Then, for all $x_1, x_2 \in V_{\mu,y}$, we have

$$|x_1 + x_2| \wedge |y| \leq |x_1| \wedge |y| + |x_2| \wedge |y| \in U_\mu + U_\mu \subseteq U_\lambda.$$

Since F is an ideal of E and U_λ is a solid subset of F, this implies that $|x_1 + x_2| \wedge |y| \in U_\lambda$, so that $x_1 + x_2 \in V_{\lambda,y}$. Hence $V_{\mu,y} + V_{\mu,y} \subseteq V_{\lambda,y}$.

An appeal to [31, p. 46, Theorem 3] now establishes our claim.

It is clear from the definition of $\mathrm{u}_S\tau_F$ that, for a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in E, $x_\alpha \xrightarrow{\mathrm{u}_S\tau_F} 0$ in E if and only if $|x_\alpha| \wedge |y| \xrightarrow{\tau_F} 0$ in F for all $y \in I_S$.

Certainly, the fact that $|x_\alpha| \wedge |y| \xrightarrow{\tau_F} 0$ in F for all $y \in I_S$ implies that $|x_\alpha| \wedge |s| \xrightarrow{\tau_F} 0$ in F for all $s \in S$. Conversely, suppose that $(x_\alpha)_{\alpha\in\mathcal{A}}$ is a net in E such that $|x_\alpha| \wedge |s| \xrightarrow{\tau_F} 0$ in F for all $s \in S$. Take $y \in I_S$. There exist $s_1, \ldots, s_n \in S$ and integers $k_1, \ldots, k_n \geq 1$ such that $|y| \leq \sum_{i=1}^n k_i |s_i|$. Hence $|x_\alpha| \wedge |y| \leq \sum_{i=1}^n k_i \left(|x_\alpha| \wedge |s_i|\right)$. Since τ_F is a locally solid additive topology on F, this implies that $|x_\alpha| \wedge |y| \xrightarrow{\tau_F} 0$ in F.

We turn to the parts (1)–(4).

Since F is an ideal of E and the U_λ are solid subsets of F, we have $U_\lambda \subseteq V_{\lambda,y}$ for all $\lambda \in \Lambda$ and $y \in I_S$. This implies that the inclusion map from F into E is τ_F–$\mathrm{u}_S\tau_F$-continuous.

The topology $\mathrm{u}_S\tau_F$ is a locally solid additive topology on E by construction.

Suppose that $\mathrm{u}_S\tau_F$ is a Hausdorff topology on E. Then so is the topology it induces on F, which is weaker than τ_F. Hence τ_F is a Hausdorff topology on F. Take $x \in E$ with $x > 0$. Then there exists a $V_{\lambda,y} \in \mathcal{N}_0$ with $x \notin V_{\lambda,y}$. In particular, $x \wedge |y| \neq 0$. Hence $0 < x \wedge |y| \leq x$. Since $x \wedge |y| \in I_S$, we see that I_S is order dense in E.

Suppose, conversely, that τ_F is a Hausdorff topology on F and that I_S is order dense in E. Take $x \neq 0$ in E. There exists a $y \in I_S$ with $0 < y \leq |x|$. Pick $U_{\lambda_0} \in \{U_\lambda\}_{\lambda\in\Lambda}$ such that $y \notin U_{\lambda_0}$. Then $|x| \wedge |y| = y \notin U_{\lambda_0}$, so that $x \notin V_{\lambda_0,y}$. Hence $\bigcap_{V\in\mathcal{N}_0} V = \{0\}$. By [31, p. 48, Theorem 4], $\mathrm{u}_S\tau_F$ is a Hausdorff additive topology on the topological group E.

We shall now verify the equivalence of the parts (i)–(iii) of (4).

We prove that (i) implies (ii). Take $x \in E$ and $y \in I_S$. There exist $s_1, \ldots, s_n \in S$ and integers $k_1, \ldots, k_n \geq 1$ such that $|y| \leq \sum_{i=1}^n k_i |s_i|$, and it follows from this that $|\varepsilon x| \wedge |y| \leq \sum_{i=1}^n k_i \left(|\varepsilon x| \wedge |s_i|\right)$ for all $\varepsilon \in \mathbb{R}$. Since τ_F is a locally solid additive topology on F, it follows that $|\varepsilon x| \wedge |y| \xrightarrow{\tau_F} 0$ in F as $\varepsilon \to 0$ in $\mathbb{R}$.

We prove that (ii) implies (iii). Fix $\lambda \in \Lambda$ and $y \in I_S$, and take $x \in E$. Since $|\varepsilon x| \wedge |y| \xrightarrow{\tau_F} 0$ in F as $\varepsilon \to 0$ in $\mathbb{R}$, there exists a $\delta > 0$ such that $|\varepsilon x| \wedge |y| \in U_\lambda$ whenever $|\varepsilon| \leq \delta$. That is, $\varepsilon x \in V_{\lambda,y}$ whenever $|\varepsilon| \leq \delta$. This implies that $V_{\lambda,y}$ is absorbing. Furthermore, since $V_{\lambda,y}$ is a solid subset of E, it is clear that $\varepsilon x \in V_{\lambda,y}$ whenever $x \in V_{\lambda,y}$ and $\varepsilon \in [-1,1]$. Hence $V_{\lambda,y}$ is balanced. Then [5, Theorem 5.6] implies that $\mathrm{u}_S\tau_F$ is a linear topology on E.

We prove that (iii) implies (i). Take $x \in E$. Then $\varepsilon x \xrightarrow{\mathrm{u}_S\tau_F} 0$ in E as $\varepsilon \to 0$ in $\mathbb{R}$. By construction, this implies (and is, in fact, equivalent to) the fact that $|\varepsilon x| \wedge |s| \xrightarrow{\tau_F} 0$ in F for all $s \in S$.

This concludes the proof of the equivalence of the three parts of (4). The proof of the theorem is now complete. □

Definition 2.3.2. The topology $\mathrm{u}_S\tau_F$ in Theorem 2.3.1 is called the *unbounded topology on E that is generated by τ_F via S.*

Remark 2.3.3. It is clear from the two equivalent criteria in Theorem 2.3.1 for a net in E to be $\mathrm{u}_S\tau_F$-convergent to zero that $\mathrm{u}_S\tau_F = \mathrm{u}_{I_S}\tau_F$ for every non-empty subset S of F. Consequently, $\mathrm{u}_{S_1}\tau_F = \mathrm{u}_{S_2}\tau_F$ whenever S_1, S_2 are non-empty subsets of F such that $I_{S_1} = I_{S_2}$.

Remark 2.3.4. In Theorem 2.3.1, suppose that the locally solid additive topology F is the restriction $\tau_E|_F$ of a locally solid additive topology on E. It is then easy to see that $\mathrm{u}_S(\tau_E|_F) = \mathrm{u}_S\tau_E$ for every non-empty subset S of F.

Remark 2.3.5. In Theorem 2.3.1, and also in the remainder of this paper, the topologies of interest are characterised by their convergent nets. It should be noted, however, that in equations (2.2) and (2.3) the proof of Theorem 2.3.1 provides an explicit neighbourhood base of zero in E for $\mathrm{u}_S\tau_F$, in terms of a neighbourhood base of zero in F for τ_F and the ideal I_S. Suppose, for example that τ_F is a (possibly non-Hausdorff) locally convex linear topology on F that is generated by a family $\{\rho_\gamma : \gamma \in \Gamma\}$ of semi-norms on F, as will be the case in Section 2.5. Then the collection of subsets of E of the form

$$\{x \in E : \rho_i(|x| \wedge |y|) < \varepsilon \text{ for } \rho_1, \dots \rho_n \in \Gamma\},$$

where $y \in I_S$, $n \geq 1$, and $\varepsilon > 0$ are arbitrary, is a neighbourhood base of zero in E for $\mathrm{u}_S\tau_F$.

Our next result is concerned with iterating the construction in Theorem 2.3.1. It generalises what is in [44, p. 997].

Proposition 2.3.6. *Let E be a vector lattice, let F_1 be an ideal of E, and let τ_{F_1} be a (not necessarily Hausdorff) locally solid additive topology on F_1. Take a non-empty subset S_1 of F_1, and consider the unbounded topology $\mathrm{u}_{S_1}\tau_{F_1}$ on E that is generated by τ_{F_1} via S_1. Let F_2 be an ideal of E, and let $(\mathrm{u}_{S_1}\tau_{F_1})|_{F_2}$ denote the topology on F_2 that is induced on F_2 by $\mathrm{u}_{S_1}\tau_{F_1}$. Then $(\mathrm{u}_{S_1}\tau_{F_1})|_{F_2}$ is a locally solid additive topology on F_2. Take a non-empty subset S_2 of F_2. Then $\mathrm{u}_{S_2}\big[(\mathrm{u}_{S_1}\tau_{F_1})|_{F_2}\big] = \mathrm{u}_{I_{S_1}\cap I_{S_2}}\tau_{F_1}$. In particular, when S is a non-empty subset of $F_1 \cap F_2$, then $\mathrm{u}_S\big[(\mathrm{u}_S\tau_{F_1})|_{F_2}\big] = \mathrm{u}_S\tau_{F_1}$.*

Proof. It is clear from Theorem 2.3.1 that $(\mathrm{u}_{S_1}\tau_{F_1})|_{F_2}$ is a locally solid additive topology on F_2. Let $(x_\alpha)_{\alpha\in\mathcal{A}}$ be a net in E. Then we have the following chain of equivalent statements:

$$x_\alpha \xrightarrow{\mathrm{u}_{S_2}\left[(\mathrm{u}_{S_1}\tau_{F_1})|_{F_2}\right]} 0 \text{ in } E$$

$$\iff |x_\alpha| \wedge |y_2| \xrightarrow{(\mathrm{u}_{S_1}\tau_{F_1})|_{F_2}} 0 \text{ in } F_2 \text{ for all } y_2 \in I_{S_2}$$

$$\iff |x_\alpha| \wedge |y_2| \xrightarrow{\mathrm{u}_{S_1}\tau_{F_1}} 0 \text{ in } E \text{ for all } y_2 \in I_{S_2}$$

$$\iff |x_\alpha| \wedge |y_2| \wedge |y_1| \xrightarrow{\tau_{F_1}} 0 \text{ in } F_1 \text{ for all } y_1 \in I_{S_1} \text{ and } y_2 \in I_{S_2}$$

$$\iff |x_\alpha| \wedge |y| \xrightarrow{\tau_{F_1}} 0 \text{ in } F_1 \text{ for all } y \in I_{S_1} \cap I_{S_2}$$

$$\iff x_\alpha \xrightarrow{\mathrm{u}_{I_{S_1}\cap I_{S_2}}\tau_{F_1}} 0 \text{ in } E.$$

Hence $\mathrm{u}_{S_2}\big[\big(\mathrm{u}_{S_1}\tau_{F_1}\big)|_{F_2}\big] = \mathrm{u}_{I_{S_1}\cap I_{S_2}}\tau_{F_1}$. $\square$

Remark 2.3.7. In Proposition 2.3.6, suppose that τ_{F_1} is a (not necessarily Hausdorff) locally solid additive topology on F_1 such that, for all $x \in E$ and $s \in S_1$, $|\varepsilon x| \wedge |s| \xrightarrow{\tau_{F_1}} 0$ in F_1 as $\varepsilon \to 0$ in $\mathbb{R}$. It is then clear from Theorem 2.3.1 that $\mathrm{u}_{S_1}\tau_{F_1}$, $\big(\mathrm{u}_{S_1}\tau_{F_1}\big)|_{F_2}$, and $\mathrm{u}_{I_{S_1}\cap I_{S_2}}\tau_{F_1}$ are (possibly non-Hausdorff) locally solid linear topologies on E, F_2, and E, respectively.

We shall now explain how Theorem 2.3.1 relates to various results already in the literature.

Example 2.3.8. When $F = E$ and τ_E is a locally solid linear topology on $F = E$, the condition in equation (2.1) is automatically satisfied for any non-empty subset S of $F = E$. According to Theorem 2.3.1, $\mathrm{u}_E\tau_E$ is a locally solid linear topology on E that is Hausdorff if and only if τ_E is Hausdorff; this is [44, Theorem 2.3]. Furthermore, when A is an ideal of E, $\mathrm{u}_A\tau_E$ is a locally solid linear topology on E that is Hausdorff if and only if τ_E is Hausdorff and A is order dense in E; this is [44, Propositions 9.3 and 9.4].

Example 2.3.9. Let E be a Banach lattice. In Theorem 2.3.1, we take $F = E$, for τ_F we take the norm topology τ_E on $F = E$, and for $S \subseteq F$ we take $S = F = E$. Then the condition in equation (2.1) is satisfied. According to Theorem 2.3.1, $\mathrm{u}_E\tau_E$ is a Hausdorff locally solid linear topology on E and, for a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in E, $x_\alpha \xrightarrow{\mathrm{u}_E\tau_E} 0$ if and only if $\||x_\alpha| \wedge |y|\| \to 0$ for all $y \in E$. In [21], this type of convergence is called *unbounded norm convergence*, or *un-convergence* for short. It was already observed in [21, Section 7] that it is topological; in [32, p. 746], $\mathrm{u}_F\tau_F$ is then called the *un-topology*.

Example 2.3.10. Let E be a vector lattice, and let F be an ideal of E that is a normed vector lattice. In Theorem 2.3.1, we take for τ_F the norm topology on F, and for $S \subseteq F$ we take $S = F$. According to Theorem 2.3.1, $\mathrm{u}_F\tau_F$ is a (possibly non-Hausdorff) additive topology on E and, for a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in E, $x_\alpha \xrightarrow{\mathrm{u}_F\tau_F} 0$ if and only if $\||x_\alpha| \wedge |y|\| \to 0$ for all $y \in F$. This type of convergence is called *un-convergence with respect to X* in [32]. It was already observed that it is topological in [32, p. 747], where $\mathrm{u}_F\tau_F$ is called the *un-topology on E induced by F*.

In [32, Example 1.3], it is shown that $\mathrm{u}_F\tau_F$ can fail to be a Hausdorff topology on E. Since τ_F is a Hausdorff topology on F, Theorem 2.3.1 shows that the pertinent ideal F in [32, Example 1.3] must fail to be order dense in E; this is indeed easily seen to be the case.

Theorem 2.3.1 implies that $\mathrm{u}_F\tau_F$ is Hausdorff if and only if F is order dense in F; this is [32, Proposition 1.4].

In [32, Example 1.5], it is shown that $\mathrm{u}_F\tau_F$ can fail to be a linear topology on E. According to Theorem 2.3.1, the condition in equation (2.1) must fail to be satisfied in the context of [32, Example 1.5]; this is indeed easily seen to be the case. Theorem 2.3.1 shows that $\mathrm{u}_F\tau_F$ always provides E with an additive topology; this was also noted in [32, p. 748] in that particular context.

In [32, p. 748], the authors observe that $\mathrm{u}_F\tau_F$ is a locally solid linear topology on the vector lattice E whenever E is a normed lattice and the norm on E extends that on F, and also whenever the norm on F is order continuous. Both facts follow from Theorem 2.3.1 because equation (2.1) is then satisfied. This is clear when E is a normed lattice and the norm on E extends that on F. Suppose that the norm on F is order continuous. Take $x \in E$ and $y \in F$. Then $|\varepsilon x| \wedge |y| \xrightarrow{\mathrm{o}} 0$ in E as $\varepsilon \to 0$. Since the net $|\varepsilon x| \wedge |y|$ is order bounded in the ideal F of E, which is a regular vector sublattice of E, Theorem 2.2.2 implies that $|\varepsilon x| \wedge |y| \xrightarrow{\mathrm{o}} 0$ in F, and then $|\varepsilon x| \wedge |y| \xrightarrow{\tau_F} 0$ as $\varepsilon \to 0$ by the order continuity of the norm on F.

Example 2.3.11. For a vector lattice E, we let $|\sigma|(E, E^\sim)$ denote its absolute weak topology; the definition of this locally solid linear topology will be recalled in Section 2.5. Taking $E = F = S$ in Theorem 2.3.1 yields the so-called *unbounded absolute weak topology* $\mathrm{u}_E|\sigma|(E, E^\sim)$ on E. It is a locally solid additive topology on E that is Hausdorff if and only if $E^\sim$ separates the points of E. When E is a Banach lattice, $\mathrm{u}_E|\sigma|(E, E^\sim)$ is a Hausdorff locally solid linear topology on E. It is studied in [52].

Example 2.3.12. In [11, p. 290], a construction is given to obtain a locally solid linear topology on a vector lattice E from a locally solid linear topology on an ideal F of E. This is done using Riesz pseudo-norms, rather than by working with neighbourhood bases of zero as we have done. The key ingredient is to start with a Riesz pseudo-norm p on F, take an element u of F^+, and introduce a map $p_u : E \to \mathbb{R}$ by setting $p_u(x) := p(|x| \wedge u)$ for $x \in E$. It is then remarked that p_u is a Riesz pseudo-norm on E. This need not always be the case, however. By way of counter-example, take for E the vector lattice of all real-valued functions on $\mathbb{R}$, and for F the ideal of E consisting of all bounded functions on $\mathbb{R}$. For p, we take the supremum norm on F. For $u \in F^+$, we choose the constant function 1. We define $x \in E$ by setting $x(t) := t$ for $t \in \mathbb{R}$. Then $p_u(\lambda x) = \| |\lambda x| \wedge u \| = 1$ for all non-zero $\lambda \in \mathbb{R}$, whereas we should have that $\lim_{\lambda \to 0} p_u(\lambda x) = 0$. This implies that the topologies on E that are thus constructed, although locally solid additive topologies, need not be linear topologies. This 'pathology' is similar to that in [32, Example 1.5] that was mentioned above; our example here is also quite similar to that in [32, Example 1.5]. Fortunately, in the continuation of the argument in [11], p is taken to be a Riesz pseudo-norm on F that is continuous with respect to a Hausdorff o-Lebesgue topology τ_F on F. In this context, p_u *is* a Riesz pseudo-norm on E. Indeed, since F, being an ideal of E, is a regular vector sublattice of E, Theorem 2.2.2 easily yields that $|\lambda x| \wedge u \xrightarrow{\mathrm{o}} 0$ in F as $\lambda \to 0$. Since τ_F is an o-Lebesgue topology on E, we have $|\lambda x| \wedge u \xrightarrow{\tau_F} 0$ in F as $\lambda \to 0$, and then the continuity of p on F yields that $p_u(\lambda x) \to 0$ as $\lambda \to 0$. Thus the construction in [11] proceeds correctly after all. The results of our systematic investigation with minimal hypotheses in Theorem 2.3.1, however, are more comprehensive than those in [11].

2.4 Hausdorff uo-Lebesgue topologies: going up and going down

In this section, we investigate how, via a going-up-going-down construction, the existence of a Hausdorff o-Lebesgue topology on an order dense ideal of a vector lattice E implies that every regular vector sublattice of E admits a (necessarily unique) Hausdorff uo-Lebesgue topology.

We start by going up.

Proposition 2.4.1. *Let E be a vector lattice, and let F be an ideal of E. Suppose that F admits an o-Lebesgue topology τ_F. Choose a non-empty subset S of F. Then $\mathrm{u}_S\tau_F$ is a uo-Lebesgue topology on E. It is a (necessarily unique) Hausdorff uo-Lebesgue topology on E if and only if τ_F is a Hausdorff topology on F and the ideal I_S that is generated by S is order dense in E.*

Proof. We know from Theorem 2.3.1 that $\mathrm{u}_S\tau_F$ is a locally solid additive topology on E. In order to see that it is a linear topology on E, we verify the condition in equation (2.1). Take x in E and s in S. Then $|\varepsilon x| \wedge |s| \xrightarrow{\mathrm{o}} 0$ in E as $\varepsilon \to 0$ in $\mathbb{R}$. Since F, being in ideal of E, is a regular vector sublattice of E, Theorem 2.2.2 shows that $|\varepsilon x| \wedge |s| \xrightarrow{\mathrm{o}} 0$ in F. Since τ_F is an o-Lebesgue topology on F, this implies that $|\varepsilon x| \wedge |s| \xrightarrow{\tau_F} 0$ in F as $\varepsilon \to 0$ in $\mathbb{R}$, as required.

To conclude the proof, suppose that $(x_\alpha)_{\alpha\in\mathcal{A}}$ is a net in E such that $x_\alpha \xrightarrow{\mathrm{uo}} 0$ in E. Take $s \in S$. Then $|x_\alpha| \wedge |s| \xrightarrow{\mathrm{o}} 0$ in E. Again, since F is a regular vector sublattice of E, Theorem 2.2.2 shows that $|x_\alpha| \wedge |s| \xrightarrow{\mathrm{o}} 0$ in F. Since τ_F is an o-Lebesgue topology on F, this implies that $|x_\alpha| \wedge |s| \xrightarrow{\tau_F} 0$ in F. It now follows from Theorem 2.3.1 that $x_\alpha \xrightarrow{\mathrm{u}_S\tau_F} 0$ in E, as required.

The uniqueness statement is clear from Theorem 2.2.4. $\square$

The combination of Theorem 2.3.1 and Proposition 2.4.1 immediately yields the following.

Theorem 2.4.2. *Let E be a vector lattice. Suppose that E has an order dense ideal F that admits a Hausdorff o-Lebesgue topology. Then E admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$. This topology $\widehat{\tau}_E$ is equal to $\mathrm{u}_S\tau_F$ for every subset S of F such that the ideal $I_S \subseteq F$ that is generated by S is order dense in E.*

For a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in E, the following are equivalent:

(1) $x_\alpha \xrightarrow{\widehat{\tau}_E} 0$ *in E;*

(2) $|x_\alpha| \wedge |s| \xrightarrow{\tau_F} 0$ *in F for all $s \in S$;*

(3) $|x_\alpha| \wedge |y| \xrightarrow{\tau_F} 0$ *in F for all $y \in F$.*

Remark 2.4.3. For the case in Theorem 2.4.2 where $S = F$ and τ_F is the restriction of a Hausdorff o-Lebesgue topology on E, it was already established in [44, Theorem 9.6] that $\mathrm{u}_F\tau_F$ is a Hausdorff uo-Lebesgue topology on E. It is, therefore, of some importance to point out that not every Hausdorff o-Lebesgue topology on an order dense ideal is the restriction of a Hausdorff o-Lebesgue topology on the enveloping vector lattice. By way of example, consider the order dense ideal c_0 of ℓ^∞. Since the supremum norm on c_0 is order continuous, the usual norm topology τ_{c_0} on c_0 is a Hausdorff o-Lebesgue topology. However, there does not even exist a possibly non-Hausdorff o-Lebesgue topology τ_{ℓ^∞} on ℓ^∞ that extends τ_{c_0}. In order to see this, consider the sequence of standard unit vectors $(e)_{n=1}^\infty$ in ℓ^∞. We have $e_n \xrightarrow{\mathrm{o}} 0$ in ℓ^∞ , which would imply that $e_n \xrightarrow{\tau_{\ell^\infty}} 0$ in ℓ^∞. Since τ_{ℓ^∞} extends τ_{c_0}, we would have that $e_n \to 0$ in norm. This contradiction shows that such an extension does not exist.

Although the terminology is not used as such, the fact that $\mathrm{u}_F\tau_F$ is a Hausdorff uo-Lebesgue topology on E is implicit in the construction in [11, p. 290].

Remark 2.4.4. We are not aware of a reference where it is noted, as in part (2), that convergence of a net in the Hausdorff uo-Lebesgue topology on E can be established by using a (presumably small and manageable) subset S of F instead of the full ideal F. This non-trivial fact, which relies on the uniqueness of a Hausdorff uo-Lebesgue topology, appears to be of some practical value.

In view of the uniqueness of a Hausdorff uo-Lebesgue topology (see Theorem 2.2.4), the following is now clear from Theorem 2.4.2.

Corollary 2.4.5. *Let E be a vector lattice, and suppose that E has order dense ideals F_1 and F_2, each of which admits a Hausdorff o-Lebesgue topology. For $i = 1, 2$, choose a Hausdorff o-Lebesgue topology τ_{F_i} on F_i, and choose a non-empty subset S_i of F_i such that the ideal $I_{S_i} \subseteq F_i$ that is generated by S_i in E is order dense in E. Then $\mathrm{u}_{S_1}\tau_{F_1}$ and $\mathrm{u}_{S_2}\tau_{F_2}$ are both equal to the (necessarily unique) uo-Lebesgue topology topology $\widehat{\tau}_E$ on E.*

Remark 2.4.6. The case in Corollary 2.4.5 where $S_1 = F_1$ and $S_2 = F_2$ is [11, Proposition 3.2].

The case where, for $i = 1, 2$, $S_i = F_i$ and τ_{F_i} is the restriction to F_i of a Hausdorff o-Lebesgue topology τ_i on E, is a part of [44, Theorem 9.6]. Note, however, that our underlying proof in Proposition 2.4.1 that $\mathrm{u}_S\tau_F$ is a uo-Lebesgue topology is direct, whereas in the proof of [44, Theorem 9.6] the identification of a Hausdorff uo-Lebesgue topology as a minimal Hausdorff locally solid topology as in Theorem 2.2.4 is used.

Complementing the preceding going-up results, we cite the following going-down result; see [44, Proposition 5.12].

Proposition 2.4.7 (Taylor). *Suppose that the vector lattice E admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$. Take a vector sublattice F of E. Then F is a regular vector sublattice of E if and only if the restriction $\widehat{\tau}_E|_F$ of $\widehat{\tau}_E$ to F is a (necessarily unique) Hausdorff uo-Lebesgue topology on F.*

A variation on this theme, with a wider range of topologies to use for testing the regularity of a vector sublattice, is the following.

Proposition 2.4.8. *Suppose that the vector lattice E admits a Hausdorff o-Lebesgue topology τ_E. Take a vector sublattice F of E. Then F is a regular vector sublattice of E if and only if the restriction $\tau_E|_F$ of τ_E to F is a Hausdorff o-Lebesgue topology on F.*

Proof. Once one recalls that, by definition, order convergence of a net to 0 in the regular vector sublattice F of E implies order convergence of the net to 0 in E, the proof is a straightforward minor adaptation of that of [44, Proposition 5.12]. □

We now have the following overview theorem concerning Hausdorff o-Lebesgue topologies and Hausdorff uo-Lebesgue topologies on a vector lattice and on its order dense ideals. It is easily established by recalling that a uo-Lebesgue topology is an o-Lebesgue topology, that an ideal is a regular vector sublattice, and by using Theorem 2.4.2, Proposition 2.4.7, and Proposition 2.4.8.

Theorem 2.4.9. *Let E be a vector lattice, and let F be an order dense ideal of E.*
(1) Suppose that E admits a Hausdorff o-Lebesgue topology τ_E. Then the restricted topology $\tau_E|_F$ is a Hausdorff o-Lebesgue topology on E.
(2) Suppose that E admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$. Then the restricted topology $\widehat{\tau}_E|_F$ is a (necessarily unique) Hausdorff uo-Lebesgue topology on F.
(3) The following are equivalent:
  (a) F admits a Hausdorff o-Lebesgue topology;
  (b) F admits a Hausdorff uo-Lebesgue topology;
  (c) E admits a Hausdorff o-Lebesgue topology;
  (d) E admits a Hausdorff uo-Lebesgue topology.

  In that case, the unique uo-Lebesgue topology $\widehat{\tau}_E$ on E equals $\mathrm{u}_S\tau_F$ for every Hausdorff o-Lebesgue topology on F and every subset S of F such that the ideal $I_S \subseteq F$ is order dense in E, and the following are equivalent:
  (i) $x_\alpha \xrightarrow{\widehat{\tau}_E} 0$ in E;
  (ii) $|x_\alpha| \wedge |s| \xrightarrow{\tau_F} 0$ in F for all $s \in S$;
  (iii) $|x_\alpha| \wedge |y| \xrightarrow{\tau_F} 0$ in F for all $y \in F$.

We conclude this section with a short discussion of Banach lattices with order continuous norms. Evidently, the norm topologies on such Banach lattices are Hausdorff o-Lebesgue topologies. As already noted in [44, p. 993], Theorem 2.4.2 allows one to identify the so-called un-topologies (see [21, Section 7] and [32, p. 746]) on such lattices as the Hausdorff uo-Lebesgue topologies that these spaces apparently admit. Consequently, we have the following result. The case where $S = E$ can be found in [44, p. 993].

Proposition 2.4.10. *Let E be a Banach lattice with an order continuous norm and norm topology τ_E. Then E admits a (necessarily unique) uo-Lebesgue topology.*

Choose a subset S of E such that the ideal I_S that is generated by S in E is order dense in E. Then:
(1) $\mathrm{u}_S\tau_E$ is the uo-Lebesgue topology $\widehat{\tau}_E$ of E;
(2) when $(x_\alpha)_{\alpha\in\mathcal{A}}$ is a net in E, then $x_\alpha \xrightarrow{\widehat{\tau}_E} 0$ in E if and only if $\| |x_\alpha| \wedge |s| \| \to 0$ for all $s \in S$; equivalently, if and only if $\| |x_\alpha| \wedge |y| \| \to 0$ for all $y \in E$.

There is an alternative reason why Banach lattices with an order continuous norms admit Hausdorff uo-Lebesgue topologies, and this results in an alternative description of these topologies; see Corollary 2.5.4, below.

Finally, suppose that E is a vector lattice that has order dense ideals F_1 and F_2 that are Banach lattices with order continuous norm topologies τ_{F_1} and τ_{F_2}, respectively. Then it is immediate from Corollary 2.4.5 that E admits a Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$, and that $\mathrm{u}_{F_1}\tau_{F_1}$ and $\mathrm{u}_{F_2}\tau_{F_2}$ are both equal to $\widehat{\tau}_E$. As discussed in Example 2.3.10, this can, using the terminology in [32], be rephrased as stating that F_1 and F_2 induce the same un-topology on E. We have thus retrieved [32, Theorem 2.6].

2.5 uo-Lebesgue topologies generated by absolute weak topologies on order dense ideals

In this section, we shall be concerned with vector lattices having order dense ideals with separating order continuous duals as a source for Hausdorff uo-Lebesgue topologies on the vector lattices themselves.

We start by recapitulating some facts from [6, p. 63–64]. Let E be a vector lattice, and let A be a non-empty subset of the order dual $E^\sim$ of E. For $\varphi \in A$, define the lattice seminorm $\rho_\varphi : E \to [0, \infty)$ by setting $\rho_\varphi(x) := |\varphi|(|x|)$ for $x \in E$. Then the locally convex-solid linear topology on E that is generated by the family $\{\rho_\varphi : \varphi \in A\}$ is called the *absolute weak topology generated by A on E*; it is denoted by $|\sigma|(E,A)$. With I_A denoting the ideal generated by A in $E^\sim$, we have $|\sigma|(E,A) = |\sigma|(E,I_A)$. Using Proposition 2.2.1, one easily concludes that $|\sigma|(E,A)$ is Hausdorff if and only if I_A separates the points of E. Although we shall not use it, let us still remark that it is not difficult to see that a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in E is $|\sigma|(E,A)$-convergent to zero if and only if $\varphi(x_\alpha) \to 0$ uniformly for φ in each fixed order interval of I_A. Thus absolute weak topologies are more natural than is perhaps apparent from their definition.

The following is now clear.

Lemma 2.5.1. *Let E be a vector lattice, and let A be a non-empty subset of $E^\sim_{\mathrm{oc}}$. Let I_A denote the ideal that is generated by A in $E^\sim_{\mathrm{oc}}$. Then $|\sigma|(E,A) = |\sigma|(E,I_A)$ is an o-Lebesgue topology on E that is even locally convex-solid. It is a Hausdorff topology if and only if I_A separates the points of E. When $(x_\alpha)_{\alpha\in\mathcal{A}}$ is a net in E, then $x_\alpha \xrightarrow{|\sigma|(E,A)} 0$ in E if and only if $|\varphi|(|x_\alpha|) \to 0$ for all $\varphi \in A$; equivalently, if and only if $|\varphi|(|x_\alpha|) \to 0$ for all $\varphi \in I_A$.*

Now that Lemma 2.5.1 provides a whole class of vector lattices admitting Hausdorff o-Lebesgue topologies, we can use these as input for Theorem 2.4.2. Taking the convergence statements in Lemma 2.5.1 into account, we arrive at the following.

Theorem 2.5.2. *Let E be a vector lattice. Suppose that E has an order dense ideal F such that $F^\sim_{\mathrm{oc}}$ separates the points of F. Then E admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$.*

Choose a subset A of $F^\sim_{\mathrm{oc}}$ such that the ideal I_A that is generated by A in $F^\sim_{\mathrm{oc}}$ separates the points of F, and choose a subset S of F such that the ideal $I_S \subseteq F$ that is generated by S is order dense in E. Then:

(1) $\mathrm{u}_S|\sigma|(F,A)$ and $\mathrm{u}_F|\sigma|(F,I_A)$ are both equal to $\widehat{\tau}_E$;

(2) *for a net* $(x_\alpha)_{\alpha\in\mathcal{A}}$ *in* E*,* $x_\alpha \xrightarrow{\widehat{\tau}_E} 0$ *in* E *if and only if* $|\varphi|(|x_\alpha| \wedge |s|) \to 0$ *for all* $\varphi \in A$ *and* $s \in S$*; equivalently, if and only if* $|\varphi|(|x_\alpha| \wedge |y|) \to 0$ *for all* $\varphi \in F^{\sim}_{\mathrm{oc}}$ *and* $y \in F$*.*

For the sake of completeness, we recall that a regular vector sublattice of a vector lattice E as in the theorem also has a (necessarily unique) Hausdorff uo-Lebesgue topology, and that this topology is the restriction of $\widehat{\tau}_E$ to the vector sublattice.

Remark 2.5.3. As noted in Remark 2.3.5, one can give an explicit neighbourhood base at zero for the topology $\widehat{\tau}_E$ in Theorem 2.5.2.

For Banach lattices with order continuous norms, the order/norm dual consists of order continuous linear functionals only. Hence we have the following result, which should be compared to Proposition 2.4.10 where the same Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$ is also identified as the un-topology.

Corollary 2.5.4. *A Banach lattice* E *with an order continuous norm admits a (necessarily unique) Hausdorff uo-Lebesgue topology* $\widehat{\tau}_E$*, namely* $\mathrm{u}_E|\sigma|(E, E^*)$*.*

The following gives a necessary condition for convergence in a Hausdorff uo-Lebesgue topology. It is essential in the proof of Theorem 2.7.6, below.

Proposition 2.5.5. *Let* E *be a vector lattice that admits a (necessarily unique) Hausdorff uo-Lebesgue topology* $\widehat{\tau}_E$*, and let* $(x_\alpha)_{\alpha\in\mathcal{A}}$ *be a net in* E *such that* $x_\alpha \xrightarrow{\widehat{\tau}_E} 0$ *in* E*. Take an ideal* F *of* E *such that* $F^{\sim}_{\mathrm{oc}}$ *separates the points of* F*. Then* $|\varphi|(|x_\alpha| \wedge |y|) \to 0$ *for all* $\varphi \in F^{\sim}_{\mathrm{oc}}$ *and* $y \in F$*.*

Proof. Take $\varphi \in F^{\sim}_{\mathrm{oc}}$ and $y \in F$. Since $\widehat{\tau}_E$ is a locally solid topology, we have $|x|_\alpha \wedge |y| \xrightarrow{\widehat{\tau}_E} 0$ in E. It follows from Proposition 2.4.7 that F has a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_F$ and that $|x|_\alpha \wedge |y| \xrightarrow{\widehat{\tau}_F} 0$. Now we apply Theorem 2.5.2 with $E = F$ to see that $|\varphi|((|x_\alpha| \wedge |y|) \wedge |y|) \to 0$. □

We shall now consider the order dual $E^{\sim}$ of a vector lattice E. For $x \in E$, we set

$$\varphi_x(\varphi) := \varphi(x)$$

for $\varphi \in E^{\sim}$. Then $\varphi_x \in (E^{\sim})^{\sim}_{\mathrm{oc}}$, and the map $\varphi : E \to E^{\sim}$ is a lattice homomorphism; see [6, p. 43]. Since $\varphi(E)$ already separates the points of $E^{\sim}$, we see that $(E^{\sim})^{\sim}_{\mathrm{oc}}$ separates the points of $E^{\sim}$.

We can now apply Theorem 2.5.2 twice. In both cases, we replace E with $E^{\sim}$, and we choose $E^{\sim}$ for both F and S. In the first application, we choose $(E^{\sim})^{\sim}_{\mathrm{oc}}$ for A; in the second, we choose $\varphi(E)$. The result is as follows.

Corollary 2.5.6. *Let* E *be a vector lattice. Then the order dual* $E^{\sim}$ *of* E *admits a (necessarily unique) Hausdorff uo-Lebesgue topology* $\widehat{\tau}_{E^{\sim}}$*.*

Moreover:

(1) $\mathrm{u}_{E^{\sim}}|\sigma|(E^{\sim}, (E^{\sim})^{\sim}_{\mathrm{oc}})$ *and* $\mathrm{u}_{E^{\sim}}|\sigma|(E^{\sim}, E)$ *are both equal to* $\widehat{\tau}_{E^{\sim}}$*;*

(2) *when* $(\varphi_\alpha)_{\alpha\in\mathcal{A}}$ *is a net in* $E^{\sim}$*, then* $\varphi_\alpha \xrightarrow{\widehat{\tau}_{E^{\sim}}} 0$ *in* E *if and only if* $|\xi|(|\varphi_\alpha| \wedge |\varphi|) \to 0$ *for all* $\xi \in (E^{\sim})^{\sim}_{\mathrm{oc}}$ *and* $\varphi \in E^{\sim}$*; equivalently, if and only if* $(|\varphi_\alpha| \wedge |\varphi|)(|x|) \to 0$ *for all* $x \in E$ *and* $\varphi \in E^{\sim}$*.*

Remark 2.5.7.

(1) As in the case of Theorem 2.5.2, Remark 2.3.5 shows how to give an explicit neighbourhood base at zero for the topology $\widehat{\tau}_{E^\sim}$ in Corollary 2.5.6.

(2) By Proposition 2.4.7, every regular sublattice of the order dual of a vector lattice also admits a (necessarily unique) Hausdorff Lebesgue topology that can be described in two ways. For an ideal, one of these descriptions is already in [44, Example 5.8].

(3) Corollary 2.5.6 shows that, in particular, the norm/order dual E^* of a Banach lattice admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_{E^*}$, namely the so-called unbounded absolute weak *-topology $\mathrm{u}_{E^*}|\sigma|(E^*, E)$. This was already observed in [44, Lemma 6.6].

2.6 Regular vector sublattices of $\mathrm{L}_0(X, \Sigma, \mu)$ for semi-finite measures

Let (X, Σ, μ) be a measure space, and write $\mathrm{L}_0(X, \Sigma, \mu)$ for the vector lattice of all real-valued Σ-measurable functions on X, with identification of two functions when they agree μ-almost everywhere. In this section we show that, for semi-finite μ, every regular sublattice of $\mathrm{L}_0(X, \Sigma, \mu)$ admits a (necessarily unique) Hausdorff uo-Lebesgue topology, and that a net converges in this topology if and only if it converges in measure on subsets of finite measure; see Theorem 2.6.3, below.

For some regular sublattices of $\mathrm{L}_0(X, \Sigma, \mu)$, it is quite obvious that they admit a Hausdorff uo-Lebesgue topology. Recall that the spaces $\mathrm{L}_p(X, \Sigma, \mu)$ for p such that $1 \leq p < \infty$ have order continuous norms for all measures μ; see [5, Theorem 13.7], for example. Hence their norm topologies are Hausdorff o-Lebesgue topologies, and then their un-topologies are the Hausdorff uo-Lebesgue topologies on these spaces. Alternatively, one can observe that their order continuous duals separate their points, and then also identify the Hausdorff uo-Lebesgue topologies on these spaces as the unbounded absolute weak topologies. In a similar vein, when μ is σ-finite, every ideal of $L_0(X, \Sigma, \mu)$ that can be supplied with a lattice norm has a separating order continuous dual. This result of Lozanovsky's (see [2, Theorem 5.25], for example) then implies that such a normed function space admits a Hausdorff uo-Lebesgue topology.

How about the spaces $\mathrm{L}_p(X, \Sigma, \mu)$ for $0 \leq p < 1$? There is no norm to work with, and it may well be the case that their order continuous duals are even trivial. Indeed, when μ is atomless, then, according to a results of Day's, the order continuous dual of $\mathrm{L}_p(X, \Sigma, \mu)$ is trivial for $0 < p < 1$; see [5, Theorem 13.31], for example. According to [51, Exercise 25.2], the order continuous dual of $\mathrm{L}_0(X, \Sigma, \mu)$ is trivial for every σ-finite measure with the property that, for any measurable subset A such that $0 < \mu(A) < \infty$ and for any α such that $0 < \alpha < \mu(A)$, there exists a measurable subset A' of A such that $\mu(A') = \alpha$. Taking [49, Exercise 10.12 on p. 67] into account, we see that, in particular, the order continuous dual of $\mathrm{L}_0(X, \Sigma, \mu)$ is trivial for all atomless σ-finite measures.

In spite of the failure of the two obvious approaches, it is still possible to show that all spaces $\mathrm{L}_p(X, \Sigma, \mu)$ for $0 \leq p < 1$ admit Hausdorff uo-Lebesgue topologies, provided that the measure is semi-finite. For such μ, this is even true for all regular vector sublattices of

$\mathrm{L}_0(X,\Sigma,\mu)$. This can be seen via the going-up-going-down approach from Section 2.4, and we shall now elaborate on this. We start with a few preliminary remarks.

Recall that a measure space (X,Σ,μ) is said to be *semi-finite* if, for any $A\in\Sigma$ with $\mu(A)=\infty$, there exists a measurable subset A' of A such that $0<\mu(A')<\infty$. Every σ-finite measure is semi-finite. For an arbitrary measure μ and an arbitrary p such that $1\leq p<\infty$, it is easy to see that the ideal $\mathrm{L}_p(X,\Sigma,\mu)$ of $\mathrm{L}_0(X,\Sigma,\mu)$ is order dense in $\mathrm{L}_0(X,\Sigma,\mu)$ if and only if μ is semi-finite. In that case, the ideal that is generated in $\mathrm{L}_0(X,\Sigma,\mu)$ by the subset $S:=\{\,1_A : A\in\Sigma \text{ has finite measure}\,\}$ of $\mathrm{L}_p(X,\Sigma,\mu)$ is obviously also order dense in $\mathrm{L}_0(X,\Sigma,\mu)$.

Let (X,Σ,μ) be a measure space. Take $f\in\mathrm{L}_0(X,\Sigma,\mu)$. Then a net $(f_\alpha)_{\alpha\in\mathcal{A}}$ in $\mathrm{L}_0(X,\Sigma,\mu)$ *converges to f in measure on subsets of finite measure* when, for all $A\in\Sigma$ such that $\mu(A)<\infty$ and for all $\varepsilon>0$, $\mu(\{\,x\in A : |f_\alpha(x)-f(x)|\geq\varepsilon\,\})\to 0$. In that case, we write $f_\alpha\xrightarrow{\mu^*}f$, using as asterisk to distinguish this convergence from the perhaps more usual global convergence in measure.

The following is the core result of this section. We recall that, as already mentioned, the spaces $\mathrm{L}_p(X,\Sigma,\mu)$ have order continuous norms for all measures μ and for all p such that $1\leq p<\infty$, so that their norm topologies are Hausdorff o-Lebesgue topologies.

Theorem 2.6.1. *Let $E=\mathrm{L}_0(X,\Sigma,\mu)$, where μ is a semi-finite measure. Then G admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$.*

Take a net $(f_\alpha)_{\alpha\in\mathcal{A}}$ in E. Then the following are equivalent for every p such that $1\leq p<\infty$:

(1) $f_\alpha\xrightarrow{\widehat{\tau}_E}0$;

(2)

$$\int_X |f_\alpha|^p\wedge 1_A\,d\mu=\|\,|f_\alpha|\wedge|1_A|\,\|_p^p\to 0$$

for every measurable subset A of X with finite measure;

(3)

$$\int_X |f_\alpha|^p\wedge|f|^p\,d\mu=\|\,|f_\alpha|\wedge|f|\,\|_p^p\to 0$$

for every $f\in\mathrm{L}_p(X,\Sigma,\mu)$;

(4) $f_\alpha\xrightarrow{\mu^*}f$.

Proof. We know from the semi-finiteness of μ that, for p such that $1\leq p\leq\infty$, $L_p(X,\Sigma,\mu)$ is an order dense ideal of $\mathrm{L}_0(X,\Sigma,\mu)$. Since $L_p(X,\Sigma,\mu)$ admits a Hausdorff o-Lebesgue topology when $1\leq p<\infty$, Theorem 2.4.2 shows that $\mathrm{L}_0(X,\Sigma,\mu)$ admits a (necessarily unique) Hausdorff uo-Lebesgue topology, and also that the statements in the parts (1), (2), and (3) of the present theorem are equivalent for all such p.

We show that part (3) implies part (4). Take a measurable subset A of X with finite measure, and let $\varepsilon>0$. Since $\varepsilon 1_A\in\mathrm{L}_p(X,\Sigma,\mu)$, we have, by assumption,

$$\int_X |f_\alpha|^p\wedge(\varepsilon^p 1_A)\,\mathrm{d}\mu\to 0.$$

Because

$$\int_X |f_\alpha|^p \wedge (\varepsilon^p 1_A)\,\mathrm{d}\mu \geq \int_{\{x\in A: |f_\alpha(x)|\geq\varepsilon\}} \varepsilon^p\,\mathrm{d}\mu = \varepsilon^p \mu(\{x\in A: |f_\alpha(x)|\geq\varepsilon\})$$

we conclude that $\mu(\{x\in A: |f_\alpha(x)|\geq\varepsilon\}) \to 0$. Hence $f_\alpha \xrightarrow{\mu^*} 0$.

We show that part (4) implies part (2). Take a measurable subset A of X with finite measure, and take $\varepsilon > 0$. Choose a $\delta > 0$ such that $\delta^p\mu(A) < \varepsilon/2$. Then

$$\begin{aligned}
\int_X |f_\alpha|^p \wedge 1_A\,\mathrm{d}\mu &= \int_{\{x\in A: |f_\alpha(x)|^p\geq\delta^p\}} |f_\alpha|^p \wedge 1_A\,\mathrm{d}\mu + \int_{\{x\in A: |f_\alpha(x)|^p<\delta^p\}} |f_\alpha|^p \wedge 1_A\,\mathrm{d}\mu \\
&\leq \int_{\{x\in A: |f_\alpha(x)|^p\geq\delta^p\}} 1\,\mathrm{d}\mu + \int_A \delta^p\,\mathrm{d}\mu \\
&\leq \mu(\{x\in A: |f_\alpha(x)|\geq\delta\}) + \varepsilon/2.
\end{aligned}$$

By our assumption, there exists an $\alpha_0 \in \mathcal{A}$ such that $\mu(\{x\in A: |f_\alpha(x)|\geq\delta\}) < \varepsilon/2$ for all $\alpha \geq \alpha_0$. Then $\int_X |f_\alpha|^p \wedge 1_A\,\mathrm{d}\mu < \varepsilon$ for all $\alpha \geq \alpha_0$. Hence $\int_X |f_\alpha| \wedge 1_A\,\mathrm{d}\mu \to 0$. $\square$

Remark 2.6.2.
(1) We are not aware of a proof of Theorem 2.6.1 in the literature. It is stated in [11, p. 292] that the parts (1) and (4) are equivalent, but there only a reference is given to [24, 65K and 63L]. Since [24, 63L] relies on the solution of the non-trivial exercise [24, Exercise 63M(j)] for which a solution is not provided, we thought it appropriate to give an independent proof in the present paper.
(2) The equivalence of the parts (3) and (4) for finite measures and sequences was also established by different methods in [45, Example 23]. Still earlier, this case was covered in [21, Corollary 4.2], with a proof in the same spirit as our proof.

As an immediate consequence of Proposition 2.4.7 and Theorem 2.6.1, we obtain the following result via our going-up-going-down approach.

Theorem 2.6.3. *Let (X,Σ,μ) be a measure space, where μ is a semi-finite measure. Take a regular vector sublattice E of $\mathrm{L}_0(X,\Sigma,\mu)$. Then E admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$. This topology $\widehat{\tau}_E$ on E is the restriction of the Hausdorff uo-Lebesgue topology on $\mathrm{L}_0(X,\Sigma,\mu)$. A net $(f_\alpha)_{\alpha\in\mathcal{A}}$ in E converges to zero in $\widehat{\tau}_E$ if and only if it satisfies one of the three equivalent criteria in the parts (2), (3), and (4) of Theorem 2.6.1. In particular, it is $\widehat{\tau}_E$-convergent to zero if and only if it converges to zero in measure on subsets of finite measure.*

Remark 2.6.4. Let p be such that $1 \leq p < \infty$. For arbitrary measures, Proposition 2.4.10 and Corollary 2.5.4 both give a description of the convergent nets in the Hausdorff uo-Lebesgue topology on $\mathrm{L}_p(X,\Sigma,\mu)$. The former as the convergent nets in the un-topology, and the latter as the convergent nets in the unbounded absolute weak topology, respectively. When μ is semi-finite, Theorem 2.6.3 gives a third description as the convergence in measure on subsets of finite measure.

Also for $p = \infty$, Theorem 2.6.3 shows that $\mathrm{L}_\infty(X, \Sigma, \mu)$ admits a (necessarily unique) Hausdorff uo-Lebesgue topology whenever μ is semi-finite, and gives a description of its convergent nets. When μ is a localisable measure, two more descriptions are possible. We refer to [25, 211G] for the definition of localisable measures, and note that σ-finite measures are localisable, and that localisable measures are semi-finite. Indeed, for localisable measures, $\mathrm{L}_\infty(X, \Sigma, \mu)$ is the order dual of $L_1(X, \Sigma, \mu)$; see [25, 243G(b)]. Hence Corollary 2.5.6 shows once more that $\mathrm{L}_\infty(X, \Sigma, \mu)$ admits a Hausdorff uo-Lebesgue topology when μ is localisable, and gives a second and third description of its convergent nets.

Remark 2.6.5. Let (X, Σ, μ) be a measure space, where μ is a semi-finite measure.

Let p be such that $0 < p < \infty$. The combination of Theorem 2.6.3 and Remark 2.2.5 shows that the topology of convergence in measure on subsets of finite measure is the *smallest* Hausdorff locally solid linear topology on $\mathrm{L}_p(X, \Sigma, \mu)$.[3] For σ-finite measures, this can already be found in [6, Theorem 7.74], where it is also established that the usual metric topology is then the largest Hausdorff locally solid linear topology.

For $p = \infty$, the combination of Theorem 2.6.3 and Theorem 2.2.4 shows that the topology of convergence in measure on subsets of finite measure is the unique *minimal* Hausdorff locally solid linear topology on $\mathrm{L}_\infty(X, \Sigma, \mu)$. It seems worthwhile to note that, when μ is, in fact, σ-finite, and also non-atomic, [6, Theorem 7.75] shows that there is now no *smallest* Hausdorff locally solid linear topology on $\mathrm{L}_\infty(X, \Sigma, \mu)$.

Remark 2.6.6. Let $(x_n)_{n=1}^\infty$ be a sequence in $\mathrm{L}_0(X, \Sigma, \mu)$, where μ is a semi-finite measure. Suppose that $f_n \longrightarrow 0$ μ-almost everywhere. Then $f_n \xrightarrow{\mu^*} 0$. This is immediate from Egoroff's theorem (see [23, Theorem 2.33], for example), but it can also be obtained (with a long detour) in the context of uo-convergence and uo-Lebesgue topologies. Indeed, by [28, Proposition 3.1], almost everywhere convergence of a sequence in $\mathrm{L}_0(X, \Sigma, \mu)$ is, for arbitrary measures, equivalent to uo-convergence in $\mathrm{L}_0(X, \Sigma, \mu)$. Since, by definition, uo-convergence implies convergence in a uo-Lebesgue topology (when this exists), an appeal to Theorem 2.6.1 also yields the desired result.

2.7 uo-convergent sequences within $\widehat{\tau}_E$-convergent nets

Let E be a vector lattice that admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$. When $(x_\alpha)_{\alpha \in \mathcal{A}}$ is a net in E such that $x_\alpha \xrightarrow{\mathrm{uo}} 0$, then, by definition, $x_\alpha \xrightarrow{\widehat{\tau}_E} 0$. The present section is concerned with results that go in the opposite direction. The main result

[3]For this conclusion, we should note here that the usual metric topology on $\mathrm{L}_p(X, \Sigma, \mu)$ is a complete o-Lebesgue topology for every measure μ and for every p such that $0 < p < \infty$. This is commonly known when $1 \leq p < \infty$. When $0 < p < 1$, then the completeness is asserted in [39, 1.47]. The fact that the metric topology is an o-Lebesgue topology for such p follows from what is stated on [6, p. 211] in the context of σ-finite measures. This implies the result for general measures. Indeed, suppose that $(f_\alpha)_{\alpha \in \mathcal{A}}$ is a net in $\mathrm{L}_p(X, \Sigma, \mu)$ such that $f_\alpha \downarrow 0$. Passing to a tail, we may suppose that the net is bounded above by an $f_{\alpha_0} \in \mathrm{L}_p(X, \Sigma, \mu)$. The support of this f_{α_0} is σ-finite. Using the fact that the elements of $\mathrm{L}_p(X, \Sigma, \mu)$ that vanish off this support form an ideal of $\mathrm{L}_p(X, \Sigma, \mu)$, it is then easily seen from the σ-finite case that the chosen tail of the net converges to zero in the metric topology of $\mathrm{L}_p(X, \Sigma, \mu)$.

is Theorem 2.7.6, below, which lies at the basis of topological considerations in Section 2.8, but we start with a few more elementary results.

For an atomic vector lattice E, the situation is as easy as can be. Recall that, by [6, Theorem 1.78], the atomic vector lattices are precisely the order dense vector sublattices of $\mathbb{R}^X$ for some set X. Combining [13, Proposition 1] and [44, Lemma 7.4], we have the following.

Proposition 2.7.1 (Taylor). *Let E be an atomic vector lattice. Then E admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$, and this topology is locally convex-solid. For a net in E, uo-convergence and $\widehat{\tau}_E$-convergence coincide, so that uo-convergence is topological. When E is an order dense vector sublattice of $\mathbb{R}^X$ for some set X, then a net in E is uo- and $\widehat{\tau}_E$-convergent if and only if it is pointwise convergent.*

For monotone nets, uo-convergence and $\widehat{\tau}_E$ convergence still always coincide, according to the following elementary lemma.

Lemma 2.7.2. *Let E be a vector lattice, and suppose that τ is a Hausdorff locally solid linear topology on E. Let $(x_\alpha)_{\alpha\in\mathcal{A}}$ be a monotone net in E and let $x \in E$. When $x_\alpha \xrightarrow{\tau} x$ in E, then $x_\alpha \xrightarrow{\text{uo}} x$ in E. When $\widehat{\tau}_E$ is a (necessarily unique) Hausdorff uo-Lebesgue topology on E, then $x_\alpha \xrightarrow{\widehat{\tau}_E} x$ in E if and only if $x_\alpha \xrightarrow{\text{uo}} x$ in E.*

Proof. We may suppose that $x_\alpha \downarrow$. Take $y \in E$. Then $|x_\alpha - x| \wedge |y| \downarrow$ and $|x - x_\alpha| \wedge |y| \xrightarrow{\tau} 0$. By [6, Theorem 2.21], we have $|x_\alpha - x| \wedge |y| \downarrow 0$. Hence $x_\alpha \xrightarrow{\text{uo}} x$. The final statement is clear. $\square$

For non-monotone nets in general vector lattices, it is not generally true that $\widehat{\tau}_E$-convergence implies uo-convergence. This can already fail for sequences in Banach lattices with order continuous norms. As an example, consider $E = \mathrm{L}_1([0,1])$. For $n = 1, 2, \ldots$ and $k = 1, 2, \ldots, n$, let f_{nk} be the characteristic function of $[\frac{k-1}{n}, \frac{k}{n}]$, and consider the sequence $f_{11}, f_{21}, f_{22}, f_{31}, f_{32}, f_{33}, f_{41}, \ldots$. It converges to zero in measure, so Theorem 2.6.1 shows that it is $\widehat{\tau}_E$-convergent to zero. On the other hand, [28, Proposition 3.1] shows that uo-convergence of a sequence in $\mathrm{L}_1([0,1])$ is the same as almost everywhere convergence. Hence the sequence is not uo-convergent to zero.

Still, something can be salvaged in the general case. As a motivating example, suppose that (X, Σ, μ) is a measure space. It is well known that a sequence in $\mathrm{L}_0(X, \Sigma, \mu)$ that converges (globally) in measure has a subsequence that converges to the same limit almost everywhere; see [23, Theorem 2.30], for example. When μ is finite, then, in view of Theorem 2.6.1 and [28, Proposition 3.1], this can be restated as saying that a $\widehat{\tau}_E$ convergent sequence in $\mathrm{L}_0(X, \Sigma, \mu)$ has a subsequence that is uo-convergent to the same limit. We shall now extend this formulation of the result to a more general context of nets and Hausdorff uo-Lebesgue topologies on vector lattices; see Theorem 2.7.6, below. In Corollary 2.7.8, below, we shall then obtain a stronger version of the motivating result for convergence in measure and convergence almost everywhere, as a specialisation of the general result.

We start with three preparatory results. The first two appear to have some independent interest.

Proposition 2.7.3. *Let E be a vector lattice with the countable sup property such that $E_{\mathrm{oc}}^{\sim}$ separates the points of E. Take $e \in E^{+}$, and let I_e denote the ideal that is generated in E by e. Then $(I_e)_{\mathrm{oc}}^{\sim}$ separates the points of I_e. In fact, there even exists a $\varphi \in (I_e)_{\mathrm{oc}}^{\sim}$ that is strictly positive on I_e.*

Proof. It is immediate from Theorem 2.2.3 that $(I_e)_{\mathrm{oc}}^{\sim}$ separates the points of I_e. It follows from Proposition 2.2.1 that the ideal of $(I_e)_{\mathrm{oc}}^{\sim}$ that is generated by a strictly positive φ in $(I_e)_{\mathrm{oc}}^{\sim}$ would already separate the points of E. We turn to the existence of such a strictly positive $\varphi \in (I_e)_{\mathrm{oc}}^{\sim}$,

Suppose first that E is Dedekind complete. For $\psi \in \left(E_{\mathrm{oc}}^{\sim}\right)^{+}$, we let

$$N_{\psi} := \{\, x \in E : \psi(|x|) = 0 \,\}$$

denote its null ideal, and we let

$$C_{\psi} := \mathrm{N}_{\psi}^{\mathrm{d}}$$

denote its carrier. Since ψ is order continuous, N_{ψ} is a band in E.

Let B_0 be the band that is generated by the subset $\{\, C_{\psi} : \psi \in \left(E_{\mathrm{oc}}^{\sim}\right)^{+} \,\}$ of E. Then

$$B_0^{\mathrm{d}} = \bigcap_{\psi \in \left(E_{\mathrm{oc}}^{\sim}\right)^{+}} C_{\psi}^{\mathrm{d}} = \bigcap_{\psi \in \left(E_{\mathrm{oc}}^{\sim}\right)^{+}} N_{\psi}^{\mathrm{dd}} = \bigcap_{\psi \in \left(E_{\mathrm{oc}}^{\sim}\right)^{+}} N_{\psi} = \{0\},$$

where in the final step we have used Proposition 2.2.1 and the fact that $E_{\mathrm{oc}}^{\sim}$ separates the points of E. We thus see that $B_0 = E$.

For $\psi \in \left(E_{\mathrm{oc}}^{\sim}\right)^{+}$, let $P_{C_{\psi}}$ denote the band projection from E onto C_{ψ}. When $\psi_1, \psi_2 \in \left(E_{\mathrm{oc}}^{\sim}\right)^{+}$ and $\psi_1 \leq \psi_2$, then $C_{\psi_1} \subseteq C_{\psi_2}$ which, by [7, Theorem 1.46], is equivalent to $P_{C_{\psi_1}} \leq P_{C_{\psi_2}}$. Therefore, the net $\{\, P_{C_{\psi}} : \psi \in \left(E_{\mathrm{oc}}^{\sim}\right)^{+} \,\}$ in $\mathscr{L}_{\mathrm{r}}(E)$ is increasing. Set

$$P := \sup \{\, P_{C_{\psi}} : \psi \in \left(E_{\mathrm{oc}}^{\sim}\right)^{+} \,\},$$

where the supremum is in $\mathscr{L}_{\mathrm{r}}(E)$. From [36, Theorem 30.5] we know that P is a band projection with B_0 as its range space. Since $B_0 = E$, it follows that $P = I$. This implies that $\{\, P_{C_{\psi}} e : \psi \in \left(E_{\mathrm{oc}}^{\sim}\right)^{+} \,\} \uparrow e$, and it follows from the fact that E has the countable sup property that there exists a sequence $(\psi_n)_{n=1}^{\infty}$ in $\left(E_{\mathrm{oc}}^{\sim}\right)^{+}$ such that $P_{C_{\psi_n}} e \uparrow e$ in E.

Consider the ideal I_e of E. Since E is Dedekind complete it is uniformly complete, so that I_e is a Banach lattice when supplied with its order unit norm $\|\cdot\|_e$. Its order dual $I_e^{\sim}$ coincides with its norm dual E^{*} and is then a Banach lattice. Choose strictly positive real numbers $\alpha_1, \alpha_2, \ldots$ such that $\sum_{n=1}^{\infty} \alpha_n \|\psi_n|_{I_e}\| < \infty$, and define $\varphi \in I_e^{\sim}$ by setting

$$\varphi := \sum_{n=1}^{\infty} \alpha_n \psi_n|_{I_e}.$$

Since I_e, being an ideal of E, is a regular vector sublattice of E, each $\psi_n|_{I_e}$ is order continuous. On observing that, being a band, $(I_e)_{\mathrm{oc}}^{\sim}$ is an order closed and, therefore, norm closed subset of the Banach lattice E^*, we see that φ is order continuous on I_e. Obviously, φ is positive.

Suppose that $x \in I_e$ is positive and that $\varphi(x) = 0$. Then $\psi_n(x) = 0$ for all $n \geq 1$. That is, $x \in N_{\psi_n}$ for all $n \geq 1$, so that $P_{C_{\psi_n}} x = 0$ for all $n \geq 1$.

Take $\lambda \geq 0$ such that $0 \leq x \leq \lambda e$. Using [7, Theorem 2.49, Theorem 2.44, and Definition 2.41], we see that there exists an order continuous operator T on E that commutes with all band projections on E and is such that $T(\lambda e) = x$. Since $P_{C_{\Psi_n}}(\lambda e) \uparrow \lambda e$ in E, we have $TP_{C_{\Psi_n}}(\lambda e) \uparrow T(\lambda e) = x$ in E. On the other hand, we know that $TP_{C_{\Psi_n}}(\lambda e) = P_{C_{\Psi_n}} T(\lambda e) = P_{C_{\Psi_n}} x = 0$ for all n. We conclude that $x = 0$. Hence φ is strictly positive on I_e. This completes the proof when E is Dedekind complete.

For general E, we note that its Dedekind completion E^δ also has the countable sup property; see [36, Theorem 32.9]. Furthermore, Theorem 2.2.3 shows that $\left(E^\delta\right)^\sim_{\mathrm{oc}}$ separates the points of E^δ. Let $I_{e,\delta}$ denote the ideal that is generated by e in E^δ. By what has been established above, there exists a $\varphi_\delta \in \left(I_{e,\delta}\right)^\sim_{\mathrm{oc}}$ that is strictly positive on $I_{e,\delta}$. Hence its restriction $\varphi_\delta|_{I_e}$ to I_e is strictly positive on I_e. This restriction is also order continuous on I_e. To see this, suppose that $(x_\alpha)_{\alpha \in \mathcal{A}}$ is a net in I_e and that $x_\alpha \xrightarrow{\mathrm{o}} 0$ in I_e. Since I_e, being an ideal of E, is a regular vector sublattice of E, and since E, being order dense in E^δ, is a regular vector sublattice of E^δ, I_e is a regular vector sublattice of E^δ. Thus $x_\alpha \xrightarrow{\mathrm{o}} 0$ in E^δ. There exists an $\alpha_0 \in \mathcal{A}$ such that the tail $(x_\alpha)_{\alpha \in \mathcal{A}, \alpha \geq \alpha_0}$ is order bounded in I_e. Since this tail is then evidently also order bounded in $I_{e,\delta}$, Theorem 2.2.2 shows that $x_\alpha \xrightarrow{\mathrm{o}} 0$ in $I_{e,\delta}$ for $\alpha \geq \alpha_0$. Then $\varphi_\delta|_{I_e}(x_\alpha) \longrightarrow 0$ for $\alpha \geq \alpha_0$ by the order continuity of φ on $I_{e,\delta}$. Consequently, $\varphi_\delta|_{I_e}(x_\alpha) \longrightarrow 0$, as required. □

Suppose that a vector lattice E has an order unit e and that $(x_\alpha)_{\alpha \in \mathcal{A}}$ is a net in E. According to [28, Corollary 3.5], the fact that $|x_\alpha| \wedge e \xrightarrow{\mathrm{o}} 0$ is already enough to imply that $x_\alpha \xrightarrow{\mathrm{uo}} 0$. This is a special case of the following.

Proposition 2.7.4. *Let E be a vector lattice, let S be a non-empty subset of E, and let B_S denote the band that is generated by S in E. Suppose that $(x_\alpha)_{\alpha \in \mathcal{A}}$ is a net in B_S such that $|x_\alpha| \wedge |y| \xrightarrow{\mathrm{o}} 0$ in E for all $y \in S$. Then $x_\alpha \xrightarrow{\mathrm{uo}} 0$ in E.*

Proof. Suppose first that E is Dedekind complete.

Let I_S denote the ideal that is generated by S in E. Take $y \in I_S$. Then there exist $y_1, \ldots, y_n \in S$ and $r_1, \ldots, r_n \geq 1$ such that $|y| \leq \sum_{i=1}^n r_i |y_i|$. This implies that $|x_\alpha| \wedge |y| \leq \sum_{i=1}^n r_i (|x_\alpha| \wedge |y_i|)$, so that $|x_\alpha| \wedge |y| \xrightarrow{\mathrm{o}} 0$ in E.

Take $y \in B_S$. Then there exists a net $(y_\beta)_{\beta \in \mathcal{B}}$ in I_S such that $0 \leq y_\beta \uparrow |y|$ in E. For $\alpha \in \mathcal{A}$, set $s_\alpha := \sup_{i \geq \alpha} (|x_i| \wedge |y|)$, where the supremum is in E. Clearly, $s_\alpha \downarrow$ in E. We claim that $\inf_\alpha s_\alpha = 0$ in E. To see this, take any $\beta \in \mathcal{B}$. Then

$$
\begin{aligned}
\inf_\alpha s_\alpha &= \inf_\alpha \sup_{i \geq \alpha} (|x_i| \wedge |y|) = \inf_\alpha \sup_{i \geq \alpha} \left(|x_i| \wedge |y_\beta + |y| - y_\beta|\right) \\
&\leq \inf_\alpha \sup_{i \geq \alpha} \left(|x_i| \wedge |y_\beta| + |x_i| \wedge ||y| - y_\beta|\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \inf_{\alpha}\sup_{i\geq\alpha}\big(|x_i|\wedge|y_\beta|+||y|-y_\beta|\big)\\
&=\inf_{\alpha}\sup_{i\geq\alpha}\big(|x_i|\wedge|y_\beta|\big)+||y|-y_\beta|.
\end{aligned}
$$

Since we have already established that $|x_\alpha|\wedge|y_\beta|\xrightarrow{\mathrm{o}}0$ in E, [28, Remark 2.2] shows that $\inf_\alpha\sup_{i\geq\alpha}\big(|x_i|\wedge|y_\beta|\big)=0$. Hence $\inf_\alpha s_\alpha\leq||y|-y_\beta|$ for all $\beta\in\mathcal{B}$. Since $y_\beta\uparrow|y|$ in E, we see that $\inf_\alpha s_\alpha=0$ in E, as claimed. Since obviously $|x_\alpha|\wedge|y|\leq s_\alpha$ for all $\alpha\in\mathcal{A}$, we conclude that $|x_\alpha|\wedge|y|\xrightarrow{\mathrm{o}}0$ in E.

Because $(x_\alpha)_{\alpha\in\mathcal{A}}\subseteq B_S$, it is immediate that $|x_\alpha|\wedge|y|\xrightarrow{\mathrm{o}}0$ in E for all $y\in B_S^{\mathrm{d}}$. Since $E=B_S+B_S^{\mathrm{d}}$, we conclude that $|x_\alpha|\wedge|y|\xrightarrow{\mathrm{o}}0$ in E for all $y\in E$. This completes the proof when E is Dedekind complete.

For a general vector lattice E, we let $B_{S,\delta}$ be the band that is generated by S in E^δ. Then $B_S\subseteq B_{S,\delta}$. By what we have just established, $x_\alpha\xrightarrow{\mathrm{uo}}0$ in E^δ, and then Theorem 2.2.2 shows that $x_\alpha\xrightarrow{\mathrm{uo}}0$ in E. □

Proposition 2.7.5. *Let E be a vector lattice, and let F be an order dense ideal of E. The following are equivalent:*
(1) E has the countable sup property;
(2) F has the countable sup property and F is super order dense in E.

Proof. Suppose that E has the countable sup property. Then F has the countable sup property, as is then true for any ideal of E; see [51, Theorem 17.6]. Since F is order dense in E, the fact that E has the countable sup property then implies that F is even super order dense in E; see [36, Theorem 29.3].

Suppose that F has the countable sup property and that F is super order dense in E. Then E has the countable sup property by [36, Theorem 29.4]. □

All preparations have now been made for the proof of the core result of this section.

Theorem 2.7.6. *Let E be a vector lattice with the countable sup property, and suppose that E has an order dense ideal F such that $F_{\mathrm{oc}}^\sim$ separates the points of F. Let G be a regular vector sublattice of E. Then G admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_G$.*

Let $(x_\alpha)_{\alpha\in\mathcal{A}}$ be a net in G and suppose that $x_\alpha\xrightarrow{\widehat{\tau}_G}x$ for some $x\in G$. Take a sequence $(\alpha_n')_{n=1}^\infty$ of indices in $\mathcal{A}$. Then there exists an increasing sequence $\alpha_1'=\alpha_1\leq\alpha_2\leq\cdots$ of indices in $\mathcal{A}$ such that $\alpha_n\geq\alpha_n'$ for all $n\geq1$ and $x_{\alpha_n}\xrightarrow{\mathrm{uo}}x$ in G. In particular, when a sequence $(x_n)_{n=1}^\infty$ in G and $x\in G$ are such that $x_n\xrightarrow{\widehat{\tau}_G}x$ in G, then there exists a subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that $x_{n_k}\xrightarrow{\mathrm{uo}}x$ in G.

Proof. In view of Proposition 2.4.7 and Theorem 2.2.2, we may (and shall) suppose that $G=E$.

We know from Theorem 2.5.2 that E admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$, The statement on subsequences is clear from the statement on nets, so we need only establish the existence of the α_n for $n\geq1$. We may suppose that $x=0$.

Suppose first that E is Dedekind complete.

For $y \in F^+$, we let $I_y \subseteq F$ denote the ideal that is generated by y in E. By Proposition 2.7.5, F inherits the countable sup property from E. Hence Proposition 2.7.3 applies to the vector lattice F. We then see that $(I_y)^\sim_{\mathrm{oc}}$ separates the points of I_y and that there even exists a strictly positive order continuous linear functional on I_y. We choose and fix such a strictly positive $\varphi_y \in (I_y)^\sim_{\mathrm{oc}}$ for each $y \in F^+$. From Proposition 2.5.5 we know that

$$\varphi_y(|x_\alpha| \wedge y) \to 0 \tag{2.4}$$

for all $y \in F^+$.

Set $\alpha_1 := \alpha'_1$. Since F is super order dense in E by Proposition 2.7.5, we can choose a sequence $\{y^1_m\}_{m=1}^\infty$ in F^+ such that $0 \leq y^1_m \uparrow_m |x_{\alpha_1}|$.

For $n \geq 2$, we shall now inductively construct an indice $\alpha_n \in \mathcal{A}$ and a sequence $\{y^n_m\}_{m=1}^\infty$ in F^+ such that, for all $n \geq 2$:

(a) $\alpha_n \geq \alpha'_n$;
(b) $\alpha_n \geq \alpha_{n-1}$;
(c) $\varphi_{y^i_m}\big(|x_{\alpha_n}| \wedge y^i_m\big) < 2^{-n}$ for $i = 1, 2, \ldots, n-1$ and $m = 1, 2, \ldots, n$;
(d) $0 \leq y^n_m \uparrow_m |x_{\alpha_n}|$ in E.

We start with $n = 2$. The elements y^1_m of F^+ are already known for all $m \geq 1$, and $\varphi_{y^1_m}\big(|x_\alpha| \wedge y^1_m\big) \longrightarrow 0$ for all $m \geq 1$ by equation (2.4). Therefore, we can choose an $\alpha_2 \in \mathcal{A}$ such that $\varphi_{y^1_m}\big(|x_{\alpha_2}| \wedge y^1_m\big) < 2^{-2}$ for $m = 1, 2$. We can arrange that also $\alpha_2 \geq \alpha'_2$ and $\alpha_2 \geq \alpha_1$. Finally, we choose a sequence $(y^2_m)_{m=1}^\infty$ in F such that $0 \leq y^2_m \uparrow_m |x_{\alpha_2}|$. This completes the construction for $n = 2$.

Suppose that $n \geq 2$ and that we have already constructed $\alpha_2, \ldots, \alpha_n \in \mathcal{A}$ and sequences $(y^1_m)_{m=1}^\infty, \ldots, (y^n_m)_{m=1}^\infty$ in F^+ satisfying the four requirements above. The elements y^i_m of F^+ are already known for all $i = 1, 2, \ldots, n$ and $m \geq 1$, and $\varphi_{y^i_m}\big(|x_\alpha| \wedge y^i_m\big) \longrightarrow 0$ for all such i and m by equation (2.4). Therefore, we can choose $\alpha_{n+1} \in \mathcal{A}$ such that $\varphi_{y^i_m}\big(|x_{\alpha_{n+1}}| \wedge y^i_m\big) < 2^{-(n+1)}$ for all $i = 1, 2, \ldots, n$ and $m = 1, 2, \ldots, n+1$. We can arrange that also $\alpha_{n+1} \geq \alpha'_{n+1}$ and $\alpha_{n+1} \geq \alpha_n$. Finally, we choose a sequence $(y^{n+1}_m)_{m=1}^\infty$ in F^+ such that $0 \leq y^{n+1}_m \uparrow_m |x_{\alpha_{n+1}}|$ in E. This completes the construction for $n+1$.

Fix $i, m \geq 1$. Since $0 \leq |x_{\alpha_j}| \wedge y^i_m \leq y^i_m$ for all $j \geq 1$, we can define elements $z^{j,m}_n$ of $I_{y^i_m}$ for $n \geq 1$ by setting $z^{i,m}_n := \bigvee_{j=n}^\infty \big(|x_{\alpha_j}| \wedge y^i_m\big)$. Here the supremum is in the ideal $I_{y^i_m}$ in E (which, although this is immaterial, happens to coincide with the supremum in E). It is clear that $z_n \geq 0$ for $n \geq 1$ and that $z^{i,m}_n \downarrow_n$; we shall show that $z^{i,m}_n \downarrow_n 0$ in $I_{y^i_m}$. For this, we start by noting that the inequality in (c) shows that $\varphi_{y^i_m}\big(|x_{\alpha_j}| \wedge y^i_m\big) < 2^{-j}$ for all $j \geq \max(i+1, m)$. Therefore, for all $n \geq \max(i+1, m)$, we can use the order continuity of $\varphi_{y^i_m}$ on $I_{y^i_m}$ to see that

$$\begin{aligned}
0 &\leq \varphi_{y^i_m}(z^{i,m}_n) \\
&= \varphi_{y^i_m}\left(\bigvee_{j=n}^\infty \big(|x_{\alpha_j}| \wedge y^m_i\big)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \varphi_{y^i_m}\left(\sup_{k\geq n}\left(\bigvee_{j=n}^{k}\left(|x_{\alpha_j}|\wedge y^m_i\right)\right)\right)\\
&= \lim_{\substack{k\to\infty\\ k\geq n}} \varphi_{y^i_m}\left(\bigvee_{j=n}^{k}\left(|x_{\alpha_j}|\wedge y^m_i\right)\right)\\
&\leq \limsup_{\substack{k\to\infty\\ k\geq n}} \varphi_{y^i_m}\left(\sum_{j=n}^{k}\left(|x_{\alpha_j}|\wedge y^m_i\right)\right)\\
&\leq \limsup_{\substack{k\to\infty\\ k\geq n}} \sum_{j=n}^{k} 2^{-j}\\
&\leq 2^{-n+1}.
\end{aligned}
$$

We see from this that for the infimum $\inf_{n\geq 1} z^{i,m}_n$ in $I_{y^i_m}$ (which, although again immaterial, happens to coincide with the infimum in E) we have

$$0 \leq \varphi_{y^i_m}\left(\inf_{n\geq 1} z^{i,m}_n\right) \leq 2^{-n+1}$$

for all $n \geq \max(i+1, m)$. Hence $\varphi_{y^i_m}\left(\inf_{n\geq 1} z^{i,m}_n\right) = 0$. Since $\varphi_{y^i_m}$ is strictly positive on $I_{y^i_m}$, this implies that $\inf_{n\geq 1} z^{i,m}_n = 0$ in $I_{y^i_m}$, as we wanted to show.

The inequalities $0 \leq |x_{\alpha_n}| \wedge y^i_m \leq z^{i,m}_n$ for all $n \geq 1$ now show that $|x_{\alpha_n}| \wedge y^i_m \xrightarrow{\text{o}} 0$ in $I_{y^i_m}$ as $n \to \infty$, and then also $|x_{\alpha_n}| \wedge y^i_m \xrightarrow{\text{o}} 0$ in E as $n \to \infty$.

We have now shown that, for all $i, m \geq 1$, $|x_{\alpha_n}| \wedge y^i_m \xrightarrow{\text{o}} 0$ in E as $n \to \infty$.

Let B denote the band that is generated by $\{\, y^i_m : i, m \geq 1 \,\}$ in E. In view of (d) above, it is clear that the sequence $(x_{\alpha_n})_{n=1}^{\infty}$ is a sequence in B. We can now conclude from Proposition 2.7.4 that $x_{\alpha_n} \xrightarrow{\text{uo}} 0$ in E. This concludes the proof when E is Dedekind complete.

For a general vector lattice E, we pass to the Dedekind completion E^δ of E. By [36, Theorem 32.9], E^δ also has the countable sup property. We let F^δ denote the ideal that is generated in E^δ by F. Then F is obviously majorising in F^δ. Since F is order dense in E and E is order dense in E^δ, F is order dense in E^δ and then also in F^δ. We see from this that, as the notation already suggests, F^δ is the Dedekind completion of F, but what we actually need is that, by Theorem 2.2.3, $\left(F^\delta\right)^\sim_{\text{oc}}$ separates the points of F^δ. The fact that F is order dense in E^δ implies that $F^\delta \supseteq F$ is order dense in E^δ. Hence E^δ also admits a (necessarily) unique Hausdorff o-Lebesgue topology $\widehat{\tau}_{E^\delta}$. Moreover, Proposition 2.4.7 shows that $x_\alpha \xrightarrow{\widehat{\tau}_{E^\delta}} 0$ in E^δ. By what has been established for the Dedekind complete case, there exist indices α_n as specified such that $x_{\alpha_n} \xrightarrow{\text{uo}} 0$ in E^δ. By Theorem 2.2.2, $x_{\alpha_n} \xrightarrow{\text{uo}} 0$ in E. □

For comparison, we include the following; see [6, Theorem 4.19]. We recall that a topology on a vector lattice E is a *Fatou topology* when it is a (not necessarily Hausdorff) locally solid linear topology on E that has a base of neighbourhoods of zero consisting of solid and order closed sets. A Lebesgue topology is a Fatou topology; see [6, Lemma 4.1], for example.

Theorem 2.7.7. *Let E be a vector lattice with the countable sup property that is supplied with a Hausdorff locally solid linear topology τ with the Fatou property. Suppose that $(x_\alpha)_{\alpha\in\mathcal{A}}$ is an order bounded net in E and that $x_\alpha \xrightarrow{\tau} x$ for some $x \in E$. Then there exist indices $\alpha_1 \leq \alpha_2 \leq \cdots$ in $\mathcal{A}$ such that $x_{\alpha_n} \xrightarrow{\mathrm{o}} x$.*

The hypotheses in Theorem 2.7.7 on the topology on the vector lattice are weaker than those in Theorem 2.7.6, and its conclusion is stronger. The big difference is, however, that the net in Theorem 2.7.7 is supposed to be order bounded, whereas there is no such restriction in Theorem 2.7.6.

Theorem 2.7.7 also holds when, instead of requiring E to have the countable sup property, it is required that there exist an at most countably infinite subset of E such that the band that it generates equals the carrier of τ; see [33, Theorem 6.7]. We refer to [6, Definition 4.15] for the definition of the carrier of a (not necessarily Hausdorff) locally solid topology on a vector lattice.

For a fourth result with a similar flavour, in the context of metrisable Hausdorff locally solid linear topologies on vector lattices that need not have the countable sup property, we refer to [44, Corollary 9.9]. This generalises a similar result (see [32, Corollary 3.2]) for Banach lattices.

We have the following consequence of Theorem 2.6.1 and Theorem 2.7.6.

Corollary 2.7.8. *Let (X, Σ, μ) be a measure space where μ is σ-finite. Suppose that $(f_\alpha)_{\alpha\in\mathcal{A}}$ is a net in $\mathrm{L}_0(X, \Lambda, \mu)$ such that $f_\alpha \xrightarrow{\mu^*} 0$. Take a sequence $(\alpha'_n)_{n=1}^\infty$ of indices in $\mathcal{A}$. Then there exists an increasing sequence $\alpha'_1 = \alpha_1 \leq \alpha_2 \leq \cdots$ of indices in $\mathcal{A}$ such that $\alpha_n \geq \alpha'_n$ for all $n \geq 1$ and $f_{\alpha_n} \to 0$ almost everywhere. In particular, when a sequence $(f_n)_{n=1}^\infty$ is a sequence in $\mathrm{L}_0(X, \Lambda, \mu)$ and $f_n \xrightarrow{\mu^*} 0$, then there exists a subsequence $(f_{n_k})_{k=1}^\infty$ of $(f_n)_{n=1}^\infty$ such that $f_{n_k} \to 0$ almost everywhere.*

Proof. It is known that $\mathrm{L}_0(X, \Sigma, \mu)$ has the countable sup property for every σ-finite measure μ; see [6, Theorem 7.73] or [37, Lemma 2.6.1], for example.

The combination of Theorem 2.6.1 and Theorem 2.7.6 yields a sequence of indices α_n as specified such that $f_{\alpha_n} \xrightarrow{\mathrm{uo}} 0$. Since, for a general measure μ, uo-convergence of a sequence in $\mathrm{L}_0(X, \Sigma, \mu)$ is equivalent to its convergence almost everywhere (see [28, Proposition 3.1]), the proof is complete. □

Remark 2.7.9.

(1) In view of its proof, the natural condition on μ in Corollary 2.7.8 is that μ be semi-finite and have the countable sup property. It is known, however, that this is equivalent to requiring that μ be σ-finite; see [33, Proposition 6.5].

(2) For every measure μ, a sequence in $\mathrm{L}_0(X, \Lambda, \mu)$ that converges (globally) in measure has a subsequence that converges almost everywhere to the same limit; see [23, Theorem 2.30], for example. Corollary 2.7.8 does not imply this result for arbitrary measures, but once the measure is known to be σ-finite, it *does* produce the desired subsequence,

and it even does so under the weaker hypothesis of convergence in measure on subsets of finite measure.

(3) Even for finite measures, we are not aware of an existing result that, as in Corollary 2.7.8, is concerned with *nets* that converge in measure.

Remark 2.7.10. The hypothesis in Theorem 2.7.6 that E have the countable sup property cannot be relaxed to merely requiring that F have this property. As a counter-example, consider the situation where F is a Banach lattice with an order continuous norm that is an order dense ideal of a vector lattice E. Then $F_{\mathrm{oc}}^{\sim} = F^*$ separates the points of F, and it is easy to see that F has the countable sup property; the latter also follows from a more general result in [6, Theorem 4.26]. Since the norm topology on F is a Hausdorff o-Lebesgue topology on F, E has a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$. It is the topology of un-convergence with respect to F. It is possible to find such F and E, and a sequence in E that is $\widehat{\tau}_E$ convergent to zero in E, yet has no subsequence that is uo-convergent to zero in E; see [32, Example 9.6].

Theorem 2.7.6 can be specified to various situations. Here is one involving an unbounded absolute weak topology.

Corollary 2.7.11. *Let E be a vector lattice with the countable sup property. Suppose that $E_{\mathrm{oc}}^{\sim}$ separates the points of E. Let $(x_\alpha)_{\alpha\in\mathcal{A}}$ be a net in E, and suppose that $x_\alpha \xrightarrow{\mathrm{u}|\sigma|(E,E_{\mathrm{oc}}^{\sim})} x$ for some $x \in E$. Then there exist indices $\alpha_1 \leq \alpha_2 \cdots$ such that $x_{\alpha_n} \xrightarrow{\mathrm{uo}} x$.*

We conclude this section by extending another classical result from measure theory to the context of Hausdorff uo-Lebesgue topologies and uo-convergence. Suppose that $(X, \Sigma.\mu)$ is a measure space, where μ is σ-finite. Then a sequence in $\mathrm{L}_0(X, \Sigma, \mu)$ is convergent in measure on subsets of finite measure if and only if every subsequence has a further subsequence that converges to the same limit almost everywhere; see [49, Exercise 18.14 on p. 132]. This is a special case of the following.

Theorem 2.7.12. *Let E be a vector lattice with the countable sup property, and suppose that E has an order dense ideal F such that $F_{\mathrm{oc}}^{\sim}$ separates the points of F. Let G be a regular sublattice of E. Then G admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_G$. For a sequence $(x_n)_{n=1}^{\infty} \subseteq G$, $x_n \xrightarrow{\widehat{\tau}_G} 0$ in G if and only if every subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ has a further subsequence $(x_{n_{k_i}})_{i=1}^{\infty}$ such that $x_{n_{k_i}} \xrightarrow{\mathrm{uo}} 0$ in G.*

Proof. In view of Proposition 2.4.7 and Theorem 2.2.2, we may (and shall) suppose that $G = E$.

The forward implication is clear from Theorem 2.7.6. We now show the converse. When it fails that $x_n \xrightarrow{\widehat{\tau}} 0$ in E, then Theorem 2.5.2 shows that there exists an $\varphi \in F_{\mathrm{oc}}^{\sim}$, an $y \in F$, a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ and an $\varepsilon > 0$ such that $|\varphi|(|x_{n_k}| \wedge |y|) > \varepsilon$ for all k. It is then clear from the order continuity of φ that it is impossible to find a further subsequence $(x_{n_{k_i}})_{i=1}^{\infty}$ of $(x_{n_k})_{k=1}^{\infty}$ such that $x_{n_{k_i}} \xrightarrow{\mathrm{uo}} 0$ in E. □

As another special case of Theorem 2.7.12, we see that a sequence in a Banach lattice with an order continuous norm is un-convergent to zero if and only if every subsequence has a further subsequence that is uo-convergent to zero. We have thus retrieved [21, Theorem 4.4].

2.8 Topological aspects of (unbounded) order convergence

In this section, we consider topological issues that are related to (sequential) order convergence and to (sequential) unbounded order convergence, with an emphasis on the latter. Theorem 2.7.6 will be seen to be an important tool.

Let E be a vector lattice, and let $A \subseteq E$. We define the *o-adherence of A* as the set of all order limits of nets in A, and denote it by $a_{\mathrm{o}}(A)$. The *σ-o-adherence of A* is the set of all order limits of sequences in A; it is denoted by $a_{\sigma\mathrm{o}}(A)$. [4] The *uo-adherence* $a_{\mathrm{uo}}(A)$ and the *σ-uo-adherence* $a_{\sigma\mathrm{uo}}(A)$ of A are similarly defined. The subset A is *o-closed* when $a_{\mathrm{o}}(A) = A$.[5] The collection of all o-closed subsets of E is easily seen to be the collection of closed sets of a topology that is called the *o-topology on E*. The closure of a subset A in the o-topology is denoted by $\overline{A}^{\mathrm{o}}$.[6] We have $a_{\mathrm{o}}(A) \subseteq \overline{A}^{\mathrm{o}}$, with equality if and only if $a_{\mathrm{o}}(A)$ is o-closed. Likewise, there are σ-o-closed subsets and a σ-o-topology, uo-closed subsets and a uo-topology, and σ-uo-closed subsets and a σ-uo-topology, with similar notations and statements about inclusions and equalities of sets. Evidently, a uo-closed subset is o-closed, and a σ-uo-closed subset is σ-o-closed.

Order convergence in a vector lattice E is hardly ever topological; according to [13, Theorem 1] or [43, Theorem 18.36], this is the case if and only if E is finite-dimensional. It is not even true that the set map $A \mapsto a_{\mathrm{o}}(A)$ is always idempotent, i.e., that the o-adherence of a set is always o-closed. It is known, for example, that in every σ-order complete Banach lattice that does *not* have an order continuous norm, there even exists a vector sublattice such that its o-adherence is not order closed; see [26, Theorem 2.7].

We know from Proposition 2.7.1 that uo-convergence in atomic vector lattices is topological. According to [43, Theorem 6.54], atomic vector lattices are, in fact, the only ones for which this is the case.

It appears to be open whether the uo-adherence of a subset of a vector lattice is always uo-closed. In [26, Problem 2.5], it is even asked whether the uo-adherence of a vector sublattice is always o-closed, which is asking for a weaker conclusion for a much more restrictive class of subsets.

Even though the topological aspects of uo-convergence are still not well understood in general, there is a class of vector lattices where we have a reasonably complete picture. In order to formulate this, we need some more notation. For a set X with a topology τ and a subset $A \subseteq X$ of X, we let $a_{\sigma\tau}(A)$ denote the σ-τ-adherence of A, i.e., $a_{\sigma\tau}(A)$ is the set consisting of all τ-limits of sequences in A. When $a_{\sigma\tau}(A) = A$, A is said to be σ-τ-closed.

[4] In [36, p. 82], our σ-o-adherence is called the pseudo order closure. In [26], our o-adherence of a subset A is called the order closure of A, and it is denoted by $\overline{A}^{\mathrm{o}}$. These two terminologies, as well as the notation $\overline{A}^{\mathrm{o}}$, could suggest that taking the (pseudo) order closure is a (sequential) closure operation for a topology. Since this is hardly ever the case, we prefer a terminology and notation that avoid this possible confusion. It is inspired by [8, Definition 1.3.1].

[5] This definition is consistent with that in [26].

[6] There is no notation for the closure operation in the o-topology in [26].

The σ-τ-closed subsets of X are the closed subsets of a topology on X that is called the σ-τ-topology on X. We let $\overline{A}^{\tau}$ and $\overline{A}^{\sigma\text{-}\tau}$ denote the τ-closure and the σ-τ-closure of a subset A of X, respectively. Then $a_{\sigma\tau}(A) \subseteq \overline{A}^{\sigma\text{-}\tau}$, with equality if and only if $a_{\sigma\tau}(A)$ is σ-τ-closed.

Theorem 2.8.1. *Let E be a vector lattice with the countable sup property, and suppose that E has an order dense ideal F such that $F_{\mathrm{oc}}^{\sim}$ separates the points of F. Let G be a regular vector sublattice of E. Then G admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_G$. For a subset A of G, the following seven subsets of G are all equal:*

(1) $a_{\sigma\widehat{\tau}_G}(A)$ and $\overline{A}^{\sigma\text{-}\widehat{\tau}_G}$;

(2) $a_{\sigma\mathrm{uo}}(A)$ and $\overline{A}^{\sigma\text{-uo}}$;

(3) $a_{\mathrm{uo}}(A)$ and $\overline{A}^{\mathrm{uo}}$;

(4) $\overline{A}^{\widehat{\tau}_G}$.

In particular, the σ-$\widehat{\tau}_G$-topology, the σ-uo-topology, and the uo-topology on G all coincide with $\widehat{\tau}_G$.

In Theorem 2.8.1, the topological closures and (σ-)adherences are to be taken with respect to the topologies and convergences in G.

Proof. The existence and uniqueness of $\widehat{\tau}_G$ are clear from Theorem 2.7.6. Using Theorem 2.7.6 for the first inclusions, we have, for an arbitrary subset A of G,

$$\overline{A}^{\widehat{\tau}_G} \subseteq a_{\sigma\mathrm{uo}}(A) \subseteq a_{\mathrm{uo}}(A) \subseteq \overline{A}^{\widehat{\tau}_G}$$

and

$$a_{\sigma\widehat{\tau}_G}(A) \subseteq a_{\sigma\mathrm{uo}}(A) \subseteq a_{\sigma\widehat{\tau}_G}(A).$$

This gives equality of $a_{\sigma\widehat{\tau}_G}(A)$, $a_{\sigma\mathrm{uo}}(A)$, $a_{\mathrm{uo}}(A)$, and $\overline{A}^{\widehat{\tau}_G}$. Since the set map $A \mapsto \overline{A}^{\widehat{\tau}_G}$ is idempotent, so is $A \mapsto a_{\sigma\widehat{\tau}_G}(A)$. Hence $a_{\sigma\widehat{\tau}_G}(A)$ is σ-$\widehat{\tau}_G$-closed, so that it coincides with the σ-τ-closure $\overline{A}^{\sigma\text{-}\widehat{\tau}_G}$ of A. A similar argument works for $\overline{A}^{\sigma\text{-uo}}$ and $\overline{A}^{\mathrm{uo}}$. □

Remark 2.8.2. Taking $G = E$ in Theorem 2.8.1, the equality of $\overline{A}^{\widehat{\tau}_G}$ and $a_{\sigma\mathrm{uo}}(A)$ implies that, for a σ-finite measure μ, a subset of $\mathrm{L}_0(X, \Sigma, \mu)$ is closed in the topology of convergence in measure on subsets of finite measure if and only if it contains the almost every limits of sequences in it. This is [25, 245L(b)].

In the context of Theorem 2.8.1, it is also possible to give a necessary and sufficient condition for sequential uo-convergence to be topological; see Corollary 2.8.5, below. The proof of the following preparatory lemma is an abstraction of the argument in [38].

Lemma 2.8.3. *Let E be a vector lattice that is supplied with a topology τ. Suppose that τ has the following properties:*

(1) for every sequence $(x_n)_{n=1}^{\infty}$ in E and for every $x \in E$, the fact that $x_n \xrightarrow{\tau} x$ implies that there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $x_{n_k} \xrightarrow{\mathrm{uo}} x$ as $k \to \infty$.

(2) there exists a sequence $(x_n)_{n=1}^{\infty}$ in E and an $x \in E$ such that $x_n \xrightarrow{\tau} x$ but $x_n \stackrel{\mathrm{uo}}{\nrightarrow} x$;

Then there does not exist a topology τ' on E such that, for every sequence $(x_n)_{n=1}^\infty$ in E and for every $x \in E$, $x_n \xrightarrow{\mathrm{uo}} x$ if and only if $x_n \xrightarrow{\tau'} x$.

Proof. Suppose that there were such a topology τ'. Take a sequence $(x_n)_{n=1}^\infty$ in E and an $x \in E$ such that $x_n \xrightarrow{\tau} x$ but $x_n \overset{\mathrm{uo}}{\not\rightarrow} x$. Then also $x_n \overset{\tau'}{\not\rightarrow} x$, so that there exists a τ'-neighbourhood V of x and a subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that $x_{n_k} \notin V$ for all $k \geq 1$. Since also $x_{n_k} \xrightarrow{\tau} x$ as $k \to \infty$, there exists a subsequence $(x_{n_{k_i}})_{i=1}^\infty$ of $(x_{n_k})_{k=1}^\infty$ such that $x_{n_{k_i}} \xrightarrow{\mathrm{uo}} x$ as $i \to \infty$. Hence also $x_{n_{k_i}} \xrightarrow{\tau'} x$ as $i \to \infty$. But this is impossible, since the entire sequence $(x_{n_{k_i}})_{i=1}^\infty$ stays outside V. $\square$

The following is a direct consequence of Lemma 2.8.3. The topology τ in it could be a uo-Lebesgue topology, but for the result to hold it need not even be a linear topology, nor need the topology τ' be.

Proposition 2.8.4. *Let E be a vector lattice that is supplied with a topology τ. Suppose that τ has the following properties:*

(1) for every sequence $(x_n)_{n=1}^\infty$ in E and for every $x \in E$, the fact that $x_n \xrightarrow{\mathrm{uo}} x$ implies that $x_n \xrightarrow{\tau} x$;

(2) for every sequence $(x_n)_{n=1}^\infty$ in E and for every $x \in E$, the fact that $x_n \xrightarrow{\tau} x$ implies that there exists a subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that $x_{n_k} \xrightarrow{\mathrm{uo}} x$ as $k \to \infty$.

Then the following are equivalent;

(1) there exists a topology τ' on E such that, for every sequence $(x_n)_{n=1}^\infty$ in E and for every $x \in E$, $x_n \xrightarrow{\mathrm{uo}} x$ if and only if $x_n \xrightarrow{\tau'} x$;

(2) for every sequence $(x_n)_{n=1}^\infty$ in E and for every $x \in E$, the fact that $x_n \xrightarrow{\tau} x$ implies that $x_n \xrightarrow{\mathrm{uo}} x$.

In that case, one can take τ for τ'.

In the appropriate context, the combination of Theorem 2.7.6 and Proposition 2.8.4 yields the following necessary and sufficient condition for sequential uo-convergence to be topological. Note that there are no assumptions at all on the topology τ in its first part.

Corollary 2.8.5. *Let E be a vector lattice with the countable sup property, and suppose that E has an order dense ideal F such that $F_{\mathrm{oc}}^\sim$ separates the points of F. Let G be a regular vector sublattice of E. Then G admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_G$, and the following are equivalent:*

(1) there exists a topology τ on G such that, for every sequence $(x_n)_{n=1}^\infty$ in G and for every $x \in G$, $x_n \xrightarrow{\mathrm{uo}} x$ in G if and only if $x_n \xrightarrow{\tau} x$;

(2) for every sequence $(x_n)_{n=1}^\infty$ in G and for every $x \in G$, the fact that $x_n \xrightarrow{\widehat{\tau}_G} x$ in G implies that $x_n \xrightarrow{\mathrm{uo}} x$ in G.

In that case, one can take $\widehat{\tau}_G$ for τ.

The proof of the following result closely follows the one in [38], where it is shown that sequential almost everywhere convergence in $L_\infty([0,1])$ is not topological.

Corollary 2.8.6. *Let (X,Σ,μ) be a measure space, where μ is σ-finite. Suppose that there exists an $A\in\Sigma$ with the property that, for every $k\geq 1$, there exist finitely many mutually disjoint $A_{k,1},\ldots,A_{k,N_k}\in\Sigma$ such that $0<\mu(A_{k,1}),\ldots,\mu(A_{k,N_k})<1/k$ and $A=\bigcup_{l=1}^{N_k}A_{k,l}$.*

Take a regular vector sublattice G of $\mathrm{L}_0(X,\Sigma,\mu)$ that contains the characteristic functions $1_{A_{k,l}}$ of all sets $A_{k,l}$ for $k=1,2,\ldots$ and $l=1,\ldots,N_k$. Then there does not exist a topology τ on G such that, for every sequence $(x_n)_{n=1}^{\infty}$ in G and for every $x\in G$, $x_n\xrightarrow{\mathrm{uo}}x$ in G if and only if $x_n\xrightarrow{\tau}x$.

Proof. We are in the situation of Corollary 2.8.5, where $\widehat{\tau}_G$-convergence is convergence in measure on subsets of finite measure by Theorem 2.6.1, and sequential uo-convergence is almost everywhere convergence by [28, Proposition 3.1]. Consider the following sequence in G:

$$A_{1,1},\ldots,A_{1,N_1},A_{2,1},\ldots,A_{2,N_2},A_{3,1},\ldots,A_{3,N_3},\ldots.$$

This sequence clearly converges to zero on subsets of finite measure, but it converges nowhere to zero on the subset A of strictly positive measure. Hence the property in part (2) of Corollary 2.8.5 does not hold, and then neither does the property in its part (1). $\square$

Remark 2.8.7. Corollary 2.8.6 provides us with a large class of examples of vector lattices where sequential uo-convergence is not topological—so that uo-convergence is certainly not topological—but where, according to Theorem 2.8.1, the set maps $A\mapsto a_{\sigma\mathrm{uo}}(A)$ and $A\mapsto a_{\mathrm{uo}}(A)$ are both still idempotent, so that $a_{\sigma\mathrm{uo}}(A)$ is σ-uo-closed and $a_{\mathrm{uo}}(A)$ is uo-closed for every subset A of G. For all p such that $0\leq p\leq\infty$, the space $L_p([0,1])$ is such an example.

We conclude with a strengthened version of [26, Theorem 2.2]. The improvement lies in the removal of the hypothesis that E be Banach lattice, and by adding eight more equal, but not obviously equal, sets to the three equal sets in the original result.

Theorem 2.8.8. *Let E be a vector lattice with the countable sup property, and suppose that $E_{\mathrm{oc}}^{\sim}$ separates the points of E. Then E admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$. Take an ideal I of $E_{\mathrm{oc}}^{\sim}$ that separates the points of E, and take a vector sublattice F of E. Then the following eleven vector sublattices of E are all equal:*

(1) $a_{\sigma\widehat{\tau}_E}(F)$ and $\overline{F}^{\sigma\text{-}\widehat{\tau}_E}$;
(2) $a_{\sigma\mathrm{uo}}(F)$ and $\overline{F}^{\sigma\text{-uo}}$;
(3) $a_{\mathrm{uo}}(F)$ and $\overline{F}^{\mathrm{uo}}$;
(4) $\overline{F}^{\widehat{\tau}_E}$, $\overline{F}^{|\sigma|(E,I)}$, and $\overline{F}^{\sigma(E,I)}$;
(5) $(a_{\mathrm{o}}(a_{\mathrm{o}}(F)))$ and $\overline{F}^{\mathrm{o}}$.

The equality of $a_{\mathrm{uo}}(F)$, $a_{\mathrm{o}}(a_{\mathrm{o}}(F))$, and $\overline{F}^{\sigma(E,I)}$ can already be found in [26, Theorem 2.2], where it also noted that these sets coincide with the smallest order closed vector sublattice of E containing F.

Proof. The equality of the first seven subsets is clear from Theorem 2.8.1. Since we know from Theorem 2.5.2 that $\widehat{\tau}_E=\mathrm{u}_E|\sigma|(E,I)$, it follows from [44, Proposition 2.12] that $\overline{F}^{\widehat{\tau}_E}=$

$\overline{F}^{|\sigma|(E,I)}$. Furthermore, from Kaplan's theorem (see [6, Theorem 2.33], for example) we know that E, when supplied with the Hausdorff locally convex $|\sigma|(E,I)$-topology, has the same topological dual as when it is supplied with the Hausdorff locally convex $\sigma(E,I)$-topology. By the convexity of F, we have $\overline{F}^{|\sigma|(E,I)} = \overline{F}^{\sigma(E,I)}$. This argument was already used in [26, Proof of Lemma 2.1].

We turn to the two sets in part (4). It was established in [26, Lemma 2.1] that $a_{\mathrm{uo}}(F) \subseteq a_{\mathrm{o}}(a_{\mathrm{o}}(F))$; this is, in fact, valid for vector sublattices of general vector lattices. It was also observed there that, obviously, the fact that $I \subseteq E_{\mathrm{oc}}^{\sim}$ implies that $\overline{F}^{\sigma(E,I)}$ is o-closed. Using also that we already know that $a_{\mathrm{uo}}(F) = \overline{F}^{\mathrm{uo}}$, we therefore have the following chain of inclusions:

$$\overline{F}^{\mathrm{uo}} = a_{\mathrm{uo}}(F) \subseteq a_{\mathrm{o}}(a_{\mathrm{o}}(F)) \subseteq \overline{F}^{\mathrm{o}} \subseteq \overline{F}^{\sigma(E,I)}.$$

Since we also already know that $\overline{F}^{\mathrm{uo}} = \overline{F}^{\sigma(E,I)}$, the proof is complete. $\square$

Chapter 3

Convergence structures and locally solid topologies on vector lattices of operators

Abstract

For vector lattices E and F, where F is Dedekind complete and supplied with a locally solid topology, we introduce the corresponding locally solid absolute strong operator topology on the order bounded operators $\mathscr{L}_{\mathrm{ob}}(E,F)$ from E into F. Using this, it follows that $\mathscr{L}_{\mathrm{ob}}(E,F)$ admits a Hausdorff uo-Lebesgue topology whenever F does.

For each of order convergence, unbounded order convergence, and—when applicable—convergence in the Hausdorff uo-Lebesgue topology, there are both a uniform and a strong convergence structure on $\mathscr{L}_{\mathrm{ob}}(E,F)$. Of the six conceivable inclusions within these three pairs, only one is generally valid. On the orthomorphisms of a Dedekind complete vector lattice, however, five are generally valid, and the sixth is valid for order bounded nets. The latter condition is redundant in the case of sequences of orthomorphisms on a Banach lattice, as a consequence of a uniform order boundedness principle for orthomorphisms that we establish.

We also show that, in contrast to general order bounded operators, the orthomorphisms preserve not only order convergence of nets, but unbounded order convergence and—when applicable—convergence in the Hausdorff uo-Lebesgue topology as well.

3.1 Introduction and overview

Let X be a non-empty set. A *convergence structure on X* is a non-empty collection $\mathscr{C}$ of pairs $((x_\alpha)_{\alpha\in\mathcal{A}}, x)$, where $(x_\alpha)_{\alpha\in\mathcal{A}}$ is a net in X and $x \in X$, such that:

(1) when $((x_\alpha)_{\alpha\in\mathcal{A}}, x) \in \mathscr{C}$, then also $((y_\beta)_{\beta\in\mathcal{B}}, x) \in \mathscr{C}$ for every subnet $(y_\beta)_{\beta\in\mathcal{B}}$ of $(x_\alpha)_{\alpha\in\mathcal{A}}$;

(2) when a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in X is constant with value x, then $((x_\alpha)_{\alpha\in\mathcal{A}}, x) \in \mathscr{C}$.

One can easily vary on this definition. For example, one can allow only sequences. There does not appear to be a consensus in the literature about the notion of a convergence structure; [8] uses filters, for example. Ours is sufficient for our merely descriptive purposes, and close in spirit to what may be the first occurrence of such a definition in [22] for sequences. Although we shall not pursue this in the present paper, let us still mention that the inclusion of the subnet criterion in the definition makes it possible to introduce an associated topology on X in a natural way. Indeed, define a subset of S of X to be *$\mathscr{C}$-closed* when $x \in S$ for all pairs $((x_\alpha)_{\alpha\in\mathcal{A}}, x) \in \mathscr{C}$ such that $(x_\alpha)_{\alpha\in\mathcal{A}} \subseteq S$. Then the collection of the complements of the $\mathscr{C}$-closed subsets of X is a topology on X.

The convergent nets in a topological space, together with their limits, are the archetypical example of a convergence structure. In the context of vector lattices, there are other ones that are rarely of a topological nature. For example, the order convergence nets with their order limits form a convergence structure, and likewise there is a convergence structure for unbounded order convergence. Taken together with the (topological) structure for convergence in a Hausdorff uo-Lebesgue topology, when this exists, there are three natural and related convergence structures on a vector lattice to consider.

Suppose that E and F are vector lattices, where F is Dedekind complete. The above then yields three convergence structures on the vector lattice $\mathscr{L}_{\mathrm{ob}}(E,F)$ of order bounded operators from E into F, but there are also three others that are derived from those in F. For example, one can consider all pairs $((T_\alpha)_{\alpha\in\mathcal{A}}, T)$, where $(T_\alpha)_{\alpha\in\mathcal{A}}$ is a net in $\mathscr{L}_{\mathrm{ob}}(E,F)$ and $T \in \mathscr{L}_{\mathrm{ob}}(E)$, such that $(T_\alpha x)_{\alpha\in\mathcal{A}}$ is order convergent to Tx in F for all $x \in E$. These pairs also form a convergence structure on $\mathscr{L}_{\mathrm{ob}}(E,F)$. Likewise, the pointwise unbounded order convergence in F and—when applicable—the pointwise convergence in a Hausdorff uo-Lebesgue topology on F yield convergence structures on $\mathscr{L}_{\mathrm{ob}}(E,F)$. Motivated by the terminology for operators between Banach spaces, we shall speak of *uniform* and *strong* convergence structures on $\mathscr{L}_{\mathrm{ob}}(E)$—with the obvious meanings.

The present paper is primarily concerned with the possible inclusions between the uniform and strong convergence structure for each of order convergence, unbounded order convergence, and—when applicable—convergence in a Hausdorff uo-Lebesgue topology. We consider these inclusions for $\mathscr{L}_{\mathrm{ob}}(E,F)$, but also for the orthomorphisms $\mathrm{Orth}(E)$ on a Dedekind complete vector lattice. This special interest in $\mathrm{Orth}(E)$ stems from representation theory. When a group acts as order automorphisms on a Dedekind complete vector lattice E, then the Boolean lattice of all invariant bands in E can be retrieved from the commutant of the group action in $\mathrm{Orth}(E)$. This commutant, therefore, plays the role of the von Neumann algebra which is the commutant of a unitary action of a group on a Hilbert space. It has been known long since that more than one topology on a von Neumann algebras is needed to understand it and its role in representation theory on Hilbert spaces, and the same holds true for the convergence structures as related to these commutants in an ordered context. Using these convergence structures, it is, for example, possible to obtain ordered versions of von Neumann's bicommutant theorem. We shall report separately on this. Apart from its intrinsic interest, the material on $\mathrm{Orth}(E)$ in the present paper is an ingredient for these next steps.

This paper is organised as follows.

Section 3.2 contains the basic notations, definitions, conventions, and references to earlier results.

In Section 3.3, we show how, given a vector lattice E, a Dedekind complete vector lattice F, and a (not necessarily Hausdorff) locally solid linear topology τ_F on F, a locally solid linear topology can be introduced on $\mathscr{L}_{\mathrm{ob}}(E,F)$ that deserves to be called the absolute strong operator topology that is generated by τ_F. This is a preparation for Section 3.4, where we show that regular vector sublattices of $\mathscr{L}_{\mathrm{ob}}(E,F)$ admit a Hausdorff uo-Lebesgue topology when F admits one.

For each of order convergence, unbounded order convergence, and—when applicable—convergence in a Hausdorff uo-Lebesgue topology, there are two conceivable implications between uniform and strong convergence of a net of order bounded operators. In Section 3.5, we show that only one of these six is generally valid. Section 3.9 will make it clear that the five failures are, perhaps, not as 'only to be expected' as one might think at first sight.

In Section 3.6, we review some material concerning orthomorphism and establish a few auxiliary result for use in the present paper and in future ones. It is shown here that a Dedekind complete vector lattice and its orthomorphisms have the same universal completion.

Section 3.7 briefly digresses from the main line of the paper. It is shown that orthomorphisms preserve not only the order convergence of nets, but also the unbounded order convergence and—when applicable—the convergence in a Hausdorff uo-Lebesgue topology. None of this is true for arbitrary order bounded operators.

In Section 3.8, we return to the main line, and we specialise the results in Sections 3.3 and 3.4 to the orthomorphisms. When restricted to $\mathrm{Orth}(E)$, the absolute strong operator topologies from Section 3.3 are simply strong operator topologies.

Section 3.9 on orthomorphisms is the companion of Section 3.5, but the results are quite in contrast. For each of order convergence, unbounded order convergence, and—when applicable—convergence in a Hausdorff uo-Lebesgue topology, both implications between uniform and strong convergence of a net of orthomorphisms *are* valid, with an order boundedness condition on the net being necessary only for order convergence. For sequences of orthomorphisms on Banach lattices, this order boundedness condition is redundant as a consequence of a uniform order boundedness principle for orthomorphisms that is also established in this section.

3.2 Preliminaries

In this section, we collect a number of definitions, notations, conventions and earlier results.

All vector spaces are over the real numbers; all vector lattices are supposed to be Archimedean. We write E^+ for the positive cone of a vector lattice E. For a non-empty subset S of E, we let I_S and B_S denote the ideal of E and the band in E, respectively, that are generated by S; we write $S^\vee$ for $\{s_1 \vee \cdots \vee s_n : s_1, \ldots, s_n \in S\}$.

Let E be a vector lattice, and let $x \in E$. We say that a net $(x_\alpha)_{\alpha \in \mathcal{A}}$ in E is *order convergent to* $x \in E$ (denoted by $x_\alpha \xrightarrow{\text{o}} x$) when there exists a net $(y_\beta)_{\beta \in \mathcal{B}}$ in E such that $y_\beta \downarrow 0$ and with the property that, for every $\beta_0 \in \mathcal{B}$, there exists an $\alpha_0 \in \mathcal{A}$ such that $|x - x_\alpha| \leq y_{\beta_0}$ whenever α in $\mathcal{A}$ is such that $\alpha \geq \alpha_0$. We explicitly include this definition to make clear that the index sets $\mathcal{A}$ and $\mathcal{B}$ need not be equal.

Let $(x_\alpha)_{\alpha \in \mathcal{A}}$ be a net in a vector lattice E, and let $x \in E$. We say that (x_α) is *unbounded order convergent to* x *in* E (denoted by $x_\alpha \xrightarrow{\text{uo}} x$) when $|x_\alpha - x| \wedge y \xrightarrow{\text{o}} 0$ in E for all $y \in E^+$. Order convergence implies unbounded order convergence to the same limit. For order bounded nets, the two notions coincide.

Let E and F be vector lattices. The order bounded operators from E into F will be denoted by $\mathscr{L}_{\text{ob}}(E, F)$. We write $E^\sim$ for $\mathscr{L}_{\text{ob}}(E, \mathbb{R})$. A linear operator $T : E \to F$ between two vector lattices E and F is *order continuous* when, for every net $(x_\alpha)_{\alpha \in \mathcal{A}}$ in E, the fact that $x_\alpha \xrightarrow{\text{o}} 0$ in E implies that $Tx_\alpha \xrightarrow{\text{o}} 0$ in F. An order continuous linear operator between two vector lattices is automatically order bounded; see [7, Lemma 1.54], for example. The order continuous linear operators from E into F will be denoted by $\mathscr{L}_{\text{oc}}(E, F)$. We write $E^\sim_{\text{oc}}$ for $\mathscr{L}_{\text{oc}}(E, \mathbb{R})$.

Let F be a vector sublattice of a vector lattice E. Then F is a *regular vector sublattice of* E when the inclusion map from F into E is order continuous. Ideals are regular vector sublattices. For a net in a regular vector sublattice F of E, its uo-convergence in F and in E are equivalent; see [28, Theorem 3.2].

When E is a vector space, a *linear topology on* E is a (not necessarily Hausdorff) topology that provides E with the structure of a topological vector space. When E is a vector lattice, a *locally solid linear topology on* E is a linear topology on E such that there exists a base of (not necessarily open) neighbourhoods of 0 that are solid subsets of E. For the general theory of locally solid linear topologies on vector lattices we refer to [6]. When E is a vector lattice, a *locally solid additive topology on* E is a topology that provides the additive group E with the structure of a (not necessarily Hausdorff) topological group, such that there exists a base of (not necessarily open) neighbourhoods of 0 that are solid subsets of E.

A topology τ on a vector lattice E is an *o-Lebesgue topology* when it is a (not necessarily Hausdorff) locally solid linear topology on E such that, for a net $(x_\alpha)_{\alpha \in \mathcal{A}}$ in E, the fact that $x_\alpha \xrightarrow{\text{o}} 0$ in E implies that $x_\alpha \xrightarrow{\tau} 0$. A vector lattice need not admit a Hausdorff o-Lebesgue topology. A topology τ on a vector lattice E is a *uo-Lebesgue topology* when it is a (not necessarily Hausdorff) locally solid linear topology on E such that, for a net $(x_\alpha)_{\alpha \in \mathcal{A}}$ in E, the fact that $x_\alpha \xrightarrow{\text{uo}} 0$ in E implies that $x_\alpha \xrightarrow{\tau} 0$. Since order convergence implies unbounded order convergence, a uo-Lebesgue topology is an o-Lebesgue topology. A vector lattice E need not admit a Hausdorff uo-Lebesgue topology, but when it does, then this topology is unique (see [11, Propositions 3.2, 3.4, and 6.2] or [44, Theorems 5.5 and 5.9]) and we denote it by $\widehat{\tau}_E$.

Let E be a vector lattice, let F be an ideal of E, and suppose that τ_F is a (not necessarily Hausdorff) locally solid linear topology on F. Take a non-empty subset S of F. Then there exists a unique (possibly non-Hausdorff) locally solid linear topology $\mathrm{u}_S\tau_F$ on E such that, for a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in E, $x_\alpha \xrightarrow{\mathrm{u}_S\tau_F} 0$ if and only if $|x_\alpha| \wedge |s| \xrightarrow{\tau_F} 0$ for all $s \in S$; see [20, Theorem 3.1] for this, which extends earlier results in this vein in, e.g., [11] and [44]. This topology $\mathrm{u}_S\tau_F$ is called the unbounded topology on E that is generated by τ_F via S. Suppose that E admits a Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$. The uniqueness of such a topology then implies that $\mathrm{u}_E\widehat{\tau}_E = \widehat{\tau}_E$. In the sequel we shall use this result from [11] and [44] a few times.

Finally, the characteristic function of a set S will be denoted by χ_S, and the identity operator on a vector space will be denoted by I.

3.3 Absolute strong operator topologies on $\mathscr{L}_{\mathrm{ob}}(E,F)$

Let E and F be vector lattices, where F is Dedekind complete. In this section, we start by showing how topologies can be introduced on vector sublattices of $\mathscr{L}_{\mathrm{ob}}(E,F)$ that can be regarded as absolute strong operator topologies; see Corollary 3.3.5 and Remark 3.3.7, below. Once this is known to be possible, it is easy to relate this to o-Lebesgue topologies and uo-Lebesgue topologies on regular vector sublattices of $\mathscr{L}_{\mathrm{ob}}(E,F)$. In particular, we shall see that every regular vector sublattice of $\mathscr{L}_{\mathrm{ob}}(E,F)$ admits a (necessarily unique) Hausdorff uo-Lebesgue topology when F admits a Hausdorff o-Lebesgue topology; see Corollary 3.4.5, below.

When restricted to the orthomorphisms on a Dedekind complete vector lattice, the picture simplifies; see Section 3.8. In particular, the restrictions of absolute strong operator topologies are then simply strong operator topologies.

The construction in the proof of the following result is an adaptation of that in the proof of [20, Theorem 3.1]. The latter construction is carried out under minimal hypotheses and uses neighbourhood bases at zero as in [44, proof of Theorem 2.3] rather than Riesz pseudo-norms. Such an approach enables one to also understand various 'pathologies' in the literature from one central result; see [20, Example 3.10]. It is for this reason of maximum flexibility that we also choose such a neighbourhood approach here.

Theorem 3.3.1. *Let E and F be vector lattices, where F is Dedekind complete, and let τ_F be a (not necessarily Hausdorff) locally solid additive topology on F. Take a non-empty subset S of E. There exists a unique (possibly non-Hausdorff) additive topology $\mathrm{ASOT}_S\tau_F$ on $\mathscr{L}_{\mathrm{ob}}(E,F)$ such that, for a net $(T_\alpha)_{\alpha\in\mathcal{A}}$ in $\mathscr{L}_{\mathrm{ob}}(E,F)$, $T_\alpha \xrightarrow{\mathrm{ASOT}_S\tau_F} 0$ if and only if $|T_\alpha||s| \xrightarrow{\tau_F} 0$ for all $s \in S$.*

Let I_S be the ideal of E that is generated by S. For a net $(T_\alpha)_{\alpha\in\mathcal{A}}$ in $\mathscr{L}_{\mathrm{ob}}(E,F)$, $T_\alpha \xrightarrow{\mathrm{ASOT}_S\tau_F} 0$ if and only if $|T_\alpha||x| \xrightarrow{\tau_F} 0$ for all $x \in I_S$; and also if and only if $|T_\alpha|x \xrightarrow{\tau_F} 0$ for all $x \in I_S$.

Furthermore:

(1) for every $x \in I_S$, the map $T \mapsto Tx$ is an $\mathrm{ASOT}_S\tau_F$–τ_F continuous map from $\mathscr{L}_{\mathrm{ob}}(E,F)$ into F;

(2) the topology $\mathrm{ASOT}_S\tau_F$ on $\mathscr{L}_{\mathrm{ob}}(E,F)$ is a locally solid additive topology;

(3) *when τ_F is a Hausdorff topology on F, the following are equivalent for an additive subgroup $\mathscr{G}$ of $\mathscr{L}_{\mathrm{ob}}(E,F)$:*
   (a) *the restriction $\mathrm{ASOT}_S\tau_F|_{\mathscr{G}}$ of $\mathrm{ASOT}_S\tau_F$ to $\mathscr{G}$ is a Hausdorff topology on $\mathscr{G}$;*
   (b) *I_S separates the points of $\mathscr{G}$.*

(4) *the following are equivalent for a linear subspace $\mathscr{V}$ of $\mathscr{L}_{\mathrm{ob}}(E,F)$:*
   (a) *for all $T \in \mathscr{V}$ and $s \in S$, $|\varepsilon T||s| \xrightarrow{\tau_F}$ as $\varepsilon \to 0$ in $\mathbb{R}$;*
   (b) *the restriction $\mathrm{ASOT}_S\tau_F|_{\mathscr{V}}$ of $\mathrm{ASOT}_S\tau_F$ to $\mathscr{V}$ is a (possibly non-Hausdorff) linear topology on $\mathscr{V}$.*

Proof. Suppose that τ_F is a (not necessarily Hausdorff) locally solid additive topology on F.

It is clear from the required translation invariance of $\mathrm{ASOT}_S\tau_F$ that it is unique, since the nets that are $\mathrm{ASOT}_S\tau_F$-convergent to zero are prescribed.

For its existence, we take a τ_F-neighbourhood base $\{U_\lambda\}_{\lambda\in\Lambda}$ of zero in F that consists of solid subsets of F. For $x \in I_S$ and $\lambda \in \Lambda$, we set

$$V_{\lambda,x} := \{\, T \in \mathscr{L}_{\mathrm{ob}}(E,F) : |T||x| \in U_\lambda \,\}.$$

The $V_{\lambda,y}$ are solid subsets of $\mathscr{L}_{\mathrm{ob}}(E,F)$ since the U_λ are solid subsets of F.

Set

$$\mathscr{N}_0 := \{\, V_{\lambda,x} : \lambda \in \Lambda, x \in I_S \,\}.$$

We shall now verify that $\mathscr{N}_0$ satisfies the necessary and sufficient conditions in [31, Theorem 3 on p. 46] to be a base of neighbourhoods of zero for an additive topology on $\mathscr{L}_{\mathrm{ob}}(E,F)$.

Take $V_{\lambda_1,x_1}, V_{\lambda_2,x_2} \in \mathscr{N}_0$. There exists a $\lambda_3 \in \Lambda$ such that $U_{\lambda_3} \subseteq U_{\lambda_1} \cap U_{\lambda_2}$, and it is easy to verify that then $V_{\lambda_3,|x_1|\vee|x_2|} \subseteq V_{\lambda_1,x_1} \cap V_{\lambda_2,x_2}$. Hence $\mathscr{N}_0$ is a filter base.

It is clear that $V_{\lambda,x} = -V_{\lambda,x}$.

Take $V_{\lambda,x} \in \mathscr{N}_0$. There exists a $\mu \in \Lambda$ such that $U_\mu + U_\mu \subseteq U_\lambda$, and it is easy to see that then $V_{\mu,x} + V_{\mu,x} \subseteq V_{\lambda,x}$.

An appeal to [31, Theorem 3 on p. 46] now yields that $\mathscr{N}_0$ is a base of neighbourhoods of zero for an additive topology on $\mathscr{L}_{\mathrm{ob}}(E,F)$ that we shall denote by $\mathrm{ASOT}_S\tau_F$. It is a direct consequence of its definition that, for a net $(T_\alpha)_{\alpha\in\mathcal{A}}$ in $\mathscr{L}_{\mathrm{ob}}(E,F)$, $T_\alpha \xrightarrow{\mathrm{ASOT}_S\tau_F} 0$ if and only if $|T_\alpha||x| \xrightarrow{\tau_F} 0$ for all $x \in I_S$. Using the fact that τ_F is a locally solid additive topology on F, it is routine to verify that the latter condition is equivalent to the condition that $|T|x \xrightarrow{\tau_F} 0$ for all $x \in I_S$, as well as to the condition that $|T_\alpha||s| \xrightarrow{\tau_F} 0$ for all $s \in S$.

We turn to the statements in the parts (1)–(4).

For part (1), suppose that $(T_\alpha)_{\alpha\in\mathcal{A}}$ is a net in $\mathscr{L}_{\mathrm{ob}}(E,F)$ such that $T_\alpha \xrightarrow{\mathrm{ASOT}_S\tau_F} 0$. Then $|T_\alpha||x| \xrightarrow{\tau_F} 0$ for all $x \in I_S$. Since $|T_\alpha x| \leq |T_\alpha||x|$, the fact that τ_F is locally solid implies that then also $T_\alpha x \xrightarrow{\tau_F} 0$ for all $x \in I_S$.

Since the topology $\mathrm{ASOT}_S\tau_F$ is a locally solid additive topology on $\mathscr{L}_{\mathrm{ob}}(E,F)$ by construction, part (2) is clear.

For part (3), we recall from [31, p. 48, Theorem 4] that an additive topology on a group is Hausdorff if and only if the intersection of the elements of a neighbourhood base of zero is trivial. Using this for F in the second step, and invoking [20, Proposition 2.1] in the third, we see that

$$
\begin{aligned}
\bigcap_{\lambda\in\Lambda, x\in I_S}\big(V_{\lambda,x}\cap\mathscr{G}\big) &= \{\, T\in\mathscr{L}_{\mathrm{ob}}(E,F) : |T||x| \in \bigcap_{\lambda\in\Lambda} U_\lambda \text{ for all } x\in I_S \,\}\cap\mathscr{G}\\
&= \{\, T\in\mathscr{L}_{\mathrm{ob}}(E,F) : |T||x| = 0 \text{ for all } x\in I_S \,\}\cap\mathscr{G}.\\
&= \{\, T\in\mathscr{L}_{\mathrm{ob}}(E,F) : Tx = 0 \text{ for all } x\in I_S \,\}\cap\mathscr{G}\\
&= \{\, T\in\mathscr{G} : Tx = 0 \text{ for all } x\in I_S \,\}.
\end{aligned}
$$

Another appeal to [31, p. 48, Theorem 4] then completes the proof of part (3).

We prove that part (4a) implies part (4b). It is clear that $\mathrm{ASOT}_S\tau_F|_{\mathscr{V}}$ is an additive topology on $\mathscr{V}$. From what we have already established, we know that the assumption implies that also $|\varepsilon T||x| \xrightarrow{\tau_F} 0$ as $\varepsilon\to 0$ in $\mathbb{R}$ for all $T\in\mathscr{V}$ and $x\in I_S$. Fix $\lambda\in\Lambda$ and $x\in I_S$, and take $T\in\mathscr{V}$. Since $|\varepsilon T||x| \xrightarrow{\tau_F} 0$ as $\varepsilon\to 0$ in $\mathbb{R}$, there exists a $\delta>0$ such that $|\varepsilon T||x|\in U_\lambda$ whenever $|\varepsilon|<\delta$. That is, $\varepsilon T\in V_{\lambda,x}\cap\mathscr{V}$ whenever $|\varepsilon|<\delta$. Hence $V_{\lambda,x}\cap\mathscr{V}$ is an absorbing subset of $\mathscr{V}$. Furthermore, since $V_{\lambda,x}$ is a solid subset of $\mathscr{L}_{\mathrm{ob}}(E,F)$, it is clear that $\varepsilon T\in V_{\lambda,x}\cap\mathscr{V}$ whenever $T\in V_{\lambda,x}\cap\mathscr{V}$ and $\varepsilon\in[-1,1]$. We conclude from [5, Theorem 5.6] that $\mathrm{ASOT}_S\tau_F|_{\mathscr{V}}$ is a linear topology on $\mathscr{V}$.

We prove that part (4b) implies part (4a). Take $T\in\mathscr{V}$. Then $\varepsilon T \xrightarrow{\mathrm{ASOT}_S\tau_F|_{\mathscr{V}}} 0$ as $\varepsilon\to 0$ in $\mathbb{R}$. By construction, this implies that (and is, in fact, equivalent to) the fact that $|\varepsilon T||s| \xrightarrow{\tau_F} 0$ for all $s\in S$. $\square$

Remark 3.3.2. It is clear from the convergence criteria for nets that the topologies $\mathrm{ASOT}_{S_1}\tau_F$ and $\mathrm{ASOT}_{S_2}\tau_F$ are equal when $I_{S_1}=I_{S_2}$. One could, therefore, work with ideals from the very start, but it seems worthwhile to keep track of a smaller set of presumably more manageable 'test vectors'. See also the comments preceding Theorem 3.4.3, below.

Remark 3.3.3. Suppose that $(T_\alpha)_{\alpha\in\mathcal{A}}$ is a net in $\mathscr{L}_{\mathrm{ob}}(E,F)$ such that $T_\alpha \xrightarrow{\mathrm{ASOT}_S\tau_F} 0$. It is easy to see that then $|T_\alpha|x \xrightarrow{\tau_F} 0$ uniformly on every order bounded subset of I_S, so that then also $T_\alpha x \xrightarrow{\tau_F} 0$ uniformly on every order bounded subset of I_S. When τ_F is a Fatou topology on F (in particular: when τ_F is an o-Lebesgue topology on F; see [6, Lemma 4.2]), then, conversely, the fact that $T_\alpha x \xrightarrow{\tau_F} 0$ uniformly on every order bounded subset of I_S implies that $T_\alpha \xrightarrow{\mathrm{ASOT}_S\tau_F} 0$. This follows readily from the Riesz-Kantorovich formula for the modulus of an operator.

Definition 3.3.4. The topology $\mathrm{ASOT}_S\tau_F$ in Theorem 3.3.1 is called the *absolute strong operator topology that is generated by τ_F via S*. We shall comment on this nomenclature in Remark 3.3.7, below.

The following result, which can also be obtained using Riesz pseudo-norms, is clear from Theorem 3.3.1.

Corollary 3.3.5. *Let E and F be vector lattices, where F is Dedekind complete, and let τ_F be a (not necessarily Hausdorff) locally solid linear topology on F. Take a vector sublattice $\mathscr{E}$ of $\mathscr{L}_{\mathrm{ob}}(E,F)$ and a non-empty subset S of E.*

There exists a unique additive topology $\mathrm{ASOT}_S\tau_F$ *on* $\mathscr{E}$ *such that, for a net* $(T_\alpha)_{\alpha\in\mathcal{A}}$ *in* $\mathscr{E}$, $T_\alpha \xrightarrow{\mathrm{ASOT}_S\tau_F} 0$ *if and only if* $|T_\alpha||s| \xrightarrow{\tau_F} 0$ *for all* $s \in S$.

Let I_S *be the ideal of* E *that is generated by* S. *For a net* $(T_\alpha)_{\alpha\in\mathcal{A}}$ *in* $\mathscr{E}$, $T_\alpha \xrightarrow{\mathrm{ASOT}_S\tau_F} 0$ *if and only if* $|T_\alpha||x| \xrightarrow{\tau_F} 0$ *for all* $x \in I_S$*; and also if and only if* $|T_\alpha|x \xrightarrow{\tau_F} 0$ *for all* $x \in I_S$.

Furthermore:

(1) *for every* $x \in I_S$, *the map* $T \mapsto Tx$ *is an* $\mathrm{ASOT}_S\tau_F$–τ_F *continuous map from* $\mathscr{E}$ *into* F;
(2) *the additive topology* $\mathrm{ASOT}_S\tau_F$ *on the group* $\mathscr{E}$ *is, in fact, a locally solid linear topology on the vector lattice* $\mathscr{E}$. *When* τ_F *is a Hausdorff topology on* F, *then* $\mathrm{ASOT}_S\tau_F$ *is a Hausdorff topology on* $\mathscr{E}$ *if and only if* I_S *separates the points of* $\mathscr{E}$.

Remark 3.3.6. Although in the sequel of this paper we shall mainly be interested in the nets that are convergent in a given topology, let us still remark that is possible to describe an explicit $\mathrm{ASOT}_S\tau_F$-neighbourhood base of zero in $\mathscr{E}$. Take a τ_F-neighbourhood base $\{U_\lambda\}_{\lambda\in\Lambda}$ of zero in F that consists of solid subsets of F. For $\lambda \in \Lambda$ and $x \in I_S$, set

$$V_{\lambda,x} := \{\, T \in \mathscr{E} : |T||x| \in U_\lambda \,\}.$$

Then $\{\, V_{\lambda,x} : \lambda \in \Lambda, x \in I_S \,\}$ is an $\mathrm{ASOT}_S\tau_F$-neighbourhood base of zero in $\mathscr{E}$.

Remark 3.3.7. It is not difficult to see that $\mathrm{ASOT}_S\tau_F$ is the weakest locally solid linear topology $\tau_{\mathscr{E}}$ on $\mathscr{E}$ such that, for every $x \in I_S$, the map $T \to Tx$ is a $\tau_{\mathscr{E}}$–τ_F continuous map from $\mathscr{E}$ into F. It is also the weakest linear topology $\tau'_{\mathscr{E}}$ on $\mathscr{E}$ such that, for every $x \in I_S$, the map $T \to |T|x$ is a $\tau'_{\mathscr{E}}$–τ_F continuous map from $\mathscr{E}$ into F. The latter characterisation is our motivation for the name 'absolute strong operator topology'.

Take $F = \mathbb{R}$ and $S = E$. Then $\mathrm{ASOT}_E\tau_{\mathbb{R}}$ is what is commonly known as the absolute weak*-topology on $E^\sim$. There is an unfortunate class of 'weak' and 'strong' here that appears to be unavoidable.

Remark 3.3.8. For comparison with Remark 3.3.7, and in order to make clear the role of the local solidness of the topologies in the present section, we mention the following, which is an easy consequence of [5, Theorem 5.6], for example. Let E and F be vector spaces, where F is supplied with a (not necessarily) Hausdorff linear topology τ_F. Take a linear subspace $\mathscr{E}$ of the vector space of all linear maps from E into F, and take a non-empty subset S of E. Then there exists a unique (not necessarily Hausdorff) linear topology $\mathrm{SOT}_S\tau_F$ on $\mathscr{E}$ such that, for a net $(T_\alpha)_{\alpha\in\mathcal{A}}$ in $\mathscr{E}$, $T_\alpha \xrightarrow{\mathrm{SOT}_S\tau_F} 0$ if and only if $T_\alpha s \xrightarrow{\tau_F} 0$ for all $s \in S$. The subsets of $\mathscr{E}$ of the form $\bigcap_{i=1}^n \{\, T \in \mathscr{E} : Ts_i \in V_{\lambda_i} \,\}$, where the s_i run over S and the V_{λ_i} run over a balanced τ_F-neighbourhood base $\{\, V_\lambda : \lambda \in \Lambda \,\}$ of zero in F, are an $\mathrm{SOT}_S\tau_F$-neighbourhood base of zero in $\mathscr{E}$. When τ_F is Hausdorff, then $\mathrm{SOT}_S\tau_F$ is Hausdorff if and only if S separates the points of $\mathscr{E}$. This strong operator topology $\mathrm{SOT}_S\tau_F$ on $\mathscr{E}$ that is generated by τ_F via S, is the weakest linear topology $\tau_{\mathscr{E}}$ on $\mathscr{E}$ such that, for every $s \in S$, the map $T \mapsto Tx$ is $\tau_{\mathscr{E}}$–τ_F-continuous.

3.4 o-Lebesgue topologies and uo-Lebesgue topologies on vector lattices of operators

In order to arrive at results concerning o-Lebesgue topologies and uo-Lebesgue topologies on regular vector sublattices of operators, we need a preparatory result for which we are not aware of a reference. Given its elementary nature, we refrain from any claim to originality. It will re-appear at several places in the sequel.

Lemma 3.4.1. *Let E and F be vector lattices, where F is Dedekind complete, and let $\mathscr{E}$ be a regular vector sublattice of $\mathscr{L}_{\mathrm{ob}}(E,F)$. Suppose that $(T_\alpha)_{\alpha\in\mathcal{A}}$ is net in $\mathscr{E}$ such that $T_\alpha \xrightarrow{\mathrm{o}} 0$ in $\mathscr{E}$. Then $T_\alpha x \xrightarrow{\mathrm{o}} 0$ for all $x \in E$.*

Proof. By the regularity of $\mathscr{E}$, we also have that $T_\alpha \xrightarrow{\mathrm{o}} 0$ in $\mathscr{L}_{\mathrm{ob}}(E,F)$. Hence there exists a net $(S_\beta)_{\beta\in\mathcal{B}}$ in $\mathscr{L}_{\mathrm{ob}}(E,F)$ such that $S_\beta \downarrow 0$ in $\mathscr{L}_{\mathrm{ob}}(E,F)$ and with the property that, for every $\beta_0 \in \mathcal{B}$, there exists an $\alpha_0 \in \mathcal{A}$ such that $|T_\alpha| \leq S_{\beta_0}$ for all $\alpha \in \mathcal{A}$ such that $\alpha \geq \alpha_0$. We know from [7, Theorem 1.18], for example, that $S_\beta x \downarrow 0$ for all $x \in E^+$. Since $|T_\alpha x| \leq |T_\alpha| x$ for $x \in E^+$, it then follows easily that $T_\alpha x \xrightarrow{\mathrm{o}} 0$ for all $x \in E^+$. Hence $T_\alpha x \xrightarrow{\mathrm{o}} 0$ for all $x \in E$. $\square$

We can now show that the o-Lebesgue property of a locally solid linear topology on the Dedekind complete codomain is inherited by the associated absolute strong operator topology on a regular vector sublattice of operators.

Proposition 3.4.2. *Let E and F be vector lattices, where F is Dedekind complete. Suppose that F admits an o-Lebesgue topology τ_F. Take a regular vector sublattice $\mathscr{E}$ of $\mathscr{L}_{\mathrm{ob}}(E,F)$ and a non-empty subset S of E. Then $\mathrm{ASOT}_S\tau_F$ is an o-Lebesgue topology on $\mathscr{E}$. When τ_F is a Hausdorff topology on F, then $\mathrm{ASOT}_S\tau_F$ is a Hausdorff topology on $\mathscr{E}$ if and only if I_S separates the points of $\mathscr{E}$.*

Proof. In view of Corollary 3.3.5, we merely need to show that, for a net $(T_\alpha)_{\alpha\in\mathcal{A}}$ in $\mathscr{E}$, the fact that $T_\alpha \xrightarrow{\mathrm{o}} 0$ in $\mathscr{E}$ implies that $T_\alpha \xrightarrow{\mathrm{ASOT}_S\tau_F} 0$. Take $s \in S$. Since also $|T_\alpha| \xrightarrow{\mathrm{o}} 0$ in $\mathscr{E}$, Lemma 3.4.1 implies that $|T_\alpha||s| \xrightarrow{\mathrm{o}} 0$ in F. Using that τ_F is an o-Lebesgue topology on F, we find that $|T_\alpha||s| \xrightarrow{\tau_F} 0$. Since this holds for all $s \in S$, Corollary 3.3.5 shows that $T_\alpha \xrightarrow{\mathrm{ASOT}_S\tau_F} 0$ in $\mathscr{E}$. $\square$

We conclude by showing that every regular vector sublattice of $\mathscr{L}_{\mathrm{ob}}(E,F)$ admits a (necessarily unique) Hausdorff uo-Lebesgue topology when the Dedekind complete codomain F admits a Hausdorff o-Lebesgue topology. It is the unbounded topology that is associated to (in general multiple) absolute strong operator topologies on the vector sublattice. Our most precise result in this direction is the following. The convergence criterion in part (2) is a 'minimal one' that is convenient when one wants to show that a net is convergent, whereas the criteria in part (3) exploits the known convergence of a net to its maximum.

Theorem 3.4.3. *Let E and F be vector lattices, where F is Dedekind complete. Suppose that F admits an o-Lebesgue topology τ_F. Take a regular vector sublattice $\mathscr{E}$ of $\mathscr{L}_{\mathrm{ob}}(E,F)$, a non-empty subset $\mathscr{S}$ of $\mathscr{E}$, and a non-empty subset S of E.*

Then $\mathrm{u}_{\mathscr{S}}\mathrm{ASOT}_S\tau_F$ *is a uo-Lebesgue topology on* $\mathscr{E}$.

We let I_S *denote the ideal of* E *that is generated by* S, *and* $I_{\mathscr{S}}$ *the ideal of* $\mathscr{E}$ *that is generated by* $\mathscr{S}$. *For a net* $(T_\alpha)_{\alpha\in\mathcal{A}}$ *in* $\mathscr{E}$, *the following are equivalent:*

(1) $T_\alpha \xrightarrow{\mathrm{u}_{\mathscr{S}}\mathrm{ASOT}_S\tau_F} 0$;

(2) $(|T_\alpha|\wedge|T|)|s| \xrightarrow{\tau_F} 0$ *for all* $T\in\mathscr{S}$ *and* $s\in S$;

(3) $(|T_\alpha|\wedge|T|)x \xrightarrow{\tau_F} 0$ *for all* $T\in I_{\mathscr{S}}$ *and* $x\in I_S$.

Suppose that τ_F *is actually a* Hausdorff *o-Lebesgue topology on* F. *Then the following are equivalent:*

(1) $\mathrm{u}_{\mathscr{S}}\mathrm{ASOT}_S\tau_F$ *is a* (*necessarily unique*) Hausdorff *uo-Lebesgue topology on* $\mathscr{E}$;

(2) I_S *separates the points of* $\mathscr{E}$ *and* $I_{\mathscr{S}}$ *is order dense in* $\mathscr{E}$.

In that case, the Hausdorff uo-Lebesgue topology $\mathrm{u}_{\mathscr{S}}\mathrm{ASOT}_S\tau_F$ *on* $\mathscr{E}$ *is the restriction of the* (*necessarily unique*) *Hausdorff uo-Lebesgue topology on* $\mathscr{L}_{\mathrm{ob}}(E,F)$, *i.e., of* $\mathrm{u}_{\mathscr{L}_{\mathrm{ob}}(E,F)}\mathrm{ASOT}_E\tau_F$, *and the criteria in* (1), (2), *and* (3) *are also equivalent to:*

(4) $(|T_\alpha|\wedge|T|)x \xrightarrow{\tau_F} 0$ *for all* $T\in\mathscr{L}_{\mathrm{ob}}(E,F)$ *and* $x\in E$.

Proof. It is clear from Proposition 3.4.2 and [20, Proposition 4.1] that $\mathrm{u}_{\mathscr{S}}\mathrm{ASOT}_S\tau_F$ is a uo-Lebesgue topology on $\mathscr{E}$. The two convergence criteria for nets follow from the combination of those in [20, Theorem 3.1] and in Corollary 3.3.5.

According to [20, Proposition 4.1], $\mathrm{u}_{\mathscr{S}}\mathrm{ASOT}_S\tau_F$ is a Hausdorff topology on $\mathscr{E}$ if and only if $\mathrm{ASOT}_S\tau_F$ is a Hausdorff topology on $\mathscr{E}$ and $I_{\mathscr{S}}$ is order dense in $\mathscr{E}$. An appeal to Proposition 3.4.2 then completes the proof of the necessary and sufficient conditions for $\mathrm{u}_{\mathscr{S}}\mathrm{ASOT}_S\tau_F$ to be Hausdorff.

Suppose that τ_F is actually also Hausdorff, that I_S separates the points of $\mathscr{E}$, and that $I_{\mathscr{S}}$ is order dense in $\mathscr{E}$. From what we have already established, it is clear that $\mathrm{u}_{\mathscr{L}_{\mathrm{ob}}(E,F)}\mathrm{ASOT}_E\tau_F$ is a (necessarily unique) Hausdorff uo-Lebesgue topology on $\mathscr{L}_{\mathrm{ob}}(E,F)$. Since the restriction of a Hausdorff uo-Lebesgue topology on a vector lattice to a regular vector sublattice is a (necessarily unique) Hausdorff uo-Lebesgue topology on the vector sublattice (see [44, Proposition 5.12]), the criterion in part (4) follows from that in part (3) applied to $\mathrm{u}_{\mathscr{L}_{\mathrm{ob}}(E,F)}\mathrm{ASOT}_E\tau_F$. □

Remark 3.4.4. Take a τ_F-neighbourhood base $\{U_\lambda\}_{\lambda\in\Lambda}$ of zero in F that consists of solid subsets of F. For $\lambda\in\Lambda$, $\widetilde{T}\in I_{\mathscr{S}}$, and $x\in I_S$, set

$$V_{\lambda,\widetilde{T},x} := \{\, T\in\mathscr{E} : (|T|\wedge|\widetilde{T}|)|x|\in U_\lambda \,\}.$$

As a consequence of the constructions of unbounded and absolute strong operator topologies, $\{\, V_{\lambda,\widetilde{T},x} : \lambda\in\Lambda, T\in I_{\mathscr{S}}, x\in I_S \,\}$ is then a $\mathrm{u}_{\mathscr{S}}\mathrm{ASOT}_S\tau_F$-neighbourhood base of zero in $\mathscr{E}$.

The following is a less precise consequence of Theorem 3.4.3 that will be sufficient in many situations.

Corollary 3.4.5. *Let* E *and* F *be vector lattices, where* F *is Dedekind complete. Suppose that* F *admits a Hausdorff o-Lebesgue topology* τ_F.

Take a regular vector sublattice $\mathscr{E}$ of $\mathscr{L}_{\mathrm{ob}}(E,F)$. Then $\mathscr{E}$ admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_{\mathscr{E}}$. This topology equals $\mathrm{u}_{\mathscr{E}}\mathrm{ASOT}_E\tau_F$, and is also equal to the restriction to $\mathscr{E}$ of the Hausdorff uo-Lebesgue topology $\mathrm{u}_{\mathscr{L}_{\mathrm{ob}}(E,F)}\mathrm{ASOT}_E\tau_F$ on $\mathscr{L}_{\mathrm{ob}}(E,F)$.

For a net $(T_\alpha)_{\alpha\in\mathcal{A}}$ in $\mathscr{E}$, the following are equivalent:

(1) $T_\alpha \xrightarrow{\widehat{\tau}_{\mathscr{E}}} 0$;

(2) $(|T_\alpha|\wedge|T|)x \xrightarrow{\tau_F} 0$ *for all* $T\in\mathscr{E}$ *and* $x\in E$;

(3) $(|T_\alpha|\wedge|T|)x \xrightarrow{\tau_F} 0$ *for all* $T\in\mathscr{L}_{\mathrm{ob}}(E,F)$ *and* $x\in E$.

Remark 3.4.6. There can, sometimes, be other ways to see that a given regular vector sublattice of $\mathscr{L}_{\mathrm{ob}}(E,F)$ admits a Hausdorff uo-Lebesgue topology. For example, suppose that $F_{\mathrm{oc}}^{\sim}$ separates the points of F. For $x\in E$ and $\varphi\in F_{\mathrm{oc}}^{\sim}$, the map $T\mapsto\varphi(Tx)$ defines an order continuous linear functional on $\mathscr{L}_{\mathrm{oc}}(E,F)$, and it is then clear that the order continuous dual of $\mathscr{L}_{\mathrm{oc}}(E,F)$ separates the points of $\mathscr{L}_{\mathrm{oc}}(E,F)$. Hence $\mathscr{L}_{\mathrm{oc}}(E,F)$ can also be supplied with a Hausdorff uo-Lebesgue topology as in [20, Theorem 5.2] which, in view of its uniqueness, coincides with the one as supplied by Corollary 3.4.5.

3.5 Comparing uniform and strong convergence structures on $\mathscr{L}_{\mathrm{ob}}(E,F)$

Suppose that E and F are vector lattices, where F is Dedekind complete. As explained in Section 3.1, there exist a uniform and a strong convergence structure on $\mathscr{L}_{\mathrm{ob}}(E,F)$ for each of order convergence, unbounded order convergence, and—when applicable—convergence in the Hausdorff uo-Lebesgue topology. In this section, we investigate what the relation is between the members of each of these three pairs. We shall show that only one of the six conceivable implications is valid in general, and that the others are not even generally valid for uniformly bounded sequences of order continuous operators on Banach lattices. Whilst the failures of such general implications may, perhaps, not come as too big a surprise, the positive results for orthomorphisms (see Theorems 3.9.4, 3.9.7, 3.9.9, and 3.9.12, below) may serve to indicate that they are less evident than one would think at first sight.

For monotone nets in $\mathscr{L}_{\mathrm{ob}}(E,F)$, however, the following result shows that then even all four (or six) convergence structures on $\mathscr{L}_{\mathrm{ob}}(E,F)$ are equal.

Proposition 3.5.1. *Let E and F be vector lattices, where F is Dedekind complete, and let $(T_\alpha)_{\alpha\in\mathcal{A}}$ be a monotone net in $\mathscr{L}_{\mathrm{ob}}(E,F)$. The following are equivalent:*

(1) $T_\alpha \xrightarrow{\mathrm{o}} 0$ *in* $\mathscr{L}_{\mathrm{ob}}(E,F)$;

(2) $T_\alpha \xrightarrow{\mathrm{uo}} 0$ *in* $\mathscr{L}_{\mathrm{ob}}(E,F)$;

(3) $T_\alpha x \xrightarrow{\mathrm{o}} 0$ *in F for all* $x\in E$;

(4) $T_\alpha x \xrightarrow{\mathrm{uo}} 0$ *in F for all* $x\in E$.

Suppose that, in addition, F admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_F$, so that $\mathscr{L}_{\mathrm{ob}}(E,F)$ also admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E,F)}$ by Corollary 3.4.5. Then (1)–(4) are also equivalent to:

(5) $T_\alpha \xrightarrow{\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E,F)}} 0$;

(6) $T_\alpha x \xrightarrow{\widehat{\tau}_F} 0$ *for all* $x \in E$.

Proof. We may suppose that $T_\alpha \downarrow 0$ and that $x \in E^+$. For order bounded nets in a vector lattice, order convergence and unbounded order convergence are equivalent. Passing to an order bounded tail of $(T_\alpha)_{\alpha\in\mathcal{A}}$, we thus see that the parts (1) and (2) are equivalent. Similarly, the parts (3) and (4) are equivalent. The equivalence of the parts (1) and (3) is well known; see [6, Theorem 1.67], for example.

Suppose that F admits a Hausdorff uo-Lebesgue topology $\widehat{\tau}_F$. In that case, it follows from [20, Lemma 7.2] that the parts (2) and (5) are equivalent, as are the parts (4) and (6). □

When $(T_\alpha)_{\alpha\in\mathcal{A}}$ is a not necessarily monotone net in $\mathscr{L}_{\mathrm{ob}}(E,F)$ such that $T_\alpha \xrightarrow{\mathrm{o}} 0$, then Lemma 3.4.1 shows that $T_\alpha x \xrightarrow{\mathrm{o}} 0$ in F for all $x \in E$. We shall now give five examples to show that each of the remaining five conceivable implications between a corresponding uniform and strong convergence structures on $\mathscr{L}_{\mathrm{ob}}(E,F)$ is not generally valid. In each of these examples, we can even take $E = F$ to be a Banach lattice, and for the net $(T_\alpha)_{\alpha\in\mathcal{A}}$ we can even take a uniformly bounded sequence $(T_n)_{n=1}^\infty$ of order continuous operators on E.

Example 3.5.2. We give an example of a *uniformly bounded sequence* $(T_n)_{n=1}^\infty$ *of positive order continuous operators on a Dedekind complete Banach lattice E with a strong order unit, such that* $T_n x \xrightarrow{\mathrm{o}} 0$ *in E for all* $x \in E$ *but* $T_n \not\xrightarrow{\mathrm{o}} 0$ *in* $\mathscr{L}_{\mathrm{ob}}(E)$ because the sequence is not even order bounded in $\mathscr{L}_{\mathrm{ob}}(E)$.

We choose $\ell_\infty(\mathbb{N})$ for $E = F$. For $n \geq 1$, we set $T_n := S^n$, where S is the right shift operator on E. The T_n are evidently positive and of norm one. A moment's thought shows that they are order continuous. Furthermore, it is easy to see that $T_n x \xrightarrow{\mathrm{o}} 0$ in E for all $x \in E$. We shall now show that $\{ T_n : n \geq 1 \}$ is not order bounded in $\mathscr{L}_{\mathrm{ob}}(E)$. For this, we start by establishing that the T_n are mutually disjoint. Let $(e_i)_{i=1}^\infty$ be the standard sequence of unit vectors in E. Take $m \neq n$ and $i \geq 1$. Since e_i is an atom, the Riesz-Kantorovich formula for the infimum of two operators shows that

$$0 \leq (T_m \wedge T_n)e_i = \inf\{ t e_{m+i} + (1-t)e_{n+i} : 0 \leq t \leq 1 \} \leq \inf\{e_{m+i}, e_{n+i}\} = 0.$$

Hence $(T_m \wedge T_n)$ vanishes on the span of the e_i. Since this span is order dense in E, and since $T_n \wedge T_m \in \mathscr{L}_{\mathrm{oc}}(E)$, it follows that $T_n \wedge T_m = 0$.

We can now show that $(T_n)_{n=1}^\infty$ is not order bounded in $\mathscr{L}_{\mathrm{ob}}(E)$. Indeed, suppose that $T \in \mathscr{L}_{\mathrm{ob}}(E)$ is a upper bound for all T_n. Set $e := \bigvee_{i=1}^\infty e_i$. Then, for all $N \geq 1$,

$$Te \geq \left(\bigvee_{n=1}^N T_n \right) e = \left(\sum_{n=1}^N T_n \right) e \geq N e_{N+1}.$$

This shows that Te cannot be an element of ℓ_∞. We conclude from this contradiction that $(T_n)_{n=1}^\infty$ is not order bounded in $\mathscr{L}_{\mathrm{ob}}(E)$.

Example 3.5.3. We give an example of a *uniformly bounded sequence* $(T_n)_{n=1}^\infty$ *of positive order continuous operators on a Dedekind complete Banach lattice E with a strong order unit, such that* $T_n \xrightarrow{\mathrm{uo}} 0$ *in* $\mathscr{L}_{\mathrm{ob}}(E)$ *but* $T_n x \not\xrightarrow{\mathrm{uo}} 0$ *for some* $x \in E$.

We choose $\ell_\infty(\mathbb{Z})$ for $E = F$. For $n \geq 1$, we set $T_n := S^n$, where S is the right shift operator on E. Just as in Example 3.5.2, the T_n are positive order continuous operators on E of norm one that are mutually disjoint. Since disjoint sequences in vector lattices are unbounded order convergent to zero (see [28, Corollary 3.6]), we have $T_n \xrightarrow{\mathrm{uo}} 0$ in $\mathscr{L}_{\mathrm{ob}}(E)$. On the other hand, if we let e be the two-sided sequence that is constant 1, then $T_n e = e$ for all $n \geq 1$. Hence $(T_n e)_{n=1}^\infty$ is not unbounded order convergent to zero in E.

For our next example, we require a preparatory lemma.

Lemma 3.5.4. *Let μ be the Lebesgue measure on the Borel σ-algebra $\mathscr{B}$ of $[0,1]$, and let $1 \leq p \leq \infty$. Take a Borel subset S of $[0,1]$, and define the positive operator $T_S : \mathrm{L}_p([0,1],\mathscr{B},\mu) \to \mathrm{L}_p([0,1],\mathscr{B},\mu)$ by setting*

$$T_S(f) := \int_S f \,\mathrm{d}\mu \cdot \chi_S$$

for $f \in \mathrm{L}_p([0,1],\mathscr{B},\mu)$. Then $T_S \wedge I = 0$.

Proof. Take an $n \geq 1$, and choose disjoint a partition $[0,1] = \bigcup_{i=1}^n A_i$ of $[0,1]$ into Borel sets A_i of measure $1/n$. Let e denote the constant function 1. Then

$$\begin{aligned}
(T_S \wedge I)e &= \sum_{i=1}^n (T_S \wedge I)\chi_{A_i} \\
&\leq \sum_{i=1}^n (T_S \chi_{A_i}) \wedge \chi_{A_i} \\
&\leq \sum_{i=1}^n (\mu(A_i)\chi_S) \wedge \chi_{A_i} \\
&\leq \sum_{i=1}^n \mu(A_i)\chi_{A_i} \\
&= \frac{1}{n} e.
\end{aligned}$$

Since n is arbitrary, we see that $(T_S \wedge I)e = 0$. Because $0 \leq T_S \wedge I \leq I$, $T_S \wedge I$ is order continuous. From the fact that the positive order continuous operator $T_S \wedge I$ vanishes on the weak order unit e of $\mathrm{L}_p([0,1],\mathscr{B},\mu)$, we conclude that $T_S \wedge I = 0$. $\square$

Example 3.5.5. We give an example of a *uniformly bounded sequence $(T_n)_{n=1}^\infty$ of order continuous operators on a separable reflexive Banach lattice E with a weak order unit, such that $T_n x \xrightarrow{\mathrm{uo}} 0$ in E for all $x \in E$ but $T_n \overset{\mathrm{uo}}{\not\to} 0$ in $\mathscr{L}_{\mathrm{ob}}(E)$* because even $T_n \xrightarrow{\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}} 0$ in $\mathscr{L}_{\mathrm{ob}}(E)$.

Let μ be the Lebesgue measure on the Borel σ-algebra $\mathscr{B}$ of $[0,1]$, and let $1 \leq p \leq \infty$. For E we choose $\mathrm{L}_p([0,1],\mathscr{B},\mu)$, so that E is reflexive for $1 < p < \infty$. For $n \geq 1$, we let $\mathscr{B}_n$ be the sub-σ-algebra of $\mathscr{B}$ that is generated by the intervals $S_{n,i} := [(i-1)/2^n, i/2^n]$ for $i = 1,\ldots,2^n$, and we let $\mathbb{E}_n : E \to E$ be the corresponding conditional expectation. By [9, Theorem 10.1.5], $\mathbb{E}_n$ is a positive norm one projection. A moment's thought shows that every open subset of $[0,1]$ is the union of the countably infinitely many $S_{n,i}$ that are contained in it, so that it follows from [9, Theorem 10.2.3] that $\mathbb{E}_n f \to f$ almost everywhere as $n \to \infty$. By [28, Proposition 3.1], we can now conclude that $\mathbb{E}_n f \xrightarrow{\mathrm{uo}} f$ for all $f \in E$.

On the other hand, it is not true that $\mathbb{E}_n \xrightarrow{\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}} I$. To see this, we note that, by [9, Example 10.1.2], every $\mathbb{E}_n$ is a linear combination of operators as in Lemma 3.5.4. Hence $\mathbb{E}_n \perp I$ for all n. Since $\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$ is a locally solid linear topology, a possible $\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$-limit of the $\mathbb{E}_n$ is also disjoint from I, hence cannot be I itself.

On setting $T_n := \mathbb{E}_n - I$ for $n \geq 1$, we have obtained a sequence of operators as desired.

Example 3.5.6. We give an example of a *uniformly bounded sequence* $(T_n)_{n=1}^{\infty}$ *of positive order continuous operators on a Dedekind complete Banach lattice E with a strong order unit that admits a Hausdorff uo-Lebesgue topology, such that* $T_n \xrightarrow{\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}} 0$ *in* $\mathscr{L}_{\mathrm{ob}}(E)$ *but* $T_n x \overset{\widehat{\tau}_E}{\not\to} 0$ *in E for some $x \in E$.*

We choose E, the $T_n \in \mathscr{L}_{\mathrm{ob}}(E)$, and $e \in E$ as in Example 3.5.3. There are several ways to see that E admits a Hausdorff uo-Lebesgue topology. This follows most easily from the fact that E is atomic (see [44, Lemma 7.4]) and also from [20, Theorem 6.3] in the context of measure spaces. By Corollary 3.4.5, $\mathscr{L}_{\mathrm{ob}}(E)$ then also admits such a topology. Since we already know from Example 3.5.3 that $T_n \xrightarrow{\mathrm{uo}} 0$, we also have that $T_n \xrightarrow{\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}} 0$. On the other hand, the fact that $T_n e = e$ for $n \geq 1$ evidently shows that $(T_n e)_{n=1}^{\infty}$ is not $\widehat{\tau}_E$-convergent to zero in E.

Example 3.5.7. We note that Example 3.5.5 also gives an example of a *uniformly bounded sequence* $(T_n)_{n=1}^{\infty}$ *of order continuous operators on a separable reflexive Banach lattice E with a weak order unit that admits a Hausdorff uo-Lebesgue topology, such that* $T_n x \xrightarrow{\widehat{\tau}_E} 0$ *in E for all $x \in E$ but* $T_n \overset{\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}}{\not\longrightarrow} 0$ *in* $\mathscr{L}_{\mathrm{ob}}(E)$.

3.6 Orthomorphisms

In this section, we review some material concerning orthomorphism and establish a few auxiliary result for use in the present paper and in future ones.

Let E be a vector lattice. We recall from [7, Definition 2.41] that an operator on E is called an *orthomorphism* when it is a band preserving order bounded operator. An orthomorphism is evidently disjointness preserving, it is order continuous (see [7, Theorem 2.44]), and its kernel is a band (see [7, Theorem 2.48]). We denote by $\mathrm{Orth}(E)$ the collection of all orthomorphism on E. Even when E is not Dedekind complete, the supremum and infimum of two orthomorphisms S and T in E always exists in $\mathscr{L}_{\mathrm{ob}}(E)$. In fact, we have

$$\begin{aligned} [S \vee T](x) &= S(x) \vee T(x) \\ [S \wedge T](x) &= S(x) \wedge T(x) \end{aligned} \tag{3.1}$$

for $x \in E^+$ and

$$|Tx| = |T||x| = |T(|x|)| \tag{3.2}$$

for $x \in E$; see [7, Theorems 2.43 and 2.40]. Consequently, $\mathrm{Orth}(E)$ is a unital vector lattice algebra for every vector lattice E. Even more is true: according to [7, Theorem 2.59], $\mathrm{Orth}(E)$ is an (obviously Archimedean) f-algebra for every vector lattice E, so it is commutative by [7, Theorem 2.56]. Furthermore, for every vector lattice E, when $T \in \mathrm{Orth}(E)$ and $T : E \to E$ is injective and surjective, then the linear map $T^{-1} : E \to E$ is again an orthomorphism. We refer to [37, Theorem 3.1.10] for a proof of this result of Huijsmans' and de Pagter's.

It follows easily from equation (3.1) that, for every vector lattice E, the identity operator is a weak order unit of $\mathrm{Orth}(E)$. When E is Dedekind complete, $\mathrm{Orth}(E)$ is the band in $\mathscr{L}_{\mathrm{ob}}(E)$ that is generated by the identity operator on E; see [7, Theorem 2.45].

Let E be a vector lattice, let $T \in \mathscr{L}_{\mathrm{ob}}(E)$, and let $\lambda \geq 0$. Using [7, Theorem 2.40], it is not difficult to see that the following are equivalent:
(1) $-\lambda I \leq T \leq \lambda I$;
(2) $|T|$ exists in $\mathscr{L}_{\mathrm{ob}}(E)$, and $|T| \leq \lambda I$;
(3) $|Tx| \leq \lambda |x|$ for all $x \in E$.
The set of all such T is a unital subalgebra $\mathscr{Z}(E)$ of $\mathrm{Orth}(E)$ consisting of ideal preserving order bounded operators on E. It is called the *ideal centre of E*.

Let E be a vector lattice, and define the *stabiliser of E*, denoted by $\mathscr{S}(E)$, as the set of linear operators on E that are ideal preserving. It is not required that these operators be order bounded, but this is nevertheless always the case. In fact, $\mathscr{S}(E)$ is a unital subalgebra of $\mathrm{Orth}(E)$ for every vector lattice E (see [47, Proposition 2.6]), so that we have the chain

$$\mathscr{Z}(E) \subseteq \mathscr{S}(E) \subseteq \mathrm{Orth}(E)$$

of unital algebras for every vector lattice E. For every Banach lattice E, we have

$$\mathscr{Z}(E) = \mathscr{S}(E) = \mathrm{Orth}(E);$$

see [47, Corollary 4.2], so that the identity operator on E is then even an order unit of $\mathrm{Orth}(E)$.

For every Banach lattice E, $\mathrm{Orth}(E)$ is a unital Banach subalgebra of the bounded linear operators on E in the operator norm. This follows easily from the facts that bands are closed and that a band preserving operator on a Banach lattice is automatically order bounded; see [7, Theorem 4.76].

Let E be a Banach lattice. Since the identity operator is an order unit of $\mathrm{Orth}(E)$, we can introduce the order unit norm $\|\cdot\|_I$ with respect to I on $\mathrm{Orth}(E)$ by setting

$$\|T\|_I := \inf\{\, \lambda \geq 0 : |T| \leq \lambda I \,\}$$

for $T \in \mathrm{Orth}(E)$. Then $\|T\| = \|T\|_I$ for all $T \in \mathrm{Orth}(E)$; see [47, Proposition 4.1]. Since we already know that $\mathrm{Orth}(E)$ is complete in the operator norm, it follows that $\mathrm{Orth}(E)$, when supplied with $\|\cdot\| = \|\cdot\|_I$, is a unital Banach lattice algebra that is also an AM-space. When E is a Dedekind complete Banach lattice, then evidently $\|T\| = \|T\|_I = \||T|\|_I = \|\,|T|\,\| = \|T\|_{\mathrm{r}}$ for $T \in \mathrm{Orth}(E)$. Hence $\mathrm{Orth}(E)$ is then also a unital Banach lattice subalgebra of the Banach lattice algebra of all order bounded operators on E in the regular norm.

Let E be Banach lattice. It is clear from the above that $(\mathrm{Orth}(E), \|\cdot\|) = (\mathrm{Orth}(E), \|\cdot\|_I)$ is a unital Banach f-algebra in which its identity element is also a (positive) order unit. The following result is, therefore, applicable with $\mathscr{A} = \mathrm{Orth}(E)$ and $e = I$. It shows, in particular, that $\mathrm{Orth}(E)$ is isometrically Banach lattice algebra isomorphic to a $C(K)$-space. Both its statement and its proof improve on the ones in [15, Proposition 2.6], [41, Proposition 1.4], and [30].

Theorem 3.6.1. *Let $\mathscr{A}$ be a unital f-algebra such that its identity element e is also a (positive) order unit, and such that it is complete in the submultiplicative order unit norm $\|\cdot\|_e$ on $\mathscr{A}$. Let $\mathscr{B}$ be a (not necessarily unital) associative subalgebra of $\mathscr{A}$. Then $\overline{\mathscr{B}}^{\|\cdot\|_e}$ is a Banach f-subalgebra of $\mathscr{A}$. When $e \in \overline{\mathscr{B}}$, then there exist a compact Hausdorff space K, uniquely determined up to homeomorphism, and an isometric surjective Banach lattice algebra isomorphism $\psi : \overline{\mathscr{B}}^{\|\cdot\|_e} \to C(K)$.*

Proof. Since $(\mathscr{A}, \|\cdot\|_I)$ is an AM-space with order unit e, there exist a compact Hausdorff space K' and an isometric surjective lattice homomorphism $\psi' : \mathscr{A} \to C(K')$ such that $\psi'(e) = 1$; see [37, Theorem 2.1.3] for this result of Kakutani's, for example. Via this isomorphism, the f-algebra multiplication on $C(K')$ provides the vector lattice $\mathscr{A}$ with a multiplication that makes $\mathscr{A}$ into an f-algebra with e as its positive multiplicative identity element. Such a multiplication is, however, unique; see [7, Theorem 2.58]. Hence ψ' also preserves multiplication, and we conclude that $\psi' : \mathscr{A} \to C(K')$ is an isometric surjective Banach lattice algebra isomorphism.

We now turn to $\mathscr{B}$. It is clear that $\overline{\mathscr{B}}^{\|\cdot\|_e}$ is Banach subalgebra of $\mathscr{A}$. After moving to the $C(K')$-model for $\mathscr{A}$ that we have obtained, [23, Lemma 4.48] shows that $\overline{\mathscr{B}}^{\|\cdot\|_e}$ is also a vector sublattice of $\mathscr{A}$. Hence $\overline{\mathscr{B}}^{\|\cdot\|_e}$ is a Banach f-subalgebra of $\mathscr{A}$. When $e \in \overline{\mathscr{B}}^{\|\cdot\|_e}$, we can then apply the first part of the proof to $\mathscr{B}$, and obtain a compact Hausdorff space K and an isometric surjective Banach lattice algebra isomorphism $\psi : \overline{\mathscr{B}}^{\|\cdot\|_e} \to C(K)$. The Banach-Stone theorem (see [12, Theorem VI.2.1], for example) implies that K is uniquely determined up to homeomorphism. $\square$

We now proceed to show that E and $\mathrm{Orth}(E)$ have isomorphic universal completions. We start with a preparatory lemma.

Proposition 3.6.2. *Let E be a Dedekind complete vector lattice, and let $x \in E$. Let I_x be the principal ideal of E that is generated by x, let B_x be the principal band in E that is generated by x, let $P_x : E \to B_x$ be the corresponding order projection, and let $\mathscr{I}_{P_x}$ be the principal ideal of $\mathscr{L}_{\mathrm{ob}}(E)$ that is generated by P_x. For $T \in \mathscr{I}_{P_x}$, set $\psi_x(T) := T|x|$. Then $\psi_x(T) \in I_x$, and:*
(1) the map $\psi_x : \mathscr{I}_{P_x} \to I_x$ is a surjective vector lattice isomorphism such that $\psi_x(P_x) = |x|$;
(2) $\mathscr{I}_{P_x} = P_x \mathscr{Z}(E)$.

Proof. Take $T \in \mathscr{I}_{P_x}$. There exists a $\lambda \geq 0$ such that $|T| \leq \lambda P_x$, and this implies that $|Ty| \leq \lambda P_x|y|$ for all $y \in E$. This shows that $T|x| \in I_x$, so that ψ_x maps $\mathscr{I}_{P_x}$ into I_x; it also shows that $T(B_x^{\mathrm{d}}) = \{0\}$. Suppose that $T|x| = 0$. Since the kernel of T is a band in E, this implies that T vanishes on B_x. We already know that it vanishes on B_x^{d}. Hence $T = 0$, and we conclude that ψ_x is injective. We show that ψ_x is surjective. Let $y \in I_x$. Take a

$\lambda > 0$ such that $0 \leq |y/\lambda| \leq |x|$. An inspection of the proof of [7, Theorem 2.49] shows that there exists a $T \in \mathscr{Z}(E)$ with $T|x| = y/\lambda$. Since $\lambda T P_x \in \mathscr{I}_{P_x}$ and $(\lambda T P_x)|x| = y$, we see that ψ_x is surjective. Finally, it is clear from equation (3.1) that ψ_x is a vector lattice homomorphism. This completes the proof of part (1).

We turn to part (2). It is clear that $\mathscr{I}_{P_x} \supseteq P_x\mathscr{Z}(E)$. Take $T \in \mathscr{I}_{P_x} \subseteq \mathscr{Z}(E)$. Then also $P_x T \in \mathscr{I}_{P_x}$. Since $\psi_x(T) = \psi_x(P_x T)$, the injectivity of ψ_x on $\mathscr{I}_{P_x}$ implies that $T = P_x T \in P_x\mathscr{Z}(E)$. □

The first part of Proposition 3.6.2 is used in the proof of our next result.

Proposition 3.6.3. *Let E be a Dedekind complete vector lattice. Then there exist an order dense ideal I of E and an order dense ideal $\mathscr{I}$ of* Orth(E) *such that I and $\mathscr{I}$ are isomorphic vector lattices.*

Proof. Choose a maximal disjoint system $\{x_\alpha : \alpha \in \mathcal{A}\}$ in E. For each $\alpha \in \mathcal{A}$, let I_{x_α}, B_{x_α}, $P_{x_\alpha} : E \to B_{x_\alpha}$, $\mathscr{I}_{P_{x_\alpha}}$, and the vector lattice isomorphism $\psi_{x_\alpha} : \mathscr{I}_{P_{x_\alpha}} \to I_{x_\alpha}$ be as in Proposition 3.6.2.

Since the x_α are mutually disjoint, it is clear that the ideal $\sum_{\alpha\in\mathcal{A}} I_{x_\alpha}$ of E is, in fact, an internal direct sum $\bigoplus_{\alpha\in\mathcal{A}} I_{x_\alpha}$. Since the disjoint system is maximal, $\bigoplus_{\alpha\in\mathcal{A}} I_{x_\alpha}$ is an order dense ideal of E.

It follows easily from equation (3.1) that the P_{x_α} are also mutually disjoint. They even form a maximal disjoint system in Orth(E). To see this, suppose that $T \in$ Orth(E) is such that $|T| \wedge P_{x_\alpha} = 0$ for all $\alpha \in \mathcal{A}$. Then $(|T|x_\alpha) \wedge x_\alpha = (|T| \wedge P_{x_\alpha})x_\alpha = 0$ for all $\alpha \in \mathcal{A}$. Since $|T|$ is band preserving, this implies that $|T|x_\alpha = 0$ for all $\alpha \in \mathcal{A}$. The fact that the kernel of $|T|$ is a band in E then yields that $|T| = 0$. Just as for E, we now conclude that the ideal $\sum_{\alpha\in\mathcal{A}} \mathscr{I}_{P_{x_\alpha}}$ of Orth(E) is an internal direct sum $\bigoplus_{\alpha\in\mathcal{A}} \mathscr{I}_{P_{x_\alpha}}$ that is order dense in Orth(E).

Since $\bigoplus_{\alpha\in\mathcal{A}} \psi_{x_\alpha} : \bigoplus_{\alpha\in\mathcal{A}} \mathscr{I}_{P_{x_\alpha}} \to \bigoplus_{\alpha\in\mathcal{A}} I_{x_\alpha}$ is a vector lattice isomorphism by Proposition 3.6.2, the proof is complete. □

It is generally true that a vector lattice and an order dense vector sublattice of it have isomorphic universal completions; see [6, Theorems 7.21 and 7.23]. Proposition 3.6.3 therefore implies the following.

Corollary 3.6.4. *Let E be a Dedekind complete vector lattice. Then the universal completions of E and of* Orth(E) *are isomorphic vector lattices.*

The previous result enables us to relate the countable sup property of E to that of Orth(E). We recall that vector lattice E *has the countable sup property* when, for every non-empty subset S of E that has a supremum in E, there exists an at most countable subset of S that has the same supremum in E as S. In parts of the literature, such as in [36] and [51], E is then said to be order separable. We also recall that a subset of a vector lattice is said to be an *order basis* when the band that it generates is the whole vector lattice.

Proposition 3.6.5. *Let E be a Dedekind complete vector lattice. The following are equivalent:*
(1) Orth(E) *has the countable sup property;*
(2) E has the countable sup property and an at most countably infinite order basis.

Proof. It is proved in [33, Theorem 6.2] that, for an arbitrary vector lattice F, F^{u} has the countable sup property if and only if F has the countable sup property as well as an at most countably infinite order basis. Since $\mathrm{Orth}(E)$ has a weak order unit I, we see that $\mathrm{Orth}(E)^{\mathrm{u}}$ has the countable sup property if and only if $\mathrm{Orth}(E)$ has the countable sup property. On the other hand, since $\mathrm{Orth}(E)^{\mathrm{u}}$ and E^{u} are isomorphic by Corollary 3.6.4, an application of this same result to E shows that $\mathrm{Orth}(E)^{\mathrm{u}}$ has the countable sup property if and only if E has the countable sup property and an at most countably infinite order basis. □

We conclude by giving some estimates for orthomorphisms in Proposition 3.6.7 that will be used in the sequel. As a preparation, we need the following extension of [7, Exercise 1.3.7].

Lemma 3.6.6. *Let E be a vector lattice with the principal projection property. Take $x, y \in E$. For $\lambda \in \mathbb{R}$, let P_λ denote the order projection in E onto the band generated by $(x-\lambda y)^+$. Then $\lambda P_\lambda y \leq P_\lambda x$. When $x, y \in E^+$ and $\lambda \geq 0$, then $x \leq \lambda y + P_\lambda x$.*

Proof. The first inequality follows from the fact that

$$0 \leq P_\lambda(x-\lambda y)^+ = P_\lambda(x-\lambda y) = P_\lambda x - \lambda P_\lambda y.$$

For the second inequality, we note that $x-\lambda y \leq (x-\lambda y)^+ = P_\lambda(x-\lambda y)^+$ for all x, y, and λ. When $x, y \in E^+$ and $\lambda \geq 0$, then $(x-\lambda y)^+ \leq x^+ = x$, so that

$$x \leq \lambda y + P_\lambda(x-\lambda y)^+ \leq \lambda y + P_\lambda x.$$

□

Proposition 3.6.7. *Let E be a Dedekind complete vector lattice, and let $T \in \mathrm{Orth}(E)^+$. For $\lambda > 0$, let $\mathcal{P}_\lambda$ be the order projection in $\mathrm{Orth}(E)$ onto the band generated by $(T-\lambda I)^+$ in $\mathrm{Orth}(E)$. There exists a unique order projection P_λ in E such that $\mathcal{P}_\lambda(S) = P_\lambda S$ for all $S \in \mathrm{Orth}(E)$. Furthermore:*

(1) $\lambda P_\lambda \leq P_\lambda T \leq T$;
(2) $T \leq \lambda I + P_\lambda T$;
(3) $(P_\lambda T x) \wedge y \leq \frac{1}{\lambda} T y$ *for all* $x, y \in E^+$.

Proof. Since $0 \leq \mathcal{P}_\lambda \leq I_{\mathrm{Orth}(E)}$, it follows from [6, Theorem 2.62] that there exists a unique $P_\lambda \in \mathrm{Orth}(E)$ with $0 \leq P_\lambda \leq I$ such that $\mathcal{P}_\lambda(S) = P_\lambda S$ for all $S \in \mathrm{Orth}(E)$. The fact that $\mathcal{P}_\lambda$ is idempotent implies that P_λ is also idempotent. Hence P_λ is an order projection.

The inequalities in the parts (1) and (2) are then a consequence of those in Lemma 3.6.6. For part (3), we note that $(P_\lambda T x) \wedge y$ is in the image of the projection P_λ. Since order projections are vector lattice homomorphisms, we have, using part (1) in the final step, that

$$(P_\lambda T x) \wedge y = P_\lambda((P_\lambda T x) \wedge y) = (P_\lambda^2 T x) \wedge P_\lambda y \leq P_\lambda y \leq \frac{1}{\lambda} T y.$$

□

3.7 Continuity properties of orthomorphisms

Orthomorphisms preserve order convergence of nets. In this short section, we show that they also preserve unbounded order convergence and, when applicable, convergence in the (necessarily unique) Hausdorff uo-Lebesgue topology.

Before doing so, let us note that this is in contrast to the case of general order bounded operators. Surely, there exist order bounded operators that are not order continuous. For the remaining two convergence structures, we consider ℓ_1 with its standard basis $(e_n)_{n=1}^{\infty}$. It follows from [28, Corollary 3.6] that $e_n \xrightarrow{\text{uo}} 0$. There are several ways to see that ℓ_1 admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_{\ell_1}$. This follows from the fact that its norm is order continuous (see [44, p. 993]), from the fact that it is atomic (see [44, Lemma 7.4]), and from a result in the context of measure spaces (see [20, Theorem 6.2]). The latter two results also show that $e_n \xrightarrow{\widehat{\tau}_{\ell_1}} 0$ which is, of course, also a consequence of the fact that $e_n \xrightarrow{\text{uo}} 0$. Define $T : \ell_1 \to \ell_1$ by setting $Tx := \left(\sum_{n=1}^{\infty} x_n\right) e_1$ for $x = \sum_{n=1}^{\infty} x_n e_n \in \ell_1$. Since $Te_n = e_1$ for all $n \geq 1$, the order continuous positive operator T on ℓ_1 preserves neither uo-convergence nor $\widehat{\tau}_{\ell_1}$-convergence of sequences in ℓ_1.

Proposition 3.7.1. *Let E be a Dedekind complete vector lattice, and let $T \in \mathrm{Orth}(E)$. Suppose that $(x_\alpha)_{\alpha \in \mathcal{A}}$ is a net in E such that $x_\alpha \xrightarrow{\text{uo}} 0$ in E. Then $Tx_\alpha \xrightarrow{\text{uo}} 0$ in E.*

Proof. Using equation (3.2), one easily sees that we may suppose that T and the x_α are positive. For $n \geq 1$, we let $\mathcal{P}_n$ be the order projection in $\mathrm{Orth}(E)$ onto the band generated by $(T - nI)^+$ in $\mathrm{Orth}(E)$. According to Proposition 3.6.7, there exists a unique order projection P_n in E such that $\mathcal{P}_n(S) = P_n S$ for all $S \in \mathrm{Orth}(E)$. Take $e \in E^+$. By applying part (2) of Proposition 3.6.7 in the first step and its part (3) in the third, we see that, for $\alpha \in \mathcal{A}$ and $n \geq 1$,

$$
\begin{aligned}
(Tx_\alpha) \wedge e &\leq (nx_\alpha + P_n T x_\alpha) \wedge e \\
&\leq n(x_\alpha \wedge e) + P_n T x_\alpha \wedge e \\
&\leq n(x_\alpha \wedge e) + \frac{1}{n} Te.
\end{aligned}
\tag{3.3}
$$

This implies that, for $n \geq 1$,

$$
0 \leq \inf_\alpha \sup_{\beta \geq \alpha} \left[(Tx_\beta) \wedge e\right] \leq n \inf_\alpha \sup_{\beta \geq \alpha} \left[x_\beta \wedge e\right] + \frac{1}{n} Te.
$$

Since $x_\alpha \wedge e \xrightarrow{\text{o}} 0$ in E, it now follows from [28, Remark 2.2] that

$$
0 \leq \inf_\alpha \sup_{\beta \geq \alpha} \left[(Tx_\beta) \wedge e\right] \leq \frac{1}{n} Te
$$

for all $n \geq 1$. Hence $\inf_\alpha \sup_{\beta \geq \alpha} \left[(Tx_\beta) \wedge e\right] = 0$, and we conclude that $(Tx_\alpha) \wedge e \xrightarrow{\text{o}} 0$ in E. Since $e \in E^+$ was arbitrary, the proof is complete. $\square$

For the case of a Hausdorff uo-Lebesgue topology, we need the following preparatory result that has some independent interest. Lemma 3.9.11 is of the same flavour.

Proposition 3.7.2. *Let E be a Dedekind complete vector lattice that admits a (not necessarily Hausdorff) locally solid linear topology τ_E, and let $T \in \mathrm{Orth}(E)$. Suppose that $(x_\alpha)_{\alpha \in \mathcal{A}}$ is a net in E such that $x_\alpha \xrightarrow{\tau_E} 0$ in E. Then $Tx_\alpha \xrightarrow{\mathrm{u}_E\tau_E} 0$ in E.*

Proof. As in the proof of Proposition 3.7.1, we may suppose that T and the x_α are positive. For $n \geq 1$, we let $\mathcal{P}_n$ be the order projection in $\mathrm{Orth}(E)$ onto the band generated by $(T-nI)^+$ in $\mathrm{Orth}(E)$ again, so that again there exists a unique order projection P_n in E such that $\mathcal{P}_n(S) = P_n S$ for all $S \in \mathrm{Orth}(E)$. Fix $e \in E^+$. Take a solid τ_E-neighbourhood U of 0 in E, and choose a τ_E-neighbourhood V of 0 such that $V + V \subseteq U$. Take an $n_0 \geq 1$ such that $Te/n_0 \in V$. As $x_\alpha \xrightarrow{\tau_E} 0$, there exists an $\alpha_0 \in \mathcal{A}$ such that $n_0 x_\alpha \in V$ for all $\alpha \geq \alpha_0$. Continuing the chain of inequalities in equation (3.3) for n_0 for one more step, we see that, for all $\alpha \geq \alpha_0$,

$$\begin{aligned}(Tx_\alpha) \wedge e &\leq n_0(x_\alpha \wedge e) + \frac{1}{n_0} Te \\ &\leq n_0 x_\alpha + \frac{1}{n_0} Te \\ &\in V + V \subseteq U\end{aligned} \tag{3.4}$$

The solidness of V then implies that $(Tx_\alpha) \wedge e \in U$ for all $\alpha \geq \alpha_0$. Since U and e were arbitrary, we conclude that $T_\alpha x \xrightarrow{\mathrm{u}_E\tau_E} 0$. ☐

Since the unbounded topology $\mathrm{u}_E\widehat{\tau}_E$ that is generated by a Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$ equals $\widehat{\tau}_E$ again, the following is now clear.

Corollary 3.7.3. *Let E be a Dedekind complete vector lattice that admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$, and let $T \in \mathrm{Orth}(E)$. Suppose that $(x_\alpha)_{\alpha \in \mathcal{A}}$ is a net in E such that $x_\alpha \xrightarrow{\widehat{\tau}_E} 0$ in E. Then $Tx_\alpha \xrightarrow{\widehat{\tau}_E} 0$ in E.*

3.8 Topologies on Orth(E)

Let E be a Dedekind complete vector lattice, and suppose that τ_E is a (not necessarily Hausdorff) locally solid additive topology on E. Take a non-empty subset S of E. According to Theorem 3.3.1, there exists a unique additive topology $\mathrm{ASOT}_S\tau_E$ on $\mathscr{L}_{\mathrm{ob}}(E)$ such that, for a net $(T_\alpha)_{\alpha \in \mathcal{A}}$ in $\mathscr{L}_{\mathrm{ob}}(E)$, $T_\alpha \xrightarrow{\mathrm{ASOT}_S\tau_E} 0$ if and only if $|T_\alpha||s| \xrightarrow{\tau_E} 0$ for all $s \in S$. When $(T_\alpha)_{\alpha \in \mathcal{A}} \subseteq \mathrm{Orth}(E)$, equation (3.2) and the local solidness of τ_E imply that this convergence criterion is also equivalent to the one that $T_\alpha s \xrightarrow{\tau_F} 0$ for all $s \in S$. Hence *on subsets of* Orth(E), *an absolute strong operator topology that is generated by a locally solid additive topology on E coincides with the corresponding strong operator topology*. In order to remind ourselves of the connection with the topology on the enveloping vector lattice $\mathscr{L}_{\mathrm{ob}}(E)$ of Orth(E), we shall keep writing $\mathrm{ASOT}_S\tau_F$ when considering the restriction of this topology to subsets of $\mathrm{Orth}(E)$, rather than switch to, e.g., $\mathrm{SOT}_S\tau_F$.

The above observation can be used in several results in Section 3.3. For the ease of reference, we include the following consequence of Corollary 3.3.5.

Corollary 3.8.1. *Let E be a Dedekind complete vector lattice, and let τ_E be a (not necessarily Hausdorff) locally solid linear topology on E. Take a vector sublattice $\mathscr{E}$ of* $\mathrm{Orth}(E)$ *and a non-empty subset S of E.*

There exists a unique locally solid linear topology $\mathrm{ASOT}_S\tau_E$ *on $\mathscr{E}$ such that, for a net $(T_\alpha)_{\alpha\in\mathcal{A}}$ in $\mathscr{E}$, $T_\alpha \xrightarrow{\mathrm{ASOT}_S\tau_E} 0$ if and only if $T_\alpha s \xrightarrow{\tau_E} 0$ for all $s \in S$.*

Let I_S be the ideal of E that is generated by S. For a net $(T_\alpha)_{\alpha\in\mathcal{A}}$ in $\mathscr{E}$, $T_\alpha \xrightarrow{\mathrm{ASOT}_S\tau_E} 0$ if and only if $T_\alpha x \xrightarrow{\tau_E} 0$ for all $x \in I_S$.

When τ_E is a Hausdorff topology on F, then $\mathrm{ASOT}_S\tau_E$ *is a Hausdorff topology on $\mathscr{E}$ if and only if I_S separates the points of $\mathscr{E}$.*

According to the next result, there is an intimate relation between the existence of Hausdorff o-Lebesgue topologies and uo-Lebesgue topologies on E and on $\mathrm{Orth}(E)$.

Proposition 3.8.2. *Let E be a Dedekind complete vector lattice. The following are equivalent:*
(1) E admits a Hausdorff o-Lebesgue topology;
(2) $\mathrm{Orth}(E)$ *admits a Hausdorff o-Lebesgue topology;*
(3) E admits a (necessarily unique) Hausdorff uo-Lebesgue topology;
(4) $\mathrm{Orth}(E)$ *admits a (necessarily unique) Hausdorff uo-Lebesgue topology.*

Proof. As E and $\mathrm{Orth}(E)$ are Dedekind complete, they are not just order dense vector sublattices of their universal completions but even order dense ideals; see [7, p.126–127]. Since these universal completions are isomorphic vector lattices by Corollary 3.6.4, the proposition follows from a double application of [20, Theorem 4.9.(3)]. □

For a Dedekind complete vector lattice E, $\mathrm{Orth}(E)$, being a band in $\mathscr{L}_{\mathrm{ob}}(E)$, is a regular vector sublattice of $\mathscr{L}_{\mathrm{ob}}(E)$. A regular vector sublattice $\mathscr{E}$ of $\mathrm{Orth}(E)$ is, therefore, also a regular vector sublattice of $\mathscr{L}_{\mathrm{ob}}(E)$, and Proposition 3.4.2 then shows how o-Lebesgue topologies on $\mathscr{E}$ can be obtained from an o-Lebesgue topology on E as (absolute) strong operator topologies. In particular, this makes the fact that part (1) of Proposition 3.8.2 implies its part (2) more concrete. The fact that part (1) implies part (2) is made more concrete as a special case of the following consequence of Theorem 3.4.3.

Theorem 3.8.3. *Let E be a Dedekind complete vector lattice. Suppose that E admits an o-Lebesgue topology τ_E. Take a regular vector sublattice $\mathscr{E}$ of* $\mathrm{Orth}(E)$*, a non-empty subset $\mathscr{S}$ of $\mathscr{E}$, and a non-empty subset S of E.*

Then $\mathrm{u}_{\mathscr{S}}\mathrm{ASOT}_S\tau_E$ *is a uo-Lebesgue topology on $\mathscr{E}$.*

We let I_S denote the ideal of E that is generated by S, and $I_{\mathscr{S}}$ the ideal of $\mathscr{E}$ that is generated by $\mathscr{S}$. For a net $(T_\alpha)_{\alpha\in\mathcal{A}}$ in $\mathscr{E}$, the following are equivalent:
(1) $T_\alpha \xrightarrow{\mathrm{u}_{\mathscr{S}}\mathrm{ASOT}_S\tau_E} 0$;
(2) $|T_\alpha s| \wedge |Ts| \xrightarrow{\tau_E} 0$ for all $T \in \mathscr{S}$ and $s \in S$;

(3) $|T_\alpha x| \wedge |Tx| \xrightarrow{\tau_E} 0$ *for all* $T \in I_{\mathscr{S}}$ *and* $x \in I_S$.

Suppose that τ_E *is actually a* Hausdorff *o-Lebesgue topology* $\widehat{\tau}_E$ *on* E. *Then the following are equivalent:*

(1) $\mathrm{u}_{\mathscr{S}}\mathrm{ASOT}_S\widehat{\tau}_E$ *is a (necessarily unique)* Hausdorff *uo-Lebesgue topology on* $\mathscr{E}$*;*

(2) I_S *separates the points of* $\mathscr{E}$ *and* $I_{\mathscr{S}}$ *is order dense in* $\mathscr{E}$.

In that case, the Hausdorff uo-Lebesgue topology $\mathrm{u}_{\mathscr{S}}\mathrm{ASOT}_S\tau_E$ *on* $\mathscr{E}$ *is the restriction of the (necessarily unique) Hausdorff uo-Lebesgue topology on* $\mathscr{L}_{\mathrm{ob}}(E,F)$*, i.e., of* $\mathrm{u}_{\mathscr{L}_{\mathrm{ob}}(E,F)}\mathrm{ASOT}_E\tau_E$*, and the criteria in (1), (2), and (3) are also equivalent to:*

(4) $(|T_\alpha| \wedge |T|)x \xrightarrow{\widehat{\tau}_E} 0$ *for all* $T \in \mathscr{L}_{\mathrm{ob}}(E)$ *and* $x \in E$.

3.9 Comparing uniform and strong convergence structures on Orth(E)

Let E and F be vector lattices, where F is Dedekind complete, and let $(T_\alpha)_{\alpha\in\mathcal{A}}$ be a net in $\mathscr{L}_{\mathrm{ob}}(E,F)$. In Section 3.5, we studied the relation between uniform and strong convergence of $(T_\alpha)_{\alpha\in\mathcal{A}}$ for order convergence, unbounded order convergence, and—when applicable—convergence in a Hausdorff uo-Lebesgue topology. In the present section, we consider the case where $(T_\alpha)_{\alpha\in\mathcal{A}}$ is actually contained in Orth(E). As we shall see, the relation between uniform and strong convergence is now much more symmetrical than in the general case of Section 3.5; see Theorem 3.9.4 (and Theorem 3.9.7), Theorem 3.9.9, and Theorem 3.9.12, below.

These positive results might, perhaps, lead one to wonder whether some of the three uniform convergence structures under consideration might actually even be identical for the orthomorphisms. This, however, is not the case. There even exist sequences of positive orthomorphisms on separable reflexive Banach lattices with weak order units showing that the two 'reverse' implications in question are not generally valid. For this, we consider $E := \mathrm{L}_p([0,1])$ for $1 < p < \infty$. In that case, Orth(E) can canonically be identified with $\mathrm{L}_\infty([0,1])$ as an f-algebra; see [7, Example 2.67], for example. The uo-convergence of a net in the regular vector sublattice $\mathrm{L}_\infty([0,1])$ of $\mathrm{L}_0([0,1])$ coincides with that in $\mathrm{L}_0([0,1])$ which, according to [28, Proposition 3.1], is simply convergence almost everywhere in the case of sequences. According to [20, Theorem 6.3], the convergence of a net in the Hausdorff uo-Lebesgue topology of $\mathrm{L}_\infty([0,1])$ is equal to the convergence in measure. For $n \geq 1$, set $f_n := n\chi_{[0,1/n]}$. Then $f_n \xrightarrow{\mathrm{uo}} 0$ in $\mathrm{L}_\infty([0,1])$, but it is not true that $f_n \xrightarrow{\mathrm{o}} 0$ in $\mathrm{L}_\infty([0,1])$ since the f_n are not even order bounded in $\mathrm{L}_\infty([0,1])$. Using $\chi_{[(k-1)2^{-n},k2^{-n}]}$ for $n \geq 1$ and $k = 1,\dots,2^n$, one easily finds a sequence that is convergent to zero in measure, but that is not convergent in any point of $[0,1]$.

We now start with uniform and strong order convergence for nets of orthomorphisms. For this, we need a few preparatory results. The first one is on general order continuous operators.

Lemma 3.9.1. *Let* E *be a Dedekind complete vector lattice, let* $(T_\alpha)_{\alpha\in\mathcal{A}}$ *be a decreasing net in* $\mathscr{L}_{\mathrm{oc}}(E)^+$*, and let* F *be an order dense vector sublattice of* E. *The following are equivalent:*

(1) $T_\alpha x \xrightarrow{\mathrm{o}} 0$ *in* E *for all* $x \in F$*;*

(2) $T_\alpha x \xrightarrow{\mathrm{o}} 0$ *in* E *for all* $x \in E$.

Proof. We need to show only that part (1) implies part (2). Suppose that $T_\alpha x \xrightarrow{\mathrm{o}} 0$ in E for all $x \in F$. By passing to a tail, we may suppose that there exists a $T \in \mathscr{L}_{\mathrm{oc}}(E)^+$ such that $T_\alpha \leq T$ for $\alpha \in \mathcal{A}$. Take $x \in E^+$. Since $(T_\alpha x)_{\alpha \in \mathcal{A}}$ is directed downwards and E is Dedekind complete, there exists a $y \in E^+$ such that $T_\alpha x \downarrow y$ in E. The order denseness of F in E implies that there exists a net $(x_\beta)_{\beta \in \mathcal{B}} \subseteq F^+$ with $x_\beta \uparrow x$ in E. For each $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$, we have

$$\begin{aligned} y \leq T_\alpha x &= T_\alpha(x - x_\beta) + T_\alpha x_\beta \\ &\leq T(x - x_\beta) + T_\alpha x_\beta. \end{aligned}$$

For each fixed $\beta \in \mathcal{B}$, the assumption then implies that

$$y \leq T(x - x_\beta) + \inf_\alpha T_\alpha x_\beta = T(x - x_\beta).$$

The order continuity of T then shows that

$$0 \leq y \leq \inf_\beta T(x - x_\beta) = 0,$$

and so $y = 0$. We conclude that $T_\alpha x \downarrow 0$ in E for every $x \in E^+$, and the statement in part (2) follows. □

Proposition 3.9.2. *Let* E *be a Dedekind complete vector lattice, let* $(T_\alpha)_{\alpha \in \mathcal{A}}$ *be a decreasing net in* $\mathrm{Orth}(E)^+$, *and let* S *be a non-empty subset of* E. *The following are equivalent:*
(1) $T_\alpha s \xrightarrow{\mathrm{o}} 0$ *in* E *for all* $s \in S$;
(2) $T_\alpha x \xrightarrow{\mathrm{o}} 0$ *in* E *for all* $x \in B_S$.
In particular, if E *has a positive weak order unit* e, *then* $T_\alpha x \xrightarrow{\mathrm{o}} 0$ *in* E *for all* $x \in E$ *if and only if* $T_\alpha e \downarrow 0$ *in* E.

Proof. We need to show only that part (1) implies part (2). Take $y \in I_S^+$. There exist $s_1, \ldots, s_n \in S$ and $\lambda_1, \ldots, \lambda_n \geq 0$ such that $0 \leq y \leq \sum_{i=1}^n \lambda_i |s_i|$. Hence $0 \leq T_\alpha y \leq \sum_{i=1}^n \lambda_i T_\alpha |s_i| = \sum_{i=1}^n \lambda_i |T_\alpha s_i|$ for $\alpha \in \mathcal{A}$, and the assumption then implies that $T_\alpha y \downarrow 0$ in E. Since orthomorphisms preserve bands, we have $T_\alpha y \in B_S$ for all α, and the fact that B_S is an ideal of E now shows that $T_\alpha y \downarrow 0$ in B_S. It follows that $T_\alpha y \xrightarrow{\mathrm{o}} 0$ in B_S for all $y \in I_S$. Since the restriction of each T_α to the regular vector sublattice B_S of E is again order continuous, and since I_S is an order dense vector sublattice of the vector lattice B_S, Lemma 3.9.1 implies that $T_\alpha y \xrightarrow{\mathrm{o}} 0$ in B_S for all $y \in B_S$. The fact that B_S is a regular vector sublattice of E then yields that $T_\alpha y \xrightarrow{\mathrm{o}} 0$ in E for all $y \in B_S$. □

Lemma 3.9.3. *Let* E *be a Dedekind complete vector lattice, and let* $\mathscr{S}$ *be a subset of* $\mathrm{Orth}(E)$ *that is bounded above in* $\mathscr{L}_{\mathrm{ob}}(E)$. *Then, for* $x \in E^+$,

$$\left(\bigvee_{T \in \mathscr{S}} T \right) x = \bigvee_{T \in \mathscr{S}} Tx.$$

Proof. Using [6, Theorem 1.67.(b)] in the second step, we see that, for $x \in E^+$,

$$\Big(\bigvee_{T\in\mathscr{S}} T\Big)x = \Big(\bigvee_{T^\vee\in\mathscr{S}^\vee} T^\vee\Big)x = \bigvee_{T^\vee\in\mathscr{S}^\vee} T^\vee x.$$

By equation (3.1), this equals

$$\bigvee_{y^\vee\in(\mathscr{S}x)^\vee} y^\vee = \bigvee_{y\in\mathscr{S}x} y = \bigvee_{T\in\mathscr{S}} Tx.$$

□

We can now establish our main result regarding uniform and strong order convergence for nets of orthomorphisms.

Theorem 3.9.4. *Let E be a Dedekind complete vector lattice, and let $(T_\alpha)_{\alpha\in\mathcal{A}}$ be a net in* Orth(E) *that is order bounded in $\mathscr{L}_{\mathrm{ob}}(E)$. Let S be a non-empty subset of E with $B_S = E$. The following are equivalent:*
(1) $T_\alpha \xrightarrow{\mathrm{o}} 0$ in Orth(E)*;*
(2) $T_\alpha \xrightarrow{\mathrm{o}} 0$ in $\mathscr{L}_{\mathrm{ob}}(E)$;
(3) $T_\alpha s \xrightarrow{\mathrm{o}} 0$ in E for all $s \in S$;
(4) $T_\alpha x \xrightarrow{\mathrm{o}} 0$ in E for all $x \in E$.
In particular, when E has a weak order unit e, then $T_\alpha \xrightarrow{\mathrm{o}} 0$ in $\mathscr{L}_{\mathrm{ob}}(E)$ if and only if $T_\alpha e \xrightarrow{\mathrm{o}} 0$ in E.

Before proceeding with the proof, we remark that, since Orth(E) is a projection band in $\mathscr{L}_{\mathrm{ob}}(E)$, the order boundedness of the net could equivalently have been required in Orth(E).

Proof. Since the net $(T_\alpha)_{\alpha\in\mathcal{A}}$ is supposed to be order bounded in the regular vector sublattice Orth(E), the equivalence of the parts (1) and (2) follows from [28, Corollary 2.12]. Lemma 3.4.1 shows that part (2) implies part (4), and evidently part (4) implies part (3). The proof will be completed by showing that part (3) implies part (1). Suppose that $T_\alpha s \xrightarrow{\mathrm{o}} 0$ in E for all $s \in S$ or, equivalently, that $|T_\alpha||s| = |T_\alpha s| \xrightarrow{\mathrm{o}} 0$ in E for all $s \in S$. For $\alpha \in \mathcal{A}$, set $\widetilde{T}_\alpha := \bigvee_{\beta\geq\alpha}|T_\beta|$ in $\mathscr{L}_{\mathrm{ob}}(E)$. Since Lemma 3.9.3 shows that $\widetilde{T}_\alpha|s| = \bigvee_{\beta\geq\alpha}|T_\beta||s|$ for $\alpha \in \mathcal{A}$ and $s \in S$, we see that $\widetilde{T}_\alpha|s| \downarrow 0$ in E for all $s \in S$. Proposition 3.9.2 then yields that $\widetilde{T}_\alpha x \xrightarrow{\mathrm{o}} 0$ for all $x \in B_{|S|} = E$. Using that $\widetilde{T}_\alpha \downarrow$, it follows that $\widetilde{T}_\alpha \downarrow 0$ in $\mathscr{L}_{\mathrm{ob}}(E)$. Since $|T_\alpha| \leq \widetilde{T}_\alpha$ for $\alpha \in \mathcal{A}$, we see that $|T_\alpha| \xrightarrow{\mathrm{o}} 0$ in $\mathscr{L}_{\mathrm{ob}}(E)$, as required. □

In view of Lemma 3.4.1, the most substantial part of Theorem 3.9.4 is the fact that the parts (3) and (4) imply the parts (1) and (2). For this to hold in general, the assumption that $(T_\alpha)_{\alpha\in\mathcal{A}}$ be order bounded is actually necessary. To see this, let Γ be an uncountable set that is supplied with the counting measure, and consider $E := \ell_p(\Gamma)$ for $1 \leq p \leq \infty$. Set

$$\mathcal{A} := \{\,(n, S) : n \geq 1,\ S \subset \Gamma \text{ is at most countably infinite}\,\}$$

and, for (n_1, S_1), $(n_2, S_2) \in \mathcal{A}$, say that $(n_1, S_2) \leq (n_2, S_2)$ when $n_1 \leq n_2$ and $S_1 \subseteq S_2$. For $(n, S) \in \mathcal{A}$, define $T_{(n,S)} \in \mathscr{Z}(E) = \mathrm{Orth}(E)$ by setting

$$T_{(n,S)}x := n\chi_{\Gamma\setminus S}x$$

for all $x : \Gamma \to \mathbb{R}$ in E. Take an $x \in E$. Then the net $(T_{(n,S)}x)_{(n,S)\in\mathcal{A}}$ has a tail that is identically zero, namely $(T_{(n,S)}x)_{(n,S)\geq(1,\mathrm{supp}\,x)}$. Hence $T_{(n,S)}x \xrightarrow{\mathrm{o}} 0$ in E for all $x \in E$. We claim that $(T_{(n,S)})_{(n,S)\in\mathcal{A}}$ is not order convergent in $\mathrm{Orth}(E)$, and not even in $\mathscr{L}_{\mathrm{ob}}(E)$. For this, it is sufficient to show that it does not have any tail that is order bounded in $\mathscr{L}_{\mathrm{ob}}(E)$. Suppose, to the contrary, that there exist an $n_0 \geq 1$, an at most countably infinite subset S_0 of Γ, and a $T \in \mathscr{L}_{\mathrm{ob}}(E)$ such that $T_{(n,A)} \leq T$ for all $(n, A) \in \mathcal{A}$ with $n \geq n_0$ and $A \supseteq A_0$. As Γ is uncountable, we can choose an $x_0 \in \Gamma \setminus A_0$; we let e_{x_0} denote the corresponding atom in E. Then, in particular, $T_{(n,A_0)}e_{x_0} \leq Te_{x_0}$ for all $n \geq n_0$. Hence $Te_{x_0} \geq ne_{x_0}$ for all $n \geq n_0$, which is impossible.

We now consider uniform and strong order convergence in the case where E is a Dedekind complete Banach lattice. In that case, a version of Theorem 3.9.4 can be obtained for sequences where the order boundedness of the sequence need to be a part of the hypotheses because it is automatic; see Theorem 3.9.7, below. Our results are based on the following ordered version of the uniform boundedness principle for orthomorphisms.

Proposition 3.9.5. *Let E be a Dedekind complete Banach lattice, and let $\{ T_\alpha : \alpha \in \mathcal{A} \}$ be a non-empty subset of* $\mathrm{Orth}(E)$*. The following are equivalent:*
(1) $\{ T_\alpha : \alpha \in \mathcal{A} \}$ is an order bounded subset of $\mathscr{L}_{\mathrm{ob}}(E)$;
(2) for each $x \in E$, $\{ T_\alpha x : \alpha \in \mathcal{A} \}$ is an order bounded subset of E.

As in Theorem 3.9.4, the order boundedness of the net could equivalently have been stated in $\mathrm{Orth}(E)$.

Proof. It is trivial that part (1) implies part (2). We give two proofs for the fact that part (2) implies part (1).

The first proof is as follows. The fact that $|T_\alpha||x| = |T_\alpha x|$ implies that we may suppose that the T_α are positive. Suppose, to the contrary, that $\{ T_\alpha : \alpha \in \mathcal{A} \}$ is not an order bounded subset of $\mathscr{L}_{\mathrm{ob}}(E)$. Using that $\mathrm{Orth}(E) = \mathscr{Z}(E)$, it is easy to see that, for every $n \geq 1$, there exists an $\alpha_n \in \mathcal{A}$ such that $(T_{\alpha_n} - 2^n I)^+ > 0$. For $n \geq 1$, we let B_n be the band generated by $(T_{\alpha_n} - 2^n I)^+$ in $\mathrm{Orth}(E)$, and we let $\mathcal{P}_n$ be the corresponding non-zero order projection onto B_n. According to Proposition 3.6.7, there exists a unique order projection P_n in E such that $\mathcal{P}_n S = P_n S$ for all $S \in \mathrm{Orth}(E)$. Furthermore, $T_{\alpha_n} \geq 2^n P_n$. As $P_n \neq 0$, we can choose an $x_n \in E^+$ such that $\|P_n x_n\| = 1/2^n$. Since $\bigvee_{n=1}^m P_n x_n \leq \sum_{n=1}^\infty P_n x_n$ for all $m \geq 1$, we can set $e := \bigvee_{n=1}^\infty P_n x_n \in E^+$. By assumption, we can choose an upper bound x of $\{ T_{\alpha_n} e : n \geq 1 \}$ in E^+. Then

$$x \geq T_{\alpha_n} e \geq 2^n P_n e \geq 2^n P_n(P_n x_n) = 2^n P_n x_n$$

for $n \geq 1$. Again by assumption, we can choose an upper bound y of $\{ T_{\alpha_n} x : n \geq 1 \}$ in E^+, and then

$$y \geq T_{\alpha_n} x \geq 2^n P_n x \geq 2^n P_n(2^n P_n x_n) = 4^n P_n x_n$$

for $n \geq 1$. This implies that $\|y\| \geq 4^n \|P_n x_n\| = 2^n$ for all n. This contradiction completes the first proof.

The second proof, which uses somewhat 'heavier' auxiliary results, is as follows. The fact that the T_α are pointwise order bounded implies that they are pointwise norm bounded. Hence, by the uniform boundedness principle, the T_α are bounded in the uniform norm on the bounded operators on E. Since they are in $\mathrm{Orth}(E) = \mathscr{Z}(E)$, where (see Section 3.6) the operator norm agrees with the order unit norm with respect to the strong order unit I of $\mathscr{Z}(E)$, the T_α are also order bounded in $\mathscr{Z}(E)$. $\square$

As a side result, we note the following consequence of Proposition 3.9.5. It is an ordered analogue of the familiar result for a sequence of bounded operators on a Banach space.

Corollary 3.9.6. *Let E be a Dedekind complete Banach lattice, and let $(T_n)_{n=1}^\infty$ be a sequence in* $\mathrm{Orth}(E)$*. Suppose that the sequence $(T_n x)_{n=1}^\infty$ is order convergent in E for all $x \in E$. Then $\{ T_n : n \geq 1 \}$ is an order bounded subset of $\mathscr{L}_{\mathrm{ob}}(E)$. For $x \in E$, define $T : E \to E$ by setting*

$$Tx := \mathrm{o} - \lim_{n \to \infty} T_n x.$$

Then $T \in$ $\mathrm{Orth}(E)$*.*

Proof. It is clear that T is linear. Since order convergent sequences are order bounded, Proposition 3.9.5 shows that there exist an $S \in \mathrm{Orth}(E)$ such that $|T_n| \leq |S|$ for $n \geq 1$. As $\mathrm{Orth}(E) = \mathscr{Z}(E)$, there exists a $\lambda \geq 0$ such that $|T_n| \leq \lambda I$ for $n \geq 1$. Using equation (3.2), one then easily sees that $|Tx| \leq \lambda |x|$ for $x \in E$. Hence $T \in \mathscr{Z}(E) = \mathrm{Orth}(E)$. $\square$

Using Theorem 3.9.4 and the order boundedness statement in Corollary 3.9.6, the following is easily established. As announced above, there is no order boundedness in the hypotheses.

Theorem 3.9.7. *Let E be a Dedekind complete Banach lattice, and let $(T_n)_{n=1}^\infty$ be a sequence in* $\mathrm{Orth}(E)$*. Let S be a non-empty subset of E such that $I_S = E$. The following are equivalent:*

(1) $T_n \xrightarrow{\mathrm{o}} 0$ in $\mathrm{Orth}(E)$*;*

(2) $T_n \xrightarrow{\mathrm{o}} 0$ in $\mathscr{L}_{\mathrm{ob}}(E)$;

(3) $T_n s \xrightarrow{\mathrm{o}} 0$ in E for all $s \in S$;

(4) $T_n x \xrightarrow{\mathrm{o}} 0$ in E for all $x \in E$.

In particular, when E has a strong order unit e, then $T_n \xrightarrow{\mathrm{o}} 0$ in $\mathrm{Orth}(E)$ *if and only if $T_n e \xrightarrow{\mathrm{o}} 0$ in E.*

Remark 3.9.8. In Theorem 3.9.7, the condition that $I_S = E$ cannot be relaxed to $B_S = E$. To see this, we choose $E := c_0$ and set $e := \bigvee_{n \geq 1} e_i / i^2$, where $(e_i)_{i=1}^\infty$ is the standard unit basis of E. It is clear that $B_e = E$. For $n \geq 1$, there exists a unique $T_n \in \mathrm{Orth}(E)$ such that, for $i \geq 1$, $T_n e_i = n e_i$ when $i = n$, and $T_n e_i = 0$ when $i \neq n$. It is clear that $T_n e \xrightarrow{\mathrm{o}} 0$ in E. However, a consideration of $T_n(\bigvee_{i \geq 1} e_i / i)$ for $n \geq 1$ shows that $(T_n)_{n=1}^\infty$ fails to be order bounded in $\mathrm{Orth}(E)$, hence cannot be order convergent in $\mathrm{Orth}(E)$.

We continue our comparison of uniform and strong convergence structures on the orthomorphisms by considering unbounded order convergence. In that case, the result is as follows.

Theorem 3.9.9. *Let E be a Dedekind complete vector lattice, and let $(T_\alpha)_{\alpha\in\mathcal{A}}$ be a net in* $\mathrm{Orth}(E)$*. Let S be a non-empty subset of E such that $B_S = E$. The following are equivalent:*
(1) $T_\alpha \xrightarrow{\mathrm{uo}} 0$ *in* $\mathrm{Orth}\,E$*;*
(2) $T_\alpha \xrightarrow{\mathrm{uo}} 0$ *in* $\mathscr{L}_{\mathrm{ob}}(E)$*;*
(3) $T_\alpha s \xrightarrow{\mathrm{uo}} 0$ *in E for all $s \in S$;*
(4) $T_\alpha x \xrightarrow{\mathrm{uo}} 0$ *in E for all $x \in E$.*
In particular, when E has a weak order unit e, then $T_\alpha \xrightarrow{\mathrm{uo}} 0$ in $\mathrm{Orth}(E)$ *if and only if $T_\alpha e \xrightarrow{\mathrm{uo}} 0$ in E.*

Proof. Since $\mathrm{Orth}(E)$ is a regular vector sublattice of $\mathscr{L}_{\mathrm{ob}}(E)$, the equivalence of the parts (1) and (2) is clear from [28, Theorem 3.2]

We prove that part (2) implies part (4). Suppose that $T_\alpha \xrightarrow{\mathrm{uo}} 0$ in $\mathscr{L}_{\mathrm{ob}}(E)$, so that, in particular, $|T_\alpha| \wedge I \xrightarrow{\mathrm{o}} 0$ in $\mathscr{L}_{\mathrm{ob}}(E)$. Take $x \in E$. Using equation (3.1) in the second step, and Lemma 3.4.1 in the third, we have

$$(|T_\alpha||x|) \wedge |x| = (|T_\alpha||x|) \wedge (I|x|) = (|T_\alpha| \wedge I)|x| \xrightarrow{\mathrm{o}} 0.$$

Since the net $(|T_\alpha||x|)_{\alpha\in\mathcal{A}}$ is contained in the band $B_{|x|}$, it now follows from [20, Proposition 7.4] that $|T_\alpha||x| \xrightarrow{\mathrm{uo}} 0$ in E. As $|T_\alpha||x| = |T_\alpha x|$, we conclude that $T_\alpha x \xrightarrow{\mathrm{uo}} 0$ in E.

It is clear that part (4) implies part (3).

We prove that part (3) implies part (2). Suppose that $T_\alpha s \xrightarrow{\mathrm{uo}} 0$ in E for all $s \in S$, so that also $|T_\alpha||s| = |T_\alpha s| \xrightarrow{\mathrm{uo}} 0$ in E for $s \in S$. Using equation (3.1) again, we have

$$(|T_\alpha| \wedge I)|s| = (|T_\alpha||s|) \wedge |s| \xrightarrow{\mathrm{o}} 0$$

in E for all $x \in S$. In view of the order boundedness of $(|T_\alpha| \wedge I)_{\alpha\in\mathcal{A}}$, Theorem 3.9.4 then yields that $|T_\alpha| \wedge I \xrightarrow{\mathrm{o}} 0$ in $\mathscr{L}_{\mathrm{ob}}(E)$. As I is a weak order unit of $\mathrm{Orth}(E)$, [29, Lemma 3.2] (or the more general [20, Proposition 7.4]) shows that $T_\alpha \xrightarrow{\mathrm{uo}} 0$ in $\mathscr{L}_{\mathrm{ob}}(E)$. ☐

We now consider uniform and strong convergence of nets of orthomorphisms for the Hausdorff uo-Lebesgue topology. Let E be a Dedekind complete vector lattice. Suppose that E admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$. We recall from Theorem 3.8.3 that $\mathrm{Orth}(E)$ then also admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_{\mathrm{Orth}(E)}$, and that this topology equals $\mathrm{u}_{\mathrm{Orth}(E)}\mathrm{ASOT}_E\widehat{\tau}_E$. Furthermore, for a net $(T_\alpha)_{\alpha\in\mathcal{A}}$ in $\mathrm{Orth}(E)$, we have that $T_\alpha \xrightarrow{\widehat{\tau}_{\mathrm{Orth}(E)}} 0$ if and only if $|T_\alpha x| \wedge |Tx| \xrightarrow{\widehat{\tau}_E} 0$ for all $T \in \mathrm{Orth}(E)$ and $x \in E$.

We need two preparatory results.

Lemma 3.9.10. *Let E be a vector lattice that admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$. Suppose that E has a positive weak order unit e. Let $(x_\alpha)_{\alpha\in\mathcal{A}}$ be a net in E. Then $x_\alpha \xrightarrow{\widehat{\tau}_E} 0$ in E if and only if $|x_\alpha| \wedge e \xrightarrow{\widehat{\tau}_E} 0$ in E.*

Proof. We need to show only the "if"-part. Suppose that $|x_\alpha| \wedge e \xrightarrow{\widehat{\tau}_E} 0$ in E. For each $x \in E$, there exists a net $(y_\beta)_{\beta\in\mathcal{B}}$ in I_e such that $y_\beta \xrightarrow{\mathrm{o}} x$, and then certainly $y_\beta \xrightarrow{\widehat{\tau}_E} x$. Hence $\overline{I_e}^{\widehat{\tau}_E} = E$. An appeal to [44, Proposition 9.8] then shows that $x_\alpha \xrightarrow{\mathrm{u}_E\widehat{\tau}_E} 0$. Since $\mathrm{u}_E\widehat{\tau}_E = \widehat{\tau}_E$, we are done. $\square$

Our second preparatory result is in the same vein as Proposition 3.7.2.

Lemma 3.9.11. *Let E be a vector lattice with the principal projection property that admits a (not necessarily Hausdorff) o-Lebesgue topology τ_E, and let $(T_\alpha)_{\alpha\in\mathcal{A}}$ be a net in* Orth(E)*. Let S be a non-empty subset of E such that $B_S = E$. Suppose that $T_\alpha s \xrightarrow{\tau_E} 0$ for all $s \in S$. Then $T_\alpha x \xrightarrow{\mathrm{u}_E\tau_E} 0$ for all $x \in E$.*

Proof. Using equation (3.2), it follows easily that $T_\alpha x \xrightarrow{\tau_E} 0$ for all $x \in I_S$. Take an $x \in E$, and let U be a solid τ_E-neighbourhood U of 0. Choose a τ_E-neighbourhood V of 0 such that $V + V \subseteq U$. There exists a net $(x_\beta)_{\beta\in\mathcal{B}}$ in I_S such that $x_\beta \xrightarrow{\mathrm{o}} x$ in E, and then we can choose a $\beta_0 \in \mathcal{B}$ such that $|x - x_{\beta_0}| \in V$. As $|T_\alpha||x_{\beta_0}| = |T_\alpha x_{\beta_0}| \xrightarrow{\tau_E} 0$, there exists an $\alpha_0 \in \mathcal{A}$ such that $|T_\alpha||x_{\beta_0}| \in V$ for all $\alpha \geq \alpha_0$. For all $\alpha \geq \alpha_0$, we then have

$$
\begin{aligned}
0 &\leq (|T_\alpha x|) \wedge |x| = (|T_\alpha| \wedge I)|x| \\
&\leq (|T_\alpha| \wedge I)|x_{\beta_0}| + (|T_\alpha| \wedge I)|x - x_{\beta_0}| \\
&\leq |T_\alpha||x_{\beta_0}| + |x - x_{\beta_0}| \\
&\in V + V \subseteq U.
\end{aligned}
$$

As U is solid, we see that $(|T_\alpha x|) \wedge |x| \in U$ for $\alpha \geq \alpha_0$, and we conclude that $(|T_\alpha x|) \wedge |x| \xrightarrow{\tau_E} 0$. Since $|T_\alpha x| \in B_{|x|}$ for $\alpha \in \mathcal{A}$, it then follows from [44, Proposition 9.8] that $|T_\alpha x| \wedge |y| \xrightarrow{\tau_E} 0$ in E for all $y \in B_{|x|}$. As $B_{|x|}$ is a projection band in E, this holds, in fact, for all $y \in E$. $\square$

Theorem 3.9.12. *Let E be a Dedekind complete vector lattice. Suppose that E admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$, so that* Orth(E) *and $\mathscr{L}_{\mathrm{ob}}(E)$ also admit (necessarily unique) Hausdorff uo-Lebesgue topologies $\widehat{\tau}_{\mathrm{Orth}(E)}$ and $\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$, respectively. Let $(T_\alpha)_{\alpha\in\mathcal{A}}$ be a net in* Orth(E)*. Let S be a non-empty subset S of E such that $B_S = E$. The following are equivalent:*

(1) $T_\alpha \xrightarrow{\widehat{\tau}_{\mathrm{Orth}(E)}} 0$ in Orth(E)*;*

(2) $T_\alpha \xrightarrow{\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}} 0$ in $\mathscr{L}_{\mathrm{ob}}(E)$;

(3) $T_\alpha s \xrightarrow{\widehat{\tau}_E} 0$ in E for all $s \in S$;

(4) $T_\alpha x \xrightarrow{\widehat{\tau}_E} 0$ in E for all $x \in E$.

In particular, when E has a weak order unit e, then $T_\alpha \xrightarrow{\widehat{\tau}_{\mathrm{Orth}(E)}} 0$ *in* $\mathrm{Orth}(E)$ *if and only if* $T_\alpha e \xrightarrow{\widehat{\tau}_E} 0$ *in* E.

Proof. The equivalence of the parts (1) and (2) follows from the final part of Theorem 3.4.3.

We prove that part (1) implies part (4). Suppose that $T_\alpha \xrightarrow{\widehat{\tau}_{\mathrm{Orth}(E)}} 0$ in $\mathrm{Orth}(E)$. Take an $x \in E$. Then certainly $|T_\alpha x| \wedge |x| = |T_\alpha x| \wedge |Ix| \xrightarrow{\widehat{\tau}_E} 0$. The net $(T_\alpha x)_{\alpha \in \mathcal{A}}$ is contained in the band $B_{|x|}$. Since, by [44, Proposition 5.12], the regular vector sublattice $B_{|x|}$ of E also admits a (necessarily unique) Hausdorff uo-Lebesgue topology (namely, the restriction of $\widehat{\tau}_E$ to $B_{|x|}$), it then follows from Lemma 3.9.10 that $T_\alpha x \xrightarrow{\widehat{\tau}_E} 0$ in E.

We prove that part (4) implies part (1). Suppose that $T_\alpha x \xrightarrow{\widehat{\tau}_E} 0$ for all $x \in E$. Since $\widehat{\tau}_E$ is locally solid, we then also have $|T_\alpha x| \wedge |Tx| \xrightarrow{\widehat{\tau}_E} 0$ for all $T \in \mathrm{Orth}(E)$ and $x \in E$. Hence $T_\alpha \xrightarrow{\widehat{\tau}_{\mathrm{Orth}(E)}} 0$ in $\mathrm{Orth}(E)$.

It is clear that part (4) implies part (3).

Since $\mathrm{u}_E \widehat{\tau}_E = \widehat{\tau}_E$, Lemma 3.9.11 shows that part (3) implies part (4). $\square$

Chapter 4

Convergence structures and Hausdorff uo-Lebesgue topologies on vector lattice algebras of operators

Abstract

A vector sublattice of the order bounded operators on a Dedekind complete vector lattice can be supplied with the convergence structures of order convergence, strong order convergence, unbounded order convergence, strong unbounded order convergence, and, when applicable, convergence with respect to a Hausdorff uo-Lebesgue topology and strong convergence with respect to such a topology. We determine the general validity of the implications between these six convergences on the order bounded operator and on the orthomorphisms. Furthermore, the continuity of left and right multiplications with respect to these convergence structures on the order bounded operators, on the order continuous operators, and on the orthomorphisms is investigated, as is their simultaneous continuity. A number of results are included on the equality of adherences of vector sublattices of the order bounded operators and of the orthomorphisms with respect to these convergence structures. These are consequences of more general results for vector sublattices of arbitrary Dedekind complete vector lattices.

The special attention that is paid to vector sublattices of the orthomorphisms is motivated by explaining their relevance for representation theory on vector lattices.

4.1 Introduction and overview

In an earlier paper [19], the authors studied aspects of locally solid linear topologies on vector lattices of order bounded linear operators between vector lattices. Particular attention was paid to the possibility of introducing a Hausdorff uo-Lebesgue topology on such vector lattices.

Such vector lattices of operators carry at least three natural convergence structures (order convergence, unbounded order convergence, and convergence with respect to a possible Hausdorff uo-Lebesgue topology), as they can be defined for arbitrary vector lattices. For vector lattices of operators, however, besides these 'uniform' convergence structures, there are also three corresponding 'strong' counterparts that can be defined in the obvious way. Several relations between the resulting six convergence structures on vector lattices of operators were also investigated in [19]. In view of their relevance for representation theory in vector lattices, special emphasis was put on the orthomorphisms on a Dedekind complete vector lattice. In that case, implications between convergences hold that do not hold for more general vector lattices of operators. Furthermore, it was shown that the orthomorphisms are not only order continuous, but also continuous with respect to unbounded order convergence on the vector lattice and with respect to a possible Hausdorff uo-Lebesgue topology on it.

Apart from their intrinsic interest, the results in [19] can be viewed as a part of the groundwork that has to be done in order to facilitate further developments of aspects of the theory of vector lattices of operators. The questions that are asked are natural and basic, but even so the answers are often more easily formulated than proved.

In the present paper, we take this one step further and study these six convergence structures in the context of vector lattice *algebras* of order bounded linear operators on a Dedekind complete vector lattice. Also here there are many natural questions of a basic nature that need to be answered before one can expect to get much further with the theory of such vector lattice algebras and with representation theory on vector lattices. For example, is the left multiplication by a fixed element continuous on the order bounded linear operators with respect to unbounded order convergence? Is the multiplication on the order continuous linear operators simultaneously continuous with respect to a possible Hausdorff uo-Lebesgue topology on it? Given a vector lattice subalgebra of the order continuous linear operators, is the closure (we shall actually prefer to speak of the 'adherence') in the order bounded linear operators with respect to strong unbounded order convergence again a vector lattice subalgebra? Is there a condition, sufficiently lenient to be of practical relevance, under which the order adherence of a vector lattice subalgebra of the orthomorphisms coincides with its closure in a possible Hausdorff uo-Lebesgue topology? Building on [19], we shall answer these questions in the present paper, together with many more similar ones. As indicated, we hope and expect that, apart from their intrinsic interest, this may serve as a stockpile of basic, but non-elementary, results that will facilitate a further development of the theory of vector lattice algebras of operators and of representation theory in vector lattices.

This paper is organised as follows.

Section 4.2 contains the necessary notations, definitions, and conventions, as well as a few preparatory results that are of interest in their own right. Corollary 4.2.3, below, shows that, in many cases of practical interest, a unital positive linear representation of a unital f-algebra on a vector lattice is always an action by orthomorphisms. Its consequence Corollary 4.2.5, below, unifies several known results in the literature on compositions with orthomorphisms.

In Section 4.3, we study the validity of each of the 36 possible implications between the 6 convergences that we consider on vector lattice algebras of order bounded linear operators on a Dedekind complete vector lattice. We do this for the order bounded linear operators as well as for the orthomorphisms. The results that are already in [19] and a few additional ones are sufficient to complete the Tables 4.3.1 and 4.3.2, below.

Section 4.4 contains our results on the continuity of the left and right multiplications by a fixed element with respect to each of the six convergence structures on the order bounded linear operators. For this, we distinguish between the multiplication by an arbitrary order bounded linear operator, by an order continuous one, and by an orthomorphism. By giving (counter) examples, we show that our results are sharp in the sense that, whenever we state that continuity holds for multiplication by, e.g., an orthomorphism, it is no longer generally true for an arbitrary order continuous linear operator, i.e., for an operator in the 'next best class'. We also consider these questions for the orthomorphisms. The results are contained in Tables 4.4.14 to 4.4.16, below.

In Section 4.5, we investigate the simultaneous continuity of the multiplication with respect to each of the six convergence structures. When there is simultaneous continuity, the adherence of a subalgebra is, of course, again a subalgebra. With only one exception (see Corollary 4.5.6 and Example 4.5.7, below), we give (counter) examples to show that our conditions for the adherence of an algebra to be a subalgebra again are 'sharp' in the sense as indicated above for Section 4.4.

Section 4.6 is dedicated to the equality of various adherences of vector sublattices and vector lattice subalgebras. It is also indicated there how representation theory in vector lattices leads quite naturally to the study of vector lattice subalgebras of the orthomorphisms (see the Theorems 4.6.1 and 4.6.2, below), thus motivating in more detail the special attention that is paid in [19] and in the present paper to the orthomorphisms.

4.2 Preliminaries

In this section, we collect a number of notations, conventions, and definitions. We also include a few preliminary results.

All vector spaces are over the real numbers and all vector lattices are supposed to be Archimedean. We let E^+ denote the positive cone of a vector lattice E. The identity operator on a vector lattice E will be denoted by I, or by I_E when the context requires this. The characteristic function of a set S is denoted by χ_S.

Let E be a vector lattice, and let $x \in E$. We say that a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in E is *order convergent to* $x \in E$ (denoted by $x_\alpha \xrightarrow{\mathrm{o}} x$) when there exists a net $(y_\beta)_{\beta\in\mathcal{B}}$ in E such that $y_\beta \downarrow 0$ and with the property that, for every $\beta_0 \in \mathcal{B}$, there exists an $\alpha_0 \in \mathcal{A}$ such that $|x - x_\alpha| \leq y_{\beta_0}$

whenever α in $\mathcal{A}$ is such that $\alpha \geq \alpha_0$. Note that, in this definition, the index sets $\mathcal{A}$ and $\mathcal{B}$ need not be equal.

A net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in a vector lattice E is said to be *unbounded order convergent* to an element x in E (denoted by $x_\alpha \xrightarrow{\mathrm{uo}} x$) when $|x_\alpha - x| \wedge y \xrightarrow{\mathrm{o}} 0$ in E for all $y \in E^+$. Order convergence implies unbounded order convergence to the same limit. For order bounded nets, the two notions coincide.

Let E and F be vector lattices. The order bounded linear operators from E into F will be denoted by $\mathscr{L}_{\mathrm{ob}}(E,F)$, and we write $E^\sim$ for $\mathscr{L}_{\mathrm{ob}}(E,\mathbb{R})$. A linear operator $T: E \to F$ between two vector lattices E and F is *order continuous* when, for every net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in E, the fact that $x_\alpha \xrightarrow{\mathrm{o}} 0$ in E implies that $Tx_\alpha \xrightarrow{\mathrm{o}} 0$ in F. An order continuous linear operator between two vector lattices is automatically order bounded; see [7, Lemma 1.54], for example. The order continuous linear operators from E into F will be denoted by $\mathscr{L}_{\mathrm{oc}}(E,F)$. We write $E_{\mathrm{oc}}^\sim$ for $\mathscr{L}_{\mathrm{oc}}(E,\mathbb{R})$.

Let F be a vector sublattice of a vector lattice E. Then F is a *regular vector sublattice of* E when the inclusion map from F into E is order continuous. Ideals are regular vector sublattices. For a net in a regular vector sublattice F of E, its unbounded order convergence in F and in E are equivalent; see [28, Theorem 3.2].

An *orthomorphism* on a vector lattice E is a band preserving order bounded linear operator. We let $\mathrm{Orth}(E)$ denote the orthomorphisms on E. Orthomorphisms are automatically order continuous; see [7, Theorem 2.44]. An overview of some basic properties of the orthomorphisms that we shall use can be found in the first part of [19, Section 6], with detailed references included.

A topology τ on a vector lattice E is a *uo-Lebesgue topology* when it is a (not necessarily Hausdorff) locally solid linear topology on E such that, for a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in E, the fact that $x_\alpha \xrightarrow{\mathrm{uo}} 0$ in E implies that $x_\alpha \xrightarrow{\tau} 0$. For the general theory of locally solid linear topologies on vector lattices we refer to [6]. A vector lattice need not admit a uo-Lebesgue topology, and it admits at most one Hausdorff uo-Lebesgue topology; see [11, Propositions 3.2, 3.4, and 6.2] or [44, Theorems 5.5 and 5.9]). In this case, this unique Hausdorff uo-Lebesgue topology is denoted by $\widehat{\tau}_E$.

The following fact will often be used in the present paper.

Theorem 4.2.1. *Let E be a Dedekind complete vector lattice. The following are equivalent:*
(1) E admits a (necessarily unique) Hausdorff uo-Lebesgue topology;
(2) $\mathrm{Orth}(E)$ admits a (necessarily unique) Hausdorff uo-Lebesgue topology;
(3) $\mathscr{L}_{\mathrm{ob}}(E)$ admits a (necessarily unique) Hausdorff uo-Lebesgue topology.

Proof. The equivalence of the parts (1) and (2) is a part of [19, Proposition 8.2]. Part (1) implies part (3) by [19, Theorem 4.3], and part (3) implies part (2) by [44, Proposition 5.12]. □

Let X be a non-empty set. As in [19], we define a *convergence structure on X* to be a non-empty collection $\mathscr{C}$ of pairs $((x_\alpha)_{\alpha\in\mathcal{A}}, x)$, where $(x_\alpha)_{\alpha\in\mathcal{A}}$ is a net in X and $x \in X$, such that:

(1) when $((x_\alpha)_{\alpha\in\mathcal{A}}, x) \in \mathscr{C}$, then also $((y_\beta)_{\beta\in\mathcal{B}}, x) \in \mathscr{C}$ for every subnet $(y_\beta)_{\beta\in\mathcal{B}}$ of $(x_\alpha)_{\alpha\in\mathcal{A}}$;

(2) when a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in X is constant with value x, then $((x_\alpha)_{\alpha\in\mathcal{A}}, x) \in \mathscr{C}$.

It is obvious how to define a sequential convergence structure by using sequences and subsequences.

Suppose that $\mathscr{C}$ is a convergence structure on a non-empty set X. For a non-empty subset $S \subseteq X$, we define the *$\mathscr{C}$-adherence of S in X* as

$$a_{\mathscr{C}}(S) := \{\, x \in E : \text{there exists a net } (x_\alpha)_{\alpha\in\mathcal{A}} \text{ in } S \text{ such that } ((x_\alpha)_{\alpha\in\mathcal{A}}, x) \in \mathscr{C}\,\}$$

We set $a_{\mathscr{C}}(\emptyset) := \emptyset$. A subset S of X is said to be *$\mathscr{C}$-closed* when $a_{\mathscr{C}}(S) = S$. It is not difficult to see that the $\mathscr{C}$-closed subsets of X are the closed subsets of a topology $\tau_{\mathscr{C}}$ on X. It is not generally true that $a_{\mathscr{C}}(S)$ is $\tau_{\mathscr{C}}$-closed. In fact, $\overline{a_{\mathscr{C}}(S)}^{\tau_{\mathscr{C}}} = \overline{S}^{\tau_{\mathscr{C}}}$ for $S \subseteq X$.

On a vector lattice E, the set of all pairs of order convergent nets and their order limits forms a convergence structure $\mathscr{C}_{\mathrm{o}}$ on E. Likewise, there is a convergence structure $\mathscr{C}_{\mathrm{uo}}$ on E and, when applicable, a convergence structure $\mathscr{C}_{\widehat{\tau}_E}$ of a topological nature. For a subset S of E, we shall write $a_{\mathrm{o}}(S)$ for $a_{\mathscr{C}_{\mathrm{o}}}(S)$, $a_{\mathrm{uo}}(S)$ for $a_{\mathscr{C}_{\mathrm{uo}}}(S)$, and, when applicable, $\overline{S}^{\widehat{\tau}_E}$ for $a_{\mathscr{C}_{\widehat{\tau}_E}}(S)$. There are self-explanatory notations $a_{\sigma\mathrm{o}}(S)$, $a_{\sigma\mathrm{uo}}(S)$, and, when applicable, $a_{\sigma\widehat{\tau}_E}(S)$. We shall also speak of the order adherence (or o-adherence) of a subset, rather than of its $\mathscr{C}_{\mathrm{o}}$-adherence; etc. Note that the order adherence $a_{\mathrm{o}}(S)$ of S is what is called the 'order closure' of S in other sources. Since this 'order closure' need not be closed in the $\tau_{\mathscr{C}_{\mathrm{o}}}$-topology on E, we shall not use this terminology that is prone to mistakes.

Let E and F be vector lattices, where F is Dedekind complete. Suppose that $\mathscr{E}$ is a vector sublattice of $\mathscr{L}_{\mathrm{ob}}(E,F)$. As for general vector lattices, we have the convergence structures $\mathscr{C}_{\mathrm{o}}(\mathscr{E})$, $\mathscr{C}_{\mathrm{uo}}(\mathscr{E})$ and, when applicable, a convergence structure $\mathscr{C}_{\widehat{\tau}_{\mathscr{E}}}$ on E. In addition to these 'uniform' convergence structures, there are in this case also 'strong' ones that we shall now define. Let $(T_\alpha)_{\alpha\in\mathcal{A}}$ be a net in $\mathscr{E}$, and let $T \in \mathscr{E}$. Then we shall say that $(T_\alpha)_{\alpha\in\mathcal{A}}$ is *strongly order convergent to T* (denoted by $T_\alpha \xrightarrow{\mathrm{SO}} T$) when $T_\alpha x \xrightarrow{\mathrm{o}} Tx$ for all $x \in E$. The set of all pairs of strongly order convergent nets in $\mathscr{E}$ and their limits forms a convergence structure $\mathscr{C}_{\mathrm{SO}}$ on $\mathscr{E}$. Likewise, the net is *strongly unbounded order convergent to T* (denoted by $T_\alpha \xrightarrow{\mathrm{SUO}} T$) when it is pointwise unbounded order convergent to T, resulting in a convergence structure $\mathscr{C}_{\mathrm{SUO}}$ on $\mathscr{E}$. When E admits a Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$, then a net is *strongly convergent with respect to $\widehat{\tau}_E$ to T* (denoted by $T_\alpha \xrightarrow{\mathrm{S}\widehat{\tau}_E} T$) when it is pointwise $\widehat{\tau}_E$-convergent to T, yielding to a convergence structure $\mathscr{C}_{S\widehat{\tau}_{\mathscr{E}}}$ on $\mathscr{E}$. As for the three convergence structures on general vector lattices, we shall simply write $a_{\mathrm{SUO}}(\mathscr{S})$ for the $\mathscr{C}_{\mathrm{SUO}}$-adherence $a_{\mathscr{C}_{\mathrm{SUO}}}(\mathscr{S})$ of a subset $\mathscr{S}$ of $\mathscr{E}$; etc. We shall use a similar simplified notation for adherences corresponding to the sequential strong convergence structures that are defined in the obvious way.

The adherence of a set in a convergence structure obviously depends on the superset, since this determines the available possible limits of nets. In an ordered context, there can be additional complications because, for example, the notion of order convergence of a net itself depends on the vector lattice that the net is considered to be a subset of. It is for this reason that, although we have not included the superset in the notation for adherences, we shall always indicate it in words.

Let $\mathscr{C}_X$ be a convergence structure on a non-empty set X, and let $\mathscr{C}_Y$ be a convergence structure on a non-empty set Y. A map $\Phi : X \to Y$ is said to be $\mathscr{C}_X$-$\mathscr{C}_Y$ *continuous* when, for every pair $((x_\alpha)_{\alpha\in\mathcal{A}}, x)$ in $\mathscr{C}_X$, the pair $((\Phi(x_\alpha))_{\alpha\in\mathcal{A}}, \Phi(x))$ is an element of $\mathscr{C}_Y$. We shall speak of $\mathrm{S}\widehat{\tau}_E$-o continuity rather than of $\mathscr{C}_{\mathrm{S}\widehat{\tau}_E}$-$\mathscr{C}_{\mathrm{o}}$ continuity; etc.

Let E be a vector lattice. For $T \in \mathscr{L}_{\mathrm{ob}}(E)$, we define $\rho_T, \lambda_T : \mathscr{L}_{\mathrm{ob}}(E) \to \mathscr{L}_{\mathrm{ob}}(E)$ by setting $\rho_T(S) := ST$ and $\lambda_T(S) := TS$ for $S \in \mathscr{L}_{\mathrm{ob}}(E)$. We shall use the same notations for the maps that result in other contexts when compositions with linear operators map one set of linear operators into another.

For later use in this paper, we establish a few preparatory results that are of some interest in their own right.

Lemma 4.2.2. *Let $\mathscr{A}$ be an f-algebra with a (not necessarily positive) identity element e, and let E be a vector lattice with the principal projection property. Let $a \in \mathscr{A}^+$, and suppose that*

$$\pi : \mathrm{Span}\{e, a, a^2\} \to \mathscr{L}_{\mathrm{ob}}(E)$$

is a positive linear map such that $\pi(e) = I$. Then $\pi(a) \in \mathrm{Orth}(E)$.

Proof. It is obvious that $\pi(a) \in \mathscr{L}_{\mathrm{ob}}(E)$, so it remains to be shown that $\pi(a)$ is band preserving on E. We know from [7, Theorem 2.57] that

$$a \leq a \wedge ne + \frac{1}{n}a^2 \leq ne + \frac{1}{n}a^2$$

for $n \geq 1$. Take $x \in E^+$. Then we have

$$\pi(a)x \leq \pi\left[ne + \frac{1}{n}a^2\right]x = nx + \frac{1}{n}\pi(a^2)x. \tag{4.1}$$

for $n \geq 1$. Let B_x be the band generated by x in E, and let $P_x \in \mathscr{L}_{\mathrm{ob}}(E)$ be the order projection onto B_x. Using that $\pi(a)x \geq 0$ and equation (4.1), we have

$$\begin{aligned} 0 &\leq (I - P_x)[\pi(a)x] \\ &\leq (I - P_x)[nx + \frac{1}{n}\pi(a^2)x] \\ &= \frac{1}{n}(I - P_x)[\pi(a^2)x] \end{aligned}$$

for all $n \geq 1$. Hence $(I - P_x)[\pi(a)x] = 0$, so that $\pi(a)x \in B_x$. Since x was arbitrary, this shows that $\pi(a)$ is band preserving. $\square$

Corollary 4.2.3. *Let $\mathscr{A}$ be an f-algebra with a (not necessarily positive) identity element e, and let E be a vector lattice with the principal projection property. Suppose that $\pi : \mathscr{A} \to \mathscr{L}_{\mathrm{ob}}(E)$ is a positive linear map such that $\pi(e) = I$. Then $\pi(\mathscr{A}) \subseteq \mathrm{Orth}(E)$.*

Remark 4.2.4. Corollary 4.2.3 shows that part (iii) of [34, Definition 4.1] is redundant in a number of cases of practical interest. The fact that the action of the f-algebra preserves multiplication is not even needed for this redundancy to be the case.

The following is immediate from Corollary 4.2.3.

Corollary 4.2.5. *Let E and F be vector lattices, where F is Dedekind complete. Let $\mathscr{E}$ be a vector sublattice of $\mathscr{L}_{\mathrm{ob}}(E,F)$ with the principal projection property.*
(1) Suppose that $ST \in \mathscr{E}$ for all $S \in \mathscr{E}$ and $T \in \mathrm{Orth}(F)$, so that there is a naturally defined map $\rho_T : \mathscr{E} \to \mathscr{E}$ for $T \in \mathrm{Orth}(F)$. Then $\rho_T \in \mathrm{Orth}(\mathscr{E})$ for $T \in \mathrm{Orth}(F)$.
(2) Suppose that $TS \in \mathscr{E}$ for all $S \in \mathscr{E}$ and $T \in \mathrm{Orth}(E)$, so that there is a naturally defined map $\lambda_T : \mathscr{E} \to \mathscr{E}$ for $T \in \mathrm{Orth}(E)$. Then $\lambda_T \in \mathrm{Orth}(\mathscr{E})$ for $T \in \mathrm{Orth}(E)$.

Remark 4.2.6.
(1) For $\mathscr{E} = \mathscr{L}_{\mathrm{ob}}(E,F)$, Corollary 4.2.5 is established in the beginning of [34, Section 2].
(2) For $\mathscr{E} = \mathscr{L}_{\mathrm{oc}}(E)$, where E is a Dedekind complete vector lattice, Corollary 4.2.5 is established in [18, Proof of Theorem 8.4].
(3) For $\mathscr{E} = \mathrm{Orth}(E)$, where E is a Dedekind complete vector lattice, [7, Theorem 2.62] provides a much stronger result than Corollary 4.2.5, also when E need not be Dedekind complete.

4.3 Implications between convergences on vector lattices of operators

In this section, we investigate the implications between the six convergences on the order bounded linear operators and on the orthomorphisms on a Dedekind complete vector lattice. Without further ado, let us simply state the answers and explain how they are obtained.

For a general net of order bounded linear operators (resp. orthomorphisms) on a general Dedekind complete vector lattice, the implications between order convergence, unbounded order convergence, convergence in a possible Hausdorff uo-Lebesgue topology, strong order convergence, strong unbounded order convergence, and strong convergence with respect to a possible Hausdorff uo-Lebesgue topology, are given in Table 4.3.1 (resp. Table 4.3.2).

Table 4.3.1: Implications between convergences of nets in $\mathscr{L}_{\mathrm{ob}}(E)$.

	o	uo	$\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$	SO	SUO	$\mathrm{S}\widehat{\tau}_E$
o	1	1	1	1	1	1
uo	0	1	1	0	0	0
$\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$	0	0	1	0	0	0
SO	0	0	0	1	1	1
SUO	0	0	0	0	1	1
$\mathrm{S}\widehat{\tau}_E$	0	0	0	0	0	1

Table 4.3.2: Implications between convergences of nets in Orth(E).

	o	uo	$\widehat{\tau}_{\mathrm{Orth}}(E)$	SO	SUO	$\mathrm{S}\widehat{\tau}_E$
o	1	1	1	1	1	1
uo	0	1	1	0	1	1
$\widehat{\tau}_{\mathrm{Orth}(E)}$	0	0	1	0	0	1
SO	0	1	1	1	1	1
SUO	0	1	1	0	1	1
$\mathrm{S}\widehat{\tau}_E$	0	0	1	0	0	1

In Orth(E), uo and SUO convergence of nets coincide, as do a possible $\widehat{\tau}_{\mathrm{Orth}}(E)$ and $\mathrm{S}\widehat{\tau}_E$ convergence.

In these tables, the value in a cell indicates whether the convergence of a net in the sense that labels the row of that cell does (value 1) or does not (value 0) in general imply its convergence (to the same limit) in the sense that labels the column of that cell. For example, the value 0 in the cell $(\mathrm{uo}, \mathrm{S}\widehat{\tau}_E)$ in Table 4.3.1 indicates that there exists a net of order bounded linear operators on a Dedekind complete vector lattice E that admits a Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$, such that this net is unbounded order convergent to zero in $\mathscr{L}_{\mathrm{ob}}(E)$, but not strongly convergent to zero with respect to $\widehat{\tau}_E$. The value 1 in the cell $(\mathrm{uo}, \mathrm{S}\widehat{\tau}_E)$ in Table 4.3.2, however, indicates that every net of *orthomorphisms* on an arbitrary Dedekind complete vector lattice E that admits a Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$, such that this net is unbounded order convergent to zero, *is* strongly convergent to zero with respect to $\widehat{\tau}_E$.

We shall now explain how these tables can be obtained.

Obviously, the order convergence of a net of operators implies its unbounded order convergence, which implies its convergence in a possible Hausdorff uo-Lebesgue topology. There are similar implications for the three associated strong convergences. Furthermore, an implication that fails for orthomorphisms also fails in the general case. Using these basic facts, it is a logical exercise to complete the tables from a few 'starting values' that we now validate.

For Table 4.3.1, we have the following 'starting values':

- the value 1 in the cell (o, SO) follows from [19, Lemma 4.1];
- the value 0 in the cell $(\mathrm{uo}, S\widehat{\tau}_E)$ follows from [19, Example 5.3], when using that, for an atomic vector lattice as in that example, the unbounded order convergence of a net and the convergence in the Hausdorff uo-Lebesgue topology coincide (see [13, Proposition 1] and [44, Lemma 7.4]);
- the value 0 in the cell $(\mathrm{SO}, \widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)})$ follows from the case where $p = \infty$ in [19, Example 5.5]. The reason is—we resort to the notation and context of that example—that, for $p = \infty$, it follows from [9, Example 10.1.2] that the sequence $\mathbb{E}_n f$ is order bounded in $\mathrm{L}_\infty([0,1])$ for all $f \in \mathrm{L}_\infty([0,1])$. Since we already know from the general case that it is almost everywhere convergent to f it is, in fact, order convergent to f in $\mathrm{L}_\infty([0,1])$. The remainder of the arguments in the example then validate the value 0 in the cell.

For Table 4.3.2, we have the following 'starting values':

- the values 0 in the cells (uo, o), (uo, SO), ($\widehat{\tau}_{\mathrm{Orth}(E)}$, uo), and ($\widehat{\tau}_{\mathrm{Orth}(E)}$, SUO) follow from the examples preceding [19, Lemma 9.1], letting the multiplication operators act on the constant function 1 for the second and fourth of these cells;
- the value 0 in the cell (SO, o) follows from the example following the proof of [19, Theorem 9.4];
- the values 1 in the cells (uo, SUO) and (SUO, uo) follow from [19, Theorem 9.9];
- the values 1 in the cells ($\widehat{\tau}_{\mathrm{Orth}(E)}$, $\mathrm{S}\widehat{\tau}_E$) and ($\mathrm{S}\widehat{\tau}_E$, $\widehat{\tau}_{\mathrm{Orth}(E)}$) follow from [19, Theorem 9.12].

The reader may check for himself that the above is, indeed, sufficient information to determine both tables.

Remark 4.3.3.

(1) Every *order bounded* net of orthomorphisms on an arbitrary Dedekind complete vector lattice E that is strongly order convergent to zero, is order convergent to zero in $\mathrm{Orth}(E)$; see [19, Theorem 9.4];

(2) Every *sequence* of orthomorphisms on a Dedekind complete Banach lattice E that is strongly order convergent to zero, is order convergent to zero in $\mathrm{Orth}(E)$; see [19, Theorem 9.7];

(3) The validity of all zeroes in Table 4.3.1 (resp. Table 4.3.2) follows from the existence of a net of order bounded linear operators (resp. orthomorphisms) on a Dedekind complete *Banach lattice* for which the implication in question does not hold. With the cell (SO, o) in Table 4.3.2 as the only exception, such a net of operators on a Banach lattice can even be taken to be a sequence. This follows from an inspection of the (counter) examples referred to above when validating the 'starting' zeroes in the tables.

4.4 Continuity of left and right multiplications

In this section, we study continuity properties of left and right multiplication operators. For example, take an arbitrary $T \in \mathscr{L}_{\mathrm{ob}}(E)$, where E is an arbitrary Dedekind complete vector lattice that admits a Hausdorff uo-Lebesgue topology $\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$. Is it then true that $\lambda_T : \mathscr{L}_{\mathrm{ob}}(E) \to \mathscr{L}_{\mathrm{ob}}(E)$ maps unbounded order convergent nets in $\mathscr{L}_{\mathrm{ob}}(E)$ to $\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$-convergent nets (with corresponding limits)? If not, is this then true when we suppose that $T \in \mathscr{L}_{\mathrm{oc}}(E)$? If not, is this true when we suppose that $T \in \mathrm{Orth}(E)$? One can ask a similar combination of questions, specifying to classes of increasingly well-behaved operators, for each of the $6 \cdot 6 = 36$ combinations of convergences of nets in $\mathscr{L}_{\mathrm{ob}}(E)$ under consideration in this paper. There are also 36 combinations to be considered for left multiplication operators. This section provides the answers in all 72 cases; the results are contained in the Tables 4.4.14 and 4.4.15, below. For the example that we gave, the answer is still negative when asking it for arbitrary $T \in \mathscr{L}_{\mathrm{oc}}(E)$, but affirmative for arbitrary $T \in \mathrm{Orth}(E)$.

For $\mathrm{Orth}(E)$, there are similar questions to be asked for its left and right regular representation, but their number is smaller. Firstly, we see no obvious better-behaved subclass of $\mathrm{Orth}(E)$ that we should also consider. Secondly, since $\mathrm{Orth}(E)$ is commutative, there is only one type of multiplication involved. Thirdly, as in Table 4.3.2, there are two pairs of coinciding convergences. All in all, there are only 4 x 4 = 16 possible combinations that actually have to be considered for the regular representation of $\mathrm{Orth}(E)$. Also in this case, all answers can be given; the results are contained in Table 4.4.16, below. As it turns out, Table 4.4.16 is identical to Table 4.3.2. There appears to be no a priori reason for this fact; it is simply the outcome.

We shall now set out to validate the Tables 4.4.14, 4.4.15, and 4.4.16. Fortunately, we do not need individual results for every cell. Upon considering the multiplications by the orthomorphism that is the identity operator, the zeroes in the Tables 4.3.1 and 4.3.2 already determine the values in many cells. For the remaining ones, the combination of the 'standard' implications that were already used for the Tables 4.3.1 and 4.3.2 and a limited number of results and (counter) examples already suffices. We shall now start to collect these.

We start with o-o and SO-SO continuity.

Proposition 4.4.1. *Let E be a Dedekind complete vector lattice. Then:*
(1) $\rho_T : \mathscr{L}_{\mathrm{ob}}(E) \to \mathscr{L}_{\mathrm{ob}}(E)$ is o-o *continuous for all $T \in \mathscr{L}_{\mathrm{ob}}(E)$;*
(2) $\lambda_T : \mathscr{L}_{\mathrm{ob}}(E) \to \mathscr{L}_{\mathrm{ob}}(E)$ is o-o *continuous for all $T \in \mathscr{L}_{\mathrm{oc}}(E)$;*
(3) $\rho_T : \mathscr{L}_{\mathrm{ob}}(E) \to \mathscr{L}_{\mathrm{ob}}(E)$ is SO-SO *continuous for all $T \in \mathscr{L}_{\mathrm{ob}}(E)$;*
(4) $\lambda_T : \mathscr{L}_{\mathrm{ob}}(E) \to \mathscr{L}_{\mathrm{ob}}(E)$ is SO-SO *continuous for all $T \in \mathscr{L}_{\mathrm{oc}}(E)$.*

Proof. We prove the parts (1) and (2). Take $T \in \mathscr{L}_{\mathrm{ob}}(E)$, and let $(T_\alpha)_{\alpha\in\mathcal{A}} \subseteq \mathscr{L}_{\mathrm{ob}}(E)$ be a net such that $S_\alpha \xrightarrow{\mathrm{o}} 0$ in $\mathscr{L}_{\mathrm{ob}}(E)$. By passing to a tail, we may assume that $(|S_\alpha|)_{\alpha\in\mathcal{A}}$ is order bounded in $\mathscr{L}_{\mathrm{ob}}(E)$. Set $R_\alpha := \bigvee_{\beta\geq\alpha}|S_\beta|$ for $\alpha \in \mathcal{A}$. Then $|S_\alpha| \leq R_\alpha$ for $\alpha \in \mathcal{A}$ and $R_\alpha \downarrow 0$ in $\mathscr{L}_{\mathrm{ob}}(E)$ (see [28, Remark 2.2]). It is immediate from [7, Theorem 1.18] that also $R_\alpha|T| \downarrow 0$ in $\mathscr{L}_{\mathrm{ob}}(E)$. Since $|\rho_T(S_\alpha)| \leq R_\alpha|T|$ for $\alpha \in \mathcal{A}$, we see that $\rho_T(S_\alpha) \xrightarrow{\mathrm{o}} 0$ in $\mathscr{L}_{\mathrm{ob}}(E)$, as desired. Suppose that, in fact, $T \in \mathscr{L}_{\mathrm{oc}}(E)$. Since $R_\alpha x \downarrow 0$ for $x \in E^+$ by [7, Theorem 1.18], we then also have that $|T|R_\alpha x \downarrow 0$ for $x \in E^+$. Hence $|T|R_\alpha \downarrow 0$ in $\mathscr{L}_{\mathrm{ob}}(E)$. The fact that $|\lambda_T(S_\alpha)| \leq |T|R_\alpha$ for $\alpha \in \mathcal{A}$ then implies that $\lambda_T(S_\alpha) \xrightarrow{\mathrm{o}} 0$ in $\mathscr{L}_{\mathrm{ob}}(E)$.

The parts (3) and (4) are immediate consequences of the definitions. □

We now show that the condition in the parts (2) and (4) of Proposition 4.4.1 that $T \in \mathscr{L}_{\mathrm{oc}}(E)$ cannot be relaxed to $T \in \mathscr{L}_{\mathrm{ob}}(E)$.

Examples 4.4.2. Take $E = \ell_\infty$, let $(e_n)_{n=1}^\infty$ be the sequence of standard unit vectors in E, and let c denote the sublattice of E consisting of the convergent sequences. We define a positive linear functional f_c on c by setting

$$f_c(x) := \lim_{n\to\infty} x_n$$

for $x = \bigvee_{i=1}^\infty x_i e_i \in c$. Since c is a majorising vector subspace of E, [7, Theorem 1.32] shows that there exists a positive functional f on E that extends f_c. We define $T : E \to E$ by setting $Tx = f(x)e_1$ for $x \in E$. Clearly, $T \in \mathscr{L}_{\mathrm{ob}}(E)$; a consideration of $T(\bigvee_{i=n}^\infty e_i)$ for $n \geq 1$ shows that $T \notin \mathscr{L}_{\mathrm{oc}}(E)$.

We define $S_n \in \mathscr{L}_{\mathrm{oc}}(E)$ for $n \geq 1$ by setting

$$S_n x := x_1 \bigvee_{i=1}^{n} e_i,$$

and $S \in \mathscr{L}_{\mathrm{oc}}(E)$ by setting

$$Sx := x_1 \bigvee_{i=1}^{\infty} e_i$$

for $x = \bigvee_{i=1}^{\infty} x_i e_i \in E$. Clearly, $S_n \uparrow S$ in $\mathscr{L}_{\mathrm{ob}}(E)$. On the other hand, $\lambda_T(S_n) = 0$ for all $n \geq 1$, while $\lambda_T(S) = P_1$, where $P_1 \in \mathscr{L}_{\mathrm{ob}}(E)$ is the order projection onto the span of e_1. This shows that $\lambda_T : \mathscr{L}_{\mathrm{ob}}(E) \to \mathscr{L}_{\mathrm{ob}}(E)$ is not o-o continuous.

The sequence $(S_n)_{n=1}^{\infty}$, being order convergent to S, is also strongly unbounded order convergent to S in $\mathscr{L}_{\mathrm{ob}}(E)$. Hence $\lambda_T : \mathscr{L}_{\mathrm{ob}}(E) \to \mathscr{L}_{\mathrm{ob}}(E)$ is not SO-SO continuous.

Remark 4.4.3. Examples 4.4.2 also shows that, already for a Banach lattice E, λ_T need not even be sequentially o-$\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$ continuous, sequentially o-S$\widehat{\tau}_E$ continuous, sequentially SO-$\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$ continuous, or sequentially SO-S$\widehat{\tau}_E$ continuous for arbitrary $T \in \mathscr{L}_{\mathrm{ob}}(E)$.

Remark 4.4.4. The o-o continuity (appropriately defined) of left and right multiplications on ordered algebras is studied in [4]. It is established on [4, p. 542–543] that, for a Dedekind complete vector lattice E, the right and left multiplication by an element T of the ordered algebra $L(E)$ of all (!) linear operators on E are both order continuous on $L(E)$ in the sense of [4] if and only if the left multiplication is, which is the case if and only if $T \in \mathscr{L}_{\mathrm{oc}}(E)$. The proof refers to [3, Example 2.9 (a)], which is concerned with multiplications by a positive operator T on the ordered Banach algebra $L(E)$ of all (!) bounded linear operators on a Dedekind complete Banach lattice E. It is established in that example that the simultaneous order continuity of the right and left multiplication by T on $L(E)$ in the sense of [3] is equivalent to T being order continuous. On [3, p. 151] it is mentioned that this criterion for the order continuity of an operator can also be presented for an arbitrary Dedekind complete vector lattice. Although it is not stated as such, and although a proof as such is not given, the author may have meant to state, and have known to be true, that, for a Dedekind complete vector lattice E and $T \in \mathscr{L}_{\mathrm{ob}}(E)$, λ_T and ρ_T are both o-o continuous on $\mathscr{L}_{\mathrm{ob}}(E)$ in the sense of the present paper if and only if λ_T is, which is the case if and only if $T \in \mathscr{L}_{\mathrm{oc}}(E)$. Using arguments as on [3, p. 151] and [4, p. 542–543], the authors of the present paper have verified that—this is the hard part—for $T \in \mathscr{L}_{\mathrm{ob}}(E)$, the o-o continuity of λ_T on $\mathscr{L}_{\mathrm{ob}}(E)$ in the sense of the present paper does implies that $T \in \mathscr{L}_{\mathrm{oc}}(E)$. Hence the three properties of $T \in \mathscr{L}_{\mathrm{ob}}(E)$ mentioned above are, indeed, equivalent; a result that is to be attributed to the late Egor Alekhno.

We use the opportunity to establish the following side result, which follows easily from combining each of [42, Satz 3.1] and [10, Proposition 2.2] with the parts (1) and (2) of Proposition 4.4.1.

Proposition 4.4.5. *Let E be a Dedekind complete vector lattice. Then:*

(1) the map $T \mapsto \rho_T$ defines an order continuous lattice homomorphism $\rho : \mathscr{L}_{\mathrm{ob}}(E) \to \mathscr{L}_{\mathrm{oc}}(\mathscr{L}_{\mathrm{oc}}(E), \mathscr{L}_{\mathrm{ob}}(E))$.

(2) *the map $T \mapsto \lambda_T$ defines an order continuous lattice homomorphism $\lambda : \mathscr{L}_{\mathrm{ob}}(E) \to \mathscr{L}_{\mathrm{ob}}(\mathscr{L}_{\mathrm{ob}}(E))$ that maps $\mathscr{L}_{\mathrm{oc}}(E)$ into $\mathscr{L}_{\mathrm{oc}}(\mathscr{L}_{\mathrm{ob}}(E))$.*

Remark 4.4.6. In [48, Problem 1], it was asked, among others, whether, for a Dedekind complete vector lattice E, the left regular representation of $\mathscr{L}_{\mathrm{ob}}(E)$ is a lattice homomorphism from $\mathscr{L}_{\mathrm{ob}}(E)$ into $\mathscr{L}_{\mathrm{ob}}(\mathscr{L}_{\mathrm{ob}}(E))$. In [14, Theorem 11.19], it was observed that the affirmative answer is, in fact, provided by [42, Satz 3.1]. Part (2) of Proposition 4.4.5 gives still more precise information.

Part (1), which relies on [10, Proposition 2.2], implies that the right regular representation of $\mathscr{L}_{\mathrm{oc}}(E)$ is an order continuous lattice homomorphism from $\mathscr{L}_{\mathrm{oc}}(E)$ into $\mathscr{L}_{\mathrm{ob}}(\mathscr{L}_{\mathrm{oc}}(E))$, with an image that is, in fact, contained in $\mathscr{L}_{\mathrm{oc}}(\mathscr{L}_{\mathrm{oc}}(E))$.

After this brief digression, we continue with the main line of this section, and consider uo-uo and SUO-SUO continuity of left and right multiplication operators.

Proposition 4.4.7. *Let E be a Dedekind complete vector lattice. Then:*
(1) *ρ_T is* uo-uo *continuous on $\mathscr{L}_{\mathrm{ob}}(E)$ for all $T \in \mathrm{Orth}(E)$;*
(2) *λ_T is* uo-uo *continuous on $\mathscr{L}_{\mathrm{ob}}(E)$ for all $T \in \mathrm{Orth}(E)$;*
(3) *ρ_T is* SUO-SUO *continuous on $\mathscr{L}_{\mathrm{ob}}(E)$ for all $T \in \mathscr{L}_{\mathrm{ob}}(E)$;*
(4) *λ_T is* SUO-SUO *continuous on $\mathscr{L}_{\mathrm{ob}}(E)$ for all $T \in \mathrm{Orth}(E)$.*

Proof. For the parts (1) and (2), we note that $\rho_T, \lambda_T \in \mathrm{Orth}(\mathscr{L}_{\mathrm{ob}}(E))$ by Corollary 4.2.5. Their uo-uo continuity then follows from [19, Proposition 7.1].

Part (3) is trivial.

We prove part (4). Let $(T_\alpha)_{\alpha\in\mathcal{A}}$ be a net in $\mathscr{L}_{\mathrm{ob}}(E)$ such that $S_\alpha \xrightarrow{\mathrm{SUO}} 0$. Take an $x \in E$. Then $S_\alpha x \xrightarrow{\mathrm{uo}} 0$ in E. It follows from [19, Proposition 7.1] that $\lambda_T(S_\alpha)x = TS_\alpha x \xrightarrow{\mathrm{uo}} 0$ in E, as desired. □

Remark 4.4.8. For the proof of the parts (1) and (2) of Proposition 4.4.7, an appeal to the beginning of [34, Section 2] can replace the use of Corollary 4.2.5. It is, however, only Corollary 4.2.5 that permits the obvious extensions of the parts (1) and (2) of Proposition 4.4.7 to (not necessarily regular) vector sublattices of $\mathscr{L}_{\mathrm{ob}}(E)$ that are invariant under left or right composition with orthomorphisms, provided that they have the principal projection property.

We now show that the condition in the parts (1), (2), and (4) of Proposition 4.4.7 that $T \in \mathrm{Orth}(E)$ cannot be relaxed to $T \in \mathscr{L}_{\mathrm{oc}}(E)$.

Examples 4.4.9.
(1) We first give an example showing that λ_T and ρ_T need not be uo-uo continuous on $\mathscr{L}_{\mathrm{ob}}(E)$ for $T \in \mathscr{L}_{\mathrm{oc}}(E)$.
Let $E = \mathrm{L}_p[0,1]$ with $1 \leq p < \infty$. We define $T \in \mathscr{L}_{\mathrm{ob}}(E) = \mathscr{L}_{\mathrm{oc}}(E)$ by setting

$$Sf := \int f \,\mathrm{d}\mu \cdot \chi_{[0,1]} \tag{4.2}$$

for $f \in E$. For $n \geq 1$, we define the positive operator S_n on E by setting

$$S_n f(t) := \begin{cases} f(t+1/n) & \text{for } t \in [0,(n-1)/n); \\ f(t-(n-1)/n) & \text{for } t \in [(n-1)/n, 1]. \end{cases}$$

We claim that $(S_n)_{n=1}^{\infty}$ is a disjoint sequence in $\mathscr{L}_{\mathrm{ob}}(E)$. Let $m, n \geq 1$ with $m > n$. Take a $k \geq 1$ such that $1/k < 1/n - 1/m$. For every $f \in E^+$, [7, Theorem 1.51] then implies that

$$0 \leq S_m \wedge S_n(f) \leq \sum_{i=1}^{k} S_m(f \cdot \chi_{[(i-1)/k, i/k]}) \wedge S_n(f \cdot \chi_{[(i-1)/k, i/k]}) = 0$$

because the supports of $S_m(f \cdot \chi_{[(i-1)/k,i/k]})$ and $S_n(f \cdot \chi_{[(i-1)/k,i/k]})$ are disjoint for $i = 1, \ldots, k$. Hence $S_m \wedge S_n = 0$, as claimed.
By [28, Corollary 3.6], the disjoint sequence $(S_n)_{n=1}^{\infty}$ is unbounded order convergent to zero in $\mathscr{L}_{\mathrm{ob}}(E)$. On the other hand, it is easy to see that $\rho_T(S_n) = \lambda_T(S_n) = S$ for all $n \geq 1$. Hence neither of $(\rho_T(S_n))_{n=1}^{\infty}$ and $(\lambda_T(S_n))_{n=1}^{\infty}$ is unbounded order convergent to zero in $\mathscr{L}_{\mathrm{ob}}(E)$. This shows that neither ρ_T nor λ_T is uo-uo continuous on $\mathscr{L}_{\mathrm{ob}}(E)$.

(2) We now give an example showing that λ_T need not be SUO-SUO continuous on $\mathscr{L}_{\mathrm{ob}}(E)$ for $T \in \mathscr{L}_{\mathrm{oc}}(E)$. Let $E = \mathrm{L}_p[0,1]$ with $1 \leq p < \infty$. For $n \geq 1$, define the positive operator S_n on E by setting $S_n f := 2^n \chi_{[1-1/2^{n-1}, 1-1/2^n]} \cdot f$ for $f \in E$. Let $T \in \mathscr{L}_{\mathrm{oc}}(E)$ be defined as in equation (4.2). For every $f \in E$, it is clear that $S_n f$ and $S_m f$ are disjoint whenever $m \neq n$, and then [28, Corollary 3.6] shows that $S_n f \xrightarrow{\mathrm{uo}} 0$ in E. That is, $(S_n)_{n=1}^{\infty}$ is strongly unbounded order convergent to zero. On the other hand, it is easily seen that $\lambda_T(S_n)\chi_{[0,1]} = \chi_{[0,1]}$ for $n \geq 1$. This implies that $(\lambda_T(S_n))_{n=1}^{\infty}$ is not strongly unbounded order convergent to zero, so that λ_T is not SUO-SUO continuous on $\mathscr{L}_{\mathrm{ob}}(E)$.

Remark 4.4.10. Examples 4.4.9 also shows that, already for a Banach lattice with an order continuous norm, ρ_T and λ_T need not even be sequentially uo-$\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$ continuous and λ_T need not be even be sequentially SUO-S$\widehat{\tau}_E$ continuous on $\mathscr{L}_{\mathrm{ob}}(E)$ for arbitrary $T \in \mathscr{L}_{\mathrm{oc}}(E) = \mathscr{L}_{\mathrm{ob}}(E)$.

We now turn to the Hausdorff uo-Lebesgue topologies. The reader may wish to recall Theorem 4.2.1.

Proposition 4.4.11. *Let E be a Dedekind complete vector lattice that admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$, so that $\mathscr{L}_{\mathrm{ob}}(E)$ also admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$. Then:*
(1) ρ_T is $\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$-$\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$ continuous on $\mathscr{L}_{\mathrm{ob}}(E)$ for all $T \in \mathrm{Orth}(E)$;
(2) λ_T is $\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$-$\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$ continuous on $\mathscr{L}_{\mathrm{ob}}(E)$ for all $T \in \mathrm{Orth}(E)$;
(3) ρ_T is S$\widehat{\tau}_E$-S$\widehat{\tau}_E$ continuous on $\mathscr{L}_{\mathrm{ob}}(E)$ for all $T \in \mathscr{L}_{\mathrm{ob}}(E)$;
(4) λ_T is S$\widehat{\tau}_E$-S$\widehat{\tau}_E$ continuous on $\mathscr{L}_{\mathrm{ob}}(E)$ for all $T \in \mathrm{Orth}(E)$.

Proof. We know from Corollary 4.2.5 that $\rho_T, \lambda_T \in \mathrm{Orth}(\mathscr{L}_{\mathrm{ob}}(E))$ when $T \in \mathrm{Orth}(E)$, and then the parts (1) and (2) follow from [19, Corollary 7.3].

Part (3) is trivial.

Part (4) follows from [19, Corollary 7.3]. □

We now show that the condition in the parts (1), (2), and (4) of Proposition 4.4.11 that $T \in \mathrm{Orth}(E)$ cannot be relaxed to $T \in \mathscr{L}_{\mathrm{oc}}(E)$.

Examples 4.4.12.
(1) We first give an example showing that λ_T and ρ_T need not be $\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$-$\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$ continuous on $\mathscr{L}_{\mathrm{ob}}(E)$ for $T \in \mathscr{L}_{\mathrm{oc}}(E)$. For this, we resort to the context and notation of part (1) of Examples 4.4.9. In that example, we know that $S_n \xrightarrow{\mathrm{uo}} 0$ in $\mathscr{L}_{\mathrm{ob}}(E)$, and then certainly $S_n \xrightarrow{\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}} 0$. Since $\rho_T(S_n) = \lambda_T(S_n)$ for all $n \geq 1$, we see that neither ρ_S nor λ_S is $\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$-$\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$ continuous on $\mathscr{L}_{\mathrm{ob}}(E)$.
(2) We give an example showing that λ_T need not be $\mathrm{S}\widehat{\tau}_E$-$\mathrm{S}\widehat{\tau}_E$ continuous on $\mathscr{L}_{\mathrm{ob}}(E)$ for $T \in \mathscr{L}_{\mathrm{oc}}(E)$. For this, we resort to the context and notation of part (2) of Examples 4.4.9. In that example, we know that $S_n f \xrightarrow{\mathrm{uo}} 0$ in E for $f \in E$. Then certainly $S_n f \xrightarrow{\widehat{\tau}_E} 0$ for $f \in E$. Since $\lambda_T(S_n)\chi_{[0,1]} = \chi_{[0,1]}$ for all $n \geq 1$, we see that λ_T is not $\mathrm{S}\widehat{\tau}_E$-$\mathrm{S}\widehat{\tau}_E$ continuous.

Remark 4.4.13. Examples 4.4.12 shows that, already for a Banach lattices with an order continuous norm, ρ_T and λ_T need not even be sequentially $\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$-$\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$ continuous and λ_T need not even be sequentially $\mathrm{S}\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$-$\mathrm{S}\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$ continuous on $\mathscr{L}_{\mathrm{ob}}(E)$ for arbitrary $T \in \mathscr{L}_{\mathrm{oc}}(E) = \mathscr{L}_{\mathrm{ob}}(E)$.

We now have sufficient material at our disposal to determine the tables mentioned at the beginning of this section.

For right multiplications on $\mathscr{L}_{\mathrm{ob}}(E)$, the results are in Table 4.4.14. The value in a cell with a row label indicating a convergence structure $\mathscr{C}_1$ and a column label indicating a convergence structure $\mathscr{C}_2$ is to be interpreted as follows:
(1) A value $\{0\}$ (resp. $\mathrm{Orth}(E)$, resp. $\mathscr{L}_{\mathrm{oc}}(E)$) means that ρ_T is $\mathscr{C}_1$-$\mathscr{C}_2$ continuous on $\mathscr{L}_{\mathrm{ob}}(E)$ for every Dedekind complete vector lattice E and for every $T \in \{0\}$ (resp. $T \in \mathrm{Orth}(E)$, resp. $T \in \mathscr{L}_{\mathrm{oc}}(E)$), but there exist a Dedekind complete vector lattice E and a $T \in \mathrm{Orth}(E)$ (resp. $T \in \mathscr{L}_{\mathrm{oc}}(E)$, resp. $T \in \mathscr{L}_{\mathrm{ob}}(E)$) for which this is not the case;
(2) A value $\mathscr{L}_{\mathrm{ob}}(E)$ means that ρ_T is $\mathscr{C}_1$-$\mathscr{C}_2$ continuous on $\mathscr{L}_{\mathrm{ob}}(E)$ for every Dedekind complete vector lattice E and for every $T \in \mathscr{L}_{\mathrm{ob}}(E)$.

Table 4.4.14: Continuity of right multiplications on $\mathscr{L}_{\mathrm{ob}}(E)$.

	o	uo	$\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$	SO	SUO	$\mathrm{S}\widehat{\tau}_E$
o	$\mathscr{L}_{\mathrm{ob}}(E)$	$\mathscr{L}_{\mathrm{ob}}(E)$	$\mathscr{L}_{\mathrm{ob}}(E)$	$\mathscr{L}_{\mathrm{ob}}(E)$	$\mathscr{L}_{\mathrm{ob}}(E)$	$\mathscr{L}_{\mathrm{ob}}(E)$
uo	$\{0\}$	$\mathrm{Orth}(E)$	$\mathrm{Orth}(E)$	$\{0\}$	$\{0\}$	$\{0\}$
$\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$	$\{0\}$	$\{0\}$	$\mathrm{Orth}(E)$	$\{0\}$	$\{0\}$	$\{0\}$
SO	$\{0\}$	$\{0\}$	$\{0\}$	$\mathscr{L}_{\mathrm{ob}}(E)$	$\mathscr{L}_{\mathrm{ob}}(E)$	$\mathscr{L}_{\mathrm{ob}}(E)$
SUO	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathscr{L}_{\mathrm{ob}}(E)$	$\mathscr{L}_{\mathrm{ob}}(E)$
$\mathrm{S}\widehat{\tau}_E$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathscr{L}_{\mathrm{ob}}(E)$

As mentioned in the beginning of this section, the zeroes in Table 4.3.1 give zeroes in Table 4.4.14. The reader may verify that the remaining values can be determined using that order convergence implies unbounded order convergence, which implies $\widehat{\tau}_E$ convergence when applicable; that analogous implications hold for their strong versions; that order convergence implies strong order convergence; combined with Proposition 4.4.1, Proposition 4.4.7, Remark 4.4.10, Proposition 4.4.11, and Remark 4.4.13.

For left multiplications on $\mathscr{L}_{\mathrm{ob}}(E)$, the results are in Table 4.4.15, with a similar interpretation of the values in the cells as for Table 4.4.14.

Table 4.4.15: Continuity of left multiplications on $\mathscr{L}_{\mathrm{ob}}(E)$.

	o	uo	$\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$	SO	SUO	$\mathrm{S}\widehat{\tau}_E$
o	$\mathscr{L}_{\mathrm{oc}}(E)$	$\mathscr{L}_{\mathrm{oc}}(E)$	$\mathscr{L}_{\mathrm{oc}}(E)$	$\mathscr{L}_{\mathrm{oc}}(E)$	$\mathscr{L}_{\mathrm{oc}}(E)$	$\mathscr{L}_{\mathrm{oc}}(E)$
uo	$\{0\}$	$\mathrm{Orth}(E)$	$\mathrm{Orth}(E)$	$\{0\}$	$\{0\}$	$\{0\}$
$\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$	$\{0\}$	$\{0\}$	$\mathrm{Orth}(E)$	$\{0\}$	$\{0\}$	$\{0\}$
SO	$\{0\}$	$\{0\}$	$\{0\}$	$\mathscr{L}_{\mathrm{oc}}(E)$	$\mathscr{L}_{\mathrm{oc}}(E)$	$\mathscr{L}_{\mathrm{oc}}(E)$
SUO	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathrm{Orth}(E)$	$\mathrm{Orth}(E)$
$\mathrm{S}\widehat{\tau}_E$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathrm{Orth}(E)$

For Table 4.4.15, the values of the cells can be determined using the zeroes in Table 4.3.1, the ‘standard implications’ as listed for Table 4.4.14, combined with Proposition 4.4.1, Remark 4.4.3, Proposition 4.4.7, Remark 4.4.10, Proposition 4.4.11, and Remark 4.4.13.

For multiplications on $\mathrm{Orth}(E)$, the continuity properties are given by Table 4.4.16. In that table, a value 1 in a cell with a row label indicating a convergence structure $\mathscr{C}_1$ and a column label indicating a convergence structure $\mathscr{C}_2$ means that the maps $\rho_T = \lambda_T : \mathrm{Orth}(E) \to \mathrm{Orth}(E)$ is $\mathscr{C}_1$-$\mathscr{C}_2$ continuous for all $T \in \mathrm{Orth}(E)$. A value 0 means that there exists a Dedekind complete vector lattice E and a $T \in \mathrm{Orth}(E)$ for which this is not the case.

Table 4.4.16: Continuity of multiplications on $\mathrm{Orth}(E)$.

	o	uo	$\widehat{\tau}_{\mathrm{Orth}}(E)$	SO	SUO	$\mathrm{S}\widehat{\tau}_E$
o	1	1	1	1	1	1
uo	0	1	1	0	1	1
$\widehat{\tau}_{\mathrm{Orth}(E)}$	0	0	1	0	0	1
SO	0	1	1	1	1	1
SUO	0	1	1	0	1	1
$\mathrm{S}\widehat{\tau}_E$	0	0	1	0	0	1

In $\mathrm{Orth}(E)$, uo and SUO convergence of nets coincide, as do a possible $\widehat{\tau}_{\mathrm{Orth}}(E)$ and $\mathrm{S}\widehat{\tau}_E$ convergence.

The values in the cells of Table 4.4.16 can be determined using the zeroes in Table 4.3.2, the ‘standard implications’ as listed for Table 4.4.14; the fact that $\mathrm{Orth}(E)$ is a regular vector sublattice of $\mathscr{L}_{\mathrm{ob}}(E)$; the facts that unbounded order convergence and strong unbounded order convergence coincide on $\mathrm{Orth}(E)$, as do a possible $\widehat{\tau}_{\mathrm{Orth}(E)}$ and $\mathrm{S}\widehat{\tau}_E$ convergence; combined with Proposition 4.4.1, Proposition 4.4.7, and Proposition 4.4.11.

4.5 Simultaneous continuity of multiplications and adherences of subalgebras of $\mathscr{L}_{\mathrm{ob}}(E)$

In this section, we study the simultaneous continuity of the multiplications in subalgebras of $\mathscr{L}_{\mathrm{ob}}(E)$ (where E is a Dedekind complete vector lattice) with respect to the six convergence structures under consideration in this paper. This is motivated by questions of the following type. Suppose that E admits a Hausdorff uo-Lebesgue topology. Take a subalgebra (not necessarily a vector lattice subalgebra) of $\mathscr{L}_{\mathrm{ob}}(E)$. Is its adherence $a_{S\widehat{\tau}_E}(\mathscr{A})$ in $\mathscr{L}_{\mathrm{ob}}(E)$ with respect to strong $\widehat{\tau}_E$ convergence again a subalgebra of $\mathscr{L}_{\mathrm{ob}}(E)$? This is not always the case, not even when $\mathscr{A} \subseteq \mathscr{L}_{\mathrm{oc}}(E)$; see Example 4.5.13, below. When $\mathscr{A} \subseteq \mathrm{Orth}(E)$, however, the answer is affirmative; see Corollary 4.5.12, below.

As the reader may verify, it follows already from the continuity of the left and right multiplications with respect to strong $\widehat{\tau}_E$ convergence (see Proposition 4.4.11) that $a_{S\widehat{\tau}_E}(\mathscr{A}) \cdot a_{S\widehat{\tau}_E}(\mathscr{A}) \subseteq a_{S\widehat{\tau}_E}(a_{S\widehat{\tau}_E}(\mathscr{A}))$ when $\mathscr{A} \subseteq \mathrm{Orth}(E)$, but that is not sufficient to show that $a_{S\widehat{\tau}_E}(\mathscr{A})$ is a subalgebra. The simultaneous continuity of the multiplication in $\mathrm{Orth}(E)$ with respect to strong $\widehat{\tau}_E$ convergence in $\mathrm{Orth}(E)$ would be sufficient to conclude this, and this can indeed be established; see Proposition 4.5.11, below.

For each of the remaining five convergence structures, we follow the same pattern. We establish (this also relies on the single variable results in Section 4.4) the simultaneous continuity of the multiplication with respect to the convergence structure under consideration, and then conclude that the pertinent adherence of a subalgebra is again a subalgebra. For the latter result it is—as the above example already indicates—essential to impose an extra condition on the subalgebra. This condition depends on the convergence structure under consideration. Natural extra conditions are that it be a subalgebra of $\mathrm{Orth}(E)$ or of $\mathscr{L}_{\mathrm{oc}}(E)$ and we obtain positive results under such conditions. We also have fairly complete results showing that the relaxation of the pertinent condition to the 'natural' next lenient one does, in fact, render the statement that the adherence is a subalgebra again invalid. This also implies that multiplication is then not simultaneously continuous.

In the cases where the lattice operations are known to be simultaneously continuous with respect to the convergence structure under consideration, it obviously also follows that the pertinent adherence of a vector lattice subalgebra is a vector lattice subalgebra again.

We shall now embark on this programme. We start with order convergence, which is the easiest case. For this, we have the following result on the simultaneous continuity of multiplication.

Proposition 4.5.1. *Let E be a Dedekind complete vector lattice. Suppose that $(T_\alpha)_{\alpha\in\mathcal{A}}$ is a net in $\mathscr{L}_{\mathrm{oc}}(E)$ such that $S_\alpha \xrightarrow{\mathrm{o}} S$ in $\mathscr{L}_{\mathrm{ob}}(E)$ for some $S \in \mathscr{L}_{\mathrm{ob}}(E)$ and that $(T_\beta)_{\beta\in\mathcal{B}} \subseteq \mathscr{L}_{\mathrm{ob}}(E)$ is a net such that $T_\beta \xrightarrow{\mathrm{o}} T$ in $\mathscr{L}_{\mathrm{ob}}(E)$ for some $T \in \mathscr{L}_{\mathrm{ob}}(E)$. Then $S \in \mathscr{L}_{\mathrm{oc}}(E)$ and $S_\alpha T_\beta \xrightarrow{\mathrm{o}} ST$ in $\mathscr{L}_{\mathrm{ob}}(E)$.*

Proof. It is clear that $S \in \mathscr{L}_{\mathrm{oc}}(E)$. By passing to a tail, we may suppose that $(|T_\beta|)_{\beta\in\mathcal{B}}$ is bounded above by some $R \in \mathscr{L}_{\mathrm{ob}}(E)^+$. Using the parts (1) and (2) of Proposition 4.4.1 for the final order convergence, we have that

$$|S_\alpha T_\beta - ST| \leq |S_\alpha T_\beta - ST_\beta| + |ST_\beta - ST|$$

$$\leq |S_\alpha - S|R + |S||T_\beta - T| \xrightarrow{\mathrm{o}} 0$$

in $\mathscr{L}_{\mathrm{ob}}(E)$. Hence $S_\alpha T_\beta \xrightarrow{\mathrm{o}} ST$ in $\mathscr{L}_{\mathrm{ob}}(E)$. $\square$

The following is now clear from Proposition 4.5.1 and the simultaneous order continuity of the lattice operations.

Corollary 4.5.2. *Let E be a Dedekind complete vector lattice. Suppose that $\mathscr{A}$ is a subalgebra of $\mathscr{L}_{\mathrm{oc}}(E)$. Then the adherence $a_{\mathrm{o}}(\mathscr{A})$ in $\mathscr{L}_{\mathrm{ob}}(E)$ is also a subalgebra of $\mathscr{L}_{\mathrm{oc}}(E)$. When $\mathscr{A}$ is a vector lattice subalgebra of $\mathscr{L}_{\mathrm{oc}}(E)$, then so is $a_{\mathrm{o}}(\mathscr{A})$.*

We now show that the condition in Corollary 4.5.2 that $\mathscr{A} \subseteq \mathscr{L}_{\mathrm{oc}}(E)$ cannot be relaxed to $\mathscr{A} \subseteq \mathscr{L}_{\mathrm{ob}}(E)$.

Example 4.5.3. Take $E = \ell_\infty$ and let $(e_n)_{n=1}^\infty$ be the standard sequence of unit vectors in E. We define $T \in \mathscr{L}_{\mathrm{ob}}(E)$ as in Examples 4.4.2. For $n \geq 1$, we now define $S_n' \in \mathscr{L}_{\mathrm{oc}}(E)$ by setting

$$S_n' x := x_2 \bigvee_{i=3}^{n+2} e_i,$$

and $S' \in \mathscr{L}_{\mathrm{oc}}(E)$ by setting

$$S' x := x_2 \bigvee_{i=3}^{\infty} e_i$$

for $x = \bigvee_{i=1}^\infty x_i e_i \in E$. It is easily verified that $T^2 = 0$, that $S_n' S_m' = 0$ for $m, n \geq 1$, and that $S_n' T = T S_n' = 0$ for $n \geq 1$. Hence $\mathscr{A} := \mathrm{Span}\{\, T, S_n' : n \geq 1 \,\}$ is a subalgebra of $\mathscr{L}_{\mathrm{ob}}(E)$. As $S_n' \uparrow S'$ in $\mathscr{L}_{\mathrm{ob}}(E)$, both S' and T are elements of $a_{\mathrm{o}}(\mathscr{A})$.

However, $TS' \notin a_{\mathrm{o}}(\mathscr{A})$. In fact, TS' is not even an element of $a_{\mathrm{SO}}(\mathscr{A}) \supseteq a_{\mathrm{o}}(\mathscr{A})$. To see this, we observe that $TS'e_2 = e_1 \neq 0$, and that, as is easily verified, $TS'e_2 \perp Re_2$ for all $R \in \mathscr{A}$. Hence there cannot exist a net $(R_\alpha)_{\alpha \in \mathcal{A}} \subseteq \mathscr{A}$ such that $R_\alpha e_2 \xrightarrow{\mathrm{o}} TS'e_2$ in E, let alone such that $R_\alpha \xrightarrow{\mathrm{SO}} TS'$ in $\mathscr{L}_{\mathrm{ob}}(E)$.

Now we turn to the strong order adherences of subalgebras of $\mathscr{L}_{\mathrm{ob}}(E)$. We start by showing that $\mathrm{Orth}(E)$ is closed in $\mathscr{L}_{\mathrm{ob}}(E)$ under the convergences under consideration in this paper. We recall from Theorem 4.2.1 that either all of E, $\mathrm{Orth}(E)$, and $\mathscr{L}_{\mathrm{ob}}(E)$ admit a Hausdorff uo-Lebesgue topology, or none does.

Lemma 4.5.4. *Let E be a Dedekind complete vector lattice. Then* $\mathrm{Orth}(E)$ *is closed in $\mathscr{L}_{\mathrm{ob}}(E)$ under order convergence, unbounded order convergence, strong order convergence, and strong unbounded order convergence. Suppose that E admits a (necessarily unique) Hausdorff uo-Lebesgue topology. Then* $\mathrm{Orth}(E)$ *is closed in $\mathscr{L}_{\mathrm{ob}}(E)$ under $\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$ convergence and strong $\widehat{\tau}_E$ convergence.*

Proof. The statements concerning order convergence, unbounded order convergence, and convergence in a possible Hausdorff uo-Lebesgue topology on $\mathscr{L}_{\mathrm{ob}}(E)$ are evident, since $\mathrm{Orth}(E)$ is a band in $\mathscr{L}_{\mathrm{ob}}(E)$. These three general properties of bands in vector lattices, but now for bands in E, also imply that, for each of the three strong convergences, a limit in $\mathscr{L}_{\mathrm{ob}}(E)$ of a net in $\mathrm{Orth}(E)$ is again a band preserving operator on E. $\square$

Proposition 4.5.5. *Let E be a Dedekind complete vector lattice. Suppose that $(T_\alpha)_{\alpha\in\mathcal{A}}$ is a net in* $\mathrm{Orth}(E)$ *such that $S_\alpha \xrightarrow{\mathrm{SO}} S$ in $\mathscr{L}_{\mathrm{ob}}(E)$ for some $S \in \mathscr{L}_{\mathrm{ob}}(E)$ and that $(T_\beta)_{\beta\in\mathcal{B}} \subseteq \mathscr{L}_{\mathrm{ob}}(E)$ is a net such that $T_\beta \xrightarrow{\mathrm{SO}} T$ in $\mathscr{L}_{\mathrm{ob}}(E)$ for some $T \in \mathscr{L}_{\mathrm{ob}}(E)$. Then $S \in$* $\mathrm{Orth}(E)$*, and $S_\alpha T_\beta \xrightarrow{\mathrm{SO}} ST$ in $\mathscr{L}_{\mathrm{ob}}(E)$.*

Proof. Lemma 4.5.4 shows that $S \in \mathrm{Orth}(E)$. Take $x \in E$. By passing to a tail, we may suppose that $(|T_\beta x|)_{\beta\in\mathcal{B}}$ is bounded above by some $y \in E^+$. By applying [7, Theorem 2.43] and the order continuity of $|S|$ for the final convergence, we see that

$$\begin{aligned}
|S_\alpha T_\beta x - STx| &\leq |(S_\alpha - S)T_\beta x| + |S(T_\beta - T)x| \\
&\leq |S_\alpha - S||T_\beta x| + |S||(T_\beta - T)x| \\
&\leq |S_\alpha - S|y + |S||T_\beta x - Tx| \\
&= |(S_\alpha - S)y| + |S||T_\beta x - Tx| \xrightarrow{\mathrm{o}} 0
\end{aligned}$$

in E. Hence $S_\alpha T_\beta \xrightarrow{\mathrm{SO}} ST$ in $\mathscr{L}_{\mathrm{ob}}(E)$. □

The following is now clear from Proposition 4.5.5.

Corollary 4.5.6. *Let E be a Dedekind complete vector lattice. Suppose that $\mathscr{A}$ is a subalgebra of* $\mathrm{Orth}(E)$*. Then the adherence $a_{\mathrm{SO}}(\mathscr{A})$ in $\mathscr{L}_{\mathrm{ob}}(E)$ is also a subalgebra of* $\mathrm{Orth}(E)$*.*

We now show that the condition in Corollary 4.5.6 that $\mathscr{A} \subseteq \mathrm{Orth}(E)$ cannot be relaxed to $\mathscr{A} \subseteq \mathscr{L}_{\mathrm{ob}}(E)$. At the time of writing, the authors do not know whether it might be relaxed to $\mathscr{A} \subseteq \mathscr{L}_{\mathrm{oc}}(E)$.

Example 4.5.7. We resort to the context and notation of Example 4.5.3. In that example, we had operators $T, S' \in a_{\mathrm{o}}(\mathscr{A})$ such that $TS' \notin a_{\mathrm{SO}}(\mathscr{A})$. Since $a_{\mathrm{o}}(\mathscr{A}) \subseteq a_{\mathrm{SO}}(\mathscr{A})$, this example also provides an example as currently needed.

We turn to unbounded order adherences and strong unbounded order adherences.

Proposition 4.5.8. *Let E be a Dedekind complete vector lattice. Suppose that $(T_\alpha)_{\alpha\in\mathcal{A}}$ is a net in* $\mathrm{Orth}(E)$ *such that $S_\alpha \xrightarrow{\mathrm{uo}} S$ in $\mathscr{L}_{\mathrm{ob}}(E)$ for some $S \in \mathscr{L}_{\mathrm{ob}}(E)$ and that $(T_\beta)_{\beta\in\mathcal{B}} \subseteq$* $\mathrm{Orth}(E)$ *is a net such that $T_\beta \xrightarrow{\mathrm{uo}} T$ in $\mathscr{L}_{\mathrm{ob}}(E)$ for some $T \in \mathscr{L}_{\mathrm{ob}}(E)$. Then $S, T \in$* $\mathrm{Orth}(E)$*, and $S_\alpha T_\beta \xrightarrow{\mathrm{uo}} ST$ in $\mathscr{L}_{\mathrm{ob}}(E)$. Seven similar statements hold that are obtained by, for each of the three occurrences of unbounded order convergence, either keeping it or replacing it with strong unbounded order convergence.*

Proof. We start with the statement for three occurrences of unbounded order convergence. For this, we first suppose that $S = T = 0$.

For $\alpha \in \mathcal{A}$, let $\mathcal{P}_\alpha$ be the order projection in $\mathrm{Orth}(E)$ onto the band B_α in $\mathrm{Orth}(E)$ that is generated by $(|S_\alpha| - I)^+$. Then $0 \leq \mathcal{P}_\alpha I \leq \mathcal{P}_\alpha |S_\alpha| \leq |S_\alpha|$ by [19, Lemma 6.6]. Hence $\mathcal{P}_\alpha I \xrightarrow{\mathrm{uo}} 0$ in $\mathscr{L}_{\mathrm{ob}}(E)$, so that also $\mathcal{P}_\alpha I \xrightarrow{\mathrm{uo}} 0$ in the regular vector sublattice $\mathrm{Orth}(E)$ of $\mathscr{L}_{\mathrm{ob}}(E)$ by [28, Theorem 3.2]. Since the net $(\mathcal{P}_\alpha I)_{\alpha\in\mathcal{A}}$ is order bounded in $\mathrm{Orth}(E)$, we see that

$$\mathcal{P}_\alpha I \xrightarrow{\mathrm{o}} 0 \tag{4.3}$$

in Orth(E). Furthermore, since $(\mathcal{P}_\alpha|S_\alpha|)T_\beta \in B_\alpha$ for $\alpha \in \mathcal{A}, \beta \in \mathcal{B}$, (see [7, Theorem 2.62] or Corollary 4.2.5), we also have that $[(\mathcal{P}_\alpha|S_\alpha|)T_\beta] \wedge I \in B_\alpha$ for $\alpha \in \mathcal{A}, \beta \in \mathcal{B}$. Hence

$$[(\mathcal{P}_\alpha|S_\alpha|)T_\beta] \wedge I = \mathcal{P}_\alpha\big([(\mathcal{P}_\alpha|S_\alpha|)T_\beta] \wedge I\big) \leq \mathcal{P}_\alpha I \tag{4.4}$$

for $\alpha \in \mathcal{A}, \beta \in \mathcal{B}$.

Combining the fact that $|S_\alpha| \leq I + \mathcal{P}_\alpha|S_\alpha|$ by [19, Proposition 6.7(2)] with equation (4.4), we have, for $\alpha \in \mathcal{A}, \beta \in \mathcal{B}$,

$$\begin{aligned} |S_\alpha T_\beta| \wedge I &\leq (|S_\alpha||T_\beta|) \wedge I \\ &\leq [(I + \mathcal{P}_\alpha|S_\alpha|)|T_\beta|] \wedge I \\ &\leq |T_\beta| \wedge I + [(\mathcal{P}_\alpha|S_\alpha|)|T_\beta|] \wedge I \\ &\leq |T_\beta| \wedge I + \mathcal{P}_\alpha I. \end{aligned}$$

The fact that $T_\beta \xrightarrow{\text{uo}} 0$ in $\mathscr{L}_{\text{ob}}(E)$ and then also in Orth(E), together with equation (4.3), now shows that $|S_\alpha T_\beta| \wedge I \xrightarrow{\text{o}} 0$ in Orth(E). Since I is a weak order unit of Orth(E), [28, Corollary 3.5] (or the more general [20, Proposition 7.4]) implies that $S_\alpha T_\beta \xrightarrow{\text{uo}} 0$ in Orth(E) and then also in $\mathscr{L}_{\text{ob}}(E)$.

For the case of general S and T, we first note that $S, T \in$ Orth(E) as a consequence of Lemma 4.5.4. On writing

$$S_\alpha T_\beta - TS = (S_\alpha - S)(T_\beta - T) + S_\alpha T + S T_\beta - 2TS,$$

the special case considered above, together with Proposition 4.4.7, then implies that $S_\alpha T_\beta \xrightarrow{\text{uo}} ST$ in $\mathscr{L}_{\text{ob}}(E)$, as desired.

On invoking Lemma 4.5.4, [19, Theorem 9.9], and [28, Theorem 3.2], the remaining seven statements follow from the case just established. □

The following is now clear from Proposition 4.5.8, [19, Theorem 9.9], and the simultaneous unbounded order continuity of the lattice operations.

Corollary 4.5.9. *Let E be a Dedekind complete vector lattice, and let $\mathscr{A}$ be a subalgebra of* Orth(E)*. Then the adherences $a_{\text{uo}}(\mathscr{A})$ and $a_{\text{SUO}}(\mathscr{A})$ in $\mathscr{L}_{\text{ob}}(E)$ are equal, and are a subalgebra of* Orth(E)*. When $\mathscr{A}$ is a vector lattice subalgebra of* Orth(E)*, then so is $a_{\text{uo}}(\mathscr{A}) = a_{\text{SUO}}(\mathscr{A})$.*

We now show that, neither for $a_{\text{uo}}(\mathscr{A})$ to be a subalgebra of $\mathscr{L}_{\text{ob}}(E)$, nor for $a_{\text{SUO}}(\mathscr{A})$ to be a subalgebra of $\mathscr{L}_{\text{ob}}(E)$, the condition in Corollary 4.5.9 that $\mathscr{A} \subseteq$ Orth(E) can be relaxed to $\mathscr{A} \subseteq \mathscr{L}_{\text{oc}}(E)$.

Example 4.5.10. Let $E = \ell_1$, and let $(e_n)_{n=1}^\infty$ be the standard sequence of unit vectors in E. For $i, j \geq 1$, we define $S_{i,j} \in \mathscr{L}_{\text{oc}}(E) = \mathscr{L}_{\text{ob}}(E)$ by setting

$$S_{i,j}e_n := \begin{cases} e_j & \text{if } n = i; \\ 0 & \text{if } n \neq i \end{cases}$$

for $n \geq 1$, and we define $T \in \mathscr{L}_{\mathrm{oc}}(E)$ by setting

$$Tx := \left(\sum_{i=2}^{\infty} x_i\right) e_3$$

for $x = \bigvee_{i=1}^{\infty} x_i e_i \in E$. Set $S_n := S_{1,2} - S_{1,n+3}$ for $n \geq 1$. It is not hard to check that $T^2 = T$, that $S_n T = T S_n = 0$ for $n \geq 1$, and that $S_m S_n = 0$ for $m, n \geq 1$. Hence $\mathscr{A} := \operatorname{Span}\{T, S_n : n \geq 1\}$ is a subalgebra of $\mathscr{L}_{\mathrm{oc}}(E)$.

Using [7, Theorem 1.51], it is easy to see that $(S_{1,n+3})_{n=1}^{\infty}$ is a disjoint sequence in $\mathscr{L}_{\mathrm{ob}}(E)$, so that $S_{1,n+3} \xrightarrow{\mathrm{uo}} 0$ in $\mathscr{L}_{\mathrm{ob}}(E)$ by [28, Corollary 3.6]. Hence $S_n \xrightarrow{\mathrm{uo}} S_{1,2}$ in $\mathscr{L}_{\mathrm{ob}}(E)$, showing that $S_{1,2} \in a_{\mathrm{uo}}(\mathscr{A})$. Obviously, $T \in a_{\mathrm{uo}}(\mathscr{A})$. We claim that, however, $TS_{1,2}$ is not even an element of $\overline{\mathscr{A}}^{\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}} \supseteq a_{\mathrm{uo}}(\mathscr{A})$. In order to see this, we observe that $TS_{1,2} = S_{1,3}$ and, using [7, Theorem 1.51], that $S_{1,3} \perp T$ and $S_{1,3} \perp S_n$ for $n \geq 1$. Hence $TS_{1,2} \perp \mathscr{A}$, which implies that $TS_{1,2} \notin \overline{\mathscr{A}}^{\widehat{\tau}_{\mathrm{Orth}(E)}}$.

For $x = \bigvee_{i=1}^{\infty} x_i e_i \in \ell_1$, we have $S_{1,n+3}x = x_1 e_{n+3}$ for $n \geq 1$. This implies that $S_{1,n+3} \xrightarrow{\mathrm{SUO}} 0$ in $\mathscr{L}_{\mathrm{ob}}(E)$, showing that $S_n \xrightarrow{\mathrm{SUO}} S_{1,2}$ in $\mathscr{L}_{\mathrm{ob}}(E)$. Hence $S_{1,2} \in a_{\mathrm{SUO}}(\mathscr{A})$. Obviously, $T \in a_{\mathrm{SUO}}(E)$. We claim that, however, $TS_{1,2}$ is not even an element of $a_{S\widehat{\tau}_E}(\mathscr{A}) \supseteq a_{\mathrm{SUO}}(\mathscr{A})$. In order to see this, it is sufficient to observe that $TSe_{1,2} = e_3 \neq 0$ and that $TS_{1,2}e_1 \perp Re_1$ for all $R \in \mathscr{A}$. This implies that there cannot exist a net $(R_\alpha)_{\alpha \in \mathcal{A}} \subseteq \mathscr{A}$ such that $R_\alpha e_1 \xrightarrow{\widehat{\tau}_E} TS' e_1$ in E, let alone such that $R_\alpha \xrightarrow{S\widehat{\tau}_E} TS'$ in $\mathscr{L}_{\mathrm{ob}}(E)$.

We turn to closures in a Hausdorff uo-Lebesgue topology and strong closures with respect to a Hausdorff uo-Lebesgue topology. We recall once more from Theorem 4.2.1 that either all of E, $\mathrm{Orth}(E)$, and $\mathscr{L}_{\mathrm{ob}}(E)$ admit a Hausdorff uo-Lebesgue topology, or none does. If they do, then, by general principles (see [44, Proposition 5.12]), $\widehat{\tau}_{\mathrm{Orth}(E)}$ is the restriction of $\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$ to $\mathrm{Orth}(E)$.

Proposition 4.5.11. *Let E be a Dedekind complete vector lattice that admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$. Suppose that $(T_\alpha)_{\alpha \in \mathcal{A}} \subseteq \mathrm{Orth}(E)$ is a net such that $S_\alpha \xrightarrow{\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}} S$ in $\mathscr{L}_{\mathrm{ob}}(E)$ for some $S \in \mathscr{L}_{\mathrm{ob}}(E)$ and that $(T_\beta)_{\beta \in \mathcal{B}} \subseteq \mathrm{Orth}(E)$ is a net such that $T_\beta \xrightarrow{\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}} T$ in $\mathscr{L}_{\mathrm{ob}}(E)$ for some $T \in \mathscr{L}_{\mathrm{ob}}(E)$. Then $S, T \in \mathrm{Orth}(E)$, and $S_\alpha T_\beta \xrightarrow{\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}} ST$ in $\mathscr{L}_{\mathrm{ob}}(E)$. Seven similar statements hold that are obtained by, for each of the three occurrences of $\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$ convergence, either keeping it or replacing it with strong $\widehat{\tau}_E$ convergence.*

Proof. We start with the statement for three occurrences of $\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$ convergence. For this, we first suppose that $S = T = 0$.

We can use parts of the proof of Proposition 4.5.8 here. In that proof, it was established that, for $\alpha \in \mathcal{A}$, there exists a band projection $\mathcal{P}_\alpha$ in $\mathrm{Orth}(E)$ such that

$$0 \leq \mathcal{P}_\alpha I \leq |S_\alpha| \tag{4.5}$$

and such that

$$|S_\alpha T_\beta| \wedge I \leq |T_\beta| \wedge I + \mathcal{P}_\alpha I \tag{4.6}$$

for $\beta \in \mathcal{B}$. It follows from equation (4.5) that also

$$\mathcal{P}_\alpha I \xrightarrow{\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}} 0$$

in $\mathscr{L}_{\mathrm{ob}}(E)$, and then equation (4.6) shows that $|S_\alpha T_\beta| \wedge I \xrightarrow{\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}} 0$ in $\mathscr{L}_{\mathrm{ob}}(E)$, so that also

$$|S_\alpha T_\beta| \wedge I \xrightarrow{\widehat{\tau}_{\mathrm{Orth}(E)}} 0 \tag{4.7}$$

in $\mathrm{Orth}(E)$. The ideal of $\mathrm{Orth}(E)$ that is generated by I in $\mathscr{L}_{\mathrm{ob}}(E)$ is order dense in $\mathscr{L}_{\mathrm{ob}}(E)$. It follows from [7, Theorem 1.36] that its order adherence in $\mathscr{L}_{\mathrm{ob}}(E)$, as well as in $Orth(E)$, is exactly $\mathrm{Orth}(E)$. Hence it is certainly $\widehat{\tau}_{\mathrm{Orth}(E)}$ dense in $\mathrm{Orth}(E)$. Since $\widehat{\tau}_{\mathrm{Orth}(E)}$ is an unbounded topology on $\mathrm{Orth}(E)$, it now follows from equation (4.7) and [33, Corollary 3.5] that $S_\alpha T_\beta \xrightarrow{\widehat{\tau}_{\mathrm{Orth}(E)}} 0$ in $\mathrm{Orth}(E)$, and then also $S_\alpha T_\beta \xrightarrow{\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}} 0$ in $\mathscr{L}_{\mathrm{ob}}(E)$.

For the case of general S and T, we first note that $S, T \in \mathrm{Orth}(E)$ as a consequence of Lemma 4.5.4. On writing

$$S_\alpha T_\beta - TS = (S_\alpha - S)(T_\beta - T) + S_\alpha T + ST_\beta - 2TS,$$

we see that the special case considered above, together with Proposition 4.4.11, implies that $S_\alpha T_\beta \xrightarrow{\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}} ST$ in $\mathscr{L}_{\mathrm{ob}}(E)$, as desired.

On invoking Lemma 4.5.4 and [19, Theorem 9.12], the remaining seven statements follow from the case just established. □

The following is now clear from Proposition 4.5.11 and the simultaneous continuity of the lattice operations with respect to the $\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}$ topology.

Corollary 4.5.12. *Let E be a Dedekind complete vector lattice that admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$. Suppose that $\mathscr{A}$ is a subalgebra of* $\mathrm{Orth}(E)$*. Then the closure $\overline{\mathscr{A}}^{\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}}$ in $\mathscr{L}_{\mathrm{ob}}(E)$ and the adherence $a_{S\widehat{\tau}_E}(\mathscr{A})$ in $\mathscr{L}_{\mathrm{ob}}(E)$ are equal, and are a subalgebra of* $\mathrm{Orth}(E)$*. When $\mathscr{A}$ is a vector lattice subalgebra of* $\mathrm{Orth}(E)$*, then so is $\overline{\mathscr{A}}^{\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}} = a_{S\widehat{\tau}_E}(\mathscr{A})$.*

We now show that, neither for $\overline{\mathscr{A}}^{\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}}$ to be a subalgebra of $\mathscr{L}_{\mathrm{ob}}(E)$, nor for $a_{S\widehat{\tau}_E}(\mathscr{A})$ to be a subalgebra of $\mathscr{L}_{\mathrm{ob}}(E)$, the condition in Corollary 4.5.12 that $\mathscr{A} \subseteq \mathrm{Orth}(E)$ can be relaxed to $\mathscr{A} \subseteq \mathscr{L}_{\mathrm{oc}}(E)$.

Example 4.5.13. We return to the context and notation of Example 4.5.10. In that example, we saw that $S_n \xrightarrow{\mathrm{uo}} S_{1,2}$ in $\mathscr{L}_{\mathrm{ob}}(E)$. Then certainly $S_n \xrightarrow{\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}}} S_{1,2}$ in $\mathscr{L}_{\mathrm{ob}}(E)$, so that both T and $S_{1,2}$ are elements of $\overline{\mathscr{A}}^{\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}}$. We saw in Example 4.5.10, however, that $TS_{1,2} \notin \overline{\mathscr{A}}^{\widehat{\tau}_{\mathscr{L}_{\mathrm{ob}}(E)}}$.

It was also observed that $S_n \xrightarrow{\mathrm{SUO}} S_{1,2}$ in $\mathscr{L}_{\mathrm{ob}}(E)$. Since E is atomic, the unbounded order convergence of a net in E and its convergence in the Hausdorff uo-Lebesgue topology on E are known to coincide (see [13, Proposition 1] and [44, Lemma 7.4]). Thus also $S_n \xrightarrow{S\widehat{\tau}_E} S$, so that both T and $S_{1,2}$ are elements of $a_{S\hat{\tau}_E}(\mathscr{A})$. We saw in Example 4.5.10, however, that $TS_{1,2} \notin a_{S\hat{\tau}_E}(\mathscr{A})$.

4.6 Equality of adherences of vector sublattices

In this section, we establish the equality of various adherences of vector sublattices with respect to convergence structures under consideration in this paper. We pay special attention to vector sublattices of the orthomorphisms on a Dedekind complete vector lattice. Apart from the intrinsic interest of the results, our research in this direction is also motivated by representation theory. We shall now explain this.

Suppose that E is a vector lattice, and that $\mathscr{S}$ is a non-empty set of order bounded linear operators on E. For example, E could be a group of order automorphisms of E, as arises naturally when considering positive representations of groups on vector lattices. Likewise, $\mathscr{S}$ could be a (vector lattice) algebra of order bounded linear operators, as arises naturally when considering positive representations of (vector lattice) algebras on vector lattices. One of the main issues in representation theory is to investigate the possible decompositions of a module into submodules. In our case, this is asking for decompositions $E = F_1 \oplus F_2$ as an order direct sum of vector sublattices F_1 and F_2 that are both invariant under $\mathscr{S}$. It is well known (see [51, Theorem 11.3] for an even stronger result) that F_1 and F_2 are then projection bands that are each other's disjoint complements. Their respective order projections then commute with all elements of $\mathscr{S}$. Conversely, when an order projection has this property, then E is the order direct sum of its range and its kernel, and both are invariant under $\mathscr{S}$. All in all, the decomposition question for the action of $\mathscr{S}$ on E is the same as asking for the order projections on E that commute with $\mathscr{S}$. This makes it natural to ask for the commutant of $\mathscr{S}$ in $\mathrm{Orth}(E)$, where these order projections reside. This commutant is obviously an associative subalgebra of $\mathrm{Orth}(E)$. Somewhat surprisingly, it is actually also a vector sublattice of $\mathrm{Orth}(E)$ in quite a few cases of interest. For example, this is always true for Banach lattices, in which case the operators in $\mathscr{S}$ need not even be regular. Being bounded is enough, as is shown by the following result, for which the Banach lattice need not even be Dedekind complete.

Theorem 4.6.1. *Let E be a Banach lattice, and let $\mathscr{S}$ be a non-empty set of bounded linear operators on E. Then the commutant*

$$\mathscr{S}'^{\circ} := \{\, T \in \mathrm{Orth}(E) : TS = ST \text{ for all } S \in \mathscr{S} \,\}$$

of $\mathscr{S}$ in $\mathrm{Orth}(E)$ *is a Banach f-subalgebra of* $\mathrm{Orth}(E)$ *that contains the identity operator I as a strong order unit; here* $\mathrm{Orth}(E)$ *is supplied with the coinciding operator norm and order unit norm $\|\cdot\|_I$.*

Proof. It is obvious that $\mathscr{S}'^{\circ}$ is an associative subalgebra of $\mathrm{Orth}(E)$ that contains I and that is closed with respect to the coinciding operator norm and order unit norm $\|\cdot\|_I$. An appeal to [19, Theorem 6.1] then finishes the proof. □

For Dedekind complete vector lattices, we have the following.

Theorem 4.6.2. *Let E be a Dedekind complete vector lattice, and let $\mathscr{S}$ be a non-empty subset of $\mathscr{L}_{\mathrm{oc}}(E)^{+} \cup \mathscr{L}_{\mathrm{oc}}(E)^{-}$. Then the commutant*

$$\mathscr{S}'^{\circ} := \{\, T \in \mathrm{Orth}(E) : TS = ST \textit{ for all } S \in \mathscr{S} \,\}$$

of $\mathscr{S}$ in Orth(E) *is a vector lattice subalgebra of* Orth(E) *that contains the identity operator I as a weak order unit. Furthermore:*

(1) $\mathscr{S}'^{\circ}$ is an order closed vector sublattice of every regular vector sublattice of $\mathscr{L}_{\mathrm{ob}}(E)$ containing $\mathscr{S}'^{\circ}$;

(2) $\mathscr{S}'^{\circ}$ is a regular vector sublattice of every Dedekind complete regular vector sublattice of $\mathscr{L}_{\mathrm{ob}}(E)$ containing $\mathscr{S}'^{\circ}$;

(3) $\mathscr{S}'^{\circ}$ is a Dedekind complete vector lattice.

Proof. We start by proving that $\mathscr{S}'^{\circ}$ is a vector sublattice of Orth(E). For this, we may suppose that $\mathscr{S}$ consists of one positive operator $S \in \mathscr{L}_{\mathrm{oc}}(E)$. It is then sufficient to show that, for $T_1, T_2 \in \mathrm{Orth}(E)$, $T_1 \vee T_2$ commutes with S whenever T_1 and T_2 do. We shall now show this. In the argument that is to follow, all left and right multiplication operators are to be viewed as order bounded linear operators on $\mathscr{L}_{\mathrm{ob}}(E)$.

Obviously, $(T_1 \vee T_2)S = \lambda_{T_1 \vee T_2}(S)$ which, by [42, Satz 3.1], equals $(\lambda_{T_1} \vee \lambda_{T_2})(S)$. We know from part (1) of Corollary 4.2.5 that left multiplications by elements of Orth(E) are orthomorphisms on $\mathscr{L}_{\mathrm{ob}}(E)$, so that [6, Theorem 2.43] can be used to conclude that $(\lambda_{T_1} \vee \lambda_{T_2})(S) = \lambda_{T_1}(S) \vee \lambda_{T_2}(S) = (T_1 S) \vee (T_2 S)$ which, as a consequence of the assumption, equals $(ST_1) \vee (ST_2) = \rho_{T_1}(S) \vee \rho_{T_2}(S)$. Part (2) of Corollary 4.2.5 and [6, Theorem 2.43] then show that this equals $[\rho_{T_1} \vee \rho_{T_2}](S)$. So far, we have not used that S is order continuous, but it is at this point that this enables us to conclude from [10, Proposition 2.2] that $[\rho_{T_1} \vee \rho_{T_2}](S) = \rho_{T_1 \vee T_2}(S)$, which is just $S(T_1 \vee T_2)$. Hence $\mathscr{S}'^{\circ}$ is a vector sublattice of Orth(E).

It is clear that $\mathscr{S}'^{\circ}$ is an associative subalgebra of Orth(E) containing I and that I, which is a weak order unit of Orth(E), is also one of the vector lattice $\mathscr{S}'^{\circ}$.

We turn to the remaining statements. Suppose that $(T_\alpha)_{\alpha \in \mathcal{A}}$ is a net in $\mathscr{S}'^{\circ}$, that $T \in \mathscr{L}_{\mathrm{ob}}(E)$, and that $T_\alpha \xrightarrow{\mathrm{o}} T$ in $\mathscr{L}_{\mathrm{ob}}(E)$. Then certainly $T \in \mathrm{Orth}(E)$. Using once more that $\mathscr{S} \subseteq \mathscr{L}_{\mathrm{oc}}(E)$, it follows from Proposition 4.4.1 that T commutes with all elements of $\mathscr{S}$. Hence $T \in \mathscr{S}'^{\circ}$, and we conclude that $\mathscr{S}'^{\circ}$ is an order closed vector sublattice of $\mathscr{L}_{\mathrm{ob}}(E)$. Obviously, it is then also order closed in every regular vector sublattice of $\mathscr{L}_{\mathrm{ob}}(E)$ containing it. We have thus established part (1).

Take a Dedekind complete regular vector sublattice $\mathscr{F}$ of $\mathscr{L}_{\mathrm{ob}}(E)$ that contains $\mathscr{S}'^{\circ}$. Since we know that $\mathscr{S}'^{\circ}$ is order closed in $\mathscr{F}$, [35, p. 303] shows that $\mathscr{S}'^{\circ}$ is a complete vector sublattice of $\mathscr{F}$ as this notion is defined on [35, p. 295-296]. It then follows from [35, p. 296] that $\mathscr{S}'^{\circ}$ is a regular vector sublattice of $\mathscr{F}$ and, on taking $\mathscr{F} = \mathscr{L}_{\mathrm{ob}}(E)$, also that $\mathscr{S}'^{\circ}$ is Dedekind complete. □

Remark 4.6.3. Theorem 4.6.2 applies, in particular, when $\mathscr{S}$ is a group of order automorphisms of E. Obviously, it holds equally well when $\mathscr{S}$ is replaced with a linear subspace of

$\mathscr{L}_{\mathrm{oc}}(E)$ that is spanned by its intersections with the positive and negative cones of $\mathscr{L}_{\mathrm{oc}}(E)$. In particular, it holds whenever $\mathscr{S}$ is a vector sublattice of $\mathscr{L}_{\mathrm{oc}}(E)$; the fact that S'° is then an order closed vector lattice subalgebra of $\mathrm{Orth}(E)$ was already established in [18, Lemma 8.9]. Likewise, it holds whenever $\mathscr{S}$ is an associative subalgebra of $\mathscr{L}_{\mathrm{oc}}(E)$ that is generated, as an associative algebra, by its intersections with the positive and negative cones of $\mathscr{L}_{\mathrm{oc}}(E)$.

In Theorem 4.6.2, the vector lattice $\mathscr{S}'^{\circ}$ is a Dedekind complete vector lattice with the identity operator I as a weak order unit. The unbounded version of Freudenthal's spectral theorem (see [36, Theorem 40.3], for example) then shows that an arbitrary element $T \in \mathscr{S}'^{\circ}$ is an order limit of a sequence of linear combinations of the components of I in $\mathscr{S}'^{\circ}$. Since the latter are precisely the order projections that commute with $\mathscr{S}$ we see that, in this case, $\mathscr{S}'^{\circ}$ does not only contain all information about the collection of bands in E that reduce $\mathscr{S}$, but that it is also completely determined by this collection.

On a later occasion, we shall report more elaborately on the procedures of taking commutants and also of taking bicommutants in vector lattices of order bounded linear operators, as well as on their relations with reducing projection bands for sets of order bounded linear operators. For the moment, we content ourselves with the general observation that the study of vector lattice subalgebras of the orthomorphisms is relevant for representation theory on vector lattices.

We shall now set out to study one particular aspect of this, namely, the equality of the adherences of vector sublattices of the orthomorphism with respect to several of the convergence structures under consideration in this paper. Although from a representation theoretical point of view it would be natural to require that they also be associative subalgebras, this does, so far, not appear to be relevant for these issues. Such results on equal adherences can then also be obtained for associative subalgebras of the orthomorphisms on a Banach lattice, as a consequence of the fact that their norm closures in the orthomorphisms are. in fact, vector sublattices to which the previous results can be applied.

Regarding the results below that are given for vector sublattices of the orthomorphisms, we recall that, for a Dedekind complete vector lattice, several adherences coincide for subsets of the orthomorphisms. Indeed, since, for nets of orthomorphisms, unbounded order convergence coincides with strong unbounded order convergence, and since the convergence in a possible Hausdorff uo-Lebesgue topology coincides with the corresponding strong convergence, the corresponding adherences of subsets of the orthomorphisms are also equal. The same holds for sequential adherences. *For reasons of brevity, we have refrained from including these 'obviously also equal' adherences in the statements.*

Although our motivation leads us to study vector sublattice of the orthomorphisms, the results as we shall derive them for these are actually consequences of more general statements for vector lattices that need not even consist of operators. These are of interest in their own right. Other such results are [20, Theorem 8.8], [26, Theorem 2.13], and [44, Proposition 2.12].

We start by establishing results showing that the closures of vector sublattices (or associative subalgebras) in a possible Hausdorff uo-Lebesgue topology coincide with their closures in other linear topologies on the vector lattices (or associative algebras) under consideration. These are based on the following result which, as the reader may verify, is established in the first paragraph of the proof of [20, Theorem 8.8]. For the definition of the absolute weak topology $|\sigma|(E,I)$ on E that occurs in it we refer to [6, p. 63].

Proposition 4.6.4. *Let E be a vector lattice such that $E_{\mathrm{oc}}^{\sim}$ separates the points of E. Then E admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$. Take an ideal I of $E_{\mathrm{oc}}^{\sim}$ that separates the points of E, and take a vector sublattice F of E. Then*

$$\overline{F}^{\widehat{\tau}_E} = \overline{F}^{\sigma(E,I)} = \overline{F}^{|\sigma|(E,I)}$$

in E.

In the following consequence of Proposition 4.6.4, the lattice $\mathscr{F}$ of operators can be taken to be $\mathrm{Orth}(E)$.

Corollary 4.6.5. *Let E be a Dedekind complete vector lattice such that $E_{\mathrm{oc}}^{\sim}$ separates the points of E, and let $\mathscr{E}$ be a regular vector sublattice of $\mathscr{L}_{\mathrm{ob}}(E)$. Then $\mathscr{E}_{\mathrm{oc}}^{\sim}$ separates the points of $\mathscr{E}$, and $\mathscr{E}$ admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_{\mathscr{E}}$. Take an ideal I of $\mathscr{E}_{\mathrm{oc}}^{\sim}$ that separates the points of $\mathscr{E}$, and take a vector sublattice $\mathscr{F}$ of $\mathscr{E}$. Then*

$$\overline{\mathscr{F}}^{\widehat{\tau}_{\mathscr{E}}} = \overline{\mathscr{F}}^{\sigma(\mathscr{E},I)} = \overline{\mathscr{F}}^{|\sigma|(\mathscr{E},I)}$$

in $\mathscr{E}$.

Proof. For $\varphi \in E_{\mathrm{oc}}^{\sim}$ and $x \in E$, define the order bounded linear functional on $\mathscr{E}$ by setting $\Phi_{\varphi,x}(T) := \varphi(Tx)$ for $T \in \mathscr{E}$. Since $\mathscr{E}$ is a regular vector sublattice of $\mathscr{L}_{\mathrm{ob}}(E)$, an appeal to [19, Lemma 4.1] shows that $\Phi_{\varphi,x} \in \mathscr{E}_{\mathrm{oc}}^{\sim}$. Is then clear that $\mathscr{E}_{\mathrm{oc}}^{\sim}$ separates the points of $\mathscr{E}$. Now Proposition 4.6.4 can be applied with E replaced by $\mathscr{E}$ and F by $\mathscr{F}$. $\square$

Proposition 4.6.4 is also used in the proof of the following.

Theorem 4.6.6. *Let $\mathscr{A}$ be a unital f-algebra such that its identity element e is also a positive strong order unit of $\mathscr{A}$, and such that it is complete in the submultiplicative order unit norm $\|\cdot\|_e$ on $\mathscr{A}$. Suppose that $\mathscr{A}_{\mathrm{oc}}^{\sim}$ separates the points of $\mathscr{A}$. Then $\mathscr{A}$ admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_{\mathscr{A}}$. Take an ideal I of $\mathscr{A}_{\mathrm{oc}}^{\sim}$ that separates the points of $\mathscr{A}$, and take a (not necessarily unital) associative subalgebra $\mathscr{B}$ of E. Then*

$$\begin{aligned}&\overline{\mathscr{B}}^{\widehat{\tau}_{\mathscr{A}}} = \overline{\mathscr{B}}^{\sigma(\mathscr{A},I)} = \overline{\mathscr{B}}^{|\sigma|(\mathscr{A},I)} = \\ &\overline{\overline{\mathscr{B}}^{\|\cdot\|_e}}^{\widehat{\tau}_{\mathscr{A}}} = \overline{\overline{\mathscr{B}}^{\|\cdot\|_e}}^{\sigma(\mathscr{A},I)} = \overline{\overline{\mathscr{B}}^{\|\cdot\|_e}}^{|\sigma|(\mathscr{A},I)}\end{aligned} \tag{4.8}$$

in $\mathscr{A}$.

Before giving the proof, we mention the following fact that is easily verified. Suppose that X is a topological space that is supplied with two topologies τ_1 and τ_2, where τ_2 is weaker than τ_1. Then $\overline{\overline{S}^{\tau_1}}^{\tau_2} = \overline{S}^{\tau_2}$ for every subset S of X.

Proof. It follows from [19, Theorem 6.1] that $\overline{\mathscr{B}}^{\|\cdot\|_e}$ is a Banach f-subalgebra of $\mathscr{A}$. Being a vector sublattice of $\mathscr{A}$, Proposition 4.6.4 shows that the sets in the second line of equation (4.8) are equal. Since the convergence of a net in the order unit norm $\|\cdot\|_e$ implies its order convergence to the same limit (and then also its convergence in $\widehat{\tau}_{\mathscr{A}}$ to the same limit), we are done by an appeal to the remark preceding the proof. □

The following is now clear from Theorem 4.6.6 and the argument in the proof of Corollary 4.6.5.

Corollary 4.6.7. *Let E be a Dedekind complete Banach lattice. Suppose that $E_{\mathrm{oc}}^{\sim}$ separates the points of E. Then* $\mathrm{Orth}(E)_{\mathrm{oc}}^{\sim}$ *separates the points of* $\mathrm{Orth}(E)$*, and* $\mathrm{Orth}(E)$ *admits a (necessarily unique) Hausdorff uo-Lebesgue topology* $\widehat{\tau}_{\mathrm{Orth}(E)}$*. Take an ideal I of* $\mathrm{Orth}(E)_{\mathrm{oc}}^{\sim}$ *that separates the points of* $\mathrm{Orth}(E)$*, and take a (not necessarily unital) associative subalgebra $\mathscr{A}$ of* $\mathrm{Orth}(E)$*. Then*

$$\overline{\mathscr{A}}^{\widehat{\tau}_{\mathrm{Orth}(E)}} = \overline{\mathscr{A}}^{\sigma(\mathrm{Orth}(E),I)} = \overline{\mathscr{A}}^{|\sigma|(\mathrm{Orth}(E),I)} =$$
$$\overline{\overline{\mathscr{A}}^{\|\cdot\|}}^{\widehat{\tau}_{\mathrm{Orth}(E)}} = \overline{\overline{\mathscr{A}}^{\|\cdot\|}}^{\sigma(\mathrm{Orth}(E),I)} = \overline{\overline{\mathscr{A}}^{\|\cdot\|}}^{|\sigma|(\mathrm{Orth}(E),I)}$$

in $\mathrm{Orth}(E)$*; here $\|\cdot\|$ denotes the coinciding operator norm, order unit norm with respect to the identity operator, and regular norm on* $\mathrm{Orth}(E)$*.*

We shall now continue by establishing results showing that the closures of vector sublattices (or associative subalgebras) in a possible Hausdorff uo-Lebesgue topology coincide with their adherences with respect to various convergence structures on the enveloping vector lattices (or vector lattice algebras) under consideration in this paper.

Needless to say, under appropriate conditions, 'topological' results as obtained above may apply at the same time as 'adherence' results to be obtained below. *For reasons of brevity, we have refrained from formulating such 'combined' results.*

Let us also notice at this point that the results below imply that the adherences of vector sublattices that occur in the statements are closed with respect to the pertinent convergence structures. Indeed, these adherences are set maps that map vector sublattices to vector sublattices. When they agree on vector sublattices with the topological closure operator that is supplied by the Hausdorff uo-Lebesgue topology, then they, too, are idempotent. For example, the unbounded order adherence of the vector sublattice F in Proposition 4.6.8, below, is unbounded order closed. *For reasons of brevity, we have refrained from including such consequences in the results.*

We start by considering two cases where the enveloping vector lattices have weak order units.

Proposition 4.6.8. *Let E be a Dedekind complete vector lattice with the countable sup property and a weak order unit. Suppose that E admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$. Let F be vector sublattice of E. Then*

$$\overline{F}^{\widehat{\tau}_E} = a_{\sigma\mathrm{uo}}(F) = a_{\mathrm{uo}}(F)$$

in E.

Proof. Clearly, we have $a_{\sigma\mathrm{uo}}(F) \subseteq a_{\mathrm{uo}}(F) \subseteq \overline{F}^{\widehat{\tau}_E}$. Let e be a positive weak order unit of E. Take $x \in \overline{F}^{\widehat{\tau}_E}$. There exists a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in F with $x_\alpha \xrightarrow{\widehat{\tau}_E} x$. Then $|x_\alpha - x| \wedge e \xrightarrow{\widehat{\tau}_E} 0$, and we conclude from [6, Theorem 4.19] that there exists an increasing sequence $(\alpha_n)_{n=1}^\infty$ of indices in $\mathcal{A}$ such that $|x_{\alpha_n} - x| \wedge e \xrightarrow{\mathrm{o}} 0$ in E. An appeal to [29, Lemma 3.2] shows that $x_{\alpha_n} \xrightarrow{\mathrm{uo}} x$ in E. Hence $x \in a_{\sigma\mathrm{uo}}(F)$. We conclude that $\overline{F}^{\widehat{\tau}_E} \subseteq a_{\sigma\mathrm{uo}}(F)$. ☐

On combining Theorem 4.2.1, Proposition 4.6.8, and [19, Proposition 6.5], the following is easily obtained. We recall that a subset of a vector lattice is said to be an order basis when the band that it generates is the whole vector lattice.

Corollary 4.6.9. *Let E be a Dedekind complete vector lattice with the countable sup property and an at most countably infinite order basis. Suppose that E admits a (necessarily unique) Hausdorff uo-Lebesgue topology. Then* $\mathrm{Orth}(E)$ *admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_{\mathrm{Orth}(E)}$. Let $\mathscr{E}$ be vector sublattice of* $\mathrm{Orth}(E)$. *Then*

$$\overline{\mathscr{E}}^{\widehat{\tau}_{\mathrm{Orth}(E)}} = a_{\sigma\mathrm{uo}}(\mathscr{E}) = a_{\mathrm{uo}}(\mathscr{E})$$

in $\mathrm{Orth}(E)$.

We continue by considering cases where the enveloping vector lattice (or vector lattice algebra) has a strong order unit.

It is known that the o-adherence of a vector sublattice of a Dedekind complete Banach lattice E with a strong order unit can be a proper sublattice of its uo-adherence; see [26, Lemma 2.6] for details. When the vector sublattice contains a strong order unit of E, however, then this cannot occur, not even in general vector lattices. This is shown by the following preparatory result.

Lemma 4.6.10. *Let E be a vector lattice with a strong order unit. Suppose that F is a vector sublattice of E that contains a strong order unit of E. Then $a_{\mathrm{o}}(F) = a_{\mathrm{uo}}(F)$ and $a_{\sigma\mathrm{o}}(F) = a_{\sigma\mathrm{uo}}(F)$ in E.*

Proof. We prove that $a_{\mathrm{o}}(F) = a_{\mathrm{uo}}(F)$. It is clear that $a_{\mathrm{o}}(F) \subseteq a_{\mathrm{uo}}(F)$. For the reverse inclusion, we choose a positive strong order unit e of E such that $e \in F$. Take $x \in a_{\mathrm{uo}}(F)$, and let $(x_\alpha)_{\alpha\in\mathcal{A}}$ be a net in F such that $x_\alpha \xrightarrow{\mathrm{uo}} x$ in E. There exists a $\lambda \in \mathbb{R}_{\geq 0}$ such that $|x| \leq \lambda e$. For $\alpha \in \mathcal{A}$, set $y_\alpha := (-\lambda e \vee x_\alpha) \wedge \lambda e$. Clearly, $(y_\alpha)_\alpha \subseteq F$ and $y_\alpha \xrightarrow{\mathrm{uo}} (-\lambda e \vee x) \wedge \lambda e = x$. Since the net $(y_\alpha)_{\alpha\in\mathcal{A}}$ is order bounded in E, we have that $y_\alpha \xrightarrow{\mathrm{o}} x$ in E. Hence $x \in a_{\mathrm{o}}(F)$. We conclude that $a_{\mathrm{uo}}(F) \subseteq a_{\mathrm{o}}(F)$.

The proof for the sequential adherences is a special case of the above one. ☐

Remark 4.6.11. For comparison, we recall that, for a *regular* vector sublattice F of a vector lattice E, it is always the case that $a_{\mathrm{o}}(F) = a_{\mathrm{uo}}(F)$ in E, and that these coinciding subsets are order closed subsets of E; see [26, Theorem 2.13]. For this to hold, no assumptions on E are necessary.

The following is immediate from Proposition 4.6.8 and Lemma 4.6.10.

Theorem 4.6.12. *Let E be a Dedekind complete vector lattice with the countable sup property and a strong order unit. Suppose that E admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_E$. Let F be vector sublattice of E that contains a strong order unit of E. Then*

$$\overline{F}^{\widehat{\tau}_E} = a_{\sigma\mathrm{o}}(F) = a_{\mathrm{o}}(F) = a_{\sigma\mathrm{uo}}(F) = a_{\mathrm{uo}}(F)$$

in E.

The following result follows from the combination of Theorem 4.2.1, Theorem 4.6.12, and [19, Proposition 6.5]. In view of [19, Proposition 6.5], the natural condition to include is that E have an at most countably infinite order basis, but it is easily verified fact that, for a Banach lattice, this property is equivalent to having a weak order unit.

Corollary 4.6.13. *Let E be a Dedekind complete Banach lattice with the countable sup property and a weak order unit. Suppose that E admits a (necessarily unique) Hausdorff uo-Lebesgue topology. Then* $\mathrm{Orth}(E)$ *admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_{\mathrm{Orth}(E)}$. Let $\mathscr{E}$ be a vector sublattice of* $\mathrm{Orth}(E)$ *that contains a strong order unit of* $\mathrm{Orth}(E)$. *Then*

$$\overline{\mathscr{E}}^{\widehat{\tau}_{\mathrm{Orth}(E)}} = a_{\sigma\mathrm{o}}(\mathscr{E}) = a_{\mathrm{o}}(\mathscr{E}) = a_{\sigma\mathrm{uo}}(\mathscr{E}) = a_{\mathrm{uo}}(\mathscr{E})$$

in $\mathrm{Orth}(E)$.

We now turn to closures and adherences of associative subalgebras of a class of f-algebras with strong order units. For this, we need the following preparatory result.

Lemma 4.6.14. *Let E be a Banach lattice, and let A be a subset of E. Then $a_{\sigma\mathrm{o}}(A) = a_{\sigma\mathrm{o}}(\overline{A})$ in E, where $\overline{A}$ denotes the norm closure of A.*

Proof. We need to prove only that $a_{\sigma\mathrm{o}}(\overline{A}) \subseteq a_{\sigma\mathrm{o}}(A)$. For this, we may suppose that $A \neq \emptyset$. Take $x \in a_{\sigma\mathrm{o}}(\overline{A})$ and a sequence $(x_n)_{n=1}^{\infty}$ in $\overline{A}$ such that $x_n \xrightarrow{\sigma\mathrm{o}} x$ in E. For $n \geq 1$, take an $y_n \in A$ such that $\|y_n - x_n\| \leq 2^{-n}$. For $n \geq 1$, define z_n by setting $z_n := \sum_{m=n}^{\infty} |y_n - x_n|$, which is meaningful since the series is absolutely convergent. It is clear that $z_n \downarrow$. Since $\|z_n\| \leq 2^{-n+1}$, we have $z_n \downarrow 0$ in E. The fact that $|y_n - x_n| \leq z_n$ for $n \geq 1$ then shows that $|y_n - x_n| \xrightarrow{\sigma\mathrm{o}} 0$ in E. From

$$0 \leq |y_n - x| \leq |y_n - x_n| + |x_n - x| \xrightarrow{\sigma\mathrm{o}} 0,$$

we then see that $y_n \xrightarrow{\sigma\mathrm{o}} x$ in E. Hence $x \in a_{\sigma\mathrm{o}}(A)$, as desired. □

Theorem 4.6.15. *Let $\mathscr{A}$ be a Dedekind complete unital f-algebra with the countable sup property, such that its identity element e is also a positive strong order unit of $\mathscr{A}$, and such that it is complete in the submultiplicative order unit norm $\|\cdot\|_e$ on $\mathscr{A}$. Suppose that $\mathscr{A}$ admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_{\mathscr{A}}$. Let $\mathscr{B}$ be an associative subalgebra* *of $\mathscr{A}$ such that $\overline{\mathscr{B}}^{\|\cdot\|_e}$ contains a strong order unit of $\mathscr{A}$. Then*

$$\begin{aligned}
&\overline{\mathscr{B}}^{\widehat{\tau}_{\mathscr{A}}} = a_{\sigma\mathrm{o}}(\mathscr{B}) = a_{\mathrm{o}}(\mathscr{B}) = a_{\sigma\mathrm{uo}}(\mathscr{B}) = a_{\mathrm{uo}}(\mathscr{B}) = \\
&\overline{\overline{\mathscr{B}}^{\|\cdot\|_e}}^{\widehat{\tau}_{\mathscr{A}}} = a_{\sigma\mathrm{o}}(\overline{\mathscr{B}}^{\|\cdot\|_e}) = a_{\mathrm{o}}(\overline{\mathscr{B}}^{\|\cdot\|_e}) = a_{\sigma\mathrm{uo}}(\overline{\mathscr{B}}^{\|\cdot\|_e}) = a_{\mathrm{uo}}(\overline{\mathscr{B}}^{\|\cdot\|_e})
\end{aligned} \tag{4.9}$$

in $\mathscr{A}$.

Proof. We know from [19, Theorem 6.1] that $\overline{\mathscr{B}}^{\|\cdot\|_e}$ is a Banach f-subalgebra of $\mathscr{A}$. Then Theorem 4.6.12 shows that all equalities in the second line of equation (4.9) hold. Furthermore, it is obvious that

$$a_{\sigma\mathrm{o}}(\mathscr{B}) \subseteq a_{\mathrm{o}}(\mathscr{B}) \subseteq a_{\mathrm{uo}}(\mathscr{B}) \subseteq \overline{\mathscr{B}}^{\widehat{\tau}_{\mathscr{A}}}$$

and that

$$a_{\sigma\mathrm{o}}(\mathscr{B}) \subseteq a_{\sigma\mathrm{uo}}(\mathscr{B}) \subseteq \mathscr{B}^{\widehat{\tau}_{\mathscr{A}}}.$$

Using that $a_{\sigma\mathrm{o}}(\mathscr{B}) = a_{\sigma\mathrm{o}}(\overline{\mathscr{B}}^{\|\cdot\|_e})$ by Lemma 4.6.14 and that, as in the proof of Theorem 4.6.6, we also know that $\overline{\mathscr{B}}^{\widehat{\tau}_{\mathscr{A}}} = \overline{\overline{\mathscr{B}}^{\|\cdot\|_e}}^{\widehat{\tau}_{\mathscr{A}}}$, it then follows that all sets in equation (4.9) are equal. □

The following is now clear from Theorem 4.2.1, Theorem 4.6.15, and [19, Proposition 6.5].

Corollary 4.6.16. *Let E be a Dedekind complete Banach lattice with the countable sup property and a weak order unit. Suppose that E admits a (necessarily unique) Hausdorff uo-Lebesgue topology. Then* $\mathrm{Orth}(E)$ *admits a (necessarily unique) Hausdorff uo-Lebesgue topology $\widehat{\tau}_{\mathrm{Orth}(E)}$. Let $\mathscr{A}$ be an associative subalgebra of* $\mathrm{Orth}(E)$ *such that $\overline{\mathscr{A}}^{\|\cdot\|}$ contains a strong order unit of* $\mathrm{Orth}(E)$. *Then*

$$\begin{aligned}
&\overline{\mathscr{A}}^{\widehat{\tau}_{\mathscr{A}}} = a_{\sigma\mathrm{o}}(\mathscr{A}) = a_{\mathrm{o}}(\mathscr{A}) = a_{\sigma\mathrm{uo}}(\mathscr{A}) = a_{\mathrm{uo}}(\mathscr{A}) = \\
&\overline{\overline{\mathscr{A}}^{\|\cdot\|}}^{\widehat{\tau}_{\mathscr{A}}} = a_{\sigma\mathrm{o}}(\overline{\mathscr{A}}^{\|\cdot\|}) = a_{\mathrm{o}}(\overline{\mathscr{A}}^{\|\cdot\|}) = a_{\sigma\mathrm{uo}}(\overline{\mathscr{A}}^{\|\cdot\|}) = a_{\mathrm{uo}}(\overline{\mathscr{A}}^{\|\cdot\|})
\end{aligned} \tag{4.10}$$

in $\mathrm{Orth}(E)$*; here $\|\cdot\|$ denotes the coinciding operator norm, order unit norm with respect to the identity operator, and regular norm on* $\mathrm{Orth}(E)$.

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Summary

This thesis consists of three papers that are centered around the common theme of Hausdorff uo-Lebesgue topologies and convergence structures on vector lattices and on vector lattices and vector lattice algebras of order bounded operators. Its origins lie in asking for possible analogues of the von Neumann bicommutant theorem in the context of Banach lattices and vector lattices. Apart from being interesting in their own right, such analogues are expected to be relevant for the study of vector lattice algebras and Banach lattice algebras of order bounded operators, as well as for representation theory in vector lattices and Banach lattices.

When contemplating a possible bicommutant theorem for vector lattices, the evident ingredient that is missing in that context is the weak (or strong) operator topology that figures in von Neumann's theorem. Fortunately, there are natural candidates that can take over this role. The first one is a possible Hausdorff uo-Lebesgue topology on a vector lattice algebra of operators. Such topologies on vector lattices have received considerable attention in recent years, and they appear to have a rather special position among the possible locally solid topologies. Apart from this, there are several natural convergence structures on vector lattices of order bounded operators to be considered. These come in pairs, consisting of a uniform and a strong version. For example, the general theory of vector lattices provides the definition of order convergence of a net for any vector lattice, hence also for a vector lattice of order bounded operators. For a net of operators on a vector lattice, however, one can also require that it be pointwise order convergent for every element of the underlying vector lattice. Thus there are a uniform and a strong order convergence structure on a vector lattice of order bounded operators. Likewise, there are a uniform unbounded order convergence structure and a strong unbounded order convergence structure, as well as a uniform convergence structure for a possible Hausdorff uo-Lebesgue topology (which is the topological structure as already mentioned) and a strong one with respect to such a topology on the underlying vector lattice. Thus we have six convergence structures. For each of these, one can speak of the corresponding adherence of a set of order bounded linear operators. These adherences (one of which is an actual topological closure) are all natural candidates that can take over the role of the closure in the weak (or strong) operator topology in von Neumann's theorem.

When trying to work with these convergence structures in the context of an attempted bicommutant theorem, one very quickly starts to feel the need for some 'basic facts to work with'. For example, is the adherence of a vector lattice subalgebra of the order bounded operators with respect to strong unbounded order convergence again a vector lattice subalgebra? As it turns out, this is always the case when it is contained in the orthomorphisms, but

not always when it is contained in the order continuous operators. The question is natural and the answer is easily formulated, but establishing this answer (including its 'sharpness' as an important ingredient) is a non-trivial matter. There are many more such basic, but very often non-trivial, issues to be resolved before one can get to more advanced parts of the theory such as a bicommutant theorem. Since these have not been considered to any substantial extent before, this is now undertaken in a systematic fashion in this thesis.

The first mathematical part of the thesis, Chapter 2, is still concerned with the general theory of Hausdorff uo-Lebesgue topologies on vector lattices. Starting from a Hausdorff o-Lebesgue topology on an order dense ideal, a Hausdorff uo-Lebesgue topology on the vector lattice itself is constructed. This results in a going-up-going-down procedure of supplying regular vector sublattices with such a topology that takes an earlier uniform construction of such topologies still one step further. Classical relations between convergence in measure and convergence almost everywhere are shown to be special cases of a more general result relating convergence in a Hausdorff uo-Lebesgue topology and unbounded order convergence.

In Chapter 3, Hausdorff uo-Lebesgue topologies on vector lattices of order bounded operators are constructed from Hausdoff o-Lebesgue topologies on the underlying vector lattices. The six convergence structures mentioned above are introduced in this chapter, and their relations are studied at the level of vector lattices of order bounded operators. Particular attention is paid to the orthomorphisms, where several implications between these convergences hold that are not generally valid.

In Chapter 4, after completing the investigation of the thirty-six possible implications between the six convergence structures on vector lattices of order bounded operators, these convergence structures are then considered in the context of vector lattice algebras of order bounded operators. The continuity with respect to the six convergence structures is investigated of the left and right multiplications, as well as of the simultaneous continuity of the multiplication. Results about adherences of vector lattice subalgebras being vector lattice subalgebras again are then an immediate consequence; with one exception it is also shown that these results are 'sharp'. Results are also included concerning the equality of various adherences of vector lattices (and of vector lattice algebras) of order bounded operators.

The material in Chapter 4, which builds on the earlier parts, is the most closely related to the original question regarding possible analogues of von Neumann's bicommutant theorem for vector lattices. It can, in fact, be used to obtain such analogues. These results will be published at a later date.

Samenvatting

Dit proefschrift bestaat uit drie artikelen rondom het gemeenschappelijke thema van Hausdorff uo-Lebesgue topologieën en convergentiestructuren op vectorroosters en op vectorroosters en vectorroosteralgebra's van ordebegrensde operatoren. De oorsprong ervan is gelegen in de vraag naar mogelijke analoga van von Neumanns bicommutantstelling in de context van Banachroosters en vectorroosters. Afgezien van hun waarde als zodanig wordt van dergelijke analoga verwacht dat ze relevant zijn voor het onderzoek naar vectorroosteralgebra's en Banachroosteralgebra's van ordebegrensde operatoren, evenals voor representatietheorie in vectorroosters en Banachroosters.

Bij het overwegen van een mogelijke bicommutantstelling voor vectorroosters ontbreekt er in die context een belangrijk ingredient, namelijk de zwakke (of sterke) operatortopologie die in von Neumanns stelling voorkomt. Gelukkig zijn er wel natuurlijke kandidaten om de rol daarvan over te nemen. De eerste hiervan is een mogelijke Hausdorff uo-Lebesgue topologie op een vectorroosteralgebra van operatoren. Dergelijke topologieën op vectorroosters hebben de afgelopen jaren sterk in de belangstelling gestaan en ze lijken een bijzondere plaats in te nemen binnen de mogelijke lokaal solide topologieën. Buiten dit zijn er verschillende natuurlijke convergentiestructuren op vectorroosters van ordebegrensde operatoren die in overweging genomen kunnen worden. Deze treden in paren op, bestaande uit een uniforme en een sterke versie. Bij wijze van voorbeeld: in de algemene theorie wordt de definitie van ordeconvergentie van een net in een vectorrooster gegeven, die dus ook van toepassing is op een net in een vectorrooster van ordebegrensde operatoren. Voor een net van operatoren op een vectorrooster kan men echter ook verlangen dat het net puntsgewijs ordeconvergent is voor ieder element van het onderliggende vectorrooster. Op die manier kennen we zowel een uniforme als een sterke ordeconvergentiestructuur voor een vectorrooster van ordebegrensde operatoren. Net zo zijn er een uniforme en een sterke onbegrensde ordeconvergentiestructuur, evenals een uniforme convergentiestructuur met betrekking tot een eventuele Hausdorff uo-Lebesgue topologie (die de topologische structuur is zoals hierboven al genoemd) en een sterke convergentiestructuur met betrekking tot een dergelijke topologie op het onderliggende vectorrooster. Op die manier worden zes convergentiestructuren verkregen. Voor ieder van deze convergentiestructuren kan men spreken van de bijbehorende aankleving van een verzameling van ordebegrensde operatoren. Deze aanklevingen (waarvan één een topologische afsluiting is) zijn ieder natuurlijke kandidaten om de rol van de afsluiting in de zwakke (of sterke) operatortopologie in von Neumann's stelling over te nemen.

Bij het nadenken over deze convergentiestructuren in de context van een mogelijke bicommutantstelling wordt al heel snel de behoefte gevoeld aan 'basisresultaten om mee te werken'. Bij wijze van voorbeeld: is de aankleving van een vectorroosterdeelalgebra van de ordebegrensde operatoren met betrekking tot de sterke onbegrensde ordeconvergentiestructuur weer een vectorroosterdeelalgebra? Naar blijkt is dat altijd het geval wanneer deze bevat is in de orthomorfismen, maar niet altijd wanneer deze bevat is in de ordecontinue operatoren. De vraag is een natuurlijke en het antwoord is gemakkelijk te formuleren, maar het aantonen van de juistheid van dit antwoord (inclusief de 'optimaliteit' ervan als een belangrijke component) is een niet-triviale aangelegenheid. Er zijn vele van dergelijke fundamentele, maar vaak niet-triviale, vragen die beantwoord moeten worden voordat het onderzoek kan beginnen aan meer geavanceerde aspecten van de theorie, zoals een bicommutantstelling. Dergelijke vragen zijn nog niet serieus onderzocht. In dit proefschrift wordt daar op een systematische manier mee begonnen.

Het eerste wiskundige gedeelte van het proefschrift, Hoofdstuk 2, is nog gericht op de algemene theorie van Hausdorff uo-Lebesgue topologieën op vectorroosters. Uitgaande van een Hausdorff o-Lebesgue topologie op een ordedicht ideaal wordt een Hausdorff uo-Lebesgue topologie op het vectorrooster als geheel geconstrueerd. Dit resulteert in een omhoog-omlaag procedure waarmee reguliere vectordeelroosters van een dergelijke topologie voorzien kunnen worden. Deze methode gaat nog een stap verder dan een al eerder bekende uniforme constructie van dergelijke topologieën. Er wordt ook aangetoond hoe klassieke relaties tussen convergentie in maat en convergentie bijna overal speciale gevallen zijn van meer algemene verbanden tusen convergentie in een Hausdorff uo-Lebesgue topologie en onbegrensde ordeconvergentie.

In Hoofdstuk 3 worden Hausdorff uo-Lebesgue topologieën op vectorroosters van onbegrensde operatoren geconstrueerd uitgaande van Hausdorff o-Lebesgue topologieën op de onderliggende vectorroosters. De zes convergentiestructuren die hierboven genoemd zijn worden in dit hoofdstuk geïntroduceerd en hun relaties worden onderzocht op het niveau van vectorroosters van ordebegrensde operatoren. Er wordt speciale aandacht besteed aan de orthomorfismen, waar implicaties tussen deze convergenties gelden die niet in het algemeen waar zijn.

In Hoofdstuk 4 worden, nadat het onderzoek naar de zesendertig mogelijke implicaties tussen de zes convergentiestructuren voor vectorroosters van ordebegrensde operatoren voltooid is, deze convergentiestructuren vervolgens beschouwd in de context van vectorroosteralgebra's van ordebegrensde operatoren. De continuïteit van de links- en rechtsvermenigvuldiging met betrekking tot deze convergentiestructuren wordt onderzocht, evenals de gelijktijdige continuïteit van de vermenigvuldiging. Resultaten over aanklevingen van vectorroosterdeelalgebra's die weer vectorroosterdeelalgebra's zijn worden als onmiddellijke gevolgen verkregen. Met één uitzondering wordt ook aangetoond dat deze resultaten 'optimaal' zijn. Verder worden ook resultaten gegeven met betrekking tot de gelijkheid van een aantal aanklevingen van vectorroosters en vectorroosteralgebra's van ordebegrensde operatoren.

Het materiaal in Hoofdstuk 4, dat voortbouwt op de eerdere onderdelen, is het nauwst gerelateerd aan de oorspronkelijke vraag met betrekking tot mogelijke analoga van von Neumanns bicommutantstelling voor vectorroosters. Het is, gebruikmakend van dit materiaal, inderdaad ook mogelijk om dergelijke analoga te verkijgen. De betreffende resultaten zullen op een later tijdstip gepubliceerd worden.

Curriculum Vitae

Yang Deng was born on 8th September 1990 in Jingshan, Hubei, China. In 2009, he moved to Chengdu to start his study in mathematics at Southwest Jiaotong University in that city. He received his B.Sc. degree in 2013 and continued with his master's studies at the same university. In 2016, he obtained his M.Sc. degree with a master's thesis titled *Limited operators on Banach lattices* under the supervision of prof. Zili Chen.

In 2016, he was awarded a scholarship by China Scholarship Council (CSC) to pursue his Ph.D. studies at Leiden University under the supervision of dr. Marcel de Jeu. During this time he attended and spoke at numerous meetings of the functional analysis seminar in Leiden. He attended the conference *Positivity IX* in Edmonton, Canada, in 2017; the *Dutch Mathematical Congress* in Veldhoven, the Netherlands, in 2018; the workshop *Functional Analysis Days* in Lancaster, United Kingdom, in 2018; the conference *50th anniversary of NBFAS* in Edinburgh, United Kingdom, in 2018; and the workshop *Recent Advances in Banach Lattices* in Oaxaca, Mexico, in 2018. He gave contributed talks at the workshop *Ordered Banach Spaces, Positive Operators and Applications* in Dresden, Germany, in 2019, and at the *Young Functional Analysts' Workshop* in Leeds, United Kingdom, in 2019. In May 2019, he visited Niushan Gao and Foivos Xanthos at Ryerson University; in May and during all of June 2019, he visited Vladimir Troitsky at the University of Alberta.

He was a teaching assistant for the Linear Analysis course in Leiden in the fall of 2019.

Since September 2020, he is a visiting researcher at Sichuan University in Chengdu, financially supported by an Erasmus+ International Credit Mobility grant for exchange between Leiden University and Sichuan University.

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