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Bayesian learning: Challenges, limitations and pragmatics

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Chapter 7

Fixed-confidence guarantees for Bayesian best-arm identification

Abstract

We investigate and provide new insights on the sampling rule called Top-Two Thompson Sampling (TTTS). In particular, we justify its use for *fixed-confidence best-arm identification*. We further propose a variant of TTTS called Top-Two Transportation Cost (T3C), which disposes of the computational burden of TTTS. As our main contribution, we provide the first sample complexity analysis of TTTS and T3C when coupled with a very natural Bayesian stopping rule, for bandits with Gaussian rewards, solving one of the open questions raised by Russo (2016). We also provide new posterior convergence results for TTTS under two models that are commonly used in practice: bandits with Gaussian and Bernoulli rewards and conjugate priors.

7.1 Introduction

In multi-armed bandits, a learner repeatedly chooses an *arm* to play, and receives a reward from the associated unknown probability distribution. When the task is *best-arm* identification (BAI), the learner is not only asked to sample an arm at each stage, but is also asked to output a recommendation (i.e., a guess for the arm with the largest mean reward) after a certain period. Unlike in another well-studied bandit setting, the learner is not interested in maximizing the sum of rewards gathered during the exploration (or minimizing *regret*), but only cares about the quality of her recommendation. As such, BAI is a particular *pure exploration* setting (Bubeck, Munos and Stoltz, 2009).

Formally, we consider a finite-arm bandit model, which is a collection of K probability distributions, called arms $\mathcal{A} \triangleq \{1, \dots, K\}$, parametrized by their means $\mu_1, \dots, \mu_K$. We assume the (unknown) best arm is unique and we denote it by $I^\star \triangleq \arg\max_i \mu_i$. A best-arm identification

strategy (I_n, J_n, τ) consists of three components. The first is a *sampling rule*, which selects an arm I_n at round n. At each round n, a vector of rewards $\mathbf{Y}_n = (Y_{n,1}, \cdots, Y_{n,K})$ is generated for all arms independently from past observations, but only Y_{n,I_n} is revealed to the learner. Let $\mathcal{F}_n$ be the σ-algebra generated by $(U_0, I_1, Y_{1,I_1}, U_1, \cdots, I_n, Y_{n,I_n}, U_n)$, then I_n is $\mathcal{F}_{n-1}$-measurable, i.e., it can only depend on the past $n-1$ observations, and some exogenous randomness, materialized into $U_{n-1} \sim \mathcal{U}([0,1])$. The second component is a $\mathcal{F}_n$-measurable *recommendation rule* J_n, which returns a guess for the best arm, and thirdly, the *stopping rule* τ, a stopping time with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$, decides when the exploration is over.

BAI is studied within several theoretical frameworks. In this chapter we consider the fixed-confidence setting, introduced by Even-dar, Mannor and Mansour, 2003. Given a risk parameter $\delta \in [0,1]$, the goal is to ensure that the probability to stop and recommend a wrong arm, $\mathbb{P}[J_\tau \neq I^\star \wedge \tau < \infty]$, is smaller than δ, while minimizing the expected total number of samples to make this accurate recommendation, $\mathbb{E}[\tau]$. The most studied alternative setting is the fixed-budget setting for which the stopping rule τ is fixed to some (known) maximal budget n, and the goal is to minimize the error probability $\mathbb{P}[J_n \neq I^\star]$ (Audibert and Bubeck, 2010). Note that these two frameworks are very different in general and do not share transferable regret bounds (see Carpentier and Locatelli 2016 for an additional discussion).

Most existing sampling rules for the fixed-confidence setting depend on the risk parameter δ. Some of them rely on confidence intervals such as `LUCB` (Kalyanakrishnan et al., 2012), `UGapE` (Gabillon, Ghavamzadeh and Lazaric, 2012), or `lil'UCB` (Jamieson et al., 2014); others are based on eliminations such as `SuccessiveElimination` (Even-dar, Mannor and Mansour, 2003) and `ExponentialGapElimination` (Karnin, Koren and Somekh, 2013). The first known sampling rule for BAI that does not depend on δ is the *tracking* rule proposed by Garivier and Kaufmann, 2016, which is proved to achieve the minimal sample complexity when combined with the Chernoff stopping rule when δ goes to zero. Such an *anytime* sampling rule (neither depending on a risk δ or a budget n) is very appealing for applications, as advocated by Jun and Nowak, 2016 who introduce the anytime best-arm identification framework. In this chapter, we investigate another anytime sampling rule for BAI: Top-Two Thompson Sampling (`TTTS`), and propose a second anytime sampling rule: Top-Two Transportation Cost (`T3C`).

Thompson Sampling (Thompson, 1933) is a Bayesian algorithm well known for regret minimization, for which it is now seen as a major competitor to `UCB`-typed approaches (Burnetas and Katehakis, 1996; Auer, Cesa-Bianchi and Fischer, 2002; Cappé et al., 2013). However, it is also well known that regret minimizing algorithms cannot yield optimal performance for BAI (Bubeck, Munos and Stoltz, 2011; Kaufmann and Garivier, 2017) and as we opt Thompson Sampling for BAI, then its adaptation is necessary. Such an adaptation, `TTTS`, was given by Russo (2016) along with two other top-two sampling rules `TTPS` and `TTVS`. By choosing between two different candidate arms in each round, these sampling rules enforce the exploration of sub-optimal arms, which would be under-sampled by vanilla Thompson sampling due to its objective of maximizing rewards.

While `TTTS` appears to be a good anytime sampling rule for fixed-confidence BAI when coupled with an appropriate stopping rule, so far there is no theoretical support for this employment. Indeed, the (Bayesian-flavored) asymptotic analysis of Russo, 2016 shows that under `TTTS`, the posterior probability that $I^\star$ is the best arm converges almost surely to 1 at the best possible

rate. However, this property does not by itself translate into sample complexity guarantees. Since the result of Russo, 2016, Qin, Klabjan and Russo (2017) proposed and analyzed TTEI, another Bayesian sampling rule, both in the fixed-confidence setting and in terms of posterior convergence rate. Nonetheless, similar guarantees for TTTS have been left as an open question by Russo, 2016. In the present chapter, we answer the question whether we can obtain fixed-confidence guarantees and optimal posterior convergence rates for TTTS. In addition, we propose T3C, a computationally more favorable variant of TTTS and extend the fixed-confidence guarantees to T3C as well.

Contributions (1) We propose a new Bayesian sampling rule, T3C, which is inspired by TTTS but easier to implement and computationally advantageous (2) We investigate two Bayesian stopping and recommendation rules and establish their δ-correctness for a bandit model with Gaussian rewards.[1] (3) We provide the first sample complexity analysis of TTTS and T3C for a Gaussian model and our proposed stopping rule. (4) Russo's posterior convergence results for TTTS were obtained under restrictive assumptions on the models and priors, which exclude the two mostly used models in practice: Gaussian bandits with Gaussian priors and bandits with Bernoulli rewards[2] with Beta priors. We prove that optimal posterior convergence rates can be obtained for those two as well.

Outline In Section 7.2, we restate TTTS and introduce T3C along with our proposed recommendation and stopping rules. Then, in Section 7.3, we describe in detail two important notions of optimality that are invoked in this chapter. The main fixed-confidence analysis follows in Section 7.4, and further Bayesian optimality results are given in Section 7.5. Numerical illustrations are given in Section 7.6.

7.2 Bayesian BAI Strategies

In this section, we give an overview of the sampling rule TTTS and introduce T3C. We provide details for Bayesian updating for Gaussian and Bernoulli models respectively, and introduce associated Bayesian stopping and recommendation rules.

7.2.1 Sampling rules

Both TTTS and T3C employ a Bayesian machinery and make use of a prior distribution Π_1 over a set of parameters Θ, which is assumed to contain the unknown true parameter vector $\boldsymbol{\mu}$. Upon acquiring observations $(Y_{1,I_1}, \cdots, Y_{n-1,I_{n-1}})$, we update our beliefs according to Bayes' rule and obtain a posterior distribution Π_n which we assume to have density π_n w.r.t. the Lebesgue measure. Russo's analysis is requires strong regularity properties on the models and priors, which exclude two important useful cases we consider in this chapter: (1) the observations of each arm i follow a Gaussian distribution $\mathcal{N}(\mu_i, \sigma^2)$ with common known variance σ^2, with imposed Gaussian prior $\mathcal{N}(\mu_{1,i}, \sigma^2_{1,i})$, (2) all arms receive Bernoulli rewards with unknown means, with a uniform ($\mathcal{B}eta(1,1)$) prior on each arm.

[1] Hereafter *Gaussian bandits* or *Gaussian model.*
[2] Hereafter *Bernoulli bandits.*

Gaussian model For Gaussian bandits with a $\mathcal{N}(0, \kappa^2)$ prior on each mean, the posterior distribution of μ_i at round n is Gaussian with mean and variance that are respectively given by

$$\frac{\sum_{\ell=1}^{n-1} \mathbb{1}\{I_\ell = i\} Y_{\ell, I_\ell}}{T_{n,i} + \sigma^2/\kappa^2} \quad \text{and} \quad \frac{\sigma^2}{T_{n,i} + \sigma^2/\kappa^2},$$

where $T_{n,i} \triangleq \sum_{\ell=1}^{n-1} \mathbb{1}\{I_\ell = i\}$ is the number of selections of arm i before round n. For the sake of simplicity, we consider improper Gaussian priors with $\mu_{1,i} = 0$ and $\sigma_{1,i} = +\infty$ for all $i \in \mathcal{A}$, for which

$$\mu_{n,i} = \frac{1}{T_{n,i}} \sum_{\ell=1}^{n-1} \mathbb{1}\{I_\ell = i\} Y_{\ell, I_\ell} \quad \text{and} \quad \sigma^2_{n,i} = \frac{\sigma^2}{T_{n,i}}.$$

Observe that in this case the posterior mean $\mu_{n,i}$ coincides with the empirical mean.

Beta-Bernoulli model For Bernoulli bandits with a uniform ($\mathcal{B}eta(1,1)$) prior on each mean, the posterior distribution of μ_i at round n is a Beta distribution with shape parameters $\alpha_{n,i} = \sum_{\ell=1}^{n-1} \mathbb{1}\{I_\ell = i\} Y_{\ell, I_\ell} + 1$ and $\beta_{n,i} = T_{n,i} - \sum_{\ell=1}^{n-1} \mathbb{1}\{I_\ell = i\} Y_{\ell, I_\ell} + 1$.

Now we briefly recall TTTS and introduce T3C. The pseudo-code of TTTS and T3C are shown in Algorithm 2.

Description of TTTS At each time step n, TTTS has two potential actions: (1) with probability β, a parameter vector $\boldsymbol{\theta}$ is sampled from Π_n, and TTTS chooses to play $I_n^{(1)} \triangleq \arg\max_{i \in \mathcal{A}} \theta_i$, (2) and with probability $1-\beta$, the algorithm continues sampling new $\boldsymbol{\theta}'$ until we obtain a *challenger* $I_n^{(2)} \triangleq \arg\max_{i \in \mathcal{A}} \theta'_i$ that is different from $I_n^{(1)}$, and TTTS chooses to play $I_n^{(2)}$.

Description of T3C One drawback of TTTS is that, in practice, when the posteriors become concentrated, it takes many Thompson samples before the challenger $I_n^{(2)}$ is obtained. We thus propose a variant of TTTS, called T3C, which alleviates this computational burden. Instead of re-sampling from the posterior until a different candidate appears, we define the challenger as the arm that has the lowest *transportation cost* $W_n(I_n^{(1)}, i)$ with respect to the first candidate (with ties broken uniformly at random).

Let $\mu_{n,i}$ be the empirical mean of arm i and $\mu_{n,i,j} \triangleq (T_{n,i}\mu_{n,i} + T_{n,j}\mu_{n,j})/(T_{n,i} + T_{n,j})$, then we define

$$W_n(i,j) \triangleq \begin{cases} 0 & \text{if } \mu_{n,j} \geq \mu_{n,i}, \\ W_{n,i,j} + W_{n,j,i} & \text{otherwise,} \end{cases} \tag{7.1}$$

where $W_{n,i,j} \triangleq T_{n,i} d\left(\mu_{n,i}, \mu_{n,i,j}\right)$ for any i, j and $d(\mu; \mu')$ denotes the Kullback-Leibler between the distribution with mean μ and that of mean μ'. In the Gaussian case, $d(\mu; \mu') = (\mu - \mu')^2/(2\sigma^2)$ while in the Bernoulli case $d(\mu; \mu') = \mu \ln(\mu/\mu') + (1-\mu)\ln(1-\mu)/(1-\mu')$. In particular, for Gaussian bandits

$$W_n(i,j) = \frac{(\mu_{n,i} - \mu_{n,j})^2}{2\sigma^2(1/T_{n,i} + 1/T_{n,j})} \mathbb{1}\{\mu_{n,j} < \mu_{n,i}\}.$$

Note that under the Gaussian model with improper priors, one should pull each arm once at the beginning for the sake of obtaining proper posteriors.

Algorithm 2 Sampling rule (TTTS/T3C)

Input: β
for $n \leftarrow 1, 2, \cdots$ **do**
  `sample` $\boldsymbol{\theta} \sim \Pi_n$
  $I^{(1)} \leftarrow \arg\max_{i \in \mathcal{A}} \theta_i$
  `sample` $b \sim \mathcal{B}ern(\beta)$
  if $b = 1$ **then**
    `evaluate arm` $I^{(1)}$
  else
    `repeat sample` $\boldsymbol{\theta}' \sim \Pi_n$ } TTTS
    $I^{(2)} \leftarrow \arg\max_{i \in \mathcal{A}} \theta'_i$ } TTTS
    `until` $I^{(2)} \neq I^{(1)}$ } TTTS
    $I^{(2)} \leftarrow \arg\min_{i \neq I^{(1)}} W_n(I^{(1)}, i)$, cf. (7.1) T3C
    `evaluate arm` $I^{(2)}$
  end if
  `update mean and variance`
  $t = t + 1$
end for

7.2.2 Rationale for T3C

In order to explain how T3C can be seen as an approximation of the re-sampling performed by TTTS, we first need to define the *optimal action probabilities*.

Optimal action probability The optimal action probability $a_{n,i}$ is defined as the posterior probability that arm i is optimal. Formally, letting Θ_i be the subset of Θ such that arm i is the optimal arm,

$$\Theta_i \triangleq \left\{ \boldsymbol{\theta} \in \Theta \;\middle|\; \theta_i > \max_{j \neq i} \theta_j \right\},$$

then we define

$$a_{n,i} \triangleq \Pi_n(\Theta_i) = \int_{\Theta_i} \pi_n(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta}. \tag{7.2}$$

With this notation, one can show that under TTTS,

$$\Pi_n\left(I_n^{(2)} = j | I_n^{(1)} = i\right) = \frac{a_{n,j}}{\sum_{k \neq i} a_{n,k}}. \tag{7.3}$$

Furthermore, when i coincides with the empirical best mean (and this will often be the case for $I_n^{(1)}$ when n is large due to posterior convergence) one can write

$$a_{n,j} \simeq \Pi_n\left(\theta_j \geq \theta_i\right) \simeq \exp\left(-W_n(i,j)\right),$$

where the last step is justified in Lemma 6 in the Gaussian case (and Lemma 32 in Appendix 7.I.3 in the Bernoulli case). Hence, T3C replaces sampling from the distribution (7.3) by an approx-

imation of its mode which is *easy to compute*. Note that directly computing the mode would require to compute $a_{n,j}$, which is much more costly than the computation of $W_n(i,j)$[3].

7.2.3 Stopping and recommendation rules

In order to use TTTS or T3C as the sampling rule for fixed-confidence BAI, we need to additionally define stopping and recommendation rules. While Qin, Klabjan and Russo, 2017 suggest to couple TTEI with the "frequentist" Chernoff stopping rule (Garivier and Kaufmann, 2016), we propose in this section natural Bayesian stopping and recommendation rules. They both rely on the optimal action probabilities defined in (7.2).

Bayesian recommendation rule At time step n, a natural candidate for the best arm is the arm with largest optimal action probability, hence we define

$$J_n \triangleq \arg\max_{i \in \mathcal{A}} a_{n,i} \,.$$

Bayesian stopping rule In view of the recommendation rule, it is natural to stop when the posterior probability that the recommended action is optimal is large, and exceeds some threshold $c_{n,\delta}$ which gets close to 1. Hence our Bayesian stopping rule is

$$\tau_\delta \triangleq \inf \left\{ n \in \colon \max_{i \in \mathcal{A}} a_{n,i} \geq c_{n,\delta} \right\} . \tag{7.4}$$

Links with frequentist counterparts Using the transportation cost $W_n(i,j)$ defined in (7.1), the Chernoff stopping rule of Garivier and Kaufmann, 2016 can actually be rewritten as

$$\tau_\delta^{\text{Ch.}} \triangleq \inf \left\{ n \in \mathbb{N} : \max_{i \in \mathcal{A}} \min_{j \in \mathcal{A} \setminus \{i\}} W_n(i,j) > d_{n,\delta} \right\} . \tag{7.5}$$

This stopping rule is coupled with the recommendation rule $J_n = \arg\max_i \mu_{n,i}$.

As explained in that paper, $W_n(i,j)$ can be interpreted as a (log) Generalized Likelihood Ratio statistic for rejecting the hypothesis $\mathcal{H}_0 : (\mu_i < \mu_j)$. Through our Bayesian lens, we rather have in mind the approximation $\Pi_n(\theta_j > \theta_i) \simeq \exp\{-W_n(i,j)\}$, valid when $\mu_{n,i} > \mu_{n,j}$, which permits to analyze the two stopping rules using similar tools, as will be seen in the proof of Theorem 7.3.

As shown later in Sec. 7.4, τ_δ and $\tau_\delta^{\text{Ch.}}$ prove to be fairly similar for some corresponding choices of the thresholds $c_{n,\delta}$ and $d_{n,\delta}$. This similarity endorses the use of the Chernoff stopping rule in practice, which does not require the (heavy) computation of optimal action probabilities. Still, our sample complexity analysis applies to the two stopping rules, and we believe that a frequentist sample complexity analysis of a fully Bayesian-flavored BAI strategy is a nice theoretical contribution.

[3] TTPS (Russo, 2016) also requires the computation of $a_{n,i}$, thus we do not report simulations for it in Sec. 7.6.

Useful notation We follow the notation of Russo (2016) and define the following measures of effort allocated to arm i up to time n,

$$\psi_{n,i} \triangleq \mathbb{P}\left[I_n = i | \mathcal{F}_{n-1}\right] \quad \text{and} \quad \Psi_{n,i} \triangleq \sum_{l=1}^{n} \psi_{l,i}.$$

In particular, for TTTS we have

$$\psi_{n,i} = \beta a_{n,i} + (1-\beta) a_{n,i} \sum_{j \neq i} \frac{a_{n,j}}{1 - a_{n,j}},$$

while for T3C

$$\psi_{n,i} = \beta a_{n,i} + (1-\beta) \sum_{j \neq i} a_{n,j} \frac{\mathbb{1}\{W_n(j,i) = \min_{k \neq j} W_n(j,k)\}}{\# \left| \arg\min_{k \neq j} W_n(j,k) \right|}.$$

7.3 Two Related Optimality Notions

In the fixed-confidence setting, we aim for building δ-correct strategies, i.e. strategies that identify the best arm with high confidence on any problem instance.

Definition 7.1. A strategy (I_n, J_n, τ) is δ-correct if for all bandit models $\boldsymbol{\mu}$ with a unique optimal arm, it holds that $\mathbb{P}_{\boldsymbol{\mu}}\left[J_\tau \neq I^\star \wedge \tau < \infty\right] \leq \delta$.

Among δ-correct strategies, we seek the one with the smallest sample complexity $\mathbb{E}\left[\tau_\delta\right]$. So far, TTTS has not been analyzed in terms of sample complexity; Russo (2016) focuses on posterior consistency and optimal convergence rates. Interestingly, both the smallest possible sample complexity and the fastest rate of posterior convergence can be expressed in terms of the following quantities.

Definition 7.2. Let $\Sigma_K = \{\boldsymbol{\omega} : \sum_{k=1}^K \omega_k = 1, \omega_k \geq 0\}$ and define for all $i \neq I^\star$

$$C_i(\omega, \omega') \triangleq \min_{x \in \mathcal{I}} \ \omega d(\mu_{I^\star}; x) + \omega' d(\mu_i; x),$$

where $d(\mu, \mu')$ is the KL-divergence defined above and $\mathcal{I} = \mathbb{R}$ in the Gaussian case and $\mathcal{I} = [0,1]$ in the Bernoulli case. We define

$$\begin{aligned} \Gamma^\star &\triangleq \max_{\boldsymbol{\omega} \in \Sigma_K} \min_{i \neq I^\star} C_i(\omega_{I^\star}, \omega_i), \\ \Gamma^\star_\beta &\triangleq \max_{\substack{\boldsymbol{\omega} \in \Sigma_K \\ \omega_{I^\star} = \beta}} \min_{i \neq I^\star} C_i(\omega_{I^\star}, \omega_i). \end{aligned} \tag{7.6}$$

The quantity $C_i(\omega_{I^\star}, \omega_i)$ can be interpreted as a "transportation cost"[4] from the original bandit instance $\boldsymbol{\mu}$ to an alternative instance in which the mean of arm i is larger than that of $I^\star$, when the proportion of samples allocated to each arm is given by the vector $\boldsymbol{\omega} \in \Sigma_K$. As shown by Russo, 2016, the $\boldsymbol{\omega}$ that maximizes (7.6) is unique, which allows us to define the *β-optimal allocation* $\boldsymbol{\omega}^\beta$ in the following proposition.

[4] for which $W_n(I^\star, i)$ is an empirical counterpart

Proposition 1. *There is a unique solution $\boldsymbol{\omega}^\beta$ to the optimization problem (7.6) satisfying $\omega^\beta_{I^\star} = \beta$, and for all $i, j \neq I^\star$, $C_i(\beta, \omega^\beta_i) = C_j(\beta, \omega^\beta_j)$.*

For models with more than two arms, there is no closed form expression for $\Gamma^\star_\beta$ or $\Gamma^\star$, even for Gaussian bandits with variance σ^2 for which we have

$$\Gamma^\star_\beta = \max_{\boldsymbol{\omega}: \omega_{I^\star} = \beta} \min_{i \neq I^\star} \frac{(\mu_{I^\star} - \mu_i)^2}{2\sigma^2(1/\omega_i + 1/\beta)}.$$

Bayesian β-optimality Russo (2016) proves that any sampling rule allocating a fraction β to the optimal arm ($\Psi_{n,I^\star}/n \to \beta$) satisfies $1 - a_{n,I^\star} \geq e^{-n(\Gamma^\star_\beta + o(1))}$ (a.s.).We define a *Bayesian β-optimal* sampling rule as a sampling rule matching this lower bound, i.e. satisfying $\Psi_{n,I^\star}/n \to \beta$ and $1 - a_{n,I^\star} \leq e^{-n(\Gamma^\star_\beta + o(1))}$.

Russo (2016) proves that TTTS with parameter β is Bayesian β-optimal. However, the result is valid only under strong regularity assumptions, excluding the two practically important cases of Gaussian and Bernoulli bandits. In this chapter, we complete the picture by establishing Bayesian β-optimality for those models in Sec. 7.5. For the Gaussian bandit, Bayesian β-optimality was established for TTEI by Qin, Klabjan and Russo, 2017 with Gaussian priors, but this remained an open problem for TTTS.

A fundamental ingredient of these proofs is to establish the convergence of the allocation of measurement effort to the β-optimal allocation: $\Psi_{n,i}/n \to \omega^\beta_i$ for all i, which is equivalent to $T_{n,i}/n \to \omega^\beta_i$ (cf. Lemma 8).

β-optimality in the fixed-confidence setting In the fixed confidence setting, the performance of an algorithm is evaluated in terms of sample complexity. A lower bound given by Garivier and Kaufmann, 2016 states that any δ-correct strategy satisfies $\mathbb{E}[\tau_\delta] \geq (\Gamma^\star)^{-1} \ln(1/(3\delta))$.

Observe that $\Gamma^\star = \max_{\beta \in [0,1]} \Gamma^\star_\beta$. Using the same lower bound techniques, one can also prove that under any δ-correct strategy satisfying $T_{n,I^\star}/n \to \beta$,

$$\liminf_{\delta \to 0} \frac{\mathbb{E}[\tau_\delta]}{\ln(1/\delta)} \geq \frac{1}{\Gamma^\star_\beta}.$$

This motivates the relaxed optimality notion that we introduce in this chapter: A BAI strategy is called *asymptotically β-optimal* if it satisfies

$$\frac{T_{n,I^\star}}{n} \to \beta \quad \text{and} \quad \limsup_{\delta \to 0} \frac{\mathbb{E}[\tau_\delta]}{\ln(1/\delta)} \leq \frac{1}{\Gamma^\star_\beta}.$$

In this chapter, we provide the first sample complexity analysis of a BAI algorithm based on TTTS (with the stopping and recommendation rules described in Sec. 7.2), establishing its asymptotic β-optimality.

As already observed by Qin, Klabjan and Russo, 2017, any sampling rule converging to the β-optimal allocation (i.e. satisfying $T_{n,i}/n \to w_i^\beta$ for all i) can be shown to satisfy

$$\limsup_{\delta \to 0} \frac{\tau_\delta}{\ln(1/\delta)} \leq (\Gamma_\beta^\star)^{-1}$$

almost surely, when coupled with the Chernoff stopping rule. The fixed confidence optimality that we define above is stronger as it provides guarantees on $\mathbb{E}[\tau_\delta]$.

7.4 Fixed-Confidence Analysis

In this section, we consider Gaussian bandits and the Bayesian rules using an improper prior on the means. We state our main result below, showing that TTTS and T3C are asymptotically β-optimal in the fixed confidence setting, when coupled with appropriate stopping and recommendation rules.

Theorem 7.2. *With $\mathcal{C}^{g_G}$ the function defined in Corollary 10 of Kaufmann and Koolen, 2018, which satisfies $\mathcal{C}^{g_G}(x) \simeq x + \ln(x)$, we introduce the threshold*

$$d_{n,\delta} = 4\ln(4 + \ln(n)) + 2\mathcal{C}^{g_G}\left(\frac{\ln((K-1)/\delta)}{2}\right). \tag{7.7}$$

The TTTS and T3C sampling rules coupled with either

- *the Bayesian stopping rule (7.4) with threshold*

$$c_{n,\delta} = 1 - \frac{1}{\sqrt{2\pi}} e^{-\left(\sqrt{d_{n,\delta}} + \frac{1}{\sqrt{2}}\right)^2}$$

  and recommendation rule $J_t = \arg\max_i a_{n,i}$, or
- *the Chernoff stopping rule (7.5) with threshold $d_{n,\delta}$ and recommendation rule $J_t = \arg\max_i \mu_{n,i}$,*

form a δ-correct BAI strategy. Moreover, if all the arms means are distinct, it satisfies

$$\limsup_{\delta \to 0} \frac{\mathbb{E}[\tau_\delta]}{\log(1/\delta)} \leq \frac{1}{\Gamma_\beta^\star}.$$

We now give the proof of Theorem 7.2, which is divided into three parts. The **first step** of the analysis is to prove the δ-correctness of the studied BAI strategies.

Theorem 7.3. *Regardless of the sampling rule, the stopping rule (7.4) with the threshold $c_{n,\delta}$ and the Chernoff stopping rule (7.5) with threshold $d_{n,\delta}$ defined in (7.7) satisfy $\mathbb{P}[\tau_\delta < \infty \wedge J_{\tau_\delta} \neq I^\star] \leq \delta$.*

To prove that TTTS and T3C allow to reach a β-optimal sample complexity, one needs to quantify how fast the measurement effort for each arm is concentrating to its corresponding optimal weight. For this purpose, we introduce the random variable

$$T_\beta^\varepsilon \triangleq \inf\left\{N \in : \max_{i \in \mathcal{A}} |T_{n,i}/n - \omega_i^\beta| \leq \varepsilon, \forall n \geq N\right\}.$$

The **second step** of our analysis is a sufficient condition for β-optimality, stated in Lemma 4. Its proof is given in Appendix 7.F. The same result was proven for the Chernoff stopping rule by Qin, Klabjan and Russo, 2017.

Lemma 4. *Let $\delta, \beta \in (0,1)$. For any sampling rule which satisfies $\mathbb{E}\left[T_\beta^\varepsilon\right] < \infty$ for all $\varepsilon > 0$, we have*

$$\limsup_{\delta \to 0} \frac{\mathbb{E}\left[\tau_\delta\right]}{\log(1/\delta)} \leq \frac{1}{\Gamma_\beta^\star},$$

if the sampling rule is coupled with stopping rule (7.4),

Finally, it remains to show that TTTS and T3C meet the sufficient condition, and therefore the **last step**, which is the core component and the most technical part our analysis, consists of showing the following.

Theorem 7.5. *Under TTTS or T3C, $\mathbb{E}\left[T_\beta^\varepsilon\right] < +\infty$.*

In the rest of this section, we prove Theorem 7.3 and sketch the proof of Theorem 7.5. But we first highlight some important ingredients for these proofs.

7.4.1 Core ingredients

Our analysis hinges on properties of the Gaussian posteriors, in particular on the following tail bounds, which follow from Lemma 1 of Qin, Klabjan and Russo, 2017.

Lemma 6. *For any $i, j \in \mathcal{A}$, if $\mu_{n,i} \leq \mu_{n,j}$*

$$\Pi_n\left[\theta_i \geq \theta_j\right] \leq \frac{1}{2} \exp\left\{-\frac{\left(\mu_{n,j} - \mu_{n,i}\right)^2}{2\sigma_{n,i,j}^2}\right\}, \tag{7.8}$$

$$\Pi_n\left[\theta_i \geq \theta_j\right] \geq \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{\left(\mu_{n,j} - \mu_{n,i} + \sigma_{n,i,j}\right)^2}{2\sigma_{n,i,j}^2}\right\}, \tag{7.9}$$

where $\sigma_{n,i,j}^2 \triangleq \sigma^2/T_{n,i} + \sigma^2/T_{n,j}$.

This lemma is crucial to control $a_{n,i}$ and $\psi_{n,i}$, the optimal action and selection probabilities.

7.4.2 Proof of Theorem 7.3

We upper bound the desired probability as follows

$$\begin{aligned}
\mathbb{P}\left[\tau_\delta < \infty \wedge J_{\tau_\delta} \neq I^\star\right] &\leq \sum_{i \neq I^\star} \mathbb{P}\left[\exists n \in \colon a_{n,i} > c_{n,\delta}\right] \\
&\leq \sum_{i \neq I^\star} \mathbb{P}\left[\exists n \in \colon \Pi_n(\theta_i \geq \theta_{I_\star}) > c_{n,\delta}, \mu_{n,I^\star} \leq \mu_{n,i}\right] \\
&\leq \sum_{i \neq I^\star} \mathbb{P}\left[\exists n \in \colon 1 - c_{n,\delta} > \Pi_n(\theta_{I^\star} > \theta_i), \mu_{n,I^\star} \leq \mu_{n,i}\right].
\end{aligned}$$

The second step uses the fact that as $c_{n,\delta} \geq 1/2$, a necessary condition for $\Pi_n(\theta_i \geq \theta_{I_\star}) \geq c_{n,\delta}$ is that $\mu_{n,i} \geq \mu_{n,I_\star}$. Now using the lower bound (7.9), if $\mu_{n,I^\star} \leq \mu_{n,i}$, the inequality $1 - c_{n,\delta} > \Pi_n(\theta_{I^\star} > \theta_i)$ implies

$$\frac{(\mu_{n,i} - \mu_{n,I^\star})^2}{2\sigma^2_{n,i,I^\star}} \geq \left(\sqrt{\ln \frac{1}{\sqrt{2\pi}(1 - c_{n,\delta})}} - \frac{1}{\sqrt{2}} \right)^2 = d_{n,\delta},$$

where the equality follows from the expression of $c_{n,\delta}$ as function of $d_{n,\delta}$. Hence to conclude the proof it remains to check that

$$\mathbb{P}\left[\exists n \in \mu_{n,i} \geq \mu_{n,I^\star}, \frac{(\mu_{n,i} - \mu_{n,I^\star})^2}{2\sigma^2_{n,i,I^\star}} \geq d_{n,\delta} \right] \leq \frac{\delta}{K-1}. \tag{7.10}$$

To prove this, we observe that for $\mu_{n,i} \geq \mu_{n,I^\star}$,

$$\begin{aligned} \frac{(\mu_{n,i} - \mu_{n,I^\star})^2}{2\sigma^2_{n,i,I^\star}} &= \inf_{\theta_i < \theta_{I^\star}} T_{n,i} d(\mu_{n,i}; \theta_i) + T_{n,I^\star} d(\mu_{n,I^\star}; \theta_{I^\star}) \\ &\leq T_{n,i} d(\mu_{n,i}; \mu_i) + T_{n,I^\star} d(\mu_{n,I^\star}; \mu_{I^\star}). \end{aligned}$$

Corollary 10 of Kaufmann and Koolen, 2018 then allows us to upper bound the probability

$$\mathbb{P}\left[\exists n \in : T_{n,i} d(\mu_{n,i}; \mu_i) + T_{n,I^\star} d(\mu_{n,I^\star}, \mu_{I^\star}) \geq d_{n,\delta} \right]$$

by $\delta/(K-1)$ for the choice of threshold given in (7.7), which completes the proof that the stopping rule (7.4) is δ-correct. The fact that the Chernoff stopping rule with the above threshold $d_{n,\delta}$ given above is δ-correct straightforwardly follows from (7.10).

7.4.3 Sketch of the proof of Theorem 7.5

We present a unified proof sketch of Theorem 7.5 for TTTS and T3C. While the two analyses follow the same steps, some of the lemmas given below have different proofs for TTTS and T3C, which can be found in Appendix 7.D and 7.E respectively.

We first state two important concentration results, that hold under any sampling rule.

Lemma 7. *[Lemma 5 of Qin, Klabjan and Russo 2017] There exists a random variable W_1, such that for all $i \in \mathcal{A}$,*

$$\forall n \in , \quad |\mu_{n,i} - \mu_i| \leq \sigma W_1 \sqrt{\frac{\log(e + T_{n,i})}{1 + T_{n,i}}} \ a.s.,$$

and $\mathbb{E}\left[e^{\lambda W_1}\right] < \infty$ for all $\lambda > 0$.

Lemma 8. *There exists a random variable W_2, such that for all $i \in \mathcal{A}$,*

$$\forall n \in , |T_{n,i} - \Psi_{n,i}| \leq W_2 \sqrt{(n+1)\log(e^2 + n)} \ a.s.,$$

and $\mathbb{E}\left[e^{\lambda W_2}\right] < \infty$ for any $\lambda > 0$.

Lemma 7 controls the concentration of the posterior means towards the true means and Lemma 8 establishes that $T_{n,i}$ and $\Psi_{n,i}$ are close. Both results rely on uniform deviation inequalities for martingales.

Our analysis uses the same principle as that of TTEI: We establish that T^{ε}_{β} is upper bounded by some random variable N which is a polynomial of the random variables W_1 and W_2 introduced in the above lemmas, denoted by $\mathrm{Poly}(W_1, W_2) \triangleq \mathcal{O}(W_1^{c_1} W_2^{c_2})$, where c_1 and c_2 are two constants (that may depend on the arms' means and the constant hidden in the $\mathcal{O}$). As all exponential moments of W_1 and W_2 are finite, N has a finite expectation as well, concluding the proof.

The first step to exhibit such an upper bound N is to establish that every arm is pulled sufficiently often.

Lemma 9. *Under TTTS or T3C, there exists* $N_1 = \mathrm{Poly}(W_1, W_2)$ *s.t.*

$$\forall n \geq N_1, \forall i,\ T_{n,i} \geq \sqrt{\frac{n}{K}},\ a.s..$$

Due to the randomized nature of TTTS and T3C, the proof of Lemma 9 is significantly more involved than for a deterministic rule like TTEI. Intuitively, the posterior of each arm would be well concentrated once the arm is sufficiently pulled. If the optimal arm is under-sampled, then it would be chosen as the first candidate with large probability. If a sub-optimal arm is under-sampled, then its posterior distribution would possess a relatively wide tail that overlaps with or cover the somehow narrow tails of other overly-sampled arms. The probability of that sub-optimal arm being chosen as the challenger would be large enough then.

Combining Lemma 9 with Lemma 7 straightforwardly leads to the following result.

Lemma 10. *Under TTTS or T3C, fix a constant* $\varepsilon > 0$*, there exists* $N_2 = \mathrm{Poly}(1/\varepsilon, W_1, W_2)$ *s.t.* $\forall n \geq N_2, \forall i \in \mathcal{A}, \quad |\mu_{n,i} - \mu_i| \leq \varepsilon$.

We can then deduce a very nice property about the optimal action probability for sub-optimal arms from the previous two lemmas. Indeed, we can show that

$$\forall i \neq I^\star, \quad a_{n,i} \leq \exp\left\{-\frac{\Delta^2_{\min}}{16\sigma^2}\sqrt{\frac{n}{K}}\right\}$$

for n larger than some $\mathrm{Poly}(W_1, W_2)$, where $\Delta_{\min}$ is the smallest mean difference among all the arms.

Plugging this in the expression of $\psi_{n,i}$, one can easily quantify how fast $\psi_{n,I^\star}$ converges to β, which eventually yields the following result.

Lemma 11. *Under TTTS or T3C, fix* $\varepsilon > 0$*, then there exists* $N_3 = \mathrm{Poly}(1/\varepsilon, W_1, W_2)$ *s.t.* $\forall n \geq N_3$,

$$\left|\frac{T_{n,I^\star}}{n} - \beta\right| \leq \varepsilon.$$

The last, more involved, step is to establish that the fraction of measurement allocation to every sub-optimal arm i is indeed similarly close to its optimal proportion ω_i^{β}.

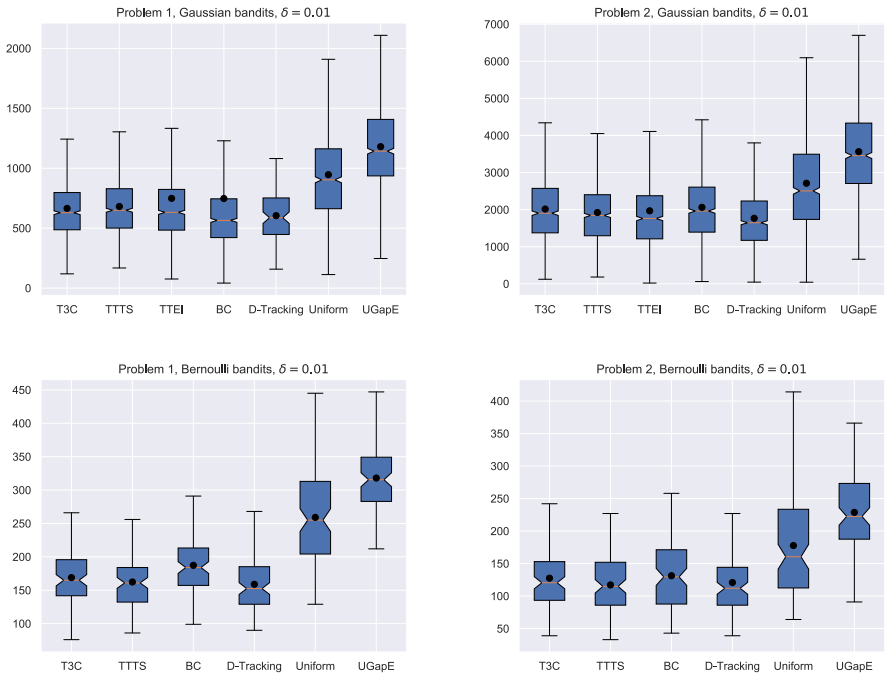

Figure 7.1: Black dots represent means and oranges lines represent medians.

Lemma 12. *Under TTTS or T3C, fix a constant $\varepsilon > 0$, there exists $N_4 = Poly(1/\varepsilon, W_1, W_2)$ s.t. $\forall n \geq N_4$,*

$$\forall i \neq I^\star, \quad \left| \frac{T_{n,i}}{n} - \omega_i^\beta \right| \leq \varepsilon.$$

The major step in the proof of Lemma 12 for each sampling rule, is to establish that if some arm is over-sampled, then its probability to be selected is exponentially small. Formally, we show that for n larger than some $\mathrm{Poly}(1/\varepsilon, W_1, W_2)$,

$$\frac{\Psi_{n,i}}{n} \geq \omega_i^\beta + \xi \quad \Rightarrow \quad \psi_{n,i} \leq \exp\left\{-f(n,\xi)\right\},$$

for some function $f(n,\xi)$ to be specified for each sampling rule, satisfying $f(n) \geq C_\xi \sqrt{n}$ (a.s.). This result leads to the concentration of $\Psi_{n,i}/n$, thus can be easily converted to the concentration of $T_{n,i}/n$ by Lemma 8.

Finally, Lemma 11 and Lemma 12 show that T_β^ε is upper bounded by $N \triangleq \max(N_3, N_4)$, which yields

$$\mathbb{E}[T_\beta^\varepsilon] \leq \max(\mathbb{E}[N_3], \mathbb{E}[N_4]) < \infty.$$

Sampling rule	Execution time (s)
T3C	1.6×10^{-5}
TTTS	2.3×10^{-4}
TTEI	1×10^{-5}
BC	1.4×10^{-5}
D-Tracking	1.3×10^{-3}
Uniform	6×10^{-6}
UGapE	5×10^{-6}

Table 7.1: Average execution time in seconds for different sampling rules.

7.5 Optimal Posterior Convergence

Recall that $a_{n,I^\star}$ denotes the posterior mass assigned to the event that action $I^\star$ (i.e. the true optimal arm) is optimal at time n. As the number of observations tends to infinity, we want the posterior distribution to converge to the truth. In this section we show equivalently that the posterior mass on the complementary event, $1 - a_{n,I^\star}$, the event that arm $I^\star$ is not optimal, converges to zero at an exponential rate, and that it does so at optimal rate $\Gamma^\star_\beta$.

Russo (2016) proves a similar theorem under three confining boundedness assumptions (see Russo 2016, Assumption 1) on the parameter space, the prior density and the (first derivative of the) log-normalizer of the exponential family. Hence, the theorems in Russo, 2016 do not apply to the two bandit models most used in practice, which we consider in this chapter: the Gaussian and Bernoulli model.

In the first case, the parameter space is unbounded, in the latter model, the derivative of the log-normalizer (which is $e^\eta/(1+e^\eta)$) is unbounded. Here we provide a theorem, proving that under TTTS, the optimal, exponential posterior convergence rates are obtained for the Gaussian model with uninformative (improper) Gaussian priors (proof in Appendix 7.H), and the Bernoulli model with $\mathcal{B}eta(1,1)$ priors (proof in Appendix 7.I).

Theorem 7.13. *Under TTTS, for Gaussian bandits with improper Gaussian priors and for Bernoulli bandits with uniform priors, it holds almost surely that*

$$\lim_{n\to\infty} -\frac{1}{n}\log(1 - a_{n,I^\star}) = \Gamma^\star_\beta.$$

7.6 Numerical Illustrations

This section is aimed at illustrating our theoretical results and supporting the practical use of Bayesian sampling rules for fixed-confidence BAI.

We experiment with 3 Bayesian sampling rules: T3C, TTTS and TTEI with $\beta = 1/2$, against the Direct Tracking (D-Tracking) of Garivier and Kaufmann, 2016 (which is adaptive to β), UGapE of Gabillon, Ghavamzadeh and Lazaric, 2012, and a uniform baseline. To make fair

comparisons, we use the stopping rule (7.5) and associated recommendation rule for all of the sampling rules except for UGapE which has its own stopping rule.

We further include a top-two variant of the Best Challenger (BC) heuristic (see Ménard, 2019). BC selects the empirical best arm $\widehat{I}_n$ with probability β and the maximizer of $W_n(\widehat{I}_n, j)$ with probability $1-\beta$, but also performs forced exploration (selecting any arm sampled less than $\sqrt{n}$ times at round n). T3C can thus be viewed as a variant of BC in which no forced exploration is needed to converge to $\boldsymbol{\omega}^\beta$, due to the noise added by replacing $\widehat{I}_n$ with $I_n^{(1)}$. This randomization is crucial as BC without forced exploration can fail: we observed that on bandit instances with two identical sub-optimal arms, BC has some probability to alternate forever between these two arms and never stop.

We consider two simple instances with arm means given by $\boldsymbol{\mu}_1 = [0.5\ 0.9\ 0.4\ 0.45\ 0.44999]$, and $\boldsymbol{\mu}_2 = [1\ 0.8\ 0.75\ 0.7]$. We run simulations for both Gaussian ($\sigma = 1$) and Bernoulli bandits, with a risk parameter $\delta = 0.01$. Fig. 7.1 reports the empirical distribution of τ_δ under the different sampling rules, estimated over 1000 independent runs. We also indicate the values of $N^\star \triangleq \log(1/\delta)/\Gamma^\star$ (resp. $N^\star_{0.5} \triangleq \log(1/\delta)/\Gamma^\star_{0.5}$), the theoretical minimal number of samples needed for any strategy (resp. any 1/2-optimal strategy). In Appendix 7.C, we further illustrate how the empirical stopping time of T3C matches the theoretical one.

These figures provide several insights: (1) T3C is competitive with, and sometimes slightly better than TTTS/TTEI in terms of sample complexity. (2) The UGapE algorithm has a larger sample complexity than the uniform sampling rule, which highlights the importance of the stopping rule in the fixed-confidence setting. (3) The fact that D-Tracking performs best is not surprising, since it converges to $\boldsymbol{\omega}^{\beta^\star}$ and achieves minimal sample complexity. However, in terms of computation time, D-Tracking is much worse than others, as shown in Table 7.1, which reports the average execution time of one step of each sampling rule for $\boldsymbol{\mu}_1$ in the Gaussian case. (4) TTTS also suffers from computational costs, whose origins are explained in Sec. 7.2, unlike T3C or TTEI. Although TTEI is already computationally more attractive than TTTS, its practical benefits are limited to the Gaussian case, since the *Expected Improvement* (EI) does not have a closed form beyond this case and its approximation would be costly. In contrast, T3C can be applied for other distributions.

7.7 Conclusion

We have advocated the use of Bayesian sampling rules for BAI. In particular, we proved that TTTS and a computationally advantageous approach T3C, are both β-optimal in the fixed-confidence setting, for Gaussian bandits. We further extended the Bayesian optimality properties (Russo, 2016) to more practical choices of models and prior distributions. In order to be optimal, these sampling rules would need the oracle tuning $\beta^\star = \arg\max_{\beta\in[0,1]} \Gamma^\star_\beta$, which is not feasible. In future work, we will investigate the efficient online tuning of β to circumvent this issue. We also wish to obtain explicit finite-time sample complexity bound for these Bayesian strategies, and justify the use of these appealing anytime sampling rules in the fixed-budget setting. The latter is often more plausible in application scenarios such as BAI for automated machine learning (Li et al., 2017; Shang, Kaufmann and Valko, 2019).

7.A Outline

The appendix of this chapter is organized as follows:

Appendix 7.C provides some further numerical illustration for better understanding of T3C.
Appendix 7.D provides the complete fixed-confidence analysis of TTTS (Gaussian case).
Appendix 7.E provides the complete fixed-confidence analysis of T3C (Gaussian case).
Appendix 7.F is dedicated to Lemma 4.
Appendix 7.G is dedicated to crucial technical lemmas.
Appendix 7.H is the proof to the posterior convergence Theorem 7.27 (Gaussian case).
Appendix 7.I is the proof to the posterior convergence Theorem 7.34 (Beta-Bernoulli case).

7.B Useful Notation

In this section, we provide a list of useful notation that is applied in appendices (including reminders of previous notation in the main text and some new ones).

- Recall that $d(\mu_1;\mu_2)$ denotes the KL-divergence between two distributions parametrized by their means μ_1 and μ_2. For Gaussian distributions, we know that

$$d(\mu_1;\mu_2) = \frac{(\mu_1-\mu_2)^2}{2\sigma^2}.$$

  When it comes to Bernoulli distributions, we denote this with kl, i.e.

$$kl(\mu_1;\mu_2) = \mu_1 \ln\left(\frac{\mu_1}{\mu_2}\right) + (1-\mu_1)\ln\left(\frac{1-\mu_1}{1-\mu_2}\right).$$

- $\mathcal{B}eta(\cdot,\cdot)$ denotes a Beta distribution.
- $\mathcal{B}ern(\cdot)$ denotes a Bernoulli distribution.
- $\mathcal{B}(\cdot)$ denotes a Binomial distribution.
- $\mathcal{N}(\cdot,\cdot)$ denotes a normal distribution.
- $Y_{n,i}$ is the reward of arm i at time n.
- Y_{n,I_n} is the observation of the sampling rule at time n.
- $\mathcal{F}_n \triangleq \sigma(I_1, Y_{1,I_1}, I_2, Y_{2,I_2}, \cdots, I_n, Y_{n,I_n})$ is the filtration generated by the first n observations.
- $\psi_{n,i} \triangleq \mathbb{P}\left[I_n = i | \mathcal{F}_{n-1}\right]$.
- $\Psi_{n,i} \triangleq \sum_{l=1}^{n} \psi_{l,i}$.
- For the sake of simplicity, we further define $\overline{\psi}_{n,i} \triangleq \frac{\Psi_{n,i}}{n}$.
- $T_{n,i}$ is the number of pulls of arm i before round n.
- $\mathbf{T}_n$ denotes the vector of the number of arm selections.
- $I_n^\star \triangleq \arg\max_{i\in\mathcal{A}} \mu_{n,i}$ denotes the empirical best arm at time n.
- For any $a, b > 0$, define a function $C_{a,b}$ s.t. $\forall y$,

$$C_{a,b}(y) \triangleq (a+b-1)kl(\frac{a-1}{a+b-1}; y).$$

- We define the minimum and the maximum means gap as

$$\Delta_{\min} \triangleq \min_{i \neq j} |\mu_i - \mu_j|; \qquad \Delta_{\max} \triangleq \max_{i \neq j} |\mu_i - \mu_j|.$$

- We introduce two indices

$$J_n^{(1)} \triangleq \arg\max_j a_{n,j}; \qquad J_n^{(2)} \triangleq \arg\max_{j \neq J_n^{(1)}} a_{n,j}.$$

  Note that $J_n^{(1)}$ coincides with the Bayesian recommendation index J_n.
- Two real-valued sequences (a_n) and (b_n) are are said to be logarithmically equivalent if

$$\lim_{n \to \infty} \frac{1}{n} \log\left(\frac{a_n}{b_n}\right) = 0,$$

  and we denote this by $a_n \doteq b_n$.

7.C Empirical vs. theoretical sample complexity

In Fig. 7.2, we plot expected stopping time of T3C for $\delta = 0.01$ as a function of $1/\Gamma_\beta^\star$ on 100 randomly generated problem instances. We see on this plot that the empirical stopping time has the right linear scaling in $1/\Gamma_\beta^\star$ (ignoring a few outliers).

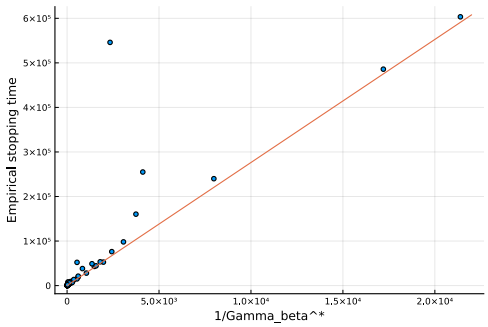

Figure 7.2: dots: empirical sample complexity, solid line: theoretical sample complexity.

7.D Fixed-Confidence Analysis for TTTS

This section is entirely dedicated to TTTS.

7.D.1 Technical novelties and some intuitions

Before we start the analysis, we first highlight some technical novelties and intuitions. The main novelty in our analysis is the proof of Lemma 9, establishing that all arms are sufficiently explored

by our randomized strategies. Although Qin, Klabjan and Russo, 2017 indeed establish a similar result, our proof is much more intricate due to the randomized nature of the two candidate arms $I^{(1)}$ and $I^{(2)}$ for TTTS (resp. $I^{(1)}$ for T3C). In the proof of Lemma 9 (in Appendix 7.D.2 and Appendix 7.E.1 respectively), we need to add a sort of 'extra layer' where we first study the behaviour of $J^{(1)}$ and $J^{(2)}$ for TTTS (resp. $J^{(1)}$ and $\widetilde{J^{(2)}}$ for T3C). We show in Lemma 14 (resp. Lemma 21 for T3C) that if there exists some under-sampled arm, then either $J^{(1)}$ or $J^{(2)}$ is also under-sampled. A link between I and J is then established using the expression of $\psi_{n,i}$, which also allows to upper bound the optimal action probability with a known rate (see Lemma 17).

7.D.2 Sufficient exploration of all arms proof of Lemma 9 under TTTS

To prove this lemma, we introduce the two following sets of indices for a given $L > 0$: $\forall n \in \mathbb{N}$ we define

$$U_n^L \triangleq \{i : T_{n,i} < \sqrt{L}\},$$
$$V_n^L \triangleq \{i : T_{n,i} < L^{3/4}\}.$$

It is seemingly non trivial to manipulate directly TTTS's candidate arms, we thus start by connecting TTTS with TTPS (top two probability sampling). TTPS is another sampling rule presented by Russo, 2016 for which the two candidate samples are defined as in Appendix 7.B, we recall them in the following.

$$J_n^{(1)} \triangleq \arg\max_j a_{n,j}, J_n^{(2)} \triangleq \arg\max_{j \neq J_n^{(1)}} a_{n,j}.$$

Lemma 9 is proved via the following sequence of lemmas.

Lemma 14. *There exists $L_1 = Poly(W_1)$ s.t. if $L > L_1$, for all n, $U_n^L \neq \varnothing$ implies $J_n^{(1)} \in V_n^L$ or $J_n^{(2)} \in V_n^L$.*

Proof. If $J_n^{(1)} \in V_n^L$, then the proof is finished. Now we assume that $J_n^{(1)} \in \overline{V_n^L}$, and we prove that $J_n^{(2)} \in V_n^L$.

Step 1 According to Lemma 7, there exists $L_2 = \text{Poly}(W_1)$ s.t. $\forall L > L_2, \forall i \in \overline{U_n^L}$,

$$\begin{aligned}
|\mu_{n,i} - \mu_i| &\leq \sigma W_1 \sqrt{\frac{\log(e + T_{n,i})}{1 + T_{n,i}}} \\
&\leq \sigma W_1 \sqrt{\frac{\log(e + \sqrt{L})}{1 + \sqrt{L}}} \\
&\leq \sigma W_1 \frac{\Delta_{\min}}{4\sigma W_1} = \frac{\Delta_{\min}}{4}.
\end{aligned}$$

The second inequality holds since $x \mapsto \frac{\log(e+x)}{1+x}$ is a decreasing function. The third inequality holds for a large $L > L_2$ with $L_2 = \ldots$.

Step 2 We now assume that $L > L_2$, and we define

$$\overline{J_n^\star} \triangleq \arg\max_{j \in \overline{U_n^L}} \mu_{n,j} = \arg\max_{j \in \overline{U_n^L}} \mu_j.$$

The last equality holds since $\forall j \in \overline{U_n^L}$, $|\mu_{n,i} - \mu_i| \leq \Delta_{\min}/4$. We show that there exists $L_3 = \mathrm{Poly}(W_1)$ s.t. $\forall L > L_3$,

$$\overline{J_n^\star} = J_n^{(1)}.$$

We proceed by contradiction, and suppose that $\overline{J_n^\star} \neq J_n^{(1)}$, then $\mu_{n,J_n^{(1)}} < \mu_{n,\overline{J_n^\star}}$, since $J_n^{(1)} \in \overline{V_n^L} \subset \overline{U_n^L}$. However, we have

$$\begin{aligned} a_{n,J_n^{(1)}} &= \Pi_n\left[\theta_{J_n^{(1)}} > \max_{j \neq J_n^{(1)}} \theta_j\right] \\ &\leq \Pi_n\left[\theta_{J_n^{(1)}} > \theta_{\overline{J_n^\star}}\right] \\ &\leq \frac{1}{2}\exp\left\{-\frac{(\mu_{n,J_n^{(1)}} - \mu_{n,\overline{J_n^\star}})^2}{2\sigma^2(1/T_{n,J_n^{(1)}} + 1/T_{n,\overline{J_n^\star}})}\right\}. \end{aligned}$$

The last inequality uses the Gaussian tail inequality (7.8) of Lemma 6. On the other hand,

$$\begin{aligned} |\mu_{n,J_n^{(1)}} - \mu_{n,\overline{J_n^\star}}| &= |\mu_{n,J_n^{(1)}} - \mu_{J_n^{(1)}} + \mu_{J_n^{(1)}} - \mu_{\overline{J_n^\star}} + \mu_{\overline{J_n^\star}} - \mu_{n,\overline{J_n^\star}}| \\ &\geq |\mu_{J_n^{(1)}} - \mu_{\overline{J_n^\star}}| - |\mu_{n,J_n^{(1)}} - \mu_{J_n^{(1)}} + \mu_{\overline{J_n^\star}} - \mu_{n,\overline{J_n^\star}}| \\ &\geq \Delta_{\min} - \left(\frac{\Delta_{\min}}{4} + \frac{\Delta_{\min}}{4}\right) \\ &= \frac{\Delta_{\min}}{2}, \end{aligned}$$

and

$$\frac{1}{T_{n,J_n^{(1)}}} + \frac{1}{T_{n,\overline{J_n^\star}}} \leq \frac{2}{\sqrt{L}}.$$

Thus, if we take L_3 s.t.

$$\exp\left\{-\frac{\sqrt{L_3}\Delta_{\min}^2}{16\sigma^2}\right\} \leq \frac{1}{2K},$$

then for any $L > L_3$, we have

$$a_{n,J_n^{(1)}} \leq \frac{1}{2K} < \frac{1}{K},$$

which contradicts the definition of $J_n^{(1)}$. We now assume that $L > L_3$, thus $J_n^{(1)} = \overline{J_n^\star}$.

Step 3 We finally show that for L large enough, $J_n^{(2)} \in V_n^L$. First note that $\forall j \in \overline{V_n^L}$, we have

$$a_{n,j} \leq \Pi_n\left[\theta_j \geq \theta_{\overline{J_n^\star}}\right] \leq \exp\left\{-\frac{L^{3/4}\Delta_{\min}^2}{16\sigma^2}\right\}. \tag{7.11}$$

This last inequality can be proved using the same argument as Step 2. Now we define another index $J_n^\star \triangleq \arg\max_{j \in U_n^L} \mu_{n,j}$ and the quantity $c_n \triangleq \max(\mu_{n,J_n^\star}, \mu_{n,\overline{J_n^\star}})$. We can lower bound $a_{n,J_n^\star}$ as follows:

$$\begin{aligned} a_{n,J_n^\star} &\geq \Pi_n\left[\theta_{J_n^\star} \geq c_n\right] \prod_{j \neq J_n^\star} \Pi_n\left[\theta_j \leq c_n\right] \\ &= \Pi_n\left[\theta_{J_n^\star} \geq c_n\right] \prod_{j \neq J_n^\star; j \in U_n^L} \Pi_n\left[\theta_j \leq c_n\right] \prod_{j \in \overline{U_n^L}} \Pi_n\left[\theta_j \leq c_n\right] \\ &\geq \Pi_n\left[\theta_{J_n^\star} \geq c_n\right] \frac{1}{2^{K-1}}. \end{aligned}$$

Now there are two cases:

- If $\mu_{n,J_n^\star} > \mu_{n,\overline{J_n^\star}}$, then we have

$$\Pi_n\left[\theta_{J_n^\star} \geq c_n\right] = \Pi_n\left[\theta_{J_n^\star} \geq \mu_{n,J_n^\star}\right] \geq \frac{1}{2}.$$

- If $\mu_{n,J_n^\star} < \mu_{n,\overline{J_n^\star}}$, then we can apply the Gaussian tail bound (7.9) of Lemma 6, and we obtain

$$\begin{aligned} \Pi_n\left[\theta_{J_n^\star} \geq c_n\right] &= \Pi_n\left[\theta_{J_n^\star} \geq \mu_{n,\overline{J_n^\star}}\right] = \Pi_n\left[\theta_{J_n^\star} \geq \mu_{n,J_n^\star} + (\mu_{n,\overline{J_n^\star}} - \mu_{n,J_n^\star})\right] \\ &\geq \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(1 - \frac{\sqrt{T_{n,J_n^\star}}}{\sigma}(\mu_{n,J_n^\star} - \mu_{n,\overline{J_n^\star}})\right)^2\right\} \\ &= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(1 + \frac{\sqrt{T_{n,J_n^\star}}}{\sigma}(\mu_{n,\overline{J_n^\star}} - \mu_{n,J_n^\star})\right)^2\right\}. \end{aligned}$$

  On the other hand, by Lemma 7, we know that

$$\begin{aligned} |\mu_{n,J_n^\star} - \mu_{n,\overline{J_n^\star}}| &= |\mu_{n,J_n^\star} - \mu_{J_n^\star} + \mu_{J_n^\star} - \mu_{\overline{J_n^\star}} + \mu_{\overline{J_n^\star}} - \mu_{n,\overline{J_n^\star}}| \\ &\leq |\mu_{J_n^\star} - \mu_{\overline{J_n^\star}}| + \sigma W_1 \sqrt{\frac{\log(e + T_{n,J_n^\star})}{1 + T_{n,J_n^\star}}} + \sigma W_1 \sqrt{\frac{\log(e + T_{n,\overline{J_n^\star}})}{1 + T_{n,\overline{J_n^\star}}}} \\ &\leq |\mu_{J_n^\star} - \mu_{\overline{J_n^\star}}| + 2\sigma W_1 \sqrt{\frac{\log(e + T_{n,J_n^\star})}{1 + T_{n,J_n^\star}}} \\ &\leq \Delta_{\max} + 2\sigma W_1 \sqrt{\frac{\log(e + T_{n,J_n^\star})}{1 + T_{n,J_n^\star}}}. \end{aligned}$$

Therefore,

$$\begin{aligned}\Pi_n\left[\theta_{J_n^\star} \geq c_n\right] &\geq \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(1+\frac{\sqrt{T_{n,J_n^\star}}}{\sigma}\left(\Delta_{\max}+2\sigma W_1\sqrt{\frac{\log(e+T_{n,J_n^\star})}{1+T_{n,J_n^\star}}}\right)\right)^2\right\} \\ &\geq \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(1+\frac{\sqrt{\sqrt{L}}}{\sigma}\left(\Delta_{\max}+2\sigma W_1\sqrt{\frac{\log(e+\sqrt{L})}{1+\sqrt{L}}}\right)\right)^2\right\} \\ &\geq \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(1+\frac{L^{1/4}\Delta_{\max}}{\sigma}+2W_1\sqrt{\log(e+\sqrt{L})}\right)^2\right\}.\end{aligned}$$

Now we have

$$a_{n,J_n^\star} \geq \max\left(\left(\frac{1}{2}\right)^K, \left(\frac{1}{2}\right)^{K-1}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(1+\frac{L^{1/4}\Delta_{\max}}{\sigma}+2W_1\sqrt{\log(e+\sqrt{L})}\right)^2\right\}\right),$$

and we have $\forall j \in \overline{V_n^L}$, $a_{n,j} \leq \exp\left\{-L^{3/4}\Delta_{\min}^2/(16\sigma^2)\right\}$, thus there exists $L_4 = \text{Poly}(W_1)$ s.t. $\forall L > L_4$, $\forall j \in \overline{V_n^L}$,

$$a_{n,j} \leq \frac{a_{n,J_n^\star}}{2},$$

and by consequence, $J_n^{(2)} \in V_n^L$.

Finally, taking $L_1 = \max(L_2, L_3, L_4)$, we have $\forall L > L_1$, either $J_n^{(1)} \in V_n^L$ or $J_n^{(2)} \in V_n^L$. □

Next we show that there exists at least one arm in V_n^L for whom the probability of being pulled is large enough. More precisely, we prove the following lemma.

Lemma 15. *There exists $L_1 = Poly(W_1)$ s.t. for $L > L_1$ and for all n s.t. $U_n^L \neq \emptyset$, then there exists $J_n \in V_n^L$ s.t.*

$$\psi_{n,J_n} \geq \frac{\min(\beta, 1-\beta)}{K^2} \triangleq \psi_{\min}.$$

Proof. Using Lemma 14, we know that $J_n^{(1)}$ or $J_n^{(2)} \in V_n^L$. On the other hand, we know that

$$\forall i \in \mathcal{A}, \psi_{n,i} = a_{n,i}\left(\beta + (1-\beta)\sum_{j\neq i}\frac{a_{n,j}}{1-a_{n,j}}\right).$$

Therefore we have

$$\psi_{n,J_n^{(1)}} \geq \beta a_{n,J_n^{(1)}} \geq \frac{\beta}{K},$$

since $\sum_{i\in\mathcal{A}} a_{n,i} = 1$, and

$$\begin{aligned}\psi_{n,J_n^{(2)}} &\geq (1-\beta)a_{n,J_n^{(2)}}\frac{a_{n,J_n^{(1)}}}{1-a_{n,J_n^{(1)}}}\\ &= (1-\beta)a_{n,J_n^{(1)}}\frac{a_{n,J_n^{(2)}}}{1-a_{n,J_n^{(1)}}}\\ &\geq \frac{1-\beta}{K^2},\end{aligned}$$

since $a_{n,J_n^{(1)}} \geq 1/K$ and $\sum_{i\neq J_n^{(1)}} a_{n,i}/(1-a_{n,J_n^{(1)}}) = 1$, thus $a_{n,J_n^{(2)}}/(1-a_{n,J_n^{(1)}}) \geq 1/K$. □

The rest of this subsection is quite similar to that of Qin, Klabjan and Russo, 2017. Indeed, with the above lemma, we can show that the set of poorly explored arms U_n^L is empty when n is large enough.

Lemma 16. *Under TTTS, there exists* $L_0 = Poly(W_1, W_2)$ *s.t.* $\forall L > L_0$, $U^L_{\lfloor KL\rfloor} = \varnothing$.

Proof. We proceed by contradiction, and we assume that $U^L_{\lfloor KL\rfloor}$ is not empty. Then for any $1 \leq \ell \leq \lfloor KL\rfloor$, U_ℓ^L and V_ℓ^L are non empty as well.

There exists a deterministic L_5 s.t. $\forall L > L_5$,

$$\lfloor L\rfloor \geq KL^{3/4}.$$

Using the pigeonhole principle, there exists some $i \in \mathcal{A}$ s.t. $T_{\lfloor L\rfloor,i} \geq L^{3/4}$. Thus, we have $|V^L_{\lfloor L\rfloor}| \leq K-1$.

Next, we prove $|V^L_{\lfloor 2L\rfloor}| \leq K-2$. Otherwise, since U_ℓ^L is non-empty for any $\lfloor L\rfloor + 1 \leq \ell \leq \lfloor 2L\rfloor$, thus by Lemma 15, there exists $J_\ell \in V_\ell^L$ s.t. $\psi_{\ell,J_\ell} \geq \psi_{\min}$. Therefore,

$$\sum_{i\in V_\ell^L} \psi_{\ell,i} \geq \psi_{\min},$$

and

$$\sum_{i\in V^L_{\lfloor L\rfloor}} \psi_{\ell,i} \geq \psi_{\min}$$

since $V_\ell^L \subset V^L_{\lfloor L\rfloor}$. Hence, we have

$$\sum_{i\in V^L_{\lfloor L\rfloor}} (\Psi_{\lfloor 2L\rfloor,i} - \Psi_{\lfloor L\rfloor,i}) = \sum_{\ell=\lfloor L\rfloor+1}^{\lfloor 2L\rfloor} \sum_{i\in V^L_{\lfloor L\rfloor}} \psi_{\ell,i} \geq \psi_{\min}\lfloor L\rfloor.$$

Then, using Lemma 8, there exists $L_6 = \mathrm{Poly}(W_2)$ s.t. $\forall L > L_6$, we have

$$\begin{aligned}\sum_{i \in V^L_{\lfloor L \rfloor}} (T_{\lfloor 2L \rfloor, i} - T_{\lfloor L \rfloor, i}) &\geq \sum_{i \in V^L_{\lfloor L \rfloor}} (\Psi_{\lfloor 2L \rfloor, i} - \Psi_{\lfloor L \rfloor, i} - 2W_2 \sqrt{\lfloor 2L \rfloor \log(e^2 + \lfloor 2L \rfloor)}) \\ &\geq \sum_{i \in V^L_{\lfloor L \rfloor}} (\Psi_{\lfloor 2L \rfloor, i} - \Psi_{\lfloor L \rfloor, i}) - 2KW_2 \sqrt{\lfloor 2L \rfloor \log(e^2 + \lfloor 2L \rfloor)} \\ &\geq \psi_{\min} \lfloor L \rfloor - 2KW_2 C_2 \lfloor L \rfloor^{3/4} \\ &\geq KL^{3/4},\end{aligned}$$

where C_2 is some absolute constant. Thus, we have one arm in $V^L_{\lfloor L \rfloor}$ that is pulled at least $L^{3/4}$ times between $\lfloor L \rfloor + 1$ and $\lfloor 2L \rfloor$, thus $|V^L_{\lfloor 2L \rfloor}| \leq K - 2$.

By induction, for any $1 \leq k \leq K$, we have $|V^L_{\lfloor kL \rfloor}| \leq K - k$, and finally if we take $L_0 = \max(L_1, L_5, L_6)$, then $\forall L > L_0$, $U^L_{\lfloor KL \rfloor} = \varnothing$. □

We can finally conclude the proof of Lemma 9 for TTTS.

Proof of Lemma 9 Let $N_1 = KL_0$ where $L_0 = \mathrm{Poly}(W_1, W_2)$ is chosen according to Lemma 16. For all $n > N_1$, we let $L = n/K$, then by Lemma 16, we have $U^L_{\lfloor KL \rfloor} = U^{n/K}_n$ is empty, which concludes the proof.

■

7.D.3 Concentration of the empirical means, proof of Lemma 10 under TTTS

As a corollary of the previous section, we can show the concentration of $\mu_{n,i}$ to μ_i for TTTS[5].

By Lemma 7, we know that $\forall i \in \mathcal{A}$ and $n \in \mathbb{N}$,

$$|\mu_{n,i} - \mu_i| \leq \sigma W_1 \sqrt{\frac{\log(e + T_{n,i})}{T_{n,i} + 1}}.$$

According to the previous section, there exists $N_1 = \mathrm{Poly}(W_1, W_2)$ s.t. $\forall n \geq N_1$ and $\forall i \in \mathcal{A}$, $T_{n,i} \geq \sqrt{n/K}$. Therefore,

$$|\mu_{n,i} - \mu_i| \leq \sqrt{\frac{\log(e + \sqrt{n/K})}{\sqrt{n/K} + 1}},$$

[5] this proof is the same as Proposition 3 of Qin, Klabjan and Russo, 2017

since $x \mapsto \log(e+x)/(x+1)$ is a decreasing function. There exists $N_2' = \mathrm{Poly}(\varepsilon, W_1)$ s.t. $\forall n \geq N_2'$,

$$\sqrt{\frac{\log(e+\sqrt{n/K})}{\sqrt{n/K}+1}} \leq \sqrt{\frac{2(n/K)^{1/4}}{\sqrt{n/K}+1}} \leq \frac{\varepsilon}{\sigma W_1}.$$

Therefore, $\forall n \geq N_2 \triangleq \max\{N_1, N_2'\}$, we have

$$|\mu_{n,i} - \mu_i| \leq \sigma W_1 \frac{\varepsilon}{\sigma W_1}.$$

7.D.4 Measurement effort concentration of the optimal arm, proof of Lemma 11 under TTTS

In this section we show that the empirical arm draws proportion of the true best arm for TTTS concentrates to β when the total number of arm draws is sufficiently large.

The proof is established upon the following lemmas. First, we prove that the empirical best arm coincides with the true best arm when the total number of arm draws goes sufficiently large.

Lemma 17. *Under TTTS, there exists $M_1 = Poly(W_1, W_2)$ s.t. $\forall n > M_1$, we have $I_n^\star = I^\star = J_n^{(1)}$ and $\forall i \neq I^\star$,*

$$a_{n,i} \leq \exp\left\{-\frac{\Delta_{min}^2}{16\sigma^2}\sqrt{\frac{n}{K}}\right\}.$$

Proof. Using Lemma 10 with $\varepsilon = \Delta_{\min}/4$, there exists $N_1' = \mathrm{Poly}(4/\Delta_{\min}, W_1, W_2)$ s.t. $\forall n > N_1'$,

$$\forall i \in \mathcal{A}, |\mu_{n,i} - \mu_i| \leq \frac{\Delta_{\min}}{4},$$

which implies that starting from a known moment, $\mu_{n,I^\star} > \mu_{n,i}$ for all $i \neq I^\star$, hence $I_n^\star = I^\star$. Thus, $\forall i \neq I^\star$,

$$\begin{aligned} a_{n,i} &= \Pi_n\left[\theta_i > \max_{j\neq i}\theta_j\right] \\ &\leq \Pi_n\left[\theta_i > \theta_{I^\star}\right] \\ &\leq \frac{1}{2}\exp\left\{-\frac{(\mu_{n,i}-\mu_{n,I^\star})^2}{2\sigma^2(1/T_{n,i}+1/T_{n,I^\star})}\right\}. \end{aligned}$$

The last inequality uses the Gaussian tail inequality of (7.8) Lemma 6. Furthermore,

$$\begin{aligned} (\mu_{n,i}-\mu_{n,I^\star})^2 &= (|\mu_{n,i}-\mu_{n,I^\star}|)^2 \\ &= (|\mu_{n,i}-\mu_i+\mu_i-\mu_{I^\star}+\mu_{I^\star}-\mu_{n,I^\star}|)^2 \\ &\geq (|\mu_i-\mu_{I^\star}|-|\mu_{n,i}-\mu_i+\mu_{I^\star}-\mu_{n,I^\star}|)^2 \\ &\geq \left(\Delta_{\min}-\left(\frac{\Delta_{\min}}{4}+\frac{\Delta_{\min}}{4}\right)\right)^2 = \frac{\Delta_{\min}^2}{4}, \end{aligned}$$

and according to Lemma 9, we know that there exists $M_2 = \mathrm{Poly}(W_1, W_2)$ s.t. $\forall n > M_2$,

$$\frac{1}{T_{n,i}} + \frac{1}{T_{n,I^\star}} \leq \frac{2}{\sqrt{n/K}}.$$

Thus, $\forall n > \max\{N_1', M_2\}$, we have

$$\forall i \neq I^\star, a_{n,i} \leq \exp\left\{-\frac{\Delta_{\min}^2}{16\sigma^2}\sqrt{\frac{n}{K}}\right\}.$$

Then, we have

$$a_{n,I^\star} = 1 - \sum_{i \neq I^\star} a_{n,i} \geq 1 - (K-1)\exp\left\{-\frac{\Delta_{\min}^2}{16\sigma^2}\sqrt{\frac{n}{K}}\right\}.$$

There exists M_2' s.t. $\forall n > M_2'$, $a_{n,I^\star} > 1/2$, and by consequence $I^\star = J_n^{(1)}$. Finally taking $M_1 \triangleq \max\{N_1', M_2, M_2'\}$ concludes the proof. □

Before we prove Lemma 11, we first show that $\Psi_{n,I^\star}/n$ concentrates to β.

Lemma 18. *Under TTTS, fix a constant $\varepsilon > 0$, there exists $M_3 = \mathrm{Poly}(\varepsilon, W_1, W_2)$ s.t. $\forall n > M_3$, we have*

$$\left|\frac{\Psi_{n,I^\star}}{n} - \beta\right| \leq \varepsilon.$$

Proof. By Lemma 17, we know that there exists $M_1' = \mathrm{Poly}(W_1, W_2)$ s.t. $\forall n > M_1'$, we have $I_n^\star = I^\star = J_n^{(1)}$ and $\forall i \neq I^\star$,

$$a_{n,i} \leq \exp\left\{-\frac{\Delta_{\min}^2}{16\sigma^2}\sqrt{\frac{n}{K}}\right\}.$$

Note also that $\forall n \in \mathbb{N}$, we have

$$\psi_{n,I^\star} = a_{n,I^\star}\left(\beta + (1-\beta)\sum_{j \neq I^\star}\frac{a_{n,j}}{1-a_{n,j}}\right).$$

We proceed the proof with the following two steps.

Step 1 We first lower bound $\Psi_{n,I^\star}$ for a given ε. Take $M_4 > M_1'$ that we decide later, we have $\forall n > M_4$,

$$\begin{aligned}
\frac{\Psi_{n,I^\star}}{n} &= \frac{1}{n}\sum_{l=1}^{n}\psi_{l,I^\star} = \frac{1}{n}\sum_{l=I^\star}^{M_4}\psi_{l,I^\star} + \frac{1}{n}\sum_{l=M_4+1}^{n}\psi_{l,I^\star} \\
&\geq \frac{1}{n}\sum_{l=M_4+1}^{n}\psi_{l,I^\star} \geq \frac{1}{n}\sum_{l=M_4+1}^{n} a_{l,I^\star}\beta \\
&= \frac{\beta}{n}\sum_{l=M_4+1}^{n}\left(1 - \sum_{j\neq I^\star} a_{l,j}\right) \\
&\geq \frac{\beta}{n}\sum_{l=M_4+1}^{n}\left(1 - (K-1)\exp\left\{-\frac{\Delta_{\min}^2}{16\sigma^2}\sqrt{\frac{l}{K}}\right\}\right) \\
&= \beta - \frac{M_4}{n}\beta - \frac{\beta}{n}\sum_{l=M_4+1}^{n}(K-1)\exp\left\{-\frac{\Delta_{\min}^2}{16\sigma^2}\sqrt{\frac{l}{K}}\right\} \\
&\geq \beta - \frac{M_4}{n}\beta - \frac{(n-M_4)}{n}\beta(K-1)\exp\left\{-\frac{\Delta_{\min}^2}{16\sigma^2}\sqrt{\frac{M_4}{K}}\right\} \\
&\geq \beta - \frac{M_4}{n}\beta - \beta(K-1)\exp\left\{-\frac{\Delta_{\min}^2}{16\sigma^2}\sqrt{\frac{M_4}{K}}\right\}.
\end{aligned}$$

For a given constant $\varepsilon > 0$, there exists M_5 s.t. $\forall n > M_5$,

$$\beta(K-1)\exp\left\{-\frac{\Delta_{\min}^2}{16\sigma^2}\sqrt{\frac{n}{K}}\right\} < \frac{\varepsilon}{2}.$$

Furthermore, there exists $M_6 = \mathrm{Poly}(\varepsilon/2, M_5)$ s.t. $\forall n > M_6$,

$$\frac{M_5}{n}\beta < \frac{\varepsilon}{2}.$$

Therefore, if we take $M_4 \triangleq \max\{M_1', M_5, M_6\}$, we have $\forall n > M_4$,

$$\frac{\Psi_{n,I^\star}}{n} \geq \beta - \varepsilon.$$

Step 2 On the other hand, we can also upper bound $\Psi_{n,I^\star}$. We have $\forall n > M_3$,

$$\begin{aligned}
\frac{\Psi_{n,I^\star}}{n} &= \frac{1}{n}\sum_{l=1}^{n}\psi_{l,I^\star} \\
&= \frac{1}{n}\sum_{l=1}^{n} a_{l,I^\star}\left(\beta + (1-\beta)\sum_{j\neq I^\star}\frac{a_{l,j}}{1-a_{l,j}}\right) \\
&\leq \frac{1}{n}\sum_{l=1}^{n} a_{l,I^\star}\beta + \frac{1}{n}\sum_{l=1}^{n} a_{l,I^\star}(1-\beta)\sum_{j\neq I^\star}\frac{a_{l,j}}{1-a_{l,j}} \\
&\leq \beta + \frac{1}{n}\sum_{l=1}^{n}(1-\beta)\sum_{j\neq I^\star}\frac{a_{l,j}}{1-a_{l,j}} \\
&\leq \beta + \frac{1}{n}\sum_{l=1}^{n}(1-\beta)\sum_{j\neq I^\star}\frac{\exp\left\{-\frac{\Delta_{\min}^2}{16\sigma^2}\sqrt{\frac{l}{K}}\right\}}{1-\exp\left\{-\frac{\Delta_{\min}^2}{16\sigma^2}\sqrt{\frac{l}{K}}\right\}}.
\end{aligned}$$

Since, for a given $\varepsilon > 0$, there exists M_8 s.t. $\forall n > M_8$,

$$\exp\left\{-\frac{\Delta_{\min}^2}{16\sigma^2}\sqrt{\frac{n}{K}}\right\} < \frac{1}{2},$$

and there exists M_9 s.t. $\forall n > M_9$,

$$(1-\beta)(K-1)\exp\left\{-\frac{\Delta_{\min}^2}{16\sigma^2}\sqrt{\frac{n}{K}}\right\} < \frac{\varepsilon}{4}.$$

Thus, $\forall n > M_{10} \triangleq \max\{M_8, M_9\}$,

$$\begin{aligned}
\frac{\Psi_{n,I^\star}}{n} &\leq \beta + \frac{1-\beta}{n}\left(\sum_{l=1}^{M_{10}}\sum_{j\neq I^\star}\frac{\exp\left\{-\frac{\Delta_{\min}^2}{16\sigma^2}\sqrt{\frac{l}{K}}\right\}}{1-\exp\left\{-\frac{\Delta_{\min}^2}{16\sigma^2}\sqrt{\frac{l}{K}}\right\}} + \sum_{l=M_{10}+1}^{n}\sum_{j\neq I^\star}\frac{\exp\left\{-\frac{\Delta_{\min}^2}{16\sigma^2}\sqrt{\frac{l}{K}}\right\}}{1-\exp\left\{-\frac{\Delta_{\min}^2}{16\sigma^2}\sqrt{\frac{l}{K}}\right\}}\right) \\
&\leq \beta + \frac{1-\beta}{n}\sum_{l=1}^{M_{10}}\sum_{j\neq I^\star}\frac{\exp\left\{-\frac{\Delta_{\min}^2}{16\sigma^2}\sqrt{\frac{l}{K}}\right\}}{1-\exp\left\{-\frac{\Delta_{\min}^2}{16\sigma^2}\sqrt{\frac{l}{K}}\right\}} + 2(1-\beta)(K-1)\exp\left\{-\frac{\Delta_{\min}^2}{16\sigma^2}\sqrt{\frac{M_{10}}{K}}\right\} \\
&\leq \beta + \frac{1-\beta}{n}\sum_{l=1}^{M_{10}}\sum_{j\neq I^\star}\frac{\exp\left\{-\frac{\Delta_{\min}^2}{16\sigma^2}\sqrt{\frac{l}{K}}\right\}}{1-\exp\left\{-\frac{\Delta_{\min}^2}{16\sigma^2}\sqrt{\frac{l}{K}}\right\}} + \frac{\varepsilon}{2}.
\end{aligned}$$

There exists $M_{11} = \mathrm{Poly}(\varepsilon/2, M_{10})$ s.t. $\forall n > M_{11}$,

$$\frac{1-\beta}{n}\sum_{l=1}^{M_{10}}\sum_{j\neq I^\star}\frac{\exp\left\{-\frac{\Delta_{\min}^2}{16\sigma^2}\sqrt{\frac{l}{K}}\right\}}{1-\exp\left\{-\frac{\Delta_{\min}^2}{16\sigma^2}\sqrt{\frac{l}{K}}\right\}} < \frac{\varepsilon}{2}.$$

Therefore, $\forall n > M_7 \triangleq \max\{M_3, M_{11}\}$, we have

$$\frac{\Psi_{n,I^\star}}{n} \leq \beta + \varepsilon.$$

Conclusion Finally, combining the two steps and define $M_3 \triangleq \max\{M_4, M_7\}$, we have $\forall n > M_3$,

$$\left|\frac{\Psi_{n,I^\star}}{n} - \beta\right| \leq \varepsilon.$$

□

With the help of the previous lemma and Lemma 8, we can finally prove Lemma 11.

Proof of Lemma 11 Fix an $\varepsilon > 0$. Using Lemma 8, we have $\forall n \in \mathbb{N}$,

$$\left|\frac{T_{n,I^\star}}{n} - \frac{\Psi_{n,I^\star}}{n}\right| \leq \frac{W_2\sqrt{(n+1)\log(e^2+n)}}{n}.$$

Thus there exists M_{12} s.t. $\forall n > M_{12}$,

$$\left|\frac{T_{n,I^\star}}{n} - \frac{\Psi_{n,I^\star}}{n}\right| \leq \frac{\varepsilon}{2}.$$

And using Lemma 18, there exists $M_3' = \mathrm{Poly}(\varepsilon/2, W_1, W_2)$ s.t. $\forall n > M_3'$,

$$\left|\frac{\Psi_{n,I^\star}}{n} - \beta\right| \leq \frac{\varepsilon}{2}.$$

Again, according to Lemma 15, there exists M_3' s.t. $\forall n > M_3'$,

$$\frac{\Psi_{n,I^\star}}{n} \leq \beta + \frac{\varepsilon}{2}.$$

Thus, if we take $N_3 \triangleq \max\{M_3', M_{12}\}$, then $\forall n > N_3$, we have

$$\left|\frac{T_{n,I^\star}}{n} - \beta\right| \leq \varepsilon.$$

■

7.D.5 Measurement effort concentration of other arms, proof of Lemma 12 under TTTS

In this section, we show that, for TTTS, the empirical measurement effort concentration also holds for other arms than the true best arm. We first show that if some arm is overly sampled at time n, then its probability of being picked is reduced exponentially.

Lemma 19. *Under TTTS, for every $\xi \in (0,1)$, there exists $S_1 = Poly(1/\xi, W_1, W_2)$ such that for all $n > S_1$, for all $i \neq I^\star$,*

$$\frac{\Psi_{n,i}}{n} \geq \omega_i^\beta + \xi \;\;\Rightarrow\;\; \psi_{n,i} \leq \exp\left\{-\varepsilon_0(\xi) n\right\},$$

where ε_0 is defined in (7.12) below.

Proof. First, by Lemma 17, there exists $M_1'' = \mathrm{Poly}(W_1, W_2)$ s.t. $\forall n > M_1''$,

$$I^\star = I_n^\star = J_n^{(1)}.$$

Then, following the similar argument as in Lemma 31, one can show that for all $i \neq I^\star$ and for all $n > M_1''$,

$$\begin{aligned}
\psi_{n,i} &= a_{n,i}\left(\beta + (1-\beta)\sum_{j\neq i}\frac{a_{n,j}}{1-a_{n,j}}\right) \\
&\leq a_{n,i}\beta + a_{n,i}(1-\beta)\frac{\sum_{j\neq i} a_{n,j}}{1-a_{n,J_n^{(1)}}} \\
&= a_{n,i}\beta + a_{n,i}(1-\beta)\frac{\sum_{j\neq i} a_{n,j}}{1-a_{n,I^\star}} \\
&\leq a_{n,i}\beta + a_{n,i}(1-\beta)\frac{1}{1-a_{n,I^\star}} \\
&\leq \frac{a_{n,i}}{1-a_{n,I^\star}} \\
&\leq \frac{\Pi_n\left[\theta_i \geq \theta_{I^\star}\right]}{\Pi_n\left[\cup_{j\neq I^\star}\theta_j \geq \theta_{I^\star}\right]} \\
&\leq \frac{\Pi_n\left[\theta_i \geq \theta_{I^\star}\right]}{\max_{j\neq I^\star}\Pi_n\left[\theta_j \geq \theta_{I^\star}\right]}.
\end{aligned}$$

Using the upper and lower Gaussian tail bounds from Lemma 6, we have

$$\begin{aligned}
\psi_{n,i} &\leq \frac{\exp\left\{-\dfrac{(\mu_{n,I^\star}-\mu_{n,i})^2}{2\sigma^2\left(1/T_{n,I^\star}+1/T_{n,i}\right)}\right\}}{\exp\left\{-\min\limits_{j\neq I^\star}\dfrac{1}{2}\left(\dfrac{(\mu_{n,I^\star}-\mu_{n,j})}{\sigma\sqrt{\left(1/T_{n,I^\star}+1/T_{n,j}\right)}}-1\right)^2\right\}} \\
&= \frac{\exp\left\{-n\dfrac{(\mu_{n,I^\star}-\mu_{n,i})^2}{2\sigma^2\left(n/T_{n,I^\star}+n/T_{n,i}\right)}\right\}}{\exp\left\{-n\left(\min\limits_{j\neq I^\star}\dfrac{(\mu_{n,I^\star}-\mu_{n,j})}{\sqrt{2\sigma^2\left(n/T_{n,I^\star}+n/T_{n,j}\right)}}-\dfrac{1}{\sqrt{2n}}\right)^2\right\}},
\end{aligned}$$

where we assume that $n > S_2 = \mathrm{Poly}(W_1, W_2)$ for which

$$\frac{(\mu_{n,I^\star} - \mu_{n,i})^2}{\sigma^2 \left(1/T_{n,I^\star} + 1/T_{n,i}\right)} \geq 1$$

according to Lemma 9. From there we take a supremum over the possible allocations to lower bound the denominator and write

$$\begin{aligned}
\psi_{n,i} &\leq \frac{\exp\left\{-n \frac{(\mu_{n,I^\star} - \mu_{n,i})^2}{2\sigma^2 \left(n/T_{n,I^\star} + n/T_{n,i}\right)}\right\}}{\exp\left\{-n \left(\sup_{\boldsymbol{\omega}: \omega_{I^\star} = T_{n,I^\star}/n} \min_{j \neq I^\star} \frac{(\mu_{n,I^\star} - \mu_{n,i})}{\sqrt{2\sigma^2 \left(1/\omega_{I^\star} + 1/\omega_j\right)}} - \frac{1}{\sqrt{2n}}\right)^2\right\}} \\
&= \frac{\exp\left\{-n \frac{(\mu_{n,I^\star} - \mu_{n,i})^2}{2\sigma^2 \left(n/T_{n,I^\star} + n/T_{n,i}\right)}\right\}}{\exp\left\{-n \left(\sqrt{\Gamma^\star_{T_{n,I^\star}/n}(\boldsymbol{\mu}_n)} - \frac{1}{\sqrt{2n}}\right)^2\right\}},
\end{aligned}$$

where $\boldsymbol{\mu}_n \triangleq (\mu_{n,1}, \cdots, \mu_{n,K})$, and $(\beta, \boldsymbol{\mu}) \mapsto \Gamma^\star_\beta(\boldsymbol{\mu})$ represents a function that maps β and $\boldsymbol{\mu}$ to the parameterized optimal error decay that any allocation rule can reach given parameter β and a set of arms with means $\boldsymbol{\mu}$. Note that this function is continuous with respect to β and $\boldsymbol{\mu}$ respectively.

Now, assuming $\Psi_{n,i}/n \geq \omega_i^\beta + \xi$ yields that there exists $S_2' \triangleq \mathrm{Poly}(2/\xi, W_2)$ s.t. for all $n > S_2'$, $T_{n,i}/n \geq \omega_i^\beta + \xi/2$, and by consequence,

$$\psi_{n,i} \leq \exp\left\{ \underbrace{-n \left(\frac{(\mu_{n,I^\star} - \mu_{n,i})^2}{2\sigma^2 \left(n/T_{n,I^\star} + 1/(\omega_i^\beta + \xi/2)\right)} - \Gamma^\star_{T_{n,I^\star}/n}(\boldsymbol{\mu}_n) - \frac{1}{2n} + \sqrt{\frac{2\Gamma^\star_{T_{n,I^\star}/n}(\boldsymbol{\mu}_n)}{n}} \right)}_{\varepsilon_n(\xi)} \right\}.$$

Using Lemma 11, we know that for any ε, there exists $S_3 = \mathrm{Poly}(1/\varepsilon, W_1, W_2)$ s.t. $\forall n > S_3$, $|T_{n,I^\star}/n - \beta| \leq \varepsilon$, and $\forall j \in \mathcal{A}, |\mu_{n,j} - \mu_j| \leq \varepsilon$. Furthermore, $(\beta, \boldsymbol{\mu}) \mapsto \Gamma^\star_\beta(\boldsymbol{\mu})$ is continuous with respect to β and $\boldsymbol{\mu}$, thus for a given ε_0, there exists $S_3' = \mathrm{Poly}(1/\varepsilon_0, W_1, W_2)$ s.t. $\forall n > S_3'$, we have

$$\left| \varepsilon_n(\xi) - \left(\frac{(\mu_{I^\star} - \mu_i)^2}{2\sigma^2 \left(1/\beta + 1/(\omega_i^\beta + \xi/2)\right)} - \Gamma^\star_\beta \right) \right| \leq \varepsilon_0.$$

Finally, define $S_1 \triangleq \max\{S_2, S_2', S_3'\}$, we have $\forall n > S_1$,

$$\psi_{n,i} \leq \exp\left\{-\varepsilon_0(\xi) n\right\},$$

where

$$\varepsilon_0(\xi) = \frac{(\mu_{I^\star} - \mu_i)^2}{2\sigma^2\left(1/\beta + 1/(\omega_i^\beta + \xi/2)\right)} - \Gamma_\beta^\star + \varepsilon_0\,. \tag{7.12}$$

□

Next, starting from some known moment, no arm is overly allocated. More precisely, we show the following lemma.

Lemma 20. *Under TTTS, for every ξ, there exists $S_4 = Poly(1/\xi, W_1, W_2)$ s.t. $\forall n > S_4$,*

$$\forall i \in \mathcal{A},\quad \frac{\Psi_{n,i}}{n} \le \omega_i^\beta + \xi.$$

Proof. From Lemma 19, there exists $S_1' = \mathrm{Poly}(2/\xi, W_1, W_2)$ such that for all $n > S_1'$ and for all $i \neq I^\star$,

$$\frac{\Psi_{n,i}}{n} \ge \omega_i^\beta + \frac{\xi}{2} \quad\Rightarrow\quad \psi_{n,i} \le \exp\left\{-\varepsilon_0(\xi/2)n\right\}.$$

Thus, for all $i \neq I^\star$,

$$\begin{aligned}\frac{\Psi_{n,i}}{n} &\le \frac{S_1'}{n} + \frac{\sum_{\ell=S_1'+1}^{n} \psi_{\ell,i}\mathbb{1}\left(\frac{\Psi_{\ell,i}}{n} \ge \omega_i^\beta + \frac{\xi}{2}\right)}{n} + \frac{\sum_{\ell=S_1'+1}^{n} \psi_{\ell,i}\mathbb{1}\left(\frac{\Psi_{\ell,i}}{n} \le \omega_i^\beta + \frac{\xi}{2}\right)}{n} \\ &\le \frac{S_1'}{n} + \frac{\sum_{\ell=1}^{n} \exp\left\{-\varepsilon_0(\xi/2)n\right\}}{n} + \frac{\sum_{\ell=S_1'+1}^{\ell_n(\xi)} \psi_{\ell,i}\mathbb{1}\left(\frac{\Psi_{\ell,i}}{n} \le \omega_i^\beta + \frac{\xi}{2}\right)}{n},\end{aligned}$$

where we let $\ell_n(\xi) = \max\left\{\ell \le n : \Psi_{\ell,i}/n \le \omega_i^\beta + \xi/2\right\}$. Then

$$\begin{aligned}\frac{\Psi_{n,i}}{n} &\le \frac{S_1'}{n} + \frac{\sum_{\ell=1}^{n} \exp\left\{-\varepsilon_0(\xi/2)n\right\}}{n} + \Psi_{\ell_n(\xi),i} \\ &\le \frac{S_1' + (1 - \exp(-\varepsilon_0(\xi/2)))^{-1}}{n} + \omega_i^\beta + \frac{\xi}{2}\end{aligned}$$

Then, there exists S_5 such that for all $n \ge S_5$,

$$\frac{S_1' + (1 - \exp(-\varepsilon_0(\xi/2)))^{-1}}{n} \le \frac{\xi}{2}.$$

Therefore, for any $n > S_4 \triangleq \max\{S_1', S_5\}$, $\Psi_{n,i} \le \omega_i^\beta + \xi$ holds for all $i \neq I^\star$. For $i = I^\star$, it is already proved for the optimal arm. □

We now prove Lemma 12 under TTTS.

Proof of Lemma 12 From Lemma 20, there exists $S_4' = \mathrm{Poly}((K-1)/\xi, W_1, W_2)$ such that for all $n > S_4'$,

$$\forall i \in \mathcal{A}, \frac{\Psi_{n,i}}{n} \leq \omega_i^\beta + \frac{\xi}{K-1}.$$

Using the fact that $\Psi_{n,i}/n$ and ω_i^β all sum to 1, we have $\forall i \in \mathcal{A}$,

$$\begin{aligned}\frac{\Psi_{n,i}}{n} &= 1 - \sum_{j \neq i} \frac{\Psi_{n,j}}{n} \\ &\geq 1 - \sum_{j \neq i} \left(\omega_j^\beta + \frac{\xi}{K-1} \right) \\ &= \omega_i^\beta - \xi.\end{aligned}$$

Thus, for all $n > S_4'$, we have

$$\forall i \in \mathcal{A}, \left| \frac{\Psi_{n,i}}{n} - \omega_i^\beta \right| \leq \xi.$$

And finally we use the same reasoning as the proof of Lemma 11 to link $T_{n,i}$ and $\Psi_{n,i}$. Fix an $\varepsilon > 0$. Using Lemma 8, we have $\forall n \in \mathbb{N}$,

$$\forall i \in \mathcal{A}, \left| \frac{T_{n,i}}{n} - \frac{\Psi_{n,i}}{n} \right| \leq \frac{W_2 \sqrt{(n+1)\log(e^2+n)}}{n}.$$

Thus there exists S_5 s.t. $\forall n > S_5$,

$$\left| \frac{T_{n,I^\star}}{n} - \frac{\Psi_{n,I^\star}}{n} \right| \leq \frac{\varepsilon}{2}.$$

And using the above result, there exists $S_4'' = \mathrm{Poly}(2/\varepsilon, W_1, W_2)$ s.t. $\forall n > S_4''$,

$$\left| \frac{\Psi_{n,i}}{n} - \omega_i^\beta \right| \leq \frac{\varepsilon}{2}.$$

Thus, if we take $N_4 \triangleq \max\{S_4'', S_5\}$, then $\forall n > N_4$, we have

$$\forall i \in \mathcal{A}, \left| \frac{T_{n,i}}{n} - \omega_i^\beta \right| \leq \varepsilon.$$

■

7.E Fixed-Confidence Analysis for T3C

This section is entirely dedicated to T3C. Note that the analysis to follow share the same proof line with that of TTTS, and some parts even completely coincide with those of TTTS. For the sake of clarity and simplicity, we shall only focus on the parts that differ and skip some redundant proofs.

7.E.1 Sufficient exploration of all arms, proof of Lemma 9 under T3C

To prove this lemma, we still need the two sets of indices for under-sampled arms like in Appendix 7.D.2. We recall that for a given $L > 0$: $\forall n \in \mathbb{N}$ we define

$$U_n^L \triangleq \{i : T_{n,i} < \sqrt{L}\},$$

$$V_n^L \triangleq \{i : T_{n,i} < L^{3/4}\}.$$

For T3C however, we investigate the following two indices,

$$J_n^{(1)} \triangleq \arg\max_j a_{n,j}; \qquad \widetilde{J_n^{(2)}} \triangleq \arg\min_{j \neq J_n^{(1)}} W_n(J_n^{(1)}, j).$$

Lemma 9 is proved via the following sequence of lemmas.

Lemma 21. *There exists $L_1 = Poly(W_1)$ s.t. if $L > L_1$, for all n, $U_n^L \neq \varnothing$ implies $J_n^{(1)} \in V_n^L$ or $\widetilde{J_n^{(2)}} \in V_n^L$.*

Proof. If $J_n^{(1)} \in V_n^L$, then the proof is finished. Now we assume that $J_n^{(1)} \in \overline{V_n^L} \subset \overline{U_n^L}$, and we prove that $J_n^{(2)} \in V_n^L$.

Step 1 Following the same reasoning as Step 1 and Step 2 of the proof of Lemma 14, we know that there exists $L_2 = \mathrm{Poly}(W_1)$ s.t. if $L > L_2$, then

$$\overline{J_n^\star} \triangleq \arg\max_{j \in \overline{U_n^L}} \mu_{n,j} = \arg\max_{j \in \overline{U_n^L}} \mu_j = J_n^{(1)}.$$

Step 2 Now assuming that $L > L_2$, and we show that for L large enough, $\widetilde{J_n^{(2)}} \in V_n^L$. In the same way that we proved (7.11) one can show that for all $\forall j \in \overline{V_n^L}$,

$$W_n(J_n^{(1)}, j) = \frac{(\mu_{n,I^\star} - \mu_{n,j})^2}{2\sigma^2\left(\frac{1}{T_{n,I^\star}} + \frac{1}{T_{n,j}}\right)} \geq \frac{L^{3/4}\Delta_{\min}^2}{16\sigma^2}.$$

Again, denote $J_n^\star \triangleq \arg\max_{j \in U_n^L} \mu_{n,j}$, we obtain

$$W_n(J_n^{(1)}, J_n^\star) = \begin{cases} 0 & \text{if } \mu_{n,J_n^\star} \geq \mu_{n,J_n^{(1)}}, \\ \dfrac{(\mu_{n,J_n^{(1)}} - \mu_{n,J_n^\star})^2}{2\sigma^2\left(\frac{1}{T_{n,J_n^{(1)}}} + \frac{1}{T_{n,J_n^\star}}\right)} & \text{else.} \end{cases}$$

In the second case, as already shown in Step 3 of Lemma 14 we have that

$$|\mu_{n,J_n^\star} - \mu_{n,\overline{J_n^\star}}| \leq \Delta_{\max} + 2\sigma W_1 \sqrt{\frac{\log(e + T_{n,J_n^\star})}{1 + T_{n,J_n^\star}}}$$
$$\leq \Delta_{\max} + 2\sigma W_1 \sqrt{\frac{\log(e + \sqrt{L})}{1 + \sqrt{L}}},$$

since $J_n^\star \in U_n^L$. We also know that

$$2\sigma^2 \left(\frac{1}{T_{n,J_n^{(1)}}} + \frac{1}{T_{n,J_n^\star}} \right) \geq \frac{2\sigma^2}{T_{n,J_n^\star}} \geq \frac{2\sigma^2}{\sqrt{L}}.$$

Therefore, we get

$$W_n(J_n^{(1)}, J_n^\star) \leq \frac{\sqrt{L}}{2\sigma^2} \left(\Delta_{\max} + 2\sigma W_1 \sqrt{\frac{\log(e + \sqrt{L})}{1 + \sqrt{L}}} \right)^2.$$

On the other hand, we know that for all $j \in \overline{V_n^L}$,

$$W_n(J_n^{(1)}, j) \geq \frac{L^{3/4} \Delta_{\min}^2}{16\sigma^2}.$$

Thus, there exists L_3 s.t. if $L > L_3$, then

$$\forall j \in \overline{V_n^L},\ W_n(J_n^{(1)}, j) \geq 2 W_n(J_n^{(1)}, J_n^\star).$$

That means $\widetilde{J_n^{(2)}} \notin \overline{V_n^L}$ and by consequence, $\widetilde{J_n^{(2)}} \in V_n^L$.

Finally, taking $L_1 = \max(L_2, L_3)$, we have $\forall L > L_1$, either $J_n^{(1)} \in V_n^L$ or $\widetilde{J_n^{(2)}} \in V_n^L$. □

Next we show that there exists at least one arm in V_n^L for whom the probability of being pulled is large enough. More precisely, we prove the following lemma.

Lemma 22. *There exists $L_1 = Poly(W_1)$ s.t. for $L > L_1$ and for all n s.t. $U_n^L \neq \emptyset$, then there exists $J_n \in V_n^L$ s.t.*

$$\psi_{n,J_n} \geq \frac{\min(\beta, 1-\beta)}{K^2} \triangleq \psi_{\min}.$$

Proof. Using Lemma 21, we know that $J_n^{(1)}$ or $\widetilde{J_n^{(2)}} \in V_n^L$. We also know that under T3C, for any arm i, $\psi_{n,i}$ can be written as

$$\psi_{n,i} = \beta a_{n,i} + (1-\beta) \sum_{j \neq i} a_{n,j} \frac{\mathbb{1}\{W_n(j,i) = \min_{k \neq j} W_n(j,k)\}}{\left| \arg\min_{k \neq j} W_n(j,k) \right|}.$$

Note that $(\psi_{n,i})_i$ sums to 1,

$$\begin{aligned}\sum_i \psi_{n,i} &= \beta + (1-\beta)\sum_j a_{n,j} \sum_{i\neq j} \frac{\mathbb{1}\{W_n(j,i) = \min_{k\neq j} W_n(j,k)\}}{\left|\arg\min_{k\neq j} W_n(j,k)\right|} \\ &= \beta + (1-\beta)\sum_j a_{n,j} = 1\,.\end{aligned}$$

Therefore, we have

$$\psi_{n,J_n^{(1)}} \geq \beta a_{n,J_n^{(1)}} \geq \frac{\beta}{K}$$

on one hand, since $\sum_{i\in\mathcal{A}} a_{n,i} = 1$. On the other hand, we have

$$\begin{aligned}\psi_{n,\widehat{J_n^{(2)}}} &\geq (1-\beta)\frac{a_{n,J_n^{(1)}}}{K} \\ &\geq \frac{1-\beta}{K^2},\end{aligned}$$

which concludes the proof. □

The rest of this subsection is exactly the same to that of TTTS. Indeed, with the above lemma, we can show that the set of poorly explored arms U_n^L is empty when n is large enough.

Lemma 23. *Under T3C, there exists* $L_0 = Poly(W_1, W_2)$ *s.t.* $\forall L > L_0$, $U^L_{\lfloor KL \rfloor} = \varnothing$.

Proof. See proof of Lemma 16 in Appendix 7.D.2. □

We can finally conclude the proof of Lemma 9 for T3C in the same way as for TTTS in Appendix 7.D.2. ■

7.E.2 Concentration of the empirical means, proof of Lemma 10 under T3C

As a corollary of the previous section, we can show the concentration of $\mu_{n,i}$ to μ_i, and the proof remains the same as that of TTTS in Appendix 7.D.3.

7.E.3 Measurement effort concentration of the optimal arm, proof of Lemma 11 under T3C

Next, we show that the empirical arm draws proportion of the true best arm for T3C concentrates to β when the total number of arm draws is sufficiently large. This proof also remains the same as that of TTTS in Appendix 7.D.4.

7.E.4 Measurement effort concentration of other arms, proof of Lemma 12 under T3C

In this section, we show that, for T3C, the empirical measurement effort concentration also holds for other arms than the true best arm. Note that this part differs from that of TTTS.

We again establish first an over-allocation implies negligible probability result as follow.

Lemma 24. *Under T3C, for every $\xi \le \varepsilon_0$ with ε_0 problem dependent, there exists $S_1 = Poly(1/\xi, W_1, W_2)$ such that for all $n > S_1$, for all $i \neq I^\star$,*

$$\frac{\Psi_{n,i}}{n} \ge \omega_i^\beta + 2\xi \;\Rightarrow\; \psi_{n,i} \le (K-1)\exp\left\{-\frac{\Delta_{min}^2}{16\sigma^2}\sqrt{\frac{n}{K}}\right\}.$$

Proof. Fix $i \neq I^\star$ s.t. $\Psi_{n,i}/n \ge \omega_i^\beta + 2\xi$, then using Lemma 8, there exists $S_2 = \mathrm{Poly}(1/\xi, W_2)$ such that for any $n > S_2$, we have

$$\frac{T_{n,i}}{n} \ge \omega_i^\beta + \xi.$$

Then,

$$\begin{aligned}
\psi_{n,i} &\le \beta a_{n,i} + (1-\beta)\sum_{j\neq i} a_{n,j}\mathbb{1}\{W_n(j,i) = \min_{k\neq j} W_n(j,k)\} \\
&\le \beta a_{n,i} + (1-\beta)\left(\sum_{j\neq i, I^\star} a_{n,j} + a_{n,I^\star}\mathbb{1}\{W_n(I^\star,i) = \min_{k\neq I^\star} W_n(I^\star,k)\}\right) \\
&\le \sum_{j\neq I^\star} a_{n,j} + \mathbb{1}\{W_n(I^\star,i) = \min_{k\neq I^\star} W_n(I^\star,k)\}.
\end{aligned}$$

Next we show that the indicator function term in the previous inequality equals 0.

Using Lemma 7 and Lemma 11 for T3C, there exists $S_3 = \mathrm{Poly}(1/\xi, W_1, W_2)$ such that for any $n > S_3$,

$$\left|\frac{T_{n,I^\star}}{n} - \beta\right| \le \xi^2 \text{ and } \forall j \in \mathcal{A}, |\mu_{n,j} - \mu_j| \le \xi^2.$$

Now if $\forall j \neq I^\star, i$, we have $T_{n,j}/n > \omega_j^\beta$, then

$$\begin{aligned}
\frac{n-1}{n} &= \sum_{j\in\mathcal{A}} \frac{T_{n,j}}{n} \\
&= \frac{T_{n,I^\star}}{n} + \frac{T_{n,i}}{n} + \sum_{j\neq I^\star, i} \frac{T_{n,j}}{n} \\
&> \beta - \varepsilon^2 + \omega_i^\beta + \varepsilon + \sum_{j\neq I^\star, i} \omega_j^\beta \ge 1,
\end{aligned}$$

which is a contradiction.

Thus there exists at least one $j_0 \neq I^\star, i$, such that $T_{n,j_0}/n \leq \omega_j^\beta$. Assuming $n > \max(S_2, S_3)$, we have

$$W_n(I^\star, i) - W_n(I^\star, j_0) = \frac{(\mu_{n,I^\star} - \mu_{n,i})^2}{2\sigma^2\left(\frac{1}{T_{n,I^\star}} + \frac{1}{T_{n,i}}\right)} - \frac{(\mu_{n,I^\star} - \mu_{n,j_0})^2}{2\sigma^2\left(\frac{1}{T_{n,I^\star}} + \frac{1}{T_{n,j_0}}\right)}$$
$$\geq \underbrace{\frac{(\mu_{I^\star} - \mu_i - 2\xi^2)^2}{2\sigma^2\left(\frac{1}{\beta - \xi^2} + \frac{1}{\omega_i^\beta + \xi}\right)} - \frac{(\mu_{I^\star} - \mu_{j_0} + 2\xi^2)^2}{2\sigma^2\left(\frac{1}{\beta + \xi^2} + \frac{1}{\omega_{j_0}^\beta}\right)}}_{W_{i,j_0}^\xi}.$$

According to Proposition 1, W_{i,j_0}^ξ converges to 0 when ξ goes to 0, more precisely we have

$$W_{i,j_0}^\xi = \frac{(\mu_{I^\star} - \mu_i)^2}{2\sigma^2}\left(\frac{\beta}{\beta + \omega_i^\beta}\right)^2 \xi + O(\xi^2),$$

thus there exists a ε_0 such that for all $\xi < \varepsilon_0$ it holds for all $i, j_0 \neq I^\star$, $W_{i,j_0}^\xi > 0$. It follows then

$$W_n(I^\star, i) - \min_{k \neq I^\star} W_n(I^\star, k) \geq W_n(I^\star, i) - W_n(I^\star, j_0) > 0,$$

and $\mathbb{1}\{W_n(I^\star, i) = \min_{k \neq I^\star} W_n(I^\star, k)\} = 0$.

Knowing that Lemma 17 is also valid for T3C, thus there exists $M_1 = \mathrm{Poly}(4/\Delta_{\min}, W_1, W_2)$ such that for all $n > M_1$,

$$\forall j \neq I^\star, a_{n,j} \leq \exp\left\{-\frac{\Delta_{\min}^2}{16\sigma^2}\sqrt{\frac{n}{K}}\right\},$$

which then concludes the proof by taking $S_1 \triangleq \max(M_1, S_2, S_3)$. □

The rest of this subsection almost coincides with that of TTTS. We first show that, starting from some known moment, no arm is overly allocated. More precisely, we show the following lemma.

Lemma 25. *Under T3C, for every ξ, there exists $S_4 = \mathrm{Poly}(1/\xi, W_1, W_2)$ s.t. $\forall n > S_4$,*

$$\forall i \in \mathcal{A}, \quad \frac{\Psi_{n,i}}{n} \leq \omega_i^\beta + 2\xi.$$

Proof. See proof of Lemma 20 in Appendix 7.D.5. Note that the previous step does not match exactly that of TTTS, so the proof would be slightly different. However, the difference is only a matter of constant, we thus still choose to skip this proof. □

It remains to prove Lemma 12 for T3C, which stays the same as that of TTTS.

Proof of Lemma 12 for T3C See proof of Lemma 12 for TTTS in Appendix 7.D.5. ■

7.F Proof of Lemma 4

Finally, it remains to prove Lemma 4 under the Gaussian case before we can conclude for Theorem 7.2 for TTTS or T3C.

Lemma 4. *Let $\delta, \beta \in (0,1)$. For any sampling rule which satisfies $\mathbb{E}\left[T_\beta^\varepsilon\right] < \infty$ for all $\varepsilon > 0$, we have*

$$\limsup_{\delta \to 0} \frac{\mathbb{E}\left[\tau_\delta\right]}{\log(1/\delta)} \leq \frac{1}{\Gamma_\beta^\star},$$

if the sampling rule is coupled with stopping rule (7.4),

For the clarity, we recall the definition of generalized likelihood ratio. For any pair of arms i, j, We first define a weighted average of their empirical means,

$$\widehat{\mu}_{n,i,j} \triangleq \frac{T_{n,i}}{T_{n,i} + T_{n,j}} \widehat{\mu}_{n,i} + \frac{T_{n,j}}{T_{n,i} + T_{n,j}} \widehat{\mu}_{n,j}.$$

And if $\widehat{\mu}_{n,i} \geq \widehat{\mu}_{n,j}$, then the generalized likelihood ratio $Z_{n,i,j}$ for Gaussian noise distributions has the following analytic expression,

$$Z_{n,i,j} \triangleq T_{n,i} d(\widehat{\mu}_{n,i}; \widehat{\mu}_{n,i,j}) + T_{n,j} d(\widehat{\mu}_{n,j}; \widehat{\mu}_{n,i,j}).$$

We further define a statistic Z_n as

$$Z_n \triangleq \max_{i \in \mathcal{A}} \min_{j \in \mathcal{A} \setminus \{i\}} Z_{n,i,j}.$$

The following lemma stated by Qin, Klabjan and Russo (2017) is needed in our proof.

Lemma 26. *For any $\zeta > 0$, there exists ε s.t. $\forall n \geq T_\beta^\varepsilon$, $Z_n \geq (\Gamma_\beta^\star - \zeta) n$.*

To prove Lemma 4, we need the Gaussian tail inequality (7.8) of Lemma 6.

Proof. We know that

$$\begin{aligned}
1 - a_{n,I^\star} &= \sum_{i \neq I^\star} a_{n,i} \\
&\leq \sum_{i \neq I^\star} \Pi_n \left[\theta_i > \theta_{I^\star}\right] \\
&= \sum_{i \neq I^\star} \Pi_n \left[\theta_i - \theta_{I^\star} > 0\right] \\
&\leq (K-1) \max_{i \neq I^\star} \Pi_n \left[\theta_i - \theta_{I^\star} > 0\right].
\end{aligned}$$

We can further rewrite $\Pi_n\left[\theta_i - \theta_{I^\star} > 0\right]$ as

$$\Pi_n\left[\theta_i - \theta_{I^\star} > \mu_{n,i} - \mu_{n,I^\star} + \mu_{n,I^\star} - \mu_{n,i}\right].$$

We choose ε sufficiently small such that the empirical best arm $I_n^\star = I^\star$. Then, for all $n \geq T_\beta^n$ and for any $i \neq I^\star$, $\mu_{n,I^\star} \geq \mu_{n,i}$. Thus, fix any $\zeta \in (0, \Gamma_\beta^\star/2)$ and apply inequality (7.8) of Lemma 6 with $\mu_{n,I^\star}$ and $\mu_{n,i}$, we have for any $n \geq T_\beta^\varepsilon$,

$$\begin{aligned} 1 - a_{n,I^\star} &\leq (K-1) \max_{i \neq I^\star} \frac{1}{2} \exp\left\{-\frac{(\mu_{n,I^\star} - \mu_{n,i})^2}{2\sigma_{n,i,I^\star}^2}\right\} \\ &= \frac{(K-1)\exp\{-Z_n\}}{2} \\ &\leq \frac{(K-1)\exp\left\{-(\Gamma_\beta^\star - \zeta)n\right\}}{2}. \end{aligned}$$

The last inequality is deduced from Lemma 26. By consequence,

$$\forall n \geq T_\beta^\varepsilon, \ln\left(1 - a_{n,I^\star}\right) \leq \ln\frac{K-1}{2} - (\Gamma_\beta^\star - \zeta)n.$$

On the other hand, we have for any n,

$$1 - c_{n,\delta} = \frac{\delta}{2n(K-1)\sqrt{2\pi e}\exp\left\{\sqrt{2\ln\frac{2n(K-1)}{\delta}}\right\}}.$$

Thus, there exists a deterministic time N s.t. $\forall n \geq N$,

$$\begin{aligned} \ln\left(1 - c_{n,\delta}\right) &= \ln\frac{\delta}{(K-1)\sqrt{8\pi e}} - \ln n - \sqrt{2\ln\frac{2n(K-1)}{\delta}} \\ &\geq \ln\frac{\delta}{2(K-1)\sqrt{2\pi e}} - \zeta n. \end{aligned}$$

Let $C_3 \triangleq (K-1)^2\sqrt{2\pi e}$, we have for any $n \geq N_0 \triangleq T_\beta^\varepsilon + N$,

$$\ln\left(1 - a_{n,I^\star}\right) - \ln\left(1 - c_{n,\delta}\right) \leq \ln\frac{C_3}{\delta} - (\Gamma_\beta^\star - 2\zeta)n, \tag{7.13}$$

and it is clear that $\mathbb{E}\left[N_0\right] < \infty$.

Let us consider the following two cases:

Case 1 There exists $n \in [1, N_0]$ s.t. $a_{n,I^\star} \geq c_{n,\delta}$, then by definition,

$$\tau_\delta \leq n \leq N_1.$$

Case 2 For any $n \in [1, N_0]$, we have $a_{n,I^\star} < c_{n,\delta}$, then $\tau_\delta \geq N_0 + 1$, thus by Equation 7.13,

$$\begin{aligned} 0 &\leq \ln\left(1 - a_{\tau_\delta - 1, I^\star}\right) - \ln\left(1 - c_{\tau_\delta - 1, \delta}\right) \\ &\leq \ln \frac{C_3}{\delta} - (\Gamma_\beta^\star - 2\zeta)(\tau_\delta - 1), \end{aligned}$$

and we obtain

$$\tau_\delta \leq \frac{\ln(C_3/\delta)}{\Gamma_\beta^\star - 2\zeta} + 1.$$

Combining the two cases, and we have for any $\zeta \in (0, \Gamma_\beta^\star/2)$,

$$\begin{aligned} \tau_\delta &\leq \max\left\{N_0, \frac{\ln(C_3/\delta)}{\Gamma_\beta^\star - 2\zeta} + 1\right\} \\ &\leq N_0 + 1 + \frac{\ln(C_3)}{\Gamma_\beta^\star - 2\zeta} + \frac{\ln(1/\delta)}{\Gamma_\beta^\star - 2\zeta}. \end{aligned}$$

Since $\mathbb{E}[N_1] < \infty$, therefore

$$\limsup_\delta \frac{\mathbb{E}[\tau_\delta]}{\log(1/\delta)} \leq \frac{1}{\Gamma_\beta^\star - 2\zeta}, \forall \zeta \in (0, \Gamma_\beta^\star/2),$$

which concludes the proof. □

7.G Technical Lemmas

The whole fixed-confidence analysis for the two sampling rules are both substantially based on two lemmas: Lemma 5 of Qin, Klabjan and Russo, 2017 and Lemma 8. We prove Lemma 8 in this section.

Lemma 8. *There exists a random variable W_2, such that for all $i \in \mathcal{A}$,*

$$\forall n \in, |T_{n,i} - \Psi_{n,i}| \leq W_2 \sqrt{(n+1)\log(e^2 + n)} \text{ a.s.},$$

and $\mathbb{E}\left[e^{\lambda W_2}\right] < \infty$ for any $\lambda > 0$.

Proof. The proof shares some similarities with that of Lemma 6 of Qin, Klabjan and Russo, 2017. For any arm $i \in \mathcal{A}$, define $\forall n \in \mathbb{N}$,

$$D_n \triangleq T_{n,i} - \Psi_{n,i},$$

$$d_n \triangleq \mathbb{1}\{I_n = i\} - \psi_{n,i}.$$

It is clear that $D_n = \sum_{l=1}^{n-1} d_l$ and $\mathbb{E}\left[d_n|\mathcal{F}_{n-1}\right] = 0$. Indeed,

$$\begin{aligned}\mathbb{E}\left[d_n|\mathcal{F}_{n-1}\right] &= \mathbb{E}\left[\mathbb{1}\{I_n = i\} - \psi_{n,i}|\mathcal{F}_{n-1}\right] \\ &= \mathbb{P}\left[I_n = i|\mathcal{F}_{n-1}\right] - \mathbb{E}\left[\mathbb{P}\left[I_n = i|\mathcal{F}_{n-1}\right]|\mathcal{F}_{n-1}\right] \\ &= \mathbb{P}\left[I_n = i|\mathcal{F}_{n-1}\right] - \mathbb{P}\left[I_n = i|\mathcal{F}_{n-1}\right] = 0.\end{aligned}$$

The second last equality holds since $\mathbb{P}\left[I_n = i|\mathcal{F}_{n-1}\right]$ is $\mathcal{F}_{n-1}$-measurable. Thus D_n is a martingale, whose increment are 1 sub-Gaussian as $d_n \in [-1, 1]$ for all n.

Applying Corollary 8 of Abbasi-Yadkori, Pál and Szepesvári, 2012[6], it holds that, with probability larger than $1 - \delta$, for all n,

$$|D_n| \leq \sqrt{2\left(1+n\right)\ln\left(\frac{\sqrt{1+n}}{\delta}\right)}$$

which yields the first statement of Lemma 8.

We now introduce the random variable

$$W_2 \triangleq \max_{n\in\mathbb{N}} \max_{i\in\mathcal{A}} \frac{|T_{n,i} - \Psi_{n,i}|}{\sqrt{(n+1)\ln(e^2+n)}}.$$

Applying the previous inequality with $\delta = e^{-x^2/2}$ yields

$$\begin{aligned}\mathbb{P}\left[\exists n \in \mathbb{N}^\star : |D_n| > \sqrt{\left(1+n\right)\left(\ln\left(1+n\right) + x^2\right)}\right] &\leq e^{-x^2/2}, \\ \mathbb{P}\left[\exists n \in \mathbb{N}^\star : |D_n| > \sqrt{\left(1+n\right)\ln\left(e^2+n\right)x^2}\right] &\leq e^{-x^2/2},\end{aligned}$$

where the last inequality uses that for all $a, b \geq 2$, we have $ab \geq a + b$.

Consequently $\forall x \geq 2$, for all $i \in \mathcal{A}$

$$\mathbb{P}\left[\max_{n\in\mathbb{N}} \frac{|T_{n,i} - \Psi_{n,i}|}{\sqrt{(n+1)\log\left(e^2+n\right)}} \geq x\right] \leq e^{-x^2/2}.$$

Now taking a union bound over $i \in \mathcal{A}$, we have $\forall x \geq 2$,

$$\begin{aligned}\mathbb{P}\left[W_2 \geq x\right] &\leq \mathbb{P}\left[\max_{i\in\mathcal{A}} \max_{n\in\mathbb{N}} \frac{|T_{n,i} - \Psi_{n,i}|}{(n+1)\log\left(\sqrt{e^2+n}\right)} \geq x\right] \\ &\leq \mathbb{P}\left[\bigcup_{i\in\mathcal{A}} \max_{n\in\mathbb{N}} \frac{|T_{n,i} - \Psi_{n,i}|}{(n+1)\log\left(\sqrt{e^2+n}\right)} \geq x\right] \\ &\leq \sum_{i\in\mathcal{A}} \mathbb{P}\left[\max_{n\in\mathbb{N}} \frac{|T_{n,i} - \Psi_{n,i}|}{(n+1)\log\left(\sqrt{e^2+n}\right)} \geq x\right] \\ &\leq Ke^{-x^2/2}.\end{aligned}$$

[6] but we could actually use several deviation inequalities that hold uniformly over time for martingales with sub-Gaussian increments

The previous inequalities imply that $\forall i \in \mathcal{A}$ and $\forall n \in \mathbb{N}$, we have

$$|T_{n,i} - \Psi_{n,i}| \leq W_2 \sqrt{(n+1)\log(e^2+n)}$$

almost surely. Now it remains to show that $\forall \lambda > 0, \mathbb{E}\left[e^{\lambda W_2}\right] < \infty$. Fix some $\lambda > 0$.

$$\begin{aligned}
\mathbb{E}\left[e^{\lambda W_2}\right] &= \int_{x=1}^{\infty} \mathbb{P}\left[e^{\lambda W_2} \geq x\right] \mathrm{d}x = \int_{y=0}^{\infty} \mathbb{P}\left[e^{\lambda W_2} \geq e^{2\lambda y}\right] 2\lambda e^{2\lambda y}\, \mathrm{d}y \\
&= 2\lambda \int_{y=0}^{2} \mathbb{P}\left[W_2 \geq 2y\right] e^{2\lambda y}\, \mathrm{d}y + 2\lambda \int_{y=2}^{\infty} \mathbb{P}\left[W_2 \geq 2y\right] e^{2\lambda y}\, \mathrm{d}y \\
&\leq \underbrace{2\lambda \int_{y=0}^{2} \mathbb{P}\left[W_2 \geq 2y\right] e^{2\lambda y}\, \mathrm{d}y}_{= e^{4\lambda - 1}} + \underbrace{2\lambda C_1 \int_{y=2}^{\infty} e^{-y^2/2} e^{2\lambda y}\, \mathrm{d}y}_{< \infty} < \infty,
\end{aligned}$$

where C_1 is some constant.

□

7.H Proof of Posterior Convergence for the Gaussian Bandit

7.H.1 Proof of Theorem 7.13, Gaussian case

Theorem 7.27. *Under TTTS, for Gaussian bandits with improper Gaussian priors, it holds almost surely that*

$$\lim_{n\to\infty} -\frac{1}{n}\log(1 - a_{n,I^\star}) = \Gamma_\beta^\star.$$

From Theorem 2 in Qin, Klabjan and Russo, 2017, any allocation rule satisfying $T_{n,i}/n \to \omega_i^\beta$ for each $i \in \mathcal{A}$, satisfies

$$\lim_{n\to\infty} -\frac{1}{n}\log(1 - a_{n,I^\star}) = \Gamma_\beta^\star.$$

Therefore, to prove Theorem 7.27, it is sufficient to prove that under TTTS,

$$\forall i \in \{1, \ldots, K\}, \quad \lim_{n\to\infty} \frac{T_{n,i}}{n} \overset{a.s}{=} \omega_i^\beta. \tag{7.14}$$

Due to the concentration result in Lemma 8 that we restate below (and proved in Appendix 7.D), which will be useful at several places in the proof, observe that

$$\lim_{n\to\infty} \frac{T_{n,i}}{n} \overset{a.s}{=} \omega_i^\beta \quad \Leftrightarrow \quad \lim_{n\to\infty} \frac{\Psi_{n,i}}{n} \overset{a.s}{=} \omega_i^\beta,$$

therefore it suffices to establish the convergence of $\overline{\psi}_{n,i} = \Psi_{n,i}/n$ to ω_i^β, which we do next. For that purpose, we need again the following maximality inequality lemma.

Lemma 8. *There exists a random variable W_2, such that for all $i \in \mathcal{A}$,*

$$\forall n \in, |T_{n,i} - \Psi_{n,i}| \leq W_2 \sqrt{(n+1)\log(e^2+n)}\ a.s.,$$

and $\mathbb{E}\left[e^{\lambda W_2}\right] < \infty$ for any $\lambda > 0$.

Step 1: TTTS draws all arms infinitely often and satisfies $T_{n,I^\star}/n \to \beta$. More precisely, we prove the following lemma.

Lemma 28. *Under TTTS, it holds almost surely that*

1. *for all $i \in \mathcal{A}$, $\lim_{n\to\infty} T_{n,i} = \infty$.*
2. *$a_{n,I^\star} \to 1$.*
3. *$T_{n,I^\star}/n \to \beta$.*

Proof. Our first ingredient is a lemma showing the implications of finite measurement, and consistency when all arms are sampled infinitely often. Its proof follows standard posterior concentration arguments and is given in Appendix 7.H.2.

Lemma 29 (Consistency and implications of finite measurement)**.**
Denote with $\overline{\mathcal{I}}$ the arms that are sampled only a finite amount of times:

$$\overline{\mathcal{I}} = \{i \in \{1, \dots, k\} : \forall n, T_{n,i} < \infty\}.$$

If $\overline{\mathcal{I}}$ is empty, $a_{n,i}$ converges almost surely to 1 when $i = I^\star$ and to 0 when $i \neq I^\star$. If $\overline{\mathcal{I}}$ is non-empty, then for every $i \in \overline{\mathcal{I}}$, we have $\liminf_{n\to\infty} a_{n,i} > 0$ a.s.

First we show that $\sum_{n\in\mathbb{N}} T_{n,j} = \infty$ for each arm j. Suppose otherwise. Let $\overline{\mathcal{I}}$ again be the set of arms to which only finite measurement effort is allocated. Under TTTS, we have

$$\psi_{n,i} = a_{n,i}\left(\beta + (1-\beta)\sum_{j\neq i}\frac{a_{n,j}}{1-a_{n,j}}\right),$$

so $\psi_{n,i} \geq \beta a_{n,i}$. Therefore, by Lemma 29, if $i \in \overline{\mathcal{I}}$, then $\liminf a_{n,i} > 0$ implies that $\sum_n \psi_{n,i} = \infty$. By Lemma 8, we then must have that $\lim_{n\to\infty} T_{n,i} = \infty$ as well: contradiction. Thus, $\lim_{n\to\infty} T_{n,i} = \infty$ for all i, and we conclude that $a_{n,I^\star} \to 1$, by Lemma 29.

For TTTS with parameter β this implies that $\overline{\psi}_{n,I^\star} \to \beta$, and since we have a bound on $|T_{n,i}/n - \overline{\psi}_{n,i}|$ in Lemma 8, we have $T_{n,I^\star}/n \to \beta$ as well. □

Step 2: Controlling the over-allocation of sub-optimal arms. The convergence of $T_{n,I^\star}/n$ to β leads to following interesting consequence, expressed in Lemma 30: if an arm is sampled more often than its optimal proportion, the posterior probability of this arm to be optimal is reduced compared to that of other sub-optimal arms.

Lemma 30 (Over-allocation implies negligible probability)**.** [7] *Fix any $\xi > 0$ and $j \neq I^\star$. With probability 1, under any allocation rule, if $T_{n,I^\star}/n \to \beta$, there exist $\xi' > 0$ and a sequence ε_n with $\varepsilon_n \to 0$ such that for any $n \in \mathbb{N}$,*

$$\frac{T_{n,j}}{n} \geq \omega_j^\beta + \xi \Rightarrow \frac{a_{n,j}}{\max_{i\neq I^\star} a_{n,i}} \leq e^{-n(\xi'+\varepsilon_n)}.$$

[7] analogue of Lemma 13 of Russo, 2016

Proof. We have $\Pi_n(\Theta_{\cup i \neq I^\star}) = \sum_{i \neq I^\star} a_{n,i} = 1 - a_{n,I^\star}$, therefore $\max_{i \neq I^\star} a_{n,i} \leq 1 - a_{n,I^\star}$. By Theorem 2 of Qin, Klabjan and Russo, 2017 we have, as $T_{n,I^\star}/n \to \beta$,

$$\limsup_{n \to \infty} -\frac{1}{n} \log \left(\max_{i \neq I^\star} a_{n,i} \right) \leq \Gamma^\star_\beta.$$

We also have the following from the standard Gaussian tail inequality, for $n \geq \tau$ after which $\mu_{n,I^\star} \geq \mu_{n,i}$, using that $\theta_i - \theta_{I^\star} \sim \mathcal{N}(\mu_{n,i} - \mu_{n,I^\star}, \sigma^2_{n,i} + \sigma^2_{n,I^\star})$ and $\sigma^2_{n,i} + \sigma^2_{n,I^\star} = \sigma^2(1/T_{n,i} + 1/T_{n,I^\star})$,

$$a_{n,i} \leq \Pi_n(\theta_i \geq \theta_{I^\star}) \leq \exp\left(\frac{-(\mu_{n,i} - \mu_{n,I^\star})^2}{2\sigma^2(1/T_{n,I^\star} + 1/T_{n,i})} \right) = \exp\left(-n \frac{(\mu_{n,i} - \mu_{n,1})^2}{2\sigma^2(n/T_{n,I^\star} + n/T_{n,i})} \right).$$

Thus, there exists a sequence $\varepsilon_n \to 0$, for which

$$\begin{aligned} \frac{a_{n,j}}{\max_{i \neq I^\star} a_{n,i}} &\leq \frac{\exp\left\{ -n \left(\frac{(\mu_{n,j} - \mu_{n,I^\star})^2}{2\sigma^2(n/T_{n,I^\star} + n/T_{n,j})} - \varepsilon_n/2 \right) \right\}}{\exp\left\{ -n \left(\Gamma^\star_\beta + \varepsilon_n/2 \right) \right\})} \\ &= \exp\left\{ -n \left(\frac{(\mu_{n,j} - \mu_{n,I^\star})^2}{2\sigma^2(n/T_{n,I^\star} + n/T_{n,j})} - \Gamma^\star_\beta - \varepsilon_n \right) \right\}. \end{aligned}$$

Now we take a look at the two terms in the middle:

$$\frac{(\mu_{n,j} - \mu_{n,I^\star})^2}{2\sigma^2(n/T_{n,I^\star} + n/T_{n,j})} - \Gamma^\star_\beta.$$

Note that the first term is increasing in $T_{n,j}/n$. We have the definition from Qin, Klabjan and Russo, 2017, for any $j \neq I^\star$,

$$\Gamma^\star_\beta = \frac{(\mu_j - \mu_{I^\star})^2}{2\sigma^2 \left(1/\omega^\beta_{I^\star} + 1/\omega^\beta_j \right)},$$

and we have the premise

$$\frac{T_{n,j}}{n} \geq \omega^\beta_j + \xi.$$

Combining these with the convergence of the empirical means to the true means (consistency, see Lemma 29), we can conclude that for all $\varepsilon > 0$, there exists a time n_0 such that for all later times $n \geq n_0$, we have

$$\frac{(\mu_{n,j} - \mu_{n,I^\star})^2}{2\sigma^2(n/T_{n,I^\star} + n/T_{n,j})} \geq \frac{(\mu_j - \mu_{I^\star})^2}{2\sigma^2 \left(1/\beta + n/T_{n,j} \right)} - \varepsilon \geq \frac{(\mu_j - \mu_{I^\star})^2}{2\sigma^2 \left(1/\beta + 1/(\omega^\beta_j + \xi) \right)} - \varepsilon > \Gamma^\star_\beta,$$

where the first inequality follows from consistency, the second from monotonicity in $T_{n,j}/n$. That means that there exist a $\xi' > 0$ such that

$$\frac{(\mu_{n,j} - \mu_{n,I^\star})^2}{2\sigma^2(n/T_{n,I^\star} + n/T_{n,j})} - \Gamma^\star_\beta > \xi',$$

and thus the claim follows that when $\frac{T_{n,j}}{n} \geq \omega_j^\beta + \xi$, we have

$$\frac{a_{n,j}}{\max_{i \neq I^\star} a_{n,i}} \leq \exp\left\{-n\left(\frac{(\mu_{n,j} - \mu_{n,I^\star})^2}{2\sigma^2(n/T_{n,I^\star} + n/T_{n,j})} - \Gamma_\beta^\star - \varepsilon_n\right)\right\} \leq e^{-n(\xi' + \varepsilon_n)}.$$

□

Step 3: $\overline{\psi}_{n,i}$ converges to ω_i^β for all arms. To establish the convergence of the allocation effort of all arms, we rely on the same sufficient condition used in the analysis of Russo, 2016, that we recall below.

Lemma 31 (Sufficient condition for optimality). [8] *Consider any adaptive allocation rule. If we have*

$$\overline{\psi}_{n,I^\star} \to \beta, \quad \text{and} \quad \sum_{n \in \mathbb{N}} \psi_{n,j} \mathbf{1}\left\{\overline{\psi}_{n,j} \geq \omega_j^\beta + \xi\right\} < \infty, \quad \forall j \neq I^\star, \xi > 0, \tag{7.15}$$

then $\overline{\psi}_n \to \psi^\beta$.

First, note that from Lemma 28 we know that $T_{n,I^\star}/n \to \beta$, an by Lemma 8 this implies $\overline{\psi}_{n,I^\star} \to \beta$, hence we can use Lemma 31 to prove convergence to the optimal proportions. Thus, we now show that (7.15) holds under TTTS. Recall that $J_n^{(1)} = \arg\max_j a_{n,j}$ and $J_n^{(2)} = \arg\max_{j \neq J_n^{(1)}} a_{n,j}$. Since $a_{n,I^\star} \to 1$ by Lemma 28, there is some finite time τ after which for all $n > \tau$, $J_n^{(1)} = I^\star$. Under TTTS,

$$\begin{aligned}
\psi_{n,i} &= a_{n,i}\left(\beta + (1-\beta)\sum_{j \neq i} \frac{a_{n,j}}{1 - a_{n,j}}\right) \\
&\leq a_{n,i}\beta + a_{n,i}(1-\beta)\frac{\sum_{j \neq i} a_{n,j}}{1 - a_{n,J_n^{(1)}}} \\
&\leq a_{n,i}\beta + a_{n,i}(1-\beta)\frac{\sum_{j \neq i} a_{n,j}}{a_{n,J_n^{(2)}}} \\
&\leq a_{n,i}\beta + a_{n,i}(1-\beta)\frac{1}{a_{n,J_n^{(2)}}} \\
&\leq \frac{a_{n,i}}{a_{n,J_n^{(2)}}},
\end{aligned}$$

where we use the fact that for $j \neq J_n^{(1)}$, we have $a_{n,J_n^{(1)}} \geq a_{n,j}$ and $a_{n,J_n^{(2)}} \leq 1 - a_{n,J_n^{(1)}}$. For $n \geq \tau$ this means that $\psi_{n,i} \leq a_{n,i}/\max_{j \neq I^\star} a_{n,i}$ for any $i \neq I^\star$.

By Lemma 30, there is a constant $\xi' > 0$ such and a sequence $\varepsilon_n \to 0$ such that

$$T_{n,i}/n \geq w_i^\beta + \xi \Rightarrow \frac{a_{n,i}}{\max_{j \neq I^\star} a_{n,j}} \leq e^{-n(\xi' - \varepsilon_n)}.$$

[8] Lemma 12 of Russo, 2016

Now take a time τ large enough, such that for $n \geq \tau$ we have $|T_{n,j}/n - \overline{\psi}_{n,j}| \leq \xi$ (which can be found by Lemma 8). Then we have

$$\mathbb{1}\left\{\overline{\psi}_{n,j} \geq \psi_j^\beta + \xi\right\} \leq \mathbb{1}\left\{\frac{T_{n,j}}{n} \geq \omega_j^\beta + 2\xi\right\}$$

Therefore, for all $i \neq I^\star$, we have

$$\sum_{n\geq\tau} \psi_{n,i}\mathbb{1}\left\{\overline{\psi}_{n,j} \geq \psi_j^\beta + \xi\right\} \leq \sum_{n\geq\tau} \psi_{n,i}\mathbb{1}\left\{\frac{T_{n,j}}{n} \geq \omega_j^\beta + 2\xi\right\} \leq \sum_{n\geq\tau} e^{-n(\xi'-\varepsilon_n)} < \infty.$$

Thus (7.15) holds and the convergence to the optimal proportions follows by Lemma 31.

7.H.2 Proof of auxiliary lemmas

Proof of Lemma 29 Let $\overline{\mathcal{I}}$ be nonempty. Define

$$\mu_{\infty,n} \triangleq \lim_{n\to\infty} \mu_{n,i}, \text{ and } \sigma^2_{\infty,i} \triangleq \lim_{n\to\infty} \sigma^2_{n,i},$$

and recall that for $i \in \mathcal{A}$ for which $T_{n,i} = 0$, we have $\mu_{n_i} = \mu_{1,i} = 0$ and $\sigma^2_{n,i} = \sigma^2_{1,i} = \infty$, and if $T_{n,i} > 0$, we have

$$\mu_{n,i} = \frac{1}{T_{n,i}} \sum_{\ell=1}^{n-1} \mathbb{1}\{I_\ell = i\} Y_{\ell,I_\ell}, \text{ and } \sigma^2_{n,i} = \frac{\sigma^2}{T_{n,i}}.$$

For all arms that are sampled infinitely often, we therefore have $\mu_{\infty,i} = \mu_i$ and $\sigma^2_{\infty,i} = 0$. For all arms that are sampled only a finite number of times, i.e. $i \in \overline{\mathcal{I}}$, we have $\sigma^2_{\infty,i} > 0$, and there exists a time n_0 after which for all $n \geq n_0$ and $i \in \overline{\mathcal{I}}$, we have $T_{n,i} = T_{n_0,i}$. Define

$$\Pi_\infty \triangleq \mathcal{N}(\mu_{\infty,1}, \sigma^2_{\infty,1}) \otimes \mathcal{N}(\mu_{\infty,2}, \sigma^2_{\infty,2}) \otimes \ldots \otimes \mathcal{N}(\mu_{\infty,k}, \sigma^2_{\infty,k}) = \bigotimes_{i\notin\overline{\mathcal{I}}} \delta_{\mu_i} \otimes \bigotimes_{i\in\overline{\mathcal{I}}} \Pi_{n_0}.$$

Then for each $i \in \mathcal{A}$ we define

$$a_{\infty,i} \triangleq \Pi_\infty\left(\theta_i > \max_{j\neq i} \theta_j\right).$$

Then we have for all $i \in \overline{\mathcal{I}}$, $a_{\infty,i} \in (0,1)$, since $\sigma^2_{\infty,i} > 0$, and thus $a_{\infty,I^\star} < 1$.

When $\overline{\mathcal{I}}$ is empty, we have $a_{n,I^\star} = \Pi_n(\theta_{I^\star} > \max_{i\neq I^\star} \theta_i)$, but since $\Pi_\infty = \bigotimes_{i\in\mathcal{A}} \delta_{\mu_i}$, we have $a_{\infty,I^\star} = 1$ and $a_{\infty,i} = 0$ for all $i \neq I^\star$.

■

7.I Proof of Posterior Convergence for the Bernoulli Bandit

7.I.1 Preliminaries

We first introduce a crucial Beta tail bound inequality. Let $F^{\text{Beta}}_{a,b}$ denote the cdf of a Beta distribution with parameters a and b, and $F^{\text{B}}_{c,d}$ the cdf of a Binomial distribution with parameters c and d, then we have the following relationship, often called the 'Beta-Binomial trick',

$$F^{\text{Beta}}_{a,b}(y) = 1 - F^{\text{B}}_{a+b-1,y}(a-1),$$

so that we have

$$\mathbb{P}\left[X \geq x\right] = \mathbb{P}\left[B_{a+b-1,x} \leq a-1\right] = \mathbb{P}\left[B_{a+b-1,1-x} \geq b\right].$$

We can bound Binomial tails with Sanov's inequality:

$$\frac{e^{-nd(k/n,x)}}{n+1} \leq \mathbb{P}\left[B_{n,x} \geq k\right] \leq e^{-nd(k/n,x)},$$

where the last inequalities hold when $k \geq nx$.

Lemma 32. *Let $X \sim \mathcal{B}eta(a,b)$ and $Y \sim \mathcal{B}eta(c,d)$ with $0 < \frac{a-1}{a+b-1} < \frac{c-1}{c+d-1}$. Then we have $\mathbb{P}\left[X > Y\right] \leq De^{-C}$ where*

$$C = \inf_{\frac{a-1}{a+b-1} \leq y \leq \frac{c-1}{c+d-1}} C_{a,b}(y) + C_{c,d}(y),$$

and

$$D = 3 + \min\left(C_{a,b}\left(\frac{c-1}{c+d-1}\right), C_{c,d}\left(\frac{a-1}{a+b-1}\right)\right).$$

Note that this lemma is the Bernoulli version of Lemma 6.

Theorem 7.33. *Consider the Beta-Bernoulli setting. For $\beta \in (0,1)$, under any allocation rule satisfying $T_{n,I^\star}/n \to \omega^\beta_{I^\star}$,*

$$\lim_{n\to\infty} -\frac{1}{n}\log(1 - a_{n,I^\star}) \leq \Gamma^\star_\beta,$$

and under any allocation rule satisfying $T_{n,i}/n \to \omega^\beta_i$ for each $i \in \mathcal{A}$,

$$\lim_{n\to\infty} -\frac{1}{n}\log(1 - a_{n,I^\star}) = \Gamma^\star_\beta.$$

Proof. Denote again with $\overline{\mathcal{I}}$ again the set of arms sampled only finitely many times. For $\overline{\mathcal{I}}$ empty, we thus have $\mu_{\infty,i} \triangleq \lim_{n\to\infty} \mu_{n,i} = \mu_i$. The posterior variance is

$$\sigma^2_{n,i} = \frac{\alpha_{n,i}\beta_{n,i}}{(\alpha_{n,i}+\beta_{n,i})^2(\alpha_{n,i}+\beta_{n,i}+1)} = \frac{(1+\sum_{\ell=1}^{n-1}\mathbb{1}\{I_\ell = i\}Y_{\ell,I_\ell})(1+T_{n,i}-\sum_{\ell=1}^{n-1}\mathbb{1}\{I_\ell = i\}Y_{\ell,I_\ell})}{(2+T_{n,i})^2(2+T_{n,i}+1)}.$$

We see that when $\overline{\mathcal{I}}$ is empty, we have $\sigma^2_{\infty,i} \triangleq \lim_{n\to\infty}\sigma^2_{n,i} = 0$, i.e., the posterior is concentrated.

Step 1: A lower bound when some arms are sampled only finitely often. First, note that when $T_{n,i} = 0$ for some $i \in \mathcal{A}$, the empirical mean for that arm equals the prior mean

$$\mu_{n,i} = \alpha_{0,i}/(\alpha_{0,i} + \beta_{0,i}),$$

and the variance is strictly positive:

$$\sigma^2_{n,i} = (\alpha_{0,i}\beta_{0,i})/\left((\alpha_{0,i} + \beta_{0,i})^2(\alpha_{0,i} + \beta_{0,i} + 1)\right) > 0.$$

When $\overline{\mathcal{I}}$ is not empty, then for every $i \in \overline{\mathcal{I}}$ we have $\sigma^2_{\infty,i} > 0$, and $a_{\infty,i} \in (0,1)$, implying $a_{\infty,I^\star} < 1$, and thus

$$\lim_{n\to\infty} -\frac{1}{n}\log(1 - a_{n,I^\star}) = -\frac{1}{n}\log(1 - a_{\infty,I^\star}) = 0.$$

Step 2: A lower bound when every arm is sampled infinitely often. Suppose now that $\overline{\mathcal{I}}$ is empty, then we have

$$\max_{i\neq I^\star} \Pi_n(\theta_i \geq \theta_{I^\star}) \leq 1 - a_{n,I^\star} \leq \sum_{i\neq I^\star} \Pi_n(\theta_i \geq \theta_{I^\star}) \leq (k-1)\max_{i\neq I^\star} \Pi_n(\theta_i \geq \theta_{I^\star}).$$

Thus, we have $1 - a_{n,I^\star} \leq (k-1)\max_{i\neq I^\star} \Pi_n(\theta_i \geq \theta_{I^\star})$ and also $1 - a_{n,I^\star} \doteq \max_{i\neq I^\star} \Pi_n(\theta_i \geq \theta_{I^\star})$. We have

$$\begin{aligned}
\Gamma^\star &= \max_{w\in W} \min_{i\neq I^\star} C_i(\omega_{I^\star}, \omega_i), \\
\Gamma^\star_\beta &= \max_{w\in W; \omega_{I^\star} = \beta} \min_{i\neq I^\star} C_i(\beta, \omega_i), \text{ with} \\
C_i(\omega_{I^\star}, \omega_i) &= \min_{x\in\mathbb{R}} \omega_{I^\star} d(\theta_{I^\star}; x) + \omega_i d(\theta_i; x) = \omega_{I^\star} d(\theta_{I^\star}; \overline{\theta}) + \omega_i d(\theta_i; \overline{\theta}),
\end{aligned}$$

where $\overline{\theta} \in [\theta_i, \theta_{I^\star}]$ is the solution to

$$A'(\overline{\theta}) = \frac{\omega_{I^\star} A'(\theta_{I^\star}) + \omega_i A'(\theta_i)}{\omega_{I^\star} + \omega_i}.$$

Since every arm is sampled infinitely often, when n is large, we have $\mu_{n,I^\star} > \mu_{n,i}$. Define $S_{n,i} \triangleq \sum_{\ell=1}^{n-1} \mathbb{1}\{I_\ell = i\} Y_{\ell,I_\ell}$. Recall that the posterior is a Beta distribution with parameters $a_{n,i} = S_{n,i} + 1$ and $\beta_{n,i} = T_{n,i} - S_{n,i} + 1$. Let $\tau \in \mathbb{N}$ be such that for every $n \geq \tau$, we have $S_{n,i}/(T_{n,i}+1) < S_{n,I^\star}/(T_{n,I^\star}+1)$. For the sake of simplicity, we define for any $i \in \mathcal{A}$ the interval

$$I_{i,I^\star} \triangleq \left[\frac{S_{n,i}}{T_{n,i}+1}, \frac{S_{n,I^\star}}{T_{n,I^\star}+1}\right].$$

Then using Lemma 32 with $a = S_{n,i}+1, b = T_{n,i} - S_{n,i} + 1, c = S_{n,I^\star} + 1, d = T_{n,I^\star} - S_{n,I^\star} + 1$, we have

$$\Pi_n(\theta_i - \theta_{I^\star} \geq 0) \leq D\exp\left\{-\inf_{y\in I_{i,I^\star}} C_{S_{n,i}+1, T_{n,i}-S_{n,i}+1}(y) + C_{S_{n,I^\star}+1, T_{n,I^\star}-S_{n,I^\star}+1}(y)\right\}.$$

This implies

$$\frac{1}{n}\log\left(\frac{\Pi_n(\theta_i \geq \theta_{I^\star})}{\exp\left\{-\inf_{y\in I_{i,I^\star}} C_{S_{n,i}+1,T_{n,i}-S_{n,i}+1}(y) + C_{S_{n,I^\star}+1,T_{n,I^\star}-S_{n,I^\star}+1}(y)\right\}}\right) \leq \frac{1}{n}\log(D),$$

which goes to zero as n goes to infinity. Indeed replacing a, b, c, d by their values in the definition of D we get

$$\begin{aligned} D &\leq 3 + (T_{n,i}-1)kl\left(\frac{S_{n,i}}{T_{n,i}+1}; \frac{S_{n,I^\star}}{T_{n,I^\star}+1}\right) \\ &\leq 3 + (n+1)kl\left(0; \frac{n}{n+1}\right) \\ &= (n+1)\log(n+1)\,. \end{aligned}$$

Hence,

$$\Pi_n(\theta_i \geq \theta_{I^\star}) \doteq \exp\left\{-\inf_{y\in I_{i,I^\star}} C_{S_{n,i}+1,T_{n,i}-S_{n,i}+1}(y) + C_{S_{n,I^\star}+1,T_{n,I^\star}-S_{n,I^\star}+1}(y)\right\}.$$

We thus have for any i,

$$\begin{aligned} 1-a_{n,i} &\doteq \max_{j\neq I^\star} \Pi_n\left[\theta_j \geq \theta_{I^\star}\right] \\ &\doteq \max_{j\neq I^\star} \exp\left\{-\inf_{y\in I_{j,I^\star}} C_{S_{n,j}+1,T_{n,j}-S_{n,j}+1}(y) + C_{S_{n,I^\star}+1,T_{n,I^\star}-S_{n,I^\star}+1}(y)\right\} \\ &\doteq \exp\left\{-n\min_{j\neq I^\star}\inf_{y\in I_{j,I^\star}} \frac{T_{n,j}+1}{n} kl\left(\frac{S_{n,j}}{T_{n,j}+1}; y\right) + \frac{T_{n,I^\star}+1}{n} kl\left(\frac{S_{n,I^\star}}{T_{n,I^\star}+1}; y\right)\right\} \\ &\geq \exp\left\{-n\max_{\boldsymbol{\omega}}\min_{j\neq I^\star}\inf_{y\in I_{j,I^\star}} \omega_i kl\left(\frac{S_{n,j}}{T_{n,j}+1}; y\right) + \omega_{I^\star} kl\left(\frac{S_{n,I^\star}}{T_{n,j}+1}; y\right)\right\}. \end{aligned}$$

Fix some $\varepsilon > 0$, then there exists some $n_0(\varepsilon)$ such that for all $n \geq n_0(\varepsilon)$, we have for any j,

$$I_{j,I^\star} = \left[\frac{S_{n,j}}{T_{n,j}+1}, \frac{S_{n,I^\star}}{T_{n,I^\star}+1},\right] \subset \left[\mu_j + \varepsilon, \mu_{I^\star} - \varepsilon\right] \triangleq I^\star_{j,\varepsilon},$$

and because KL-divergence is uniformly continuous on the compact interval $I^\star_{j,\varepsilon}$, there exists an n_1 such that for every $n \geq n_1$ we have

$$kl\left(\frac{S_{n,j}}{T_{n,j}+1}; y\right) \geq (1-\varepsilon)kl\left(\mu_j; y\right),$$

for any y and for all $j \in \mathcal{A}$. Therefore, we have

$$\begin{aligned} 1-a_{n,i} &\doteq \exp\left\{-n\max_{\boldsymbol{\omega}}\min_{j\neq I^\star}\inf_{y\in I_{j,I^\star}} \omega_j kl\left(\frac{S_{n,j}}{T_{n,j}+1}; y\right) + \omega_{I^\star} kl\left(\frac{S_{n,I^\star}}{T_{n,I^\star}+1}; y\right)\right\} \\ &\geq \exp\left\{-n\max_{\boldsymbol{\omega}}\min_{i\neq I^\star}\inf_{y\in I^\star_{j,\varepsilon}} \omega_i kl(\mu_j; y) + \omega_{I^\star} kl(\mu_{I^\star}; y)\right\}. \end{aligned}$$

Therefore, we have

$$\limsup_{n\to\infty} -\frac{1}{n}\log(1-a_{n,i}) \le \Gamma^\star.$$

If $T_{n,i}/n \to \omega_i^\star$ for each $i \in \mathcal{A}$, we have

$$\begin{aligned}\lim_{n\to\infty} \inf_{y\in I_{i,I^\star}} \frac{T_{n,i}+1}{n} kl\left(\frac{S_{n,i}}{T_{n,i}+1}; y\right) + \frac{T_{n,I^\star}+1}{n} kl\left(\frac{S_{n,I^\star}}{T_{n,i}+1}; y\right)\\ = \inf_{y\in[\mu_i,\,\mu_{I^\star}]} \omega_i^\star kl(\mu_i; y) + \omega_{I^\star}^\star kl(\mu_{I^\star}; y)\\ = \Gamma^\star,\end{aligned}$$

and thus

$$\begin{aligned}1-a_{n,i} &\doteq \exp\left\{-n \max_{\boldsymbol{\omega}} \min_{j\neq I^\star} \inf_{y\in I_\varepsilon^\star} \omega_i kl(\mu_j; y) + \omega_{I^\star} kl(\mu_{I^\star}; y)\right\}\\ &\doteq \exp\left\{-n\Gamma^\star\right\},\end{aligned}$$

implying

$$\lim_{n\to\infty} -\frac{1}{n}\log\left(1-a_{n,i}\right) = \Gamma^\star.$$

Everything goes similarly when $\omega_{I^\star} = \beta \in (0,1)$, so under any sampling rule satisfying $T_{n,I^\star}/n \to \beta$ we have

$$\limsup_{n\to\infty} -\frac{1}{n}\log(1-a_{n,i}) \le \Gamma_\beta^\star$$

and under any sampling rule satisfying $T_{n,i}/n \to \omega_i^\beta$ for each $i \in \mathcal{A}$, we have

$$\lim_{n\to\infty} -\frac{1}{n}\log(1-a_{n,i}) = \Gamma_\beta^\star.$$

□

7.I.2 Proof of Theorem 7.13, Bernoulli case

Theorem 7.34. *Under TTTS, for Bernoulli bandits and uniform priors, it holds almost surely that*

$$\lim_{n\to\infty} -\frac{1}{n}\log(1-a_{n,I^\star}) = \Gamma_\beta^\star.$$

From Theorem 7.33 we know that under any allocation rule satisfying $T_{n,i}/n \to \omega_i^\beta$ for every $i \in \mathcal{A}$, we have

$$\lim_{n\to\infty} -\frac{1}{n}\log\left(1-a_{n,I^\star}\right) = \Gamma_\beta^\star.$$

Thus, we only need to prove that under TTTS, for all $i \in \mathcal{A}$, we have

$$\lim_{n\to\infty} \frac{T_{n,i}}{n} \overset{a.s}{=} \omega_i^\beta.$$

Just as for the proof of the Gaussian case, we can use Lemma 8 (proof in Appendix 7.H.2), which implies

$$\lim_{n\to\infty} \frac{T_{n,i}}{n} \overset{a.s}{=} \omega_i^\beta \quad \Leftrightarrow \quad \lim_{n\to\infty} \frac{\Psi_{n,i}}{n} \overset{a.s}{=} \omega_i^\beta.$$

Therefore, it suffices to show convergence for $\overline{\psi}_{n,i} = \Psi_{n,i}/n$ to ω_i^β, which we will do next, following the same steps as in the proof for the Gaussian case.

Step 1: TTTS draws all arms infinitely often and satisfies $T_{n,I^\star}/n \to \beta$. We prove the following lemma.

Lemma 35. *Under TTTS, it holds almost surely that*

1. *for all $i \in \mathcal{A}$, $\lim_{n\to\infty} T_{n,i} = \infty$.*
2. *$a_{n,I^\star} \to 1$.*
3. *$\frac{T_{n,I^\star}}{n} \to \beta$.*

Proof. First, we give a lemma showing the implications of finite measurement, and consistency when all arms are sampled infinitely often, which provides a proof for 2. The proof of this lemma follows from the proof of Theorem 7.33, and is given in Appendix 7.I.3.

Lemma 36 (Consistency and implications of finite measurement)**.**
Denote with $\overline{\mathcal{I}}$ the arms that are sampled only a finite amount of times:

$$\overline{\mathcal{I}} = \{i \in \{1, \dots, k\} : \forall n, T_{n,i} < \infty\}.$$

If $\overline{\mathcal{I}}$ is empty, $a_{n,i}$ converges almost surely to 1 when $i = I^\star$ and to 0 when $i \neq I^\star$. If $\overline{\mathcal{I}}$ is non-empty, then for every $i \in \overline{\mathcal{I}}$, we have $\liminf_{n\to\infty} a_{n,i} > 0$ a.s.

Now we can show 1. of Lemma 35: we show that under TTTS, for each $j \in A$, we have $\sum_{n\in\mathbb{N}} T_{n,j} = \infty$. The proof is exactly equal to the proof for Gaussian arms.

Under TTTS, we have

$$\psi_{n,i} = a_{n,i}\left(\beta + (1-\beta)\sum_{j\neq i} \frac{a_{n,j}}{1-a_{n,j}}\right),$$

so $\psi_{n,i} \geq \beta a_{n,i}$, therefore, by Lemma 29, if $i \in \overline{\mathcal{I}}$, then $\liminf a_{n,i} > 0$ implies that $\sum_n \psi_{n,i} = \infty$. By Lemma 8, we then must have that $\lim_{n\to\infty} T_{n,i} = \infty$ as well: contradiction. Thus, $\lim_{n\to\infty} T_{n,i} = \infty$ for all i, and we conclude that $a_{n,I^\star} \to 1$, by Lemma 29.

Lastly we prove point 3. of Lemma 35. For TTTS with parameter β, the above implies that $\overline{\psi}_{n,I^\star} \to \beta$, and since we have a bound on $|T_{n,i}/n - \overline{\psi}_{n,i}|$ in Lemma 8, we have $T_{n,I^\star}/n \to \beta$ as well.

□

Step 2: Controlling the over-allocation of sub-optimal arms. Following the proof for the Gaussian case again, we can establish a consequence of the convergence of $T_{n,I^\star}/n$ to β : if an arm is sampled more often than its optimal proportion, the posterior probability of this arm to be optimal is reduced compared to that of other sub-optimal arms. We can prove this by using ingredients from the proof of the lower bound in Theorem 7.33.

Lemma 37 (Over-allocation implies negligible probability)**.** [9]
Fix any $\xi > 0$ and $j \neq I^\star$. With probability 1, under any allocation rule, if $T_{n,I^\star}/n \to \beta$, there exist $\xi' > 0$ and a sequence ε_n with $\varepsilon_n \to 0$ such that for any $n \in \mathbb{N}$,

$$\frac{T_{n,j}}{n} \geq \omega_j^\beta + \xi \implies \frac{a_{n,j}}{\max_{i \neq I^\star} a_{n,i}} \leq e^{-n(\xi' + \varepsilon_n)}.$$

Proof. By Theorem 7.33, we have, as $T_{n,I^\star}/n \to \beta$,

$$\limsup_{n \to \infty} -\frac{1}{n} \log\left(\max_{i \neq I^\star} a_{n,i}\right) \leq \Gamma_\beta^\star,$$

since $\max_{i \neq I^\star} a_{n,i} \leq 1 - a_{n,I^\star}$. We also have from Lemma 32 a deviation inequality, so that we can establish the following logarithmic equivalence:

$$a_{n,j} \leq \Pi_n(\theta_j \geq \theta_{I^\star}) \doteq \exp\left\{-nC_j\left(w_{n,I^\star}, \omega_{n,j}\right)\right\} \doteq \exp\left\{-nC_j\left(\beta, \omega_{n,j}\right)\right\},$$

where we denote $\omega_{n,j} \triangleq \frac{T_{n,j}}{n}$. We can combine these results, which implies that there exists a non-negative sequence $\varepsilon_n \to 0$ such that

$$\frac{a_{n,j}}{\max_{i \neq I^\star} a_{n,i}} \leq \frac{\exp\left\{-nC_j\left(\beta, \omega_{n,j}\right) - \varepsilon_n/2\right\}}{\exp\left\{-n(\Gamma_\beta^\star + \varepsilon/2)\right\}} = \exp\left\{-n\left(C_j\left(\beta, \omega_{n,j}\right) - \Gamma_\beta^\star\right) - \varepsilon_n\right\}.$$

We know that $C_j\left(\beta, \omega_j^\beta\right)$ is strictly increasing in ω_j^β, and $C_j\left(\beta, \omega_j^\beta\right) = \Gamma_\beta^\star$, thus, there exists some $\xi' > 0$ such that

$$\omega_{n,j} \geq \omega_j^\beta + \xi \implies C_j\left(\beta, \omega_{n,j}\right) - \Gamma_\beta^\star > \xi'.$$

□

[9] analogue of Lemma 13 of Russo, 2016

Step 3: $\overline{\psi}_{n,i}$ converges to ω_i^β for all arms. To establish the convergence of the allocation effort of all arms, we rely on the same sufficient condition used in the analysis of Russo, 2016, restated above in Lemma 31, and we will restate it here again for convenience.

Lemma 38 (Sufficient condition for optimality)**.**
Consider any adaptive allocation rule. If

$$\overline{\psi}_{n,I^\star} \to \beta, \quad \textit{and} \quad \sum_{n\in\mathbb{N}} \psi_{n,j} \mathbf{1}\left\{\overline{\psi}_{n,j} \geq \omega_j^\beta + \xi\right\} < \infty, \ \ \forall\, j \neq I^\star, \xi > 0, \tag{7.16}$$

then $\overline{\psi}_n \to \psi^\beta$.

First, note that from Lemma 35 we know that $\frac{T_{n,I^\star}}{n} \to \beta$, and by Lemma 8 this implies $\overline{\psi}_{n,I^\star} \to \beta$, hence we can use the lemma above to prove convergence to the optimal proportions. This proof is already given in Step 3 of the proof for the Gaussian case, and since it does not depend on the specifics of the Gaussian case, except for invoking Lemma 29 (consistency), which for the Bernoulli case we replace by Lemma 36, it gives a proof for the Bernoulli case as well. We conclude that (7.15) holds, and the convergence to the optimal proportions follows by Lemma 31.

7.I.3 Proof of auxiliary lemmas

Lemma 32. *Let $X \sim \mathcal{B}eta(a,b)$ and $Y \sim \mathcal{B}eta(c,d)$ with $0 < \frac{a-1}{a+b-1} < \frac{c-1}{c+d-1}$. Then we have $\mathbb{P}[X > Y] \leq De^{-C}$ where*

$$C = \inf_{\frac{a-1}{a+b-1} \leq y \leq \frac{c-1}{c+d-1}} C_{a,b}(y) + C_{c,d}(y),$$

and

$$D = 3 + \min\left(C_{a,b}\left(\frac{c-1}{c+d-1}\right), C_{c,d}\left(\frac{a-1}{a+b-1}\right)\right).$$

Proof

$$\begin{aligned}
\mathbb{P}[X > Y] = \mathbb{E}[\mathbb{P}[X > Y|Y]] &\leq \mathbb{E}\left[\mathbb{1}\{Y < \frac{a-1}{a+b-1}\} + \mathbb{1}\{Y \geq \frac{a-1}{a+b-1}\}\mathbb{P}[X > Y|Y]\right] \\
&\leq \exp\left\{-(c+d-1)kl\left(\frac{c-1}{c+d-1}; \frac{a-1}{a+b-1}\right)\right\} \\
&\quad + \underbrace{\mathbb{E}\left[\exp\left\{-(a+b-1)kl\left(\frac{a-1}{a+b-1}; Y\right)\right\}\mathbb{1}\{Y \geq \frac{a-1}{a+b-1}\}\right]}_{A},
\end{aligned}$$

Using the Beta-Binomial trick in the second inequality. Furthermore, we have

$$A \leq \underbrace{\mathbb{E}\left[\mathbb{1}\{\frac{a-1}{a+b-1} \leq Y \leq \frac{c-1}{c+d-1}\}\right] \exp\left\{-(a+b-1)kl\left(\frac{a-1}{a+b-1}; Y\right)\right\}}_{B} + \exp\left\{-(a+b-1)kl\left(\frac{a-1}{a+b-1}; \frac{c-1}{c+d-1}\right)\right\}$$

Denote with f the density of Y, then

$$B = \int_{\frac{a-1}{a+b-1}}^{\frac{c-1}{c+d-1}} \exp\left\{-(a+b-1)kl\left(\frac{a-1}{a+b-1}; y\right)\right\} f(y)\,\mathrm{d}y.$$

Via integration by parts we obtain

$$\begin{aligned} B &= \left[\exp\left\{-(a+b-1)kl\left(\frac{a-1}{a+b-1}; y\right)\right\}\mathbb{P}\left[Y \leq y\right]\right]_{\frac{a-1}{a+b-1}}^{\frac{c-1}{c+d-1}} \\ &\quad + \int_{\frac{a-1}{a+b-1}}^{\frac{c-1}{c+d-1}} (a+b-1)\frac{\mathrm{d}}{\mathrm{d}y}kl\left(\frac{a-1}{a+b-1}; y\right)\exp\left\{-C_{a,b}(y)\right\}P(Y \leq y)\,\mathrm{d}y \\ &\leq \int_{\frac{a-1}{a+b-1}}^{\frac{c-1}{c+d-1}} (a+b-1)\frac{\mathrm{d}}{\mathrm{d}y}kl\left(\frac{a-1}{a+b-1}; y\right)\exp\left\{-(C_{a,b}(y)+C_{c,d}(y))\right\}\mathrm{d}y \\ &\quad + \exp\left\{-(a+b-1)kl\left(\frac{a-1}{a+b-1}; \frac{c-1}{c+d-1}\right)\right\}, \end{aligned}$$

where the first inequality uses the Binomial trick again. Let

$$\begin{aligned} C &= \inf_{\frac{a-1}{a+b-1} \leq y \leq \frac{c-1}{c+d-1}} (a+b-1)kl\left(\frac{a-1}{a+b-1}; y\right) + (c+d-1)kl\left(\frac{c-1}{c+d-1}; y\right) \\ &= \inf_{\frac{a-1}{a+b-1} \leq y \leq \frac{c-1}{c+d-1}} C_{a,b}(y) + C_{c,d}(y), \end{aligned}$$

then note that in particular we have

$$\begin{aligned} C &\leq \min\left((a+b-1)kl\left(\frac{a-1}{a+b-1}; \frac{c-1}{c+d-1}\right), (c+d-1)kl\left(\frac{c-1}{c+d-1}; \frac{a-1}{a+b-1}\right)\right) \\ &= \min\left(C_{a,b}\left(\frac{c-1}{c+d-1}\right), C_{c,d}\left(\frac{a-1}{a+b-1}\right)\right). \end{aligned}$$

Then

$$\begin{aligned} B &\leq e^{-C}\int_{\frac{a-1}{a+b-1}}^{\frac{c-1}{c+d-1}} (a+b-1)\frac{\mathrm{d}}{\mathrm{d}y}kl\left(\frac{a-1}{a+b-1}; y\right)\mathrm{d}y + e^{-C} \\ &= \left[(a+b-1)kl\left(\frac{a-1}{a+b-1}; \frac{c-1}{c+d-1}\right)+1\right]e^{-C}. \end{aligned}$$

Thus we have

$$\mathbb{P}\left[X > Y\right] \leq \left(3 + (a+b-1)kl\left(\frac{a-1}{a+b-1}; \frac{c-1}{c+d-1}\right)\right)e^{-C}.$$

By symmetry, we have

$$\mathbb{P}\left[X > Y\right] \leq \left(3 + \min\left(C_{a,b}\left(\frac{c-1}{c+d-1}\right), C_{c,d}\left(\frac{a-1}{a+b-1}\right)\right)\right)e^{-C},$$

where

$$C = \inf_{\frac{a-1}{a+b-1} \leq y \leq \frac{c-1}{c+d-1}} (a+b-1)kl\left(\frac{a-1}{a+b-1}; y\right) + (c+d-1)kl\left(\frac{c-1}{c+d-1}; y\right).$$

■

Proof of Lemma 36 Let $\overline{\mathcal{I}}$ be empty, then we have $\mu_{\infty,i} \triangleq \lim_{n\to\infty} \mu_{n,i} = \mu_i$. The posterior variance is

$$\begin{aligned}\sigma^2_{n,i} &= \frac{\alpha_{n,i}\beta_{n,i}}{(\alpha_{n,i}+\beta_{n,i})^2(\alpha_{n,i}+\beta_{n,i}+1)} \\ &= \frac{(1+\sum_{\ell=1}^{n-1}\mathbb{1}\{I_\ell = i\}Y_{\ell,I_\ell})(1+T_{n,i}-\sum_{\ell=1}^{n-1}\mathbb{1}\{I_\ell = i\}Y_{\ell,I_\ell})}{(2+T_{n,i})^2(2+T_{n,i}+1)},\end{aligned}$$

We see that when $\overline{\mathcal{I}}$ is empty, we have $\sigma^2_{\infty,i} \triangleq \lim_{n\to\infty} \sigma^2_{n,i} = 0$, i.e., the posterior is concentrated.

When $T_{n,i} = 0$ for some $i \in \mathcal{A}$, the empirical mean for that arm equals the prior mean

$$\mu_{n,i} = \alpha_{1,i}/(\alpha_{1,i}+\beta_{1,i}),$$

and the variance is strictly positive:

$$\sigma^2_{n,i} = (\alpha_{n,i}\beta_{n,i})/\left((\alpha_{1,i}+\beta_{1,i})^2(\alpha_{1,i}+\beta_{1,i}+1)\right) > 0.$$

When $\overline{\mathcal{I}}$ is not empty, then for every $i \in \overline{\mathcal{I}}$ we have $\sigma^2_{\infty,i} > 0$, and $\alpha_{\infty,i} \in (0,1)$, implying $\alpha_{\infty,I^\star} < 1$, hence the posterior is not concentrated.

■