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Feature network models for proximity data : statistical inference, model selection, network representations and links with related models

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Feature Network Models for Proximity Data

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Feature Network Models for Proximity Data

*Statistical inference, model selection, network representations
and links with related models*

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To my parents

“On ne peut se flatter d’avoir le dernier mot d’une théorie, tant qu’on ne peut pas l’expliquer en peu de paroles à un passant dans la rue.”

[It is not possible to feel satisfied at having said the last word about some theory as long as it cannot be explained in a few words to any passer-by encountered in the street.]

Joseph Diaz Gergonne, French mathematician (Chasles, 1875, p. 115).

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Notation and Symbols

Notation conventions

matrices:	bold capital
vectors:	bold lowercase
scalars, integers:	lowercase

Symbols

<i>Symbol</i>	<i>Description</i>
O	an object or stimulus
m	the number of objects, stimuli
i	index $i = 1, \dots, m$
j	index $j = 1, \dots, m$
k	index $k = 1, \dots, m$
n	the number of object pairs = $\frac{1}{2}m(m - 1)$
l	index $l = 1, \dots, n$
N	the number of replications of samples of size $n \times 1$
ℓ	index $\ell = 1, \dots, N$
f	a frequency value associated with an object pair
δ	a dissimilarity value associated with an object pair
$\hat{\delta}$	an estimated dissimilarity value associated with an object pair
$\boldsymbol{\delta}$	an $n \times 1$ vector with dissimilarities between all object pairs
$\hat{\boldsymbol{\delta}}$	an $n \times 1$ vector with estimated dissimilarities between all object pairs
$\boldsymbol{\Delta}$	an $m \times m$ matrix with dissimilarities
$\tilde{\Delta}_{l\ell}$	a random variable producing realisations $\tilde{\delta}_{l\ell}$
$\tilde{\delta}_{l\ell}$	a realisation of random variable $\tilde{\Delta}_{l\ell}$
$\tilde{\boldsymbol{\Delta}}$	an $n \times N$ matrix of random variables $\tilde{\Delta}_{l\ell}$
$\bar{\Delta}_l$	mean of a row l of $\tilde{\boldsymbol{\Delta}}$
ζ	a similarity value associated with an object pair
$\boldsymbol{\zeta}$	an $n \times 1$ vector with similarities between all object pairs
$\boldsymbol{\Sigma}_{\zeta}$	an $m \times m$ matrix with similarities
F	a feature, which is a binary $(0, 1)$ vector of size $m \times 1$
F_C	a cluster feature, which is a binary $(0, 1)$ vector of size $m \times 1$
F_U	a unique feature, which is a binary $(0, 1)$ vector of size $m \times 1$
T	the number of features

T_C	the number of <i>cluster</i> features
T_U	the number of <i>unique</i> features
T_D	the total number of distinctive features = $\frac{1}{2}(2^m) - 1$
t	index for the features: $t = 1, \dots, T$
t_C	index for the <i>cluster</i> features: $t_C = 1, \dots, T_C$
t_U	index for the <i>unique</i> features: $t_U = 1, \dots, T_U$
S_i	the set of features that represents object O_i
\mathbf{E}	an $m \times T$ matrix with columns representing features
\mathbf{e}	a row vector from the matrix \mathbf{E}
e	an element of the matrix \mathbf{E}
\mathbf{E}_T	an \mathbf{E} matrix with special feature structure that yields a tree representation
\mathbf{E}_C	the part of \mathbf{E}_T (size $m \times T_C$) that represents the set of <i>cluster</i> features
\mathbf{E}_U	the part of \mathbf{E}_T (size $m \times T_U$) that represents the set of <i>unique</i> features
\mathbf{X}	an $n \times T$ matrix with featurewise distances obtained with $\mathbf{x}' = \mathbf{e}_{it} - \mathbf{e}_{jt} $
\mathbf{x}'	a row vector from the matrix \mathbf{X}
\mathbf{x}	a column vector from the matrix \mathbf{X}
\mathbf{X}_T	an $n \times T_C + T_U$ matrix with featurewise distances obtained with \mathbf{E}_T
\mathcal{D}	the complete set of featurewise distances
d	a distance between an object pair
\mathbf{d}	an $n \times 1$ vector of distances between all object pairs
$\hat{\mathbf{d}}$	an $n \times 1$ vector of estimated distances between all object pairs
$\hat{\mathbf{d}}_T$	an $n \times 1$ vector of estimated distances between all object pairs for a tree structure
η	feature discriminability parameter
η_{OLS}	true value of ordinary least squares feature discriminability parameter
η_{ICLS}	true value of inequality constrained least squares feature discriminability parameter
η_L	true value of Lasso feature discriminability parameter
η_{PL}	true value of Positive Lasso feature discriminability parameter
$\boldsymbol{\eta}$	an $T \times 1$ vector of feature discriminability parameters
$\boldsymbol{\eta}_{OLS}$	an $T \times 1$ vector of true values η_{OLS}
$\boldsymbol{\eta}_{ICLS}$	an $T \times 1$ vector of true values η_{ICLS}
$\boldsymbol{\eta}_L$	an $T \times 1$ vector of true values η_L
$\boldsymbol{\eta}_{PL}$	an $T \times 1$ vector of true values η_{PL}
$\hat{\eta}, \hat{\eta}_{OLS}$	estimated values of $\eta, \eta_{OLS}, \eta_{ICLS}, \eta_L, \eta_{PL}$
C	the number of constraints necessary to obtain $\hat{\boldsymbol{\eta}}_{ICLS}$
c	index $c = 1, \dots, C$
\mathbf{r}	a $C \times 1$ vector with constraints
\mathbf{A}	a $C \times T$ matrix of constraints of rank c
$\boldsymbol{\lambda}_{KT}$	a $m \times 1$ vector with Kuhn-Tucker multipliers
$\boldsymbol{\epsilon}$	a $n \times 1$ vector with error values ($\boldsymbol{\epsilon} = \boldsymbol{\delta} - \mathbf{X}\boldsymbol{\eta}$)
$\hat{\boldsymbol{\epsilon}}$	a $n \times 1$ vector with estimated error values ($\hat{\boldsymbol{\epsilon}} = \boldsymbol{\delta} - \mathbf{X}\hat{\boldsymbol{\eta}}$)
$\hat{\epsilon}$	an element from the vector $\hat{\boldsymbol{\epsilon}}$
σ^2, σ	true variance and standard deviation of $\boldsymbol{\epsilon}$
$\hat{\sigma}^2, \hat{\sigma}$	estimated variance and standard deviation of $\hat{\boldsymbol{\epsilon}}$
$\sigma_{\eta}^2, \sigma_{\eta}$	true variance and standard error of η
$\hat{\sigma}_{\eta}^2, \hat{\sigma}_{\eta}$	estimated nominal variance and estimated nominal standard error of η
$\hat{\sigma}_{\hat{\eta}}^2, \hat{\sigma}_{\hat{\eta}}$	estimated nominal variance and nominal standard error of $\hat{\eta}$
$\sigma_{OLS}^2, \sigma_{OLS}$	true variance and standard error of $\hat{\eta}_{OLS}$

$\hat{\sigma}_{\text{OLS}}^2, \hat{\sigma}_{\text{OLS}}$	estimated variance and standard error of $\hat{\eta}_{\text{OLS}}$
$\sigma_{\text{ICLS}}^2, \sigma_{\text{ICLS}}$	true variance and standard error of $\hat{\eta}_{\text{ICLS}}$
$\hat{\sigma}_{\text{ICLS}}^2, \hat{\sigma}_{\text{ICLS}}$	estimated variance and standard error of $\hat{\eta}_{\text{ICLS}}$
B	number of bootstrap samples
b	index $b = 1, \dots, B$
\mathbf{b}_b	a bootstrap sample ($n \times 1$ vector)
\mathbf{b}_b^*	a bootstrap sample, multivariate
$\tilde{\mathbf{b}}_b$	a bootstrap sample, with sampled residuals
sd_B	standard deviation of B bootstrap samples
S	number of simulation samples
a	index $a = 1, \dots, S$
\mathbf{s}^*	a simulation sample ($n \times 1$ vector)
κ, p	parameters binomial distribution
GCV	generalized cross-validation statistic
GCV_{FNM}	GCV using inequality constrained least squares estimation

