Invariant Hilbert subspaces of the oscillator representation

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belonging to the thesis

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1. Let G be equal to $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{O}(1,1)$ and let $\omega_{1,1}$ be the oscillator representation of G on $L^2(\mathbb{R}^2)$. The explicit decomposition of this representation was given by B. Ørsted and G. Zhang in [1]. They also obtained that the oscillator representation of G on $L^2(\mathbb{R}^2)$ decomposes multiplicity free. This result can be extended to every $\omega_{1,1}(G)$-invariant Hilbert subspace of $\mathcal{S}'(\mathbb{R}^2)$.

   See Chapter 7 of this thesis

2. Let $\pi_{\delta,s}$ be the representations of $\mathrm{SO}(n,\mathbb{C})$ with $n \geq 3$ induced by a maximal parabolic subgroup of $\mathrm{SO}(n,\mathbb{C})$. Let us consider $s \in \mathbb{C}$ and $\delta \in \mathbb{Z}$.

   a) If $n = 3, 4, 5$ and $\delta < 0$, $\pi_{\delta,s}$ is irreducible for s real and $s \in [\delta, -\delta]$.

   b) If $n > 4$, even and $n + \delta < 6$, $\pi_{\delta,s}$ is irreducible for s real and $s \in (n + \delta - 6, -n - \delta + 6)$.

   c) If $n > 5$, odd.

      c1) If $n + \delta \geq 6$ and $\delta < 0$ $\pi_{\delta,s}$ is irreducible for s real, $s \in [\delta, -\delta]$ and $s - \delta \notin \mathbb{Z}_{odd}$.

      c2) If $n+\delta < 6$ and $\delta > 0$ $\pi_{\delta,s}$ is irreducible for s real, $s \in (n+\delta-6, -n-\delta+6)$ and $s - \delta \notin \mathbb{Z}_{even}$.

      c3) If $n + \delta < 6$ and $\delta < 0$ $\pi_{\delta,s}$ is irreducible for s real and $s \in (n+\delta-6, -n-\delta+6)$ or for s real, $s \in [\delta, n+\delta-6] \cup [-n-\delta+6, -\delta]$ and $s - \delta \notin \mathbb{Z}_{odd}$.

   See Appendix B of this thesis

3. Let G be the group $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(2n)$ with $n > 1$ and let ω_{2n} be the oscillator representation of G on $L^2(\mathbb{R}^{2n})$. Any minimal $\omega_{2n}(G)$-invariant Hilbert subspace of $\mathcal{S}'(\mathbb{R}^{2n})$ occurs in the decomposition of $L^2(\mathbb{R}^{2n})$.

   See Chapter 6 of this thesis

4. Let $f \in \mathcal{D}\left(\mathrm{O}(n,\mathbb{C})/\mathrm{O}(n-1,\mathbb{C})\right)$. Then it follows
$$||f||^2 = C\sum_{\delta\in\mathbb{Z}}\int_{\mathbb{R}}\frac{1}{|c(\delta,is)|^2}||\mathcal{F}_{\delta,is}f||^2 ds$$
with C a positive constant, $\mathcal{F}_{\delta,is}f$ the Fourier transform of f and
$$c(\delta,is) = 2^{\rho-1}\frac{\Gamma(\frac{n}{2})\Gamma(\frac{1+\rho}{2})\Gamma(\frac{-is+|\delta|-\rho+2}{2})\Gamma(\frac{-is+|\delta|}{2})}{\sqrt{\pi}\Gamma(\frac{-is+|\delta|-\rho+n}{2})\Gamma(\frac{-is+|\delta|+\rho}{2})}.$$
See Chapter 8 of this thesis

5. Let G be the group $\mathrm{U}(1,1)\times\mathrm{U}(n)$ with $n\geq 1$ and let ω_n be the oscillator representation of G on $L^2(\mathbb{C}^n)$. Any $\omega_n(G)$-invariant Hilbert subspace of $\mathcal{S}'(\mathbb{C}^n)$ decomposes multiplicity free into minimal invariant Hilbert subspaces of $\mathcal{S}'(\mathbb{C}^n)$.

6. The pairs $(\mathrm{SL}(n,\mathbb{C}),\mathrm{GL}(n-1,\mathbb{C}))$ and $(Sp(n,\mathbb{C}),Sp(n-1,\mathbb{C})\times Sp(1,\mathbb{C}))$ with $n\geq 3$ are generalized Gelfand pairs.

7. Let us consider the space $X = \mathrm{SO}(n,\mathbb{C})/\mathrm{SO}(n-1,\mathbb{C})$, the function $Q(x) = x_1$ on X and the holomorphic differential operator $\square$ on X associated with the Casimir operator. If F is $\mathcal{C}^2$-function on $\mathbb{C}$, then
$$\square(F\circ Q) = LF\circ Q$$
where L is the second order differential operator on $\mathbb{C}$ given by
$$L = a(z)\frac{d^2}{dz^2} + b(z)\frac{d}{dz}$$
with $a(z) = z^2-1$ and $b(z) = (n-1)z$.

8. In the case of $\mathrm{SL}(2,\mathbb{R})\times\mathrm{O}(2n)$ and $\mathrm{U}(1,1)\times\mathrm{U}(n)$ with $n\geq 1$ any minimal Hilbert subspace of the space of tempered distributions invariant under the oscillator representation, occurs in the Plancherel formula.

   Conjecture: In the case of $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SO}(n,\mathbb{C})$ with $n\geq 3$ and odd there are minimal invariant Hilbert subspaces which do not occur in the Plancherel formula.

9. The Spanish occupation in Holland was terrible. But it had at least a good consequence, the foundation of the University of Leiden.

References

[1] B. Ørsted and G. Zhang. L^2-versions of the Howe correspondence. I. *Math. Scand.*, 80(1):125–160, 1997.