Invariant Hilbert subspaces of the oscillator representation

Aparicio, S.

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promotor:	Prof. dr. G. van Dijk
referent:	Dr. E. P. van den Ban (Universiteit Utrecht)
overige leden:	Prof. dr. E. G. F. Thomas (Rijksuniversiteit Groningen) Dr. M. F. E. de Jeu Prof. dr. M. N. Spijker Prof. dr. S. M. Verduyn Lunel

A mi familia y a Guillermo

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Contents

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Chapter 1

Introduction

The metaplectic representation -also called the oscillator representation, harmonic representation, or Segal-Shale-Weil representation- is a double-valued unitary representation of the symplectic group $Sp(n, \mathbb{R})$ (or, if one prefers, a unitary representation of the double cover of $Sp(n, \mathbb{R})$) on $L^2(\mathbb{R}^n)$. It appears implicitly in a number of contexts going back at least as far as Fresnel's work on optics around 1820. However, it was first rigorously constructed on the Lie algebra level -as a representation of $sp(n, \mathbb{R})$ by essentially skew-adjoint operators on a common invariant domain- by van Hove in 1950, and on the group level by Segal and Shale a decade later. These authors were motivated by quantum mechanics. At about the same time, Weil developed analogues of the metaplectic representation over arbitrary local fields, with a view to applications in number theory. Since then the metaplectic representation has attracted the attention of many people.

In this thesis we study the decomposition of the oscillator representation for some subgroups of the symplectic group. By oscillator representation of these subgroups we mean the restriction of the metaplectic representation of the symplectic group to these subgroups. The main results of this thesis are the Plancherel formula of these representations and the multiplicity free decomposition of every invariant Hilbert subspace of the space of tempered distributions.

In Section 1.1 we introduce the common concepts required for a proper understanding of this work. In Section 1.2 we explain the connection between representation theory and quantum mechanics. In Section 1.3 we provide a brief description of each chapter.

1.1 Some basic definitions

Let F be either the field $\mathbb{R}$ of real numbers, or the field $\mathbb{C}$ of complex numbers. A **Lie group** G over F is a group G endowed with an analytic structure such that the group operations $g \to g^{-1}$ and $(g_1, g_2) \to g_1 g_2$ are analytic operations.

In particular, a Lie group G over F is a locally compact group, and admits a positive left-invariant measure dg, which is unique up to a positive scalar, i.e.

$$\int_G f(xg)dg = \int_G f(g)dg,$$

for all $x \in G$ and all continuous complex valued functions f on G which vanish outside a compact subset. Such a measure is called a **Haar measure** of G. From here on we shall assume that G is **unimodular**, i.e. the measure dg is both left- and right-invariant. Let H be a closed unimodular subgroup of G, and let X denote the associated quotient space G/H. The group G acts on X in a natural way. It is well known that, as G and H are unimodular, there exists a positive G-invariant measure dx on X, which is unique up to a positive scalar.

Let G denote a Lie group over F, and V a complex topological vector space. By a **representation** π of G on V, which will often be denoted by (π, V) we shall mean a homomorphism from G into the group $GL(V)$ of invertible continuous endomorphisms of V such that for each $v \in V$ the map

$$g \longmapsto \pi(g)v, \quad g \in G,$$

is continuous on G.

A subspace W of V is called **invariant** under π if $\pi(g)W \subset W$ for each $g \in G$.

The representation (π, V) is called **irreducible** if the only closed invariant subspaces of V are the trivial ones: (0) and V itself.

Two representations (π, V) and (π', V') are called **equivalent** if there is a continuous linear isomorphism $A : V \to V'$ such that

$$A\pi(g) = \pi'(g)A, \quad g \in G.$$

Let $\mathcal{H}$ denote a Hilbert space. A representation $(\pi, \mathcal{H})$ of G is called **unitary** if each endomorphism $\pi(g)$, $g \in G$, of $\mathcal{H}$ is unitary, i.e. $\pi(g)$ is surjective and preserves the Hilbert norm.

By Schur's lemma the representation π of G on $\mathcal{H}$ is irreducible if and only if the only bounded linear operators A on $\mathcal{H}$, for which $A\pi(g) = \pi(g)A$ for all $g \in G$, are of the form $A = \lambda I$ for some $\lambda \in \mathbb{C}$.

One of the main goals of harmonic analysis on the space X is to find a decomposition of the Hilbert space $L^2(X, dx)$ into minimal subspaces which are invariant under the left regular action λ of G, this action being defined by

$$\lambda(g)f(x) = f(g^{-1}x), \;\; x \in X, \; f \in L^2(X, dx), \; g \in G.$$

This decomposition is known as the **Plancherel formula** of X.

The continuous part that occurs in the Plancherel formula of X is called the **principal or continuous series** of X and the discrete part is called the **discrete series** of X. Additional irreducible unitary representations which play no role in the Plancherel formula of X are called **complementary or supplementary series** of X.

As an example, let G be the circle-group $T = \{z \in \mathbb{C} : |z| = 1\}$. Let dt denoted the Haar measure on T normalized so that $\mathrm{vol}(T) = 1$. It is known that the Hilbert space $L^2(T, dt)$ has a decomposition

$$L^2(T, dt) = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}\chi_n,$$

where for each integer $n \in \mathbb{Z}$ the function χ_n on T is defined by $\chi_n(t) = t^n$, $t \in T$. Observe that each function χ_n $(n \in \mathbb{Z})$ defines a unitary character of the group T, i.e. a homomorphism from the group T into T itsef, the range being considered as a subgroup of the multiplicative group $\mathbb{C}^*$. Each function $f \in L^2(T, dt)$ has an expansion

$$f = \sum_{n \in \mathbb{Z}} c_n \chi_n, \tag{1.1}$$

where

$$c_n = \int_T f(t)\overline{\chi_n(t)}dt = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}d\theta, \quad n \in \mathbb{Z},$$

the sum on the right-hand side of (1.1) being convergent in the Hilbert norm of the space $L^2(T, dt)$. One recognizes the numbers c_n, $n \in \mathbb{Z}$, as the classical Fourier coefficients of the function f, and (1.1) as the inversion formula for the classical Fourier transform on the space of periodic functions on the interval $[0, 2\pi]$. The correspondig Plancherel formula, which is in this case also called the Parseval equality, is written as follows:

$$\int_T |f(t)|^2 dt = \sum_{n \in \mathbb{Z}} |c_n|^2.$$

An important source of motivation for this aspect of harmonic analysis, which can be viewed as a generalization of the classical Fourier theory, lies in this example.

1.2 Connection between representation theory and quantum mechanics

The fields of representation theory and quantum mechanics are the result of creative interaction between mathematics and physics. Ever since the foundation of both disciplines, some 75 years ago, they have continuously influenced each other, and nowadays they are well established theories of key importance for the development of other fields of research. In mathematics, unitary representation theory generalises Fourier analysis, turns out to be relevant to practically all fields of mathematics ranging from probability theory to number theory, and in addition has created a field of its own. In quantum physics, unitary representations come in whenever symmetry plays a role, from solid state physics to elementary particles and quantum field theory. As an example of its multiple applications, many of the known subatomic particles were predicted by the standard model of quantum mechanics more than 30 years before their detection.

The contribution of quantum mechanics to representation theory is two-fold: Firstly, it enriches mathematics with new constructions of representations, and secondly, it provides examples and applications of the theory developped by mathematicians.

A brief look at some key dates in the history of physics and mathematics shows that quantum mechanics and unitary representation theory were born together. In fact, this is no coincidence, as the concept of a unitary representation on a possibly infinite-dimensional Hilbert space was directly inspired by quantum mechanics. Hermann Weyl asked himself which type of infinite-dimensional topological vector space would give him the best possible generalization of the decomposition theory of the regular representation of a finite group to the compact case, and found the answer in John von Neumann's brand new concept of abstract Hilbert space. This concept, in turn, was directly inspired by the recent development of quantum mechanics, with which Weyl was thoroughly familiar.

1.3 Overview of this thesis

1.3.1 Representations of $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$

In Chapter 2 and Chapter 3, the irreducible unitary representations of $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$, respectively, are given in the 'non-compact' model. By 'non-compact' model we mean, in the case of $\mathrm{SL}(2,\mathbb{R})$, the realization of the representations on a space of functions on $\mathbb{R}$, and in the case of $\mathrm{SL}(2,\mathbb{C})$, on $\mathbb{C}$. In order to give a good analysis we require a 'compact' model. By 'compact' model we mean in Chapter 2 the realization of the representations on a space of functions on the unit circle, and in Chapter 3 on the unit sphere. The representations of $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$ are certainly well known but our treatment is a little different from the usual ones. It is also well-suited for the discussion of the oscillator representations in the next chapters.

1.3.2 The metaplectic representation

In Chapter 4 we give an overview of the construction of this representation and we examine it from several viewpoints. The contents of this chapter are adapted from [10]. First we introduce the definition of the Heisenberg group H_n. After that we define an irreducible unitary representations of H_n on $L^2(\mathbb{R}^n)$, we call this the Schrödinger representation, and then we give another realization on the Fock space, the Fock-Bargmann representation. Since the symplectic group, which is invariant under the symplectic form, acts on the Heisenberg group, we get another representation of the Heisenberg group. The theorem of Stone-von Neumann classifies all irreducible unitary representations of H_n. Applying this theorem we get our double-valued unitary representation of the symplectic group, and we call it the metaplectic representation. We consider two models for the metaplectic representation. In the Schrödinger model we can not give an explicit formula. For this purpose we use the Fock model, where we can express our representation by means of integral operators.

1.3.3 Theory of invariant Hilbert subspaces

In Chapter 5 we recall an important tool for the representation theory of Lie groups, namely the theory of Hilbert subspaces invariant under a group of automorphisms. This theory is due to L. Schwartz [27], Thomas [34] and Pestman [25]. A main subject is the study of the theory of the reproducing kernels, associated with Hilbert subspaces. We give a criterion (due to Thomas) which assures multiplicity free decomposition of representations. This criterion will be applied in the next chapters.

1.3.4 The oscillator representation of $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{O}(2n)$

In the first part of Chapter 6 we compute the Plancherel formula of the oscillator representation ω_{2n} of $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{O}(2n)$ for $n > 1$. The main tool we use is a Fourier integral operator, which was introduced by M. Kashiwara and M. Vergne, see [18]. After that we can conclude that any minimal invariant Hilbert subspace of $\mathcal{S}'(\mathbb{R}^{2n})$ occurs in the decomposition of $L^2(\mathbb{R}^{2n})$.

Finally, we study the oscillator representation ω_{2n} in the context of the theory of invariant Hilbert subspaces. The oscillator representation acts on the Hilbert space $L^2(\mathbb{R}^{2n})$. It is well-known that $\mathcal{S}(\mathbb{R}^{2n})$ is $\omega_{2n}(G)$-stable, so, by duality, ω_{2n} acts on $\mathcal{S}'(\mathbb{R}^{2n})$ as well, and $L^2(\mathbb{R}^{2n})$ can thus be considered as an invariant Hilbert subspace of $\mathcal{S}'(\mathbb{R}^{2n})$. Our main result is that any $\omega_{2n}(G)$-stable Hilbert subspace of $\mathcal{S}'(\mathbb{R}^{2n})$ decomposes multiplicity free.

The case $n = 1$ is treated in a similar way. Here a non-discrete series representation occurs in the decomposition of $L^2(\mathbb{R}^2)$.

1.3.5 The oscillator representation of $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{O}(\mathrm{p}, \mathrm{q})$

The explicit decomposition of the oscillator representation $\omega_{p,q}$ for the dual pair $G = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{O}(\mathrm{p}, \mathrm{q})$ was given by B. Ørsted and G. Zhang in [23]. In Chapter 7 we only study the multiplicity free decomposition of any $\omega_{p,q}(G)$-stable Hilbert subspace of $\mathcal{S}'(\mathbb{R}^{p+q})$. The oscillator representation acts on the Hilbert space $L^2(\mathbb{R}^{p+q})$. It is well-known that $\mathcal{S}(\mathbb{R}^{p+q})$, the space of Schwartz functions on $\mathbb{R}^{p+q}$, is $\omega_{p,q}(G)$-stable. Thus, by duality it follows that $\omega_{p,q}$ acts on $\mathcal{S}'(\mathbb{R}^{p+q})$, the space of tempered distributions on $\mathbb{R}^{p+q}$, as well, and $L^2(\mathbb{R}^{p+q})$ can thus be considered as an invariant Hilbert subspace of $\mathcal{S}'(\mathbb{R}^{p+q})$.

According to Howe [16], $L^2(\mathbb{R}^{p+q})$ decomposes multiplicity free into minimal invariant Hilbert subspaces of $\mathcal{S}'(\mathbb{R}^{p+q})$. This is a special case of our result, which states that any $\omega_{p,q}(G)$-stable Hilbert subspace of $\mathcal{S}'(\mathbb{R}^{p+q})$ decomposes multiplicity free.

We restrict to the case $p + q$ even for simplicity of the presentation of the main results. In addition we have to assume $p \geq 1$, $q \geq 2$, since we apply results from [9] where this condition is imposed. Our result is however true in general.

The contents of this chapter have appeared in [39].

1.3.6 The oscillator representation of $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SO}(n,\mathbb{C})$

In Chapter 8 we determine the explicit decomposition of the oscillator representation for the groups $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{O}(1, \mathbb{C})$ and $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SO}(n, \mathbb{C})$ with $n \geq 2$. For this, we again make use of a Fourier integral operator introduced by M. Kashiwara and M. Vergne for real matrix groups, see [18].

For the group $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{O}(1, \mathbb{C})$ we construct two intertwining operators. One from the space of even Schwartz functions to the space of functions where the complementary series of $\mathrm{SL}(2, \mathbb{C})$ are defined, and the other one, from the space of odd Schwartz functions to $L^2(\mathbb{C})$. So in this case a complementary series of $\mathrm{SL}(2,\mathbb{C})$ occurs in the decomposition of $L^2(\mathbb{C})$.

For the group $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SO}(n,\mathbb{C})$ with $n \geq 2$, given that $\mathrm{SO}(2,\mathbb{C})$ is an abelian group, we split the proof into the cases $n = 2$ and $n \geq 3$. The computation of the Plancherel measure for $\mathrm{SO}(n,\mathbb{C})/\mathrm{SO}(n-1,\mathbb{C})$ is required in order to compute the explicit decomposition for the case $n \geq 3$. For this purpose we follow the approach of Van den Ban [36]. However, the computation of the Plancherel measure is not so obvious, as it involves a good deal of non-trivial steps.

In Chapter 9 we study the oscillator representation ω_n for the groups $\mathrm{SL}(2,\mathbb{C})\times \mathrm{O}(1,\mathbb{C})$, $\mathrm{SL}(2,\mathbb{C})\times\mathrm{O}(2,\mathbb{C})$ and $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SO}(n,\mathbb{C})$ with $n \geq 3$ in the context of the theory invariant Hilbert subspaces. Our main result is that any $\omega_n(G)$-stable Hilbert subspace of $\mathcal{S}'(\mathbb{C}^n)$ decomposes multiplicity free.

In case $n = 1$ is easy to prove that every invariant Hilbert subspace of $\mathcal{S}'(\mathbb{C})$ decomposes multiplicity free. We leave to the reader to check this result using techniques similar to those applied for other cases.

The case $n = 2$ is slightly different from the other cases. The cone Ξ consists in two disjoint pieces and we have to study the $\mathrm{O}(2,\mathbb{C})$-invariant distributions on $\Xi\times\Xi$ for each of the four components.

For the case $n \geq 3$ we first need to prove that $(\mathrm{SO}(n,\mathbb{C}),\mathrm{SO}(n-1,\mathbb{C}))$ are generalized Gelfand pairs, and to compute the MN-invariant distributions on the cone.

To prove that $(\mathrm{SO}(n,\mathbb{C}),\mathrm{SO}(n-1,\mathbb{C}))$ are generalized Gelfand pairs, for $n \geq 2$ we first introduce a brief resume of the theory of generalized Gelfand pairs, which is connected with the theory of invariant Hilbert subspaces presented in Chapter 5. We also introduce a criterion that was given by Thomas to determine generalized Gelfand pairs. We will use this criterion for our own aim.

To compute the MN-invariant distributions on the cone for $n \geq 3$ we follow the same method as in [9]. Since every distribution T on the cone invariant under MN can be written as $T = \mathcal{M}'S + T_1$, where S is a continuous linear form on $\mathcal{J}$, $\mathcal{M}$ is the average map and T_1 is a singular MN-invariant distribution, see Section 9.2.2 for definitions, we need to compute the singular MN-invariant distributions on the cone. To do this we have to split in two cases $n = 3$ and $n > 3$, since for $n = 3$ the group M is equal to the identity.

1.3.7 Additional results

At the end of this thesis we include two appendixes, A and B.

In Appendix A we compute the conical distributions associated with the orthogonal complex group $\mathrm{SO}(n,\mathbb{C})$ with $n \geq 3$. The group $\mathrm{SO}(n,\mathbb{C})$ acts transitively on the isotropic cone of the quadratic form associated with it. The action of $\mathrm{SO}(n,\mathbb{C})$ in the space of homogeneous functions on this cone defines a family of representations of $\mathrm{SO}(n,\mathbb{C})$. Their study leads to the conical distributions. We follow the same method as in [9].

In Appendix B we study the irreducibility and unitarity of the representations of $\mathrm{SO}(n,\mathbb{C})$ with $n \geq 3$ induced by a maximal parabolic subgroup and defined in Section 8.3.1 in more detail. We follow the same method as in [41].

CHAPTER 2

Representations of SL(2, $\mathbb{R}$)

In this chapter we give the irreducible unitary representations of SL(2, $\mathbb{R}$) in the 'non-compact' model. By 'non-compact' model we mean the realization of the representations on a space of functions on $\mathbb{R}$. In order to give a good analysis we require a 'compact' model. By 'compact' model we mean the realization of the representations on a space of functions on the unit circle S. The representations of SL(2, $\mathbb{R}$) are well known, of course, but our treatment is a little different from the usual ones. It is also well-suited for the discussion of the oscillator representations in the next chapters.

2.1 The principal (non-unitary) series

2.1.1 A 'non-compact' model

Set

$$P = MAN = \left\{ \begin{pmatrix} t & 0 \\ x & t^{-1} \end{pmatrix} : t \in \mathbb{R}^*, x \in \mathbb{R} \right\}$$

where $M = \{\pm I\}$,

$$A = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t > 0 \right\} \text{ and } N = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} : x \in \mathbb{R} \right\}.$$

Put $\bar{N} = \left\{ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} : y \in \mathbb{R} \right\}$ and $K = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : 0 \le \theta < 2\pi \right\}$. Then $\bar{N}P$ is open, dense in $G = \mathrm{SL}(2, \mathbb{R})$ and its complement has Haar measure zero. Any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in G can be written in the form $\bar{n}p = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ x & t^{-1} \end{pmatrix}$ with $t = 1/d, y = b/d$ and $x = c$, provided $d \neq 0$.
The principal series $\pi_{\lambda,\eta}$ ($\lambda \in \mathbb{C}, \eta \in \hat{M}$) acts on the space V of $\mathcal{C}^\infty$-functions f with

$$f(gma_t n) = t^{\lambda+1}\eta(m)f(g)$$

with inner product

$$\int_K |f(k)|^2 dk = ||f||_2^2.$$

$\pi_{\lambda,\eta}$ is given by

$$\pi_{\lambda,\eta}(g_0)f(g) = f(g_0^{-1}g).$$

For this inner product, $\pi_{\lambda,\eta}(g_0)$ is a bounded transformation and the representation $\pi_{\lambda,\eta}$ is continuous.
Identifying V with a space of functions on $\bar{N} \simeq \mathbb{R}$, we can write $\pi_{\lambda,\eta}(g_0)$ in these terms. The inner product becomes

$$||f||_2^2 = \int_{-\infty}^{\infty} |f(y)|^2 (1+y^2)^{\mathrm{Re}\ \lambda} dy.$$

If $g_0^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then,

$$g_0^{-1} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & ay+b \\ c & cy+d \end{pmatrix} = \begin{pmatrix} 1 & y' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ x & t^{-1} \end{pmatrix}$$

with $y' = \frac{ay+b}{cy+d}$ and $t = (cy+d)^{-1}$, so that

$$\pi_{\lambda,\eta}(g_0)f(y) = |cy+d|^{-(\lambda+1)} \left(\frac{cy+d}{|cy+d|} \right)^{\frac{1}{2}(1-\eta)} f\left(\frac{ay+b}{cy+d} \right).$$

The representations $\pi_{\lambda,\eta}$ are unitary for λ imaginary. The converse is also true, see [32].

2.1.2 A 'compact' model

This model enables us more easily to answer questions about irreducibility, equivalence, unitarity, etc.

$G = \mathrm{SL}(2, \mathbb{R})$ acts on the unit circle $S = \{(s_1, s_2) \in \mathbb{R}^2 : s_1^2 + s_2^2 = 1\}$ by

$$g \cdot s = \frac{g(s)}{||g(s)||}.$$

The stabilizer of $e_2 = (0,1)$ is AN. So $\pi_{\lambda,\eta}$ can be realized (depending on η) on V_η, the space of $\mathcal{C}^\infty$-functions φ with

$$\varphi(m \cdot s) = \eta^{-1}(m)\varphi(s)$$

with $m \in M$, $s \in S$ and

$$\pi_{\lambda,\eta}(g)\varphi(s) = \varphi(g^{-1} \cdot s)||g^{-1}(s)||^{-(\lambda+1)}.$$

[If $g^{-1}k = k'a_t n$, then $g^{-1} \cdot s = k' \cdot e_2$ and $||g^{-1}(s)|| = t^{-1}$]. We shall also write $\pi_{\lambda,+}$ if $\eta \equiv 1$ and $\pi_{\lambda,-}$ if $\eta(\pm I) = \pm 1$.

Observe that

$$\pi_{\lambda,\eta}|_K \simeq \underset{M\uparrow K}{\operatorname{ind}} \eta.$$

V_+ is spanned by the functions $(s_1 + is_2)^l$ with l even, V_- by the same functions with l odd. Let (φ, ψ) be the usual inner product on $L^2(S)$

$$(\varphi, \psi) = \int_S \varphi(s)\overline{\psi(s)}ds \tag{2.1}$$

where ds is the normalized measure on S. The measure ds is transformed by $g \in G$ as follows: $d\tilde{s} = ||g(s)||^{-2}ds$ if $\tilde{s} = g \cdot s$. It implies that the inner product (φ, ψ) is invariant with respect to $(\pi_{\lambda,\eta}, \pi_{-\bar{\lambda},\eta})$, so that $\pi_{\lambda,\eta}$ is unitary for $\lambda \in i\mathbb{R}$.

Now we want to study for $\pi_{\lambda,\eta}$ the following questions: irreducibility, composition series, intertwining operators, unitarity.

2.2 Irreducibility

Since G is generated by K and the subgroup $A = \{\exp(tZ_0)\}$ with $Z_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, in order to study the irreducibility of $\pi_{\lambda,\eta}$ we have to know how the latter subgroup transforms the functions $\psi_l(s) = (s_1 + is_2)^l$, $l \in \mathbb{Z}$. An easy computation shows:

$$\pi_{\lambda,\eta}(Z_0)\psi_l = \frac{1}{2}(\lambda + 1 + l)\psi_{l+2} + \frac{1}{2}(\lambda + 1 - l)\psi_{l-2}. \tag{2.2}$$

This immediately leads to a complete analysis of the reducibility properties of the representations $\pi_{\lambda,\pm}$. The results are as follows.

Theorem 2.1.

a) *If $\lambda \notin \mathbb{Z}$, $\pi_{\lambda,+}$ and $\pi_{\lambda,-}$ are irreducible.*

b) *If λ is an even integer, $\pi_{\lambda,+}$ is irreducible while $\pi_{\lambda,-}$ is not irreducible. For $\pi_{\lambda,-}$ the decomposition is as follows:*

  (i) *$\lambda = 0$. In this case, $V_-^{(1)}$, spanned by $\psi_{-1}, \psi_{-3}, \ldots$, and $V_-^{(2)}$, spanned by $\psi_1, \psi_3, \ldots$, are invariant. No other closed invariant subspaces of V_- exist.*

  (ii) *$\lambda = 2k$, $k \geq 1$. In this case, $V_-^{(1)}$, spanned by $\psi_{-2k-1}, \psi_{-2k-3}, \ldots$, and $V_-^{(2)}$, spanned by $\psi_{2k+1}, \psi_{2k+3}, \ldots$, are invariant. $V_-^{(1)}$, $V_-^{(2)}$ and $V_-^{(1)} \oplus V_-^{(2)}$ are the only proper closed invariant subspaces. $V_-/V_-^{(1)} \oplus V_-^{(2)}$ is finite-dimensional and defines the irreducible representation with highest weight 2k-1.*

  (iii) *$\lambda = -2k$, $k \geq 1$. In this case, $V_-^{(1)}$, spanned by $\psi_{2k-1}, \psi_{2k-3}, \ldots$, and $V_-^{(2)}$, spanned by $\psi_{-(2k-1)}, \psi_{-(2k-3)}, \ldots$, are invariant; and these, along* *with* $V_-^{(1)} \cap V_-^{(2)}$ *are all the proper closed invariant subspaces.* $V_-^{(1)} \cap V_-^{(2)}$ *is finite-dimensional and defines the irreducible representation with highest weight 2k-1.*

c) *If* λ *is an odd integer,* $\pi_{\lambda,-}$ *is irreducible while* $\pi_{\lambda,+}$ *is reducible. For* $\pi_{\lambda,+}$ *the splitting is as follows:*

(i) $\lambda = 2k+1$, $k \geq 0$. $V_+^{(1)}$, *spanned by* $\psi_{-2k-2}, \psi_{-2k-4}, \ldots$, *and* $V_+^{(2)}$, *spanned by* $\psi_{2k+2}, \psi_{2k+4}, \ldots$, *are invariant. These and their direct sum are the only proper closed invariant subspaces.* $V_+/V_+^{(1)} \oplus V_+^{(2)}$ *is finite-dimensional and defines the irreducible representation with highest weight 2k.*

(ii) $\lambda = -2k-1$, $k \geq 0$. $V_+^{(1)}$, *spanned by* $\psi_{2k}, \psi_{2k-2}, \ldots$, *and* $V_+^{(2)}$, *spanned by* $\psi_{-2k}, \psi_{-2k+2}, \ldots$, *are invariant; these, together with their intersection, exhaust all proper closed invariant subspaces.* $V_+^{(1)} \cap V_+^{(2)}$ *is finite-dimensional and defines the irreducible representation of highest weight 2k.*

For the *reducible* case, the restriction of $\pi_{\lambda,\pm}$ to $V_\pm^{(1)}$ and $V_\pm^{(2)}$ are denoted by $\pi_{\lambda,\pm}^{(1)}$ and $\pi_{\lambda,\pm}^{(2)}$ respectively. Furthermore, $V_\pm^{(1)}$ and $V_\pm^{(2)}$ depend on λ. We shall occasionally write therefore $V_{\lambda,\pm}^{(1)}$ and $V_{\lambda,\pm}^{(2)}$.

2.3 Intertwining operators

Now we want to find (non zero) continuous linear operators $A : V_\eta \longrightarrow V_{\eta_1}$ intertwining the representations $\pi_{\lambda,\eta}$ and π_{λ_1,η_1} (and their subrepresentations and subquotiens), i.e.

$$A\pi_{\lambda,\eta}(g) = \pi_{\lambda_1,\eta_1}(g)A \qquad (g \in G).$$

Theorem 2.2. *A non-zero non-trivial intertwining operator as above exists if and only if* $\eta = \eta_1$, $\lambda_1 = -\lambda$. *Such as operator is unique up to a factor. For the reducible case, this operator vanishes on* $V_{\lambda,\pm}^{(1)} \oplus V_{\lambda,\pm}^{(2)}$ ($\lambda = 2k$ *or* $\lambda = 2k+1$) *and gives rise to an isomorphism of* $V_\pm/V_{\lambda,\pm}^{(1)} \oplus V_{\lambda,\pm}^{(2)}$ *onto* $V_{-\lambda,\pm}^{(1)} \cap V_{-\lambda,\pm}^{(2)}$. *Restricting* A *to* $V_{2k,-}^{(1)}$ (*or* $V_{2k,-}^{(2)}$) *gives an isomorphism onto* $V_-/V_{-2k,-}^{(2)}$ (*or* $V_-/V_{-2k,-}^{(1)}$). *Similarly for* $\lambda = 2k+1$:

$$V_{2k+1,+}^{(1)} \simeq V_+/V_{-2k-1,+}^{(2)} \quad \textit{and} \quad V_{2k+1,+}^{(2)} \simeq V_+/V_{-2k-1,+}^{(1)}.$$

Also the 'dual' isomorphisms are true:

$$\begin{aligned} V^{(1)}_{-2k,-} &\simeq V_-/V^{(2)}_{2k,-}, \\ V^{(2)}_{-2k,-} &\simeq V_-/V^{(1)}_{2k,-}, \\ V^{(1)}_{-2k-1,+} &\simeq V_+/V^{(2)}_{2k+1,+}, \\ V^{(2)}_{-2k-1,+} &\simeq V_+/V^{(1)}_{2k+1,+}. \end{aligned}$$

Proof. Restricting to K we obtain $\eta = \eta_1$ and since $\pi_{\lambda,\eta}|_K$ is multiplicity free, ψ_l is an eigenvector of A with eigenvalue, say a_l. These numbers depend on λ, λ_1 and η. They should satisfy the system of equations:

$$\begin{aligned} (\lambda+1+l)a_{l+2} &= (\lambda_1+1+l)a_l \\ (\lambda+1-l)a_{l-2} &= (\lambda_1+1-l)a_l \end{aligned}$$

Here we applied (2.2). Combining these equations gives

$$(\lambda_1+1+l)(\lambda_1-1-l) = (\lambda+1+l)(\lambda-1-l),$$

so $\lambda_1 = \pm\lambda$. If $\lambda = \lambda_1$, then all a_l coincide, so A is an scalar operator. In the second case ($\lambda = -\lambda_1$) we get

$$(\lambda+1+l)a_{l+2} = (-\lambda+1+l)a_l. \tag{2.3}$$

For the irreducible case, equation (2.3) has a, up to a factor, unique solution, which can be written in one of the following three forms:

$$\begin{aligned} a_l &= c_1 \frac{(-1)^{l/2}}{\Gamma(\frac{\lambda+1+l}{2})\Gamma(\frac{\lambda+1-l}{2})} & (2.4) \\ &= c_2(-1)^{l/2}\Gamma(\frac{-\lambda+1+l}{2})\Gamma(\frac{-\lambda+1-l}{2}) & (2.5) \\ &= c_3 \frac{\Gamma(\frac{-\lambda+1-l}{2})}{\Gamma(\frac{\lambda+1-l}{2})}. & (2.6) \end{aligned}$$

In the reducible cases, there is also a unique solution, but one has to start at another base value, e.g. $l = -2k-1$ for $V^{(1)}_{2k,-}$. This is proven in the same way.

The formulae obtained for the solutions of (2.3) show, when the solution is defined on an unbounded set, that it is of polynomial growth at infinity, see [6]:

$$a_l \sim const.\ |l|^{-\mathrm{Re}\ \lambda} \qquad (|l| \to \infty).$$

This implies that the operator A having a_l as eigenvalues is continuous in the $\mathcal{C}^\infty$-topology of V_η.□

Let us produce the intertwining operator in integral form. Define the operator $A_{\lambda,\nu}$ on V_η by the formula

$$A_{\lambda,\nu}\varphi(s) = \int_S [s,t]^{\lambda-1,\nu}\varphi(t)dt$$

where $\nu = 0$ if $\eta = 1$, $\nu = 1$ if $\eta = -1$ and $u^{\lambda,\nu} = |u|^\lambda \left(\frac{u}{|u|}\right)^\nu$ if $u \in \mathbb{R}$, $u \neq 0$. Furthermore, $[s,t] = s_1 t_2 - s_2 t_1$ if $s = (s_1, s_2)$, $t = (t_1, t_2)$. Observe that $[g(s), g(t)] = [s,t]$ for all $g \in G$. This integral converges for Re $\lambda > 0$ and can be extended analytically on the whole complex λ-plane to a meromorphic function. Clearly $A_{\lambda,\nu}$ carries V_η into itself.
It is easily checked that $A_{\lambda,\nu}$ is an intertwining operator:

$$A_{\lambda,\nu}\pi_{\lambda,\eta}(g) = \pi_{-\lambda,\eta}(g)A_{\lambda,\nu} \qquad (g \in G).$$

For the eigenvalues $a_l(\lambda,\nu)$ we have an explicit expression:

$$a_l(\lambda,\nu) = 2^{-\lambda+1}\frac{\Gamma(\lambda)e^{il\frac{\pi}{2}}}{\Gamma(\frac{\lambda+1+l}{2})\Gamma(\frac{\lambda+1-l}{2})}.$$

This formula is proven in the following way. We start from:

$$A_{\lambda,\nu}\psi_l(s) = a_l(\lambda,\nu)\psi_l(s).$$

Taking $s = e_1$, we obtain:

$$a_l(\lambda,\nu) = \frac{1}{2\pi}\int_0^{2\pi} (\sin\theta)^{\lambda-1,\nu} e^{il\theta} d\theta.$$

This last integral is computed with the help of [12].

As a function of λ the operator $A_{\lambda,\nu}$ has poles of the first order at $\lambda \in -2\mathbb{N}$ $(\nu = 0)$ and $\lambda \in -1 - 2\mathbb{N}$ $(\nu = 1)$, so $\lambda \in -\nu - 2\mathbb{N}$. Let $\pi_{\lambda,\eta}$ be reducible, $\lambda \neq 0$; If $\lambda > 0$, so $\lambda = \lambda_0 = 2k$ $(\nu = 1)$ or $\lambda_0 = 2k + 1$ $(\nu = 0)$, then the operator $A_{\lambda_0,\nu}$ has not a pole at these points. Moreover $A_{\lambda_0,\nu}$ vanishes on the irreducible subspaces. On each of the irreducible subspaces V it has a zero of the first order. Its derivative $\frac{\partial A_{\lambda,\nu}}{\partial\lambda} |_{\lambda=\lambda_0}$ intertwines the restriction of $\pi_{\lambda_0,\eta}$ to V and the factor representation $\pi_{-\lambda_0,\eta}$ on $V^* = V_\eta/V^\perp$.

With the Hermitian form (2.1) the operator $A_{\lambda,\nu}$ interacts as follows:

$$(A_{\lambda,\nu}\varphi, \psi) = (\varphi, A_{\bar{\lambda},\nu}\psi).$$

2.4 Invariant Hermitian forms and unitarity

In this section we determine all invariant Hermitian forms on V_η and its subfactors with respect to the representations $\pi_{\lambda,\eta}$ and determine which of these forms are positive (negative) definite, so that the corresponding representations are unitarizable. A continuous Hermitian form $H(\varphi,\psi)$ on V_η is called *invariant* with respect to $\pi_{\lambda,\eta}$ if

$$H(\pi_{\lambda,\eta}(g)\varphi, \psi) = H(\varphi, \pi_{\lambda,\eta}(g^{-1})\psi)$$

for all $g \in G$. Any such a form can be written in the form

$$H(\varphi, \psi) = (A\varphi, \psi)$$

with A an operator on V_η and the inner product (2.1) on the right-hand side. It implies that A intertwines $\pi_{\lambda,\eta}$ and $\pi_{-\bar{\lambda},\eta}$. So by Theorem 2.2, there are two possibilities: $\lambda = -\bar{\lambda}$ and $\lambda = \bar{\lambda}$. In the first case, we have Re $\lambda = 0$ and, provided $\lambda \neq 0$, the operator A is equal to cE, so that H is c times (2.1). In the second case, A intertwines $\pi_{\lambda,\eta}$ and $\pi_{-\lambda,\eta}$. The case when H is defined on a subfactor is treated in a similar way. So we get:

Theorem 2.3. *A non-zero invariant Hermitian form $H(\varphi, \psi)$ on V_η exists only if:* *(a) Re $\lambda = 0$, or (b) Im $\lambda = 0$. In case (a) the form is proportional to the L^2-inner product* (2.1)*. In case (b) the form $H(\varphi, \psi)$ on V_η has the form*

$$H(\varphi, \psi) = (A\varphi, \psi)$$

where A is an intertwining operator between $\pi_{\lambda,\eta}$ and $\pi_{-\lambda,\eta}$. On an irreducible subfactor V/W the form H looks the same with A an operator $V \to W^\perp/V^\perp$ vanishing on W and intertwining the subfactors of $\pi_{\lambda,\eta}$ on V/W and of $\pi_{-\lambda,\eta}$ on $W^\perp/V^\perp$.

In particular, the Hermitian form

$$(A_{\lambda,\nu}\varphi, \psi) \tag{2.7}$$

with $\lambda \in \mathbb{R}$, where $A_{\lambda,\nu}$ is the operator, defined in Section 2.3, defined on V_η and invariant with respect to $\pi_{\lambda,\nu}$. At singular points one has to take residues of (2.7). If $\pi_{\lambda,\nu}$ is reducible, then (2.7) vanishes on each irreducible invariant subspace V. Its derivative with respect to λ on the subspace V is an invariant Hermitian form on V. In this way we obtain Hermitian forms on all irreducible subfactors.

Now we determine when the Hermitian forms above are positive or negative definite.

Theorem 2.4. *The unitarizable irreducible representations $\pi_{\lambda,\nu}$ or their irreducible subfactors belong to the following series.*

(i) $\pi_{\lambda,\nu}$ with $\lambda \neq 0$, Re $\lambda = 0$ and $\pi_{0,+}$: the continuous series.

(ii) $\pi_{0,-}^{(1)}$ and $\pi_{0,-}^{(2)}$.

(iii) The complementary series consisting of the representations $\pi_{\lambda,+}$ with $0 < \lambda < 1$.

(iv) The trivial representation, acting on $V_{-1,+}^{(1)} \cap V_{-1,+}^{(2)}$ and also on the factor space $V_{1,+}/V_{1,+}^{(1)} \oplus V_{1,+}^{(2)}$.

(v) The analytic discrete series, consisting of the subrepresentations $\pi_{\lambda,-}^{(1)}$ on the subspaces $V_{\lambda,-}^{(1)}$ ($\lambda = 2k$, $k = 1, 2, \dots$) and $\pi_{\lambda,+}^{(1)}$ on the subspaces $V_{\lambda,+}^{(1)}$ ($\lambda = 2k+1$, $k = 0, 1, 2, \dots$). These representations are equivalent to the factor representations $\pi_{\lambda,-}$ on $V_-/V_{\lambda,-}^{(2)}$ ($\lambda = -2k$, $k = 1, 2, \dots$) and $\pi_{\lambda,+}$ on $V_+/V_{\lambda,+}^{(2)}$ ($\lambda = -(2k+1)$, $k = 0, 1, 2, \dots$).

(vi) The anti-analytic discrete series, consisting of the subrepresentations $\pi^{(2)}_{\lambda,-}$ *on the subspaces* $V^{(2)}_{\lambda,-}$ *(*$\lambda = 2k$*,* $k = 1, 2, \dots$*) and* $\pi^{(2)}_{\lambda,+}$ *on the subspaces* $V^{(2)}_{\lambda,+}$ *(*$\lambda = 2k + 1$*,* $k = 0, 1, 2, \dots$*). These representations are equivalent to the factor representations* $\pi_{\lambda,-}$ *on* $V_-/V^{(1)}_{\lambda,-}$ *(*$\lambda = -2k$*,* $k = 1, 2, \dots$*) and* $\pi_{\lambda,+}$ *on* $V_+/V^{(1)}_{\lambda,+}$ *(*$\lambda = -(2k + 1)$*,* $k = 0, 1, 2, \dots$*).*

Proof. In the basis ψ_l an invariant Hermitian form $H(\varphi, \psi)$ has a diagonal matrix, with real scalars h_l on the diagonal. The h_l have the same expression as a_l. We have to determine when the numbers a_l have the same sign for all $l \in L$, where L is the set of weights of our representation. This leads to the representations in the theorem. For example, when $0 < \lambda < 1$ we easily see that (2.4) is positive for all l, if $c_1 > 0$, l even. For odd l this is not the case, since $a_{-l} = -a_l$. □

Let us indicate the invariant inner products for these series of representations.

For the continuous series, $\pi^{(1)}_{0,-}$ and $\pi^{(2)}_{0,-}$, the inner product is just (2.1). For the trivial representation the inner product is clear.

For the complementary series it is:

$$(A_{\lambda,\nu}\varphi, \psi).$$

For the representations $\pi^{(2)}_{\lambda_0,-}$ ($\lambda_0 = 2k$, $k = 1, 2, \dots$) and $\pi^{(2)}_{\lambda_0,+}$ ($\lambda_0 = 2k + 1$, $k = 0, 1, 2, \dots$) the inner product is:

$$\left.\frac{\partial}{\partial\lambda}\right|_{\lambda=\lambda_0} (A_{\lambda,\nu}\varphi, \psi) \tag{2.8}$$

on V_-, respectively V_+. If we take the square norm of ψ_l in the sense of (2.8) for the lowest weight $l_0 = \lambda_0 + 1$ is equal to 1, then the square norms of the other ψ_l, $l = \lambda_0 + 1 + 2m$, $m \in \mathbb{N}$ are equal to

$$\frac{m!}{(\lambda_0 + 1)^{[m]}} \tag{2.9}$$

where we used the notation $a^{[m]} = a(a + 1) \cdots (a + m - 1)$.

For the representations $\pi^{(1)}_{\lambda_0,-}$ ($\lambda_0 = 2k$, $k = 1, 2, \dots$) and $\pi^{(1)}_{\lambda_0,+}$ ($\lambda_0 = 2k + 1$, $k = 0, 1, 2, \dots$) the inner product is the same as (2.8). The normalization by 1 at the highest weight $l_0 = -\lambda_0 - 1$ gives the same formula (2.9) for all other square norms, for the weights $l = l_0 - 2m$.

Let us denote the unitary completions of the representations by the same symbols. These unitary completions exhaust all irreducible unitary representations of G up to equivalence, by Harish-Chandra's subquotient theorem [13].

2.5 The ‘non-compact’ models of the irreducible unitary representations of $\mathbf{SL(2,\mathbb{R})}$

$B_{\lambda,\eta}$ is defined by

$$B_{\lambda,\eta}\varphi(x) = \varphi\left(\frac{x+i}{|x+i|}\right)|x-i|^{-\lambda-1}.$$

Observe that the operator is an intertwining operator between the ‘compact’ model and the ‘non-compact’ model. (Here S is identified with the complex numbers of absolute value one.) So it is possible to describe the spaces $V_{\lambda,\eta}$ in the ‘non-compact’ model by the functions $B_{\lambda,\eta}\psi_l$.

2.5.1 The continuous series: $\pi_{\lambda,\pm}$, $\lambda \neq 0$, $\lambda \in i\mathbb{R}$ and $\pi_{0,+}$

We just refer to Section 2.1.1. The space is $L^2(\mathbb{R})$ with the usual inner product and

$$\pi_{\lambda,\eta}(g)f(y) = |cy+d|^{-\lambda-1}\left(\frac{cy+d}{|cy+d|}\right)^{\frac{1}{2}(1-\eta)} f\left(\frac{ay+b}{cy+d}\right) \tag{2.10}$$

if $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

The space can also be described as follows. Calling

$$\phi^k_{\lambda,\nu} = \left(\frac{x+i}{x-i}\right)^k (x-i)^{-\lambda-1,\nu},$$

$V_{\lambda,+}$ and $V_{\lambda,-}$ are spanned by $\phi^k_{\lambda,0} = B_{\lambda,+}\psi_{2k}$ and $\phi^k_{\lambda,1} = B_{\lambda,-}\psi_{2k-1}$ with $k \in \mathbb{Z}$ respectively.

2.5.2 The representation: $\pi^{(1)}_{0,-}$ and $\pi^{(2)}_{0,-}$

These representations act on the closed subspaces $V^{(1)}_{0,-}$ and $V^{(2)}_{0,-}$ of $L^2(\mathbb{R})$ by formula (2.10). As before we can see that the spaces $V^{(1)}_{0,-}$ and $V^{(2)}_{0,-}$ are spanned by the functions $\phi^{-k}_{0,1}$ with $k \in \mathbb{N}$ and $\phi^k_{0,1}$ with $k \in \mathbb{N}$, $k \neq 0$ respectively.

The representations on these spaces are given by

$$\pi^{(i)}_{0,-}(g)f(y) = (cy+d)^{-1} f\left(\frac{ay+b}{cy+d}\right)$$

for i=1,2; $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

2.5.3 The complementary series: $\pi_{\lambda,+}$ $(0 < \lambda < 1)$

The inner product is given in the ‘compact model’ by

$$(\varphi,\phi) = \int_S\int_S |[s,t]|^{\lambda-1}\varphi(t)\overline{\phi}(s)dsdt$$

($\varphi, \phi \in V_+$). The spaces are defined as in Section 2.5.1 and the representation is again given by (2.10):

$$\pi_{\lambda,+}(g)f(y) = |cy+d|^{-\lambda-1} f\left(\frac{ay+b}{cy+d}\right)$$

if $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Let us rewrite (φ, ϕ) in terms of functions on IR. This is easily seen to be:

$$(f,g)_\lambda = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x-y|^{\lambda-1} f(x)\overline{g}(y)dxdy.$$

2.5.4 The analytic discrete series

Let us define the following functions by

$$\phi_n(x) = \left(\frac{x-i}{x+i}\right)^n (x+i)^{-m}$$

where $n \in \mathbb{N}$ and $m \in \mathbb{N}$, $m \geq 2$. The spaces can be described by these functions. $V^{(1)}_{2k,-}$ and $V^{(1)}_{2k+1,+}$ are spanned by $\phi_j = B_{m-1,-}\psi_{-m-2j}$ with $m-1 = 2k$, $j \in \mathbb{N}$ and $\phi_j = B_{m-1,+}\psi_{-m-2j}$ with $m-1 = 2k+1$, $j \in \mathbb{N}$ respectively.

We call these spaces as $V^+_{\lambda_0}$ where $\lambda_0 = m-1$, $V^+_{\lambda_0} = V^{(1)}_{\lambda_0,-}$ if λ_0 is even and $V^+_{\lambda_0} = V^{(1)}_{\lambda_0,+}$ if λ_0 is odd .

The representations $\pi^{(1)}_{\lambda_0}$ with $\lambda_0 = 1, 2, 3, \ldots$, where $\pi^{(1)}_{\lambda_0} = \pi^{(1)}_{\lambda_0,-}$ if λ_0 is even and $\pi^{(1)}_{\lambda_0} = \pi^{(1)}_{\lambda_0,+}$ if λ_0 is odd, act by means of the formula

$$\pi^{(1)}_{\lambda_0}(g)f(y) = (cy+d)^{-\lambda_0-1} f\left(\frac{ay+b}{cy+d}\right) \qquad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The inner product in this model is given by

$$\left.\frac{\partial}{\partial\lambda}\right|_{\lambda=\lambda_0} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x-y|^{\lambda-1,\nu} f(x)\overline{g}(y)dxdy \qquad (*)$$

($\nu = 1$ if λ_0 even, $\nu = 0$ if λ_0 odd).
So

$$(*) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x-y|^{\lambda_0-1,\nu} \log|x-y| f(x)\overline{g}(y)dxdy. \qquad (2.11)$$

2.5.5 The anti-analytic discrete series

This is similar to the Section 2.5.4. Only $V^+_{\lambda_0}$ is different. To describe these spaces we have to take $\overline{\phi_n}$.

Remark 2.5. *Extending the results of Kashiwara and Vergne [18] we can define the spaces* $V^{\pm}_{\lambda_0}$ *as Sobolev spaces. For example for the analytic discrete series:*

$$V^{+}_{\lambda_0} = \left\{ f \in L^2(\mathbb{R}, (1+x^2)^{\lambda_0} dx) : \hat{f} \textit{ has support in } \overline{\mathbb{R}_+} \right\}$$

and for the anti-analytic discrete series:

$$V^{-}_{\lambda_0} = \left\{ f \in L^2(\mathbb{R}, (1+x^2)^{\lambda_0} dx) : \hat{f} \textit{ has support in } \overline{\mathbb{R}_-} \right\}.$$

2.6 The analytic discrete series: realization on the complex upper half plane

[See e.g. [21]].
Let m be an integer ≥ 2. On $\mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Im}\ z > 0\}$ we have the usual fractional linear action of $G = \mathrm{SL}(2, \mathbb{R})$:

$$g \cdot z = \frac{az+b}{cz+d} \quad \text{if} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and, if $z = x + iy$, $\frac{dxdy}{y^2}$ is a G-invariant measure on $\mathbb{C}$.

Let $\mathcal{H}_m$ be the Hilbert space of holomorphic functions f on $\mathbb{C}^+$ with satisfying

$$\int_0^{\infty} \int_{-\infty}^{\infty} |f(z)|^2 y^{m-2} dxdy < \infty.$$

G acts in $\mathcal{H}_m$ by

$$\pi_m(g) f(z) = (cz+d)^{-m} f\left(\frac{az+b}{cz+d}\right) \tag{2.12}$$

if $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, as a continuous unitary representation.

Let n be an integer ≥ 0 and

$$\phi_n(z) = \left(\frac{z-i}{z+i}\right)^n (z+i)^{-m}.$$

Then $\phi_n \in \mathcal{H}_m$ for all n.

Theorem 2.6. *(See [21]). The representation* π_m *on* $\mathcal{H}_m$ *is irreducible. Let* V_{m+2n} *be the one-dimensional subspace generated by* ϕ_n. *Then* V_{m+2n} *is an eigenspace of* K, *with weight* $-m-2n$ *and*

$$\mathcal{H}_m = \bigoplus_{n \geq 0} V_{m+2n}$$

is an orthogonal decomposition with highest weight vector ϕ_0 *of weight* $-m$.

Computation of $||\phi_n||^2$ in $\mathcal{H}_m$ gives $c_m \frac{n!}{m^{[n]}}$, for all n. We conclude: the map

$$\sum_i \lambda_i \phi_i(z) \longrightarrow \sum_i \lambda_i \phi_i(x) \qquad (x \ \in \ \mathrm{I\!R})$$

extends to a *unitary equivalence between* π_m and the analytic discrete series representation $\pi^{(1)}_{\lambda_0}$ with $\lambda_0 = m-1$.

The inverse map is given by (2.13).

2.7 The limit of the analytic discrete series

For '$m = 1$', we consider the space $\mathcal{H}_1$ of holomorphic functions f on $\mathbb{C}^+$ satisfying:

$$||f||^2 = \lim_{\epsilon \downarrow 0} \frac{1}{\Gamma(\epsilon)} \int_0^\infty \int_{-\infty}^\infty |f(z)|^2 y^{-1+\epsilon} dxdy < \infty.$$

This is a Hilbert space and G acts on it by (2.12), with $m = 1$. Theorem 2.6 holds with m=1. Moreover, π_1 is unitarily equivalent to $\pi^{(1)}_{0,-}$, as above.

It is clear that the anti-analytic discrete series and $\pi^{(2)}_{0,-}$ can be treated in a similar way.

2.8 Explicit intertwining operator

We shall now describe the explicit form of the unitary equivalence of π_1 and $\pi^{(1)}_{0,-}$ from Section 2.7.

Observe that $V^{(1)}_{0,-}$ is spanned by the functions

$$\phi_n(x) = \left(\frac{x-i}{x+i}\right)^n (x+i)^{-1}; \qquad n = 0, 1, 2, \ldots$$

For $n = 0$, we have $\hat{\phi_0}(y) = -2\pi i Y(y) e^{-2\pi y}$ where Y is the Heaviside function, so all ϕ_n clearly have Fourier transform in $[0, \infty)$, so that obviously (see Remark 2.5)

$$V^{(1)}_{0,-} = \left\{ f \in L^2(\mathrm{I\!R}) : \hat{f} \text{ has support in } \ [0, \infty) \right\}.$$

Theorem 2.7. *Let* $f \in L^2(\mathrm{I\!R})$ *be such that* $\hat{f}$ *has support in* $[0, \infty)$. *Define*

$$F(z) = \int_0^\infty \hat{f}(y) e^{2\pi i y z} dy \tag{2.13}$$

for $z \in \mathbb{C}$, *Im* $z > 0$. *Then* F *is holomorphic on the upper half plane and satisfies*

a. $\lim_{y \downarrow 0} F(x+iy) = f(x)$ *in* L^2*- sense*

b. $||f||_2^2 = \lim_{\epsilon \downarrow 0} \frac{1}{\Gamma(\epsilon)} \int_0^\infty \int_{-\infty}^\infty |F(z)|^2 y^{-1+\epsilon} dxdy.$

Proof. $F(z)$ is well-defined for Im $z > 0$ and clearly holomorphic there.

To show a, let $g \in \mathcal{C}_c^\infty(\mathbb{R})$. Then we have:

$$\begin{aligned}\int_{-\infty}^{\infty} [F(u+iv) - f(u)]\overline{g}(u)du &= \int_{-\infty}^{\infty}\int_0^\infty \hat{f}(y)e^{2\pi iuy}e^{-2\pi vy}\overline{g}(u)dydu - \\ &\quad \int_{-\infty}^{\infty} f(u)\overline{g}(u)du \\ &= \int_0^\infty \hat{f}(y)\overline{\hat{g}(y)}e^{-2\pi vy}dy \\ &\quad -\int_{-\infty}^{\infty} f(u)\overline{g}(u)du \\ &= \int_0^\infty \hat{f}(y)\overline{\hat{g}(y)}[e^{-2\pi vy} - 1]dy.\end{aligned}$$

Select $M > 0$ such that $\left(\int_M^\infty |\hat{f}(y)|^2 dy\right)^{1/2} < \epsilon/4$. Splitting the integral into $\int_0^\infty = \int_0^M + \int_M^\infty$ gives:

$$\left|\int_{-\infty}^{\infty} [F(u+iv) - f(u)]\overline{g}(u)du\right| \leq \epsilon ||g||_2,$$

ϵ independent of g. This proves a.

Now we show that F satisfies b.

One has:

$$\int_{\mathbb{C}^+} |F(u+iv)|^2 v^{-1+\epsilon}dudv = \int_0^\infty\int_0^\infty |\hat{f}(y)|^2 e^{-4\pi vy}v^{-1+\epsilon}dydv$$

Now,

$$\int_0^\infty e^{-4\pi vy}v^{-1+\epsilon}dv = (4\pi y)^{-\epsilon}\int_0^\infty e^{-t}t^{\epsilon-1}dt = (4\pi y)^{-\epsilon}\Gamma(\epsilon).$$

Hence:

$$\lim_{\epsilon\downarrow 0}\frac{1}{\Gamma(\epsilon)}\int_0^\infty\int_{-\infty}^\infty |F(z)|^2 y^{-1+\epsilon}dxdy = \lim_{\epsilon\downarrow 0}\int_0^\infty |\hat{f}(y)|^2(4\pi y)^{-\epsilon}dy = ||f||^2.\Box$$

The map $f \to F$ is 1-1 from $V_{0,-}^{(1)}$ into $\mathcal{H}_1$. It is even onto, since $\phi_n(x)$ clearly corresponds to $\phi_n(z)$, since $\hat{\phi}_n \in L^1$. Moreover, the inverse map, restricted to linear combination of $\phi_n(z)$ is just the restriction to $\mathbb{R}$. This map commutes with the respective representations, so $f \to F$ is an intertwining operator for $\pi_{0,-}^{(1)}$ and π_1. The inverse map is given by a.

CHAPTER 3

Representations of SL$(2, \mathbb{C})$

In this chapter we give the irreducible unitary representations of SL$(2, \mathbb{C})$ in the 'non-compact' model. By 'non-compact' model we mean the realization of the representations on a space of functions on $\mathbb{C}$. In order to give a smooth analysis we apply again a 'compact' model. Compare, for appreciating our approach, Knapp's treatment in chapter XVI of his book [19].

3.1 The principal (non-unitary) series

3.1.1 A 'non-compact' model

Set

$$P = MAN = \left\{ \begin{pmatrix} t & 0 \\ x & t^{-1} \end{pmatrix} : t \in \mathbb{C}^*, x \in \mathbb{C} \right\}$$

where

$$M = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in \mathbb{R} \right\}, A = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t > 0, t \in \mathbb{R} \right\}$$

and

$$N = \left\{ \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} : w \in \mathbb{C} \right\}.$$

Put $\bar{N} = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \right\}$. Then $\bar{N}P$ is open, dense in $G = \mathrm{SL}(2, \mathbb{C})$ and its complement has Haar measure zero. Any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in G can be written in the form $\bar{n}p = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ w & t^{-1} \end{pmatrix}$ with $t = 1/d, z = b/d$ and $w = c$, provided $d \neq 0$. The principal series $\pi_{\lambda,l}$ $(\lambda \in \mathbb{C}, l \in \mathbb{Z})$ acts on the space V of $\mathcal{C}^\infty$-functions f with

$$\begin{aligned} f(gm_\theta a_t n) &= t^{\lambda+2}\chi_{-l}(m_\theta)f(g) \\ &= t^{\lambda+2}e^{-il\theta}f(g) \end{aligned}$$

with $\chi_l \in \hat{M}$

$$\chi_l(m_\theta) = \chi_l \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = e^{il\theta}$$

and with inner product

$$\int_{K/M} |f(k)|^2 dk = ||f||_2^2,$$

where $K = SU(2)$. $\pi_{\lambda,l}$ is given by

$$\pi_{\lambda,l}(g_0)f(g) = f(g_0^{-1}g)$$

with g_0, $g \in \mathrm{SL}(2,\mathbb{C})$.

Identifying V with a space of functions on $\bar{N} \simeq \mathbb{C}$, we can write $\pi_{\lambda,l}(g_0)$ in these terms. The inner product becomes:

$$||f||_2^2 = \int_{\mathbb{C}} |f(z)|^2(1 + |z|^2)^{\mathrm{Re}\ \lambda} dz.$$

If $g_0^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then,

$$g_0^{-1} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & az+b \\ c & cz+d \end{pmatrix} = \begin{pmatrix} 1 & z' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ w & t^{-1} \end{pmatrix}$$

with $z' = \frac{az+b}{cz+d}$ and $t = (cz+d)^{-1}$, so that

$$\pi_{\lambda,l}(g_0)f(z) = |cz+d|^{-(\lambda+2)} \left(\frac{cz+d}{|cz+d|}\right)^l f\left(\frac{az+b}{cz+d}\right)$$

with $f \in L^2(\mathbb{C})$, $z \in \mathbb{C}$ and $g_0 \in \mathrm{SL}(2,\mathbb{C})$.

The representations $\pi_{\lambda,l}$ are unitary for λ imaginary. The converse is also true.

3.1.2 A 'compact' model

This model enables us more easily to answer questions about irreducibility, equivalence, unitarity, etc.

$G = \mathrm{SL}(2,\mathbb{C})$ acts on the unit sphere $S = \{(s_1, s_2) \in \mathbb{C}^2 : |s_1|^2 + |s_2|^2 = 1\}$ by

$$g \cdot s = \frac{g(s)}{||g(s)||}$$

transitively. The stabilizer of $e_2 = (0,1)$ is AN. So $\pi_{\lambda,l}$ can be realized (depending on l) on V_l, the space of $\mathcal{C}^\infty$-functions φ on S satisfying

$$\varphi(\gamma s) = \gamma^l \varphi(s)$$

with $\gamma \in \mathbb{C}$, $|\gamma| = 1$, $s \in S$ and

$$\pi_{\lambda,l}(g)\varphi(s) = \varphi(g^{-1} \cdot s)||g^{-1}(s)||^{-(\lambda+2)}.$$

Observe that

$$\pi_{\lambda,l}|_K \simeq \operatorname*{ind}_{M\uparrow K} \chi_l.$$

Let (φ, ψ) be the usual inner product on $L^2(S)$:

$$(\varphi, \psi) = \int_S \varphi(s)\overline{\psi(s)}ds \tag{3.1}$$

where ds is the normalized measure on S. The measure ds is transformed by $g \in G$ as follows: $d(g \cdot s) = ||g(s)||^{-4}ds$. It implies that the inner product (φ, ψ) is invariant with respect to $(\pi_{\lambda,l}, \pi_{-\overline{\lambda},l})$, so that $\pi_{\lambda,l}$ is unitary for $\lambda \in i\mathbb{R}$.

Now we want to study for $\pi_{\lambda,l}$ the following questions: irreducibility, composition series, intertwining operators, unitarity.

3.2 Irreducibility

If $l \geq 0$, V_l is spanned by harmonic polynomials homogeneous of degree $l + j$ in s_1, s_2 and degree j in $\bar{s}_1$, $\bar{s}_2$, i.e.

$$V_l = \bigoplus_{j\geq 0} \mathcal{H}_{l+j,j}.$$

Since dim $\mathcal{H}_{l+j,j} = l + 2j + 1$ this K-splitting of V_l is multiplicity free. Moreover, since

$$\pi_{\lambda,l}|_K \simeq \operatorname*{ind}_{M\uparrow K} \chi_l,$$

we have by Frobenius reciprocity any $\mathcal{H}_{l+j,j}$ occurring in the decomposition of V_l contains an element $\psi_{l+j,j}$ with

$$\psi_{l+j,j}(m^{-1} \cdot s) = \chi_l(m)\psi_{l+j,j}(s),$$

so

$$\psi_{l+j,j}(e^{-i\varphi}s_1, e^{i\varphi}s_2) = e^{il\varphi}\psi_{l+j,j}(s_1, s_2).$$

$[\psi_{l+j,j}$ is unique up to scalars]. It is easily seen that $\psi_{l+j,j}(s_1, s_2)$ depends only on s_2. More precisely, one even has

$$\psi_{l+j,j}(s_1, s_2) = s_2^l F_{l+j,j}(|s_2|^2).$$

Then:

$$F_{l+j,j}(z) =_2 F_1(-j, l + j + 1; l + 1; |z|)$$

see [22].

Since G is generated by K and the subgroup $A = \{\exp(tZ_0)\}$ with

$$Z_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

in order to study the irreducibility of $\pi_{\lambda,l}$ we have to know how the latter subgroup transforms the functions $\psi_{l+j,j}$.

On functions ϕ of the form $\phi(s_1, s_2) = s_2^l f(|s_2|^2)$, $\pi_{\lambda,l}(Z_0)$ acts as an ordinary differential operator $\mathcal{L}$:

$$\pi_{\lambda,l}(Z_0)\phi(s_1, s_2) = s_2^l \mathcal{L} f(|s_2|^2)$$

with

$$\mathcal{L}(f)(u) = 4u(1-u)\frac{df}{du}(u) + ((\lambda + 2 + l)(1-2u) + l)\, f(u).$$

This follows by an easy computation. Applying it to ${}_2F_1(-j, l+j+1; l+1; u)$, we obtain:

$$\begin{aligned} \mathcal{L}{}_2F_1(-j, l+j+1; l+1; u) &= c_0(j,l;\lambda){}_2F_1(-j, l+j+1; l+1; u) + \\ &\quad c_-(j,l;\lambda){}_2F_1(-j+1, l+j; l+1; u) + \\ &\quad c_+(j,l;\lambda){}_2F_1(-j-1, l+j+2; l+1; u) \end{aligned}$$

using relations between Gauss hypergeometric functions (see [6]). This implies also

$$\begin{aligned} \pi_{\lambda,l}(Z_0)\psi_{l+j,j} &= c_0(j,l;\lambda)\psi_{l+j,j} + c_-(j,l;\lambda)\psi_{l+j-1,j-1} \\ &\quad + c_+(j,l;\lambda)\psi_{l+j+1,j+1} \end{aligned}$$

If $l < 0$ we can do the same,

$$V_l = \bigoplus_{j \geq 0} \mathcal{H}_{j,|l|+j}$$

and we obtain

$$\begin{aligned} \pi_{\lambda,l}(Z_0)\psi_{j,|l|+j} &= c_0(j,l;\lambda)\psi_{j,|l|+j} + c_-(j,l;\lambda)\psi_{j-1,|l|+j-1} \\ &\quad + c_+(j,l;\lambda)\psi_{j+1,|l|+j+1} \end{aligned}$$

with

$$\begin{aligned} c_0(j,l;\lambda) &= \frac{-\lambda l^2}{(|l|+2j)(|l|+2j+2)}\ , \\ c_-(j,l;\lambda) &= \frac{2j^2}{(|l|+2j+1)(|l|+2j)}(\lambda - |l| - 2j) \text{ and} \\ c_+(j,l;\lambda) &= \frac{2(|l|+j+1)^2}{(|l|+2j+1)(|l|+2j+2)}(2j + \lambda + 2 + |l|). \end{aligned}$$

This immediately leads to a complete analysis of the reducibility properties of the representations $\pi_{\lambda,l}$. The results are as follows:

Theorem 3.1.

a) If $\lambda \notin \mathbb{Z}$, $l \in \mathbb{Z}$ $\pi_{\lambda,l}$ is irreducible.

b) If λ, l are integer and $\lambda \neq |l| + 2j$ and $\lambda \neq -2j - 2 - |l|$ for all $j \in \mathbb{N}$, $\pi_{\lambda,l}$ is irreducible. For $\lambda = |l| + 2j$ for some $j \in \mathbb{N}$ the decomposition of $\pi_{\lambda,l}$ is as follows:

(i) $l \geq 0$. *In this case,*

$$V_{\lambda,l} = \bigoplus_{i=\frac{\lambda-l}{2}}^{\infty} \mathcal{H}_{l+i,i}$$

is an irreducible infinite dimensional subspace of V_l *and* $V_l/V_{\lambda,l}$ *is irreducible and finite dimensional.*

(ii) $l < 0$. *In this case,*

$$V_{\lambda,l} = \bigoplus_{i=\frac{\lambda+l}{2}}^{\infty} \mathcal{H}_{i,-l+i}$$

is an irreducible infinite dimensional subspace of V_l *and* $V_l/V_{\lambda,l}$ *is irreducible and finite dimensional.*

For $\lambda = -2j - 2 - |l|$ *for some* $j \in \mathbb{N}$ *the decomposition of* $\pi_{\lambda,l}$ *is as follows:*

(i) $l \geq 0$. *In this case,*

$$V_{\lambda,l} = \bigoplus_{i=0}^{\frac{-\lambda-2-l}{2}} \mathcal{H}_{l+i,i}$$

is an irreducible finite dimensional subspace of V_l *and* $V_l/V_{\lambda,l}$ *is irreducible and infinite dimensional.*

(ii) $l < 0$. *In this case,*

$$V_{\lambda,l} = \bigoplus_{i=0}^{\frac{-\lambda-2+l}{2}} \mathcal{H}_{i,-l+i}$$

is an irreducible finite dimensional subspace of V_l *and* $V_l/V_{\lambda,l}$ *is irreducible and infinite dimensional.*

3.3 Intertwining operators

Now we want to find (non zero) continuous linear operators $A : V_l \longrightarrow V_{l_1}$ intertwining the representations $\pi_{\lambda,l}$ and π_{λ_1,l_1} , i.e.

$$A\pi_{\lambda,l}(g) = \pi_{\lambda_1,l_1}(g)A \qquad (g \;\in\; G).$$

Theorem 3.2. *A non-zero non-trivial intertwining operator as above exists if and only if* $l = -l_1$, $\lambda = -\lambda_1$.

Proof. (a) We suppose that $l, l_1 \geq 0$.

$$A : V_l = \bigoplus_{j\geq 0} \mathcal{H}_{l+j,j} \longrightarrow V_{l_1} = \bigoplus_{j\geq 0} \mathcal{H}_{l_1+j,j}$$

is a continuous linear operator. Then $l = l_1$, because dim $\mathcal{H}_{l+j,j} = 2j + l + 1$, dim $\mathcal{H}_{l_1+j,j} = 2j + l_1 + 1$ and for $j = 0$ we have the same dimension, so $l = l_1$.

$A = a_j I$ on each $\mathcal{H}_{l+j,j}$ for some complex constant a_j by Schur. Moreover,

$$A \circ \pi_{\lambda,l}(X) = \pi_{\lambda_1,l}(X) \circ A$$

for all $X \in \mathfrak{g}$. By specializing to $X = Z_0$ and by letting act the left and right-hand side on $\psi_{l+j,j}$, we obtain:

$$A\pi_{\lambda,l}(Z_0)\psi_{l+j,j} = \pi_{\lambda_1,l}(Z_0)A\psi_{l+j,j}.$$

If $l \neq 0$,

$$\begin{array}{rcl} -\lambda a_j &=& -\lambda_1 a_j \\ (\lambda - l - 2j)a_{j-1} &=& (\lambda_1 - l - 2j)a_j \\ (\lambda + l + 2j + 2)a_{j+1} &=& (\lambda_1 + l + 2j + 2)a_j. \end{array}$$

Then $\lambda = \lambda_1$ and A is an scalar operator.
If $l = l_1 = 0$,

$$\begin{array}{rcl} (\lambda - 2j)a_{j-1} &=& (\lambda_1 - 2j)a_j \\ (\lambda + 2j + 2)a_{j+1} &=& (\lambda_1 + 2j + 2)a_j. \end{array}$$

Combining these equations gives,

$$(2j + \lambda + 2)(\lambda - 2j - 2) = (2j + \lambda_1 + 2)(\lambda_1 - 2j - 2)$$

so $\lambda = \pm\lambda_1$.
If $\lambda = \lambda_1$, then all a_j coincide, so A is an scalar operator. If $\lambda = -\lambda_1$ we get:

$$(2j + \lambda + 2)a_{j+1} = (2j - \lambda + 2)a_j. \tag{3.2}$$

For the irreducible case, the equation has, up to a factor, a unique solution which can be written in one of the following three forms:

$$\begin{array}{rcll} a_j &=& c_1 \dfrac{\Gamma(j - \frac{\lambda}{2} + 1)}{\Gamma(j + \frac{\lambda}{2} + 1)} & (3.3) \\ &=& c_2(-1)^j \Gamma(-\dfrac{\lambda}{2} + 1 + j)\Gamma(-\dfrac{\lambda}{2} + 1 - j) & (3.4) \\ &=& c_3 \dfrac{(-1)^j}{\Gamma(\frac{\lambda}{2} + 1 + j)\Gamma(\frac{\lambda}{2} + 1 - j)}. & (3.5) \end{array}$$

In the reducible cases, there is also a unique solution, but one has to start at another base value. This is proven in the same way.

The formulae obtained for the solutions of (3.2) show, when the solution is defined on an unbounded set, that it is of polynomial growth at infinity,

$$a_j \sim const.\ |j|^{-\mathrm{Re}\,\lambda} \qquad (|j| \to \infty).$$

This implies that the operator A having a_j as eigenvalues is continuous in the $\mathcal{C}^{(\infty}$-topology of V_l.

(b) We suppose that $l, l_1 < 0$.

$$A : V_l = \bigoplus_{j\geq 0} \mathcal{H}_{j,|l|+j} \longrightarrow V_{l_1} = \bigoplus_{j\geq 0} \mathcal{H}_{j,|l_1|+j}$$

is an intertwining operator between $\pi_{\lambda,l}$ and π_{λ_1,l_1} if and only if $l = l_1$ and $\lambda = \lambda_1$. The proof is the same as $l, l_1 > 0$.

(c) We suppose that $l > 0$ and $l_1 < 0$.

$$A : V_l = \bigoplus_{j\geq 0} \mathcal{H}_{l+j,j} \longrightarrow V_{l_1} = \bigoplus_{j\geq 0} \mathcal{H}_{j,|l_1|+j}$$

is an intertwining operator between $\pi_{\lambda,l}$ and π_{λ_1,l_1}. Then $l = |l_1|$, because dim $\mathcal{H}_{l+j,j} = l + 2j + 1$, dim $\mathcal{H}_{j,|l_1|+j} = |l_1| + 2j + 1$. For $j = 0$ they have to be the same, $l + 1 = |l_1| + 1$ hence $l = |l_1|$. We want to see that $\lambda_1 = -\lambda$.

Let

$$\begin{array}{rcl} A_0 : V_l & \longrightarrow & V_{-l} \\ \varphi(s) & \longmapsto & \varphi(\bar{s}) \end{array}$$

be an isomorphism and

$$\begin{array}{rcl} A_0\pi_{\lambda,l}(g)\varphi(s) & = & \varphi(\bar{g}^{-1}\cdot\bar{s})||\bar{g}^{-1}(\bar{s})||^{-\lambda-2} \\ & = & \pi_{\lambda,-l}(\bar{g})\varphi(\bar{s}) \\ & = & \pi_{\lambda,-l}(\bar{g})A_0\varphi(s). \end{array}$$

Hence

$$A_0\pi_{\lambda,l}(g) = \pi_{\lambda,-l}(\bar{g})A_0$$

and as ${}^t\bar{g}^{-1} = w\bar{g}w^{-1}$ with $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ we have, if $A_0' = \pi_{\lambda,-l}(w)A_0$,

$$A_0'\pi_{\lambda,l}(g) = \pi_{\lambda,-l}({}^t\bar{g}^{-1})A_0'.$$

Notice that A_0' is given by $A_0'\varphi(s_1, s_2) = \varphi(\bar{s}_2, -\bar{s}_1)$ if $\varphi \in V_l$. Let us define $A_{\lambda,-l}\varphi(s) = \displaystyle\int_S [s,t]^{\lambda-2,-l}\varphi(t)dt$ where $[s,t] = s_1\bar{t_1} + s_2\bar{t_2}$. This is an intertwining operator between $\pi_{\lambda,-l}({}^t\bar{g}^{-1})$ and $\pi_{-\lambda,-l}(g)$ (see Lemma 3.3), i.e.

$$A_{\lambda,-l}\pi_{\lambda,-l}({}^t\bar{g}^{-1}) = \pi_{-\lambda,-l}(g)A_{\lambda,-l} \quad (g \in G).$$

Then $A_{\lambda,-l}A_0'$ is an intertwining operator between $\pi_{\lambda,l}(g)$ and $\pi_{-\lambda,-l}(g)$ since

$$\begin{array}{rcl} A_{\lambda,-l}A_0'\pi_{\lambda,l}(g) & = & A_{\lambda,-l}\pi_{\lambda,-l}({}^t\bar{g}^{-1})A_0' \\ & = & \pi_{-\lambda,-l}(g)A_{\lambda,-l}A_0'. \end{array}$$

We have proved that $\pi_{\lambda,l} \sim \pi_{-\lambda,-l}$ with $l > 0$. Suppose that $\pi_{\lambda,l} \sim \pi_{\lambda_1,-l}$ then $\pi_{-\lambda,-l} \sim \pi_{\lambda_1,-l}$ and this can only happen if $\lambda_1 = -\lambda$. So the only possibility is $l_1 = -l$ and $\lambda_1 = -\lambda$. Observe that $A_{\lambda,-l}A'_0$ is given by

$$(A_{\lambda,-l}A'_0)\varphi(s) = \int_S (s_2t_1 - s_1t_2)^{\lambda-2,-l}\varphi(t)dt.$$

(d) We suppose that $l < 0$ and $l_1 > 0$. We obtain the same as in (c), $l_1 = -l$ and $\lambda_1 = -\lambda$.$\square$

Lemma 3.3. *The integral operator on* V_l

$$A_{\lambda,l}\varphi(s) = \int_S [s,t]^{\lambda-2,l}\varphi(t)dt$$

is an intertwining operator between $\pi_{\lambda,l}(g)$ *and* $\pi_{-\lambda,l}({}^t\bar{g}^{-1})$.

Proof. $A_{\lambda,l}$ is not defined for all λ. It is defined for Re $\lambda > 1$ and there is holomorphic. It can be meromorphically continued to the whole complex plane.

First, we prove that $A_{\lambda,l}$ is an intertwining operator between $\pi_{\lambda,l}$ and $\pi^-_{-\lambda,l}$ the induced representation from

$$P^- = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$$

given by

$$\pi^-_{-\lambda,l}(g)\varphi(s) = \varphi(\theta(g^{-1})\cdot s)||\theta(g^{-1})(s)||^{\lambda-2}$$

with $\theta(g) = (g^*)^{-1}$, the Cartan involution and $g^* = {}^t\bar{g}$.

We shall show that

$$A_{\lambda,l}\pi_{\lambda,l}(g) = \pi^-_{-\lambda,l}(g)A_{\lambda,l}.$$

Indeed,

$$\begin{aligned} A_{\lambda,l}\pi_{\lambda,l}(g)\varphi(s) &= \int_S [s,t]^{\lambda-2,l}\pi_{\lambda,l}(g)\varphi(t)dt \\ &= \int_S [s,t]^{\lambda-2,l}\varphi(g^{-1}\cdot t)||g^{-1}(t)||^{-\lambda-2}dt. \end{aligned}$$

On the other hand,

$$\begin{aligned} \pi^-_{-\lambda,l}(g)A\varphi(s) &= (A\varphi)(\theta(g^{-1})\cdot s)||\theta(g^{-1})(s)||^{\lambda-2} \\ &= ||\theta(g^{-1})(s)||^{\lambda-2}\int_S [\theta(g^{-1})\cdot s, t]^{\lambda-2,l}\varphi(t)dt \\ &= \int_S [s,gt]^{\lambda-2,l}\varphi(t)dt \\ &= \int_S [s,\tilde{t}]^{\lambda-2,l}\frac{1}{||g^{-1}(\tilde{t})||^{\lambda-2}}\varphi(g^{-1}\cdot\tilde{t})||g^{-1}(\tilde{t})||^{-4}d\tilde{t} \\ &= \int_S [s,\tilde{t}]^{\lambda-2,l}\varphi(g^{-1}\cdot\tilde{t})||g^{-1}(\tilde{t})||^{-\lambda-2}d\tilde{t}. \end{aligned}$$

because being $\theta(g) = (g^*)^{-1} = ({}^t\bar{g})^{-1} = \begin{pmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix}$ if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\theta(g^{-1}) = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$, it is obtained

$$[\theta(g^{-1}) \cdot s, t] = \frac{1}{||\theta(g^{-1})(s)||}[\theta(g^{-1})(s), t] = \frac{1}{||\theta(g^{-1})(s)||}[s, gt].$$

Doing a change of variable $t = g^{-1} \cdot \tilde{t}$ and $dt = ||g^{-1}(\tilde{t})||^{-4} d\tilde{t}$.

So $A_{\lambda,l}\pi_{\lambda,l}(g) = \pi^{-}_{-\lambda,l}(g)A_{\lambda,l}$ and $\pi^{-}_{-\lambda,l}(g) = \pi_{-\lambda,l}({}^t\bar{g}^{-1})$. Hence

$$A_{\lambda,l}\pi_{\lambda,l}(g) = \pi_{-\lambda,l}({}^t\bar{g}^{-1})A_{\lambda,l}.\Box$$

3.4 Invariant Hermitian forms and unitarity

In this section we determine all invariant Hermitian forms on V_l with respect to the representations $\pi_{\lambda,l}$ and determine which of these forms are positive (negative) definite, so that the corresponding representations are unitarizable. A continuous Hermitian form $H(\varphi, \psi)$ on V_l is called *invariant* with respect to $\pi_{\lambda,l}$ if

$$H(\pi_{\lambda,l}(g)\varphi, \psi) = H(\varphi, \pi_{\lambda,l}(g^{-1})\psi)$$

for all $g \in G$. Any such a form can be written in the form

$$H(\varphi, \psi) = (A\varphi, \psi)$$

with A an operator on V_l and the inner product (3.1) on the right-hand side. It implies that A intertwines $\pi_{\lambda,l}$ and $\pi_{-\overline{\lambda},l}$. So by Theorem 3.2, there are two possibilities: $\lambda = -\overline{\lambda}$ and $\lambda = \overline{\lambda}$. In the first case, we have Re $\lambda = 0$ and, provided $\lambda \neq 0$, the operator A is equal to cE, so that H is c times (3.1). In the second case, we must have $l = 0$ and A intertwines $\pi_{\lambda,0}$ and $\pi_{-\lambda,0}$. So we get:

Theorem 3.4. *A non-zero invariant Hermitian form $H(\varphi, \psi)$ on V_l exists only if:*
(a) Re $\lambda = 0$, or (b) Im $\lambda = 0$ and $l = 0$. In case (a) the form is proportional to the L^2-inner product (3.1). *In case (b) the form $H(\varphi, \psi)$ on V_l has the form*

$$H(\varphi, \psi) = (A\varphi, \psi)$$

where A is an intertwining operator between $\pi_{\lambda,0}$ and $\pi_{-\lambda,0}$.

Now we determine when the Hermitian forms above are positive or negative definite.

Theorem 3.5. *The unitarizable irreducible representation $\pi_{\lambda,l}$ belong to the following series.*

(i) the unitary principal series: $\pi_{\lambda,l}$ with Re $\lambda = 0$.

(ii) the complementary series consisting of the representations $\pi_{\lambda,0}$ with $0 < \lambda < 2$.

(iii) the trivial representation.

Proof. Like in case SL(2, ℝ). In the basis $\psi_{l+j,j}$ an invariant Hermitian form $H(\varphi, \psi)$ has a diagonal matrix, with real scalars h_j on the diagonal. The h_j have the same expression as a_j. We have to determine when the numbers a_j have the same sign for all $j \in J$, where J is the set of weights of our representations. This leads to the representations in the theorem. For example, when $0 < \lambda < 2$ we easily see that (3.4) is positive for all j, if $c_2 > 0$. □

Let us denote the unitary completions of the representations by the same symbols. These unitary completions exhaust all irreducible unitary representations of G up to equivalence, see [13].

3.5 The 'non-compact' models of the irreducible unitary representations of SL(2, ℂ)

3.5.1 The continuous series: $\pi_{\lambda,l}$, $\lambda \in i\mathbb{R}$

We just refer to Section 3.1.1. The space is $L^2(\mathbb{C})$ with the usual inner product and

$$\pi_{\lambda,l}(g)f(z) = |cz+d|^{-(\lambda+2)} \left(\frac{cz+d}{|cz+d|}\right)^l f\left(\frac{az+b}{cz+d}\right) \tag{3.6}$$

if $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

3.5.2 The complementary series: $\pi_{\lambda,0}$ $(0 < \lambda < 2)$

The inner product is given in the 'compact' model by

$$(\varphi, \phi) = \int_S \int_S |s_2 t_1 - s_1 t_2|^{\lambda-2} \varphi(t) \overline{\phi}(s) ds dt$$

$(\varphi, \phi \in V_0)$.

The representation in the 'non-compact' model is again given by (3.6):

$$\pi_{\lambda,0}(g)f(z) = |cz+d|^{-(\lambda+2)} f\left(\frac{az+b}{cz+d}\right)$$

if $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Let us rewrite (φ, ϕ) in terms of functions on $\mathbb{C}$. This is easily seen to be:

$$(f, g)_\lambda = \int_{\mathbb{C}} \int_{\mathbb{C}} |z-w|^{\lambda-2} f(z) \overline{g}(w) dz dw.$$

3.6 Finite-dimensional irreducible representations

Choose $\lambda = -l-2$ in the 'non-compact' picture, $\mathrm{span}(1, z, \ldots, z^l)$ is an irreducible invariant subspace of V_l under $\pi_{\lambda,l}$. These spaces exhaust all finite-dimensional irreducible representations of $\mathrm{SL}(2, \mathbb{C})$.
In the 'compact' picture the spaces are $\mathcal{H}_{l,0}$.

CHAPTER 4

The metaplectic representation

The metaplectic representation -also called the oscillator representation, harmonic representation, or Segal-Shale-Weil representation- is a double-valued unitary representation of the symplectic group $Sp(n, \mathbb{R})$ on $L^2(\mathbb{R}^n)$. In this chapter we give an overview of the construction of this representation and we examine it from several viewpoints. The contents of this chapter are adapted from [10].

4.1 The Heisenberg group

4.1.1 Definition

First we introduce some notations. Let us denote the product of two vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$ by simple juxtaposition

$$xy = \sum_1^n x_j y_j \quad (x, y \in \mathbb{R}^n \text{ or } \mathbb{C}^n).$$

Thus, the Hermitian inner product of z, $w \in \mathbb{C}^n$ is $z\overline{w}$. We also set

$$x^2 = xx = \sum_1^n x_j^2 \quad (x \in \mathbb{R}^n \text{ or } \mathbb{C}^n),$$

$$|z|^2 = z\overline{z} = \sum_1^n |z_j|^2 \quad (z \in \mathbb{C}^n).$$

When linear mappings intervene in such products we denote by

$$xAy = y\ {}^tAx = \sum x_j A_{jk} y_k \quad (x, y \in \mathbb{C}^n, A \in M_n(\mathbb{C})).$$

We consider $\mathbb{R}^{2n+1}$ with coordinates

$$(p_1, \ldots, p_n, q_1, \ldots, q_n, t) = (p, q, t),$$

and we define a Lie bracket on $\mathbb{R}^{2n+1}$ by

$$[(p,q,t),(p',q',t')] = (0,0,\ pq' - qp'). \tag{4.1}$$

It is easily verified that the bracket (4.1) makes $\mathbb{R}^{2n+1}$ into a Lie algebra, called the **Heisenberg Lie algebra** and denoted by h_n.

In order to identify the Lie group corresponding to h_n, it is convenient to use a matrix representation. Given $(p,q,t) \in \mathbb{R}^{2n+1}$, we define the matrix $m(p,q,t) \in M_{n+2}(\mathbb{R})$ by

$$m(p,q,t) = \begin{pmatrix} 0 & p_1 & \cdots & p_n & t \\ 0 & 0 & \cdots & 0 & q_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & q_n \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Moreover, we define

$$M(p,q,t) = I + m(p,q,t).$$

It is easily verified that

$$m(p,q,t)m(p',q',t') = m(0,0,pq'), \tag{4.2}$$

$$M(p,q,t)M(p',q',t') = M(p+p',q+q',t+t'+pq'). \tag{4.3}$$

From (4.2) it follows that

$$[m(p,q,t),m(p',q',t')] = m(0,0,pq'-qp'),$$

where the bracket now denotes the commutator. Hence the correspondence $X \to m(X)$ is a Lie algebra isomorphism from h_n to $\left\{m(X) : X \in \mathbb{R}^{2n+1}\right\}$ and to obtain the corresponding Lie group we can simply apply the matrix exponential map. So,

$$e^{m(p,q,t)} = M(p,q,t+\frac{1}{2}pq).$$

Thus the exponential map is a bijection from $\{m(X) : X \in \mathbb{R}^{2n+1}\}$ to $\{M(X) : X \in \mathbb{R}^{2n+1}\}$, and the latter is a group with group law (4.3). We could take this to be the Lie group corresponding to h_n, but we prefer to use a different model. It is easily verified that

$$\exp m(p,q,t) \exp m(p',q',t') = \exp m(p+p',q+q',t+t'+\frac{1}{2}(pq'-qp')).$$

Therefore, if we identify $X \in \mathbb{R}^{2n+1}$ with the matrix $e^{m(X)}$, we make $\mathbb{R}^{2n+1}$ into a group with group law

$$(p,q,t)(p',q',t') = (p+p',q+q',t+t'+\frac{1}{2}(pq'-qp')).$$

We call this group the **Heisenberg group** and denote it by H_n. We observe that

$$\mathcal{Z} = \{(0,0,t) : t \in \mathbb{R}\}$$

is the center of H_n.

4.1.2 The Schrödinger representation

Let X_j and D_j be the differential operators on $\mathbb{R}^n$ defined by

$$(X_j f)(x) = x_j f(x), \quad D_j f = \frac{1}{2\pi i}\frac{\partial f}{\partial x_j}.$$

We may regard these operators as continuous operators on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. The map $d\rho_h$ from the Heisenberg algebra h_n to the set of skew-Hermitian operators on $\mathcal{S}(\mathbb{R}^n)$ defined by

$$d\rho_h(p,q,t) = 2\pi i(hpD + qX + tI)$$

is a Lie algebra homomorphism. We exponentiate this representation of h_n to obtain a unitary representation of the Heisenberg group H_n.

The map defined by

$$\rho_h(p,q,t) = e^{2\pi iht} e^{2\pi i(hpD+qX)}$$

that is,

$$\rho_h(p,q,t)f(x) = e^{2\pi iht+2\pi iqx+\pi ihpq} f(x+hp)$$

is a unitary representation of H_n on $L^2(\mathbb{R}^n)$, for any real number h. Moreover, ρ_h and ρ'_h are inequivalent for $h \neq h'$. ρ_h is irreducible for $h \neq 0$.

We call ρ_h the **Schrödinger representation** of H_n with parameter h. Generally we shall take $h = 1$ and restrict attention to the representation $\rho = \rho_1$. Since the central variable t always acts in a simple-minded way, as multiplication by the scalar $e^{2\pi it}$, it is often convenient to disregard it entirely; we therefore define

$$\rho(p,q) = \rho(p,q,0) = e^{2\pi i(pD+qX)}.$$

4.1.3 The Fock-Bargmann representation

There is a particularly interesting realization of the infinite-dimensional irreducible unitary representations of H_n in a Hilbert space of entire functions.

Let us define the **Fock Space** as

$$\mathcal{F}_n = \left\{ F : F \text{ is entire on } \mathbb{C}^n \text{ and } ||F||^2_{\mathcal{F}} = \int |F(z)|^2 e^{-\pi|z|^2} dz < \infty \right\}$$

and for $z \in \mathbb{C}^n$

$$Bf(z) = 2^{n/4} \int f(x) e^{2\pi xz - \pi x^2 - (\pi/2)z^2} dx.$$

Bf is called the **Bargmann transform** of f and it is an isometry from $L^2(\mathbb{R}^n)$ into the Fock Space.

The Schrödinger representation can be transferred via the Bargmann transform to a representation β of H_n on $\mathcal{F}_n$. To describe this representation, it will be convenient to identify the underlying manifold of H_n with $\mathbb{C}^n \times \mathbb{R}$:

$$(p,q,t) \longleftrightarrow (p+iq,t).$$

In this parametrization of H_n the group law is given by

$$(z,t)(z',t') = (z+z', t+t' + \frac{1}{2}\mathrm{Im}\ \overline{z}z').$$

The group H_n can also be seen inside U(1, n) as the subgroup N, see [8].

The transferred representation β is then defined by

$$\beta(p+iq,t)B = B\rho(p,q,t),$$

in other words,

$$\beta(w,t)F(z) = e^{-(\pi/2)|w|^2 - \pi z\overline{w} + 2\pi i t}F(z+w).$$

This β is called the **Fock-Bargmann representation**.

4.2 The metaplectic representation

4.2.1 Symplectic linear algebra and symplectic group

In this section we shall be working with $2n \times 2n$ matrices, which we shall frequently write in block form:

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C and D are $n \times n$ matrices. Let $\mathcal{J}$ be the matrix

$$\mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

which describes the **symplectic form** on $\mathbb{R}^{2n}$:

$$[w_1, w_2] = w_1 \mathcal{J} w_2.$$

The **symplectic group** $Sp(n,\mathbb{R})$ is the group of all $2n \times 2n$ real matrices which, as operators on $\mathbb{R}^{2n}$, preserve the symplectic form:

$$\mathcal{A} \in Sp(n,\mathbb{R}) \Longleftrightarrow [\mathcal{A}w_1, \mathcal{A}w_2] = [w_1, w_2] \text{ for all } w_1,\ w_2 \in \mathbb{R}^{2n}.$$

The **symplectic Lie algebra** $sp(n,\mathbb{R})$ is the set of all $\mathcal{A} \in M_{2n}(\mathbb{R})$ such that $e^{t\mathcal{A}} \in Sp(n,\mathbb{R})$ for all $t \in \mathbb{R}$. When the dimension n is fixed, we shall abbreviate

$$Sp = Sp(n,\mathbb{R})\ ,\ sp = sp(n,\mathbb{R}).$$

The group Sp is generated by $D \cup \overline{N} \cup \{\mathcal{J}\}$ and also by $D \cup N \cup \{\mathcal{J}\}$ with

$$N = \left\{ \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} : A = {}^tA \right\},\ \overline{N} = \left\{ \begin{pmatrix} I & 0 \\ A & I \end{pmatrix} : A = {}^tA \right\}$$

and

$$D = \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} : A \in GL(n, \mathbb{R}) \right\}.$$

In connection with the Fock model we want to use complex coordinates $z = p + iq$ corresponding to the real coordinates p, q on $\mathbb{R}^{2n}$. Since the action of Sp on $\mathbb{R}^{2n}$ is not complex linear, however, it will be more appropriate to map $\mathbb{R}^{2n}$ into $\mathbb{C}^{2n}$ by

$$W_0(p, q) = (p + iq, p - iq).$$

Under this mapping, any linear transformation T on $\mathbb{R}^{2n}$ turns into the linear transformation $T_c = W_0 T W_0^{-1}$ on $\mathbb{C}^{2n}$, so on the level of matrices we have the map

$$\mathcal{A} \in M_{2n}(\mathbb{R}) \longrightarrow \mathcal{A}_c \in M_{2n}(\mathbb{C})$$

given by

$$\mathcal{A}_c = \mathcal{W}_0 \mathcal{A} \mathcal{W}_0^{-1} = \mathcal{W} \mathcal{A} \mathcal{W}^{-1}, \tag{4.4}$$

where

$$\mathcal{W}_0 = \begin{pmatrix} I & iI \\ I & -iI \end{pmatrix}, \quad \mathcal{W} = \frac{1}{\sqrt{2}} \mathcal{W}_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} I & iI \\ I & -iI \end{pmatrix}.$$

We denote the images of $M_{2n}(\mathbb{R})$ and Sp under the map (4.4) by $M_{2n}(\mathbb{R})_c$ and Sp_c:

$$M_{2n}(\mathbb{R})_c = \{\mathcal{A}_c : \mathcal{A} \in M_{2n}(\mathbb{R})\}, \quad Sp_c = \{\mathcal{A}_c : \mathcal{A} \in S_p\}.$$

One can easily show that

$$M_{2n}(\mathbb{R})_c = \left\{ \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} : P,\ Q \in M_n(\mathbb{C}) \right\}.$$

4.2.2 Construction of the metaplectic representation

4.2.2.1 The Schrödinger model

The symplectic group Sp acts on the Heisenberg group H_n by

$$\mathcal{A} \cdot (p, q, t) = (\mathcal{A}(p, q), t) \text{ with } \mathcal{A} \in Sp \text{ and } (p, q, t) \in H_n.$$

Composing with the Schrödinger representation ρ we have a new representation $\rho \circ \mathcal{A}$ of H_n on $L^2(\mathbb{R}^n)$ such that $\rho \circ \mathcal{A}(0, 0, t) = e^{2\pi i t} I$. By the Stone-von Neumann theorem, ρ and $\rho \circ \mathcal{A}$ are equivalent: there exists a unitary operator $\mu(\mathcal{A})$ on $L^2(\mathbb{R}^n)$ such that

$$\rho \circ \mathcal{A}(X) = \mu(\mathcal{A}) \rho(X) \mu(\mathcal{A})^{-1}, \qquad X \in H_n. \tag{4.5}$$

Moreover, by Schur's lemma, $\mu(\mathcal{A})$ is determined up to a scalar factor of modulus one. It follows that

$$\mu(\mathcal{A}\mathcal{B}) = c_{\mathcal{A},\mathcal{B}} \mu(\mathcal{A}) \mu(\mathcal{B}), \qquad |c_{\mathcal{A},\mathcal{B}}| = 1,$$

so that μ is a projective unitary representation of Sp in $L^2(\mathbb{R}^n)$. We shall prove later that the scalar factors of modulus one can be chosen in one and only one way up to factors of ±1 so that μ becomes a double-valued unitary representation of Sp, i.e.,

$$\mu(\mathcal{AB}) = \pm\mu(\mathcal{A})\mu(\mathcal{B}).$$

With this choice of the scalar factors of modulus one, μ is called the **metaplectic representation** of Sp.

If we pass to the double covering group Sp_2, μ defines a unitary representation of Sp_2.

The explicit calculation of $\mu(\mathcal{A})$ for a general $\mathcal{A} \in Sp$ is rather complicated, but it is easy to use (4.5) to find $\mu(\mathcal{A})$ up to a scalar factor of modulus one when $\mathcal{A}$ belongs to certain subgroups of Sp. Doing this we obtain,

$$\mu\left[\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}\right] f(x) = |\det A|^{-1/2} f(A^{-1}x), \tag{4.6}$$

$$\mu\left[\begin{pmatrix} I & 0 \\ C & I \end{pmatrix}\right] f(x) = e^{-\pi i x C x} f(x), \tag{4.7}$$

$$\mu(\mathcal{J}) = \mathcal{F}^{-1} \tag{4.8}$$

where $\mathcal{F}$ is the Fourier transform defined by $\mathcal{F}f(y) = \displaystyle\int e^{-2\pi i x y} f(x) dx$.

As Sp is generated by matrices of these three types, we have in some sense computed the metaplectic representation up to scalar factors of modulus one.

4.2.2.2 The Fock model

The easiest way to construct the metaplectic representation globally is to move it over to Fock space. We recall that on the Fock Space $\mathcal{F}_n$ we have the representation β of H_n obtained from the Schrödinger representation ρ by conjugation with the Bargmann transform B:

$$\beta(p + iq) = B\rho(p, q)B^{-1}, \qquad \beta(w)F(z) = e^{-(\pi/2)|w|^2 - \pi z \overline{w}} F(z + w).$$

Since we use the complex coordinates $w = p + iq$ for describing β, we also use the complex form Sp_c of Sp. We adopt the following notational convention: if $\mathcal{A} \in Sp_c$ and $w \in \mathbb{C}^n$, we define $\mathcal{A}w$ to be the vector $z \in \mathbb{C}^n$ such that $\mathcal{A}(w, \overline{w}) = (z, \overline{z})$. That is,

$$\mathcal{A}w = z \Longleftrightarrow z = Pw + Q\overline{w} \text{ where } \mathcal{A} = \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}.$$

The Fock-metaplectic representation ν of Sp_c is then defined up to scalar factors of modulus one by the condition

$$\beta(\mathcal{A}w) = \nu(\mathcal{A})\beta(w)\nu(\mathcal{A})^{-1}. \tag{4.9}$$

The advantage of the Fock space is that the operators $\nu(\mathcal{A})$ are integral operators whose kernels are rather easily computable. We have

$$\nu(\mathcal{A})F(z) = \int K_{\mathcal{A}}(z, \overline{w}) F(w) e^{-\pi |w|^2} dw \tag{4.10}$$

where $K_{\mathcal{A}}(z,\overline{w}) =< \nu(\mathcal{A})E_w, E_z >_{\mathcal{F}}$ and $E_w(z) = e^{\pi z\overline{w}}$. We proceed to calculate $K_{\mathcal{A}}$.

Theorem 4.1. *For* $\mathcal{A} = \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \in Sp_c$*, define the operator* $\nu(\mathcal{A})$ *(modulo* ± 1*) on* $\mathcal{F}_n$ *given by* (4.10),

$$K_{\mathcal{A}}(z,\overline{w}) = (\det{}^{-1/2} P) \exp\left\{\frac{1}{2}\pi(z\overline{Q}P^{-1}z + 2\overline{w}P^{-1}z - \overline{w}P^{-1}Q\overline{w})\right\}.$$

Then for all $\mathcal{A}$, $\mathcal{B} \in Sp_c$ *and* $w \in \mathbb{C}^n$,

a) $\nu(\mathcal{A})$ *is unitary,*

b) $\nu(\mathcal{A})\beta(w)\nu(\mathcal{A})^{-1} = \beta(\mathcal{A}w)$,

c) $\nu(\mathcal{A})\nu(\mathcal{B}) = \pm\nu(\mathcal{A}\mathcal{B})$.

Moreover, if $\{\nu'(\mathcal{A}) : \mathcal{A} \in Sp\}$ *is any family of operators satisfying these three conditions, then* $\nu'(\mathcal{A}) = \pm\nu(\mathcal{A})$ *for all* $\mathcal{A} \in Sp$.

Remark 4.2. *The uniqueness part of this theorem shows, in particular, that* ν *can not be made into a single-valued representation of* Sp_c*, as it is impossible to define a continuous single-valued branch of* $\det^{-1/2} P$ *on* Sp_c.

We can now go back to the Schrödinger picture and define the metaplectic representation μ by

$$\mu(\mathcal{A}) = B^{-1}\nu(\mathcal{A}_c)B, \qquad \mathcal{A}_c \text{ as in (4.4).}$$

Theorem 4.1 translates into a corresponding result for μ.

Proposition 4.3. *With the correct scalar factors of modulus one the equation* (4.6), (4.7) *and* (4.8) *are equal*

$$\begin{aligned} \mu\left[\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}\right] f(x) &= (\det A)^{-1/2} f(A^{-1}x), \\ \mu\left[\begin{pmatrix} I & 0 \\ C & I \end{pmatrix}\right] f(x) &= \pm e^{-\pi i x C x} f(x), \\ \mu(\mathcal{J}) &= i^{n/2}\mathcal{F}^{-1}. \end{aligned}$$

Chapter 5

Theory of invariant Hilbert subspaces

In this chapter we recall an important tool for the representation theory of Lie groups, namely the theory of Hilbert subspaces invariant under a group of automorphisms. This theory is due to L. Schwartz [27], Thomas [34] and Pestman [25]. A main subject is the study of the theory of the reproducing kernels, associated with Hilbert subspaces. We give a criterion (due to Thomas) which assures multiplicity free decomposition of representations. This criterion will be applied in the next chapters.

5.1 Kernels and Hilbert subspaces

Let E be a quasi-complete locally convex space over $\mathbb{C}$, e.g. $\mathcal{S}'(\mathbb{R}^n)$, the space of tempered distributions on $\mathbb{R}^n$, and E^* the anti-dual of E, being the linear space of continuous, anti-linear forms on E provided with the strong topology. Let H be a Hilbert space. It is called a Hilbert subspace of E if $H \subset E$ and the linear inclusion $j : H \to E$ is continuous. Note that the image of $j^* : E^* \to H$ is a dense subspace of H.

For $e \in E$, $\psi \in E^*$ we set $\langle e, \psi\rangle$ for the value of ψ at e. The inner product of H will be denoted by $(\cdot\,|\,\cdot)$.

Let $K = jj^*$. Then K is a continuous linear operator from E^* to E, called the *reproducing kernel* of H. So, for $\phi, \psi \in E^*$ one has

$$\langle K\phi, \psi\rangle = (j^*\phi \,|\, j^*\psi), \tag{5.1}$$

which shows that K is a Hermitian kernel:

$$\langle K\phi, \psi\rangle = \overline{\langle K\psi, \phi\rangle}.$$

Putting $\phi = \psi$ in (5.1), we see that K is even a positive-definite kernel. Let Γ be the convex cone of positive-definite kernels. One has (L. Schwartz [27]):

Lemma 5.1. *To each $K \in \Gamma$ there corresponds a unique Hilbert subspace H of E of which K is the reproducing kernel.*

5.2 Invariant Hilbert subspaces

Let u be a continuous automorphism of E. We say that a Hilbert subspace H of E is u-invariant if $u(H) = H$ and $u|_H$ is unitary. An equivalent statement is : $uKu^* = K$, if K is the reproducing kernel of H, so

$$\langle Ku^*\phi, u^*\psi\rangle = \langle K\phi, \psi\rangle \qquad (\phi, \psi \in E^*).$$

We call K a u-invariant kernel. Let $GL(E)$ be the group of all continuous automorphisms of E and let G be a subgroup of $GL(E)$. A Hilbert subspace H of E is said to be G-invariant if H is u-invariant for all $u \in G$. Similarly, a kernel K is said to be G-invariant if K is u-invariant for all $u \in G$. It is clear that the set of all G-invariant reproducing kernels is a closed convex cone in Γ. We will denote this cone by Γ_G.

Proposition 5.2. *A Hilbert subspace H is a minimal (irreducible) G-invariant Hilbert subspace of E if and only if K, the reproducing kernel of H, lies on an extremal ray of Γ_G.*

If E is a conuclear space, i.e. the strong dual of a barrelled nuclear space ([35] Ch. 33), then Γ_G is a well-capped cone (in the sense of Choquet [4]), so in this case Γ_G is the closed convex hull of its extremal rays ([4]). We then always have minimal G-invariant Hilbert subspace provided $\Gamma_G \neq \{0\}$. A fine example for E is the space $E = \mathcal{S}'(\mathbb{R}^n)$.

Denote by $ext(\Gamma_G)$ the set of extremal rays of Γ_G. Let S_o be a section of $ext(\Gamma_G)$. That is a set, not containing 0, having exactly one point on each extremal ray. A section is said to be *admissible* if the function equal to 1 on S_o and homogeneous of degree 1, is universally measurable.

One can prove that such sections always exist. An *admissible parameterization* of $ext(\Gamma_G)$ is a topological Hausdorff space S with a continuous one-to-one map $\mathcal{P}$ from S to $ext(\Gamma_G)$, $\mathcal{P} : s \to K_s \in ext(\Gamma_G)$, such that the image is an admissible section and the inverse map is universally measurable. Such parameterizations exist ([34]). We have ([27], [34]):

Theorem 5.3. *Let E be a conuclear space, G a subgroup of $GL(E)$ and $s \to K_s$ an admissible parameterization of $ext(\Gamma_G)$. Then for every $K \in \Gamma_G$ there is a positive Radon measure m on S such that*

$$K = \int_S K_s dm(s).$$

The next theorem describes the conditions under which the decomposition in Theorem 5.3 is unique ([25], [34]).

Theorem 5.4. *Let E be a quasi-complete locally convex space, G a subgroup of $GL(E)$ and S an admissible parameterization of $ext(\Gamma_G)$. Then the following statements are equivalent:*

1. For every G-invariant Hilbert subspace H of E the commutant of G in $\mathcal{L}(H)$, the algebra of continuous linear operators on H, is commutative.

2. The cone Γ_G is a lattice (i.e. any two elements in the cone have a smallest common majorant in the cone).

If, in addition, E is conuclear, then the above statements are each equivalent with:

3. If H_1 and H_2 are minimal G-invariant Hilbert subspaces of E which are not proportional then the irreducible representations of G on H_1 and H_2 are inequivalent.

4. For every $K \in \Gamma_G$ there is a unique positive Radon measure m on S such that

$$K = \int_S^{\oplus} K_s dm(s).$$

5.3 Multiplicity free decomposition

Definition 5.5. We say that the action of G on E is multiplicity free if one of the conditions 1.,2.,3. of Theorem 5.4 is satisfied.

Criterion 5.6. *(Thomas). Let $J : E \to E$ be an anti-automorphism. If $JH = H$ (i.e. $J|_H$ is anti-unitary) for every G-invariant Hilbert subspace H of E, then G acts multiplicity free on E.*

Let K be the reproducing kernel of H. Then the condition $JH = H$ is equivalent with $K = JKJ^*$, so it suffices to show that

$$\langle KJ^*\phi, J^*\psi\rangle = \langle K\phi, \psi\rangle$$

for all $\phi, \psi \in E^*$.

For the proof of the criterion, we refer to [25], Theorem I.5.4.

5.4 Representations

Assume that G (instead of a subgroup of $GL(E)$) is a locally compact topological group. Let E, for the moment, be an arbitrary locally convex space. A map $\pi : G \to GL(E)$ is said to be a representation of G on E if:

(i) $\pi(g_1 g_2) = \pi(g_1)\pi(g_2) \quad (g_1, g_2 \in G)$,

(ii) the map $(g, e) \to \pi(g)e$ from $G \times E$ to E is continuous.

If E is a *barrelled* space then condition (ii) is already satisfied if the map $g \to \pi(g)e$ is continuous for every $e \in E$.

Definition 5.7. If the map $\pi : G \to GL(E)$ is a representation of G on E, we say that a Hilbert subspace H is G-invariant if H is invariant under $\pi(G)$.

Of course this notion of G-invariance depends upon the representation π. This will give no problems to us because mostly there is only one representation under consideration.

Proposition 5.8. *Suppose* $\pi : G \to GL(E)$ *is a representation of* G *on* E *and* H *is a* G*-invariant Hilbert subspace of* E*. Then the map* $g \to \pi(g)|_H$ *is a continuous unitary representation of* G*.*

For a proof, see [25], Prop. I.5.4.

Clearly $\pi(g) = \pi(g)|_H$ is irreducible if and only if H is a minimal G-invariant subspace.

Denote E_c^* (E_b^*) the space E^* equipped with the topology of uniform convergence on compact (bounded) sets in E. If $\pi : G \to GL(E)$ is a representation of G on E, then the map $g \to \pi(g^{-1})^*$ is a representation of G on E_c^*. This representation is said to be the *contragradient representation* of π if the map $(g, e^*) \to \pi(g^{-1})^* e^*$ is a continuous map $G \times E_b^* \to E_b^*$. This map is in general not continuous. For a Montel space E (see [35] for definition), the spaces E_b^* and E_c^* are equal however. In particular it is true for $E = \mathcal{S}(\mathbb{R}^n)$, the space of Schwartz functions on $\mathbb{R}^n$, and $E = \mathcal{S}'(\mathbb{R}^n)$.

5.5 Schwartz's kernel theorem for tempered distributions

Let $\mathcal{L}(E, E^*)$ denote the space of continuous linear maps $E \to E^*$, where E is a locally convex space and E^* its dual, provided with the strong topology. According to [35], Theorem 51.7 we have the following canonical isomorphism.

Theorem 5.9. *The space of tempered distributions on* $\mathbb{R}^{m+n}$ *is canonically isomorphic to* $\mathcal{L}(\mathcal{S}(\mathbb{R}^m), \mathcal{S}'(\mathbb{R}^n))$*. So given* $K \in \mathcal{L}(\mathcal{S}(\mathbb{R}^m), \mathcal{S}'(\mathbb{R}^n))$*, there is a unique* $T \in \mathcal{S}'(\mathbb{R}^{m+n})$ *such that*

$$\langle K\phi, \psi\rangle = \langle T, \phi \otimes \psi\rangle \qquad (\phi \in \mathcal{S}(\mathbb{R}^m), \psi \in \mathcal{S}(\mathbb{R}^n)).$$

where $(\phi \otimes \psi)(x, y) = \phi(x)\psi(y)$ $(x \in \mathbb{R}^m,\ y \in \mathbb{R}^n)$*.*

Chapter 6

The oscillator representation of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{O}(2n)$

In the first part of this chapter we compute the Plancherel formula of the oscillator representation ω_{2n} of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{O}(2n)$. The main tool we use is a Fourier integral operator which was introduced by M. Kashiwara and M. Vergne, see [18]. After that we can conclude that any minimal invariant Hilbert subspace of $\mathcal{S}'(\mathbb{R}^{2n})$ occurs in the decomposition of $L^2(\mathbb{R}^{2n})$.

Finally we study the oscillator representation ω_{2n} in the context of the theory of invariant Hilbert subspaces. The oscillator representation acts on the Hilbert space $L^2(\mathbb{R}^{2n})$. It is well-known that $\mathcal{S}(\mathbb{R}^{2n})$ is $\omega_{2n}(G)$-stable, so, by duality, ω_{2n} acts on $\mathcal{S}'(\mathbb{R}^{2n})$ as well, and $L^2(\mathbb{R}^{2n})$ can thus be considered as an invariant Hilbert subspace of $\mathcal{S}'(\mathbb{R}^{2n})$. Our main result is that any $\omega_{2n}(G)$-stable Hilbert subspace of $\mathcal{S}'(\mathbb{R}^{2n})$ decomposes multiplicity free.

6.1 The definition of the oscillator representation

Let G be the group $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{O}(2n)$ with $n > 1$ and let ω_{2n} be the unitary representation of G on $H = L^2(\mathbb{R}^{2n})$ defined by:

$$\begin{aligned}
\omega_{2n}(g)f(x) &= f(g^{-1}\cdot x), \qquad g \in \mathrm{O}(2n)\\
\omega_{2n}(g(a))f(x) &= a^n f(ax), \qquad g(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\\
\omega_{2n}(t(b))f(x) &= e^{-i\pi b||x||^2} f(x), \qquad t(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\\
\omega_{2n}(\sigma)f(x) &= i^n \int_{\mathbb{R}^{2n}} e^{2i\pi[x,y]} f(y)dy, \qquad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\end{aligned}$$

where $[x,y] = x_1y_1 + x_2y_2 + \cdots + x_{2n}y_{2n}$ for $x = (x_1, x_2, \ldots, x_{2n})$, $y = (y_1, y_2, \ldots, y_{2n})$.

This is a well-defined representation for $2n$. If we consider $\mathrm{SL}(2,\mathbb{R})\times\mathrm{O}(n)$ with n odd, this is a unitary representation of $\widetilde{\mathrm{SL}}(2,\mathbb{R})\times\mathrm{O(n)}$ with $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ a double covering of $\mathrm{SL}(2,\mathbb{R})$ [10],[29],[30],[44].

We call ω_{2n} the oscillator representation of G. More precisely, it is the natural extension of the restriction of the metaplectic representation of the double covering of the group $Sp(2n,\mathbb{R})$ to $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(2n)$, defined by Shale, Segal and Weil.

Let $\mathcal{S}(\mathbb{R}^{2n})$ be the space of Schwartz functions on $\mathbb{R}^{2n}$. Note that $\mathcal{S}(\mathbb{R}^{2n})$ is stable under the action of $\omega_{2n}(G)$, so is the space $\mathcal{S}'(\mathbb{R}^{2n})$ of tempered distributions on $\mathbb{R}^{2n}$.

6.2 Some minimal invariant Hilbert subspaces of $\mathcal{S}'(\mathbb{R}^{2n})$

We consider $L^2(\mathbb{R}^{2n})$ as a Hilbert subspace of $\mathcal{S}'(\mathbb{R}^{2n})$ and we will try to decompose it into minimal invariant Hilbert subspaces. Irreducible unitary representations of G are of the form $\pi\otimes\rho$ where π is an irreducible representation of $\mathrm{SL}(2,\mathbb{R})$ and ρ one of $\mathrm{O}(2n)$. We will now construct intertwining operators from $\mathcal{S}(\mathbb{R}^{2n})$ to the Hilbert space of such representations. Not all combinations (π,ρ) occur. Here are some of these operators.

Let $\mathcal{H}_l$ denote the space of harmonic polynomials on $\mathbb{R}^{2n}$, homogeneous of degree l. This space has a reproducing kernel, say $K_l(x,y)$. For any $f\in\mathcal{H}_l$ one has:

$$f(x)=\int_{S^{2n-1}} f(s)K_l(s,x)ds \qquad (x\in\mathbb{R}^{2n}) \tag{6.1}$$

with $S^{2n-1}=\{x\in\mathbb{R}^{2n}:||x||^2=x_1^2+\cdots+x_{2n}^2=1\}$. The reproducing kernel K_l is known to be real-valued and is positive-definite. It satisfies $K_l(gx,gy)=K_l(x,y)$ for all $g\in\mathrm{O}(2n)$, $K_l(\lambda x,\lambda y)=\lambda^{2l}K_l(x,y)$, $K_l(x,\cdot)$ and $K_l(\cdot,y)$ are in $\mathcal{H}_l$.

The group $\mathrm{O}(2n)$ acts irreducibly on $\mathcal{H}_l$ by

$$\rho_l(g)f(x)=f(g^{-1}\cdot x)$$

for all $g\in\mathrm{O}(2n)$ and $f\in\mathcal{H}_l$.

Let $\mathbb{C}^+$ denote the upper half plane $\{z=x+iy:\ y>0\}$ with invariant measure (under $\mathrm{SL}(2,\mathbb{R})$):

$$\frac{dxdy}{y^2}.$$

Let $L^2_{\mathrm{hol},m}(\mathbb{C}^+)$ be the space of holomorphic functions on $\mathbb{C}^+$ satisfying,

$$\int_{\mathbb{C}^+}|f(z)|^2y^{m-2}dxdy<\infty$$

with m an integer ≥ 2 (in Chapter 2 this space is called $\mathcal{H}_m$). The group $\mathrm{SL}(2,\mathbb{R})$ acts irreducibly on this space by

$$\pi_m(g)f(z)=f(g^{-1}\cdot z)(cz+d)^{-m}$$

with $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $g^{-1} \cdot z = \frac{az+b}{cz+d}$. ($\pi_m$ belongs to the so-called analytic discrete series, see Chapter 2)

Let us define an operator $\mathcal{F}_l$ from $\mathcal{S}(\mathbb{R}^{2n})$ into the Hilbert space $L^2_{\text{hol},l+n}(\mathbb{C}^+)\hat{\otimes}_2\mathcal{H}_l$ as follows:

$$(\mathcal{F}_l\varphi)(\xi, x) = \int_{\mathbb{R}^{2n}} e^{i\pi\xi||y||^2}\varphi(y)K_l(y, x)dy$$

This integral exists for all $x \in \mathbb{R}^{2n}$ and $\xi \in \mathbb{C}^+$. We shall show:

1) The operator $\mathcal{F}_l$ is well defined, i.e. $\mathcal{F}_l\varphi \in L^2_{\text{hol},l+n}(\mathbb{C}^+)\hat{\otimes}_2\mathcal{H}_l$.

2) $\mathcal{F}_l$ is continuous as operator from $\mathcal{S}(\mathbb{R}^{2n})$ to $L^2_{\text{hol},l+n}(\mathbb{C}^+)\hat{\otimes}_2\mathcal{H}_l$.

3) For $l = 0, 1, 2, \ldots$ the operator $\mathcal{F}_l$ intertwines the action of ω_{2n} and $\pi_{l+n} \otimes \rho_l$.

Let us assume for the moment property 1) and 2), and let us show property 3). It is clear that $\mathcal{F}_l$ intertwines the O$(2n)$-action. The intertwining relations for the elements $g(a)$ and $t(b)$ are easy to check. We now check the σ-intertwining relation:

$$\mathcal{F}_l\omega_{2n}(\sigma)\varphi(\xi, x) = \pi_{l+n}(\sigma)(\mathcal{F}_l\varphi)(\xi, x) \tag{6.2}$$

for $\varphi \in \mathcal{S}(\mathbb{R}^{2n})$. By definition,

$$\mathcal{F}_l\omega_{2n}(\sigma)\varphi(\xi, x) = i^n \int_{\mathbb{R}^{2n}} e^{\pi i\xi||y||^2}\hat{\varphi}(-y)K_l(y, x)dy$$

with $\hat{\varphi}$ being the Fourier transform defined by $\hat{\varphi}(x) = \int_{\mathbb{R}^{2n}} \varphi(y)e^{-2i\pi[x,y]}dy.$

We apply the following lemma, see [26], Proposition 5-1.

Lemma 6.1. *For* $P \in \mathcal{H}_l$*, a harmonic polynomial of degree* l*, and* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *one has*

$$\int_{\mathbb{R}^n} e^{i\xi\pi||x||^2}P(x)\hat{\varphi}(x)dx = \xi^{-l-n/2}i^{n/2}\int_{\mathbb{R}^n} e^{-i\pi||x||^2/\xi}P(x)\varphi(x)dx$$

with $\xi \in \mathbb{C}^+$.

The proof is straightforward by induction on l, recalling that $\mathcal{H}_l$ is spanned by the polynomials of the form $[y, z]^l$ with $z \in \mathbb{C}^n$ satisfying $[z, z] = z_1^2 + \cdots + z_n^2 = 0$. For $l = 0$ the result is classical and is related to the theory of the Fourier transform of quadratic characters.
The lemma easily implies (6.2).

Now we prove property 1) and 2). Let us compute the square norm of $\mathcal{F}_l\varphi$ in $L^2_{\text{hol},l+n}(\mathbb{C}^+)\hat{\otimes}_2\mathcal{H}_l$ for $\varphi \in \mathcal{S}(\mathbb{R}^{2n})$. We have:

$$\begin{aligned} ||\mathcal{F}_l\varphi||^2 &= \int_{\mathbb{C}^+}\int_{S^{2n-1}} |(\mathcal{F}_l\varphi)(\xi, s)|^2(\text{Im }\xi)^{n+l-2}dsd\xi \\ &= \int_{\mathbb{C}^+}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} e^{\pi i\xi||y||^2-\pi i\bar{\xi}||z||^2}\varphi(y)\overline{\varphi(z)}K_l(y, z)(\text{Im }\xi)^{n+l-2}dzdyd\xi \end{aligned}$$

here we have used the reproducing property (6.1) of $K_l(y,\cdot)$.

Writing $\varphi(y)=\varphi(rs)$, $\varphi(z)=\varphi(ts')$ with $r=||y||$, $t=||z||$, $s,s'\in S^{2n-1}$, $\xi=u+iv$ and $u\in\mathbb{R}$, $v>0$, we get,

$$\begin{aligned}||\mathcal{F}_l\varphi||^2 &= \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_{S^{2n-1}}\int_{S^{2n-1}} e^{\pi iu(r^2-t^2)}e^{-\pi v(r^2+t^2)}\varphi(rs)\overline{\varphi(ts')}K_l(s,s')\,\cdot\\ &\quad\cdot r^{l+2n-1}t^{l+2n-1}v^{n+l-2}dsds'drdtdvdu\\ &= \int_0^{\infty}\int_0^{\infty}\int_{S^{2n-1}}\int_{S^{2n-1}} e^{-2\pi vr^2}\varphi(rs)\overline{\varphi(rs')}K_l(s,s')r^{2l+4n-3}v^{n+l-2}dsds'drdv\\ &= \int_0^{\infty}\left(\int_0^{\infty}e^{-2\pi vr^2}v^{n+l-2}dv\right)\left(\int_{S^{2n-1}}\int_{S^{2n-1}}\varphi(rs)\overline{\varphi(rs')}K_l(s,s')dsds'\right)\cdot\\ &\quad\cdot r^{2l+4n-3}dr \end{aligned}\tag{6.3}$$

Setting $2\pi r^2v=w$ we obtain

$$\int_0^{\infty}e^{-2\pi vr^2}v^{n+l-2}dv=\frac{\Gamma(n+l-1)}{(2\pi r^2)^{n+l-1}}.$$

Defining

$$(P_l\varphi)(rs')=\int_{S^{2n-1}}\varphi(rs)K_l(s,s')ds,$$

we get

$$\int_{S^{2n-1}}\int_{S^{2n-1}}\varphi(rs)\overline{\varphi(rs')}K_l(s,s')dsds'=\int_{S^{2n-1}}|(P_l\varphi)(rs')|^2ds'$$

since $P_l\varphi(r\cdot)$ is in $\mathcal{H}_l$ and $\varphi(rs)=\sum_{l=0}^{\infty}(P_l\varphi)(rs)$. By substitution we obtain from (6.3),

$$||\mathcal{F}_l\varphi||^2=\frac{\Gamma(n+l-1)}{(2\pi)^{n+l-1}}\int_0^{\infty}\int_{S^{2n-1}}|(P_l\varphi)(rs')|^2r^{2n-1}ds'dr$$

By Parseval's theorem,

$$\sum_{l\geq 0}\int_{S^{2n-1}}|(P_l\varphi)(rs')|^2ds'=\int_{S^{2n-1}}|\varphi(rs')|^2ds'$$

so finally we obtain

$$\sum_{l=0}^{\infty}\frac{(2\pi)^{n+l-1}}{\Gamma(n+l-1)}\cdot||\mathcal{F}_l\varphi||^2=\int_0^{\infty}\int_{S^{2n-1}}|\varphi(rs')|^2r^{2n-1}ds'dr=||\varphi||_2^2\tag{6.4}$$

This proves property 1) and also property 2) since (6.4) implies the inequality

$$||\mathcal{F}_l\varphi||^2\leq\frac{\Gamma(n+l-1)}{(2\pi)^{n+l-1}}||\varphi||_2^2,$$

which actually shows that $\mathcal{F}_l$ is continuous in the L^2-topology on $\mathcal{S}(\mathbb{R}^{2n})$, so on $\mathcal{S}(\mathbb{R}^{2n})$ itself, since the embedding of $\mathcal{S}(\mathbb{R}^{2n})$ into $L^2(\mathbb{R}^{2n})$ is continuous.

So we have shown:

Theorem 6.2. *For* $l = 0, 1, 2, \dots$ *the operator* $\mathcal{F}_l$ *is a continuous intertwining operator for the action of* ω_{2n} *on* $\mathcal{S}(\mathbb{R}^{2n})$ *and* $\pi_{l+n} \otimes \rho_l$ *on* $L^2_{hol,l+n}(\mathbb{C}^+)\hat{\otimes}_2\mathcal{H}_l$.

6.3 Decomposition of $L^2(\mathbb{R}^{2n})$

Clearly the proof of Theorem 6.2 in Section 6.2 also gives the decomposition of $L^2(\mathbb{R}^{2n})$ into minimal invariant subspaces. The decomposition is multiplicity free (see [16]) and the irreducible unitary representations which occur are given by $\pi_{l+n} \otimes \rho_l$. Observe that the factor $d_l = \frac{\Gamma(n+l-1)}{(2\pi)^{n+l-1}}$ is the formal degree of π_{n+l}. In conclusion we have,

Corollary 6.3. *Let* φ *be a function in* $\mathcal{S}(\mathbb{R}^{2n})$. *We have the following Plancherel formula*

$$||\varphi||_2^2 = \sum_{l=0}^{\infty} \frac{(2\pi)^{n+l-1}}{\Gamma(n+l-1)} \cdot ||\mathcal{F}_l\varphi||^2.$$

As representations of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{O}(2n)$, *we have*

$$L^2(\mathbb{R}^{2n}) \cong \sum_{l=0}^{\infty} \pi_{l+n} \otimes \rho_l.$$

6.4 Classification of all minimal invariant Hilbert subspaces of $\mathcal{S}'(\mathbb{R}^{2n})$

Instead of classifying the minimal $\omega_{2n}(G)$-invariant Hilbert subspaces of $\mathcal{S}'(\mathbb{R}^{2n})$ we shall classify the extremal reproducing kernels.

Let us first compute the reproducing kernel of $\pi_{l+n} \otimes \rho_l$. It is given by the distribution T_l on $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$, satisfying:

$$< T_l, \varphi \otimes \psi >= (\mathcal{F}_l\varphi, \mathcal{F}_l\psi) = \int_{\mathbb{C}^+} \int_{S^{2n-1}} \mathcal{F}_l\varphi(\xi, s)\overline{\mathcal{F}_l\psi(\xi, s)}(\mathrm{Im}\ \xi)^{l+n-2} ds d\xi.$$

T_l is equal to the distribution:

$$T_l(x, y) = d_l \delta(||x||^2 - ||y||^2)||x||^{-2n+2} K_l(s, s')$$

on $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \setminus (0, 0)$. This easily follows from the computation of $||\mathcal{F}_l\varphi||^2$ in Section 6.2.

Let now H be a Hilbert subspace of $\mathcal{S}'(\mathbb{R}^{2n})$, invariant under $\omega_{2n}(G)$, and let K be its reproducing kernel. By Schwartz's kernel theorem we can associate to it a unique tempered distribution T on $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$. This reproducing distribution satisfies the following conditions:

1. T is a positive-definite kernel, in particular a Hermitian kernel, i.e.
2. $T(x,y) = \overline{T}(y,x)$.
3. T is $\omega_{2n}(G)$-invariant: $(\omega_{2n}(g) \times \omega_{2n}(g))T = T$ for all $g \in G$. The latter property implies, in more detail,

   3a. $T(g \cdot x, g \cdot y) = T(x,y)$ for all $g \in \mathrm{O}(2n)$.

   3b. If $a \in \mathbb{R}^*$, then
   $$T(ax, ay) = |a|^{-2n} T(x,y).$$

   3c. If $b \in \mathbb{R}$, then
   $$e^{-i\pi b(||x||^2 - ||y||^2)} T(x,y) = T(x,y).$$

   3d. $\hat{T}(x,y) = T(x,y)$.

By condition 3c, we obtain

$$\mathrm{Supp}\, T \subset \Xi_{2n} = \{(x,y) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n} \ : \ ||x|| = ||y||\}.$$

Let $\Xi'_{2n} = \Xi_{2n} \setminus (0,0)$, being a cone in $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$. In a neighborhood of Ξ'_{2n} in $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ we take as coordinates $s = ||x||^2 - ||y||^2$ and $w \in \Xi'_{2n}$. So we can write locally there

$$T = \sum_{i \in I} S_i(w) \otimes \delta^{(i)}(s)$$

where I is some finite subset of $\mathbb{N}$ and the S_i are distributions on Ξ'_{2n}. Applying condition 3c again, we get that only the term with $i = 0$ survives, so

$$T(x,y) = \delta(||x||^2 - ||y||^2) R(w)$$

outside $(x,y) = (0,0)$. It remains to study the distributions R on Ξ'_{2n}. We get by properties 3a and 3b,

$$R(x,y) = ||x||^{-2n+2} \sigma(s, s')$$

where σ is a positive-definite distribution on $S^{2n-1} \times S^{2n-1}$ which is $\mathrm{O}(2n)$-invariant.

Now we have the following theorems. Let K_l be, as before, the reproducing kernel of $\mathcal{H}_l$, the space of harmonic polynomials on $\mathbb{R}^n$.

Theorem 6.4. *Let U be a positive definite distribution on $S = S^{n-1} \times S^{n-1}$ $\mathrm{O}(n)$-invariant. Then*

$$U = \sum_{l \geq 0} m_l K_l$$

with $m_l \geq 0$ and m_l of polynomial growth in l.

Proof. First we shall prove that $C^\infty(S)$ has a topology also given by semi-norms $\left(\sum_{j=0}^{m} ||\Delta_S^j f||_2^2\right)^{1/2}$ with m an integer ≥ 0 and Δ_S the Laplacian on S. Let $\mathcal{H}$ be equal to $L^2(S)$, π the continuous representation of $\mathrm{O}(n)$ on $\mathcal{H}$ by left translations and $\mathcal{H}_\infty$ the space of C^∞ vectors for π. We know that $C^\infty(S) \subset \mathcal{H}_\infty$. By [5], Theorem 3.3 we even have that

$$C^\infty(S) = \mathcal{H}_\infty. \tag{6.5}$$

Applying a result of Goodman (see [2], Theorem 1.2) and (6.5) we obtain that

$$C^\infty(S) = \{f \in L^2(S) : \left(\sum_{j=0}^{m} ||\Delta_S^j f||_2^2\right)^{1/2} < \infty\}$$

and thus, by Banach's Theorem, the topology of $C^\infty(S)$ is also given by the semi-norms $\left(\sum_{j=0}^{m} ||\Delta_S^j f||_2^2\right)^{1/2}$.

By [7], Theorem Bochner-Schwartz page 375, $U = \sum_{l\geq 0} m_l K_l$ with $m_l \geq 0$ and $< U, K_l >= m_l$. Since U is $\mathrm{O}(n)$-invariant

$$U(s,s') = U(s,ks^0) = U(k^{-1}s,s^0) = \sum_{l\geq 0} m_l K_l(k^{-1}s,s^0)$$

with $s^0 = (1,0,\ldots,0)$ and $k \in \mathrm{O}(n)$. So there exists a constant C such that,

$$\begin{aligned} m_l &= \; < U(\cdot,s^0), K_l(\cdot,s^0) > \\ &\leq C\left(\sum_{j=0}^{m} ||\Delta_S^j K_l(\cdot,s^0)||_2^2\right)^{1/2} \\ &= C\left(\sum_{j=0}^{m} ||l^j(l+n-2)^j K_l(\cdot,s^0)||_2^2\right)^{1/2} \\ &= C(\dim\mathcal{H}_l)^{1/2}\left(\sum_{j=0}^{m} l^{2j}(l+n-2)^{2j}\right)^{1/2} \end{aligned}$$

Since $\dim\mathcal{H}_l \sim l^{n-1}$, m_l has polynomial growth.□

Theorem 6.5. *i) The extremal positive-definite $\omega_{2n}(G)$-invariant reproducing distributions are given by $T_l(x,y)$ up to a positive constant. The associated irreducible unitary representations are $\pi_{n+l} \otimes \rho_l$ where $l = 0,1,2,\ldots$*

ii) Any $\omega_{2n}(G)$*-invariant Hilbert subspace of* $\mathcal{S}'(\mathbb{R}^{2n})$ *decomposes multiplicity free into minimal invariant Hilbert subspaces of* $\mathcal{S}'(\mathbb{R}^{2n})$*.*

Proof. i) Let T be an extremal positive-definite $\omega_{2n}(G)$-invariant reproducing distribution. We saw before that a reproducing distribution outside of $(0,0)$ is equal to

$$T(x,y)=\delta(||x||^2-||y||^2)||x||^{-2n+2}\sigma(s,s')$$

where σ is a positive definite distribution on $S^{2n-1}\times S^{2n-1}$ which is $\mathrm{O}(2n)$-invariant. One has

$$\sigma(s,s')=\sum_{l\geq 0} m_l K_l(s,s')$$

with $m_l\geq 0$, m_l of polynomial growth in l by the last theorem.

Let us now formally consider $\mathcal{T}=\sum\limits_{l\geq 0}\frac{m_l}{d_l}T_l$. This is a convergent series and $\mathcal{T}$ a tempered distribution satisfying the conditions a, b, c, d. Indeed, let Δ_S be the Laplacian on S^{2n-1}. Then we have:

$$\int_{S^{2n-1}}\varphi\cdot\Delta_S P ds=\int_{S^{2n-1}}\Delta_S\varphi\cdot P ds$$

for all $\varphi\in\mathcal{D}(S^{2n-1})$ and $P\in\mathcal{H}_l$ (all l). Notice that Δ_S commutes with the orthogonal projection of $\mathcal{D}(S^{2n-1})$ on $\mathcal{H}_l$. Moreover

$$\Delta_S P=-l(l+2n-2)P$$

for all $P\in\mathcal{H}_l$. By the computations in Section 6.2 we now get:

$$|<\mathcal{T},\varphi\otimes\varphi>|\leq c||\Delta_S^k\varphi||_2^2$$

for some constant $c>0$ and some integer $k\geq 0$, where

$$||\Delta_S^k\varphi||_2^2=\int_0^\infty\int_{S^{2n-1}}|\Delta_S^k\varphi(rs)|^2r^{2n-1}drds.$$

So $\mathcal{T}$ is tempered and clearly satisfies a, b, c, d since all T_l do.

From property 3c we conclude that

$$(||x||^2-||y||^2)^k\,(T-\mathcal{T})=0$$

for $k=1,2,\ldots$, hence, combining it with property 3d, we have

$$(\Delta_x-\Delta_y)^k\,(T-\mathcal{T})=0$$

for $k=1,2,\ldots$, in particular $(\Delta_x-\Delta_y)\,(T-\mathcal{T})=0$. Here Δ_x and Δ_y denote

$$\Delta_x=\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_{2n}^2},$$

and

$$\Delta_y = \frac{\partial^2}{\partial y_1^2} + \cdots + \frac{\partial^2}{\partial y_{2n}^2}.$$

The difference $T - \mathcal{T}$ is easily seen to be zero since this difference has support in $(0,0)$ and satisfies $(\Delta_x - \Delta_y)(T - \mathcal{T}) = 0$. So $T = \mathcal{T}$. So, if T is minimal, T is a positive scalar multiple of some T_l.

Observe that any minimal invariant Hilbert subspace of $\mathcal{S}'(\mathbb{R}^{2n})$ occurs in the decomposition of $L^2(\mathbb{R}^{2n})$. Moreover, given l, or $\mathcal{H}_l$, the $\mathrm{SL}(2,\mathbb{R})$ component of the irreducible representation $\pi \otimes \rho_l$ is determined, namely $\pi = \pi_{n+l}$. This is the well-known Howe-correspondence.

ii) As preparation for the multiplicity free decomposition of the oscillator representation, we shall show that any of the above reproducing distributions T is symmetric: $T(x,y) = T(y,x)$. We have seen that

$$T(x,y) = \delta(||x||^2 - ||y||^2)||x||^{-2n+2}\sigma(s,s')$$

outside $(x,y) = (0,0)$ where σ is a positive-definite distribution on $S^{2n-1} \times S^{2n-1}$ which is $\mathrm{O}(2n)$-invariant. Observe that σ is symmetric, i.e. $\sigma(s,s') = \sigma(s',s)$ since $\mathrm{O}(2n)$ acts doubly transitivily on S^{2n-1}, hence

$$T(x,y) = T(y,x)$$

on $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \setminus (0,0)$.

We easily show that $T(x,y) - T(y,x)$, having support in $(0,0)$ and satisfying $(\Delta_x - \Delta_y)(T(x,y) - T(y,x)) = 0$ as before, is actually zero. Hence $T(x,y) = T(y,x)$, i.e. T is symmetric.

The multiplicity free decomposition of the oscillator representation now easily follows from Criterion 5.6 with $JT = \overline{T}$. $\square$

Remark 6.6. *The case $n = 1$ is treated in a similar way. The contribution for $l = 0$ is given by $\pi_1 \otimes id$, where π_1 is the limit of the analytic discrete series representation, discussed in Chapter 2. So here a non-discrete series representation occurs in the decomposition of $L^2(\mathbb{R}^2)$.*

Chapter 7

Invariant Hilbert subspaces of the oscillator representation of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{O}(\mathrm{p},\mathrm{q})$

The explicit decomposition of the oscillator representation $\omega_{p,q}$ for the dual pair $G = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{O}(\mathrm{p},\mathrm{q})$ was given by B. Ørsted and G. Zhang in [23]. In this chapter we only study the multiplicity free decomposition of any $\omega_{p,q}(G)$-stable Hilbert subspace of $\mathcal{S}'(\mathbb{R}^{p+q})$. The oscillator representation acts on the Hilbert space $L^2(\mathbb{R}^{p+q})$. It is well-known that $\mathcal{S}(\mathbb{R}^{p+q})$, the space of Schwartz functions on $\mathbb{R}^{p+q}$ is $\omega_{p,q}(G)$-stable, so, by duality, $\omega_{p,q}$ acts on $\mathcal{S}'(\mathbb{R}^{p+q})$, the space of tempered distributions on $\mathbb{R}^{p+q}$, as well, and $L^2(\mathbb{R}^{p+q})$ can thus be considered as an invariant Hilbert subspace of $\mathcal{S}'(\mathbb{R}^{p+q})$.

According to Howe [16], $L^2(\mathbb{R}^{p+q})$ decomposes multiplicity free into minimal invariant Hilbert subspaces of $\mathcal{S}'(\mathbb{R}^{p+q})$.

We restrict to the case $p+q$ even for simplicity of the presentation of the main results. In addition we have to assume $p \geq 1$, $q \geq 2$, since we apply results from [9] where this condition is imposed. Our result is however true in general.

The contents of this chapter have appeared in [39].

7.1 The definition of the oscillator representation

Let $G = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{O}(\mathrm{p},\mathrm{q})$ and let $\omega_{p,q}$ be the unitary representation of G on $H = L^2(\mathbb{R}^{p+q})$ defined by:

$$\omega_{p,q}(g)f(x) = f(g^{-1} \cdot x), \qquad g \in \mathrm{O}(\mathrm{p},\mathrm{q}),$$

$$\omega_{p,q}(g(a))f(x) = |a|^{\frac{p+q}{2}} \operatorname{sgn}^{\frac{p-q}{2}}(a) f(ax), \qquad g(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

$$\omega_{p,q}(t(b))f(x) = e^{-i\pi b[x,x]} f(x), \qquad t(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

$$\omega_{p,q}(\sigma)f(x) = i^{\frac{p-q}{2}} \int_{\mathbb{R}^{p+q}} e^{2i\pi[x,y]} f(y)dy, \qquad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where $[x,y] = x_1y_1 + ... + x_py_p - x_{p+1}y_{p+1} - ... - x_ny_n$ for $x = (x_1, ..., x_n), y = (y_1, ..., y_n)$ and $n = p + q$.

This is a well-defined representation for $p + q$ even. If $p + q$ is odd, this is a unitary representation of $\widetilde{\mathrm{SL}}(2, \mathbb{R}) \times \mathrm{O(p, q)}$ with $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ a double covering of $\mathrm{SL}(2, \mathbb{R})$ [10],[29],[30],[44].

We call $\omega_{p,q}$ the oscillator representation of G. More precisely, it is the natural extension of the restriction of the metaplectic representation of the double covering of the group $Sp(n, \mathbb{R})$ to $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO(p, q)}$, defined by Shale, Segal and Weil.

We shall assume from now on that $n = p+q$ is even. Let $\mathcal{S}(\mathbb{R}^n)$ be the space of Schwartz functions on $\mathbb{R}^n$. Note that $\mathcal{S}(\mathbb{R}^n)$ is stable under the action of $\omega_{p,q}(G)$, so is the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions on $\mathbb{R}^n$.

7.2 Invariant Hilbert subspaces of $\mathcal{S}'(\mathbb{R}^n)$

Given a function $f \in \mathcal{S}(\mathbb{R}^n)$, we define its Fourier transform $\hat{f}$ by

$$\hat{f}(x) = \int_{\mathbb{R}^n} f(y) e^{-2\pi i[x,y]} dy.$$

This definition naturally gives rise to a Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$. We denote $\hat{T}$ this Fourier transform of a tempered distribution $T \in \mathcal{S}'(\mathbb{R}^n)$. Observe that $E = \mathcal{S}'(\mathbb{R}^n)$ is a quasi-complete, barrelled, locally convex, conuclear space and $\omega_{p,q}(G)$ is a group of continuous automorphisms of E. So the theory of Chapter 5 applies in its full strength. Moreover $\omega_{p,q}$ is a representation of G on $\mathcal{S}(\mathbb{R}^n)$ and its contragradient is a representation of G on $\mathcal{S}'(\mathbb{R}^n)$. The first statement follows from [10]: the space $\mathcal{S}(\mathbb{R}^n)$ is precisely the space of C^∞-vectors for the oscillator representation of the double cover of $Sp(n, \mathbb{R})$ on $L^2(\mathbb{R}^n)$. The second statement follows since $\mathcal{S}(\mathbb{R}^n)$ is a Montel space.

Let H be a Hilbert subspace of $\mathcal{S}'(\mathbb{R}^n)$, invariant under $\omega_{p,q}(G)$, and let K be its reproducing kernel. By Schwartz's kernel theorem (see Theorem 5.9) we can associate to it a unique tempered distribution T on $\mathbb{R}^n \times \mathbb{R}^n$. This distribution satisfies the following conditions:

1. T is a positive-definite kernel, in particular a Hermitian kernel, i.e.
2. $T(x, y) = \bar{T}(y, x)$.
3. T is $\omega_{p,q}(G)$-invariant : $(\omega_{p,q}(g) \times \omega_{p,q}(g))T = T$ for all $g \in G$. The latter property implies, in more detail,

3a. $T(g \cdot x, g \cdot y) = T(x, y)$ for all $g \in \mathrm{O(p, q)}$.

3b. If $a \in \mathbb{R}^*$, then

$$T(ax,\, ay) = |a|^{-n} T(x, y).$$

3c. If $b \in \mathbb{R}$, then

$$e^{-i\pi b([x,x]-[y,y])} T(x, y) = T(x, y).$$

3d. $\hat{T}(x,y) = T(x,y)$.

A straightforward example is given by $T(x,y) = \delta(x-y)$, the reproducing distribution of $H = L^2(\mathbb{R}^n)$.

As preparation for our main result, the multiplicity free decomposition of the oscillator representation, we shall show that any of the above distributions T is symmetric: $T(x,y) = T(y,x)$. We shall do this in several steps. Observe that $T(y,x)$ satisfies the same conditions as $T(x,y)$.

Step I

By condition 3c, we obtain

$$\operatorname{Supp} T \subset \Xi_{n,n} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \;:\; [x,x] - [y,y] = 0\}.$$

Step II

Let $\Xi'_{n,n} = \Xi_{n,n} \backslash (0,0)$, being an isotropic cone in $\mathbb{R}^n \times \mathbb{R}^n$. In a neighborhood of $\Xi'_{n,n}$ in $\mathbb{R}^n \times \mathbb{R}^n$ we take as coordinates $s = [x,x] - [y,y]$ and $\omega \in \Xi'_{n,n}$. So we can write locally there

$$T = \sum_{i \in I} S_i(\omega) \otimes \delta^{(i)}(s)$$

where I is some finite subset of $\mathbb{N}$ and the S_i are distributions on $\Xi'_{n,n}$. Applying condition 3c again, we get that only the term with $i = 0$ survives, so

$$T(x,y) = \delta([x,x] - [y,y])\, S_0(\omega)$$

outside $(x,y) = (0,0)$. It remains to study the distribution S_0 on $\Xi'_{n,n}$.

Step III

On the open subset of $\Xi'_{n,n}$ given by $[x,x] \neq 0$, we get by properties 3a and 3b, with $\rho = \frac{1}{2}(n-2)$,

$$S_0(x,y) = [x,x]^{-\rho}\, \sigma_0(\omega_1,\, \omega_2)$$

where σ_0 is an $\mathrm{O(p,q)}$-invariant distribution on $\mathcal{X}^+ \times \mathcal{X}^+$ or $\mathcal{X}^- \times \mathcal{X}^-$, with $\mathcal{X}^\pm = \{x \in \mathbb{R}^n \mid [x,x] = \pm 1\}$. It is known that σ_0 is symmetric: $\sigma_0(\omega_1,\, \omega_2) = \sigma_0(\omega_2,\, \omega_1)$ (see [38]), hence

$$T(x,y) = T(y,x)$$

on the open subset of $\mathbb{R}^n \times \mathbb{R}^n$ defined by $[x,x] \neq 0$ (or, what is the same, $[y,y] \neq 0$).

So $S(x,y) = T(x,y) - T(y,x)$ has support contained in $[x,x] = [y,y] = 0$.

Step IV

Set $\Xi_n = \{x \in \mathbb{R}^n \mid [x,x] = 0\}$. The distribution S has support in $\Xi_n \times \Xi_n$. With the coordinates $s_1 = [x,x]$, $s_2 = [y,y]$ near $\Xi_n \times \Xi_n$, which can be taken, provided $x \neq 0$ and $y \neq 0$, so on $\Xi'_n \times \Xi'_n$ (in obvious notation), we get

$$S(x,y) = \delta([x,x],[y,y])\, U(\xi_1,\, \xi_2) \quad (\xi_1,\, \xi_2 \in \Xi'_n)$$

where U is an $\mathrm{O(p,q)}$-invariant distribution on $\Xi'_n \times \Xi'_n$, homogeneous of degree $-2\rho + 2$. Here we applied 3c again. Since $S(x,y) = -S(y,x)$ it easily follows that $U(\xi_1,\, \xi_2) = -U(\xi_2,\, \xi_1)$.

Before we continue the preparation, we will recall now some structure theory of $\mathrm{O(p,q)}$ and some results of [9] about distributions on Ξ'_n. We therefore assume (as in [9]) $p \geq 1$, $q \geq 2$. Let P be the stabilizer in $\mathrm{O(p,q)}$ of the line generated by $\xi^0 = (1,0,\ldots,0,1)$. P is a maximal parabolic subgroup and has Langlands decomposition $P = MAN$.

The subgroup M consists of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where m is a matrix of the group $\mathrm{O(p-1,q-1)}$.

The group A is the one-parameter subgroup of matrices

$$a_u = \begin{pmatrix} \frac{1}{2}(u + \dfrac{1}{u}) & 0 & \frac{1}{2}(u - \dfrac{1}{u}) \\ 0 & 1 & 0 \\ \frac{1}{2}(u - \dfrac{1}{u}) & 0 & \frac{1}{2}(u + \dfrac{1}{u}) \end{pmatrix}$$

where $u \in \mathbb{R}^*$.

The subgroup N consists of matrices of the form

$$n_\alpha = \begin{pmatrix} 1 - \frac{1}{2}Q(\alpha) & \alpha^* & \frac{1}{2}Q(\alpha) \\ \alpha & 1 & -\alpha \\ -\frac{1}{2}Q(\alpha) & \alpha^* & 1 + \frac{1}{2}Q(\alpha) \end{pmatrix}$$

where $\alpha \in \mathbb{R}^{n-2}$. If $\alpha = (\alpha_2, \ldots, \alpha_{n-1})$, then

$$\alpha^* = -\alpha^t \cdot I_{p-1,q-1},$$

with

$$I_{p-1,q-1} = \begin{pmatrix} I_{p-1} & 0 \\ 0 & -I_{q-1} \end{pmatrix}$$

and $Q(\alpha) = \alpha_2^2 + \cdots + \alpha_p^2 - \alpha_{p+1}^2 - \cdots - \alpha_{p+q-1}^2$.

The group N is isomorphic to $\mathbb{R}^{n-2}$. Moreover:

$$a_u \, n_\alpha \, a_u^{-1} = n_{u\alpha}.$$

Step V

Since $\mathrm{O(p,q)}$ acts transitively on Ξ'_n, we can conclude that to U corresponds a MN-invariant distribution on Ξ'_n. Call it V.

We recall some results of [9] about such distributions, see ([9], III).

Define $u(\xi) = \frac{1}{2}(\xi_1 - \xi_n)$ on Ξ'_n and set for $t \in \mathbb{R}$, $\Gamma_t = \{\xi \mid u(\xi) = t\}$.

For $f \in \mathcal{D}(\Xi'_n)$, the space of C_c^∞-functions on Ξ'_n, one can define the function $\mathcal{M}f$ on $\mathbb{R}^*$ by

$$\mathcal{M}f(t) = \int f(\xi)\, \delta\,(u(\xi) - t)\,.$$

$\mathcal{M}$ maps $\mathcal{D}(\Xi'_n)$ continuously onto some topological vector space $\mathcal{F}$ of functions on $\mathbb{R}^*$ with singularities at $t = 0$ (see also [30]). And if W is a continuous linear form on $\mathcal{F}$, the distribution V defined on Ξ'_n by

$$V(f) = W(\mathcal{M}f) \quad (f \in \mathcal{D}(\Xi'_n)),$$

i.e., $V = \mathcal{M}'W$, is invariant under MN. $\mathcal{M}'$ is not surjective. One has ([9], Proposition III.1): if V is a distribution on Ξ'_n invariant under MN, then

$$V = \mathcal{M}'W + V_1$$

where W is a continuous linear form on $\mathcal{F}$ and V_1 is a distribution on Ξ'_n, MN-invariant and with support contained in Γ_0.

The structure of the distributions V_1 is as follows. We use the local chart in a neighborhood of Γ_0 given by the map from $A \times \overline{N}$ to Ξ'_n,

$$(a_u, \overline{n}_\alpha) \to a_u \, \overline{n}_\alpha \xi^0,$$

where $\overline{N}$ consists of the matrices

$$\overline{n}_\alpha = \begin{pmatrix} 1 - \frac{1}{2}Q(\alpha) & -\alpha^* & \frac{1}{2}Q(\alpha) \\ -\alpha & 1 & -\alpha \\ \frac{1}{2}Q(\alpha) & \alpha^* & 1 + \frac{1}{2}Q(\alpha) \end{pmatrix}, \quad \alpha \in \mathbb{R}^{n-2}.$$

We denote by Δ the differential operator in this chart given by

$$\Delta = \frac{\partial^2}{\partial \alpha_2^2} + \cdots + \frac{\partial^2}{\partial \alpha_p^2} - \frac{\partial^2}{\partial \alpha_{p+1}^2} - \cdots - \frac{\partial^2}{\partial \alpha_{p+q-1}^2}.$$

We quote [9], Théorème III.2 here:

Let V_1 be a MN-invariant distribution on Ξ'_n with support in Γ_0. Then there exist a distribution T_0 on $\mathbb{R}^n$ and constants A_k, B_k, $k = 1, 2, \ldots, m$ such that, in the above chart,

$$V_1 = T_0 \otimes \delta + \sum_{k=1}^{m} (A_k + B_k \operatorname{sgn}(u)) \, |u|^{\rho+k} \, \frac{du}{|u|} \otimes \Delta^k \delta$$

where δ is the Dirac measure at $\alpha = 0$.

Step VI

We return to our distribution V from Step V, and write it in the form $V = \mathcal{M}'W + V_1$, as above. By abuse of notation we have

$$V(\xi) = V(g\xi^0) = U(g\xi^0, \, \xi^0).$$

Because $u(\xi) = u(g\xi^0) = \frac{1}{2} \, [g\xi^0, \, \xi^0]$ satisfies $u(g\xi^0) = u(g^{-1}\xi^0)$, we easily get that

$$\mathcal{M}'W(g\xi^0) = \mathcal{M}'W(g^{-1}\xi^0).$$

Since $V(g\xi^0) = -V(g^{-1}\xi^0)$, we see that $2V(g\xi^0) = V_1(g\xi^0) - V_1(g^{-1}\xi^0)$. Now

$$V(a_{u_0}^{-1} g a_{u_0}\xi^0) = U(g a_{u_0}\xi^0,\, a_{u_0}\xi^0) =$$

$$U(u_0 g\xi^0,\, u_0\xi^0) = |u_0|^{-2\rho+2} V(g\xi^0).$$

Let us apply this property to the distribution $V_2(g\xi^0) = V_1(g\xi^0) - V_1(g^{-1}\xi^0)$, supported by Γ_0. We get with $g = a_u \overline{n}_\alpha \xi^0$, that

$$a_{u_0}^{-1} a_u \overline{n}_\alpha a_{u_0} = a_u a_{u_0}^{-1} \overline{n}_\alpha a_{u_0} = a_u \overline{n}_{u_0\alpha},$$

hence

$$V_2(a_u\, \overline{n}_{u_0\alpha}\, \xi^0) = |u_0|^{-n+2} T_0 \otimes \delta(\alpha)$$

$$+ \sum_{k=1}^{m} |u_0|^{-n+2-2k} \left(A_k + B_k \operatorname{sgn}(u)\right) |u|^{\rho+k} \frac{du}{|u|} \otimes \Delta^k \delta.$$

Since $V_2(a_u\, \overline{n}_{u_0\alpha}\, \xi^0) = |u_0|^{-n+4} V_2(a_u\, \overline{n}_\alpha\, \xi^0)$, we get $V_2 = 0$. So $V = 0$, and hence $S = 0$ in a neighborhood of $\Xi'_n \times \Xi'_n$, and therefore $\operatorname{Supp} S \subset \{(0) \times \Xi_n\} \cup \{\Xi_n \times (0)\}$.

Step VII

From property 3c we conclude that

$$([x,x] - [y,y])^k\, S = 0$$

for $k = 1, 2, \ldots$, hence, combining it with property 3d, we have

$$\Delta^k\, S = 0$$

for $k = 1, 2, \ldots$, in particular $\Delta\, S = 0$. Here Δ denotes the d'Alembertian

$$(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_n^2}) - (\frac{\partial^2}{\partial y_1^2} + \cdots + \frac{\partial^2}{\partial y_p^2} - \frac{\partial^2}{\partial y_{p+1}^2} - \cdots - \frac{\partial^2}{\partial y_n^2}).$$

If S has support in $\{(0) \times \Xi_n\} \cup \{\Xi_n \times (0)\}$, and $\Delta\, S = 0$, it easily follows, by recalling the local structure of such distributions, being basically a finite linear combination of tensor products of distributions supported by Ξ_n and the origin, that $S = 0$ (see [28],Ch.III, §10). So finally we have shown that $T(x,y) = T(y,x)$.

7.3 Multiplicity free decomposition of the oscillator representation

The multiplicity free decomposition of the oscillator representation, i.e. the multiplicity free decomposition into irreducible invariant Hilbert subspaces of any $\omega_{p,q}(G)$-invariant Hilbert subspace of $\mathcal{S}'(\mathbb{R}^n)$, is now easily proved by applying Criterion 5.6 with $JT = \bar{T}$. If H is any invariant Hilbert subspace of $\mathcal{S}'(\mathbb{R}^n)$ with reproducing kernel $T \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$, then $\bar{T}$ is the kernel of the space $J\,H$ and the property $J\,H = H$ comes down to $\bar{T}(x,y) = T(x,y)$. Since T is positive-definite, this is equivalent with $T(x,y) = T(y,x)$ which immediately follows from Section 7.2. So we have the following result.

Theorem 7.1. *Any $\omega_{p,q}(G)$-invariant Hilbert subspace of $\mathcal{S}'(\mathbb{R}^n)$ decomposes multiplicity free into irreducible invariant Hilbert subspaces of $\mathcal{S}'(\mathbb{R}^n)$.*

Theorem 7.1 is well-known for $H = L^2(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$, see [16]. We have shown that the result extends to any $\omega_{p,q}(G)$-invariant Hilbert subspace of $\mathcal{S}'(\mathbb{R}^n)$.

Chapter 8

Decomposition of the oscillator representation of $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SO}(n,\mathbb{C})$

In this chapter we determine the explicit decomposition of the oscillator representation for the groups $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{O}(1,\mathbb{C})$ and $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SO}(n,\mathbb{C})$ with $n \geq 2$. The main tool we use is a Fourier integral operator introduced by M. Kashiwara and M. Vergne for real matrix groups, see [18]. For the group $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SO}(n,\mathbb{C})$ with $n \geq 2$, given that $\mathrm{SO}(2,\mathbb{C})$ is an abelian group, we split the proof into the cases $n = 2$ and $n \geq 3$. The computation of the Plancherel measure for $\mathrm{SO}(n,\mathbb{C})/\mathrm{SO}(n-1,\mathbb{C})$ is required in order to compute the explicit decomposition for the case $n \geq 3$.

8.1 The case $n = 1$

8.1.1 The definition of the oscillator representation

Let G be the group $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{O}(1,\mathbb{C})$ and let ω_1 be the unitary representation of G on $H = L^2(\mathbb{C})$ defined by:

$$\begin{aligned}
\omega_1(g)f(z) &= f(g^{-1} \cdot z), \qquad g \in \mathrm{O}(1,\mathbb{C}) \\
\omega_1(g(a))f(z) &= |a|f(az), \qquad g(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad a \in \mathbb{C}^* \\
\omega_1(t(b))f(z) &= e^{-i\pi \mathrm{Re}\ (bz^2)} f(z), \qquad t(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad b \in \mathbb{C} \\
\omega_1(\sigma)f(z) &= \int_{\mathbb{C}} e^{2\pi i \mathrm{Re}\ (zw)} f(w)dw, \qquad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\end{aligned}$$

where the Fourier transform is defined by

$$\hat{f}(z) = \int_{\mathbb{C}} e^{-2\pi i \mathrm{Re}\ (zw)} f(w) dw.$$

We call ω_1 the oscillator representation of G. More precisely, it is the natural extension of the restriction of the metaplectic representation of the group $Sp(1, \mathbb{C}) \subset Sp(2, \mathbb{R})$ to $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SO}(1, \mathbb{C})$.

Let $\mathcal{S}(\mathbb{C})$ be the space of Schwartz functions on $\mathbb{C}$. Note that $\mathcal{S}(\mathbb{C})$ is stable under the action of ω_1, so is the space $\mathcal{S}'(\mathbb{C})$ of tempered distributions on $\mathbb{C}$.

8.1.2 Some minimal invariant Hilbert subspaces of $\mathcal{S}'(\mathbb{C})$

We consider $L^2(\mathbb{C})$ as a Hilbert subspace of $\mathcal{S}'(\mathbb{C})$. This space can be decomposed as $L^2(\mathbb{C}) = L^2(\mathbb{C})_+ \oplus L^2(\mathbb{C})_-$ with $L^2(\mathbb{C})_+ = \{f \in L^2(\mathbb{C}) : f \text{ is even}\}$ and $L^2(\mathbb{C})_- = \{f \in L^2(\mathbb{C}) : f \text{ is odd}\}$. Irreducible unitary representations of G are of the form $\pi \otimes \rho$ where π is an irreducible representation of $\mathrm{SL}(2, \mathbb{C})$ and ρ one of $\mathrm{O}(1, \mathbb{C})$. We will now construct intertwining operators from $\mathcal{S}(\mathbb{C})$ to the Hilbert space of such representations. Not all combinations (π, ρ) occur. Here are some of these operators.

Let ρ_s be the irreducible representation of $\mathrm{O}(1, \mathbb{C})$ defined by,

$$\rho_s(g) = g^s$$

with $s = \{0, 1\}$ and $g \in \mathrm{O}(1, \mathbb{C})$.

Let $V_{m,0}$ be the space of functions on $\mathbb{C}$ satisfying,

$$\int_{\mathbb{C}} \int_{\mathbb{C}} \frac{|f(z)|\overline{|f(w)|}}{|z-w|^{2-m}} dz dw < \infty$$

with $0 < m < 2$, $m \in \mathbb{R}$. The group $\mathrm{SL}(2, \mathbb{C})$ acts irreducibly on this space by

$$\pi_{m,0}(g) f(z) = |cz+d|^{-m-2} f\left(\frac{az+b}{cz+d}\right)$$

with $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $g \in \mathrm{SL}(2, \mathbb{C})$ and $0 < m < 2$. ($\pi_{m,0}$ belongs to the so-called complementary series, see Chapter 3)

Let us now define an operator $\mathcal{F}_{1,0}$ from $\mathcal{S}(\mathbb{C})_+$ into the Hilbert space of $V_{1,0}$ as follows:

$$(\mathcal{F}_{1,0}\varphi)(z) = \int_{\mathbb{C}} e^{i\pi \mathrm{Re}\ (zy^2)} |y|^2 \varphi(y) dy$$

This integral exists for all $z \in \mathbb{C}$.

The group $\mathrm{SL}(2, \mathbb{C})$ acts irreducibly on $L^2(\mathbb{C})$ by

$$\pi_{0,l}(g) f(z) = |cz+d|^{-2} \left(\frac{cz+d}{|cz+d|}\right)^{-l} f\left(\frac{az+b}{cz+d}\right)$$

with $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $g \in \mathrm{SL}(2, \mathbb{C})$. ($\pi_{0,l}$ belongs to the so-called principal series, see Chapter 3)

Let us now define an operator $\mathcal{F}_{0,1}$ from $\mathcal{S}(\mathbb{C})_-$ into the Hilbert space of $L^2(\mathbb{C})$ as follows:

$$(\mathcal{F}_{0,1}\varphi)(z) = \int_{\mathbb{C}} e^{i\pi \mathrm{Re}\ (zy^2)} y\varphi(y)dy$$

This integral exists for all $z \in \mathbb{C}$.

We shall show:

1) The operators $\mathcal{F}_{1,0}$, $\mathcal{F}_{0,1}$ are well defined, i.e. $\mathcal{F}_{1,0}\varphi \in V_{1,0}$ and $\mathcal{F}_{0,1}\varphi \in L^2(\mathbb{C})$.
2) $\mathcal{F}_{1,0}$, $\mathcal{F}_{0,1}$ are continuous as operators from $\mathcal{S}(\mathbb{C})_+$, $\mathcal{S}(\mathbb{C})_-$ to $V_{1,0}$, $L^2(\mathbb{C})$ respectively.
3) The operator $\mathcal{F}_{1,0}$ intertwines the action of ω_1 and $\pi_{1,0} \otimes id$. $\mathcal{F}_{0,1}$ intertwines the action of ω_1 and $\pi_{0,1} \otimes \rho_1$.

So we have:

Theorem 8.1. *The operator $\mathcal{F}_{1,0}$ on $\mathcal{S}(\mathbb{C})_+$ into $V_{1,0}$ intertwines the action ω_1 with $\pi_{1,0} \otimes id$ of* $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{O}(1, \mathbb{C})$. *$\mathcal{F}_{0,1}$ on $\mathcal{S}(\mathbb{C})_-$ into $L^2(\mathbb{C})$ intertwines the action ω_1 with $\pi_{0,1} \otimes \rho_1$*

Proof. Let us assume 1) and 2), we show 3).

It is clear that $\mathcal{F}_{1,0}$, $\mathcal{F}_{0,1}$ intertwine the $\mathrm{O}(1, \mathbb{C})$-action. The intertwining relations for the elements $g(a)$ and $t(b)$ are easy to check. We now check the σ-intertwining relation:

$$\mathcal{F}_{1,0}\omega_1(\sigma)\varphi(z) = \pi_{1,0}(\sigma)(\mathcal{F}_{1,0}\varphi)(z)$$

for $\varphi \in \mathcal{S}(\mathbb{C})_+$. By definition,

$$\mathcal{F}_{1,0}\omega_1(\sigma)\varphi(z) = \int_{\mathbb{C}} e^{i\pi \mathrm{Re}\ (zy^2)} |y|^2 \hat{\varphi}(-y)dy.$$

Applying Proposition 6-1 from [26] we obtain,

$$\begin{aligned} \mathcal{F}_{1,0}\omega_1(\sigma)\varphi(z) &= |z|^{-3} \int_{\mathbb{C}} e^{-i\pi \mathrm{Re}\ (y^2/z)} |y|^2 \varphi(y)dy \\ &= \pi_{1,0}(\sigma)\varphi(z). \end{aligned}$$

Doing the same for $\mathcal{F}_{0,1}$ we have,

$$\begin{aligned} \mathcal{F}_{0,1}\omega_1(\sigma)\varphi(z) &= \int_{\mathbb{C}} e^{i\pi \mathrm{Re}\ (zy^2)} y\hat{\varphi}(-y)dy \\ &= -|z|^{-1}z^{-1} \int_{\mathbb{C}} e^{-i\pi \mathrm{Re}\ (y^2/z)} y\varphi(y)dy \\ &= \pi_{0,1}(\sigma)\varphi(z). \end{aligned}$$

Then we have proved that $\mathcal{F}_{1,0}$ intertwines ω_1 and $\pi_{1,0} \otimes id$ and $\mathcal{F}_{0,1}$ intertwines ω_1 and $\pi_{0,1} \otimes \rho_1$. □

Now we prove 1) and 2). Let us compute $||\mathcal{F}_{1,0}\varphi||^2$ for $\varphi \in \mathcal{S}(\mathbb{C})_+$ and $||\mathcal{F}_{0,1}\varphi||^2$ for $\varphi \in \mathcal{S}(\mathbb{C})_-$. First we need the following lemma

Lemma 8.2. *Let φ be an even function in $\mathcal{S}(\mathbb{C})$ and Ψ a continuous bounded funtion*

$$\int_{\mathbb{C}} \Psi(z^2)\varphi(z)dz = \frac{1}{2}\int_{\mathbb{C}} \Psi(z)\varphi(\sqrt{z})\frac{dz}{|z|}$$

Proof. As φ is an even function

$$\int_{\mathbb{C}} \Psi(z^2)\varphi(z)dz \quad = \quad 2\int_{\text{Re } z\geq 0} \Psi(z^2)\varphi(z)dz$$

Doing the change of variables $z^2 = w$ we obtain

$$= \quad 2\frac{1}{4}\int_{\mathbb{C}} \Psi(w)\varphi(\sqrt{w})\frac{dw}{|w|}$$

and we have the result. □

Now we compute $||\mathcal{F}_{1,0}\varphi||^2$ for $\varphi \in \mathcal{S}(\mathbb{C})_+$

$$\begin{aligned} ||\mathcal{F}_{1,0}\varphi||^2 &= \int_{\mathbb{C}^2} \frac{(\mathcal{F}_{1,0}\varphi)(z)\overline{(\mathcal{F}_{1,0}\varphi)(w)}}{|z-w|}dzdw \\ &= \int_{\mathbb{C}^2} \frac{(\mathcal{F}_{1,0}\varphi)(z+w)\overline{(\mathcal{F}_{1,0}\varphi)(w)}}{|z|}dzdw \end{aligned}$$

Applying the last lemma

$$\begin{aligned} &= \frac{1}{4}\int_{\mathbb{C}^4} \frac{e^{i\pi \text{Re } ((z+w)y-wx)}}{|z|}\varphi(\sqrt{y})\overline{\varphi(\sqrt{x})}dxdydzdw \\ &= \frac{1}{4}\int_{\mathbb{C}^3} \frac{e^{i\pi \text{Re } (zy)}}{|z|}\varphi(\sqrt{y})\left(\int_{\mathbb{C}} e^{i\pi \text{Re } ((y-x)w)}\overline{\varphi(\sqrt{x})}dx\right)dydzdw \\ &= \int_{\mathbb{C}^2} \frac{e^{i\pi \text{Re } (zy)}}{|z|}\varphi(\sqrt{y})\overline{\varphi(\sqrt{y})}dzdy \end{aligned}$$

Writing $z = re^{i\theta}$, $y = se^{i\psi}$ with $r, s \geq 0$, $\theta, \psi \in [0, 2\pi]$. We then get,

$$\begin{aligned} &= \int_0^\infty \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} e^{i\pi rs\cos(\theta+\psi)}\varphi(\sqrt{se^{i\psi}})\overline{\varphi(\sqrt{se^{i\psi}})}sd\theta d\psi drds \\ &= \int_0^\infty \int_0^{2\pi} \varphi(\sqrt{se^{i\psi}})\overline{\varphi(\sqrt{se^{i\psi}})}s\left(\int_0^\infty \left(\int_0^{2\pi} e^{i\pi rs\cos(\theta+\psi)}d\theta\right)dr\right)d\psi ds \end{aligned}$$

Using the following properties of Bessel functions

$$\int_0^{2\pi} e^{i\pi rs\cos(\theta+\psi)}d\theta = 2\pi\mathcal{J}_0(\pi rs)$$

and

$$\int_0^{\infty} \mathcal{J}_0(r)dr = 1$$

we have

$$\begin{aligned}
||\mathcal{F}_{1,0}\varphi||^2 &= 2\pi\int_0^{\infty}\int_0^{2\pi} \varphi(\sqrt{se^{i\psi}})\overline{\varphi(\sqrt{se^{i\psi}})}s\left(\int_0^{\infty}\mathcal{J}_0(\pi rs)dr\right)d\psi ds \\
&= 2\int_0^{\infty}\int_0^{2\pi}\varphi(\sqrt{se^{i\psi}})\overline{\varphi(\sqrt{se^{i\psi}})}d\psi ds \\
&= 2\int_{\mathbb{C}}\varphi(\sqrt{z})\overline{\varphi(\sqrt{z})}\frac{dz}{|z|} \\
&= 4\int_{\mathbb{C}}\varphi(z)\overline{\varphi(z)}dz \\
&= 4||\varphi||^2.
\end{aligned}$$

Now we compute $||\mathcal{F}_{0,1}\varphi||^2$ for $\varphi \in \mathcal{S}(\mathbb{C})_-$

$$\begin{aligned}
||\mathcal{F}_{0,1}\varphi||^2 &= \int_{\mathbb{C}}(\mathcal{F}_{0,1}\varphi)(z)\overline{(\mathcal{F}_{0,1}\varphi)(z)}dz \\
&= \int_{\mathbb{C}^3} e^{i\pi\mathrm{Re}\ (zy^2-zx^2)}y\overline{x}\varphi(y)\overline{\varphi(x)}dxdydz \\
&= \frac{1}{4}\int_{\mathbb{C}^2}\frac{\sqrt{y}\varphi(\sqrt{y})}{|y|}\left(\int_{\mathbb{C}} e^{i\pi\mathrm{Re}\ ((y-x)z)}\frac{\sqrt{\overline{x}}\overline{\varphi(\sqrt{x})}}{|x|}dx\right)dydz \\
&= \int_{\mathbb{C}}\frac{\varphi(\sqrt{y})\overline{\varphi(\sqrt{y})}}{|y|}dy \\
&= 2\int_{\mathbb{C}}\varphi(y)\overline{\varphi(y)}dy \\
&= 2||\varphi||_2^2.
\end{aligned}$$

This proves 1). Moreover we have 2) since actually $\mathcal{F}_{1,0}$, $\mathcal{F}_{0,1}$ are continuous in the L^2-topology on $\mathcal{S}(\mathbb{C})$, so on $\mathcal{S}(\mathbb{C})$ itself, since the embedding of $\mathcal{S}(\mathbb{C})$ into $L^2(\mathbb{C})$ is continuous.

By the Parseval's theorem as $L^2(\mathbb{C}) = L^2(\mathbb{C})_+ \oplus L^2(\mathbb{C})_-$,

$$||\varphi||^2 = ||\varphi_+||^2 + ||\varphi_-||^2 = \frac{1}{4}||\mathcal{F}_{1,0}\varphi_+||^2 + \frac{1}{2}||\mathcal{F}_{0,1}\varphi_-||^2.$$

8.1.3 Decomposition of $L^2(\mathbb{C})$

Clearly the subsection 8.1.2 also gives the decomposition of $L^2(\mathbb{C})$ into a minimal invariant suspaces. The decomposition is multiplicity free (see next chapter) and the irreducible unitary representations which occur are given by $\pi_{1,0} \otimes id$ and $\pi_{0,1} \otimes \rho_1$. In conclusion we have,

Corollary 8.3. *Let φ be a function in $\mathcal{S}(\mathbb{C})$. We have the following Plancherel formula*

$$||\varphi||_2^2 = ||\varphi_+||^2 + ||\varphi_-||^2 = \frac{1}{4}||\mathcal{F}_{1,0}\varphi_+||^2 + \frac{1}{2}||\mathcal{F}_{0,1}\varphi_-||^2.$$

As representations of $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{O}(1,\mathbb{C})$, *we have*

$$L^2(\mathbb{C}) \cong \pi_{1,0} \otimes id + \pi_{0,1} \otimes \rho_1.$$

8.2 The case $n = 2$

8.2.1 The definition of the oscillator representation

Let G be the group $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SO}(2,\mathbb{C})$ and let ω_2 be the unitary representation of G on $H = L^2(\mathbb{C}^2)$ defined by:

$$\begin{aligned}
\omega_2(g)f(z) &= f(g^{-1}\cdot z), \qquad g \in \mathrm{SO}(2,\mathbb{C})\\
\omega_2(g(a))f(z) &= |a|^2 f(az), \qquad g(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad a \in \mathbb{C}^*\\
\omega_2(t(b))f(z) &= e^{-i\pi \mathrm{Re}\ (b[z,z])} f(z), \qquad t(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad b \in \mathbb{C}\\
\omega_2(\sigma)f(z) &= \int_{\mathbb{C}^2} e^{2\pi i \mathrm{Re}\ ([z,w])} f(w)dw, \qquad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\end{aligned}$$

where $[z,w] = z_1w_1 + z_2w_2$ if $z = (z_1,z_2)$, $w = (w_1,w_2)$. The Fourier transform is defined by

$$\hat{f}(z) = \int_{\mathbb{C}^2} e^{-2\pi i \mathrm{Re}\ ([z,w])} f(w)dw.$$

We call ω_2 the oscillator representation of G. More precisely, it is the restriction of the metaplectic representation of the group $Sp(2,\mathbb{C}) \subset Sp(4,\mathbb{R})$ to G.

Let $\mathcal{S}(\mathbb{C}^2)$ be the space of Schwartz functions on $\mathbb{C}^2$. Note that $\mathcal{S}(\mathbb{C}^2)$ is stable under the action of ω_2, so is the space $\mathcal{S}'(\mathbb{C}^2)$ of tempered distributions on $\mathbb{C}^2$.

8.2.2 Some minimal invariant Hilbert subspaces of $\mathcal{S}'(\mathbb{C}^2)$

We consider $L^2(\mathbb{C}^2)$ as a Hilbert subspace of $\mathcal{S}'(\mathbb{C}^2)$. Irreducible unitary representations of G are of the form $\pi \otimes \rho$ where π is an irreducible representation of $\mathrm{SL}(2,\mathbb{C})$ and ρ one of $\mathrm{SO}(2,\mathbb{C})$. We will now construct intertwining operators from $\mathcal{S}(\mathbb{C}^2)$ to the Hilbert space of such representations. Not all combinations (π,ρ) occur. Some of these operators are given below.

Let $\rho_{is,\delta}$ be the irreducible unitary representation of $\mathrm{SO}(2,\mathbb{C})$ defined by,

$$\rho_{is,\delta}(g) = (a+ib)^{is,\delta}$$

with $s \in \mathbb{R}$, $\delta \in \mathbb{Z}$, $g \in \mathrm{SO}(2,\mathbb{C})$, $g = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. We set $z^{is,\delta} = |z|^{is}\left(\frac{z}{|z|}\right)^{\delta}$ for $z \in \mathbb{C}^*$.

The group $\mathrm{SL}(2,\mathbb{C})$ acts irreducibly on $L^2(\mathbb{C})$ by

$$\pi_{\lambda,m}(g)f(z) = |cz+d|^{-\lambda-2}\left(\frac{cz+d}{|cz+d|}\right)^{-m} f\left(\frac{az+b}{cz+d}\right)$$

with $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $g \in \mathrm{SL}(2,\mathbb{C})$, $\lambda \in i\mathbb{R}$, $m \in \mathbb{Z}$. ($\pi_{\lambda,m}$ belongs to the so-called principal series, see Chapter 3)

Let us now define an operator $\mathcal{F}_{is,\delta}$ from $\mathcal{S}(\mathbb{C}^2)$ into the Hilbert space $L^2(\mathbb{C})$ as follows:

$$(\mathcal{F}_{is,\delta}\varphi)(z) = \int_{\mathbb{C}^2} e^{i\pi \mathrm{Re}\,(z(y_1^2+y_2^2))}(y_1+iy_2)^{is,\delta}\varphi(y_1,y_2)dy_1dy_2$$

with $s \in \mathbb{R}$ and $\delta \in \mathbb{Z}$. This integral exists for all $z \in \mathbb{C}$.

We will show that:

1) The operator $\mathcal{F}_{is,\delta}$ is well defined, i.e. $\mathcal{F}_{is,\delta}\varphi \in L^2(\mathbb{C})$,

2) $\mathcal{F}_{is,\delta}$ is continuous as operator from $\mathcal{S}(\mathbb{C}^2)$ to $L^2(\mathbb{C})$.

3) The operator $\mathcal{F}_{is,\delta}$ intertwines the action of ω_2 and $\pi_{is,\delta} \otimes \rho_{is,\delta}$.

First we want to prove 1) and 2). We have:

Proposition 8.4. *For all $s \in \mathbb{R}$ and $\delta \in \mathbb{Z}$ the operator $\mathcal{F}_{is,\delta}$ maps $\mathcal{S}(\mathbb{C}^2)$ into $L^2(\mathbb{C})$ and $\mathcal{F}_{is,\delta}$ is continuous.*

Proof. Observe that $w_1^2 + w_2^2 = 1$ can be parametrized by the two real parameters $t \in \mathbb{R}$, $\theta \in \mathbb{R}$, $0 \le \theta < 2\pi$ as follows: any $g \in \mathrm{SO}(2,\mathbb{C})$ can be uniquely written as

$$g = h_t k_\theta$$

where $h_t = \begin{pmatrix} \cosh t & i\sinh t \\ -i\sinh t & \cosh t \end{pmatrix}, k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

Hence $w_1^2 + w_2^2 = 1$ is precisely $\{h_t k_\theta x^0 |\ t \in \mathbb{R}, 0 \le \theta < 2\pi\}$ where $x^0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Then $dtd\theta$ is both an invariant measure on $\mathrm{SO}(2,\mathbb{C})$ and on $w_1^2+w_2^2 = 1$. More precisely, if dg is the thus defined invariant measure on $G = \mathrm{SO}(2,\mathbb{C})$, then

$$\int_G f(g)dg = \int_{-\infty}^{\infty}\int_0^{2\pi} f(h_t k_\theta)d\theta dt \quad (f \in \mathcal{C}_c(G)).$$

Then it follows:

$$\int_{\mathbb{C}^2} f(z)dz = \int_{\mathbb{C}} \int_0^{2\pi} \int_{-\infty}^{\infty} f(\lambda w_{t,\theta})|\lambda|^2 dt d\theta d\lambda \tag{8.1}$$

where $w_{t,\theta} = h_t k_\theta x^0$.

Let $\varphi \in \mathcal{S}(\mathbb{C}^2)$. Applying (8.1) we have

$$\int_{\mathbb{C}^2} |\varphi(z_1, z_2)|^2 dz = \frac{1}{4} \int_{\mathbb{C}} \int_{-\infty}^{\infty} \int_0^{2\pi} |\varphi(\sqrt{\lambda} w_{t,\theta})|^2 d\theta dt d\lambda$$

Applying (8.1) to the intertwining operator we obtain

$$\begin{aligned} \mathcal{F}_{is,\delta}\varphi(z) &= \int_{\mathbb{C}^2} e^{i\pi \mathrm{Re}\ (z(y_1^2+y_2^2))}(y_1+iy_2)^{is,\delta}\varphi(y_1,y_2)dy_1dy_2 \\ &= \int_{\mathbb{C}} e^{i\pi \mathrm{Re}\ (z\lambda)} F(\lambda, is, \delta) d\lambda \end{aligned} \tag{8.2}$$

where $F(\lambda, is, \delta) = \frac{1}{4}\lambda^{is/2,\delta/2} \int_{-\infty}^{\infty} \int_0^{2\pi} (\cosh t + \sinh t)^{is,\delta} e^{-i\delta\theta} \varphi(\sqrt{\lambda} w_{t,\theta}) d\theta dt$.

Since φ is a Schwartz class function

$$\begin{aligned} |F(\lambda, is, \delta)| &\leq \frac{1}{4} \int_0^{2\pi} \int_{-\infty}^{\infty} |\varphi(\sqrt{\lambda} w_{t,\theta})| dt d\theta \\ &\leq C \int_0^{\infty} (1+|\lambda|e^{2t})^{-N} dt \\ &\leq C \int_{|\lambda|}^{\infty} (1+u)^{-N} u^{-1} du \\ &\leq C(1+|\lambda|)^{\frac{-N}{2}} \ln|\lambda| \end{aligned}$$

by an easy calculation with $N > 4$. Thus $F(\lambda, is, \delta)$ as a function of λ is in $L^2(\mathbb{C}) \cap L^1(\mathbb{C})$ and therefore (8.2) is everywhere defined and $\mathcal{F}_{is,\delta}\varphi$ is in $L^2(\mathbb{C})$.

We can see that the value of the constant C is equal to $C' \sum_{|\alpha| \leq 2N} ||\varphi||_{\alpha,0}$ where $||\varphi||_{\alpha,0} = \sup_{z \in \mathbb{C}^2} (|z^\alpha \varphi(z)|)$ are norms in $\mathcal{S}(\mathbb{C}^2)$ and C' is a constant. Then we have

$$||\mathcal{F}_{is,\delta}\varphi|| \leq C'' \sum_{|\alpha| \leq 2N} ||\varphi||_{\alpha,0}$$

for all $\varphi \in \mathcal{S}(\mathbb{C}^2)$ where C'' is another constant. Therefore we have proved that $\mathcal{F}_{is,\delta}$ is continuous. □

Theorem 8.5. *The operator $\mathcal{F}_{is,\delta}$ from $\mathcal{S}(\mathbb{C}^2)$ into $L^2(\mathbb{C})$ intertwines the action of ω_2 with $\pi_{is,\delta} \otimes \rho_{is,\delta}$ of* $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SO}(2,\mathbb{C})$.

Proof. First we prove the intertwining relation for $\mathrm{SO}(2, \mathbb{C})$. Let g be an element of this group $g = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, then

$$\mathcal{F}_{is,\delta}\omega_2(g)\varphi(z) \;=\; \int_{\mathbb{C}^2} e^{i\pi \mathrm{Re}\ (z(y_1^2+y_2^2))}(y_1 + iy_2)^{is,\delta}\varphi(g^{-1} \cdot (y_1, y_2))dy_1dy_2.$$

Making the change of variables $g^{-1} \cdot y = w$ we obtain

$$\begin{aligned} &= (a + ib)^{is,\delta} \int_{\mathbb{C}^2} e^{i\pi \mathrm{Re}\ (z(w_1^2+w_2^2))}(w_1 + iw_2)^{is,\delta}\varphi(w_1, w_2)dw_1dw_2 \\ &= \rho_{is,\delta}(g)\mathcal{F}_{is,\delta}\varphi(z) \end{aligned}$$

The intertwining relations for the elements $g(a)$ and $t(b)$ are easy to check. We now check the σ-intertwining relation:

$$\mathcal{F}_{is,\delta}\omega_2(\sigma)\varphi(z) = \pi_{is,\delta}(\sigma)(\mathcal{F}_{s,\delta}\varphi)(z)$$

for $\varphi \in \mathcal{S}(\mathbb{C}^2)$. By definition,

$$\mathcal{F}_{is,\delta}\omega_2(\sigma)\varphi(z) = \int_{\mathbb{C}^2} e^{i\pi \mathrm{Re}\ (z(y_1^2+y_2^2))}(y_1 + iy_2)^{is,\delta}\hat{\varphi}(-y_1, -y_2)dy.$$

Here we insert the following lemma:

Lemma 8.6. *For every function $\varphi \in \mathcal{S}(\mathbb{C}^2)$ one has*

$$\begin{aligned} &\int_{\mathbb{C}^2} e^{i\pi Re\ (z(y_1^2+y_2^2))}(y_1 + iy_2)^{is,\delta}\hat{\varphi}(y_1, y_2)dy_1dy_2 \\ &\qquad = z^{-is,-\delta}|z|^{-2} \int_{\mathbb{C}^2} e^{-i\pi Re\ ((y_1^2+y_2^2)/z)}(y_1 + iy_2)^{is,\delta}\varphi(y_1, y_2)dy_1dy_2 \end{aligned}$$

with $s \in \mathbb{R}$, $\delta \in \mathbb{Z}$ and the Fourier transform is defined by

$$\hat{f}(z) = \int_{\mathbb{C}^2} e^{-2\pi i Re\ ([z,w])} f(w)dw.$$

Proof. We consider the auxiliary function

$$\phi_{\tau_1,\tau_2}(y_1, y_2) = e^{-\tau_1|y_1+iy_2|^2-\tau_2|y_1-iy_2|^2}$$

with τ_1, $\tau_2 > 0$.

By definition of the Fourier transform and the dominated convergence theorem

$$\begin{aligned} &\int_{\mathbb{C}^2} e^{i\pi \mathrm{Re}\ (z(y_1^2+y_2^2))}(y_1 + iy_2)^{is,\delta}\hat{\varphi}(y_1, y_2)dy \\ &= \int_{\mathbb{C}^2} e^{i\pi \mathrm{Re}\ (z(y_1^2+y_2^2))}(y_1 + iy_2)^{is,\delta} \left(\int_{\mathbb{C}^2} e^{-2i\pi \mathrm{Re}\ (y_1w_1+y_2w_2)}\varphi(w_1, w_2)dw\right) dy \\ &\quad = \lim_{\tau_2\to 0} \lim_{\tau_1\to 0} \int_{\mathbb{C}^2} e^{i\pi \mathrm{Re}\ (z(y_1^2+y_2^2))}(y_1 + iy_2)^{is,\delta}\phi_{\tau_1,\tau_2}(y_1, y_2) \\ &\qquad\qquad \cdot \left(\int_{\mathbb{C}^2} e^{-2i\pi \mathrm{Re}\ (y_1w_1+y_2w_2)}\varphi(w_1, w_2)dw\right) dy \end{aligned}$$

Using Fubini's theorem we get that the integral above is equal to

$$\int_{\mathbb{C}^2} \varphi(w_1, w_2) \left(\int_{\mathbb{C}^2} e^{i\pi \mathrm{Re}\ \left(z(y_1^2+y_2^2)-2(y_1w_1+y_2w_2)\right)} (y_1+iy_2)^{is,\delta} \phi_{\tau_1,\tau_2}(y_1,y_2) dy \right) dw$$

Let us perform the change of variables

$$y = Su,\ w = Sv,$$

where $S = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$. The above integral then equals

$$\frac{1}{4^2} \int_{\mathbb{C}^2} \varphi(Sv) \left(\int_{\mathbb{C}^2} e^{i\pi \mathrm{Re}\ ((zu_1u_2)-(u_1v_2+u_2v_1))} u_1^{is,\delta} e^{-\tau_1|u_1|^2-\tau_2|u_2|^2} du \right) dv \qquad (8.3)$$

The inner integral becomes

$$\begin{aligned} &\int_{\mathbb{C}} u_1^{is,\delta} e^{-i\pi \mathrm{Re}\ (u_1v_2)} e^{-\tau_1|u_1|^2} \left(\int_{\mathbb{C}} e^{i\pi \mathrm{Re}\ ((zu_1-v_1)u_2)} e^{-\tau_2|u_2|^2} du_2 \right) du_1 \\ &= \frac{\pi}{\tau_2} \int_{\mathbb{C}} u_1^{is,\delta} e^{-i\pi \mathrm{Re}\ (u_1v_2)} e^{-\tau_1|u_1|^2} e^{-\frac{\pi^2|zu_1-v_1|^2}{4\tau_2}} du_1. \end{aligned}$$

Taking the limit $\tau_1 \to 0$, by the dominated convergence theorem, we see that (8.3) converges to

$$\begin{aligned} &\frac{\pi}{4^2\tau_2} \int_{\mathbb{C}^2} \varphi(Sv) \left(\int_{\mathbb{C}} u_1^{is,\delta} e^{-i\pi \mathrm{Re}\ (u_1v_2)} e^{-\frac{\pi^2|zu_1-v_1|^2}{4\tau_2}} du_1 \right) dv \\ &= \frac{1}{4}|z|^{-2} \int_{\mathbb{C}} \left[\frac{\pi}{4\tau_2} \int_{\mathbb{C}} e^{-\frac{\pi^2|u_1-v_1|^2}{4\tau_2}} \left(\int_{\mathbb{C}} \varphi(Sv) e^{-i\pi \mathrm{Re}\ (\frac{u_1v_2}{z})} (z^{-1}u_1)^{is,\delta} dv_2 \right) du_1 \right] dv_1 \end{aligned}$$

Now the term in the inner parenthesis, as a function of v_1 is in $L^1(\mathbb{C})$ and taking the limit $\tau_2 \to 0$ we know that the term in the outer parenthesis tends to this term:

$$\int_{\mathbb{C}} \varphi(Sv) e^{-i\pi \mathrm{Re}\ (\frac{v_1v_2}{z})} (z^{-1}v_1)^{is,\delta} dv_2$$

in the space $L^1(\mathbb{C}, dv_1)$. Thus the integral becomes

$$\begin{aligned} &\frac{1}{4}|z|^{-2} \int_{\mathbb{C}^2} \varphi(Sv) e^{-i\pi \mathrm{Re}\ (\frac{v_1v_2}{z})} (z^{-1}v_1)^{is,\delta} dv \\ &= z^{-is,-\delta}|z|^{-2} \int_{\mathbb{C}^2} e^{-i\pi \mathrm{Re}\ ((y_1^2+y_2^2)/z)} (y_1+iy_2)^{is,\delta} \varphi(y_1,y_2) dy \end{aligned}$$

We have thus proved the lemma. □

Applying this lemma it follows

$$\begin{aligned} \mathcal{F}_{is,\delta}\omega_2(\sigma)\varphi(z) &= (-1)^{\delta} z^{-is,-\delta}|z|^{-2}\mathcal{F}_{is,\delta}\varphi(\frac{-1}{z}) \\ &= \pi_{is,\delta}(\sigma)\mathcal{F}_{is,\delta}\varphi(z) \end{aligned}$$

and Theorem 8.5 is proven. □

8.2.3 Decomposition of $L^2(\mathbb{C}^2)$

Let us compute $||\varphi||^2$. Making use of Plancherel's theorem on the abelian group $\mathrm{SO}(2,\mathbb{C})$ and (8.2) it follows

$$\begin{aligned}
\int_{\mathbb{C}^2} |\varphi(z_1,z_2)|^2 dz &= \frac{1}{4}\int_{\mathbb{C}}\int_{-\infty}^{\infty}\int_0^{2\pi} |\varphi(\sqrt{\lambda}w_{t,\theta})|^2 d\theta dt d\lambda \\
&= \frac{4}{\pi^2}\int_{\mathbb{C}}\left(\sum_{\delta\in\mathbb{Z}}\int_{\mathbb{R}} |F(\lambda,is,\delta)|^2 ds\right) d\lambda \\
&= \frac{4}{\pi^2}\sum_{\delta\in\mathbb{Z}}\int_{\mathbb{R}}\int_{\mathbb{C}} |F(\lambda,is,\delta)|^2 d\lambda ds \\
&= \frac{1}{\pi^2}\sum_{\delta\in\mathbb{Z}}\int_{\mathbb{R}} ||\mathcal{F}_{is,\delta}\varphi||^2 ds
\end{aligned}$$

The above result also gives the decomposition of $L^2(\mathbb{C}^2)$ into minimal invariant subspaces. The decomposition is multiplicity free and the irreducible unitary representations which occur are given by $\pi_{is,\delta}\otimes\rho_{is,\delta}$. In conclusion, we have

Corollary 8.7. *Let φ be a function in $\mathcal{S}(\mathbb{C}^2)$. We have the following Plancherel formula*

$$||\varphi||_2^2 = \frac{1}{\pi^2}\sum_{\delta\in\mathbb{Z}}\int_{\mathbb{R}} ||\mathcal{F}_{is,\delta}\varphi||^2 ds.$$

As representations of $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SO}(2,\mathbb{C})$*, we have*

$$L^2(\mathbb{C}^2) \cong \sum_{\delta\in\mathbb{Z}}\int_{\mathbb{R}} \pi_{is,\delta}\otimes\rho_{is,\delta} ds.$$

8.3 The case $n \geq 3$

To determine the explicit decomposition of the oscillator representation for the dual pair $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SO}(n,\mathbb{C})$ with $n\geq 3$ we compute first the Plancherel formula for $\mathrm{SO}(n,\mathbb{C})/\mathrm{SO}(n-1,\mathbb{C})$, $n\geq 3$.

8.3.1 Plancherel formula for $\mathrm{SO}(n,\mathbb{C})/\mathrm{SO}(n-1,\mathbb{C})$, $n \geq 3$

8.3.1.1 The symmetric spaces $\mathrm{SO}(n,\mathbb{C})/\mathrm{SO}(n-1,\mathbb{C})$ for $n \geq 3$

Let G be the group $\mathrm{SO}(n,\mathbb{C})$. G acts on $\mathbb{C}^n$ in the usual manner. Let $[,]$ be the G-invariant bilinear form on $\mathbb{C}^n$, given by,

$$[z,z'] = z_1z_1' + \cdots + z_nz_n'$$

if $z=(z_1,\ldots,z_n)$, $z'=(z_1',\ldots,z_n')$. The (algebraic) manifold $[z,z]=z_1^2+\cdots+z_n^2=1$, called X, is invariant under this action, and G acts transitively on X. Let x^0 be the vector $(1,0,\ldots,0)$ in X. The stabilizer of x^0 in G is equal to

$H = \mathrm{SO}(n-1,\mathbb{C})$. The space X is a complex manifold, homeomorphic to G/H. Set

$$J = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

J is not in G, but $-J$ is as soon as n is odd. Define the involutive automorphism σ of G by $\sigma(g) = JgJ$. Then H is of index 2 in the stabilizer H_σ of σ. So G/H is a complex symmetric space. Observe that σ is an inner automorphism if n is odd. We will pass now to the Lie algebra $\mathfrak{g}$ of G. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ be the decomposition of $\mathfrak{g}$ into eigenspaces for the eigenvalues $+1$ and -1 for σ. Then we have:

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & Y & \\ 0 & & \end{pmatrix} : Y \text{complex, antisymmetric} \right\},$$

$$\mathfrak{q} = \left\{ \begin{pmatrix} 0 & z_2 & \cdots & z_n \\ -z_2 & & & \\ \vdots & & 0 & \\ -z_n & & & \end{pmatrix} : z_2, \ldots, z_n \in \mathbb{C} \right\}.$$

Write $L = \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{pmatrix} \in \mathfrak{q}$. This element actually belongs to $\mathfrak{q} \cap \mathfrak{k}$ where $\mathfrak{k} = \mathfrak{so}(n,\mathbb{R})$, the real antisymmetric matrices. The Lie algebra $\mathfrak{k}$ belongs to $K = \mathrm{SO}(n,\mathbb{R})$. Clearly $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$. Call $\mathfrak{p} = i\mathfrak{k}$. Then $iL \in \mathfrak{q} \cap \mathfrak{p}$.
The Lie algebra $\mathfrak{a} = \mathbb{C}L$ is a maximal abelian subspace of $\mathfrak{q}$; $\mathfrak{a}$ is a Cartan subspace of $\mathfrak{q}$ with respect to σ. Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{h}$. Then

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & Y & \\ 0 & 0 & & \end{pmatrix} : Y \in \mathfrak{so}(n-2,\mathbb{C}) \right\}.$$

Let $\mathfrak{g}_\pm$ be the eigenspaces of $\mathrm{ad}(iL)$ for the eigenvalues ± 1. Then

$$\mathfrak{g}_- = \left\{ X(p) = \begin{pmatrix} 0 & 0 & {}^tp \\ 0 & 0 & i{}^tp \\ -p & -ip & 0 \end{pmatrix} : p \in \mathbb{C}^{n-2} \right\}$$

and $\mathfrak{g}_- = \sigma(\mathfrak{g}_+)$. We have the following decomposition of $\mathfrak{g}$ into eigenspaces of $\mathrm{ad}(iL)$:

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{g}_+.$$

Set $\mathfrak{n} = \mathfrak{g}_-$. Then $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$. The connected subgroup A and N of G corresponding to the Lie subalgebras $\mathfrak{a}$ and $\mathfrak{n}$ are given by:

$$A = \left\{ a_z = \exp zL = \begin{pmatrix} \cos z & \sin z & 0 \\ -\sin z & \cos z & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} : z \in \mathbb{C} \right\}$$

and

$$N = \left\{ n(p) = \begin{pmatrix} 1 + \frac{1}{2}[p,p] & \frac{1}{2}i[p,p] & i^t p \\ \frac{1}{2}i[p,p] & 1 - \frac{1}{2}[p,p] & -^t p \\ -ip & p & I_{n-2} \end{pmatrix} : p \in \mathbb{C}^{n-2} \right\}.$$

Let ξ^0 be the vector $(1, i, 0, \ldots, 0)$. Define $P_0 : X \longrightarrow \mathbb{C}$ by $P_0(x) = [x, \xi^0]$. Observe that $P_0(na_z hx^0) = e^{-iz}$ for $n \in N$, $a_z \in A$, $h \in H$.

Proposition 8.8. *(i) The set NAH is open in G and dense. One has: $g \in NAH$ if and only if $P_0(gx^0) \neq 0$.*

(ii) The map $(n, a, h) \longrightarrow nah$ is a submersion and an immersion.

(iii) $nah = n'a'h'$ if and only if $n = n'$, $a = a'$, $h = h'$.

8.3.1.2 The cone $\Xi = G/MN$ and the Poisson kernel

The centralizer M of $\mathfrak{a}$ in H consists of the elements of G of the form:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & h & \\ 0 & 0 & & & \end{pmatrix}, \text{ with } h \in \mathrm{SO}(n-2, \mathbb{C}).$$

Furthermore $Z_G(\mathfrak{a}) = MA$, $M \cap A = \{e\}$. Put $\Xi = \{z \in \mathbb{C}^n : [z, z] = 0, z \neq 0\}$, the isotropic cone. The group G acts transitively on Ξ and $\mathrm{Stab}(\xi^0) = MN$, so Ξ can be identified with G/MN. The stabilizer of $\{\lambda \xi^0 : \lambda \in \mathbb{C}\}$ is equal to $P = MAN$, a parabolic subgroup of G.

Let $K = \mathrm{SO}(n, \mathbb{R})$ as before, a maximal compact subgroup of G. Then $G = KP$, $K \cap P = (K \cap M)\exp(\mathbb{R}L)$. So $G \cdot \xi^0 = K \exp(i\mathbb{R}L) \cdot \xi^0$. Define for $x \in X$ and $\xi \in \Xi$,

$$P(x, \xi) = [x, \xi].$$

P is called the Poisson kernel and assumes all complex values. The following properties are immediate:

(i) $P(gx, g\xi) = P(x, \xi)$ for $g \in G$,

(ii) $P(x, \xi^0) = P_0(x)$.

(iii) Define $P : \Xi \times \Xi \longrightarrow \mathbb{C}$ by $P(\xi, \xi') = [\xi, \xi']$. Then one has:

$$P(g\xi^0, \xi) = 2 \lim_{t \to \infty} e^{-t} P(ga_{-it}x^0, \xi) \quad (g \in G).$$

8.3.1.3 The representations $\pi_{\delta,s}$

Write $A = A_{\mathbb{R}}A_I$ with $A_{\mathbb{R}} = \exp(\mathbb{R}L)$, $A_I = \exp(i\mathbb{R}L)$. The subgroup $P = MAN$ is a maximal parabolic subgroup of G with Langlands decomposition

$$P = (MA_{\mathbb{R}})A_I N.$$

Observe that $A_{\mathbb{R}}$ is compact.

Define

$$\tau_{\delta,s}(ma_x a_{iy} n) = e^{-i\delta x}e^{sy}$$

with $\delta \in \mathbb{Z}$, $s \in \mathbb{C}$ a representation of P and define

$$\pi_{\delta,s} = \operatorname*{ind}_{P\uparrow G} \tau_{\delta,s}.$$

The space of $\pi_{\delta,s}$ is called $E_{\delta,s}$ and consists of complex-valued $\mathcal{C}^\infty$-functions f on G satisfying

$$f(gma_{x+iy}n) = e^{i\delta x-(s-\rho)y}f(g) \quad (g \in G)$$

where $\rho = n - 2$.

The representation $\pi_{\delta,s}$ is then given by

$$\pi_{\delta,s}(g)f(x) = f(g^{-1}x) \quad (x, g \in G; f \in E_{\delta,s}).$$

The following lemma is standard.

Lemma 8.9. *Let $f \in \mathcal{D}(G)$. Then, with suitable normalization of Haar measures,*

$$\int_G f(g)dg = \int_K \int_M \int_A \int_N f(kma_z n)e^{-2\rho Im\ z}dkdmdzdn.$$

It follows that the non-degenerate sesquilinear form $<,>$ on $E_{\delta,s} \times E_{\delta,-\bar{s}}$ defined by

$$< f, h >= \int_K f(k)\bar{h}(k)dk = \int_{B=K/(M\cap K)} f(b)\bar{h}(b)db$$

is G-invariant. Hence $\pi_{\delta,s}$ is pre-unitary for $s \in i\mathbb{R}$.

8.3.1.4 Realization on the cone Ξ

Since the functions in $E_{\delta,s}$ are right MN-invariant they can be viewed as functions on Ξ: to $f \in E_{\delta,s}$ we associate $f(\xi) = f(g)$ if $\xi = g\xi^0$. Since $a_z\xi^0 = e^{iz}\xi^0$, we easily get the precise properties of the functions on Ξ:

$$f(\lambda\xi) = \left(\frac{\lambda}{|\lambda|}\right)^\delta |\lambda|^{s-\rho}f(\xi) \quad (\xi \in \Xi, \lambda \in \mathbb{C}^*). \tag{8.4}$$

G acts on $E_{\delta,s}(\Xi)$ by $\pi_{\delta,s}(g)f(\xi) = f(g^{-1}\xi)$. According to Theorem 8.3 in [36], $\pi_{\delta,s}$ ($s \in i\mathbb{R}$) is irreducible and pre-unitary for $s \neq 0$.

We will not make a more detailed study of the representations $\pi_{\delta,s}$ concerning irreducibility, equivalence, unitarity etc.

The cases $n = 3, 4$ are well-known through the isomorphism

$$\mathrm{SO}(3, \mathbb{C}) \simeq \mathrm{SL}(2, \mathbb{C})/\{\pm 1\} \quad \text{and} \quad \mathrm{SO}(4, \mathbb{C}) \simeq \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}).$$

But the cases $n > 4$ should be treated, and we should consider $\mathrm{SO}(3, \mathbb{C})$ and $\mathrm{SO}(4, \mathbb{C})$ as part of these cases. A major point in the analysis of the representations $\pi_{\delta,s}$ would be the description of their K-types. We will give it only for reference here.

8.3.1.5 The K-types of $\pi_{\delta,s}$

Clearly any function in $E_{\delta,s}$ is completely determined by its restriction to K. Actually we get $\mathcal{C}^\infty_\delta(K)$, the space of $\mathcal{C}^\infty$-functions on K satisfying:

- $f(ka_x) = e^{i\delta x} f(k)$
- $f(km) = f(k)$ if $m \in M \cap K \cong \mathrm{SO}(n-2, \mathbb{R})$.

If we set $B = K \cdot \xi^0$, then $B \simeq K/(M \cap K)$. So, $E_{\delta,s} \simeq \mathcal{C}^\infty_\delta(B)$, the space of $\mathcal{C}^\infty$-functions on B satisfying $f(\lambda b) = \lambda^\delta f(b)$ ($\lambda \in \mathbb{C}$, $|\lambda| = 1$). As a representation of K (so restricting $\pi_{\delta,s}$ to K), this space is just the space of

$$\operatorname*{ind}_{(K\cap M)A_{\mathbb{R}}\uparrow K} e^{-i\delta x}.$$

Clearly

$$L^2(B) = \bigoplus_{\delta \in \mathbb{Z}} L^2_\delta(B) \quad \text{(orthogonal direct sum)}.$$

It is well-known that $L^2_\delta(B)$ splits multiplicity free into irreducible representations of K. We need a detailed description of the K-types. We refer therefore to [31].

We introduce the following notation. Let $\mathcal{H}_j(z)$ be the space of harmonic polynomials on $\mathbb{C}^n$, homogeneous of degree j; harmonic meaning:

$$\left(\frac{\partial^2}{\partial z_1^2} + \cdots + \frac{\partial^2}{\partial z_n^2}\right)p(z) = 0 \quad \text{if } p \in \mathcal{H}_j(z)$$

with $p(z) = p(z_1, \ldots, z_n)$. So $\mathcal{H}_j(z)$ is the complexification of $\mathcal{H}_j(\mathbb{R}^n)$, the harmonic polynomials on $\mathbb{R}^n$, homogeneous of degree j. These polynomials are completely determined by their values on $S^{n-1} = \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\}$. The group K acts irreducibly on $\mathcal{H}_j$, as well as G. So we can speak of highest and lowest weight vectors (for the G-action). Clearly $p(z) = [z, \xi^0]^j$ is a highest weight vector with weight e^{-jt}.

We also consider the spaces $\overline{\mathcal{H}_j(z)}$ (complex conjugates of elements in $\mathcal{H}_j(z)$). The K representation is "the same". Set $\mathcal{H}_{m_1,m_2} = \mathcal{H}_{m_1}(z) \otimes \overline{\mathcal{H}_{m_2}(z)}$. The space B can be seen as the variety $[z, z] = 0$, $[z, \overline{z}] = 2$ in $\mathbb{C}^n$. Let ξ^0 as before and $\xi^1 = (0, 0, 1, i, 0, \cdots, 0)$ is $n \geq 4$. According to [31] we have the following. The irreducible representations of $\mathrm{SO}(n)$, which occur in the decomposition of $L^2(\mathrm{SO}(n)/\mathrm{SO}(n-2))$ are of the form R_{m_1,m_2} with $m = (m_1, m_2, 0, \ldots, 0)$ highest

weight, $m_1 \geq m_2 \geq 0$ $(n > 4)$.
The representation R_{m_1,m_2} occurs $m_1 - m_2 + 1$ times and has a unique representative in $\mathcal{H}_{l_1+m_2,l_2+m_2}$ with $l_1 + l_2 = m_1 - m_2$, $l_1 \geq 0$, $l_2 \geq 0$. The highest weight in these spaces is:

$$[z, \xi^0]^{l_1} [\overline{z}, \xi^0]^{l_2} \left([z, \xi^0][\overline{z}, \xi^1] - [z, \xi^1][\overline{z}, \xi^0] \right)^{m_2}$$

Clearly, if $l_1 - l_2 = \delta$, we are in $L^2_\delta(B)$; this occurs once. So we get for $n > 4$:

Proposition 8.10. *If $n > 4$, then*

$$\begin{aligned} L^2_\delta(B) &= \bigoplus_{l \geq 0, m \geq 0} R_{2l+\delta+m,m} \quad \textit{if} \quad \delta \geq 0 \\ &= \bigoplus_{l \geq 0, m \geq 0} R_{2l-\delta+m,m} \quad \textit{if} \quad \delta < 0. \end{aligned}$$

The highest weight vectors are:

$$Y^\delta_{l,m}(z) = [z, \xi^0]^{l+\delta} [\overline{z}, \xi^0]^l \left([z, \xi^0][\overline{z}, \xi^1] - [z, \xi^1][\overline{z}, \xi^0] \right)^m \quad (\delta \geq 0)$$

and

$$Y^\delta_{l,m}(z) = [z, \xi^0]^l [\overline{z}, \xi^0]^{l-\delta} \left([z, \xi^0][\overline{z}, \xi^1] - [z, \xi^1][\overline{z}, \xi^0] \right)^m \quad (\delta < 0)$$

respectively.

Remark 8.11. *For $n = 3$ we have to replace ξ^1 by $e_3 = (0, 0, 1)$ and only take $m_2 = 0$ or 1; for $n = 4$ we have to take in addition the vectors where ξ^1 is replaced by $\overline{\xi}^1$. We refer to [31], for more details.*

Let us denote the representations $R_{2l+|\delta|+m,m}$ just by $\rho^\delta_{l,m}$; observe that $\rho^\delta_{l,m} \simeq \rho^{-\delta}_{l,m}$, the identification is given by the map $\varphi(z) \longrightarrow \varphi(\overline{z})$.

8.3.1.6 Casimir operators

Let $\mathbb{D}(X)$ be the algebra of invariant differential operators on X. Observe that the (real) rank of $X = G/H$ is equal 2. So $\mathbb{D}(X)$ has two generators. We shall present now these generators. Both belong to the center of $U(\mathfrak{g})$. We shall also study the effect of their action on $E_{\delta,s}$. The Lie algebra $\mathfrak{g}$ is complex with Killing form $F(X, Y) = (n-2)\operatorname{tr} XY$. This form F is non-degenerate, $\operatorname{Ad}(G)$-invariant and complex bilinear. This gives rise, by well-known techniques, to a holomorphic and an anti-holomorphic invariant differential operator; call them Ω_1 and Ω_2. Then $\Omega = \Omega_1 + \Omega_2$ is the Casimir operator. According to [20] we have:

Proposition 8.12. *The holomorphic invariant operator is equal to the following*

$$\Omega_1 = \frac{1}{2(n-2)} \left((iL)^2 - \rho(iL) \right) + 4 \sum_{j=1}^{n-2} \gamma_j \sigma(N_j) N_j + \sum_{j=1}^{m} \epsilon_j Z_j^2$$

where $Y \in \mathfrak{g}$ corresponds to $(Yf)(g) = \frac{\partial}{\partial z} f(g \exp zY)\big|_{z=0}$, f holomorphic. For notation see [20].

Let $f \in E_{\delta,s}$. Then

$$\begin{aligned}(\Omega_1 f)(g) &= \frac{1}{2(n-2)}\left(\frac{\partial^2}{\partial z^2} - \rho\frac{\partial}{\partial z}\right) f(ga_{iz})\big|_{z=0} \\ &= \frac{1}{2(n-2)}\left(\frac{\partial^2}{\partial z^2} - \rho\frac{\partial}{\partial z}\right) e^{-(s-\rho)x}e^{-i\delta y} f(g)\big|_{z=0} \\ &= \frac{1}{8(n-2)}\left((s+\delta)^2 - \rho^2\right) f(g)\end{aligned}$$

where $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$.
Similarly,

$$(\Omega_2 f)(g) = \frac{1}{8(n-2)}\left((s-\delta)^2 - \rho^2\right) f(g).$$

Let Ω be the Casimir operator of the real Lie algebra $\mathfrak{g}$. Then

$$(\Omega f)(g) = \frac{1}{4(n-2)}\left(s^2 + \delta^2 - \rho^2\right) f(g) \quad \text{for } f \in E_{\delta,s}.$$

8.3.1.7 Construction of the spherical distributions

Define $P_1 : \Xi \longrightarrow \mathbb{C}$ by $P_1(\xi) = [x^0, \xi] = \xi_1$. If $f \in E_{\delta,s}(\Xi)$, define

$$Z_{\delta,s}(f) = \int_B P_1(b)^{-s-\rho,-\delta} f(b) db$$

with $y^{\lambda,l} = |y|^\lambda \left(\frac{y}{|y|}\right)^l$.

We want to extend the analytic distribution-valued function $s \longrightarrow Z_{\delta,s}$, defined on $\{s \mid \text{Re } s < -\rho + 1\}$ to a meromorphic distribution-valued function on $\mathbb{C}$.

Proposition 8.13. *If $f \in E_{\delta,s}(\Xi)$, the integral defining $Z_{\delta,s}(f)$ is absolutely convergent for Re $(s) < -\rho + 1$ which has a meromorphic continuation with at most simple poles at $\{-\rho + |\delta| + 2, -\rho + |\delta| + 4, -\rho + |\delta| + 6, \dots\}$.*

Proof. We use the identification of B with $\Sigma = \{\sigma = (p,q) |\ p, q \in S(\mathbb{R}^n), (p,q) = 0\}$. The integral $Z_{\delta,s}(f)$ can be written as

$$\begin{aligned}Z_{\delta,s}(f) &= \int_B P_1(b)^{-s-\rho,-\delta} f(b) db \\ &= \int_\Sigma (p_1 + iq_1)^{-s-\rho,-\delta} f(p_1 + iq_1, \dots, p_n + iq_n) d(p,q)\end{aligned}$$

where $d(p,q)$ is a measure on Σ. By (8.4) a function f is a function on Σ with the property

$$\forall \lambda \in \mathbb{C}, \lambda \neq 0 \quad f(\lambda\sigma) = |\lambda|^{s-\rho}\left(\frac{\lambda}{|\lambda|}\right)^\delta f(\sigma). \tag{8.5}$$

Then we use a very important lemma.

Lemma 8.14. *The following equality holds for $x \in \mathbb{C}^*$, $Re\ \lambda > -1$*

$$e(\nu, \lambda) \cdot |x|^\lambda \cdot \delta_\nu(\frac{x}{|x|}) = \int_{U(1,\mathbb{C})} |Re\ (ux)|^\lambda \cdot \operatorname{sgn}^\nu(Re\ (ux)) \cdot \delta_\nu^{-1}(u)du,$$

where $e(\nu, \lambda) = \frac{2^{-\lambda}\Gamma(1+\lambda)}{\Gamma(\frac{\lambda-\nu+2}{2})\Gamma(\frac{\lambda+\nu+2}{2})}$. *Here* $\delta_\nu(e^{i\theta}) = e^{i\nu\theta}$ *for some* $\nu \in \mathbb{Z}$.

The proof is given in [3], Appendix B, section B.2.

This lemma permits to write down the integral $Z_{\delta,s}(f)$ in the form

$$\begin{aligned} Z_{\delta,s}(f) &= \frac{\Gamma(\frac{-s-\rho+\delta+2}{2})\Gamma(\frac{-s-\rho-\delta+2}{2})}{2^{s+\rho}\Gamma(1-s-\rho)} \int_{U(1,\mathbb{C})} \int_\Sigma [\mathrm{Re}\ ((p_1+iq_1)u)]^{-s-\rho,\delta} u^\delta \cdot \\ & \quad f(p_1+iq_1, \ldots, p_n+iq_n)d(p,q)du \end{aligned}$$

since for $a \in \mathbb{R}$, $|a|^\lambda \cdot \operatorname{sgn}^l a = a^{\lambda,l}$. We change the variables $p_j + iq_j \longrightarrow (p_j + iq_j)u^{-1}$ with $j = 1, \ldots, n$. By (8.5) one has

$$\begin{aligned} Z_{\delta,s}(f) &= \frac{\Gamma(\frac{-s-\rho+\delta+2}{2})\Gamma(\frac{-s-\rho-\delta+2}{2})}{2^{s+\rho}\Gamma(1-s-\rho)} \cdot \\ & \quad \cdot \int_{U(1,\mathbb{C})} \left[\int_\Sigma p_1^{-s-\rho,\delta} f(p_1+iq_1, \ldots, p_n+iq_n)d(p,q) \right] du \end{aligned}$$

Using classical results for the distributions $|x|^\lambda$ and $|x|^\lambda \operatorname{sgn} x$ ([11]), we see that the integral in square brackets can be analytically continued to the entire s-plane with at most simple poles at

$$\begin{aligned} s &= -\rho+1, -\rho+3, -\rho+5, \ldots && \text{if } \delta \text{ is even;} \\ s &= -\rho+2, -\rho+4, -\rho+6, \ldots && \text{if } \delta \text{ is odd.} \end{aligned}$$

The Gamma function $\Gamma(\epsilon)$ is known to have simple poles at $\epsilon = 0, -1, -2, \ldots$ Thus, $Z_{\delta,s}(f)$ admits a meromorphic continuation with at most simple poles at $s = -\rho+\delta+2, -\rho+\delta+4, -\rho+\delta+6, \ldots$

If we change $\delta \mapsto -\delta$ and consider $Z_{-\delta,s}(f)$ we obtain the same answer for $Z_{-\delta,s}(f)$. So, in fact, $Z_{\delta,s}(f)$ admits a meromorphic continuation with at most simple poles at $s = -\rho+|\delta|+2, -\rho+|\delta|+4, -\rho+|\delta|+6, \ldots$ □

Let us write for $f \in E_{\delta,s}(\Xi)$

$$u_{\delta,s}(f) = \frac{1}{\Gamma(\frac{-s-\rho+|\delta|+2}{2})} Z_{\delta,s}(f).$$

Then by the above proposition, the function $u_{\delta,s}(f)$ is an entire function of $s \in \mathbb{C}$. Furthermore:

$$\begin{aligned} & u_{\delta,s} \in E_{\delta,s}(\Xi)' \quad (s \in \mathbb{C}) \\ & \pi_{\delta,s}^{-\infty}(h)u_{\delta,s} = u_{\delta,s} \ \text{ for all } \ h \in H. \end{aligned}$$

If $\varphi \in \mathcal{D}(G)$ we define:

$$\zeta_{\delta,s}(\varphi) =< u_{\delta,-\bar{s}}, \pi_{\delta,s}^{-\infty}(\varphi)u_{\delta,s} > .$$

Proposition 8.15. *The distribution $\zeta_{\delta,s}$ is an H-bi-invariant eigendistribution of $\mathbb{D}(X)$:*

$$\begin{aligned}\Omega_1\zeta_{\delta,s} &= \frac{1}{8(n-2)}\left((s+\delta)^2-\rho^2\right)\zeta_{\delta,s}\\ \Omega_2\zeta_{\delta,s} &= \frac{1}{8(n-2)}\left((s-\delta)^2-\rho^2\right)\zeta_{\delta,s}.\end{aligned}$$

8.3.1.8 Intertwining operators

Let $f \in E_{\delta,s}(\Xi)$. Define

$$\left(A_{\delta,s}f\right)(\xi) = \int_B P(\xi,b)^{-s-\rho,-\delta} f(b)db.$$

This is a meromorphic function of $s \in \mathbb{C}$.

Lemma 8.16. *(i) If $f \in E_{\delta,s}$, then $A_{\delta,s}f \in E_{-\delta,-s}$.*

(ii) $A_{\delta,s} : E_{\delta,s} \longrightarrow E_{-\delta,-s}$ is continuous.

(iii) $A_{\delta,s}$ intertwines the action of G:

$$A_{\delta,s}\circ\pi_{\delta,s}(g) = \pi_{-\delta,-s}(g)\circ A_{\delta,s} \quad \textit{for all} \ \ g\in G.$$

Let $w \in K$ be the element

$$\begin{pmatrix} 0 & 1 & 0 & \\ 1 & 0 & 0 & \\ 0 & 0 & -1 & \\ & & & I \end{pmatrix}.$$

Then $wLw^{-1} = -L$ and $wNw^{-1} = \overline{N}$. An easy calculation shows that

$$\left(A_{\delta,s}f\right)(g\xi^0) = 2^{-s-\rho}i^{-\delta}\int_{\overline{N}} f(gw\overline{n}\xi^0)d\overline{n}$$

provided $d\overline{n}$ is suitably normalized. This is the standard intertwining operator, see [19].

8.3.1.9 The Fourier transform

We start with the Cartan-Berger decomposition of X.

Lemma 8.17. *(i) There holds $G = K\overline{A_I^+}H$, with $A_I^+ = \{a_{-it} = \exp -itL : t > 0\}$. More precisely: to any $x \in G$ corresponds a unique $t \geq 0$ such that $x \in Ka_{-it}H$.*

(ii) The set $\mathcal{C}^\infty(K\setminus G/H)$ is (by restriction to A_I) in bijective correspondence with $\mathcal{C}^\infty(\mathbb{R})^{even}$.

(iii) The map $K/M \times \mathbb{R}_+ \longrightarrow G/H$ *defined by* $(k,t) \mapsto ka_{-it}H$*, is a diffeomorphism onto an open subset of* G/H*.*

Choose from now on the G-invariant measure dx on X such that:

$$\int_X \varphi(x)dx = \int_{K/M}\int_0^\infty \varphi(ka_{-it}x^0)A(t)dkdt,$$

where dk is the Haar measure of K with total volume 1 and $A(t) = (\sinh 2t)^{n-2}$.

Definition 8.18. If $\varphi \in \mathcal{D}(X)$, then the Fourier transform $\mathcal{F}_{\delta,s}\varphi$ is given by

$$\left(\mathcal{F}_{\delta,s}\varphi\right)(\xi) = \int_X P(x,\xi)^{s-\rho,\delta}\varphi(x)dx \qquad (\text{Re }(s) > \rho).$$

If $\varphi \in \mathcal{D}(G)$, then

$$\left(\mathcal{F}_{\delta,s}\varphi\right)(\xi) = \left(\mathcal{F}_{\delta,s}\varphi_0\right)(\xi)$$

where

$$\varphi_0(x) = \int_H \varphi(gh)dh \qquad (x = gx^0),$$

where dh is the Haar measure of H such as $dg = dhdx$.

We have:

(i) For fixed δ, the map $s \longrightarrow \mathcal{F}_{\delta,s}(\varphi)$ is meromorphic in s.

(ii) $\mathcal{F}_{\delta,s}(\varphi) \in E_{\delta,s}$.

(iii) $\mathcal{F}_{\delta,s}$ commutes with the G-action.

(iv)

$$\begin{aligned}
\mathcal{F}_{\delta,s}(\Omega_1\varphi) &= \frac{1}{8(n-2)}\left((s+\delta)^2-\rho^2\right)\mathcal{F}_{\delta,s}(\varphi)\\
\mathcal{F}_{\delta,s}(\Omega_2\varphi) &= \frac{1}{8(n-2)}\left((s-\delta)^2-\rho^2\right)\mathcal{F}_{\delta,s}(\varphi)
\end{aligned}$$

Remark 8.19. *The spherical distributions are related to the Fourier transform. If* φ *is a function in* $\mathcal{D}(G)$*, then we have the following relation:*

$$\zeta_{\delta,is}(\tilde{\varphi}\star\varphi) = \frac{1}{\Gamma(\frac{-is-\rho+|\delta|+2}{2})\Gamma(\frac{is-\rho+|\delta|+2}{2})}\int_B |\mathcal{F}_{\delta,is}\varphi(b)|^2 db$$

with $\tilde{\varphi}(g) = \overline{\varphi(g^{-1})}$ *and* $s \in \mathbb{R}$*.*

8.3.1.10 The Plancherel formula

In this section we will formulate the Plancherel theorem. For this we use the same method as Van den Ban. In our case only one parabolic subgroup occurs. To define the Plancherel measure, we need, as Van den Ban has done, to construct a specific intertwining operator $T_{\delta,s}$ between $\pi_{\delta,s}(g)$ and $\pi_{\delta,-s}(\overline{g})$. Composing $T_{\delta,-s}$ with $T_{\delta,s}$ we have an intertwining operator from $E_{\delta,s}(\Xi)$ to $E_{\delta,s}(\Xi)$. Then using the Schur's lemma we arrive to the conclussion that this must be a constant. For more details see [36].

We begin with the construction of $T_{\delta,s}$. Let $A_0 : E_{\delta,s}(\Xi) \longrightarrow E_{-\delta,s}(\Xi)$ be given by $\varphi(\xi) \mapsto \varphi(\overline{\xi})$. Then

$$A_0 \circ \pi_{\delta,s}(g) = \pi_{-\delta,s}(\overline{g}) \circ A_0$$

where $\overline{g}$ is the complex conjugate of g. Observe that K is fixed under this conjugation. $A_0 \circ A_{\delta,s}$ is an intertwining operator between $\pi_{\delta,s}(g)$ and $\pi_{\delta,-s}(\overline{g})$ with $A_{\delta,s}$ as before. Calling $A'_{\delta,s} = A_0 \circ A_{\delta,s}$, the specific operator $T_{\delta,s}$ is equal to $2^{s+\rho}A'_{\delta,s}$. We have $T_{\delta,-s} \circ T_{\delta,s}$ is an intertwining operator between $E_{\delta,s}(\Xi)$ and $E_{\delta,s}(\Xi)$. By Schur's lemma $T_{\delta,-s} \circ T_{\delta,s} = \lambda I$. Taking $\xi = \overline{\xi^0}$, $f(b) = [\xi^0, b]^\delta$ if $\delta \geq 0$ and $\xi = \xi^0$, $f(b) = [\xi^0, \overline{b}]^{-\delta}$ if $\delta < 0$. Applying Schur's lemma for $\pi_{\delta,s}$ restricted to K it follows $T_{\delta,s}f(\xi) = c(\delta,s)f(\xi)$ and $c(\delta,s) = c(0, s - |\delta|)$ for all δ. Therefore $\lambda = c(\delta,s)c(\delta,-s)$.

We compute the value of $c(0,s)$. Taking $\xi = \overline{\xi^0}$ and $f = 1$, we obtain

$$\begin{aligned} c(0,s) &= (T_{0,s}f)(\overline{\xi^0}) \\ &= 2^{s+\rho}(A'_{0,s}f)(\overline{\xi^0}) \\ &= 2^{s+\rho}\int_B |P(\xi^0, b)|^{-s-\rho}db \\ &= 2^{s+\rho}\int_K |P(\xi^0, k\xi^0)|^{-s-\rho}dk \\ &= 2^{s+\rho}\int_K |2\lim_{t\to\infty} e^{-t}P(ka_{-it}x^0, \xi^0)|^{-s-\rho}dk \\ &= \lim_{t\to\infty} e^{(s+\rho)t}\int_K |P(ka_{-it}x^0, \xi^0)|^{-s-\rho}dk. \end{aligned} \qquad (8.6)$$

Let $\phi(t,s)$ be equal to $\int_K |P(ka_{-it}x^0, \xi^0)|^{-s-\rho}dk$. Using property (iv) of the Fourier transform, we obtain that $u = \phi(.\,, -s)$ is a solution of

$$\frac{1}{A(t)}\frac{d}{dt}A(t)\frac{du}{dt} = (s^2 - \rho^2)u.$$

We can solve this equation by first deriving a differential equation for $v(t) = (\cosh 2t)^{(-s+\rho)/2}u(t)$, and then substituting $\tanh^2 2t$ as a new variable in the equation thus found, to obtain a hypergeometric differential equation. Solving the latter we find (using the fact that $\phi(t,-s)$ is regular for $t=0$)

$$u(t) = \beta(s)(\cosh 2t)^{(s-\rho)/2} F(\frac{\rho-s}{4}, \frac{\rho-s+2}{4}; \frac{1+\rho}{2}; \tanh^2 2t).$$

We now determine $\beta(s)$

$$\begin{aligned}
\beta(s) &= u(0) = \int_K |P(kx^0,\xi^0)|^{s-\rho} dk \\
&= \int_{S^{n-1}} |s_1^2+s_2^2|^{(s-\rho)/2} ds \\
&= \frac{2\pi}{\mathrm{vol}(S^{n-1})} \int_0^\pi \ldots \int_0^\pi (\sin\theta_2)^{s-\rho+1} \ldots (\sin\theta_{n-1})^{s-\rho+n-2} d\theta_2 \ldots d\theta_{n-1}.
\end{aligned}$$

Using $\displaystyle\int_0^\pi \sin^\alpha t dt = \sqrt{\pi}\frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{2+\alpha}{2})}$, we obtain

$$\begin{aligned}
\beta(s) &= \frac{2\pi}{\mathrm{vol}(S^{n-1})} (\sqrt{\pi})^{n-2} \frac{\Gamma(\frac{s-\rho+2}{2})}{\Gamma(\frac{s-\rho+n}{2})} \\
&= \frac{\Gamma(\frac{n}{2})\Gamma(\frac{s-\rho+2}{2})}{\Gamma(\frac{s-\rho+n}{2})}.
\end{aligned}$$

Substituting this value in (8.6) and using relations between hypergeometric functions see [6], we get

$$\begin{aligned}
c(0,s) &= \lim_{t\to\infty} e^{(s+\rho)t}\beta(-s)(\cosh 2t)^{(-s-\rho)/2} F(\frac{\rho+s}{4}, \frac{\rho+s+2}{4}; \frac{1+\rho}{2}; \tanh^2 2t) \\
&= \beta(-s)2^{(s+\rho)/2} \frac{\Gamma(\frac{1+\rho}{2})\Gamma(\frac{-s}{2})}{\Gamma(\frac{\rho-s}{4})\Gamma(\frac{2+\rho-s}{4})} \\
&= 2^{(s+\rho)/2} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{-s-\rho+2}{2})\Gamma(\frac{1+\rho}{2})\Gamma(\frac{-s}{2})}{\Gamma(\frac{-s-\rho+n}{2})\Gamma(\frac{\rho-s}{4})\Gamma(\frac{2+\rho-s}{4})} \\
&= 2^{\rho-1} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{1+\rho}{2})\Gamma(\frac{-s-\rho+2}{2})\Gamma(\frac{-s}{2})}{\sqrt{\pi}\Gamma(\frac{-s-\rho+n}{2})\Gamma(\frac{-s+\rho}{2})}.
\end{aligned}$$

Remark 8.20. *During the calculation we had to assume Re* $(s) < -\rho$*, but the result is valid for all* s*, because both sides of the equality are meromorphic functions of* s*.*

Now we can compute the value of $c(\delta,s)$

$$c(\delta,s) = 2^{\rho-1} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{1+\rho}{2})\Gamma(\frac{-s+|\delta|-\rho+2}{2})\Gamma(\frac{-s+|\delta|}{2})}{\sqrt{\pi}\Gamma(\frac{-s+|\delta|-\rho+n}{2})\Gamma(\frac{-s+|\delta|+\rho}{2})}.$$

Theorem 8.21. *Let $f \in \mathcal{D}(X)$. Then it follows*

$$||f||^2 = C \sum_{\delta \in \mathbb{Z}} \int_{\mathbb{R}} \frac{1}{|c(\delta, is)|^2} ||\mathcal{F}_{\delta,is} f||^2 ds$$

with C a positive constant.

Remark 8.22. *Taking the same normalization of the measures as Van den Ban, the value of the constant C is equal to π.*

Remark 8.23. *For $n = 3$ the value of $c(\delta, s)$ is $\dfrac{-1}{s - |\delta|}$ and we obtain the Plancherel formula of* $\mathrm{SL}(2,\mathbb{C})/\mathrm{SO}(2,\mathbb{C})$. *For $n = 4$ we get $c(\delta, s) = \dfrac{4}{(s - |\delta|)^2}$ and we obtain the Plancherel formula of* $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})/D$, *where D stands for the diagonal of* $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$.

8.3.2 Decomposition of $L^2(\mathbb{C}^n)$

8.3.2.1 The definition of the oscillator representation

Let G be the group $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SO}(n,\mathbb{C})$ with $n \geq 3$ and let ω_n be the unitary representation of G on $H = L^2(\mathbb{C}^n)$ defined by:

$$\begin{aligned}
\omega_n(g)f(z) &= f(g^{-1} \cdot z), \qquad g \in \mathrm{SO}(n,\mathbb{C}) \\
\omega_n(g(a))f(z) &= |a|^n f(az), \qquad g(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, a \in \mathbb{C}^* \\
\omega_n(t(b))f(z) &= e^{-i\pi \mathrm{Re}\ (b[z,z])} f(z), \qquad t(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, b \in \mathbb{C} \\
\omega_n(\sigma)f(z) &= \int_{\mathbb{C}^n} e^{2i\pi \mathrm{Re}\ ([z,w])} f(w) dw, \qquad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\end{aligned}$$

where $[z, w] = z_1 w_1 + z_2 w_2 + \cdots + z_n w_n$ if $z = (z_1, z_2, \ldots, z_n)$, $w = (w_1, w_2, \ldots, w_n)$. We call ω_n the oscillator representation of G. More precisely, it is the restriction of the metaplectic representation of the group $Sp(n,\mathbb{C}) \subset Sp(2n,\mathbb{R})$ to G.

Let $\mathcal{S}(\mathbb{C}^n)$ be the space of Schwartz functions on $\mathbb{C}^n$. Note that $\mathcal{S}(\mathbb{C}^n)$ is stable under the action of ω_n, so is the space $\mathcal{S}'(\mathbb{C}^n)$ of tempered distributions on $\mathbb{C}^n$.

8.3.2.2 Some minimal invariant Hilbert subspaces of $\mathcal{S}'(\mathbb{C}^n)$

We consider $L^2(\mathbb{C}^n)$ as a Hilbert subspace of $\mathcal{S}'(\mathbb{C}^n)$ and we will try to decompose it into minimal invariant Hilbert subspaces. Irreducible unitary representations of G are of the form $\pi \otimes \rho$ where π is an irreducible representation of $\mathrm{SL}(2,\mathbb{C})$ and ρ one of $\mathrm{SO}(n,\mathbb{C})$. We will now construct intertwining operators from $\mathcal{S}(\mathbb{C}^n)$ to the Hilbert space of such representations. Not all combinations (π, ρ) occur. Here are some of these operators.

Let $E_{\delta,s}(\Xi)$ be the space of $\mathcal{C}^\infty$-functions on $\Xi = \{z \in \mathbb{C}^n : [z,z] = 0, z \neq 0\}$ such that:

$$f(\lambda\xi) = \left(\frac{\lambda}{|\lambda|}\right)^\delta |\lambda|^{s-\rho} f(\xi) \quad (\xi \in \Xi, \lambda \in \mathbb{C}^*)$$

with $\delta \in \mathbb{Z}$, $s \in \mathbb{C}$.

The group $\mathrm{SO}(n,\mathbb{C})$ acts irreducibly on $E_{\delta,s}(\Xi)$ by

$$\rho_{\delta,s}(g)f(z) = f(g^{-1} \cdot z)$$

for $s \neq 0$, with $g \in \mathrm{SO}(n,\mathbb{C})$ and $f \in E_{\delta,s}(\Xi)$. The representation is pre-unitary for $s \in i\mathbb{R}$, see Section 8.3.1. Let us denote the unitary completion of this space by $\mathcal{H}_{\delta,s}$.

The group $\mathrm{SL}(2,\mathbb{C})$ acts irreducibly on $L^2(\mathbb{C})$ by

$$\pi_{\lambda,m}(g)f(z) = |cz+d|^{-\lambda-2} \left(\frac{cz+d}{|cz+d|}\right)^{-m} f\left(\frac{az+b}{cz+d}\right)$$

with $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $g \in \mathrm{SL}(2,\mathbb{C})$, $\lambda \in i\mathbb{R}$, $m \in \mathbb{Z}$. ($\pi_{\lambda,m}$ belongs to the so-called principal series, see Chapter 3).

Let us now define an operator $\mathcal{F}_{is,\delta}$ from $\mathcal{S}(\mathbb{C}^n)$ into the Hilbert space $L^2(\mathbb{C}) \hat{\otimes}_2 \mathcal{H}_{\delta,is}$ as follows:

$$(\mathcal{F}_{is,\delta}\varphi)(z,\xi) = \int_{\mathbb{C}^n} e^{i\pi \mathrm{Re}\ (z[w,w])} [w,\xi]^{is-\rho,\delta} \varphi(w) dw \tag{8.7}$$

with $s \in \mathbb{R}$, $\delta \in \mathbb{Z}$, $z \in \mathbb{C}$ and $\xi \in \Xi$.

We will show that:

1) The operator $\mathcal{F}_{is,\delta}$ is well defined, i.e. $\mathcal{F}_{is,\delta}\varphi \in L^2(\mathbb{C})\hat{\otimes}_2\mathcal{H}_{\delta,is}$.

2) $\mathcal{F}_{is,\delta}$ is continuous as operator from $\mathcal{S}(\mathbb{C}^n)$ to $L^2(\mathbb{C})\hat{\otimes}_2\mathcal{H}_{\delta,is}$.

3) The operator $\mathcal{F}_{is,\delta}$ intertwines the action of ω_n and $\pi_{is,\delta} \otimes \rho_{is,\delta}$.

First, we want to prove 1) and 2).

We have:

Proposition 8.24. *For all $s \in \mathbb{R}$ and $\delta \in \mathbb{Z}$ the operator $\mathcal{F}_{is,\delta}$ maps $\mathcal{S}(\mathbb{C}^n)$ into $L^2(\mathbb{C})\hat{\otimes}_2\mathcal{H}_{\delta,is}$ and $\mathcal{F}_{is,\delta}$ is continuous.*

Proof. We take $\xi = \xi^0 = (1, i, 0, \ldots, 0)$ in (8.7). To get the result we divide the proof in several steps.

1. Consider the following map:

$$(F\varphi)(z) = \int_{\mathbb{C}} e^{i\pi \mathrm{Re}\ (zw^2)} \varphi(w) dw$$

for $z \in \mathbb{C}$, $\varphi \in \mathcal{S}(\mathbb{C})$. We will make some estimations for $|(F\varphi)(z)|$, which we need later on.

We may assume $z \in \mathbb{R}$. Indeed, writing $z = |z|e^{i\theta}$ we obtain:

$$(F\varphi)(z) = (F\varphi_1)(|z|)$$

with $\varphi_1(w) = \varphi(e^{-i\theta/2}w)$.

So assume $z \in \mathbb{R}$. Then

$$(F\varphi)(z) = \int_{\mathbb{R}^2} e^{i\pi z(x^2-y^2)} \varphi(x,y)dxdy.$$

Set $x + y = u$, $x - y = v$. Then

$$(F\varphi)(z) = \frac{1}{2}\int_{\mathbb{R}^2} e^{i\pi zuv} \varphi\left(\frac{u+v}{2}, \frac{u-v}{2}\right) dudv.$$

Write $g(t,v) = \frac{1}{2}\int_{\mathbb{R}} e^{i\pi tu} \varphi\left(\frac{u+v}{2}, \frac{u-v}{2}\right) du$. This is again a Schwartz function in $\mathcal{S}(\mathbb{R}^2)$, and

$$(F\varphi)(z) = \int_{\mathbb{R}} g(zv, v)dv.$$

So we have:

$$|(F\varphi)(z)| \leq \sup_{t,r} \left[(1+t^2+r^2)^N |g(t,r)|\right] \int_{\mathbb{R}} (1+|zv|^2+v^2)^{-N} dv$$

with N a positive integer.

The Schwartz-norm

$$||g|| = \sup_{t,r} \left[(1+t^2+r^2)^N |g(t,r)|\right]$$

is bounded by a Schwartz-norm of $\varphi_1\left(\frac{u+v}{2}, \frac{u-v}{2}\right)$, hence by a Schwartz-norm of φ_1, hence by a Schwartz-norm of φ, since all operations which occur are continuous $\mathcal{S}(\mathbb{R}^2) \longrightarrow \mathcal{S}(\mathbb{R}^2)$.

So we get for $z \in \mathbb{C}$:

$$\begin{aligned} |(F\varphi)(z)| &\leq C||\varphi|| \int_{\mathbb{R}} \left(1+(1+|z|^2)v^2\right)^{-N} dv \\ &\leq C||\varphi||(1+|z|^2)^{-1/2} \int_{\mathbb{R}} (1+v^2)^{-N} dv \\ &\leq C'||\varphi||(1+|z|^2)^{-1/2} \end{aligned}$$

with C and C' constants.

2. Consider now

$$(\mathcal{F}_{is,\delta}\varphi)(z) = \int_{\mathbb{C}^2} e^{i\pi \mathrm{Re}\ (z(w_1^2+w_2^2))}(w_1+iw_2)^{is-\rho,\delta}\varphi(w_1,w_2)dw_1dw_2$$

for $is \in \mathbb{C}$, $\rho \in \mathbb{N}$, $\delta \in \mathbb{Z}$, $z \in \mathbb{C}$, $\varphi \in \mathcal{S}(\mathbb{C}^2)$.

The integral exists anyway for Re $(is) \geq \rho$. For other is we use analytic continuation.

To estimate $(\mathcal{F}_{is,\delta}\varphi)(z)$ also for these is, we need an explicit description of this analytic continuation. We make use of [24], p. 138,139.

Perform the change of variables $y_1 = w_1 + iw_2$, $y_2 = w_1 - iw_2$. Then we get

$$\begin{aligned}(\mathcal{F}_{is,\delta}\varphi)(z) &= \frac{1}{4}\int_{\mathbb{C}^2} e^{i\pi \mathrm{Re}\ (zy_1y_2)} y_1^{is-\rho,\delta}\varphi\left(y_1+y_2, \frac{y_1-y_2}{i}\right)dy_1dy_2 \\ &= \int_{\mathbb{C}} |y_1|^{is-\rho}\left(\frac{y_1}{|y_1|}\right)^{\delta} g(zy_1,y_1)dy_1 \qquad (8.8)\end{aligned}$$

where $g(u,y_1) = \frac{1}{4}\int_{\mathbb{C}} e^{i\pi \mathrm{Re}\ (y_2u)}\varphi\left(y_1+y_2, \frac{y_1-y_2}{i}\right)dy_2$. Now g is again in $\mathcal{S}(\mathbb{C}^2)$ and the operation $\varphi\left(y_1+y_2, \frac{y_1-y_2}{i}\right) \longrightarrow g(u,y_1)$ is continuous $\mathcal{S}(\mathbb{C}^2) \longrightarrow \mathcal{S}(\mathbb{C}^2)$.

We write this integral using polar coordinates $y_1 = re^{i\theta}$,

$$(\mathcal{F}_{is,\delta}\varphi)(z) = \int_0^\infty r^{is-\rho+1}h(z,r)dr$$

where $h(z,r) = \int_0^{2\pi} g(zre^{i\theta}, re^{i\theta})e^{i\delta\theta}d\theta$.

Now the meromorphic continuation reads:

$$(\mathcal{F}_{is,\delta}\varphi)(z) = \frac{(-1)^{\rho-1}}{(is-\rho+2)\dots(is-1)is}\int_0^\infty r^{is}\frac{\partial^{\rho-1}}{\partial r^{\rho-1}}h(z,r)dr.$$

One has:

$$\begin{aligned}\frac{\partial}{\partial r}h(z,r) = \int_0^{2\pi} &\left[ze^{i\theta}\frac{\partial}{\partial y_1}g(zre^{i\theta},re^{i\theta}) + \overline{z}e^{-i\theta}\frac{\partial}{\partial \overline{y}_1}g(zre^{i\theta},re^{i\theta})\right. \\ &\left. + e^{i\theta}\frac{\partial}{\partial y_2}g(zre^{i\theta},re^{i\theta}) + e^{-i\theta}\frac{\partial}{\partial \overline{y}_2}g(zre^{i\theta},re^{i\theta})\right]e^{i\delta\theta}d\theta.\end{aligned}$$

Hence, for some $s \in \mathbb{R}$,

$$|(\mathcal{F}_{is,\delta}\varphi)(z)| \leq C||g||(1+|z|^2)^{\frac{\rho-1}{2}}\int_0^\infty \left(1+r^2(1+|z|^2)\right)^{-N}dr$$

for some Schwartz-norm of g, depending on a positive integer N and with C a constant.

This gives:

$$\begin{aligned} |(\mathcal{F}_{is,\delta}\varphi)(z)| &\leq C'||g||(1+|z|^2)^{\frac{\rho-1}{2}}(1+|z|^2)^{-1/2} \\ &\leq C'||g||(1+|z|^2)^{\rho/2-1} \\ &\leq C''||\varphi||(1+|z|^2)^{\rho/2-1} \end{aligned}$$

for some Schwartz-norm of φ and C, C', C'' constants.

Remark 8.25. *It follows that* $\dfrac{\partial^k}{\partial r^k}h(z,0) = 0$ *for* $k = 0, 1, \ldots, |\delta|-1$. *So* (8.8) *is the actual analytic continuation to Re* $(is) > \rho - |\delta| - 2$.

Define by analytic continuation

$$(\mathcal{F}_{is,\delta}\varphi)(z,\xi^0) = \int_{\mathbb{C}^n} e^{i\pi \text{Re } (z[w,w])}(w_1 + iw_2)^{is-\rho,\delta}\varphi(w)dw$$

with $s \in \mathbb{R}$, $\rho = n-2$, $\delta \in \mathbb{Z}$, $z \in \mathbb{C}$, $\varphi \in \mathcal{S}(\mathbb{C}^n)$.

Applying 1. and 2. step by step we get:

$$\begin{aligned} |(\mathcal{F}_{is,\delta}\varphi)(z,\xi^0)| &\leq C||\varphi||(1+|z|^2)^{\rho/2-1}(1+|z|^2)^{-\frac{n-2}{2}} \\ &\leq C||\varphi||(1+|z|^2)^{-1} \end{aligned}$$

for some Schwartz-norm of φ and with C a constant.

Define by analytic continuation:

$$(\mathcal{F}_{is,\delta}\varphi)(z,\xi) = \int_{\mathbb{C}^n} e^{i\pi \text{Re } (z[w,w])}[w,\xi]^{is-\rho,\delta}\varphi(w)dw$$

$\xi \in B = K\xi^0$, $K = \text{SO}(n,\mathbb{R})$. This is a $\mathcal{C}^\infty$-function of (z,ξ). Then clearly, writing $\xi = k\xi^0$ for some $k \in K$, we get:

$$(\mathcal{F}_{is,\delta}\varphi)(z,\xi) = (\mathcal{F}_{is,\delta}(L_k\varphi))(z,\xi^0),$$

so

$$|(\mathcal{F}_{is,\delta}\varphi)(z,\xi)| \leq C||\varphi||(1+|z|^2)^{-1}$$

for all ξ, for some Schwartz-norm of φ and with C a constant. Hence $(\mathcal{F}_{is,\delta}\varphi)(z,\xi) \in L^2(\mathbb{C} \times B)$, and $\varphi \longrightarrow \mathcal{F}_{is,\delta}\varphi$ is continuous $\mathcal{S}(\mathbb{C}^n) \longrightarrow L^2(\mathbb{C} \times B)$. □

In order to prove the intertwining relation we introduce the following lemma.

Lemma 8.26. *For every function* $\varphi \in \mathcal{S}(\mathbb{C}^n)$ *one has*

$$\int_{\mathbb{C}^n} e^{i\pi Re\ (z[w,w])}[w,\xi]^{is,\delta}\hat{\varphi}(w)dw$$
$$= z^{-is,-\delta}|z|^{-n}\int_{\mathbb{C}^n} e^{-i\pi Re\ ([w,w]/z)}[w,\xi]^{is,\delta}\varphi(w)dw$$

with $\xi \in \Xi$, $s \in \mathbb{R}$, $\delta \in \mathbb{Z}$ *and the Fourier transform is defined by*

$$\hat{f}(z) = \int_{\mathbb{C}^n} e^{-2\pi i Re\ ([z,w])} f(w) dw.$$

Proof. By definition

$$\int_{\mathbb{C}^n} e^{i\pi \mathrm{Re}\ (z[w,w])} [w,\xi]^{is,\delta} \hat{\varphi}(w) dw$$
$$= \int_{\mathbb{C}^n} \varphi(x) \left(\int_{\mathbb{C}^n} e^{i\pi \left(\mathrm{Re}\ (z[w,w]) - 2\mathrm{Re}\ ([w,x])\right)} [w,\xi]^{is,\delta} dw \right) dx$$

We claim that the inner integral is equal to

$$\int_{\mathbb{C}^n} e^{i\pi \left(\mathrm{Re}\ (z[w,w]) - 2\mathrm{Re}\ ([w,x])\right)} [w,\xi]^{is,\delta} dw$$
$$= z^{-is,-\delta} |z|^{-n} [x,\xi]^{is,\delta} e^{-i\pi \mathrm{Re}\ ([x,x]/z)} \tag{8.9}$$

By the $\mathrm{SO}(n,\mathbb{C})$-invariance of the equality it suffices to prove it for $\xi = (1, i, 0, \dots, 0) \in \Xi$. In that case, writing $w = (w_1, w_2, \bar{w})$ in $\mathbb{C} \times \mathbb{C} \times \mathbb{C}^{n-2}$, we have

$$\int_{\mathbb{C}^n} e^{i\pi \left(\mathrm{Re}\ (z[w,w]) - 2\mathrm{Re}\ ([w,x])\right)} [w,\xi]^{is,\delta} dw$$
$$= \left(\int_{\mathbb{C}^2} e^{i\pi \left(\mathrm{Re}\ (z(w_1^2+w_2^2)) - 2\mathrm{Re}\ (w_1x_1+w_2x_2)\right)} (w_1 + iw_2)^{is,\delta} dw_1 dw_2 \right)$$
$$\cdot \left(\int_{\mathbb{C}^{n-2}} e^{i\pi \left(\mathrm{Re}\ (z[\bar{w},\bar{w}]) - 2\mathrm{Re}\ ([\bar{w},x])\right)} d\bar{w} \right)$$

It follows from Lemma 8.6 in Section 8.2 that the first integral in the above is:

$$\int_{\mathbb{C}^2} e^{i\pi \left(\mathrm{Re}\ (z(w_1^2+w_2^2)) - 2\mathrm{Re}\ (w_1x_1+w_2x_2)\right)} (w_1 + iw_2)^{is,\delta} dw_1 dw_2$$
$$= z^{-is,-\delta} |z|^{-2} (x_1 + ix_2)^{is,\delta} e^{-i\pi \mathrm{Re}\ \left((x_1^2+x_2^2)/z\right)}.$$

The second integral, by the Bochner formula (see Lemma 4.5 in [17]) is:

$$\int_{\mathbb{C}^{n-2}} e^{i\pi \left(\mathrm{Re}\ (z[\bar{w},\bar{w}]) - 2\mathrm{Re}\ ([\bar{w},x])\right)} d\bar{w} = |z|^{-n+2} e^{-i\pi \mathrm{Re}\ ([\bar{x},\bar{x}]/z)}.$$

Multiplying these two we have thus proved the equality (8.9) and thus the lemma. □

Remark 8.27. *We note here that we can, as in the proof of Lemma 8.6 in Section 8.2, introduce the Gaussian kernel and prove the above lemma rigorously.*

So we have:

Theorem 8.28. *For* $s \in \mathbb{R}$, $\delta \in \mathbb{Z}$ *the operator* $\mathcal{F}_{is,\delta}$ *intertwines the action* ω_n *with* $\pi_{is,\delta} \otimes \rho_{\delta,is}$ *of* $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SO}(n,\mathbb{C})$.

Proof. It is clear that $\mathcal{F}_{is,\delta}$ intertwines the $\mathrm{SO}(n, \mathbb{C})$-action. The intertwining relations for the elements $g(a)$ and $t(b)$ are easy to check. We now check the σ-intertwining relation:

$$\mathcal{F}_{is,\delta}\omega_n(\sigma)\varphi(z, \xi) = \pi_{is,\delta}(\sigma)(\mathcal{F}_{is,\delta}\varphi)(z, \xi) \tag{8.10}$$

for $\varphi \in \mathcal{S}(\mathbb{C}^n)$. By definition,

$$\mathcal{F}_{is,\delta}\omega_n(\sigma)\varphi(z, \xi) = \int_{\mathbb{C}^n} e^{i\pi \mathrm{Re}\ (z[w,w])}[w, \xi]^{is-\rho,\delta}\hat{\varphi}(-w)dw.$$

Lemma 8.26 easily implies (8.10) and thus the theorem. □

8.3.2.3 Decomposition of $L^2(\mathbb{C}^n)$

First we introduce the following lemma.

Lemma 8.29. *Let f be a function in $\mathcal{S}(\mathbb{C}^n)$. Then the relation*

$$\int_{\mathbb{C}^n} f(z)dz = \int_{\mathbb{C}} \int_{w_1^2+\cdots+w_n^2=1} f(\lambda w)|\lambda|^{2n-2} d\lambda d\mu(w)$$

holds where $d\mu(w)$ is a suitably normalized $\mathrm{SO}(n, \mathbb{C})$-invariant measure on $X = \{w \in \mathbb{C}^n | \ w_1^2 + \cdots + w_n^2 = 1\}$.

Proof. Let T be the map from $\mathbb{C}^n \setminus \{z_1^2 + \cdots + z_n^2 = 0\}$ into $\mathbb{C} \times X$ where $X = \{w \in \mathbb{C}^n : w_1^2 + \cdots + w_n^2 = 1\}$ such that for every z we have $T(z) = (\frac{z}{\sqrt{z_1^2+\cdots+z_n^2}}, \sqrt{z_1^2 + \cdots + z_n^2})$. Then T is a diffeomorphism and we can apply the theorem of change of variables. Therefore we obtain

$$\int_{\mathbb{C}^n} f(z)dz = \int_{\mathbb{C}\times X} f(\lambda w)J(\lambda, w)d\mu(w)d\lambda \tag{8.11}$$

Now we want to compute the value of this function $J(\lambda, w)$. First we see that the function J does not depend on w, i.e. $J(\lambda, w) = J(\lambda)$. Let g be an element of $\mathrm{SO}(n, \mathbb{C})$. Then we get for all functions f

$$\begin{aligned}
\int_{\mathbb{C}^n} f(z)dz &= \int_{\mathbb{C}^n} f(g \cdot z)dz \\
&= \int_{\mathbb{C}\times X} f(g \cdot (\lambda w))J(\lambda, w)d\mu(w)d\lambda \\
&= \int_{\mathbb{C}\times X} f(\lambda(g \cdot w))J(\lambda, w)d\mu(w)d\lambda \\
&= \int_{\mathbb{C}\times X} f(\lambda w)J(\lambda, g^{-1}w)d\mu(w)d\lambda
\end{aligned}$$

since $d\mu(w)$ is an $\mathrm{SO}(n,\mathbb{C})$-invariant measure on X. Comparing this relation with (8.11) we see that $J(\lambda, w) = J(\lambda, g\cdot w)$ for all $g \in \mathrm{SO}(n,\mathbb{C})$. This means $J(\lambda, w) = J(\lambda)$. Therefore it is proven that

$$\int_{\mathbb{C}^n} f(z)dz = \int_{\mathbb{C}\times X} f(\lambda w)J(\lambda)d\mu(w)d\lambda$$

Now we will compute the value of $J(\lambda)$. Let $\alpha \in \mathbb{C}^*$. Then we have

$$\begin{aligned}\int_{\mathbb{C}^n} f(\alpha z)dz &= \int_{\mathbb{C}\times X} f(\alpha\lambda w)J(\lambda)d\mu(w)d\lambda \\ &= \int_{\mathbb{C}\times X} f(\lambda w)J(\lambda\alpha^{-1})|\alpha|^{-2}d\mu(w)d\lambda\end{aligned}$$

On the other hand we can write the above integral as follows

$$\begin{aligned}\int_{\mathbb{C}^n} f(\alpha z)dz &= |\alpha|^{-2n}\int_{\mathbb{C}^n} f(z)dz \\ &= |\alpha|^{-2n}\int_{\mathbb{C}\times X} f(\lambda w)J(\lambda)d\mu(w)d\lambda\end{aligned}$$

We can conclude that

$$J(\lambda\alpha^{-1}) = |\alpha|^{-2n+2}J(\lambda).$$

If $\lambda = 1$ then we have $J(\alpha) = c|\alpha|^{2n-2}$ with c a positive constant and the result is proven. □

Let us compute $||\varphi||^2$. Let $\varphi \in \mathcal{S}(\mathbb{C}^n)$. Applying Lemma 8.29 we have

$$\int_{\mathbb{C}^n} |\varphi(z_1,\ldots,z_n)|^2dz = C\int_{\mathbb{C}}\int_{x_1^2+\cdots+x_n^2=1} |\varphi(\sqrt{\lambda}x_1,\ldots,\sqrt{\lambda}x_n)|^2|\lambda|^{n-2}d\mu(x)d\lambda$$

with C a constant. Applying the lemma to the intertwining operator and by analytic continuation we obtain

$$\begin{aligned}\mathcal{F}_{is,\delta}\varphi(z,\xi) &= \int_{\mathbb{C}^n} e^{i\pi\mathrm{Re}\ (z[w,w])}[w,\xi]^{is-\rho,\delta}\varphi(w)dw \\ &= C\int_{\mathbb{C}} e^{i\pi\mathrm{Re}\ (z\lambda)}F(\lambda, is, \delta, \xi)d\lambda \qquad (8.12)\end{aligned}$$

where $F(\lambda, is, \delta, \xi) = \lambda^{(is-\rho)/2+n-2,\delta/2}\int_{x_1^2+\cdots+x_n^2=1}[x,\xi]^{is-\rho,\delta}\varphi(\sqrt{\lambda}x)d\mu(x).$

Making use of Plancherel's theorem on $\mathrm{SO}(n,\mathbb{C})/\mathrm{SO}(n-1,\mathbb{C})$ see Theorem

8.21 in Section 8.3.1 and (8.12) it follows

$$\begin{aligned}\int_{\mathbb{C}^n} |\varphi(z)|^2 dz &= C\int_{\mathbb{C}}\int_{x_1^2+\cdots+x_n^2=1} |\varphi(\sqrt{\lambda}x)|^2|\lambda|^{n-2}d\mu(x)d\lambda \\ &= C'\int_{\mathbb{C}}\left(\sum_{\delta\in\mathbb{Z}}\int_{\mathbb{R}}\frac{1}{|c(\delta,is)|^2}||F(\lambda,is,\delta,\cdot)||^2ds\right)d\lambda \\ &= C'\sum_{\delta\in\mathbb{Z}}\int_{\mathbb{R}}\frac{1}{|c(\delta,is)|^2}\left(\int_{\mathbb{C}}||F(\lambda,is,\delta,\cdot)||^2d\lambda\right)ds \\ &= C''\sum_{\delta\in\mathbb{Z}}\int_{\mathbb{R}}\frac{1}{|c(\delta,is)|^2}||\mathcal{F}_{is,\delta}\varphi||^2ds\end{aligned}$$

where $c(\delta, is) = 2^{\rho-1}\dfrac{\Gamma(\frac{n}{2})\Gamma(\frac{1+\rho}{2})\Gamma(\frac{-is+|\delta|-\rho+2}{2})\Gamma(\frac{-is+|\delta|}{2})}{\sqrt{\pi}\Gamma(\frac{-is+|\delta|-\rho+n}{2})\Gamma(\frac{-is+|\delta|+\rho}{2})}$ and C, C', C'' are constants.

The above result also gives the decomposition of $L^2(\mathbb{C}^n)$ into a minimal invariant subspaces. The decomposition is multiplicity free (see next chapter) and the irreducible unitary representations which occur are given by $\pi_{is,\delta}\otimes\rho_{is,\delta}$. In conclusion, we have

Corollary 8.30. *Let φ be a function in $\mathcal{S}(\mathbb{C}^n)$. We have the following Plancherel formula*

$$||\varphi||_2^2 = D\sum_{\delta\in\mathbb{Z}}\int_0^\infty \frac{1}{|c(\delta,is)|^2}||\mathcal{F}_{is,\delta}\varphi||^2ds$$

with D a constant.

As representations of $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SO}(n,\mathbb{C})$, *we have*

$$L^2(\mathbb{C}^n) \cong \sum_{\delta\in\mathbb{Z}}\int_0^\infty \pi_{is,\delta}\otimes\rho_{is,\delta}ds.$$

CHAPTER 9

Invariant Hilbert subspaces of the oscillator representation of $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SO}(n, \mathbb{C})$

In this chapter we study the oscillator representation ω_n for the groups $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{O}(1, \mathbb{C})$, $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{O}(2, \mathbb{C})$ and $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SO}(n, \mathbb{C})$ with $n \geq 3$ in the context of the theory invariant Hilbert subspaces. Our main result is that any $\omega_n(G)$-stable Hilbert subspace of $\mathcal{S}'(\mathbb{C}^n)$ decomposes multiplicity free. For this purpose, we first need to prove, in case $n \geq 3$, that $(\mathrm{SO}(n, \mathbb{C}), \mathrm{SO}(n-1, \mathbb{C}))$ are generalized Gelfand pairs and to compute the MN-invariant distributions on the cone.

We do not know if the result holds for the cases $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SO}(1, \mathbb{C})$ and $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SO}(2, \mathbb{C})$.

9.1 The cases $n = 1$ and $n = 2$

If G is equal to the group $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{O}(1, \mathbb{C})$ then it is easy to prove that every invariant Hilbert subspace of $\mathcal{S}'(\mathbb{C})$ decomposes multiplicity free. We leave this to the reader to check by similar techniques as applied in this section. We are going to prove here the case $n = 2$.

9.1.1 The definition of the oscillator representation

Let G be the group $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{O}(2, \mathbb{C})$ and let ω_2 be the unitary representation of G on $H = L^2(\mathbb{C}^2)$ defined by:

$$\begin{aligned}
\omega_2(g)f(z) &= f(g^{-1} \cdot z), \qquad g \in \mathrm{O}(2, \mathbb{C}) \\
\omega_2(g(a))f(z) &= |a|^2 f(az), \qquad g(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad a \in \mathbb{C}^* \\
\omega_2(t(b))f(z) &= e^{-i\pi \mathrm{Re}\ (b[z,z])} f(z), \qquad t(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad b \in \mathbb{C}
\end{aligned}$$

$$\omega_2(\sigma)f(z) \;=\; \int_{\mathbb{C}^2} e^{2\pi i \mathrm{Re}\,([z,w])} f(w)dw, \qquad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

where $[z, w] = z_1w_1 + z_2w_2$ if $z = (z_1, z_2)$, $w = (w_1, w_2)$. The Fourier transform is defined by

$$\hat{f}(z) = \int_{\mathbb{C}^2} e^{-2\pi i \mathrm{Re}\,([z,w])} f(w)dw.$$

We call ω_2 the oscillator representation of G. More precisely, it is the natural extension of the restriction of the metaplectic representation of the group $Sp(2, \mathbb{C}) \subset Sp(4, \mathbb{R})$ to $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SO}(2, \mathbb{C})$.

Let $\mathcal{S}(\mathbb{C}^2)$ be the space of Schwartz functions on $\mathbb{C}^2$. Note that $\mathcal{S}(\mathbb{C}^2)$ is stable under the action of ω_2, so is the space $\mathcal{S}'(\mathbb{C}^2)$ of tempered distributions on $\mathbb{C}^2$.

9.1.2 Invariant Hilbert subspaces of $\mathcal{S}'(\mathbb{C}^2)$

The definition of the Fourier transform of a function on $\mathcal{S}(\mathbb{C}^2)$ naturally gives rise to a Fourier transform on $\mathcal{S}'(\mathbb{C}^2)$. We denote $\hat{T}$ this Fourier transform of a tempered distribution $T \in \mathcal{S}'(\mathbb{C}^2)$.

Let H be a Hilbert subspace of $\mathcal{S}'(\mathbb{C}^2)$, invariant under $\omega_2(G)$, and let K be its reproducing kernel. By Schwartz's kernel theorem we can associate to it a unique tempered distribution T on $\mathbb{C}^2 \times \mathbb{C}^2$. This distribution satisfies the following conditions:
1. T is a positive-definite kernel, in particular a Hermitian kernel, i.e.
2. $T(x, y) = \bar{T}(y, x)$.
3. T is $\omega_2(G)$-invariant : $(\omega_2(g) \times \omega_2(g))\, T = T$ for all $g \in G$. The latter property implies, in more detail,

3a. $T(g \cdot x, g \cdot y) = T(x, y)$ for all $g \in \mathrm{O}(2, \mathbb{C})$.

3b. If $a \in \mathbb{C}^*$, then

$$T(ax,\, ay) = |a|^{-4} T(x, y).$$

3c. If $b \in \mathbb{C}$, then

$$e^{-i\pi \mathrm{Re}\,(b([x,x]-[y,y]))} T(x, y) = T(x, y).$$

3d. $\hat{T}(x, y) = T(x, y)$.

As preparation for our main result, the multiplicty free decomposition of the oscillator representation, we shall show that any of the above distributions T is symmetric: $T(x, y) = T(y, x)$. We shall do this in several steps. Observe that $T(y, x)$ satisfies the same conditions as $T(x, y)$.

Step I

By condition 3c, we obtain

$$\mathrm{Supp}\, T \subset \Xi_{2,2} = \{(x, y) \in \mathbb{C}^2 \times \mathbb{C}^2 \,:\, [x, x] - [y, y] = 0\}.$$

Step II

Let $\Xi'_{2,2} = \Xi_{2,2} \backslash (0,0)$, being an isotropic cone in $\mathbb{C}^2 \times \mathbb{C}^2$. In a neighborhood of $\Xi'_{2,2}$ in $\mathbb{C}^2 \times \mathbb{C}^2$ we take as coordinates $s = [x,x] - [y,y]$ and $\omega \in \Xi'_{2,2}$. So we can write locally there

$$T = \sum_{i \in I} S_i(\omega) \otimes \delta^{(i)}(s)$$

where I is some finite subset of $\mathbb{N}$ and the S_i are distributions on $\Xi'_{2,2}$. Applying condition 3c again, we get that only the term with $i = 0$ survives, so

$$T(x,y) = \delta([x,x] - [y,y])\, S_0(\omega)$$

outside $(x,y) = (0,0)$. It remains to study the distribution S_0 on $\Xi'_{2,2}$.

Step III

On the open subset of $\Xi'_{2,2}$ given by $[x,x] \neq 0$, we get by properties 3a and 3b,

$$S_0(x,y) = \sigma_0(\omega_1,\, \omega_2)$$

where σ_0 is an $\mathrm{O}(2,\mathbb{C})$-invariant distribution on $\mathcal{X} \times \mathcal{X}$, with $\mathcal{X} = \{x \in \mathbb{C}^2 \,|\, [x,x] = 1\}$. The distribution σ_0 is symmetric: $\sigma_0(\omega_1,\, \omega_2) = \sigma_0(\omega_2,\, \omega_1)$. We are going to prove this. Observe that $\mathcal{X} \simeq \mathrm{SO}(2,\mathbb{C})$. Let $g_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $g = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, with $a^2 + b^2 = 1$, then

$$g_0 \cdot (g \cdot x^0) = g^{-1} \cdot x^0$$

with $x^0 = (1,0)$.

So

$$\begin{aligned} \sigma_0(\omega_1,\, \omega_2) &= \sigma_0(g \cdot x^0,\, h \cdot x^0) = \sigma_0(g_0 \cdot (g \cdot x^0),\, g_0 \cdot (h \cdot x^0)) \\ &= \sigma_0(h \cdot x^0,\, g \cdot x^0) = \sigma_0(\omega_2,\, \omega_1) \end{aligned}$$

with $g,\, h \in \mathrm{SO}(2,\mathbb{C})$.

Hence

$$T(x,y) = T(y,x)$$

on the open subset of $\mathbb{C}^2 \times \mathbb{C}^2$ defined by $[x,x] \neq 0$ (or, what is the same, $[y,y] \neq 0$).

So $S(x,y) = T(x,y) - T(y,x)$ has support contained in $[x,x] = [y,y] = 0$.

Step IV

Set $\Xi_2 = \{x \in \mathbb{C}^2 \,:\, [x,x] = 0\}$. The distribution S has support in $\Xi_2 \times \Xi_2$. With the coordinates $s_1 = [x,x]$, $s_2 = [y,y]$ near $\Xi_2 \times \Xi_2$, which can be taken, provided $x \neq 0$ and $y \neq 0$, so on $\Xi'_2 \times \Xi'_2$ (in obvious notation), we get

$$S(x,y) = \delta([x,x],[y,y])\, U(\xi_1,\, \xi_2) \quad (\xi_1,\, \xi_2 \in \Xi'_2)$$

where U is an $\mathrm{O}(2,\mathbb{C})$-invariant distribution on $\Xi'_2 \times \Xi'_2$, homogeneous of degree 4. Since $S(x,y) = -S(y,x)$ it easily follows that $U(\xi_1,\, \xi_2) = -U(\xi_2,\, \xi_1)$.

Step V

The cone $\Xi'_2 = \{x \in \mathbb{C}^2 \,:\, x_1^2 + x_2^2 = 0, x \neq 0\}$ consists of two disjoint pieces: $\mathbb{C}^* \xi^0$ and $\mathbb{C}^* \overline{\xi^0}$ with $\xi^0 = (1,i)$.

So the $\mathrm{O}(2,\mathbb{C})$-invariant distribution U on $\Xi_2' \times \Xi_2'$ has 4 components:

$$U(\lambda\xi^0, \mu\xi^0),\ U(\lambda\overline{\xi^0}, \mu\overline{\xi^0}),\ U(\lambda\xi^0, \mu\overline{\xi^0}),\ U(\lambda\overline{\xi^0}, \mu\xi^0)$$

with λ, $\mu \in \mathbb{C}^*$.

By homogeneity and $\mathrm{O}(2,\mathbb{C})$-invariance, the first two components are zero, i.e.

$$U(t\lambda\xi^0, t\mu\xi^0) = |t|^4 U(\lambda\xi^0, \mu\xi^0) = U(\lambda\xi^0, \mu\xi^0)$$

for all $t \neq 0$.

Consider the third component. By $\mathrm{O}(2,\mathbb{C})$-invariance we have

$$U(t\lambda\xi^0, t^{-1}\mu\overline{\xi^0}) = U(\lambda\xi^0, \mu\overline{\xi^0})$$

for all $t \neq 0$. By homogeneity,

$$U(t\lambda\xi^0, t^{-1}\mu\overline{\xi^0}) = |t|^4 U(\lambda\xi^0, t^{-2}\mu\overline{\xi^0}) = U(\lambda\xi^0, \mu\overline{\xi^0})$$

for all $t \neq 0$. So $U(\lambda\xi^0, t\mu\overline{\xi^0}) = |t|^2 U(\lambda\xi^0, \mu\overline{\xi^0})$. Hence $U(\lambda\xi^0, \mu\overline{\xi^0}) = d\mu \otimes V(\lambda)$.

Similarly we get $U(t^2\lambda\xi^0, \mu\overline{\xi^0}) = |t|^4 U(\lambda\xi^0, \mu\overline{\xi^0})$. Hence $V(t\lambda) = |t|^2 V(\lambda)$, so $V(\lambda) = d\lambda$.

By consequence,

$$U(\lambda\xi^0, \mu\overline{\xi^0}) = U(\mu\xi^0, \lambda\overline{\xi^0}).$$

If we apply $g_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{O}(2,\mathbb{C})$, we get

$$U(\lambda\xi^0, \mu\overline{\xi^0}) = U(\mu\overline{\xi^0}, \lambda\xi^0).$$

We have seen before that the distribution U also satisfies

$$U(\xi_1,\, \xi_2) = -U(\xi_2,\, \xi_1).$$

Then we obtain that also the third component is zero. We can do the same for the fourth component.

Then we obtain that the $\mathrm{O}(2,\mathbb{C})$-invariant distribution U on $\Xi_2' \times \Xi_2'$ is equal to zero. Hence $S = 0$ in a neighborhood of $\Xi_2' \times \Xi_2'$, and therefore $\operatorname{Supp} S \subset \{(0) \times \Xi_2\} \cup \{\Xi_2 \times (0)\}$.

Step VII

From property 3c we conclude that

$$([x,x] - [y,y])^k\, S = 0$$

for $k = 1, 2, \ldots$, hence, combining it with property 3d, we have

$$\Box^k\, S = 0$$

for $k = 1, 2, \ldots$, in particular $\Box\, S = 0$. Here $\Box$ denotes the d'Alembertian

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial y_2^2}.$$

If S has support in $\{(0) \times \Xi_2\} \cup \{\Xi_2 \times (0)\}$, and $\Box\, S = 0$, it easily follows, by recalling the local structure of such distributions, being basically a finite linear combination of tensor products of distributions supported by Ξ_2 and the origin, that $S = 0$ (see [28],Ch.III, §10). So finally we have shown that $T(x,y) = T(y,x)$.

9.1.3 Multiplicity free decomposition

The multiplicity free decomposition of the oscillator representation, i.e. the multiplicity free decomposition into irreducible invariant Hilbert subspaces of any $\omega_2(G)$-invariant Hilbert subspace of $\mathcal{S}'(\mathbb{C}^2)$, is now easily proved by applying Criterion 5.6 with $JT = \bar{T}$. If H is any invariant Hilbert subspace of $\mathcal{S}'(\mathbb{C}^2)$ with reproducing kernel $T \in \mathcal{S}'(\mathbb{C}^2 \times \mathbb{C}^2)$, then $\bar{T}$ is the kernel of the space $J\,H$ and the property $J\,H = H$ comes down to $\bar{T}(x,y) = T(x,y)$. Since T is positive-definite, this is equivalent with $T(x,y) = T(y,x)$ which immediately follows from the last section. So we have the following result.

Theorem 9.1. *Any $\omega_2(G)$-invariant Hilbert subspace of $\mathcal{S}'(\mathbb{C}^2)$ decomposes multiplicity free into irreducible invariant Hilbert subspaces of $\mathcal{S}'(\mathbb{C}^2)$.*

9.2 The case $n \geq 3$

9.2.1 Complex generalized Gelfand pairs

In this section we show that the pairs $(\mathrm{SO}(n,\mathbb{C}), \mathrm{SO}(n-1,\mathbb{C}))$ are generalized Gelfand pairs for $n \geq 2$.

9.2.1.1 Definition of generalized Gelfand pairs

We shall give a brief summary of the theory of generalized Gelfand pairs which is connected with the theory of invariant Hilbert subspaces that we have introduced in Chapter 5, for more details see [40]. Let G be a Lie group and H a closed subgroup of G. We shall assume both G and H to be unimodular. Denote by $\mathcal{D}(G)$, $\mathcal{D}(G/H)$ the space of $\mathcal{C}^\infty$-functions with compact support on G and G/H respectively, endowed with the usual topology. Let $\mathcal{D}'(G)$, $\mathcal{D}'(G/H)$ be the topological antidual of $\mathcal{D}(G)$ and $\mathcal{D}(G/H)$ respectively, provided with the strong topology.

A continuous unitary representation π of G on a Hilbert space $\mathcal{H}$ is said to be realizable on G/H if there is a continuous linear injection $j : \mathcal{H} \longrightarrow \mathcal{D}'(G/H)$ such that

$$j\pi(g) = L_g j$$

for all $g \in G$ (L_g denotes left translation by g). The space $j(\mathcal{H})$ is called an invariant Hilbert subspace of $\mathcal{D}'(G/H)$. We shall take all scalar products antilinear in the first and linear in the second factor.

Definition 9.2. The pair (G,H) is called a generalized Gelfand pair if for each continuous unitary representation π on a Hilbert space $\mathcal{H}$, which can be realized on G/H, the commutant of $\pi(G)$ in the algebra $\mathrm{End}(\mathcal{H})$ of all continuous linear operators of $\mathcal{H}$ into itself, is abelian.

For equivalent definitions we refer to [37] and [33]. A large class of examples is given by the Riemannian semisimple symmetric pairs and by the nilpotent symmetric pairs [37], [1].

A usefull criterion for determining generalized Gelfand pairs was given by Thomas ([33], Theorem E). We shall apply it throughout this section. Its proof is easy and straightforward (1.c.).

Denote by $\mathcal{D}'(G,H)$ the space of right H-invariant distributions on G provided with the relative topology of $\mathcal{D}'(G)$. It is well-known that $\mathcal{D}'(G,H)$ can be identified with $\mathcal{D}'(G/H)$.

Criterion 9.3. *Let $J : \mathcal{D}'(G,H) \longrightarrow \mathcal{D}'(G,H)$ be an anti-automorphism. If $J\mathcal{H} = \mathcal{H}$ (i.e. $(J|_{\mathcal{H}})$ anti-unitary) for all G-invariant or minimal G-invariant Hilbert subspaces of $\mathcal{D}'(G,H)$, then (G,H) is a generalized Gelfand pair.*

We shall apply it in the following form.

Criterion 9.4. *Let τ be an involutive automorphism of G which leaves H stable. Define $JT = \bar{T}^{\tau}$ for all $T \in \mathcal{D}'(G,H)$. If $JT = T$ for all bi-H-invariant positive-definite (or extremal positive-definite) distributions on G, then (G,H) is a generalized Gelfand pair.*

Remark 9.5. *T^{τ} is defined by $< T^{\tau}, f > = < T, f^{\tau} >$ ($f \in \mathcal{D}(G)$) and $f^{\tau}(g) = f(\tau(g))$ ($g \in G$). $\bar{T}$ is defined by $< \bar{T}, f > = \overline{< T, \bar{f} >}$ ($f \in \mathcal{D}(G)$).*

An important consequence of being a generalized Gelfand pair is the multiplicity -free desintegration of the left regular representation of G on $L^2(G/H)$. So one could, more or less without ambiguity, call this the Plancherel formula for G/H. If a fixed parametrization is used for the set of irreducible unitary representations realized on G/H, there is no ambiguity at all.

Let Z denote the algebra of all analytic differential operators on G which commute with left and right translations by elements of G. Any bi-H-invariant common eigendistribution of all elements of Z is called a spherical distribution. It is a well-known consequence of Schur's Lemma that any bi-H-invariant extremal positive-definite distribution on G is spherical. Spherical distributions play an important role in the harmonic analysis on G/H.

9.2.1.2 Morse's lemma

Let X be a complex analytic manifold of dimension n ($n \in \mathbb{N}$), and $f : X \longrightarrow \mathbb{C}$ an analytic function on X. The tangent space of X at a point x^0 will be denoted by TX_{x^0}. A point $x^0 \in X$ is called a critical point of f if the induced map $f_* : TX_{x^0} \longrightarrow T\mathbb{C}_{f(x^0)}$ is zero. If we choose a local coordinate system $(x_1, \dots, x_n)$ in a neighborhood U of x^0 this means that

$$\frac{\partial f}{\partial x_1}(x^0) = \cdots = \frac{\partial f}{\partial x_n}(x^0) = 0.$$

A critical point x^0 is called non-degenerate if and only if the matrix

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x^0) \right)$$

is non-singular.

Theorem 9.6. *(Morse's lemma) Let $f : X \longrightarrow \mathbb{C}$ be an analytic function from a complex manifold X into $\mathbb{C}$ and let x^0 be a non-degenerate critical point of f. There are local coordinates $(x_1, \dots, x_n)$ at x^0 with $(0, \dots, 0)$ corresponding to x^0 such that f can be written as*

$$f(x_1, \dots, x_n) = f(x^0) + x_1^2 + \cdots + x_n^2.$$

The proof of the theorem is similar to the proof of Morse's lemma in [15] p.146.

9.2.1.3 The pairs $(\mathbf{SO}(n, \mathbb{C}), \mathbf{SO}(n-1, \mathbb{C}))$

Assume $n \geq 3$.

Let G and H be the groups $\mathrm{SO}(n, \mathbb{C})$ and $\mathrm{SO}(n-1, \mathbb{C})$ respectively. The space $X = G/H$ can be clearly identified with the set of all points $x = (x_1, \dots, x_n)$ in $\mathbb{C}^n$ satisfying $x_1^2 + \cdots + x_n^2 = 1$.

We consider the following function Q on the space X which parametrizes the H-orbits on X:

$$Q(x) = x_1.$$

Q is an H-invariant complex analytic function on X with $Q(x^0) = 1$ where $x^0 = (1, 0, \dots, 0)$.

Define $X(z) = \{x \in X : Q(x) = z\}$ for $z \in \mathbb{C}$. Now the H-orbit structure on X is as follows:

Lemma 9.7. *a) Let $z \in \mathbb{C}$, $z \neq 1, -1$. Then $X(z)$ is a H-orbit.*

b) $X(1)$ consits of two H-orbits: $\{x^0\}$ and $\Gamma_1 = \{x^0 + v \in X : v \in \{x^0\}^\perp \setminus 0\}$.

c) $X(-1)$ consits of two H-orbits: $\{-x^0\}$ and $\Gamma_{-1} = \{-x^0 + v \in X : v \in \{x^0\}^\perp \setminus 0\}$.

In order to treat the sets $X(1)$ and $X(-1)$ separately, we choose open H-invariant sets X_{-1} and X_1 such that $X(-1) \subset X_{-1}$, $X(1) \not\subset X_{-1}$, $X(1) \subset X_1$, $X(-1) \not\subset X_1$ and $X_{-1} \cup X_1 = X$. These sets clearly exist.

The critical points of Q are x^0 and $-x^0$. Both critical points are non-degenerate.

We examine Q in the neighborhood of a critical point. Firstly, near x^0 there exists a coordinate system $\{w_1, \dots, w_{n-1}\}$ such that

$$Q(w_1, \dots, w_{n-1}) = 1 + w_1^2 + \cdots + w_{n-1}^2,$$

x^0 corresponding to $(0, \dots, 0)$. Secondly near $-x^0$ there exists a coordinate system $\{w_1, \dots, w_{n-1}\}$ such that

$$Q(w_1, \dots, w_{n-1}) = -1 + w_1^2 + \cdots + w_{n-1}^2,$$

$-x^0$ corresponding to $(0, \dots, 0)$. This is due to Morse's lemma.

From the properties of Q, we deduce applying [26] the existence of a linear map $\mathcal{M}$, which assigns to every $f \in \mathcal{D}(X)$ a function $\mathcal{M}f$ on $\mathbb{C}$ such that

$$\int_X F(Q(x))f(x)dx = \int_{\mathbb{C}} F(z)\mathcal{M}f(z)dz$$

for all $F \in \mathcal{D}(\mathbb{C})$. Here dx is an invariant measure on X, $dz = dxdy$ $(z = x + iy)$. $\mathcal{M}f(z)$ gives the mean of f over the set $X(z)$. Let $\mathcal{H} = \mathcal{M}(\mathcal{D}(X))$ and $\mathcal{H}_i = \mathcal{M}(\mathcal{D}(X_i))$ $(i = -1, 1)$. Using the nature of the critical points of Q and the results of [26], §6 we get:

$$\begin{aligned} \mathcal{H} &= \{\phi + \eta_0\psi_0 + \eta_1\psi_1 \,:\, \phi, \psi_0, \psi_1 \in \mathcal{D}(\mathbb{C})\} \\ \mathcal{H}_{-1} &= \{\phi_0 + \eta_0\psi_0 \,:\, \phi_0, \psi_0 \in \mathcal{D}(Q(X_{-1})\} \\ \mathcal{H}_1 &= \{\phi_1 + \eta_1\psi_1 \,:\, \phi_1, \psi_1 \in \mathcal{D}(Q(X_1)\}, \end{aligned}$$

where

$$\eta_0(z) = \begin{cases} |z+1|^{n-2} & \text{if } n \text{ is even} \\ |z+1|^{n-2}\mathrm{Log}|z+1| & \text{if } n \text{ is odd} \end{cases}$$

and

$$\eta_1(z) = \begin{cases} |z-1|^{n-2} & \text{if } n \text{ is even} \\ |z-1|^{n-2}\mathrm{Log}|z-1| & \text{if } n \text{ is odd} \end{cases} .$$

If we topologize $\mathcal{H}$, $\mathcal{H}_{-1}$ and $\mathcal{H}_1$ as in [26] we have for $i = -1, 1$:

a) $\mathcal{M} : \mathcal{D}(X_i) \longrightarrow \mathcal{H}_i$ is continuous.

b) The image of the transpose map $\mathcal{M}' : \mathcal{H}_i' \longrightarrow \mathcal{D}'(X_i)$ is the space of H-invariant distributions on X_i. $\mathcal{M}'$ is injective on $\mathcal{H}_i'$, because $\mathcal{M}$ is surjective.

Similar properties hold for $\mathcal{M} : \mathcal{D}(X) \longrightarrow \mathcal{H}$.

So, given an H-invariant distribution on X, there exists an element $S \in \mathcal{H}'$ such that

$$< T, \phi >=< S, \mathcal{M}\phi > \tag{9.1}$$

for all $\phi \in \mathcal{D}(X)$. Fix Haar measures dg on G and dh on H in such a way that $dg = dxdh$, symbolically. For $f \in \mathcal{D}(G)$ put

$$f^\sharp(x) = \int_H f(gh)dh \qquad (x = gH).$$

Given a bi-H-invariant distribution T_0 on G, there is an unique H-invariant distribution T on X satisfying $< T_0, f >=< T, f^\sharp >$ $(f \in \mathcal{D}(G))$. This is a well-known fact.

We are now prepared to prove that (G,H) is a generalized Gelfand pair. We apply Criterion 9.4 with $JT = \bar{T}$ $(T \in \mathcal{D}'(G, H))$. We have to show that $\bar{T} = T$ for all bi-H-invariant positive-definite distributions T on G. Since $\bar{T} = \check{T}$ for such T, we shall show the following: for any bi-H-invariant distribution T on G one has $T = \check{T}$. Here $< \check{T}, f >=< T, \check{f} >$, $\check{f}(g) = f(g^{-1})$ $(g \in G,\ f \in \mathcal{D}(G))$. In view of the relation between bi-H-invariant distributions on G and H-invariant distributions on X, and because of (9.1), this amounts to the relation

$$\mathcal{M}[(\check{f})^\sharp] = \mathcal{M}(f^\sharp)$$

for all $f \in \mathcal{D}(G)$. For all $F \in \mathcal{D}(\mathbb{C})$ one has

$$\begin{aligned}\int_{\mathbb{C}} F(z)\mathcal{M}[(\check{f})^\sharp](z)dz &= \int_X F(Q(x))(\check{f})^\sharp(x)dx \\ &= \int_G F(Q(g))\check{f}(g)dg \\ &= \int_G F(Q(g^{-1}))f(g)dg.\end{aligned}$$

Since $Q(g) = Q(g^{-1})$ $(g \in G)$ we get the result.

So we have shown:

Theorem 9.8. *The pairs* $(\mathrm{SO}(n,\mathbb{C}), \mathrm{SO}(n-1,\mathbb{C}))$ *are generalized Gelfand pairs for* $n \geq 3$.

The case $n = 2$ is easily seen to provide a generalized Gelfand pair too, since $\mathrm{SO}(2,\mathbb{C})$ is an abelian group.

9.2.2 MN-invariant distributions on the cone

9.2.2.1 The symmetric spaces $\mathrm{SO}(n,\mathbb{C})/\mathrm{SO}(n-1,\mathbb{C})$ for $n \geq 3$

We repeat here some facts from Section 8.3.1.

Let G be the group $\mathrm{SO}(n,\mathbb{C})$. G acts on $\mathbb{C}^n$ in the usual manner. Let $[,]$ be the G-invariant bilinear form on $\mathbb{C}^n$, given by,

$$[z, z'] = z_1z'_1 + \cdots + z_nz'_n$$

if $z = (z_1, \ldots, z_n)$, $z' = (z'_1, \ldots, z'_n)$. The (algebraic) manifold $[z, z] = z_1^2 + \cdots + z_n^2 = 1$, called X, is invariant under this action, and G acts transitively on X. Let x^0 be the vector $(1, 0, \ldots, 0)$ in X. The stabilizer of x^0 in G is equal to $H = \mathrm{SO}(n-1,\mathbb{C})$. The space X is a complex manifold, homeomorphic to G/H. Set

$$J = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

J is not in G, but $-J$ is as soon as n is odd. Define the involutive automorphism σ of G by $\sigma(g) = JgJ$. Then H is of index 2 in the stabilizer H_σ of σ. So G/H is a complex symmetric space. Observe that σ is an inner automorphism if n is odd. We will pass now to the Lie algebra $\mathfrak{g}$ of G. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ be the decomposition of $\mathfrak{g}$ into eigenspaces for the eigenvalues $+1$ and -1 for σ. Then we have:

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & Y & \\ 0 & & \end{pmatrix} : Y \text{complex, antisymmetric} \right\},$$

$$\mathfrak{q} = \left\{ \begin{pmatrix} 0 & z_2 & \cdots & z_n \\ -z_2 & & & \\ \vdots & & 0 & \\ -z_n & & & \end{pmatrix} : z_2, \ldots, z_n \in \mathbb{C} \right\}.$$

Write $L = \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{pmatrix} \in \mathfrak{q}$, actually in $\mathfrak{q} \cap \mathfrak{k}$ where $\mathfrak{k} = \mathfrak{so}(n, \mathbb{R})$, the real antisymmetric matrices. The Lie algebra $\mathfrak{k}$ belongs to $K = \mathrm{SO}(n, \mathbb{R})$. Clearly $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$. Call $\mathfrak{p} = i\mathfrak{k}$. Then $iL \in \mathfrak{q} \cap \mathfrak{p}$.

The Lie algebra $\mathfrak{a} = \mathbb{C}L$ is a maximal abelian subspace of $\mathfrak{q}$; $\mathfrak{a}$ is a Cartan subspace of $\mathfrak{q}$ with respect to σ. Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{h}$. Then

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & Y & \\ 0 & 0 & & \end{pmatrix} : Y \in \mathfrak{so}(n-2, \mathbb{C}) \right\}.$$

Let $\mathfrak{g}_\pm$ be the eigenspaces of $\mathrm{ad}(iL)$ for the eigenvalues ±1. Then

$$\mathfrak{g}_- = \left\{ X(p) = \begin{pmatrix} 0 & 0 & {}^tp \\ 0 & 0 & i{}^tp \\ -p & -ip & 0 \end{pmatrix} : p \in \mathbb{C}^{n-2} \right\}$$

and $\mathfrak{g}_- = \sigma(\mathfrak{g}_+)$. We have the following decomposition of $\mathfrak{g}$ into eigenspaces of $\mathrm{ad}(iL)$:

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{g}_+.$$

Set $\mathfrak{n} = \mathfrak{g}_-$. Then $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$. The connected subgroup A and N of G corresponding to the Lie subalgebras $\mathfrak{a}$ and $\mathfrak{n}$ are given by:

$$A = \left\{ a_z = \exp zL = \begin{pmatrix} \cos z & \sin z & 0 \\ -\sin z & \cos z & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} : z \in \mathbb{C} \right\}$$

and

$$N = \left\{ n(p) = \begin{pmatrix} 1 + \frac{1}{2}[p,p] & \frac{1}{2}i[p,p] & i{}^tp \\ \frac{1}{2}i[p,p] & 1 - \frac{1}{2}[p,p] & -{}^tp \\ -ip & p & I_{n-2} \end{pmatrix} : p \in \mathbb{C}^{n-2} \right\}.$$

9.2.2.2 The cone $\Xi = G/MN$

The centralizer M of $\mathfrak{a}$ in H consists of the elements of G of the form:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & h & \\ 0 & 0 & & & \end{pmatrix}, \text{ with } h \in \mathrm{SO}(n-2, \mathbb{C}).$$

Furthermore $Z_G(\mathfrak{a}) = MA$, $M \cap A = \{e\}$. Put $\Xi = \{z \in \mathbb{C}^n : [z, z] = 0, z \neq 0\}$, the isotropic cone. The group G acts transitively on Ξ and $\mathrm{Stab}(\xi^0) = MN$, so Ξ can be identified with G/MN. Let ξ^0 be the vector $(1, i, 0, \ldots, 0)$. The stabilizer of $\left\{\lambda\xi^0 : \lambda \in \mathbb{C}\right\}$ is equal to $P = MAN$, a parabolic subgroup of G.

The group MN is unimodular, so Ξ has a G-invariant measure which we shall denote by $d\xi$. Let u be the map from Ξ to $\mathbb{C}$ defined by $u(\xi) = [\xi, \xi^0] = \xi_1 + i\xi_2$, u is MN-invariant and holomorphic, since Ξ is a complex subvariety of $\mathbb{C}^n$.

Proposition 9.9. *The MN-orbits on Ξ are: $\Xi_{0,\lambda} = \{\lambda\xi^0\}$ with $\lambda \in \mathbb{C}^*$, $\Xi_t = \{\xi \in \Xi : u(\xi) = t\}$ for $t \in \mathbb{C}^*$ and $\Xi_{0,0} = \{\xi : u(\xi) = 0, \xi \notin \mathbb{C}\xi^0\}$ if $n \geq 3$.*

The open set $\Xi^* = \bigcup_{t \neq 0} \Xi_t = \{\xi : \ u(\xi) \neq 0\}$ is dense in Ξ. The open set of regular points for u is given by $\Xi' = \Xi^* \cup \Xi_{0,0}$. So the singular points are the points of $\Xi_0 = \bigcup_{\lambda \neq 0} \Xi_{0,\lambda}$.

9.2.2.3 MN-invariant distributions on Ξ

Denote by $\mathcal{D}'_{MN}(\Xi)$ the space of distributions on Ξ, invariant with respect to MN. Recall that for $f \in \mathcal{D}(\Xi)$ we can define the function $\mathcal{M}f$ on $\mathbb{C}^*$ by :

$$\mathcal{M}f(t) = \int f(\xi)\delta(u(\xi) - t).$$

With the help of the results of Rallis-Schiffmann we want to describe the image of $\mathcal{M}$. For this we introduce the following space.

Definition 9.10. Let $\mathcal{J}$ be the space of functions from $\mathbb{C}^*$ to $\mathbb{C}$ such that

$$\varphi(t) = \varphi_1(t) + \eta(t)\varphi_2(t)$$

with φ_1, $\varphi_2 \in \mathcal{D}(\mathbb{C})$ and

$$\eta(t) = \begin{cases} |t|^{n-4} & \text{if } n \text{ is odd} \\ |t|^{n-4}\mathrm{Log}|t| & \text{if } n \text{ is even.} \end{cases}$$

If we topologize $\mathcal{J}$ as in [26]. $\mathcal{M}$ is a map from $\mathcal{D}(\Xi)$ to a space $\mathcal{J}$, which is continuous and surjective. If S is a continuous linear form on $\mathcal{J}$, the distribution T defined by

$$T(f) = S(\mathcal{M}f) \quad (f \in \mathcal{D}(\Xi)),$$

so $T = \mathcal{M}'S$, is invariant under MN. We shall see later on that $\mathcal{M}'$ is not surjective.

Proposition 9.11. *Let T be a distribution on Ξ invariant under MN then*

$$T = \mathcal{M}'S + T_1$$

where S is a continuous linear form on $\mathcal{J}$ and T_1 is a MN-invariant distribution on Ξ with Supp $T_1 \subset \Xi_0$.

Let us assume $n \geq 3$ (as usual). Then clearly, if the support of $f \in \mathcal{D}(\Xi)$ is contained in Ξ', the complement of Ξ_0, we have $\mathcal{M}f \in \mathcal{D}(\mathbb{C})$. The restriction $\mathcal{M}_0$ of $\mathcal{M}$ to $\mathcal{D}(\Xi')$ has a image $\mathcal{D}(\mathbb{C})$, and the transposed map $\mathcal{M}'_0 : \mathcal{D}'(\mathbb{C}) \longrightarrow \mathcal{D}'(\Xi')$ has as image $\mathcal{D}'_{MN}(\Xi')$. Let T be a distribution on Ξ, invariant under MN. By the above, the restriction T' to Ξ' is of the form $T' = \mathcal{M}'_0 S_0$, where S_0 is a distribution on $\mathbb{C}$. But every distribution on $\mathbb{C}$ can be extended to a continuous linear form on $\mathcal{J}$. Let S be such an extension of S_0, and consider the distribution T_1 defined by

$$T_1 = T - \mathcal{M}'S.$$

Since the distributions T and $\mathcal{M}'S$ coincide on Ξ', the support of T_1 is contained in Ξ_0, so we get

$$T = \mathcal{M}'S + T_1.$$

We are now going to study the structure of the MN-invariant distributions on Ξ with support in Ξ_0.

9.2.2.4 Singular MN-invariant distributions on Ξ

We use local coordinates in a neighborhood of Ξ_0 provided by the map:

$$\begin{array}{ccc} A \times \overline{N} & \longrightarrow & \Xi \\ (a, \overline{n}) & \longmapsto & a\overline{n}\xi^0 \end{array}$$

We have:

$$a_u = \begin{pmatrix} \frac{1}{2}(u + \frac{1}{u}) & \frac{1}{2i}(u - \frac{1}{u}) & 0 \\ -\frac{1}{2i}(u - \frac{1}{u}) & \frac{1}{2}(u + \frac{1}{u}) & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}, \quad u \in \mathbb{C}^*$$

(if $u = e^{iz}$, then we get the usual expression for a_z)

$$\overline{n}_\alpha = \begin{pmatrix} 1 + \frac{1}{2}[\alpha, \alpha] & -\frac{i}{2}[\alpha, \alpha] & -i{}^t\alpha \\ -\frac{i}{2}[\alpha, \alpha] & 1 - \frac{1}{2}[\alpha, \alpha] & -{}^t\alpha \\ i\alpha & \alpha & I_{n-2} \end{pmatrix}, \quad \alpha \in \mathbb{C}^{n-2}.$$

So we have:

$$a_u\overline{n}_\alpha\xi^0 = \begin{pmatrix} u + \frac{1}{u}[\alpha, \alpha] \\ \left(u - \frac{1}{u}[\alpha, \alpha]\right) i \\ 2i\alpha \end{pmatrix}.$$

We denote by Δ the differential operator which is given by (in these coordinates):

$$\Delta = \frac{\partial^2}{\partial\alpha_3^2} + \cdots + \frac{\partial^2}{\partial\alpha_n^2}.$$

Let T be a distribution on Ξ which is MN-invariant and whose support is contained in Ξ_0. Consider a function f in $\mathcal{D}(\Xi)$ which can be written, on the above local coordinates, as

$$f(a_u\overline{n}_\alpha\xi^0) = \Phi(u, \overline{u})\Psi(\alpha, \overline{\alpha})$$

where $\Phi \in \mathcal{D}(\mathbb{C}^*)$ and $\Psi \in \mathcal{D}(\mathbb{C}^{n-2})$, $f = \Phi \otimes \Psi$.

For Φ fixed, the map

$$\Psi \longrightarrow T(\Phi \otimes \Psi)$$

is a distribution on $\mathbb{C}^{n-2}$, supported by the origin and $\mathrm{SO}(n-2, \mathbb{C})$-invariant.

Let us now assume that $n > 3$. (The case $n = 3$ has to be treated separately since in this case $M = 1$). We may then conclude that there are constants $C_{k,l}$ such that

$$T(\Phi \otimes \Psi) = \sum_{k=0}^{m} \sum_{l=0}^{k} C_{k,l} \Delta^{k-l} \overline{\Delta}^{l} \Psi(0)$$

where the constants depend on Φ: $C_{k,l} = T_{k,l}(\Phi)$, and $T_{k,l}$ are distributions on $\mathbb{C}^*$:

$$T(\Phi \otimes \Psi) = \sum_{k=0}^{m} \sum_{l=0}^{k} T_{k,l}(\Phi) \Delta^{k-l} \overline{\Delta}^{l} \Psi(0),$$

so we have:

$$T(u, \alpha) = \sum_{k=0}^{m} \sum_{l=0}^{k} T_{k,l}(u) \otimes \Delta^{k-l} \overline{\Delta}^{l} \delta(\alpha).$$

We are now going to use that T is also N-invariant.

So let

$$n(x) = \begin{pmatrix} 1 + \frac{1}{2}[x,x] & \frac{1}{2}i[x,x] & i\,{}^t x \\ \frac{1}{2}i[x,x] & 1 - \frac{1}{2}[x,x] & -{}^t x \\ -ix & x & I_{n-2} \end{pmatrix}, x \in \mathbb{C}^{n-2}.$$

For x small we get:

$$n_x a_u \overline{n}_\alpha \xi^0 = a_{u'} \overline{n}_{\alpha'} \xi^0.$$

A simple calculation gives:

$$\begin{aligned} \alpha' &= \alpha - \frac{x}{u}[\alpha, \alpha] \\ u' &= u - 2[\alpha, x] + \frac{1}{u}[\alpha, \alpha][x, x]. \end{aligned}$$

The distribution T is invariant under N if and only if

$$\left.\frac{\partial}{\partial x_j}\right|_{x=0} < T, f \circ n_x >= \left.\frac{\partial}{\partial \overline{x}_j}\right|_{x=0} < T, f \circ n_x >= 0$$

for $j = 3, 4, \ldots, n$.

Lemma 9.12. *Let f be a $\mathcal{C}^\infty$-function on Ξ. Set*

$$f(a_u \overline{n}_\alpha \xi^0, \overline{a_u \overline{n}_\alpha \xi^0}) = F(u, \overline{u}, \alpha, \overline{\alpha}).$$

We have:

$$\begin{aligned} \left.\frac{\partial}{\partial x_j} f(n_x \xi, \overline{n_x \xi})\right|_{x=0} &= -2\alpha_j \frac{\partial F}{\partial u}(u, \overline{u}, \alpha, \overline{\alpha}) - \frac{1}{u}[\alpha, \alpha] \frac{\partial F}{\partial \alpha_j}(u, \overline{u}, \alpha, \overline{\alpha}) \\ \left.\frac{\partial}{\partial \overline{x}_j} f(n_x \xi, \overline{n_x \xi})\right|_{x=0} &= -2\overline{\alpha}_j \frac{\partial F}{\partial \overline{u}}(u, \overline{u}, \alpha, \overline{\alpha}) - \frac{1}{\overline{u}} \overline{[\alpha, \alpha]} \frac{\partial F}{\partial \overline{\alpha}_j}(u, \overline{u}, \alpha, \overline{\alpha}) \end{aligned}$$

where $\xi = a_u \overline{n}_\alpha \xi^0$ *and* $3 \leq j \leq n$.

So, using this lemma, we get:

$$\begin{aligned} < T, -2\alpha_j \frac{\partial F}{\partial u} - \frac{1}{u}[\alpha,\alpha]\frac{\partial F}{\partial \alpha_j} > &= 0 \\ < T, -2\overline{\alpha}_j \frac{\partial F}{\partial \overline{u}} - \frac{1}{\overline{u}}\overline{[\alpha,\alpha]}\frac{\partial F}{\partial \overline{\alpha}_j} > &= 0 \end{aligned}$$

Suppose F is of the form $F(u,\overline{u},\alpha,\overline{\alpha}) = \Phi(u,\overline{u})\Psi(\alpha,\overline{\alpha})$. Then we get:

$$\sum_{k=0}^{m}\sum_{l=0}^{k}\Big(2 < T_{k,l}, \frac{\partial \Phi}{\partial u} > \Delta^{k-l}\overline{\Delta}^{l}(\alpha_j \Psi)(0) \\ + < T_{k,l}, \frac{\Phi}{u} > \Delta^{k-l}\overline{\Delta}^{l}\big([\alpha,\alpha]\frac{\partial \Psi}{\partial \alpha_j})(0)\Big) = 0$$

and

$$\sum_{k=0}^{m}\sum_{l=0}^{k}\Big(2 < T_{k,l}, \frac{\partial \Phi}{\partial \overline{u}} > \Delta^{k-l}\overline{\Delta}^{l}(\overline{\alpha}_j \Psi)(0) \\ + < T_{k,l}, \frac{\Phi}{\overline{u}} > \Delta^{k-l}\overline{\Delta}^{l}\big(\overline{[\alpha,\alpha]}\frac{\partial \Psi}{\partial \overline{\alpha}_j})(0)\Big) = 0.$$

Applying the relations:

$$\begin{aligned} \alpha_j \Delta^k \delta &= -2k \frac{\partial}{\partial \alpha_j}\Delta^{k-1}\delta \\ [\alpha,\alpha]\Delta^k \delta &= 2k(2k+n-4)\Delta^{k-1}\delta \end{aligned}$$

and similar relations for $\overline{\Delta}$, we conclude:

$$\sum_{k=0}^{m}\sum_{l=0}^{k}\Big(4(k-l) < T_{k,l}, \frac{\partial \Phi}{\partial u} > \frac{\partial}{\partial \alpha_j}\overline{\Delta}^{l}\Delta^{k-l-1}\Psi(0) \\ + 2(k-l)\big(2(k-l)+n-4\big) < T_{k,l}, \frac{\Phi}{u} > \frac{\partial}{\partial \alpha_j}\overline{\Delta}^{l}\Delta^{k-l-1}\Psi(0)\Big) = 0$$

and

$$\sum_{k=0}^{m}\sum_{l=0}^{k}\Big(4l < T_{k,l}, \frac{\partial \Phi}{\partial \overline{u}} > \frac{\partial}{\partial \overline{\alpha}_j}\overline{\Delta}^{l-1}\Delta^{k-l}\Psi(0) \\ + 2l(2l+n-4) < T_{k,l}, \frac{\Phi}{\overline{u}} > \frac{\partial}{\partial \overline{\alpha}_j}\overline{\Delta}^{l-1}\Delta^{k-l}\Psi(0)\Big) = 0.$$

Therefore, we obtain the following two equations

$$\frac{\partial T_{k,l}}{\partial u} = \left(k - l + \frac{n-4}{2}\right)\frac{T_{k,l}}{u} \quad \text{for} \quad k > l$$

and

$$\frac{\partial T_{k,l}}{\partial \overline{u}} = \left(l + \frac{n-4}{2}\right)\frac{T_{k,l}}{\overline{u}} \quad \text{for} \quad l \geq 1, \ \forall k.$$

So we have:

$$\begin{aligned} T_{k,l} &= A_{k,l} u^{\frac{n-4}{2}+k-l} \overline{u}^{\frac{n-4}{2}+l} \text{ if } k > l, \ l \geq 1 \\ &= S_{k,k}(u) \otimes \overline{u}^{\frac{n-4}{2}+k} \text{ if } k = l, \ l \geq 1 \\ &= u^{\frac{n-4}{2}+k} \otimes S_{k,0}(\overline{u}) \text{ if } l = 0, \ k > l \end{aligned}$$

where $S_{k,k}$, $S_{k,0}$ are distributions on $\mathbb{C}^*$ depending on u, $\overline{u}$ only respectively and $A_{k,l}$ are constants.

Now we consider the case $n = 3$. In this case $M = 1$ then we can not use the M-invariant as we did before.

Let T be a distribution on Ξ which is MN-invariant and whose support is contained in Ξ_0. Consider a function f in $\mathcal{D}(\Xi)$ which can be written as

$$f(a_u \overline{n}_\alpha \xi^0) = \Phi(u, \overline{u}) \Psi(\alpha, \overline{\alpha})$$

where $\Phi \in \mathcal{D}(\mathbb{C}^*)$ and $\Psi \in \mathcal{D}(\mathbb{C})$, $f = \Phi \otimes \Psi$.

For Φ fixed, the map

$$\Psi \longrightarrow T(\Phi \otimes \Psi)$$

is a distribution on $\mathbb{C}$, supported by the origin. We may then conclude that there are constants $C_{k,l}$ such that

$$T(\Phi \otimes \Psi) = \sum_{k=0}^{m} \sum_{l=0}^{k} (-1)^k C_{k,l} \frac{\partial^{k-l}}{\partial \alpha^{k-l}} \frac{\partial^l}{\partial \overline{\alpha}^l} \Psi(0)$$

where the constants depend on Φ: $C_{k,l} = T_{k,l}(\Phi)$, and $T_{k,l}$ are distributions on $\mathbb{C}^*$:

$$T(\Phi \otimes \Psi) = \sum_{k=0}^{m} \sum_{l=0}^{k} (-1)^k T_{k,l} \frac{\partial^{k-l}}{\partial \alpha^{k-l}} \frac{\partial^l}{\partial \overline{\alpha}^l} \Psi(0),$$

so we have:

$$T(u, \alpha) = \sum_{k=0}^{m} \sum_{l=0}^{k} T_{k,l}(u) \otimes \frac{\partial^{k-l}}{\partial \alpha^{k-l}} \frac{\partial^l}{\partial \overline{\alpha}^l} \delta(\alpha).$$

We are now going to use that T is also N-invariant. Using as before the last lemma, we get:

$$\begin{aligned} < T, -2\alpha \frac{\partial F}{\partial u} - \frac{1}{u}\alpha^2 \frac{\partial F}{\partial \alpha} > &= 0 \\ < T, -2\overline{\alpha} \frac{\partial F}{\partial \overline{u}} - \frac{1}{\overline{u}}\overline{\alpha}^2 \frac{\partial F}{\partial \overline{\alpha}} > &= 0 \end{aligned}$$

Suppose F is of the form $F(u,\overline{u},\alpha,\overline{\alpha}) = \Phi(u,\overline{u})\Psi(\alpha,\overline{\alpha})$. Then we get:

$$\sum_{k=0}^{m}\sum_{l=0}^{k}(-1)^k\Big(2 < T_{k,l},\frac{\partial\Phi}{\partial u} > \frac{\partial^{k-l}}{\partial\alpha^{k-l}}\frac{\partial^l}{\partial\overline{\alpha}^l}(\alpha\Psi)(0) \\ + < T_{k,l},\frac{\Phi}{u} > \frac{\partial^{k-l}}{\partial\alpha^{k-l}}\frac{\partial^l}{\partial\overline{\alpha}^l}(\alpha^2\frac{\partial\Psi}{\partial\alpha})(0)\Big) = 0$$

and

$$\sum_{k=0}^{m}\sum_{l=0}^{k}(-1)^k\Big(2 < T_{k,l},\frac{\partial\Phi}{\partial\overline{u}} > \frac{\partial^{k-l}}{\partial\alpha^{k-l}}\frac{\partial^l}{\partial\overline{\alpha}^l}(\overline{\alpha}\Psi)(0) \\ + < T_{k,l},\frac{\Phi}{\overline{u}} > \frac{\partial^{k-l}}{\partial\alpha^{k-l}}\frac{\partial^l}{\partial\overline{\alpha}^l}(\overline{\alpha}^2\frac{\partial\Psi}{\partial\overline{\alpha}})(0)\Big) = 0.$$

Applying the relations:

$$\alpha\frac{\partial^k}{\partial\alpha^k}\delta = -k\frac{\partial^{k-1}}{\partial\alpha^{k-1}}\delta \\ \alpha^2\frac{\partial^k}{\partial\alpha^k}\delta = k(k-1)\frac{\partial^{k-2}}{\partial\alpha^{k-2}}\delta$$

and similar relations for $\frac{\partial^k}{\partial\overline{\alpha}^k}$, we conclude:

$$\sum_{k=0}^{m}\sum_{l=0}^{k}(-1)^k\Big(2(k-l) < T_{k,l},\frac{\partial\Phi}{\partial u} > \frac{\partial^{k-l-1}}{\partial\alpha^{k-l-1}}\frac{\partial^l}{\partial\overline{\alpha}^l}\Psi(0) \\ + (k-l)(k-l-1) < T_{k,l},\frac{\Phi}{u} > \frac{\partial^{k-l-1}}{\partial\alpha^{k-l-1}}\frac{\partial^l}{\partial\overline{\alpha}^l}\Psi(0)\Big) = 0$$

and

$$\sum_{k=0}^{m}\sum_{l=0}^{k}(-1)^k\Big(2l < T_{k,l},\frac{\partial\Phi}{\partial\overline{u}} > \frac{\partial^{k-l}}{\partial\alpha^{k-l}}\frac{\partial^{l-1}}{\partial\overline{\alpha}^{l-1}}\Psi(0) \\ + l(l-1) < T_{k,l},\frac{\Phi}{\overline{u}} > \frac{\partial^{k-l}}{\partial\alpha^{k-l}}\frac{\partial^{l-1}}{\partial\overline{\alpha}^{l-1}}\Psi(0)\Big) = 0.$$

Therefore, we obtain the following two equations

$$\frac{\partial T_{k,l}}{\partial u} = \left(\frac{k-l-1}{2}\right)\frac{T_{k,l}}{u} \quad \text{for} \ \ k > l$$

and

$$\frac{\partial T_{k,l}}{\partial\overline{u}} = \left(\frac{l-1}{2}\right)\frac{T_{k,l}}{\overline{u}} \quad \text{for} \ \ l \geq 1, \ \forall k.$$

So we have:

$$\begin{aligned} T_{k,l} &= A_{k,l} u^{\frac{k-l-1}{2}} \overline{u}^{\frac{l-1}{2}} \quad \text{if } k > l,\ l \geq 1 \\ &= S_{k,k}(u) \otimes \overline{u}^{\frac{k-1}{2}} \quad \text{if } k = l,\ l \geq 1 \\ &= u^{\frac{k-1}{2}} \otimes S_{k,0}(\overline{u}) \quad \text{if } l = 0,\ k > l \end{aligned}$$

where $S_{k,k}$, $S_{k,0}$ are distributions on $\mathbb{C}^*$ depending on u, $\overline{u}$ only respectively and $A_{k,l}$ are constants.

Then we get the following theorem:

Theorem 9.13. *Let T be a MN-invariant distribution on Ξ with Supp $T \subset \Xi_0$. Then there exist distributions $T_{0,0} = T_{0,0}(u, \overline{u})$, $S_{k,k}(u, \overline{u}) = S_{k,k}(u)$, $S_{k,0}(u, \overline{u}) = S_{k,0}(\overline{u})$ on $\mathbb{C}^*$ and constants $A_{k,l}$ such that*

$$\begin{aligned} T &= T_{0,0} \otimes \delta + \sum_{k=1}^{m} u^{\frac{k-1}{2}} \otimes S_{k,0}(\overline{u}) \otimes \frac{\partial^k}{\partial \alpha^k}\delta + \sum_{k=1}^{m} S_{k,k}(u) \otimes \overline{u}^{\frac{k-1}{2}} \otimes \frac{\partial^k}{\partial \overline{\alpha}^k}\delta \\ &\quad + \sum_{k=1}^{m} \sum_{l=1}^{k-1} A_{k,l} |u|^{\frac{k}{2}+1} \left(\frac{u}{|u|}\right)^{\frac{k}{2}-l} \frac{du}{|u|^2} \otimes \frac{\partial^{k-l}}{\partial \alpha^{k-l}} \frac{\partial^l}{\partial \overline{\alpha}^l}\delta \quad \textit{for } n = 3 \end{aligned}$$

and

$$\begin{aligned} T &= T_{0,0} \otimes \delta + \sum_{k=1}^{m} u^{\frac{\rho-2}{2}+k} \otimes S_{k,0}(\overline{u}) \otimes \Delta^k \delta + \sum_{k=1}^{m} S_{k,k}(u) \otimes \overline{u}^{\frac{\rho-2}{2}+k} \otimes \overline{\Delta}^k \delta \\ &\quad + \sum_{k=1}^{m} \sum_{l=1}^{k-1} A_{k,l} |u|^{k+\rho} \left(\frac{u}{|u|}\right)^{k-2l} \frac{du}{|u|^2} \otimes \Delta^{k-l} \overline{\Delta}^l \delta \quad \textit{for } n > 3. \end{aligned}$$

9.2.3 Invariant Hilbert subspaces of the oscillator representation

9.2.3.1 The oscillator representation

Let $G = \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SO}(n, \mathbb{C})$ with $n \geq 3$ and let ω_n be the unitary representation of G on $H = L^2(\mathbb{C}^n)$ defined by:

$$\begin{aligned} &\omega_n(g) f(z) = f(g^{-1} \cdot z), \qquad g \in \mathrm{SO}(n, \mathbb{C}), \\ &\omega_n(g(a)) f(z) = |a|^n f(az), \qquad g(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, a \in \mathbb{C}^* \\ &\omega_n(t(b)) f(z) = e^{-i\pi \mathrm{Re}\ (b[z,z])} f(z), \qquad t(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, b \in \mathbb{C} \\ &\omega_n(\sigma) f(z) = \int_{\mathbb{C}^n} e^{2i\pi \mathrm{Re}\ ([z,w])} f(w) dw, \qquad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

where $[z, w] = z_1 w_1 + ... + z_n w_n$ for $z = (z_1, ..., z_n), w = (w_1, ..., w_n)$.
We call ω_n the oscillator representation of G.

9.2.3.2 Invariant Hilbert subspaces of $\mathcal{S}'(\mathbb{C}^n)$

Given a function $f \in \mathcal{S}(\mathbb{C}^n)$, we define its Fourier transform $\hat{f}$ by

$$\hat{f}(z) = \int_{\mathbb{C}^n} f(w) e^{-2\pi i \mathrm{Re}\ ([z,w])} dw.$$

This definition naturally gives rise to a Fourier transform on $\mathcal{S}'(\mathbb{C}^n)$. We denote $\hat{T}$ this Fourier transform of a tempered distribution $T \in \mathcal{S}'(\mathbb{C}^n)$.

Let H be a Hilbert subspace of $\mathcal{S}'(\mathbb{C}^n)$, invariant under $\omega_n(G)$, and let K be its reproducing kernel. By Schwartz's kernel theorem we can associate to it a unique tempered distribution T on $\mathbb{C}^n \times \mathbb{C}^n$. This distribution satisfies the following conditions:
1. T is a positive-definite kernel, in particular a Hermitian kernel, i.e.
2. $T(x,y) = \bar{T}(y,x)$.
3. T is $\omega_n(G)$-invariant : $(\omega_n(g) \times \omega_n(g))\, T = T$ for all $g \in G$. The latter property implies, in more detail,

3a. $T(g\cdot x, g \cdot y) = T(x,y)$ for all $g \in \mathrm{SO}(n,\mathbb{C})$.

3b. If $a \in \mathbb{C}^*$, then

$$T(ax,\, ay) = |a|^{-2n} T(x,y).$$

3c. If $b \in \mathbb{C}$, then

$$e^{-i\pi \mathrm{Re}\ (b([x,x]-[y,y]))} T(x,y) = T(x,y).$$

3d. $\hat{T}(x,y) = T(x,y)$.

As preparation for our main result, the multiplicty free decomposition of the oscillator representation, we shall show that any of the above distributions T is symmetric: $T(x,y) = T(y,x)$. We shall do this in several steps. Observe that $T(y,x)$ satisfies the same conditions as $T(x,y)$.

Step I

By condition 3c, we obtain

$$\mathrm{Supp}\, T \subset \Xi_{n,n} = \{(x,y) \in \mathbb{C}^n \times \mathbb{C}^n \ :\ [x,x] - [y,y] = 0\}.$$

Step II

Let $\Xi'_{n,n} = \Xi_{n,n} \backslash (0,0)$, being an isotropic cone in $\mathbb{C}^n \times \mathbb{C}^n$. In a neighborhood of $\Xi'_{n,n}$ in $\mathbb{C}^n \times \mathbb{C}^n$ we take as coordinates $s = [x,x] - [y,y]$ and $\omega \in \Xi'_{n,n}$. So we can write locally there

$$T = \sum_{i \in I} S_i(\omega) \otimes \delta^{(i)}(s)$$

where I is some finite subset of $\mathbb{N}$ and the S_i are distributions on $\Xi'_{n,n}$. Applying condition 3c again, we get that only the term with $i = 0$ survives, so

$$T(x,y) = \delta([x,x] - [y,y])\, S_0(\omega)$$

outside $(x, y) = (0, 0)$. It remains to study the distribution S_0 on $\Xi'_{n,n}$.

Step III

On the open subset of $\Xi'_{n,n}$ given by $[x, x] \neq 0$, we get by properties 3a and 3b, with $\rho = n - 2$,

$$S_0(x, y) = |[x, x]|^{-\rho}\, \sigma_0(\omega_1, \omega_2)$$

where σ_0 is an $\mathrm{SO}(n, \mathbb{C})$-invariant distribution on $\mathcal{X} \times \mathcal{X}$, with $\mathcal{X} = \{x \in \mathbb{C}^n \,|\, [x, x] = 1\}$. It is known that σ_0 is symmetric: $\sigma_0(\omega_1, \omega_2) = \sigma_0(\omega_2, \omega_1)$ (see Section 9.2.1), hence

$$T(x, y) = T(y, x)$$

on the open subset of $\mathbb{C}^n \times \mathbb{C}^n$ defined by $[x, x] \neq 0$ (or, what is the same, $[y, y] \neq 0$).

So $S(x, y) = T(x, y) - T(y, x)$ has support contained in $[x, x] = [y, y] = 0$.

Step IV

Set $\Xi_n = \{x \in \mathbb{C}^n \,:\, [x, x] = 0\}$. The distribution S has support in $\Xi_n \times \Xi_n$. With the coordinates $s_1 = [x, x]$, $s_2 = [y, y]$ near $\Xi_n \times \Xi_n$, which can be taken, provided $x \neq 0$ and $y \neq 0$, so on $\Xi'_n \times \Xi'_n$ (in obvious notation), we get

$$S(x, y) = \delta([x, x], [y, y])\, U(\xi_1, \xi_2) \quad (\xi_1, \xi_2 \in \Xi'_n)$$

where U is an $\mathrm{SO}(n, \mathbb{C})$-invariant distribution on $\Xi'_n \times \Xi'_n$, homogeneous of degree $-2\rho + 4$. Since $S(x, y) = -S(y, x)$ it easily follows that $U(\xi_1, \xi_2) = -U(\xi_2, \xi_1)$.

Before we continue the preparation, we will recall now some structure theory of $\mathrm{SO}(n, \mathbb{C})$ and some results of Section 9.2.2 about distributions on Ξ'_n. We therefore assume (as in Section 9.2.2) $n \geq 3$. Let P be the stabilizer in $\mathrm{SO}(n, \mathbb{C})$ of the line generated by $\xi^0 = (1, i, 0, \ldots, 0)$. $P = MAN$ is a maximal parabolic subgroup.

The subgroup M consists of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & m \end{pmatrix}$$

where m is a matrix of the group $\mathrm{SO}(n - 2, \mathbb{C})$.

The group A is the one-parameter subgroup of matrices

$$a_u = \begin{pmatrix} \frac{1}{2}(u + \dfrac{1}{u}) & \frac{1}{2i}(u - \dfrac{1}{u}) & 0 \\ -\frac{1}{2i}(u - \dfrac{1}{u}) & \frac{1}{2}(u + \dfrac{1}{u}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $u \in \mathbb{C}^*$.

The subgroup N consists of matrices of the form

$$n_\alpha = \begin{pmatrix} 1 + \frac{1}{2}[\alpha, \alpha] & \frac{i}{2}[\alpha, \alpha] & i\,{}^t\alpha \\ \frac{i}{2}[\alpha, \alpha] & 1 - \frac{1}{2}[\alpha, \alpha] & -{}^t\alpha \\ -i\alpha & \alpha & 1 \end{pmatrix}$$

where $\alpha \in \mathbb{C}^{n-2}$.

The group N is isomorphic to $\mathbb{C}^{n-2}$. Moreover:

$$a_u\, n_\alpha\, a_u^{-1} = n_{u\alpha}.$$

Step V

Since $\mathrm{SO}(n,\mathbb{C})$ acts transitively on Ξ'_n, we can conclude that to U corresponds a MN-invariant distribution on Ξ'_n. Call it V.

We recall some results of Section 9.2.2 about such distributions.

Define $u(\xi) = \xi_1 + i\xi_2$ on Ξ'_n and $\Gamma_t = \{\xi : u(\xi) = t\}$ for $t \in \mathbb{C}^*$, $\Gamma_{0,\lambda} = \{\lambda\xi^0\}$ with $\lambda \in \mathbb{C}^*$, $\Gamma_{0,0} = \{\xi : u(\xi) = 0, \xi \notin \mathbb{C}\xi^0\}$.

The open set $\Gamma^* = \bigcup_{t\neq 0} \Gamma_t = \{\xi : u(\xi) \neq 0\}$ is dense in Ξ'. The open set of regular points for u is given by $\Gamma' = \Gamma^* \cup \Gamma_{0,0}$. So the singular points are the points of $\Gamma_0 = \bigcup_{\lambda\neq 0} \Gamma_{0,\lambda}$.

For $f \in \mathcal{D}(\Xi'_n)$, the space of C_c^∞-functions on Ξ'_n, one can define the function $\mathcal{M}f$ on $\mathbb{C}^*$ by

$$\mathcal{M}f(t) = \int f(\xi)\, \delta\left(u(\xi) - t\right).$$

$\mathcal{M}$ maps $\mathcal{D}(\Xi'_n)$ continuously onto some topological vector space $\mathcal{J}$ of functions on $\mathbb{C}^*$ with singularities at $t = 0$, see Section 9.2.2. And if W is a continuous linear form on $\mathcal{J}$, the distribution V defined on Ξ'_n by

$$V(f) = W(\mathcal{M}f) \quad (f \in \mathcal{D}(\Xi'_n)),$$

i.e., $V = \mathcal{M}'W$, is invariant under MN. $\mathcal{M}'$ is not surjective. One has (Section 9.2.2, Proposition 9.11): if V is a distribution on Ξ'_n invariant under MN, then

$$V = \mathcal{M}'W + V_1$$

where W is a continuous linear form on $\mathcal{J}$ and V_1 is a MN-invariant distribution on Ξ'_n with support contained in Γ_0.

The structure of the distributions V_1 is as follows. We use the local chart in a neighborhood of Γ_0 given by the map from $A \times \overline{N}$ to Ξ'_n,

$$(a_u, \overline{n}_\alpha) \to a_u\, \overline{n}_\alpha \xi^0,$$

where $\overline{N}$ consists of the matrices

$$\overline{n}_\alpha = \begin{pmatrix} 1 + \frac{1}{2}[\alpha,\alpha] & -\frac{i}{2}[\alpha,\alpha] & -i\,{}^t\alpha \\ -\frac{i}{2}[\alpha,\alpha] & 1 - \frac{1}{2}[\alpha,\alpha] & -{}^t\alpha \\ -i\alpha & \alpha & 1 \end{pmatrix}, \quad \alpha \in \mathbb{C}^{n-2}.$$

We denote by Δ the differential operator in this chart given by

$$\Delta = \frac{\partial^2}{\partial\alpha_3^2} + \cdots + \frac{\partial^2}{\partial\alpha_n^2}.$$

We quote 9.2.2, Theorem 9.13 here:

Let V_1 be a MN-invariant distribution on Ξ'_n with support in Γ_0. Then there exist distributions $T_{0,0}$, $S_{k,k}$, $S_{k,0}$ on $\mathbb{C}^*$ and constants $A_{k,l}$ such that, in the above chart,

$$\begin{aligned} V_1 \quad = \quad & T_{0,0} \otimes \delta + \sum_{k=1}^{m} u^{\frac{k-1}{2}} \otimes S_{k,0}(\overline{u}) \otimes \frac{\partial^k}{\partial \alpha^k}\delta + \sum_{k=1}^{m} S_{k,k}(u) \otimes \overline{u}^{\frac{k-1}{2}} \otimes \frac{\partial^k}{\partial \overline{\alpha}^k}\delta \\ & + \sum_{k=1}^{m} \sum_{l=1}^{k-1} A_{k,l} |u|^{\frac{k}{2}+1} \left(\frac{u}{|u|}\right)^{\frac{k}{2}-l} \frac{du}{|u|^2} \otimes \frac{\partial^{k-l}}{\partial \alpha^{k-l}} \frac{\partial^l}{\partial \overline{\alpha}^l}\delta \quad \text{for } n = 3 \end{aligned}$$

and

$$\begin{aligned} V_1 \quad = \quad & T_{0,0} \,\otimes\, \delta + \sum_{k=1}^{m} u^{\frac{\rho-2}{2}+k} \otimes S_{k,0}(\overline{u}) \,\otimes\, \Delta^k \delta + \sum_{k=1}^{m} S_{k,k}(u) \otimes \overline{u}^{\frac{\rho-2}{2}+k} \,\otimes\, \overline{\Delta}^k \delta \\ & + \sum_{k=1}^{m} \sum_{l=1}^{k-1} A_{k,l}\, |u|^{\rho+k} \left(\frac{u}{|u|}\right)^{k-2l} \frac{du}{|u|^2} \,\otimes\, \Delta^{k-l} \overline{\Delta}^l \delta \quad \text{for } n > 3 \end{aligned}$$

where δ is the Dirac measure at $\alpha = 0$.

Step VI

We return to our distribution V from Step V, and write it in the form $V = \mathcal{M}'W + V_1$, as above. By abuse of notation we have

$$V(\xi) = V(g\xi^0) = U(g\xi^0,\, \xi^0).$$

Because $u(\xi) = u(g\xi^0) = [g\xi^0,\, \xi^0]$ satisfies $u(g\xi^0) = u(g^{-1}\xi^0)$, we easily get that

$$\mathcal{M}'W(g\xi^0) = \mathcal{M}'W(g^{-1}\xi^0).$$

Since $V(g\xi^0) = -V(g^{-1}\xi^0)$, we see that $2V(g\xi^0) = V_1(g\xi^0) - V_1(g^{-1}\xi^0)$. Now

$$\begin{gathered} V(a_{u_0}^{-1} g a_{u_0} \xi^0) = U(g a_{u_0} \xi^0,\, a_{u_0} \xi^0) = \\ U(u_0 g \xi^0,\, u_0 \xi^0) = |u_0|^{-2\rho+4} V(g\xi^0). \end{gathered}$$

Let us apply this property to the distribution $V_2(g\xi^0) = V_1(g\xi^0) - V_1(g^{-1}\xi^0)$, supported by Γ_0. We get with $g = a_u \overline{n}_\alpha \xi^0$, that

$$a_{u_0}^{-1} a_u \overline{n}_\alpha a_{u_0} = a_u a_{u_0}^{-1} \overline{n}_\alpha a_{u_0} = a_u \overline{n}_{u_0 \alpha},$$

hence

$$\begin{aligned} & V_2(a_u\, \overline{n}_{u_0\alpha}\, \xi^0) = |u_0|^{-2} T_{0,0} \otimes \delta + \sum_{k=1}^{m} |u_0|^{-2} u_0^{-k} u^{\frac{k-1}{2}} \otimes S_{k,0}(\overline{u}) \otimes \frac{\partial^k}{\partial \alpha^k}\delta \\ & + \sum_{k=1}^{m} |u_0|^{-2} \overline{u}_0^{-k} S_{k,k}(u) \otimes \overline{u}^{\frac{k-1}{2}} \otimes \frac{\partial^k}{\partial \overline{\alpha}^k}\delta \\ & + \sum_{k=1}^{m} \sum_{l=1}^{k-1} |u_0|^{-2} u_0^{-(k-l)} \overline{u}_0^{-l} A_{k,l} |u|^{\frac{k}{2}+1} \left(\frac{u}{|u|}\right)^{\frac{k}{2}-l} \frac{du}{|u|^2} \otimes \frac{\partial^{k-l}}{\partial \alpha^{k-l}} \frac{\partial^l}{\partial \overline{\alpha}^l}\delta \quad \text{for } n = 3 \end{aligned}$$

and

$$
\begin{aligned}
V_2(a_u\,\overline{n}_{u_0\alpha}\,\xi^0) &= |u_0|^{-2\rho}T_{0,0}\,\otimes\,\delta + \sum_{k=1}^{m} |u_0|^{-2\rho}u_0^{-2k}u^{\frac{\rho-2}{2}+k}\otimes S_{k,0}(\overline{u})\,\otimes\,\Delta^k\delta \\
&+ \sum_{k=1}^{m}|u_0|^{-2\rho}\overline{u}_0^{-2k}\,S_{k,k}(u)\otimes\overline{u}^{\frac{\rho-2}{2}+k}\,\otimes\,\overline{\Delta}^k\delta \\
&+ \sum_{k=1}^{m}\sum_{l=1}^{k-1}|u_0|^{-2\rho}u_0^{-2(k-l)}\overline{u}_0^{-2l}A_{k,l}\,|u|^{\rho+k}\left(\frac{u}{|u|}\right)^{k-2l}\frac{du}{|u|^2}\otimes\Delta^{k-l}\overline{\Delta}^l\delta \quad \text{for } n>3.
\end{aligned}
$$

Since $V_2(a_u\,\overline{n}_{u_0\alpha}\,\xi^0) = |u_0|^{-2\rho+4}V_2(a_u\,\overline{n}_\alpha\,\xi^0)$, we get $V_2 = 0$. So $V = 0$, and hence $S = 0$ in a neighborhood of $\Xi_n'\times\Xi_n'$, and therefore $\operatorname{Supp} S \subset \{(0)\times\Xi_n\}\cup\{\Xi_n\times(0)\}$.

Step VII

From property 3c we conclude that

$$([x,x]-[y,y])^k\,S = 0$$

for $k = 1, 2, \ldots$, hence, combining it with property 3d, we have

$$\Box^k\,S = 0$$

for $k = 1, 2, \ldots$, in particular $\Box\,S = 0$. Here $\Box$ denotes the d'Alembertian

$$\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_n^2}-\frac{\partial^2}{\partial y_1^2}-\cdots-\frac{\partial^2}{\partial y_n^2}.$$

If S has support in $\{(0)\times\Xi_n\}\cup\{\Xi_n\times(0)\}$, and $\Box\,S = 0$, it easily follows, by recalling the local structure of such distributions, being basically a finite linear combination of tensor products of distributions supported by Ξ_n and the origin, that $S = 0$ (see [28],Ch.III, §10). So finally we have shown that $T(x,y) = T(y,x)$.

9.2.3.3 Multiplicity free decomposition of the oscillator representation

The multiplicity free decomposition of the oscillator representation, i.e. the multiplicity free decomposition into irreducible invariant Hilbert subspaces of any $\omega_n(G)$-invariant Hilbert subspace of $\mathcal{S}'(\mathbb{C}^n)$, is now easily proved by applying Criterion 5.6 with $JT = \bar{T}$. If H is any invariant Hilbert subspace of $\mathcal{S}'(\mathbb{C}^n)$ with reproducing kernel $T \in \mathcal{S}'(\mathbb{C}^n\times\mathbb{C}^n)$, then $\bar{T}$ is the kernel of the space $J\,H$ and the property $J\,H = H$ comes down to $\bar{T}(x,y) = T(x,y)$. Since T is positive-definite, this is equivalent with $T(x,y) = T(y,x)$ which immediately follows from the last section. So we have the following result.

Theorem 9.14. *Any $\omega_n(G)$-invariant Hilbert subspace of $\mathcal{S}'(\mathbb{C}^n)$ decomposes multiplicity free into irreducible invariant Hilbert subspaces of $\mathcal{S}'(\mathbb{C}^n)$.*

Appendix A

Conical distributions associated with the orthogonal complex group $\mathrm{SO}(n,\mathbb{C})$, $n \geq 3$

The group G acts transitively on the isotropic cone of the quadratic form associated with it. The action of G in the space of homogeneous functions on this cone defines a family of representations of G. Their study leads to the conical distributions. In this appendix we compute the conical distributions in the case of the group $\mathrm{SO}(n,\mathbb{C})$ with $n \geq 3$. We follow the same method as in [9].

A.1 The cone $\Xi = G/MN$

Let G be the orthogonal group $\mathrm{SO}(n,\mathbb{C})$ of the quadratic form

$$[z,z] = z_1^2 + \cdots + z_n^2.$$

Let P be the stabilizer of $\left\{\lambda\xi^0 : \lambda \in \mathbb{C}\right\}$ with $\xi^0 = (1, i, 0, \ldots, 0)$. The subgroup P is a maximal parabolic subgroup and is equal to $P = MAN$ where M is the subgroup of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & h & \\ 0 & 0 & & & \end{pmatrix}$$

with $h \in \mathrm{SO}(n-2,\mathbb{C})$,

$$A = \left\{ a_z = \begin{pmatrix} \cos z & \sin z & 0 \\ -\sin z & \cos z & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} : \text{ with } z \in \mathbb{C} \right\}$$

and

$$N = \left\{ n_p = \begin{pmatrix} 1+\frac{1}{2}[p,p] & \frac{1}{2}i[p,p] & i\,{}^tp \\ \frac{1}{2}i[p,p] & 1-\frac{1}{2}[p,p] & -{}^tp \\ -ip & p & I_{n-2} \end{pmatrix} : p \in \mathbb{C}^{n-2} \right\}.$$

Let $\tau_{s,\delta}$ be the character of $\mathbb{C}^*$ defined by

$$\tau_{s,\delta}(z) = z^{s,\delta}$$

with $\delta \in \mathbb{Z}$, $s \in \mathbb{C}$. The representation $\pi_{s,\delta}$ of G induced by $\tau_{s,\delta}$ can be realized in a space of homogeneous functions defined on the cone:

$$\Xi = \{\xi \in \mathbb{C}^n : \ \xi \neq 0, [\xi,\xi] = 0\}.$$

G acts transitively on Ξ and $\operatorname{Stab}\xi^0 = MN$, so Ξ can be identified to G/MN. The space of $\pi_{s,\delta}$ is called $E_{s,\delta}$ and consists of complex-valued $\mathcal{C}^\infty$-functions f on Ξ satisfying

$$f(\lambda\xi) = \tau_{s,\delta}(\lambda)|\lambda|^{-\rho}f(\xi) = \left(\frac{\lambda}{|\lambda|}\right)^{\delta}|\lambda|^{s-\rho}f(\xi) \quad (\xi \in \Xi, \lambda \in \mathbb{C}^*)$$

where $\rho = n-2$.

The representation $\pi_{s,\delta}$ is then given by

$$\pi_{s,\delta}(g)f(\xi) = f(g^{-1}\xi) \quad (\xi \in \Xi, g \in G; f \in E_{s,\delta}).$$

The group MN is unimodular, so Ξ has a G-invariant measure which we shall denote by $d\xi$. Let u be the map from Ξ to $\mathbb{C}$ defined by $u(\xi) = [\xi,\xi^0] = \xi_1 + i\xi_2$, u is MN-invariant and holomorphic, since Ξ is a complex subvariety of $\mathbb{C}^n$.

Proposition A.1. *The MN-orbits on Ξ are: $\Xi_{0,\lambda} = \{\lambda\xi^0\}$ with $\lambda \in \mathbb{C}^*$, $\Xi_t = \{\xi \in \Xi : u(\xi) = t\}$ for $t \in \mathbb{C}^*$ and $\Xi_{0,0} = \{\xi : u(\xi) = 0, \xi \notin \mathbb{C}\xi^0\}$ if $n \geq 3$.*

The open set $\Xi^* = \bigcup_{t\neq 0} \Xi_t = \{\xi : \ u(\xi) \neq 0\}$ is dense in Ξ. The open set of regular points for u is given by $\Xi' = \Xi^* \cup \Xi_{0,0}$. So the singular points are the points of $\Xi_0 = \bigcup_{\lambda\neq 0} \Xi_{0,\lambda}$.

A.2 Definition of conical distribution

Let $\tau_{s,\delta}$ be the character of $\mathbb{C}^*$ defined by

$$\tau_{s,\delta}(z) = z^{s,\delta} \quad \text{with } \delta \in \mathbb{Z}, \ s \in \mathbb{C}.$$

A function f defined on Ξ is τ-homogeneous if

$$f(z\xi) = \tau_{s,\delta}(z)|z|^{-\rho}f(\xi).$$

Let f be a continuous function on Ξ, τ-homogeneous. We can define the distribution T on Ξ by

$$\mathrm{T}(\Phi) = \int_{\Xi} f(\xi)\Phi(\xi)d\xi$$

where $d\xi$ is the invariant measure on Ξ.

Calling

$$\Phi_z(\xi) = \Phi(\frac{\xi}{z})$$

we have $d(z\xi) = |z|^{2\rho}d\xi$ and

$$\mathrm{T}(\Phi_z) = |z|^{2\rho}\int f(z\xi)\Phi(\xi)d\xi = |z|^{\rho}\tau_{s,\delta}(z)\mathrm{T}(\Phi).$$

Definition A.2. A distribution T on Ξ is called τ-homogeneous if

$$\mathrm{T}(\Phi_z) = \tau_{s,\delta}(z)|z|^{\rho}\mathrm{T}(\Phi).$$

Definition A.3. A distribution T on Ξ is called τ-conical if

1. T is τ-homogeneous
2. T is MN-invariant.

We can associate to Ξ_0 the τ-conical distribution $\psi_{\tau,0}$ defined as follows

$$< \psi_{\tau,0}, f > = \int_{\mathbb{C}^*} f(u\xi^0)\tau(u)|u|^{\rho}\frac{du}{|u|^2}.$$

We call

$$R_{k,l} = |u|^{\frac{k}{2}+1}\left(\frac{u}{|u|}\right)^{\frac{k}{2}-l}\frac{du}{|u|^2}\otimes\frac{\partial^{k-l}}{\partial\alpha^{k-l}}\frac{\partial^l}{\partial\overline{\alpha}^l}\delta$$

which is a $(-\frac{k}{2}, -\frac{k}{2}+l)$-conical distribution with support on Ξ_0 if $n=3$ and

$$L_{k,l} = |u|^{\rho+k}\left(\frac{u}{|u|}\right)^{k-2l}\frac{du}{|u|^2}\otimes\Delta^{k-l}\overline{\Delta}^l\delta$$

which is $(-k, -k+2l)$-conical distribution with support on Ξ_0 if $n>3$.

A.3 The function $\mathcal{M}$

We can define the integral of a function $f \in \mathcal{D}(\Xi)$ on Ξ_t as follows: exists a continuous function $\mathcal{M}f$ on $\mathbb{C}^*$, integrable on $\mathbb{C}$, such that for all continuous function F on $\mathbb{C}$,

$$\int_{\Xi} F(u(\xi))f(\xi)d\xi = \int_{\mathbb{C}} F(t)\mathcal{M}f(t)dt$$

with

$$\mathcal{M}f(t) = \int f(\xi)\delta(u(\xi)-t).$$

The function $\mathcal{M}f \in \mathcal{C}^\infty(\mathbb{C}^*)$ and has a singularity in $t = 0$. This is due to the critical points of the function $\xi \longrightarrow u(\xi)$ which are the points of Ξ_0.

We will use the following chart in a neighborhood of Ξ_0.

$$\begin{aligned} A \times \overline{N} &\longrightarrow \Xi \\ (a, \overline{n}) &\longrightarrow a\overline{n}\xi^0 \end{aligned}$$

We have:

$$a_u = \begin{pmatrix} \frac{1}{2}(u + \frac{1}{u}) & \frac{1}{2i}(u - \frac{1}{u}) & 0 \\ -\frac{1}{2i}(u - \frac{1}{u}) & \frac{1}{2}(u + \frac{1}{u}) & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}, \quad u \in \mathbb{C}^*$$

(if $u = e^{iz}$, then we get the usual expression for a_z)

$$\overline{n}_\alpha = \begin{pmatrix} 1 + \frac{1}{2}[\alpha, \alpha] & -\frac{i}{2}[\alpha, \alpha] & -i^t\alpha \\ -\frac{i}{2}[\alpha, \alpha] & 1 - \frac{1}{2}[\alpha, \alpha] & -^t\alpha \\ i\alpha & \alpha & I_{n-2} \end{pmatrix}, \quad \alpha \in \mathbb{C}^{n-2}.$$

So we have:

$$a_u\overline{n}_\alpha\xi^0 = \begin{pmatrix} u + \frac{1}{u}[\alpha, \alpha] \\ \left(u - \frac{1}{u}[\alpha, \alpha]\right) i \\ 2i\alpha \end{pmatrix}$$

and

$$u(a_u\overline{n}_\alpha\xi^0) = \frac{2}{u}[\alpha, \alpha].$$

In this chart the invariant measure is

$$d\xi = 2^{2(n-1)} \frac{du}{|u|^2} d\alpha.$$

Let f be a function in $\mathcal{D}(\Xi)$ whith the support contains in $A\overline{N}\xi^0$. Calling

$$\mathcal{N}f(u, \tau) = \int f(a_u\overline{n}_\alpha\xi^0)\delta([\alpha, \alpha] - \tau)$$

such that, if F is a continuous function on $\mathbb{C}$

$$\begin{aligned} \int_\Xi F(u(\xi))f(\xi)d\xi =& 2^{2(n-1)} \int_{A\times\overline{N}} F(\frac{2}{u}[\alpha, \alpha])f(a_u\overline{n}_\alpha\xi^0)\frac{du}{|u|^2}d\alpha \\ =& \int_{\mathbb{C}^*\times\mathbb{C}} F(t)\mathcal{N}f(u, \frac{tu}{2})dudt \end{aligned}$$

and

$$\mathcal{M}f(t) = \int_{\mathbb{C}^*} \mathcal{N}f(u, \frac{tu}{2})du.$$

We summarize here some results of Rallis-Schiffmann [26].

Let Q be a non-degenerate quadratic form

$$Q(z) = z_1^2 + \cdots + z_n^2$$

and let f be a function of $\mathcal{D}(\mathbb{C}^n)$. There exists a function $\mathcal{N}f$ on $\mathbb{C}$ such that for all continuous function F on $\mathbb{C}$

$$\int_{\mathbb{C}^n} F(Q(z))f(z)dz = \int_{\mathbb{C}} F(t)\mathcal{N}f(t)dt.$$

Definition A.4. Let $\mathcal{J}$ be the space of functions from $\mathbb{C}^*$ to $\mathbb{C}$ such that

$$\varphi(t) = \varphi_1(t) + \eta(t)\varphi_2(t)$$

with φ_1, $\varphi_2 \in \mathcal{D}(\mathbb{C})$ and

$$\eta(t) = \begin{cases} |t|^{n-2} & \text{if } n \text{ is odd} \\ |t|^{n-2}\mathrm{Log}|t| & \text{if } n \text{ is even.} \end{cases}$$

Calling

$$B_{u,v}(\varphi) = \frac{\partial^{u+v}}{\partial t^u \partial \overline{t}^v}\varphi_2(0)$$

if $u, v = 0, 1, 2, 3, \ldots$, we obtain that $B_{u,v}(\mathcal{N}f)$ is a multiple of $\Delta^u\overline{\Delta}^v\delta$.

A.4 Conical distributions with support on Ξ_0

Proposition A.5. *A. The case n=3.*

(1) If $s = -\frac{k}{2}$, with $k = 1, 2, \ldots$.

a) If s and δ satisfy $s + \delta = -k$ and $s - \delta = -k$ with $k = 3, 4, \ldots$ the space of τ-conical distributions with support on Ξ_0 has dimension 4. These distributions are as follows

$$C_1\psi_{\tau,0} + C_2 u^{\frac{k-1}{2}}\overline{u}^{\frac{1-\delta}{2}-\frac{k}{4}} + C_3 u^{\frac{1+\delta}{2}-\frac{k}{4}}\overline{u}^{\frac{k-1}{2}} + C_4 R_{k,\delta+\frac{k}{2}}.$$

b) If s and δ satisfy $s + \delta = -k$ and $s - \delta \neq -k$ with $k = 3, 4, \ldots$ the space of τ-conical distributions with support on Ξ_0 has dimension 3. These distributions are as follows

$$C_1\psi_{\tau,0} + C_2 u^{\frac{k-1}{2}}\overline{u}^{\frac{1-\delta}{2}-\frac{k}{4}} + C_3 R_{k,\delta+\frac{k}{2}}.$$

c) If s and δ satisfy $s + \delta \neq -k$ and $s - \delta = -k$ with $k = 3, 4, \ldots$ the space of τ-conical distributions with support on Ξ_0 has dimension 3. These distributions are as follows

$$C_1\psi_{\tau,0} + C_2 u^{\frac{1+\delta}{2}-\frac{k}{4}}\overline{u}^{\frac{k-1}{2}} + C_3 R_{k,\delta+\frac{k}{2}}.$$

- *d) If s and δ satisfy $s+\delta \neq -k$ and $s-\delta \neq -k$ with $k = 3, 4, \ldots$ the space of τ-conical distributions with support on Ξ_0 has dimension 2. These distributions are as follows*

$$C_1\psi_{\tau,0} + C_2 R_{k,\delta+\frac{k}{2}}.$$

(2) If $s \neq -\frac{k}{2}$, with $k = 1, 2, \ldots$.

- *a) If s and δ satisfy $s+\delta = -k$ with $k = 3, 4, \ldots$ the space of τ-conical distributions with support on Ξ_0 has dimension 2. These distributions are as follows*

$$C_1\psi_{\tau,0} + C_2 u^{\frac{k-1}{2}}\overline{u}^{\frac{1+s-\delta}{2}}.$$

- *b) If s and δ satisfy $s-\delta = -k$ with $k = 3, 4, \ldots$ the space of τ-conical distributions with support on Ξ_0 has dimension 2. These distributions are as follows*

$$C_1\psi_{\tau,0} + C_2 u^{\frac{s+1+\delta}{2}}\overline{u}^{\frac{k-1}{2}}.$$

- *c) Otherwise the space of τ-conical distributions with support on Ξ_0 has dimension 1. These distributions are as follows*

$$C_1\psi_{\tau,0}.$$

B. The case $n > 3$.

(1) If $s = -k$, with $k = 1, 2, \ldots$.

- *a) If s and δ satisfy $s+\delta = -2k$ and $s-\delta = -2k$ with $k = 2, 3, \ldots$ the space of τ-conical distributions with support on Ξ_0 has dimension 4. These distributions are as follows*

$$C_1\psi_{\tau,0} + C_2 u^{\frac{\rho-2}{2}+k}\overline{u}^{\frac{\rho-k-\delta}{2}} + C_3 u^{\frac{-k+\rho+\delta}{2}}\overline{u}^{\frac{\rho-2}{2}+k} + C_4 L_{k,\frac{\delta+k}{2}}.$$

- *b) If s and δ satisfy $s+\delta = -2k$ and $s-\delta \neq -2k$ with $k = 2, 3, \ldots$ the space of τ-conical distributions with support on Ξ_0 has dimension 3. These distributions are as follows*

$$C_1\psi_{\tau,0} + C_2 u^{\frac{\rho-2}{2}+k}\overline{u}^{\frac{\rho-k-\delta}{2}} + C_3 L_{k,\frac{\delta+k}{2}}.$$

- *c) If s and δ satisfy $s+\delta \neq -2k$ and $s-\delta = -2k$ with $k = 2, 3, \ldots$ the space of τ-conical distributions with support on Ξ_0 has dimension 3. These distributions are as follows*

$$C_1\psi_{\tau,0} + C_2 u^{\frac{-k+\rho+\delta}{2}}\overline{u}^{\frac{\rho-2}{2}+k} + C_3 L_{k,\frac{\delta+k}{2}}.$$

- *c) If s and δ satisfy $s+\delta \neq -2k$ and $s-\delta \neq -2k$ with $k = 2, 3, \ldots$ the space of τ-conical distributions with support on Ξ_0 has dimension 2. These distributions are as follows*

$$C_1\psi_{\tau,0} + C_2 L_{k,\frac{\delta+k}{2}}.$$

(2) If $s \neq -k$ with $k = 1, 2, \dots$.

a) If s and δ satisfy $s + \delta = -2k$ with $k = 2, 3, \dots$ the space of τ-conical distributions with support on Ξ_0 has dimension 2. These distributions are as follows

$$C_1 \psi_{\tau,0} + C_2 u^{\frac{\rho-2}{2}+k} \overline{u}^{\frac{\rho-k-\delta}{2}}.$$

b) If s and δ satisfy $s - \delta = -2k$ with $k = 2, 3, \dots$ the space of τ-conical distributions with support on Ξ_0 has dimension 2. These distributions are as follows

$$C_1 \psi_{\tau,0} + C_2 u^{\frac{-k+\rho+\delta}{2}} \overline{u}^{\frac{\rho-2}{2}+k}.$$

c) Otherwise the space of τ-conical distributions with support on Ξ_0 has dimension 1. These distributions are as follows

$$C_1 \psi_{\tau,0}.$$

Proof. Let T be a τ-conical distribution with support on Ξ_0. Since T is MN-invariant, Theorem 9.13 of Section 9.2.2 can be applied and we have:

$$\begin{aligned} T &= T_{0,0} \otimes \delta + \sum_{k=1}^{m} u^{\frac{k-1}{2}} \otimes S_{k,0}(\overline{u}) \otimes \frac{\partial^k}{\partial \alpha^k} \delta + \sum_{k=1}^{m} S_{k,k}(u) \otimes \overline{u}^{\frac{k-1}{2}} \otimes \frac{\partial^k}{\partial \overline{\alpha}^k} \delta \\ &\quad + \sum_{k=1}^{m} \sum_{l=1}^{k-1} A_{k,l} |u|^{\frac{k}{2}+1} \left(\frac{u}{|u|} \right)^{\frac{k}{2}-l} \frac{du}{|u|^2} \otimes \frac{\partial^{k-l}}{\partial \alpha^{k-l}} \frac{\partial^l}{\partial \overline{\alpha}^l} \delta \quad \text{for } n = 3 \end{aligned}$$

and

$$\begin{aligned} T &= T_{0,0} \otimes \delta + \sum_{k=1}^{m} u^{\frac{\rho-2}{2}+k} \otimes S_{k,0}(\overline{u}) \otimes \Delta^k \delta + \sum_{k=1}^{m} S_{k,k}(u) \otimes \overline{u}^{\frac{\rho-2}{2}+k} \otimes \overline{\Delta}^k \delta \\ &\quad + \sum_{k=1}^{m} \sum_{l=1}^{k-1} A_{k,l} |u|^{k+\rho} \left(\frac{u}{|u|} \right)^{k-2l} \frac{du}{|u|^2} \otimes \Delta^{k-l} \overline{\Delta}^l \delta \quad \text{for } n > 3 \end{aligned}$$

where $T_{0,0} = T_{0,0}(u, \overline{u})$, $S_{k,k}(u, \overline{u}) = S_{k,k}(u)$, $S_{k,0}(u, \overline{u}) = S_{k,0}(\overline{u})$ are distributions on $\mathbb{C}^*$ and $A_{k,l}$ constants.

If f is a function of $\mathcal{D}(\Xi)$ such that

$$f(\xi) = f(a_u \overline{n}_\alpha \xi^0) = \Phi(u) \Psi(\alpha)$$

where $\Phi \in \mathcal{D}(\mathbb{C}^*)$ and $\Psi \in \mathcal{D}(\mathbb{C}^{n-2})$, then

$$f_\lambda(\xi) = f\left(\frac{\xi}{\lambda} \right) = \Phi\left(\frac{u}{\lambda} \right) \Psi\left(\frac{\alpha}{\lambda} \right) = \Phi_\lambda(u) \Psi_\lambda(\alpha).$$

If follows that

$$<T, f_\lambda >=< T_{0,0}, \Phi_\lambda > \Psi(0) + \sum_{k=1}^{m} < u^{\frac{k-1}{2}} \otimes S_{k,0}(\overline{u}), \Phi_\lambda >< \frac{\partial^k}{\partial \alpha^k}\delta, \Psi_\lambda >$$
$$+ \sum_{k=1}^{m} < S_{k,k}(u) \otimes \overline{u}^{\frac{k-1}{2}}, \Phi_\lambda >< \frac{\partial^k}{\partial \overline{\alpha}^k}\delta, \Psi_\lambda >$$
$$+ \sum_{k=1}^{m} \sum_{l=1}^{k-1} A_{k,l} < |u|^{\frac{k}{2}+1} \left(\frac{u}{|u|}\right)^{\frac{k}{2}-l} \frac{du}{|u|^2}, \Phi_\lambda >< \frac{\partial^{k-l}}{\partial \alpha^{k-l}} \frac{\partial^l}{\partial \overline{\alpha}^l}\delta, \Psi_\lambda > \quad \text{for } n = 3$$

and

$$<T, f_\lambda >=< T_{0,0}, \Phi_\lambda > \Psi(0) + \sum_{k=1}^{m} < u^{\frac{\rho-2}{2}+k} \otimes S_{k,0}(\overline{u}), \Phi_\lambda >< \Delta^k \delta, \Psi_\lambda >$$
$$+ \sum_{k=1}^{m} < S_{k,k}(u) \otimes \overline{u}^{\frac{\rho-2}{2}+k}, \Phi_\lambda >< \overline{\Delta}^k \delta, \Psi_\lambda >$$
$$+ \sum_{k=1}^{m} \sum_{l=1}^{k-1} A_{k,l} < |u|^{k+\rho} \left(\frac{u}{|u|}\right)^{k-2l} \frac{du}{|u|^2}, \Phi_\lambda >< \Delta^{k-l}\overline{\Delta}^l \delta, \Psi_\lambda > \quad \text{for } n > 3.$$

Since T is τ-homogeneous, i.e.

$$< T, f_\lambda >= \tau(\lambda)|\lambda|^\rho < T, f >$$

we obtain for $n = 3$

$$< T_{0,0}, \Phi_\lambda >= |\lambda|^{1+s} \left(\frac{\lambda}{|\lambda|}\right)^\delta < T_{0,0}, \Phi > \tag{A.1}$$

$$< u^{\frac{k-1}{2}} \otimes S_{k,0}(\overline{u}), \Phi_\lambda >= |\lambda|^{1+s} \left(\frac{\lambda}{|\lambda|}\right)^\delta \lambda^k < u^{\frac{k-1}{2}} \otimes S_{k,0}(\overline{u}), \Phi > \tag{A.2}$$

$$< S_{k,k}(u) \otimes \overline{u}^{\frac{k-1}{2}}, \Phi_\lambda >= |\lambda|^{1+s} \left(\frac{\lambda}{|\lambda|}\right)^\delta \overline{\lambda}^k < S_{k,k}(u) \otimes \overline{u}^{\frac{k-1}{2}}, \Phi > \tag{A.3}$$

$$A_{k,l}|\lambda|^{1-\frac{k}{2}} \left(\frac{\lambda}{|\lambda|}\right)^{-\frac{k}{2}+l} = A_{k,l}|\lambda|^{1+s} \left(\frac{\lambda}{|\lambda|}\right)^\delta \tag{A.4}$$

and for $n > 3$

$$< T_{0,0}, \Phi_\lambda >= |\lambda|^{\rho+s} \left(\frac{\lambda}{|\lambda|}\right)^\delta < T_{0,0}, \Phi > \tag{A.5}$$

$$< u^{\frac{\rho-2}{2}+k} \otimes S_{k,0}(\overline{u}), \Phi_\lambda >= |\lambda|^{\rho+s} \left(\frac{\lambda}{|\lambda|}\right)^\delta \lambda^{2k} < u^{\frac{\rho-2}{2}+k} \otimes S_{k,0}(\overline{u}), \Phi > \tag{A.6}$$

$$< S_{k,k}(u) \otimes \overline{u}^{\frac{\rho-2}{2}+k}, \Phi_\lambda >= |\lambda|^{\rho+s} \left(\frac{\lambda}{|\lambda|}\right)^\delta \overline{\lambda}^{2k} < S_{k,k}(u) \otimes \overline{u}^{\frac{\rho-2}{2}+k}, \Phi > \tag{A.7}$$

$$A_{k,l}|\lambda|^{\rho-k}\left(\frac{\lambda}{|\lambda|}\right)^{-k+2l} = A_{k,l}|\lambda|^{\rho+s}\left(\frac{\lambda}{|\lambda|}\right)^{\delta}. \tag{A.8}$$

Let us first assume that $n = 3$. Then we get from equation (A.1) that $T_{0,0} = C\psi_{\tau,0}$. Equation (A.2) leads to $s + \delta = -k - 2$ and $S_{k,0}$ is a distribution on $\mathbb{C}^*$ homogeneous of degree $\frac{s+1-\delta}{2}$, i.e.

$$S_{k,0} = C\overline{u}^{\frac{s+1-\delta}{2}}$$

where C is a constant.
From equation (A.3) one finds that $s - \delta = -k - 2$ and $S_{k,k}$ is a distribution on $\mathbb{C}^*$ homogeneous of degree $\frac{s+1+\delta}{2}$, i.e.

$$S_{k,k} = Cu^{\frac{s+1+\delta}{2}}$$

where C is a constant.
Finally, from equation (A.4) we derive that $A_{k,l} = 0$ if $(k, l) \neq (-2s, \delta - s)$.

For the case $n > 3$ we have from equation (A.5) that $T_{0,0} = C\psi_{\tau,0}$. Equation (A.6) leads to $s + \delta = -2k - 2$ and that $S_{k,0}$ is a distribution on $\mathbb{C}^*$ homogeneous of degree $\frac{s+\rho-\delta}{2}$, i.e.

$$S_{k,0} = C\overline{u}^{\frac{s+\rho-\delta}{2}}$$

where C is a constant.
Equation (A.7) shows that $s - \delta = -2k - 2$ and that $S_{k,k}$ is a distribution on $\mathbb{C}^*$ homogeneous of degree $\frac{s+\rho+\delta}{2}$, i.e.

$$S_{k,k} = Cu^{\frac{s+\rho+\delta}{2}}$$

where C is a constant.
Finally, equation (A.8) results in $A_{k,l} = 0$ if $(k, l) \neq (-s, \frac{\delta-s}{2})$.

Therefore we have proven the proposition.□

A.5 Conical distributions

Let Φ be a function of $\mathcal{J}$ and Re $s > \rho - 1$ then we define

$$Z_\tau(\Phi) = \frac{1}{\Gamma\left(\frac{s-\rho+|\delta|+2}{2}\right)}\int_{\mathbb{C}} |z|^{s-\rho}\left(\frac{z}{|z|}\right)^{\delta}\Phi(z)dz.$$

If $\Phi \in \mathcal{D}(\mathbb{C})$, the function $s \longrightarrow Z_\tau(\Phi)$ can be extended to an entire function on $\mathbb{C}$. The distribution Z_τ is $(s-\rho, \delta)$-homogeneous. All distributions which are $(s-\rho, \delta)$-homogeneous on $\mathbb{C}$ are proportional to Z_τ. If $\Phi \in \mathcal{J}$, the function $s \longrightarrow Z_\tau(\Phi)$ can be extended to a meromorphic function on $\mathbb{C}$ with simple poles in $s = -2k - |\delta| - 2$ with $k = 0, 1, 2, \ldots$.

Let Φ be a function in $\mathcal{J}$, Φ can be written as follows

$$\Phi(z) = \sum_{j,k=0}^{\infty} a_{j,k}(\Phi) z^j \overline{z}^k + \eta(z,\overline{z}) \sum_{j=0}^{\infty} b_{j,k}(\Phi) z^j \overline{z}^k$$

where

$$\eta(z,\overline{z}) = \begin{cases} |z|^{n-2} & \text{if } n \text{ is odd} \\ |z|^{n-2}\mathrm{Log}|z| & \text{if } n \text{ is even.} \end{cases}$$

First I assume that n is odd. If $\delta > 0$ the residue of Z_τ in $s = -2k - |\delta| - 2$ for $k = 0, 1, 2, \ldots$ is

$$\text{Res } (Z_\tau, s = -2k - \delta - 2) = -\frac{4\pi i}{\Gamma\left(\frac{-2k-\rho}{2}\right)} b_{k,k+\delta}.$$

If $\delta < 0$ the residue of Z_τ in $s = -2k - |\delta| - 2$ for $k = 0, 1, 2, \ldots$ is

$$\text{Res } (Z_\tau, s = -2k + \delta - 2) = -\frac{4\pi i}{\Gamma\left(\frac{-2k-\rho}{2}\right)} b_{k-\delta,k}.$$

Finally, if n is even. If $\delta > 0$ the residue of Z_τ in $s = -2k - |\delta| - 2$ for $k = 0, 1, 2, \ldots$ is

$$\text{Res } (Z_\tau, s = -2k - \delta - 2) = 4\pi i \left(\frac{1}{\Gamma}\right)' \left(\frac{-2k-\rho}{2}\right) b_{k,k+\delta}.$$

If $\delta < 0$ the residue of Z_τ in $s = -2k - |\delta| - 2$ for $k = 0, 1, 2, \ldots$ is

$$\text{Res } (Z_\tau, s = -2k + \delta - 2) = 4\pi i \left(\frac{1}{\Gamma}\right)' \left(\frac{-2k-\rho}{2}\right) b_{k-\delta,k}.$$

For the above computations see [11].

For Re $s > \rho - 1$ we have

$$< \mathcal{M}' Z_\tau, f >= \frac{1}{\Gamma\left(\frac{s-\rho+|\delta|+2}{2}\right)} \int_\Xi |u(\xi)|^{s-\rho} \left(\frac{u(\xi)}{|u(\xi)|}\right)^\delta f(\xi) d\xi.$$

Calling $\psi_{\tau,1} = \mathcal{M}' Z_\tau$. Therefore, the function $s \longrightarrow \psi_{\tau,1}$ can be meromorphically extended to $\mathbb{C}$ with poles at $s = -2k - |\delta| - 2$ for $k = 0, 1, 2, \ldots$. The residues in these poles are

$$\text{Res } (\psi_{\tau,1}, -2k - |\delta| - 2) = \mathcal{M}' \left(\text{Res } (Z_\tau, -2k - |\delta| - 2)\right).$$

These residues are equal to

$$\text{Res } (\psi_{\tau,1}, -2k - \delta - 2) = \mu_{k,\delta} L_{2k+\delta,k+\delta} \quad \text{if } \delta \geq 0$$

and

$$\text{Res } (\psi_{\tau,1}, -2k + \delta - 2) = \mu_{k,\delta} L_{2k-\delta,k} \quad \text{if } \delta < 0$$

where $\mu_{k,\delta}$ is a constant different of 0.

If $s=-2$ is a pole of $\psi_{\tau,1}$ then

$$\mathrm{Res}\,(\psi_{\tau,1},-2)=\mu_{0,0}L_{0,0}$$

where $\mu_{0,0}$ is a constant different of 0.

The distribution $\tilde{\psi}$ defined by

$$\tilde{\psi}=\lim_{s\to-2}\left(\psi_{\tau,1}-\frac{\mu_{0,0}}{s+2}\psi_{\tau,0}\right)$$

is $(-2,0)$-conical.

Proposition A.6. *A. The case n=3.*

- *(1) If $s=-\frac{k}{2}$, with $k=1,2,\ldots$.*
  - *a) If s is not a pole of $\psi_{\tau,1}$.*
    - *i) If s and δ satisfy $s+\delta=-k$ and $s-\delta=-k$ with $k=3,4,\ldots$ the space of τ-conical distributions has dimension 5. These distributions are as follows*

$$C_1\psi_{\tau,1}+C_2\psi_{\tau,0}+C_3u^{\frac{k-1}{2}}\overline{u}^{\frac{1-\delta}{2}-\frac{k}{4}}+C_4u^{\frac{1+\delta}{2}-\frac{k}{4}}\overline{u}^{\frac{k-1}{2}}+C_5R_{k,\delta+\frac{k}{2}}.$$

    - *ii) If s and δ satisfy $s+\delta=-k$ and $s-\delta\neq-k$ with $k=3,4,\ldots$ the space of τ-conical distributions has dimension 4. These distributions are as follows*

$$C_1\psi_{\tau,1}+C_2\psi_{\tau,0}+C_3u^{\frac{k-1}{2}}\overline{u}^{\frac{1-\delta}{2}-\frac{k}{4}}+C_4R_{k,\delta+\frac{k}{2}}.$$

    - *iii) If s and δ satisfy $s+\delta\neq-k$ and $s-\delta=-k$ with $k=3,4,\ldots$ the space of τ-conical distributions has dimension 4. These distributions are as follows*

$$C_1\psi_{\tau,1}+C_2\psi_{\tau,0}+C_3u^{\frac{1+\delta}{2}-\frac{k}{4}}\overline{u}^{\frac{k-1}{2}}+C_4R_{k,\delta+\frac{k}{2}}.$$

    - *iv) If s and δ satisfy $s+\delta\neq-k$ and $s-\delta\neq-k$ with $k=3,4,\ldots$ the space of τ-conical distributions has dimension 3. These distributions are as follows*

$$C_1\psi_{\tau,1}+C_2\psi_{\tau,0}+C_3R_{k,\delta+\frac{k}{2}}.$$

  - *b) If s is a pole of $\psi_{\tau,1}$.*
    - *i) If $s=-2$ the space of τ-conical distributions has dimension 2. These distributions are as follows*

$$\psi=C_1\psi_{\tau,1}+C_2\tilde{\psi}.$$

ii) If $s \neq -2$.

ii.1) If s *and* δ *satisfy* $s+\delta = -k$ *and* $s-\delta = -k$ *with* $k = 3, 4, \ldots$ *the space of* τ*-conical distributions has dimension 4. These distributions are as follows*

$$C_1\psi_{\tau,0} + C_2 u^{\frac{k-1}{2}} \overline{u}^{\frac{1-\delta}{2}-\frac{k}{4}} + C_3 u^{\frac{1+\delta}{2}-\frac{k}{4}} \overline{u}^{\frac{k-1}{2}} + C_4 R_{k,\delta+\frac{k}{2}}.$$

ii.2) If s *and* δ *satisfy* $s+\delta = -k$ *and* $s-\delta \neq -k$ *with* $k = 3, 4, \ldots$ *the space of* τ*-conical has dimension 3. These distributions are as follows*

$$C_1\psi_{\tau,0} + C_2 u^{\frac{k-1}{2}} \overline{u}^{\frac{1-\delta}{2}-\frac{k}{4}} + C_3 R_{k,\delta+\frac{k}{2}}.$$

ii.3) If s *and* δ *satisfy* $s+\delta \neq -k$ *and* $s-\delta = -k$ *with* $k = 3, 4, \ldots$ *the space of* τ*-conical distributions has dimension 3. These distributions are as follows*

$$C_1\psi_{\tau,0} + C_2 u^{\frac{1+\delta}{2}-\frac{k}{4}} \overline{u}^{\frac{k-1}{2}} + C_3 R_{k,\delta+\frac{k}{2}}.$$

ii.4) If s *and* δ *satisfy* $s+\delta \neq -k$ *and* $s-\delta \neq -k$ *with* $k = 3, 4, \ldots$ *the space of* τ*-conical distributions has dimension 2. These distributions are as follows*

$$C_1\psi_{\tau,0} + C_2 R_{k,\delta+\frac{k}{2}}.$$

(2) If $s \neq -\frac{k}{2}$*, with* $k = 1, 2, \ldots$.

a) If s *and* δ *satisfy* $s+\delta = -k$ *with* $k = 3, 4, \ldots$ *the space of* τ*-conical distributions has dimension 3. These distributions are as follows*

$$C_1\psi_{\tau,1} + C_2\psi_{\tau,0} + C_3 u^{\frac{k-1}{2}} \overline{u}^{\frac{1+s-\delta}{2}}.$$

b) If s *and* δ *satisfy* $s-\delta = -k$ *with* $k = 3, 4, \ldots$ *the space of* τ*-conical distributions has dimension 3. These distributions are as follows*

$$C_1\psi_{\tau,1} + C_2\psi_{\tau,0} + C_3 u^{\frac{s+1+\delta}{2}} \overline{u}^{\frac{k-1}{2}}.$$

c) Otherwise the space of τ*-conical distributions has dimension 2. These distributions are as follows*

$$C_1\psi_{\tau,1} + C_2\psi_{\tau,0}.$$

B. The case $n > 3$.

(1) If $s = -k$*, with* $k = 1, 2, \ldots$.

a) If s *is not pole of* $\psi_{\tau,1}$.

i) *If s and δ satisfy $s+\delta=-2k$ and $s-\delta=-2k$ with $k=2,3,\dots$ the space of τ-conical distributions has dimension 5. These distributions are as follows*

$$C_1\psi_{\tau,1}+C_2\psi_{\tau,0}+C_3u^{\frac{\rho-2}{2}+k}\overline{u}^{\frac{\rho-k-\delta}{2}}+C_4u^{\frac{-k+\rho+\delta}{2}}\overline{u}^{\frac{\rho-2}{2}+k}+C_5L_{k,\frac{\delta+k}{2}}.$$

ii) *If s and δ satisfy $s+\delta=-2k$ and $s-\delta\neq-2k$ with $k=2,3,\dots$ the space of τ-conical distributions has dimension 4. These distributions are as follows*

$$C_1\psi_{\tau,1}+C_2\psi_{\tau,0}+C_3u^{\frac{\rho-2}{2}+k}\overline{u}^{\frac{\rho-k-\delta}{2}}+C_4L_{k,\frac{\delta+k}{2}}.$$

iii) *If s and δ satisfy $s+\delta\neq-2k$ and $s-\delta=-2k$ with $k=2,3,\dots$ the space of τ-conical distributions has dimension 4. These distributions are as follows*

$$C_1\psi_{\tau,1}+C_2\psi_{\tau,0}+C_3u^{\frac{-k+\rho+\delta}{2}}\overline{u}^{\frac{\rho-2}{2}+k}+C_4L_{k,\frac{\delta+k}{2}}.$$

iv) *If s and δ satisfy $s+\delta\neq-2k$ and $s-\delta\neq-2k$ with $k=2,3,\dots$ the space of τ-conical distributions has dimension 3. These distributions are as follows*

$$C_1\psi_{\tau,1}+C_2\psi_{\tau,0}+C_3L_{k,\frac{\delta+k}{2}}.$$

b) *If s is a pole of $\psi_{\tau,1}$.*

i) *If $s=-2$ the space of τ-conical distributions has dimension 2 and these distributions are as follows*

$$\psi=C_1\psi_{\tau,1}+C_2\tilde{\psi}.$$

ii) *If $s\neq-2$.*

ii.1) *If s and δ satisfy $s+\delta=-2k$ and $s-\delta=-2k$ with $k=2,3,\dots$ the space of τ-conical distributions has dimension 4. These distributions are as follows*

$$C_1\psi_{\tau,0}+C_2u^{\frac{\rho-2}{2}+k}\overline{u}^{\frac{\rho-k-\delta}{2}}+C_3u^{\frac{-k+\rho+\delta}{2}}\overline{u}^{\frac{\rho-2}{2}+k}+C_5L_{k,\frac{\delta+k}{2}}.$$

ii.2) *If s and δ satisfy $s+\delta=-2k$ and $s-\delta\neq-2k$ with $k=2,3,\dots$ the space of τ-conical distributions has dimension 3. These distributions are as follows*

$$C_1\psi_{\tau,0}+C_2u^{\frac{\rho-2}{2}+k}\overline{u}^{\frac{\rho-k-\delta}{2}}+C_3L_{k,\frac{\delta+k}{2}}.$$

ii.3) *If s and δ satisfy $s+\delta\neq-2k$ and $s-\delta=-2k$ with $k=2,3,\dots$ the space of τ-conical distributions has dimension 3. These distributions are as follows*

$$C_1\psi_{\tau,0}+C_2u^{\frac{-k+\rho+\delta}{2}}\overline{u}^{\frac{\rho-2}{2}+k}+C_3L_{k,\frac{\delta+k}{2}}.$$

ii.4) If s and δ satisfy $s+\delta\neq -2k$ and $s-\delta\neq -2k$ with $k=2,3,\ldots$ the space of τ-conical distributions has dimension 2. These distributions are as follows

$$C_1\psi_{\tau,0}+C_2L_{k,\frac{\delta+k}{2}}.$$

(2) If $s\neq -k$ with $k=1,2,\ldots$.

a) If s and δ satisfy $s+\delta=-2k$ with $k=2,3,\ldots$ the space of τ-conical distributions has dimension 3. These distributions are as follows

$$C_1\psi_{\tau,1}+C_2\psi_{\tau,0}+C_3u^{\frac{\rho-2}{2}+k}\overline{u}^{\frac{\rho-k-\delta}{2}}.$$

b) If s and δ satisfy $s-\delta=-2k$ with $k=2,3,\ldots$ the space of τ-conical distributions has dimension 3. These distributions are as follows

$$C_1\psi_{\tau,1}+C_2\psi_{\tau,0}+C_3u^{\frac{-k+\rho+\delta}{2}}\overline{u}^{\frac{\rho-2}{2}+k}.$$

c) Otherwise the space of τ-conical distributions has dimension 2. These distributions are as follows

$$C_1\psi_{\tau,1}+C_2\psi_{\tau,0}.$$

Proof. Let τ be equal to (s,δ) and let ψ be a τ-conical distribution. We call ψ' the restriction to Ξ'. The distribution ψ' is MN-invariant and τ-homogeneous. There exists a distribution S' on $\mathbb{C}$ which is $(s-\rho,\delta)$-homogeneous such that

$$\psi'=\mathcal{M}'(S').$$

The distribution S' is proportional to Z_τ, i.e. $S'=C_1Z_\tau$.

a) If s is not a pole of $\psi_{\tau,1}$ then the distribution $\psi-C_1\psi_{\tau,1}$ is τ-conical and with support in Ξ_0. We only have to apply last proposition to get the result.

b) If s is a pole of $\psi_{\tau,1}$. Then $s=-2k-|\delta|-2$ for $k=0,1,\ldots$ and let Z^0_τ be equal to

$$Z^0_\tau=\frac{d}{ds}(s+2k+|\delta|+2)Z_\tau\mid_{s=-2k-|\delta|-2}.$$

Z^0_τ is not an homogeneous linear form. Calling $f_\lambda(t)=f\left(\frac{t}{\lambda}\right)$ its follows

$$Z_\tau(f_\lambda)=|\lambda|^{s-\rho+2}\left(\frac{\lambda}{|\lambda|}\right)^\delta Z_\tau(f)$$

and

$$Z^0_\tau(f_\lambda)=|\lambda|^{-2k-|\delta|-\rho}\left(\frac{\lambda}{|\lambda|}\right)^\delta Z^0_\tau(f)$$
$$+|\lambda|^{-2k-|\delta|-\rho}\left(\frac{\lambda}{|\lambda|}\right)^\delta \mathrm{Log}\,|\lambda|\mathrm{Res}\,(Z_\tau(f),-2k-|\delta|-2).$$

Calling $\psi^0_{\tau,1} = \mathcal{M}'(Z^0_\tau)$ we get

$$\begin{aligned}\psi^0_{\tau,1}(f_\lambda) =& |\lambda|^{-2k-|\delta|-2+\rho}\left(\frac{\lambda}{|\lambda|}\right)^\delta \psi^0_{\tau,1}(f)\\ &+ |\lambda|^{-2k-|\delta|-2+\rho}\left(\frac{\lambda}{|\lambda|}\right)^\delta \text{Log}\ |\lambda|\text{Res}\ (\psi_{\tau,1}(f), -2k-|\delta|-2).\end{aligned}$$

The distribution $\psi_1 = \psi - C_1\psi^0_{\tau,1}$ is MN-invariant and satisfies

$$\psi_1(f_\lambda) - |\lambda|^\rho\tau(\lambda)\psi_1(f) = -C_1|\lambda|^\rho\tau(\lambda)\text{Log}\ |\lambda|\text{Res}\ (\psi_{\tau,1}(f), -2k-|\delta|-2).$$

Proposition A.7. *Let τ be equal to $(-2k-|\delta|-2, \delta)$ with $k = 0, 1, \dots$. Let T be a distribution on Ξ such that*

(1) T is MN-invariant

(2) T has support in Ξ_0

(3) T satisfies

$$T(f_\lambda) - |\lambda|^\rho\tau(\lambda)T(f) = C|\lambda|^\rho\tau(\lambda)Log\ |\lambda|L_{2k+\delta,k+\delta}(f) \quad \text{if } \delta \geq 0$$

or

$$T(f_\lambda) - |\lambda|^\rho\tau(\lambda)T(f) = C|\lambda|^\rho\tau(\lambda)Log\ |\lambda|L_{2k-\delta,k}(f) \quad \text{if } \delta < 0.$$

If $k \neq 0$ or $\delta \neq 0$ then $C = 0$ and T is τ-conical. If $k = 0$ and $\delta = 0$ then $T - C\left.\dfrac{d}{ds}\psi_{\tau,0}\right|_{s=-2}$ is τ-conical.

For the proof we follow the same method as in [9] using Theorem 9.13 of Section 9.2.2.

From this proposition we get the result. If $k \neq 0$ or $\delta \neq 0$ then $C_1 = 0$ and ψ is a conical distribution with support on Ξ_0.

If $k = 0$ and $\delta = 0$ we obtain,

$$\psi - C_1\psi^0_{\tau,1} = -C_1\mu_{0,0}\left.\frac{d}{ds}\psi_{\tau,0}\right|_{s=-2} + C_0\psi_{\tau,1}$$

and

$$\begin{aligned}\tilde{\psi} &= \lim_{s\to-2}\left(\psi_{\tau,1} - \frac{\mu_{0,0}}{s+2}\psi_{\tau,0}\right)\\ &= \psi^0_{\tau,1} - \mu_{0,0}\left.\frac{d}{ds}\psi_{\tau,0}\right|_{s=-2}.\square\end{aligned}$$

Appendix B

Irreducibility and unitarity

In this appendix we study the irreducibility and unitarity of the representations of $\mathrm{SO}(n,\mathbb{C})$ induced by a maximal parabolic subgroup and defined in Section 8.3.1 in more detail. We follow the same method as in [41].

B.1 The standard minimal parabolic subgroup

We refer to [19].

Let G be the group $\mathrm{SO}(n,\mathbb{C})$ with $n \geq 3$ and let $\mathfrak{g}$ be the Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$. Let $\mathfrak{a}_0$ be the maximal abelian subalgebra of $i\mathfrak{k}$ where

$$\mathfrak{a}_0 = \left\{ \begin{pmatrix} 0 & ix_1 & & & & & & \\ -ix_1 & 0 & & & & & & \\ & & 0 & ix_2 & & & & \\ & & -ix_2 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & 0 & ix_k & \\ & & & & & -ix_k & 0 & \\ & & & & & & & 0 \end{pmatrix}, \quad x_i \in \mathbb{R} \right\}$$

if $n = 2k+1$ and

$$\mathfrak{a}_0 = \left\{ \begin{pmatrix} 0 & ix_1 & & & & & \\ -ix_1 & 0 & & & & & \\ & & 0 & ix_2 & & & \\ & & -ix_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & ix_k \\ & & & & & -ix_k & 0 \end{pmatrix}, \quad x_i \in \mathbb{R} \right\}$$

if $n = 2k$.

Let $\mathfrak{m}_0$ be the centralizer of $\mathfrak{a}_0$ in $\mathfrak{k}$, then it is equal to

$$\mathfrak{m}_0 = \left\{ \begin{pmatrix} 0 & y_1 & & & & & & & \\ -y_1 & 0 & & & & & & & \\ & & 0 & y_2 & & & & & \\ & & -y_2 & 0 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 0 & y_k & \\ & & & & & -y_k & 0 & \\ & & & & & & & 0 \end{pmatrix}, \quad y_i \in \mathbb{R} \right\}$$

if $n = 2k+1$ and

$$\mathfrak{m}_0 = \left\{ \begin{pmatrix} 0 & y_1 & & & & & \\ -y_1 & 0 & & & & & \\ & & 0 & y_2 & & & \\ & & -y_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & y_k \\ & & & & & -y_k & 0 \end{pmatrix}, \quad y_i \in \mathbb{R} \right\}$$

if $n = 2k$.

Let h_j be the function such that $h_j(H) = x_j$, $1 \leq j \leq k$ where $H \in \mathfrak{a}_0$.
The set of roots is

$$\Delta = \{\pm h_i \pm h_j \text{ with } i \neq j\} \cup \{\pm h_l\} \text{ if } \quad n = 2k+1$$

and

$$\Delta = \{\pm h_i \pm h_j \text{ with } i \neq j\} \text{ if } \quad n = 2k.$$

The root space decomposition is

$$\mathfrak{g} = \mathfrak{a}_0 \oplus \mathfrak{m}_0 \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha = \mathbb{C} H_\alpha$. To define H_α, first let $i < j$ and $\alpha = \pm h_i \pm h_j$. Then H_α is zero except in the entries corresponding to the i^{th} and j^{th} pairs of indices, where it is equal to

$$H_\alpha = \begin{array}{c} \begin{array}{cc} i & j \end{array} \\ \begin{pmatrix} 0 & X_\alpha \\ -{}^t X_\alpha & 0 \end{pmatrix} \end{array}$$

with

$$X_{h_i - h_j} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \; X_{h_i + h_j} = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix},$$

$$X_{-h_i + h_j} = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \; X_{-h_i - h_j} = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$

To define H_α for $\alpha = \pm h_l$ in case $n = 2k+1$, write

$$\begin{array}{ccc} & \text{pair} & \text{entry} \\ & l & 2k+1 \\ H_\alpha = & \left(\begin{array}{c} 0 \\ -{}^tX_\alpha \end{array}\right. & \left.\begin{array}{c} X_\alpha \\ 0 \end{array}\right) \end{array}$$

with zeros elsewhere and with

$$X_{h_l} = \begin{pmatrix} 1 \\ -i \end{pmatrix} \text{ and } X_{-h_l} = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

The set of positive roots is

$$\Delta^+ = \{h_i \pm h_j \text{ with } i < j\} \cup \{h_l\} \text{ if } \quad n = 2k+1$$

and

$$\Delta^+ = \{h_i \pm h_j \text{ with } i < j\} \text{ if } \quad n = 2k.$$

We define $\mathfrak{n}_0 = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ and $\mathfrak{p}_0 = \mathfrak{m}_0 + \mathfrak{a}_0 + \mathfrak{n}_0$. We call $P_0 = M_0 A_0 N_0$ the standard minimal parabolic subgroup where $A_0 = \exp \mathfrak{a}_0$, $N_0 = \exp \mathfrak{n}_0$ and M_0 is the centralizer of $\mathfrak{a}_0$ in $K = \mathrm{SO}(n, \mathbb{R})$.

The set of simple roots is

$$\Sigma = \{h_1 - h_2, h_2 - h_3, \ldots, h_{k-1} - h_k, h_k\} \text{ if } \quad n = 2k+1$$

and

$$\Sigma = \{h_1 - h_2, h_2 - h_3, \ldots, h_{k-1} - h_k, h_{k-1} + h_k\} \text{ if } \quad n = 2k.$$

For any subset $F \subset \Sigma$ let $P_F = M_F A_F N_F$ be the Langlands decomposition of the parabolic subgroup P_F associated to F, then $P_0 \subset P_F$, $M_0 \subset M_F$, $A_F \subset A_0$ and $N_F \subset N_0$. For details on parabolic subgroups see [42]. Taking $F = \Sigma - \{h_1 - h_2\}$ we get if $n = 2k$ or $n = 2k+1$ and $n > 4$

$$\mathfrak{a}_F = \left\{ \begin{pmatrix} 0 & ix & \\ -ix & 0 & \\ & & 0 \end{pmatrix} \text{ with } x \in \mathbb{R} \right\}$$

and if $n = 4$

$$\mathfrak{a}_F = \left\{ \begin{pmatrix} 0 & ix & 0 & 0 \\ -ix & 0 & 0 & 0 \\ 0 & 0 & 0 & -ix \\ 0 & 0 & ix & 0 \end{pmatrix} \text{ with } x \in \mathbb{R} \right\}.$$

If $n = 3$ then $P_F = P_0$.

We define $\Delta_F = \Delta \cap \mathbb{N} \cdot F$, in our case $\Delta_F = F$, then

$$\mathfrak{m}_{1F} = \mathfrak{m}_0 + \mathfrak{a}_0 + \sum_{\alpha \in \Delta_F} \mathfrak{g}_\alpha, \qquad \mathfrak{n}_F = \sum_{\alpha \in \Delta^+ \setminus \Delta_F} \mathfrak{g}_\alpha$$

and

$$\mathfrak{p}_F = \mathfrak{m}_{1F} + \mathfrak{n}_F.$$

Let $\mathfrak{m}_F$ be the orthogonal complement of $\mathfrak{a}_F$ in $\mathfrak{m}_{1F}$ with respect the inner product $< X, Y >= -\text{Re } F(X, Y)$, where F is the Killing form defined in Section 8.3.1.6, then

$$\mathfrak{m}_{1F} = \mathfrak{m}_F + \mathfrak{a}_F.$$

Let M_{1F} be the centralizer of $\mathfrak{a}_F$ in G, $N_F = \exp \mathfrak{n}_F$, $A_F = \exp \mathfrak{a}_F$, $M_{1F} = M_F A_F$ and $P_F = M_F A_F N_F$. We can see that $P_F = P$ where $P = MA\overline{N}$ is a parabolic subgroup with M, A and N defined in Section 8.3.1.

Put $\rho = n - 2$ and define for $s \in \mathbb{C}$, $\delta \in \mathbb{Z}$

$$\begin{aligned} E_{\delta,s} =& \{f \in \mathcal{C}^\infty(G) \,:\, f(gma_{x+iy}n) = e^{i\delta x} e^{-(s-\rho)y} f(g), \\ & \text{for all } \ (g, m, a_{x+iy}, n) \in G \times M \times A \times N\}. \end{aligned}$$

B.2 Irreducibility

To prove the irreducibility we are going to follow the same method as in [41]. First we will study the irreducibility of $E_{0,s}$, by determining sufficient conditions under which the, up to scalars, unique K-fixed vector is cyclic in $E_{0,s}$. Put $\mu_s = sh_1$, $\rho_1 = (n-2)h_1$ then it is easily seen that

$$\begin{aligned} E_{\delta,s} =& \{f \in \mathcal{C}^\infty(G) \,:\, f(gman) = \chi_\delta(m^{-1}) e^{-(\mu_s+\rho_1)\log a} f(g), \\ & \text{for all } \ (g, m, a, n) \in G \times M_F \times A_F \times N_F\}. \end{aligned}$$

where

$$\chi_\delta(m) = \chi_\delta \begin{pmatrix} \cos x & \sin x & \\ -\sin x & \cos x & \\ & & t \end{pmatrix} = e^{-i\delta x}$$

where $t \in \text{SO}(n-2, \mathbb{C})$.

We have endowed $E_{\delta,s}$ with a pre-Hilbert space structure with norm

$$||f||^2 = \int_K |f(k)|^2 dk$$

where $K = \text{SO}(n, \mathbb{R})$ and dk the normalized Haar measure on K. Furthermore the sesqui-linear form

$$< f, h >= \int_K f(k)\overline{h}(k) dk \tag{B.1}$$

defines a non-degenerate G-invariant pairing between $E_{\delta,s}$ and $E_{\delta,-\overline{s}}$ with $s \in \mathbb{C}$, $\delta \in \mathbb{Z}$.

Now define for $\lambda \in \mathfrak{a}_{0c}^*$ arbitrary

$$\begin{aligned} E_\delta(G/P_0, \lambda) =& \{f \in \mathcal{C}^\infty(G) \,:\, f(gman) = \chi_\delta(m^{-1}) e^{-(\lambda+\rho_0)\log a} f(g), \\ & \text{for all } \ (g, m, a, n) \in G \times M_0 \times A_0 \times N_0\}. \end{aligned}$$

Define the following special elements of $\mathfrak{a}_{0c}^*$:

$$\begin{aligned}
&\rho_0 = 2\sum_{i=1}^{k-1}(k-i)h_i \quad \text{if } n = 2k,\\
&\rho_0 = \sum_{i=1}^{k-1}\left(2(k-i)+1\right)h_i + h_k \quad \text{if } n = 2k+1,\\
&\lambda(s) = \mu_s + \rho_1 - \rho_0 \quad (s \in \mathbb{C}) \text{ if } n = 2k \text{ or } n = 2k+1.
\end{aligned}$$

These elements can also be written as

$$\begin{aligned}
&h_i = e_i, && \text{(B.2)}\\
&\lambda(s) = (s, -2k+4, -2k+6, \dots, -2, 0) \quad \text{if } n = 2k\\
&\lambda(s) = (s, -2k+3, -2k+5, \dots, -3, -1) \quad \text{if } n = 2k+1,\ k \neq 1\\
&\lambda(s) = (s) \quad \text{if } n = 3
\end{aligned}$$

whith respect to $\{e_i : i = 1, \dots, k\}$, the standard basis of $\mathbb{C}^k$.

Lemma B.1. *Let $s \in \mathbb{C}$, then $E_{\delta,s} \subset E_\delta(G/P_0, \lambda(s))$, $\delta \in \mathbb{Z}$.*

By the duality (B.1) we deduce

Corollary B.2. *Let $s \in \mathbb{C}$, then $E_{\delta,-s}$ is a quotient of $E_\delta(G/P_0, -\lambda(s))$ for $\delta \in \mathbb{Z}$.*

For $\lambda \in \mathfrak{a}_{0c}^*$ define

$$e(\lambda)^{-1} = \prod_{\alpha\in\Delta^+} \Gamma\left(1 + \frac{(\lambda,\alpha)}{2(\alpha,\alpha)}\right)\Gamma\left(\frac{1}{2} + \frac{(\lambda,\alpha)}{2(\alpha,\alpha)}\right),$$

where $(\cdot,\cdot)$ is the usual invariant bilinear form for $\mathrm{SO}(n,\mathbb{C})$.

$e(\lambda)$ is the denominator of Harish-Chandra's c-function, see [14]. The Killing form is up to a constant equal to the invariant bilinear form $(\cdot,\cdot)$ for $\mathrm{SO}(n,\mathbb{C})$, but this formula is independent of that constant.

The Iwasawa decomposition of G is given by $G = KA_0N_0$. Define

$$1_\lambda(kan) = e^{-(\lambda+\rho_0)\log a} \quad (k \in K, a \in A_0, n \in N_0),$$

then $1_\lambda \in E_0(G/P_0, \lambda)$.

Theorem B.3. *(Helgason [14]) 1_λ is a cyclic vector in $E_0(G/P_0, \lambda)$ if and only if $e(\lambda) \neq 0$.*

Let us call $\mathbb{Z}_{even}$ the set of even integers and $\mathbb{Z}_{odd}$ the set of odd integers.

Lemma B.4. *$e(-\lambda(-s)) \neq 0$ for*

a) for all $s \in \mathbb{C} \setminus (\mathbb{Z} \cap (-\infty, 0))$ when $n = 3, 5$

b) for all $s \in \mathbb{C} \setminus (\mathbb{Z}_{even} \cap (-\infty, 0))$ when $n = 4$

- *c) for all* $s \in \mathbb{C} \setminus (\mathbb{Z}_{even} \cap (-\infty, n-6])$ *when* n *is even* > 4
- *d) for all* $s \in \mathbb{C}$ *with* $s \notin \mathbb{Z}_{odd} \cap (-\infty, n-6]$ *and* $s \notin \mathbb{Z}_{even} \cap (-\infty, 0)$ *when* n *is odd* > 5.

Proof. Using the identification (B.2) one easily gets for $n = 2k$

$$-\frac{(\lambda(-s), e_i + e_j)}{(e_i + e_j, e_i + e_j)} = \begin{cases} -j - i + 2k & i \neq 1, j \neq k \\ \frac{s}{2} & i = 1, j = k \\ \frac{s+2(k-j)}{2} & i = 1, j \neq k \\ k - i & i \neq 1, j = k \end{cases},$$

$$-\frac{(\lambda(-s), e_i - e_j)}{(e_i - e_j, e_i - e_j)} = \begin{cases} j - i & i \neq 1, j \neq k \\ \frac{s}{2} & i = 1, j = k \\ \frac{s-2(k-j)}{2} & i = 1, j \neq k \\ k - i & i \neq 1, j = k \end{cases},$$

for $n = 2k + 1$ with $k \neq 1$

$$-\frac{(\lambda(-s), e_i + e_j)}{(e_i + e_j, e_i + e_j)} = \begin{cases} -i - j + 2k + 1 & i \neq 1, j \neq k \\ \frac{s+1}{2} & i = 1, j = k \\ \frac{s+2(k-j)+1}{2} & i = 1, j \neq k \\ k - i + 1 & i \neq 1, j = k \end{cases},$$

$$-\frac{(\lambda(-s), e_i - e_j)}{(e_i - e_j, e_i - e_j)} = \begin{cases} j - i & i \neq 1, j \neq k \\ \frac{s-1}{2} & i = 1, j = k \\ \frac{s-2(k-j)-1}{2} & i = 1, j \neq k \\ k - i & i \neq 1, j = k \end{cases},$$

$$-\frac{(\lambda(-s), e_i)}{(e_i, e_i)} = \begin{cases} 2(k - i) + 1 & i \neq 1, k \\ s & i = 1 \\ 1 & i = k \end{cases},$$

and for $n = 3$

$$-\frac{(\lambda(-s), e_1)}{(e_1, e_1)} = s,$$

and we obtain the result. □

Since also $G = KMAN$, the function $1_s(kma_{x+iy}n) = e^{-(s-\rho)y}$ ($k \in K$, $m \in M$, $y \in \mathbb{R}$ and $n \in N$) is well-defined and is in $E_{0,s}$. Corollary B.2 combined with the above results gives

Lemma B.5. 1_s *is cyclic in* $E_{0,s}$ *as soon as* $e(-\lambda(-s)) \neq 0$.

Corollary B.6. $E_{0,s}$ *is irreducible for*

- *a) for all* $s \in \mathbb{C} \setminus \mathbb{Z}_{even}$ *when* n *is even* > 4 *and for all* $s \in \mathbb{C} \setminus \mathbb{Z}^*_{even}$ *when* $n = 4$
- *b) for all* $s \in \mathbb{C} \setminus \mathbb{Z}^*$ *when* n *is odd* ≥ 3.

Proof. This follows easily from Lemma B.5 and the observation that $\pi_{0,s}$ is irreducible if and only if 1_s is cyclic in $E_{0,s}$ and 1_{-s} is cyclic in $E_{0,-s}$. □

Next we are going to study the irreducibility of $E_{1,s}$, this is done by applying the method of "translation of parameters" introduced by Wallach [43].

Consider the action of $\mathrm{SO}(n,\mathbb{C})$ on $\mathbb{C}^n$. Put

$$v_0 = (1,-i,0,\dots,0), \quad w_0 = (1,i,0,\dots,0).$$

$$\begin{aligned} &a_z \cdot v_0 = e^{-iz} v_0 \\ &a_z \cdot w_0 = e^{iz} w_0 \quad (z \in \mathbb{C}) \\ &m \cdot v_0 = v_0 \\ &m \cdot w_0 = w_0 \quad (m \in M) \\ &\overline{n} \cdot v_0 = v_0 \\ &n \cdot w_0 = w_0 \quad (n \in N, \overline{n} \in \overline{N}). \end{aligned}$$

Define

$$F_0(g) = (g \cdot v_0, w_0), \quad (g \in G)$$

whith $(\cdot,\cdot)$ defined as before. For any $v \in \mathbb{C}^n$ define the matrix coefficient

$$c_v(g) = (g \cdot v_0, v), \quad (g \in G).$$

The map $v \longrightarrow c_v$ is a G-equivariant injective linear map from $\mathbb{C}^n$ into $E_{-1,-1-\rho}$.

Lemma B.7. *(Wallach [43], Ch. 8.13.9) $F_0 \cdot 1_s$ is cyclic in $E_{-1,s-1}$ as soon as 1_s is cyclic in $E_{0,s}$.*

Proof. Since $G = KMAN$ the G-invariant subspace generated by $F_0 \cdot 1_s$ clearly contains $[\pi_{o,s}(G)1_s] \cdot F_0$. Hence if $f \in E_{-1,s-1}$ is orthogonal to $\pi_{-1,s-1}(G)(F_0 \cdot 1_s)$ then $f \cdot F_0 = 0$, so $f = 0$.□

To obtain $\delta = -n$ in the above lemma we have to do the same as before but taking $F_0(g)^n$ and if we want $\delta = n$ we take $\overline{F_0(g)^n}$. Calling

$$F_\delta(g) = \begin{cases} F_0(g)^n & \text{if } \delta = -n \\ \overline{F_0(g)^n} & \text{if } \delta = n \end{cases}.$$

We obtain

Lemma B.8. *$F_\delta \cdot 1_s$ is cyclic in $E_{\delta,s+\delta}$ as soon as 1_s is cyclic in $E_{0,s}$.*

For the main corollary we need another lemma.

Lemma B.9. *$E_{\delta,s}$ is irreducible if $F_\delta \cdot 1_{s-\delta}$ is cyclic in $E_{\delta,s}$ and $F_\delta \cdot 1_{-s-\delta}$ is cyclic in $E_{\delta,-s}$.*

Proof. We assume that $\delta = -n$, and consider an invariant subspace $V \subset E_{\delta,s}$.

Any invariant subspace under G, is invariant under K. Recall that F_δ, in this case, is equal to $F_0(g)^n = (g \cdot (v_0 \otimes v_0 \otimes \cdots \otimes v_0), (w_0 \otimes w_0 \otimes \cdots \otimes w_0))$, and we have

$$\mathrm{span}\{K - \text{translates of } F_\delta\} \simeq \mathrm{span}\{K \cdot (w_0 \otimes w_0 \otimes \cdots \otimes w_0)\} \subset \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n.$$

We will see that the space $\mathrm{span}\{K - \text{translates of } F_\delta\}$ is irreducible and occurs once. If we prove that there is, up to scalars, only one vector v in $\mathrm{span}\{K \cdot (w_0 \otimes w_0 \otimes \cdots \otimes w_0)\}$ such that

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & h \end{pmatrix} v = e^{in\theta} v \tag{B.3}$$

with $h \in \mathrm{SO(n-2,\mathbb{R})}$, we have proven, applying Frobenius reciprocity theorem, both statements. The vector $w_0 \otimes w_0 \otimes \cdots \otimes w_0$ is, up to scalars, the only vector that satisfies (B.3). Therefore $\mathrm{span}\{K - \text{translates of } F_\delta\}$ is either in V or in $E_{\delta,s}/V$, hence by the assumptions of the lemma $V = E_{\delta,s}$ or $V = (0)$, so $E_{\delta,s}$ is irreducible.

For $\delta = n$ we have to do the same. □

Corollary B.10. *Let us consider $s \in \mathbb{C}$ and $\delta \in \mathbb{Z}$.*

a) If $n \geq 3$ and n odd, $E_{\delta,s}$ is irreducible when $s \notin \mathbb{Z}$.

b) If $n \geq 4$ and n even, $E_{\delta,s}$ is irreducible when $s - \delta \notin \mathbb{Z}_{even}$.

For the proof we have to apply the two last lemmas.

B.3 Unitarity

When are our representations $\pi_{\delta,s}$ unitary? We know that our representations are unitary if $s \in i\mathbb{R}$ and we want to compute when $\pi_{0,s}$ with $s \in \mathbb{R}$ is unitary. To do this we need to construct an inner product and to see when it is positive definite. Let $A_{0,s}$ be the intertwining operator between $E_{0,s}$ and $E_{0,-s}$ defined in Section 8.3.1. We have

$$A_{0,s} 1_s = c(0,s) 1_{-s}$$

where $c(0,s) = -2^\rho \frac{\Gamma(\frac{n}{2})\Gamma(\frac{1+\rho}{2})\Gamma(\frac{-s-\rho+2}{2})}{\sqrt{\pi}\Gamma(\frac{-s+\rho}{2})s}$. Let us call

$$\mathcal{A}_{0,s} = \frac{A_{0,s}}{c(0,s)},$$

then $\mathcal{A}_{0,s}$ is well defined for $s < \rho$.

We take n odd then we know that $\mathcal{A}_{0,0} = I$ on $E_{0,0}$ since $\pi_{0,0}$ is an irreducible representation on $E_{0,0}$. On each K-type we have that

$$\begin{aligned} &\mathcal{A}^*_{0,s} = \mathcal{A}_{0,s} \\ &\mathcal{A}_{0,-s}\mathcal{A}_{0,s} = I \quad \text{for } -1 < s < 1, \end{aligned}$$

what means that $\mathcal{A}_{0,s}$ is Hermitian and invertible on each K-type. To obtain the second equality we apply that $\pi_{0,s}$ is irreducible for $-1 < s < 1$ see Corollary B.6. This is the reason for requiring the bound being equal to 1.

Now we need a lemma.

Lemma B.11. *(Knapp [19], Lemma 16.1) Let $F(x)$ be a continuous function from a topological space X to the vector space of n-by-n Hermitian matrices such that $F(x_0)$ is positive definite for some x_0 and such that* $\det F(x)$ *is nonvanishing for x in a dense connected subset Y. Then $F(x)$ is positive definite for x in Y, and $F(x)$ is positive semidefinite for all x in X.*

Applying the latter lemma with n odd, $X = \{s \in \mathbb{R} : -1 < s < 1\}$, $Y = X$ and $F(x) = \mathcal{A}_{0,s}|_{K-type}$ we obtain that $< \mathcal{A}_{0,s}\varphi, \psi >$ is positive definite for $-1 < s < 1$. We have proven that $\pi_{0,s}$ is unitary for $-1 < s < 1$ if n is odd.

We can do the same for the case $n = 4$ since the representation $\pi_{0,0}$ and $\pi_{0,s}$ for $-2 < s < 2$ are irreducible, see Corollary B.6. We obtain that $\pi_{0,s}$ is unitary for $-2 < s < 2$ if $n = 4$.

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Samenvatting

De resultaten die we in dit proefschrift presenteren behoren tot het gebied van de harmonische analyse en zijn sterk beinvloed door ontwikkelingen in de natuurkunde, vooral in de quantummechanica. We houden ons vooral bezig met eigenschappen van een bijzondere unitaire representatie, de metaplectische representatie.

De metaplectische representatie –ook bekend als oscillator representatie, harmonische representatie, oftewel Segal-Shale-Weil representatie– is een dubbel-waardige unitaire representatie van de symplectische groep $Sp(n, \mathbb{R})$ op $L^2(\mathbb{R}^n)$.

Is, in het algemeen G een Lie groep en $\mathcal{H}$ een Hilbert ruimte waar G unitair op werkt, dan verstaan we onder een unitaire representatie van G een homomorphisme π van G in de groep van unitaire operatoren op de Hilbert ruimte $\mathcal{H}$ zodat de afbeelding $g \rightarrow \pi(g)x$ continu is van G naar $\mathcal{H}$ voor elke x in $\mathcal{H}$. Een van de centrale onderwerpen in de harmonische analyse is het ontbinden van de representatie van G op $\mathcal{H}$ in irreducibele representaties: het vinden van de zogenaamde Plancherel formule voor de gegeven G-actie.

In dit proefschrift beschouwen we de ontbinding van de een-waardige oscillator representatie voor enkele ondergroepen van de symplectische groep, namelijk $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{O}(2n)$, $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{O}(\mathrm{p}, \mathrm{q})$ en $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SO}(n, \mathbb{C})$.

De belangrijkste resultaten zijn de Plancherel formule van deze representaties en de multipliciteitsvrije ontbinding van elke invariante Hilbert deelruimte van de ruimte van getempereerde distributies.

De ontbinding van een invariante Hilbert deelruimte van de ruimte van getempereerde distributies heet multipliciteitsvrij als voor elk paar minimale invariante Hilbert deelruimten $\mathcal{H}_1$ en $\mathcal{H}_2$, die voorkomen in de ontbinding en niet evenredig zijn, de irreducibele representaties op $\mathcal{H}_1$ en $\mathcal{H}_2$ niet equivalent zijn.

Een essentieel gereedschap dat wordt gebruikt voor de berekening van de Plancherel formule is een Fourier integraal operator, voor het eerst geintroduceerd door M. Kashiwara en M. Vergne.

Om de multipliciteitsvrije ontbinding te bewijzen hebben we gebruik gemaakt van een criterium van Thomas.

Curriculum Vitae

Sofía Aparicio Secanellas was born in June 22$^{\mathrm{nd}}$, 1977, in Zaragoza, Spain. She completed primary and started secondary studies in Escatrón, a small village in the province of Zaragoza facing the Ebro river. She finished secondary studies in the city of Zaragoza, where she started her studies in mathematics. During the last year of her studies she got an Erasmus scholarship from the European Union to go to Naples, Italy. She graduated in Applied Mathematics at the University of Zaragoza in 2001. From January 2002 until October 2005, she did her PhD research under supervision of prof. dr. Gerrit van Dijk at the Mathematical Institute of the University of Leiden, The Netherlands, which resulted in this thesis.

S'ha feito de nuey

S'ha feito de nuey
tu má guardas ya
lo peito me brinca
tornate a besar.

Lo nuestro querer
no se crebará
aunque charren muito
y te fagan plorar.

Yo no quiero vier
güellos de cristal
mullaus por glarimas
que culpa no hay.

Escuita muller
disxa de plorar
yo siempre e'stau tuyo
tu mia has d'estar.

Dicen qu'en querer
ye de dos no mas
y que ye mas facil
ferlo caminar
cuando l'uno caye
l'otro a devantar,
cuando l'uno caye
l'otro a devantar.

S'ha feito de nuey
tu m'a guarda ya
lo peito me brinca
te quiero besar.

—Jota en fabla, antigua lengua aragonesa, cantada por mis primas Maria Luisa y Gloria.