Geometry and arithmetic of del Pezzo surfaces of degree 1

Winter, R.L.

Author: Winter, R.L.
Title: Geometry and arithmetic of del Pezzo surfaces of degree 1
Issue Date: 2021-01-05

Stellingen

behorende bij het proefschrift

Geometry and arithmetic of del Pezzo surfaces of degree 1

van Rosa Winter

For a del Pezzo surface S of degree 1, we denote by K_S its canonical divisor, and by $\mathcal{E}_S$ the surface obtained by blowing up the base point of the linear system $|-K_S|$. This surface admits an elliptic fibration $\nu\colon \mathcal{E}_S \longrightarrow \mathbb{P}^1$ coming from the map $S \dashrightarrow \mathbb{P}^1$ induced by $|-K_S|$.

1. (Theorem 2.2.1). Let k be a number field, $A, B \in k^*$, and S the del Pezzo surface of degree 1 over k in the weighted projective space $\mathbb{P}(2,3,1,1)$ with coordinates $(x : y : z : w)$ defined by

   $$y^2 = x^3 + Az^6 + Bw^6.$$

   The set $S(k)$ of k-rational points on S is dense in S with respect to the Zariski topology if and only if S contains a k-rational point with non-zero z, w coordinates, such that the corresponding point on $\mathcal{E}_S$ lies on a smooth fiber of ν, and is non-torsion on that fiber.

2. (Theorems 4.1.1 and 4.1.2). Let S be a del Pezzo surface of degree 1, and P a point on S. The number of exceptional curves on S that contain P is at most 16 in characteristic 2, at most 12 in characteristic 3, and at most 10 otherwise.

3. (Theorem 5.1.1). Let S be a del Pezzo surface of degree 1. If at least 9 exceptional curves on S all contain the same point P, then the point on $\mathcal{E}_S$ corresponding to P is torsion on its fiber of ν.

Let Γ be the weighted graph of which the vertices are the 240 roots in the E_8 root system, where two distinct vertices are connected by an edge of weight w if the corresponding roots have dot product w, and where no vertex is connected by an edge to itself.

4. (Theorem 3.1.3 (iii)). For each subset c of the set $\{-2, -1, 0, 1\}$ of weights of the edges of Γ, let Γ_c be the subgraph of Γ consisting of the same vertices as in Γ, and all edges of Γ with weights in c. For all $c \neq \{-1, 0, 1\}$, two complete subgraphs in Γ_c that are maximal with respect to inclusion are isomorphic as weighted graphs if and only if they are conjugate under the action of the automorphism group of Γ.

5. A weighted graph is defined to be *k-ultrahomogeneous* if every isomorphism between two of its induced weighted subgraphs of at most k vertices can be extended to an automorphism of the whole weighted graph. The graph Γ is 3-ultrahomogeneous, but not 4-ultrahomogeneous.

6. Let $Q_1, \ldots, Q_8$ be eight points in $\mathbb{P}^2$ in general position, i.e., no three are on a line, no six are on a conic, and no eight are on a cubic that is singular at one of them. Let $\mathcal{C}$ be the pencil of cubics through these points, and denote by Q the unique base point of $\mathcal{C}$. Let C_1 and C_2 be two conics that together contain all 8 points and that intersect in two of them, say Q_i, Q_j, and let D be a curve of degree 5 that contains all 8 points and is singular in all of them except in Q_i and Q_j. If C_1, C_2, and D intersect in a ninth point $P \neq Q$, then the point P has order 3 on the cubic in $\mathcal{C}$ that contains P and whose zero point is Q.

In what follows, k is a number field, and $\overline{k}$ an algebraic closure of k.

7. *Joint with V. Cantoral-Farfán, A. Garbagnati, C. Salgado, and A. Trbović.*

   Let R be a geometrically rational relatively minimal elliptic surface over k with no singular fibers of additive type, and with geometric Mordell–Weil rank 0. Let m be the order of the geometric Mordell–Weil group. The following hold.

   - If m is odd and R has a unique reducible fiber over $\overline{k}$, then R can be contracted over k to $\mathbb{P}^1 \times \mathbb{P}^1$.
   - If m is odd and R has at least two reducible fibers over $\overline{k}$, then R can be contracted over k to $\mathbb{P}^2$.
   - If m is even, then R can be contracted over k to $\mathbb{P}^1 \times \mathbb{P}^1$.

8. *Joint with V. Cantoral-Farfán, A. Garbagnati, C. Salgado, and A. Trbović.*

   Let R be a geometrically rational relatively minimal elliptic surface over k with geometric Mordell–Weil rank 0 and exactly one reducible fiber over $\overline{k}$, which is of type I_9. Let X be a K3 surface obtained as a double cover of R branched in two smooth $\mathrm{Gal}(\overline{k}/k)$-conjugate fibers. We call two elliptic fibrations on X *equivalent* if for each singular fiber type F they have the same number of fibers of type F over $\overline{k}$, and their geometric Mordell–Weil groups are isomorphic. There are exactly 12 equivalence classes of elliptic fibrations on X, and for each elliptic fibration on X, its geometric Mordell–Weil group is defined over at most a quartic extension of k.