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Stellingen

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Geometry and arithmetic of del Pezzo surfaces of degree 1

van Rosa Winter

For a del Pezzo surface S of degree 1, we denote by K_S its canonical divisor, and by $\mathcal{E}_S$ the surface obtained by blowing up the base point of the linear system $|-K_S|$. This surface admits an elliptic fibration $\nu: \mathcal{E}_S \rightarrow \mathbb{P}^1$ coming from the map $S \dashrightarrow \mathbb{P}^1$ induced by $|-K_S|$.

1. (Theorem 2.2.1). Let k be a number field, $A, B \in k^*$, and S the del Pezzo surface of degree 1 over k in the weighted projective space $\mathbb{P}(2, 3, 1, 1)$ with coordinates $(x : y : z : w)$ defined by

$$y^2 = x^3 + Az^6 + Bw^6.$$

The set $S(k)$ of k -rational points on S is dense in S with respect to the Zariski topology if and only if S contains a k -rational point with non-zero z, w coordinates, such that the corresponding point on $\mathcal{E}_S$ lies on a smooth fiber of ν , and is non-torsion on that fiber.

2. (Theorems 4.1.1 and 4.1.2). Let S be a del Pezzo surface of degree 1, and P a point on S . The number of exceptional curves on S that contain P is at most 16 in characteristic 2, at most 12 in characteristic 3, and at most 10 otherwise.
3. (Theorem 5.1.1). Let S be a del Pezzo surface of degree 1. If at least 9 exceptional curves on S all contain the same point P , then the point on $\mathcal{E}_S$ corresponding to P is torsion on its fiber of ν .

Let Γ be the weighted graph of which the vertices are the 240 roots in the E_8 root system, where two distinct vertices are connected by an edge of weight w if the corresponding roots have dot product w , and where no vertex is connected by an edge to itself.

4. (Theorem 3.1.3 (iii)). For each subset c of the set $\{-2, -1, 0, 1\}$ of weights of the edges of Γ , let Γ_c be the subgraph of Γ consisting of the same vertices as in Γ , and all edges of Γ with weights in c . For all $c \neq \{-1, 0, 1\}$, two complete subgraphs in Γ_c that are maximal with respect to inclusion are isomorphic as weighted graphs if and only if they are conjugate under the action of the automorphism group of Γ .

5. A weighted graph is defined to be *k-ultrahomogeneous* if every isomorphism between two of its induced weighted subgraphs of at most k vertices can be extended to an automorphism of the whole weighted graph. The graph Γ is 3-ultrahomogeneous, but not 4-ultrahomogeneous.
6. Let $Q_1, \dots, Q_8$ be eight points in $\mathbb{P}^2$ in general position, i.e., no three are on a line, no six are on a conic, and no eight are on a cubic that is singular at one of them. Let $\mathcal{C}$ be the pencil of cubics through these points, and denote by Q the unique base point of $\mathcal{C}$. Let C_1 and C_2 be two conics that together contain all 8 points and that intersect in two of them, say Q_i, Q_j , and let D be a curve of degree 5 that contains all 8 points and is singular in all of them except in Q_i and Q_j . If C_1, C_2 , and D intersect in a ninth point $P \neq Q$, then the point P has order 3 on the cubic in $\mathcal{C}$ that contains P and whose zero point is Q .

In what follows, k is a number field, and $\bar{k}$ an algebraic closure of k .

7. *Joint with V. Cantoral-Farfán, A. Garbagnati, C. Salgado, and A. Trbović.*

Let R be a geometrically rational relatively minimal elliptic surface over k with no singular fibers of additive type, and with geometric Mordell–Weil rank 0. Let m be the order of the geometric Mordell–Weil group. The following hold.

- If m is odd and R has a unique reducible fiber over $\bar{k}$, then R can be contracted over k to $\mathbb{P}^1 \times \mathbb{P}^1$.
- If m is odd and R has at least two reducible fibers over $\bar{k}$, then R can be contracted over k to $\mathbb{P}^2$.
- If m is even, then R can be contracted over k to $\mathbb{P}^1 \times \mathbb{P}^1$.

8. *Joint with V. Cantoral-Farfán, A. Garbagnati, C. Salgado, and A. Trbović.*

Let R be a geometrically rational relatively minimal elliptic surface over k with geometric Mordell–Weil rank 0 and exactly one reducible fiber over $\bar{k}$, which is of type I_9 . Let X be a K3 surface obtained as a double cover of R branched in two smooth $\text{Gal}(\bar{k}/k)$ -conjugate fibers. We call two elliptic fibrations on X *equivalent* if for each singular fiber type F they have the same number of fibers of type F over $\bar{k}$, and their geometric Mordell–Weil groups are isomorphic. There are exactly 12 equivalence classes of elliptic fibrations on X , and for each elliptic fibration on X , its geometric Mordell–Weil group is defined over at most a quartic extension of k .