Geometry and arithmetic of del Pezzo surfaces of degree 1

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Summary

This thesis contains results on the arithmetic and geometry of del Pezzo surfaces of degree 1. These are exactly the smooth surfaces in the weighted projective space $\mathbb{P}(2,3,1,1)$ with coordinates x, y, z, w given by an equation of the form

$$y^2 + a_1(z,w)xy + a_3(z,w)y = x^3 + a_2(z,w)x^2 + a_4(z,w)x + a_6(z,w),$$

where $a_i \in k[z,w]$ is homogeneous of degree i. Such a surface contains 240 curves with negative self-intersection, called exceptional curves.

In Chapter 1 we give the necessary background, assuming the reader is familiar with algebraic geometry. Two main points that we cover are the elliptic surface that is constructed from a del Pezzo surface of degree 1 by blowing up the base point of the anticanonical linear system, and the connection between the exceptional curves on a del Pezzo surface of degree 1 and the $\mathbf{E}_8$ root system.

In Chapter 2, which is joint work with Julie Desjardins, we prove that for a del Pezzo surface S over a number field k, of the form

$$y^2 = x^3 + Az^6 + Bw^6$$

with $A, B \in k$ non-zero, the set $S(k)$ of k-rational points on S is dense with respect to the Zariski topology if and only if S contains a point with non-zero z, w coordinates such that the corresponding point on the elliptic surface constructed from S lies on a smooth fiber and is non-torsion on that fiber. We do this by constructing an infinite family of multisections, and showing that at least one of them has infinitely many k-rational points. This is the first result that gives necessary and sufficient conditions for the set of k-rational points of this family to be Zariski-dense, where k is any number field.

In Chapter 3, which is an adaptation of the preprint [vLWa], we study the action of the Weyl group W_8 on the $\mathbf{E}_8$ root system. The 240 roots in $\mathbf{E}_8$ are in one-to-one correspondence with the 240 exceptional curves on a del Pezzo surface of degree 1, and we use results from this chapter in Chapters 4 and 5. However, this chapter is also interesting for the reader that wants to know about the $\mathbf{E}_8$ root system without any interest in del Pezzo surfaces of degree 1. We define the complete weighted graph Γ where each vertex represents a root, and two vertices are connected by an edge of weight w if the corresponding roots have dot product w. The group of symmetries of Γ is the Weyl group W_8. We prove that for a large class of subgraphs of Γ, any two subgraphs from this class are isomorphic if and only if there is a symmetry of Γ that maps one to the other. We also give invariants that determine the isomorphism type of a subgraph. Moreover, we show that for two isomorphic subgraphs G_1, G_2 from this class that do not contain one of 7 specific subgraphs, any isomorphism between G_1 and G_2 extends to a symmetry of the whole graph Γ. These results reduce computations on the graph Γ significantly.

In Chapter 4, which is an adaptation of the preprint [vLWb], we study the configurations of the 240 exceptional curves on a del Pezzo surface of degree 1, using results from Chapter 3. We prove that a point on a del Pezzo surface of degree 1 is contained in at most 16 exceptional curves in characteristic 2, at most 12 exceptional curves in characteristic 3, and at most 10 exceptional curves in all other characteristics. We give examples that show that the upper bounds are sharp in all characteristics, except possibly in characteristic 5.

Finally, in Chapter 5 we show that if at least 9 exceptional curves intersect in a point on a del Pezzo surface S of degree 1, the corresponding point on the elliptic surface constructed from S is torsion on its fiber. This is less trivial than some experts thought. We use a list of all possible configurations of at least 9 pairwise intersecting exceptional curves computed in Chapter 3, and with an example from Chapter 4 we show that the analogue statement is false for 6 or fewer exceptional curves.