Geometry and arithmetic of del Pezzo surfaces of degree 1

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Exceptional curves and torsion points

The del Pezzo surface of degree 1 in Example 4.5.5 contains a point that is contained in the intersection of 10 exceptional curves, and whose corresponding point on the elliptic surface associated to the del Pezzo surface (obtained by blowing up the base point of the anticanonical linear system, see Section 1.4.3) is torsion on its fiber. This example comes from [SvL14, Section 4], where we find several examples of a point on a del Pezzo surface of degree 1 that is contained in the intersection of at least 6 exceptional curves, and, in all cases, corresponds to a point that is torsion on its fiber. Moreover, we do not know any example of a point that is contained in more than 6 exceptional curves and that corresponds to a point that is *not* torsion on its fiber. A natural question is therefore whether a point on a del Pezzo surface of degree 1 that is contained in 'many' exceptional curves always corresponds to a point that is torsion on its fiber (where 'many' of course needs to be specified). In this final and short chapter we give a positive answer to this question where we take 'many' to be 9 (Theorem 5.1.1), using results from Chapter 3. We also show that if we take 'many' to be 6, the answer to this question is negative in most characteristics, by providing a counterexample that comes from Chapter 4 (Example 5.1.5). Computations were done in `magma` [BCP97], and the code that we used can be found in [Code].

5.1 Main results

Let S be a del Pezzo surface of degree 1 with canonical divisor K_S, and let $\mathcal{E}$ be the associated elliptic surface obtained by blowing up the basepoint $\mathcal{O}$ of the linear system $|-K_S|$. For a point P in $S \setminus \{\mathcal{O}\}$, we denote by $P_{\mathcal{E}}$ the corresponding point on $\mathcal{E}$, and by the fiber of $P_{\mathcal{E}}$ we mean the fiber of the elliptic fibration $\mathcal{E} \longrightarrow \mathbb{P}^1$ that contains $P_{\mathcal{E}}$. The main result of this chapter is the following.

Theorem 5.1.1. *If at least 9 exceptional curves on S are concurrent in a point P, then $P_{\mathcal{E}}$ is torsion on its fiber.*

Remark 5.1.2. For del Pezzo surfaces of degree 2, the situation is simpler, and a result similar to our theorem is known [Kuw05, Proposition 7.1]. A del Pezzo surface of degree 2 is a double cover of $\mathbb{P}^2$ ramified along a smooth quartic curve. On such a surface, a point is contained in at most 4 exceptional curves, and this happens exactly when its projection to $\mathbb{P}^2$ is in the intersection of 4 bitangents of the quartic curve. In [Kuw05], Kuwata gives a construction for an elliptic surface by blowing up twice on the del Pezzo surface, and he shows that for a point contained in 4 exceptional curves, the corresponding point on the elliptic surface is torsion on its fiber. The situation in Theorem 5.1.1 is more complex, since there are a priori many different sets of 9 or more exceptional curves on a del Pezzo surface of degree 1 that can be concurrent in a point.

Remark 5.1.3. Theorem 5.1.1 seems intuitively true by the following argument, which was pointed out to us by several people. Let P be a point on S that is contained in at least 9 exceptional curves, say $L_1, \ldots, L_n$. These curves correspond to sections $\tilde{L}_1, \ldots, \tilde{L}_n$ of the elliptic surface $\mathcal{E}$ associated to S (Remark 1.4.20), which in turn correspond to elements in the Mordell–Weil group of $\mathcal{E}$ (i.e., the Mordell–Weil group of the generic fiber, which is an elliptic curve over the function field $k(t)$ of $\mathbb{P}^1$). This Mordell–Weil group has rank at most 8 over k (Remark 1.4.17), so in this group there must be a relation $a_1\tilde{L}_1 + \cdots + a_n\tilde{L}_n = 0$, where $a_1, \ldots, a_n \in \mathbb{Z}$ are not all zero. Since all n exceptional curves contain the point P, this specializes to $(a_1 + \cdots + a_n)P_{\mathcal{E}} = 0$ on the fiber of P on $\mathcal{E}$. If one reasons too quickly, it seems that this proves that $P_{\mathcal{E}}$ is torsion of order dividing $a_1 + \cdots + a_n$ on its fiber. However, it might be the case that $a_1 + \cdots + a_n = 0$, so this does not prove Theorem 5.1.1. The key part in our proof is therefore that we show, using results from Chapter 3, that there is always a relation between $\tilde{L}_1, \ldots, \tilde{L}_n$ in the Mordell–Weil group of $\mathcal{E}$ that specializes to a *non-trivial* relation on the fiber of $P_{\mathcal{E}}$; see Lemma 5.2.2.

Remark 5.1.4. Recall that S can be embedded in the weighted projective space $\mathbb{P}(2,3,1,1)$ as the set of solutions to the equation

$$y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6 = 0, \tag{5.1}$$

where $a_i \in k[z,w]$ is homogeneous of degree i for each i in $\{1,\ldots,6\}$. The linear system $|-2K_S|$ of the bi-anticanonical divisor of S induces a morphism φ, which is the composition of the projection to $\mathbb{P}(2,1,1)$ and the 2-uple embedding in $\mathbb{P}^3$; this morphism realizes S as a double cover of a cone in $\mathbb{P}^3$ ramified over a sextic curve (see also Section 1.4.1). It follows that points on S that are on the ramification curve of φ correspond to points on $\mathcal{E}$ that are 2-torsion on their fiber.

The following example shows that if S is defined over a field of characteristic 0, for a point P on S that is contained in 6 exceptional curves, the point $P_{\mathcal{E}}$ is not guaranteed to be torsion on its fiber.

Example 5.1.5. Let k be a field of characteristic 0, and consider the eight points in $\mathbb{P}^2_k$ given by

$$\begin{aligned} P_1 &= (1:0:1); & P_2 &= (889:0:823);\\ P_3 &= (2600:101:2551); & P_4 &= (325:12:287);\\ P_5 &= (0:1:1); & P_6 &:= (0:-1:1);\\ P_7 &= (4005:2464:3499); & P_8 &= (195:22:-113). \end{aligned}$$

We check that these points are in general position, by verifying that the determinants of the matrices in Lemma 3.3.12 that determine whether three of the points are on a line, or six of the points are on a conic, or seven of them are on a cubic that is singular at one of them, are non-zero. Let X be the blow-up of $\mathbb{P}^2$ in these points, which is a del Pezzo surface of degree 1. Let L_1 be the line through P_1 and P_2, which is given by $y = 0$, and let L_2 be the line through P_3 and P_4, which is given by $51y = x + z$. Finally, let C_1 be the conic through P_1, P_3, P_5, P_6, P_7, let C_2 be the conic through P_1, P_4, P_5, P_6, P_8, let C_3 be the conic through P_2, P_3, P_5, P_7, P_8, and C_4 the conic through P_2, P_4, P_6, P_7, and P_8. Note that $L_1, L_2, C_1, \ldots, C_4$ are 6 of the 10 curves in Proposition 4.4.6; using the proof of this proposition, we chose $P_1, \ldots, P_8$ such that these 6 curves are concurrent in a point. The conics $C_1, \dots, C_4$ are defined by the following equations.

$$\begin{aligned}
&C_1\colon x^2 + y^2 - z^2 = xy;\\
&C_2\colon x^2 + y^2 - z^2 = 6xy;\\
&C_3\colon 823x^2 - 1884xy - 66xz - 3739y^2 + 4628yz - 889z^2;\\
&C_4\colon 823x^2 - 4038xy - 66xz + 3139y^2 + 2250yz - 889z^2.
\end{aligned}$$

Indeed, the curves $L_1, L_2, C_1, \dots, C_4$ all contain the point $(-1 : 0 : 1)$, so the strict transforms of these six curves, which are exceptional curves on X, are concurrent in a point P on X. Let $\mathcal{C}$ be the pencil of cubics through $P_1, \dots, P_8$. This has a unique base point, which is

$$B = (3453493845425 : -16508630016087 : 20919196389638).$$

The fiber of $P_{\mathcal{E}}$ on the elliptic surface $\mathcal{E}$ is given by the element of $\mathcal{C}$ that contains P, and it is an elliptic curve with base point B. With `magma` it is quick to check that the point $P_{\mathcal{E}}$ is non-torsion on its fiber; see [Code] for the code that we used.

Remark 5.1.6. The previous example also holds if the characteristic of k is p for all but a finite number of primes p. In fact, the only characteristics for which this does not hold are the ones for which $P_1, \dots, P_8$ are not in general position, which form a set of 42 primes. Using the proof of Proposition 4.4.6, it is not hard to generate similar examples that hold in some of those 42 characteristics; for example, the eight points in $\mathbb{P}^2$ given by

$$\begin{aligned}
&Q_1 = (1 : 0 : 1); && Q_5 = (0 : 1 : 1);\\
&Q_2 = (-236857 : 0 : 402962); && Q_6 = (0 : -1 : 1);\\
&Q_3 = (666 : 5 : -301); && Q_7 = (-2337353334 : 1829935 : 2432407789);\\
&Q_4 = (222 : 5 : 143); && Q_8 = (-101872359 : 3659870 : 141722269);
\end{aligned}$$

are in general position in all but 55 characteristics, and this gives, together with Example 5.1.5, an example of six exceptional curves that are concurrent in a point P such that $P_{\mathcal{E}}$ is not torsion on its fiber for each characteristic except for 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 71, 101, and 113.

From Theorem 5.1.1 and Example 5.1.5 it is clear that there are still open questions: if a point P on S is contained in 7 exceptional curves, is the point $P_{\mathcal{E}}$ then torsion on its fiber? And what about points contained in 8 exceptional curves? We have not yet found a proof nor a counterexample to these questions.

5.2 Proof of the main theorem

In this section we prove Theorem 5.1.1. We first describe a pairing on the Mordell–Weil group of $\mathcal{E}$, and use this pairing to state and prove two lemmas.

Let $L_1, \ldots, L_n$ be at least 9 exceptional curves on S that are concurrent in a point P that lies outside the ramification curve of φ. Let $\tilde{L}_1, \ldots, \tilde{L}_n$ be the corresponding sections on $\mathcal{E}$. Let $\langle \cdot, \cdot \rangle_h$ be the symmetric and bilinear pairing on the Mordell–Weil group of $\mathcal{E}$ as defined in [Shi90, Theorem 8.4]; that is, for C_1, C_2 in $E(k(t))$, we have $\langle C_1, C_2 \rangle_h = -(\varphi_h(C_1) \cdot \varphi_h(C_2))$, where $\varphi_h \colon E(k(t)) \longrightarrow \operatorname{Pic} \mathcal{E}$ is the map given in [Shi90, Lemmas 8.1 and 8.2], and $\cdot$ is the intersection pairing in the Picard group of $\mathcal{E}$. We call $\langle \cdot, \cdot \rangle_h$ the *height pairing* on $E(k(t))$.

Lemma 5.2.1. *For two exceptional curves in Pic S, the height pairing of the corresponding sections in the Mordell–Weil group of $\mathcal{E}$ is the same as the dot product of the roots in the root system $\boldsymbol{E}_8$ associated to these exceptional curves under the bijection in Remark 1.4.9.*

Proof. Let C_1, C_2 be two sections of $\mathcal{E}$ that are strict transforms of exceptional curves c_1, c_2 in S. Since $\mathcal{E}$ has no reducible fibers, by [Shi90, Lemma 8.1] we have

$$\varphi_h(C_1) \cdot \varphi_h(C_2) = ([C_1] - [\tilde{\mathcal{O}}] - F) \cdot ([C_2] - [\tilde{\mathcal{O}}] - F),$$

where $[C_1], [C_2], [\tilde{\mathcal{O}}]$ are the classes of C_1, C_2, and the zero section, respectively, and F is the class of a fiber. This gives

$$\varphi_h(C_1) \cdot \varphi_h(C_2) = [C_1] \cdot [C_2] - 1,$$

where we use that the zero section is an exceptional curve, and it is disjoint from C_1 and C_2. We conclude that we have $\langle C_1, C_2 \rangle_h = 1 - [C_1] \cdot [C_2]$. Since C_1, C_2 are disjoint from $\tilde{\mathcal{O}}$, the intersection pairing of C_1 and C_2 in Pic $\mathcal{E}$ is the same as the intersection pairing of c_1 and c_2 in Pic S. The statement now follows from the bijection in Remark 1.4.9. $\square$

Let M be the *height pairing matrix* of $L_1, \ldots, L_n$, that is, M is the $n \times n$ matrix with $M_{ij} = \langle \tilde{L}_i, \tilde{L}_j \rangle_h$ for $i, j \in \{1, \ldots, n\}$.

Lemma 5.2.2. *The kernel of the matrix M contains a vector $(a_1, \ldots, a_n)$ in $\mathbb{Z}^n$ with $a_1 + \cdots + a_n \neq 0$.*

Proof. Recall the complete weighted graphs G and Γ as defined in Definition 1.4.12. Since P lies outside the ramification curve of φ, the exceptional curves $L_1, \ldots, L_n$ correspond to a clique of size n in G that is contained in a maximal clique in G with only edges of weights 1 and 2 (Remark 4.2.5), which corresponds to a maximal clique C in $\Gamma_{\{-1,0\}}$ by the bijection given in Remark 1.4.13. Since $n \geq 9$, the clique C has size at least 9. The table in Appendix A contains all isomorphism types of maximal cliques in $\Gamma_{\{-1,0\}}$ of size at least 9 (Proposition 3.5.28); there are 11 maximal cliques of size 9, which we call $\alpha_1, \ldots, \alpha_{11}$ in the order that they appear in the table, there are 6 maximal cliques of size 10, which we call $\beta_1, \ldots, \beta_6$ in the order that they appear in the table, and there is 1 maximal clique of size 12, which we call γ. For each of these 18 cliques, whose elements correspond to roots in $\mathbf{E}_8$, we compute its Gram matrix, which is the matrix where the entry (i, j) is the dot product of the roots corresponding to the i-th and j-th vertex in the clique after choosing an ordering on the vertices. With `magma` we find the generators for the kernels of these matrices (see [Code]). The results are in Table 5.1. Let r be the number of vertices of C, and let N be the Gram matrix of C; then the kernel of N is equal to one of the 18 kernels in the table, after rearranging the order of the vertices in C if necessary. Since $n \geq 9$, we see from Table 5.1 that for any subset of n vertices in C, there is a vector $(a_1, \ldots, a_r)$ in the kernel of N which is 0 outside the entries corresponding to the n vertices, and such that $a_1 + \cdots + a_r \neq 0$. By Lemma 5.2.1, this gives a vector in the kernel of M as claimed. □

Proof of Theorem 5.1.1. Let P be a point on S. If P is contained in the ramification curve of the morphism induced by the linear system of the bi-anticanonical divisor, then $P_{\mathcal{E}}$ is torsion (Remark 5.1.4), and we are done. Now assume that P is not contained in this ramification curve, and that there is a set of at least 9 exceptional curves that are concurrent in P. Let $K_1, \ldots, K_n$ be the corresponding sections of $\mathcal{E}$, and let N be the height pairing matrix of these sections. Let $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ be a vector in the kernel of N such that $a_1 + \cdots + a_n \neq 0$, which exists by

Clique	Basis for the kernel
α_1	$\{(1,1,0,0,0,0,1,0,1),(0,0,1,1,1,1,0,2,0)\}$
α_2	$\{(1,0,1,0,0,1,0,0,1),(0,0,0,1,1,0,1,0,0)\}$
α_3	$\{(1,1,1,0,0,1,0,0,1),(0,0,0,1,1,0,1,1,0)\}$
α_4	$\{(1,1,0,1,0,0,1,0,1),(0,0,1,0,0,1,0,1,0)\}$
α_5	$\{(2,1,1,0,2,0,0,1,1),(0,0,0,1,0,1,1,0,0)\}$
α_6	$\{(1,1,1,1,1,1,1,1,1)\}$
α_7	$\{(1,1,1,0,1,1,1,1,1)\}$
α_8	$\{(0,1,1,2,2,2,1,1,0)\}$
α_9	$\{(2,1,1,1,1,2,2,2,2))\}$
α_{10}	$\{(2,2,0,3,1,4,2,3,1)\}$
α_{11}	$\{(6,3,1,4,4,2,2,5,3)\}$
β_1	$\{(1,0,1,0,0,2,1,0,0,1),(0,1,0,1,2,0,0,1,1,0)\}$
β_2	$\{(1,1,0,0,0,0,0,0,1,1),(0,0,0,0,1,1,1,1,0,0)\}$
β_3	$\{(1,1,0,1,0,0,0,1,0,1),(0,0,1,0,1,1,1,0,1,0)\}$
β_4	$\{(1,1,0,1,0,1,0,0,1,1),(0,0,0,0,1,0,1,1,0,0)\}$
β_5	$\{(1,1,0,0,0,0,0,0,1,1),(0,0,1,1,1,1,2,2,0,0)\}$
β_6	$\{(2,1,3,0,2,0,2,0,1,1),(0,0,0,1,0,1,0,1,0,0)\}$
γ	$\{(1,1,0,0,0,0,0,0,0,0,0,1),(0,0,1,0,0,1,0,0,0,0,1,0),$ $(0,0,0,1,0,0,0,1,1,0,0,0),(0,0,0,0,1,0,1,0,0,1,0,0)\}$

Table 5.1: Bases

Lemma 5.2.2. Then we have for all $i \in \{1,\ldots,n\}$ we have that

$$a_1\langle K_i, K_1\rangle_h + \cdots + a_n\langle K_i, K_n\rangle_h = 0,$$

and since the height pairing is bilinear this implies

$$\langle K_i, a_1K_1 + a_2K_2 + \cdots + a_nK_n\rangle_h = 0 \text{ for all } i \in \{1,\ldots,n\}, \tag{5.2}$$

which implies

$$\langle a_1K_1 + a_2K_2 + \cdots + a_nK_n, a_1K_1 + a_2K_2 + \cdots + a_nK_n\rangle_h = 0.$$

From the latter we conclude that $a_1K_1 + a_2K_2 + \cdots + a_nK_n$ is torsion in the Mordell–Weil group of $\mathcal{E}$ [Shi90, Theorem 8.4], and since the torsion subgroup is trivial [Shi90, Theorem 10.4], we conclude that

$$a_1K_1 + a_2K_2 + \cdots + a_nK_n = 0.$$

Since for all i in $\{1, \ldots, n\}$, the section K_i contains the point $P_{\mathcal{E}}$, we have, on the fiber of $P_{\mathcal{E}}$, the equality $(a_1 + \cdots + a_n) P_{\mathcal{E}} = 0$. Since $a_1 + \cdots + a_n \neq 0$, this implies that $P_{\mathcal{E}}$ is torsion on its fiber. $\square$