Author: Winter, R.L.
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Concurrent exceptional curves on del Pezzo surfaces of degree 1

This chapter is an adaptation of the preprint [vLWb], which is at the moment of this writing submitted for publication. Moreover, part of this chapter is already in the master thesis [Win14] by the same author. We decided to copy those parts here for completion. See Remark 4.1.3 for a comparison with [Win14].

Recall that a del Pezzo surface of degree d over an algebraically closed field contains a fixed number of exceptional curves, depending on d (Table 1.1). The configuration of these curves can play a role in arithmetic questions; we have seen this in Chapter 2. For example, one of the conditions on the point Q that is used to show that the set of rational points on a del Pezzo surface of degree 1 is dense in [SvL14], is for Q not to lie on 6 exceptional curves, if its order is 3 or 5. Another example is found in [STVA14, Corollary 18], where Salgado, Testa and Várilly-Alvarado show that a del Pezzo surface of degree 2 is unirational if and only if it contains a point that is not contained in 4 exceptional curves, and lies outside the ramification curve of the anticanonical map. In this chapter we study the configuration of the exceptional curves on a del Pezzo surface of degree 1, and determine the maximal number of these curves that can go through one point.

4.1 Main results

We call a set of exceptional curves *concurrent* in a point on the surface if that point is contained in all of them. It is well known that on del Pezzo surfaces of degree 3, the number of exceptional curves that are concurrent in a point is at most 3. This can be seen by looking at the graph on the 27 exceptional curves, where two vertices are connected by an edge if the corresponding exceptional curves intersect. For all del Pezzo surfaces of degree 3 this gives the same graph G. A set of concurrent exceptional curves corresponds in this way to a complete subgraph of G, and the maximal size of complete subgraphs in G is 3. On a del Pezzo surface of degree 2, the number of concurrent exceptional curves in a point is at most 4. As in the case for degree 3, this can be derived directly from the intersection graph on the 56 exceptional curves. A geometric argument why 4 is an upper bound is given in [TVAV09], in the proof of Lemma 4.1. An example where this upper bound is reached is given in [STVA14], Example 2.4. For del Pezzo surfaces of degree 1, the situation is more complex. Contrary to the case of del Pezzo surfaces of degree ≥ 2, for char $k \neq 2$, the maximal size of complete subgraphs of the intersection graph on the 240 exceptional curves, which we will show is 16, is not equal to the maximal number of exceptional curves that are concurrent in a point.

Let X be a del Pezzo surface of degree 1 over an algebraically closed field k, and let K_X be the canonical divisor on X. The linear system $|-2K_X|$ gives X the structure of a double cover of a cone Q in $\mathbb{P}^3$, ramified over a sextic curve that is cut out by a cubic surface (Section 1.4.1). Let φ be the morphism associated to this linear system. In this chapter we prove the following two theorems.

Theorem 4.1.1. *Let $P \in X(k)$ be a point on the ramification curve of φ. The number of exceptional curves that go through P is at most ten if char $k \neq 2$, and at most sixteen if char $k = 2$.*

Theorem 4.1.2. *Let $Q \in X(k)$ be a point outside the ramification curve of φ. The number of exceptional curves that go through Q is at most ten if char $k \neq 3$, and at most twelve if char $k = 3$.*

Using the ramification divisor of φ, we obtain with a simple geometrical argument an upper bound of 12 outside characteristic 2 for Theorem 4.1.1, which was pointed out to us by Niels Lubbes. An anonymous referee even suggested that with some more work, this same argument can be improved to give the upper bound of 10 outside characteristic 2. See Remark 4.3.1.

In [SvL14, Example 4.1], for any field of characteristic unequal to 2, 3, or 5, a del Pezzo surface of degree 1 is defined that contains a point outside the ramification curve that is contained in 10 exceptional curves. This shows that the upper bound for char $k \neq 2, 3, 5$ in Theorem 4.1.2 is sharp. In Section 4.5 we show in all characteristics except for characteristic 5 in the case of Theorem 4.1.2, that the upper bounds in Theorems 4.1.1 and 4.1.2 are sharp. Theorems 4.1.1 and 4.1.2 are proved by using results on the automorphism group of the graph on the 240 exceptional curves, and by Propositions 4.3.6 and 4.4.6, which are purely geometrical and show that certain curves in $\mathbb{P}^2$ do not go through the same point.

Remark 4.1.3. Most of the results in Section 4.3 are proved by the same author in the master thesis [Win14]; more specifically, Theorem 4.1.1 and Proposition 4.3.6 are equal to Theorem 1 and Proposition 4.22 in [Win14], and Lemma 4.3.4 is almost the same as Lemma 4.21 in [Win14]. We decided to include these results here for completeness.
In [Win14], Theorem 4.1.2 is stated for char $k = 0$. In this chapter we extend this to a result for all characteristics. Moreover, we added several geometrical arguments (Lemmas 4.4.8 – 4.4.13, Proposition 4.4.15), that heavily reduce the usage of `magma` in the proof of Proposition 4.4.6, which is key to Theorem 4.1.2.
Examples 4.5.1 and 4.5.2 are the same as Exmples 4.24 and 4.23 in [Win14], where it was shown that the upper bounds of Theorem 4.1.1 are sharp in characteristic 0. In Section 4.5 we give extra examples, showing that the upper bounds in Theorem 4.1.1 are sharp in all characteristics, and that the upper bounds in Theorem 4.1.2 are sharp except possibly in characteristic 5.

We use `magma` [BCP97] for our computations, which is the case only in Propositions 4.3.6 and 4.4.6. The proofs of Propositions 4.2.2, 4.4.2, 4.4.3, and 4.4.4 rely on results in Chapter 3 that also make use of `magma`.

We want to thank Niels Lubbes for useful discussions, and Igor Dolgachev for useful comments. We also want to thank an anonymous referee for

giving useful remarks that improved the quality of the paper, and a second anonymous referee for suggesting a shorter proof of the upper bound of 10 outside characteristic 2 on the ramification curve.

4.2 The weighted graph on exceptional classes

We use the same notation as in Definition 1.4.12 and in Chapter 3: we denote the set of exceptional classes in Pic X by I; by G we denote the complete weighted graph whose vertex set is I, and where the weight function is the intersection pairing in Pic X.

When two exceptional curves intersect in a point on X, their corresponding classes in Pic X are connected by an edge of positive weight in G. Therefore, an upper bound on the number of exceptional curves on X that are concurrent in a point is given by the maximal size of cliques in G that have only edges of positive weight. To study these cliques, we use the correspondence between the set I and the root system $\mathbf{E}_8$ as in Remark 1.4.9. In particular, if Γ is the weighted graph where the vertices are the roots in $\mathbf{E}_8$ and the weights are induces by de dot product in $\mathbf{E}_8$, there is an isomorphism of weighted graphs between G and Γ, that sends a vertex c in G to the corresponding vertex $c + K_X$ in Γ, and an edge $d = \{c_1, c_2\}$ in G with weight w to the edge $\delta = \{c_1 + K_X, c_2 + K_X\}$ in Γ with weight $1 - w$ (Remark 1.4.13). The different weights that occur in G are $0, 1, 2$, and 3, and they correspond to weights $1, 0, -1$, and -2, respectively, in Γ. From the bijection between Γ and G we immediately obtain the following results.

Lemma 4.2.1. *(i) Let e be an exceptional class. Then there is exactly one exceptional class f with $e \cdot f = 3$, there are 56 exceptional classes f with $e \cdot f = 0$, there are 126 exceptional classes f with $e \cdot f = 1$, and 56 exceptional classes f with $e \cdot f = 2$.*

(ii) For two exceptional classes e_1, e_2 with $e_1 \cdot e_2 = 2$, there is a unique exceptional class f such that $e_1 \cdot f = e_2 \cdot f = 2$.

(iii) For every pair e_1, e_2 of exceptional classes such that $e_1 \cdot e_2 = 1$, there are exactly 60 exceptional classes f with $e_1 \cdot f = e_2 \cdot f = 1$, and 32 exceptional classes f with $e_1 \cdot f = 1$ and $e_2 \cdot f = 0$.

(iv) For e_1, e_2 two exceptional classes with $e_1 \cdot e_2 = 3$, and f a third exceptional class, we have $e_1 \cdot f = 1$ if and only if $e_2 \cdot f = 1$, and $e_1 \cdot f = 0$

if and only if $e_2 \cdot f = 2$.

Proof. Using the fact that two exceptional classes have intersection pairing a if and only if their corresponding roots in E have inner product $1-a$, we see that (i) is Proposition 4.2.1, (ii) is Lemma 3.3.9, and (iii) is Lemma 3.3.27 and Lemma 3.3.13. Finally, (iv) follows from the fact that two classes e_1, e_2 with $e_1 \cdot e_2 = 3$ correspond to two roots in E with inner product -2, which implies they are each other's inverse as vectors (Proposition 3.2.2). □

We also obtain a first upper bound for the number of exceptional curves that are concurrent in a point on X.

Proposition 4.2.2. *The number of exceptional curves that are concurrent in a point on X is at most 16.*

Proof. Cliques with edges of positive weight in G correspond to cliques with edges of weights $-2, -1, 0$ in Γ. The maximal size of such cliques in Γ is 16 by Proposition 3.5.33 and Appendix A. □

Definition 4.2.3. For an exceptional class e in Pic X, we call the unique exceptional class e' with $e \cdot e' = 3$ its *partner*.

The graph in Figure 4.1 is a translation of Figure 3.1, and summarizes Lemma 4.2.1. Vertices are exceptional classes, and the number in a subset is its cardinality. The number on an edge between two subsets is the intersection pairing of two classes, one from each subset. For $i, j \in \{1, 2, 3\}$, the exceptional class e_i' is the partner of the class e_i, and for $e_i \cdot e_j = 2$, the class $e_{i,j}$ is the unique one that intersects both e_i and e_j with multiplicity 2.
Let φ be the morphism associated to the linear system $|-2K_X|$, which realizes X as a double cover of a cone Q in $\mathbb{P}^3$. We want to distinguish cliques in G corresponding to exceptional curves that intersect in a point on the ramification curve of φ from those intersecting in a point outside the ramification curve of φ. To this end we use Proposition 4.2.4.

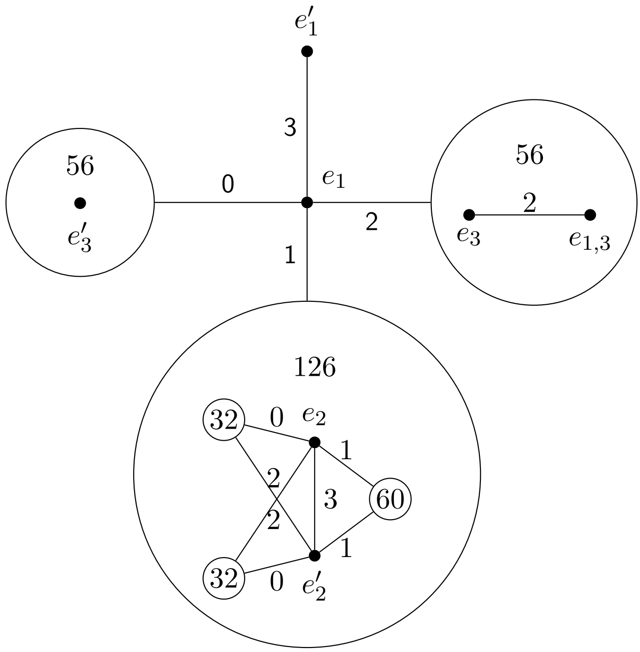

Figure 4.1: Graph G

Proposition 4.2.4.

(i) If e is an exceptional curve on X, then $\varphi(e)$ is a smooth conic, the intersection of Q with a plane in $\mathbb{P}^3$ not containing the vertex of Q. Moreover $\varphi|_e\colon e \longrightarrow \varphi(e)$ is one-to-one.

(ii) If H is a hyperplane section of Q not containing the vertex of Q, then φ^*H has an exceptional curve as component if and only if it has at least three (maybe infinitely near) singular points. If this is the case, then $\varphi^*H = e_1 + e_2$ with $e_1,\ e_2$ exceptional curves, and $e_1 \cdot e_2 = 3$. Every exceptional curve arises this way.

Proof. [CO99, Proposition 2.6 and Key-lemma 2.7]. □

Remark 4.2.5. Let e be an exceptional curve on X, and let e' be its partner. Let H be a hyperplane section of Q with $\varphi^*H = e + e'$, which exists by Proposition 4.2.4 (ii). Since $\varphi|_f$ is one-to-one for $f = e, e'$ by part (i) of the same proposition, it follows that $\varphi(e) = \varphi(e') = H$. So every point on H has two preimages under φ, except for the points with a preimage in $e \cap e'$. We conclude that the points where e intersects the ramification curve of φ are exactly the points in $e \cap e'$, hence are also contained in e'. Conversely, if a set of exceptional curves is concurrent in a point P, and this set contains an exceptional curve and its partner, then P lies on the ramification curve of φ.

4.3 Proof of Theorem 4.1.1

In this section we prove Theorem 4.1.1. We first determine which cliques in G may correspond to sets of exceptional curves intersecting on the ramification curve of φ (Remark 4.3.2). We then show that the automorphism group of G acts transitively on certain cliques of that form (Proposition 4.3.3), which allows us to reduce to specific curves on X. In Proposition 4.3.6, which is key to the proof of Theorem 4.1.1, we show that seven curves in $\mathbb{P}^2$ in a specific configuration are not concurrent.

Remark 4.3.1. From Remark 4.2.5 it follows that there is a bijection between planes in $\mathbb{P}^3$ that are tritangent to the branch curve of φ and do not contain the vertex of Q, and pairs of exceptional curves e_1, e_2 with $e_1 \cdot e_2 = 3$. Using this, we can find an upper bound for the number of exceptional curves that are concurrent in a point on the ramification curve. Let P be a point on the branch curve of φ. From Lemma 4.5 in [TVAV09], it follows that over a field of characteristic unequal to 2, there are at most 7 planes that are tangent to the branch curve at P and two other points. Moreover, Niels Lubbes gave us the insight that exactly one of those planes contains the vertex of Q, so we find an upper bound of 6 planes that are tritangent to the branch curve, that contain P, and that do not contain the vertex of Q. This gives an upper bound of 12 exceptional curves that contain the point $\varphi^{-1}(P)$ on the ramification curve of φ, if char $k \neq 2$.
Consider the map $\lambda \colon R \longrightarrow \mathbb{P}^1$, where R is the ramification curve of φ, and $\mathbb{P}^1$ parametrizes the planes through the tangent line to R at $\varphi^{-1}(P)$: λ sends each point x in $R \setminus \varphi^{-1}(P)$ to the unique plane containing x. This map has degree 4, and if char $k \neq 2$, then R is smooth, and λ extends to a morphism. The upper bound of 7 planes that was found in Lemma 4.5 in [TVAV09] comes from the fact that the ramification divisor of λ has degree 14. An anonymous referee gave us the hint that this idea could even be used to give the upper bound of 10 in char $k \neq 2$ directly, by showing that a morphism of degree 4 to $\mathbb{P}^1$ can not have 7 ramification patterns all equal to $(2, 2)$. Therefore there are at most 6 planes that are

tangent to P and two other points on the branch curve of φ. Since one of them is the plane through the vertex of Q, this gives the upper bound of 10 exceptional curves through $\varphi^{-1}(P)$. We are currently working out the details of this argument.

Remark 4.3.2. From Remark 4.2.5 it follows that a maximal set of exceptional curves that are concurrent in a point on the ramification curve consists of exceptional curves and their partners, hence has even size. Moreover, from Lemma 4.2.1 (iv) it follows that such a clique only has edges of weights 1 and 3. We conclude that all cliques in G corresponding to a maximal set of exceptional curves that are concurrent in a point on the ramification curve are of the following form.

$$K_n = \left\{ \{e_1, \ldots, e_n, e_1', \ldots, e_n'\} \;\middle|\; \begin{array}{c} \forall i:\ e_i, e_i' \in I;\ e_i \text{ is the partner of } e_i'; \\ \forall i \neq j : e_i \cdot e_j = e_i \cdot e_j' = e_i' \cdot e_j' = 1 \end{array} \right\}$$

Let W be the group of permutations of I that preserve the intersection pairing, and recall that W is isomorphic to the Weyl group of the $\mathbf{E}_8$ root system (Corollary 1.4.10).

Proposition 4.3.3. *For $n \in \{2, 3, 5, 6, 7, 8\}$, the group W acts transitively on the set K_n.*

Proof. This is Proposition 3.5.13. □

We now set up notation for Lemma 4.3.4; this lemma will be used in Propositions 4.3.6 and 4.4.6. Lemma 4.3.5 is used in Proposition 4.3.6.

Let $\mathbb{P}^2$ be the projective plane over k with coordinates x, y, z, and let $R_1, \ldots, R_9$ be nine points in $\mathbb{P}^2$, with $R_i = (x_i : y_i : z_i)$ for $i \in \{1, \ldots, 9\}$. For $i \in \{1, 2, 3, 4\}$, we define Mon_i to be the decreasing sequence of $r_i = \binom{i+2}{2} = \frac{1}{2}(i+1)(i+2)$ monomials of degree i in x, y, z, ordered lexicographically with $x > y > z$, and for $j \in \{1, \ldots, r_i\}$, let $\mathrm{Mon}_i[j]$ be the j^{th} entry of Mon_i. For $\delta \in \{x, y, z\}$, let Mon_i^δ be the list of derivatives of the entries in Mon_i with respect to δ. We will define matrices M, N, L, H. Note that each row is well defined up to scaling. This means that for all these matrices, the determinant is well defined up to scaling, so asking for the determinant to vanish is well defined.

$$
\begin{aligned}
M &= (a_{i,j})_{i,j\in\{1,2,3\}} && \text{with } a_{i,j} = \mathrm{Mon}_1[j](R_i);\\
N &= (b_{i,j})_{i,j\in\{1,\dots,6\}} && \text{with } b_{i,j} = \mathrm{Mon}_2[j](R_i);\\
L &= (c_{i,j})_{i,j\in\{1,\dots,10\}} && \text{with } c_{i,j} = \begin{cases} \mathrm{Mon}_3[j](R_i) & \text{for } i \le 8\\ \mathrm{Mon}_3^x[j](R_8) & \text{for } i = 9\\ \mathrm{Mon}_3^z[j](R_8) & \text{for } i = 10 \end{cases}.
\end{aligned}
$$

For $\alpha_7, \alpha_8, \alpha_9 \in \{x, y, z\}$, we define the matrix

$$H_{\alpha_7,\alpha_8,\alpha_9} = (d_{i,j})_{i,j\in\{1,\dots,15\}},$$

$$
\text{with } d_{i,j} = \begin{cases}
\mathrm{Mon}_4[j](R_i) & \text{for } i \le 9\\
\mathrm{Mon}_4^{\beta_7}[j](R_7) & \text{for } i = 10\\
\mathrm{Mon}_4^{\gamma_7}[j](R_7) & \text{for } i = 11\\
\mathrm{Mon}_4^{\beta_8}[j](R_8) & \text{for } i = 12\\
\mathrm{Mon}_4^{\gamma_8}[j](R_8) & \text{for } i = 13\\
\mathrm{Mon}_4^{\beta_9}[j](R_9) & \text{for } i = 14\\
\mathrm{Mon}_4^{\gamma_9}[j](R_9) & \text{for } i = 15
\end{cases},
$$

where for $i \in \{7, 8, 9\}$, we have $\{\beta_i, \gamma_i\} = \{x, y, z\} \setminus \{\alpha_i\}$, with $\beta_i > \gamma_i$ with respect to lexicographic ordering.

Lemma 4.3.4. *The following hold.*

(i) The points R_1, R_2, and R_3 are collinear if and only if det$(M) = 0$.

(ii) The points $R_1, \dots, R_6$ are on a conic if and only if det$(N) = 0$.

(iii) If the points $R_1, \dots, R_8$ are on a cubic with a singular point at R_8, then det$(L) = 0$. If $y_8 \neq 0$, then the converse also holds.

(iv) For all $\alpha_7, \alpha_8, \alpha_9$, if the points $R_1, \dots, R_9$ are on a quartic that is singular at R_7, R_8 and R_9, then det$(H_{\alpha_7,\alpha_8,\alpha_9}) = 0$. If for all i in $\{7, 8, 9\}$, the α_i-coordinate of R_i is non-zero, then the converse also holds.

Proof.

(i) The determinant of M is zero if and only if there is a non-zero element in the nullspace of M, that is, there is a non-zero vector (m_1, m_2, m_3)

such that for all $i \in \{1, 2, 3\}$, we have $m_1 a_{i,1} + m_2 a_{i,2} + m_3 a_{i,3} = 0$. But this is the case if and only if the line defined by $m_1 x + m_2 y + m_3 z$ contains all three points.

(ii) This proof goes analogously to the proof of (i).

(iii) The determinant of L is zero if and only if there is a non-zero vector $(l_1, \ldots, l_{10})$ in k^{10} such that for all $i \in \{1, \ldots, 10\}$, we have $l_1 c_{i,1} + \cdots + l_{10} c_{i,10} = 0$. This is the case if and only if the cubic C defined by $\lambda = \sum_{i=1}^{10} l_i \mathrm{Mon}_3[i]$ contains all eight points $R_1, \ldots, R_8$, and moreover, the derivatives λ_x, λ_z of λ with respect to x and z vanish in R_8. So if $R_1, \ldots, R_8$ are on a cubic with a singular point at R_8, the determinant of L vanishes. Conversely, if $\det(L) = 0$ and $y_8 \neq 0$, since we have $x\lambda_x + y\lambda_y + z\lambda_z = 3\lambda$, this implies that also the derivative λ_y of λ with respect to y vanishes in R_8, hence C is singular in R_8.

(iv) Take $\alpha_7, \alpha_8, \alpha_9 \in \{x, y, z\}$. The determinant of $H_{\alpha_7,\alpha_8,\alpha_9}$ is zero if and only if there exists a non-zero vector given by $(h_1, \ldots, h_{15})$ such that for all $i \in \{1, \ldots, 15\}$, we have $h_1 d_{i,1} + \cdots + h_{15} d_{i,15} = 0$. This is the case if and only if the quartic K defined by $\lambda = \sum_{i=1}^{15} h_i \mathrm{Mon}_4[i]$ contains $R_1, \ldots, R_9$, and moreover, for $i \in \{7, 8, 9\}$, the derivatives λ_δ for $\delta \in \{x, y, z\} \setminus \{\alpha_i\}$ vanish in R_i. So if $R_1, \ldots, R_9$ are on a quartic that is singular at R_7, R_8 and R_9, the determinant of $H_{\alpha_7,\alpha_8,\alpha_9}$ vanishes. Conversely, if $\det(H_{\alpha_7,\alpha_8,\alpha_9}) = 0$ and the α_i-coordinate of R_i is non-zero for $i \in \{7, 8, 9\}$, then, since we have $x\lambda_x + y\lambda_y + z\lambda_z = 4\lambda$, this implies that also λ_{α_i} vanishes in R_i for $i \in \{7, 8, 9\}$. So K is singular in R_7, R_8, and R_9. □

We recall that k is an algebraically closed field, and $\mathbb{P}^2$ is the projective plane over k.

Lemma 4.3.5. *If $R_1, \ldots, R_7$ are seven distinct points in $\mathbb{P}^2$ such that $R_1, \ldots, R_6$ are in general position, and the line L containing R_1 and R_7 contains none of the other points, then there is a unique cubic containing all seven points that is singular in R_1, which does not contain L.*

Proof. The linear system of cubics containing $R_1, \ldots, R_7$ is at least two-dimensional. Requiring that a cubic in this linear system is singular in R_1 gives two linear conditions, defining a linear subsystem $\mathcal{C}$ of dimension at least 0, so there is at least one cubic containing $R_1, \ldots, R_7$ that is singular at R_1.

Let D be an element of $\mathcal{C}$; we claim that D does not contain the line L that contains R_1 and R_7. Indeed, if D were the union of L and a conic C, then R_1 would be contained in C since it is a singular point of D. Since the points $R_2, \ldots, R_6$ are not on L by assumption, they would also be contained in C, contradicting the fact that $R_1, \ldots, R_6$ are in general position. So D does not contain L. Note that this implies that D is smooth in R_7, since if it were singular, then D would intersect L with multiplicity at least 4, hence D would contain L.
Now assume that there is more than one element in $\mathcal{C}$. Then there are two cubics D_1 and D_2 that contain $R_1, \ldots, R_7$ with a singularity at R_1, and whose defining polynomials are linearly independent. By what we just showed, they are not singular in R_7. For $i = 1, 2$, let l_i be the tangent line to D_i at R_7. If the equations defining l_1 and l_2 are not linearly independent, then there is an element F of $\mathcal{C}$ that is singular in R_7, giving a contradiction. We conclude that the equations defining l_1 and l_2 must be linearly independent. Therefore, there is an element G in $\mathcal{C}$ such that the line L through R_1 and R_7 is the tangent line to G at R_7. But then L intersects G in four points counted with multiplicity, so it is contained in G. This contradicts the fact that G is in $\mathcal{C}$. We conclude that there is a unique cubic through $R_1, \ldots, R_7$ that is singular in R_1, and which does not contain the line through R_1 and R_7. $\square$

Proposition 4.3.6. *Assume that the characteristic of k is not 2. Let $Q_1, \ldots, Q_8$ be eight points in $\mathbb{P}^2$ in general position. For $i \in \{1, 2, 3, 4\}$, let L_i be the line through Q_{2i} and Q_{2i-1}, and for $i, j \in \{1, \ldots, 8\}$, with $i \neq j$, let $C_{i,j}$ be the unique cubic through $Q_1, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_8$ that is singular in Q_j, which exists by Lemma 4.3.5. Assume that the four lines L_1, L_2, L_3 and L_4 are concurrent in a point P. Then the three cubics $C_{7,8}$, $C_{8,7}$, and $C_{6,5}$ do not all contain P.*

Proof. First note that if P were equal to one of the Q_i, then three of the eight Q_i would be on a line, which would contradict the fact that $Q_1, \ldots, Q_8$ are in general position. We conclude that P is not equal to one of the Q_i. Moreover, if P were collinear with any two of the three points Q_1, Q_3, Q_5, say for example with Q_1 and Q_3, then, since P is also contained in L_1 and L_2, it would follow that L_1 and L_2 are equal, giving a contradiction. So Q_1, Q_3, Q_5 and P are in general position.
Let $(x : y : z)$ be the coordinates in $\mathbb{P}^2$. Without loss of generality, after

applying an automorphism of $\mathbb{P}^2$ if necessary, we can define

$$Q_1 = (0:1:1); \qquad Q_3 = (1:0:1)$$
$$Q_5 = (1:1:1); \qquad P = (0:0:1).$$

Then we have the following.

L_1 is the line given by $x = 0$;
L_2 is the line given by $y = 0$;
L_3 is the line given by $x = y$.

Since L_4 contains P, and is unequal to L_1 and L_2, there is an $m \in k^*$ such that L_4 is the line given by $my = x$. Since Q_2, Q_7 and Q_8 are not in L_2, and Q_4 is not in L_1, there are $a, b, c, u, v \in k$ such that

$$Q_2 = (0:1:a); \qquad Q_7 = (m:1:v);$$
$$Q_4 = (1:0:b); \qquad Q_8 = (m:1:c).$$
$$Q_6 = (1:1:u);$$

We define $\mathbb{A}^6$ to be the affine space with coordinate ring T_6 given by $T_6 = k[a, b, c, m, u, v]$. Points in $\mathbb{A}^6$ correspond to configurations of the points $Q_1, \ldots, Q_8$.

Assume by contradiction that $C_{7,8}$, $C_{8,7}$, and $C_{6,5}$ all contain P. This assumption gives polynomial equations in the variables a, b, c, m, u, v, and hence defines an algebraic set A_0 in $\mathbb{A}^6$. We define S_0 to be the algebraic set of all points in $\mathbb{A}^6$ that correspond to the configurations where three of the points $Q_1, \ldots, Q_8$ lie on a line, or six of the points lie on a conic. We want to show that A_0 is contained in S_0, which proves the proposition.

Note that the line containing P and Q_5, which is L_3, does not contain any of the points Q_1, Q_2, Q_3, Q_4, Q_8. From Lemma 4.3.5, after substituting $(R_1, \ldots, R_7) = (Q_5, Q_1, Q_2, Q_3, Q_4, Q_8, P)$, it follows that there is a unique cubic D containing $Q_1, Q_2, Q_3, Q_4, Q_5, Q_8$ and P that is singular in Q_5, and that D does not contain L_3. By uniqueness, D must be equal to $C_{6,5}$, and therefore also contains Q_7. By Lemma 4.3.4, the equation expressing that Q_7 is contained in D (or equivalently, that P is contained in $C_{6,5}$) is given by $\det(L) = 0$, where L is the matrix used in the lemma, associated to the points $(R_1, \ldots, R_8) = (Q_1, Q_2, Q_3, Q_4, Q_7, Q_8, P, Q_5)$. We have

$$\det(L) = -m(m-1)(c-v)(b-1)(a-1)f,$$

where $f = \alpha v + \beta$, with

$$\alpha = a - ac - bc + bm, \quad \beta = b(a-1)m^2 + b(c-2a)m + a(b+c-1).$$

The first five factors of $\det(L)$ define subsets of S_0, and do not correspond to configurations where $Q_1, \ldots, Q_8$ are in general position. Therefore, $C_{6,5}$ contains P if and only if $f = 0$. Define the algebraic set $V = Z(\alpha)$, and let $(a_0, b_0, c_0, m_0, u_0, v_0)$ be an element in $V \cap A_0$. Then we have $\alpha(a_0, b_0, c_0, m_0, u_0, v_0) = f(a_0, b_0, c_0, m_0, u_0, v_0) = 0$, so we find $\beta(a_0, b_0, c_0, m_0, u_0, v_0) = 0$. But α and β do not depent on v, so this implies that we have $f(a_0, b_0, c_0, m_0, u_0, v') = 0$ for every v'. So every element in $V \cap A_0$ corresponds to a configuration of $Q_1, \ldots, Q_8$ such that every point $(m : 1 : v')$ on L_4 is also contained in D. But if this is the case, then D consists of L_4 and a conic, which is singular, since Q_5 is a singular point of D that is not contained in L_4. Since L_4 contains none of the points Q_1, Q_2, Q_3, Q_4, these four points are then on the singular conic, which implies that Q_5 is collinear with at least two other points. We conclude that $V \cap A_0$ is a subset of S_0.

Analogously, the fact that $C_{7,8}$ contains P is expressed by $\det(L') = 0$, where L' is the matrix denoted by L in Lemma 4.3.4 with

$$(R_1, \ldots, R_8) = (Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, P, Q_8).$$

We have

$$\det(L') = -m(u-1)(m-1)(b-1)(a-1)g,$$

where $g = \gamma u + \delta$ with

$$\gamma = bm^3 + (1 - bc - c)m^2 + (c^2 - 2c + 1)m + a(1-c) + c^2 - c,$$

and

$$\begin{aligned}\delta = -abm^3 + (abc + ab + ac - a + b - 2bc)m^2 + \\ (ab - 2abc + a + 2bc^2 - b - ac^2 + 2c^2 - 2c)m \\ + a(bc - b + 2c^2 - 2c) - bc^2 + bc - 2c^3 + 2c^2.\end{aligned}$$

The first five factors of $\det(L')$ correspond to configurations where the eight points are not in general position, so $C_{7,8}$ contains P if and only if $g = 0$. Define $U = Z(\gamma)$. By the same reasoning as for $V \cap A_0$ (now using the fact that D does not contain the line L_3), we have $U \cap A_0 \subseteq S_0$. Set

$$v' = \frac{-\beta}{\alpha} \quad \text{and} \quad u' = \frac{-\delta}{\gamma}.$$

Define $\mathbb{A}^4$ to be the affine space with coordinate ring $T_4 = k[m, a, b, c]$, and let K_4 be its fraction field. Let $Y \subset \mathbb{A}^4$ be the set defined by $\alpha = \gamma = 0$.

Consider the ring homomorphism $\psi \colon T_6 \longrightarrow K_4$ defined by

$$(m, a, b, c, u, v) \longmapsto (m, a, b, c, u', v').$$

This defines a morphism $i \colon \mathbb{A}^4 \backslash Y \longrightarrow \mathbb{A}^6 \backslash (V \cup U)$, which is a section of the projection $\mathbb{A}^6 \longrightarrow \mathbb{A}^4$ to the first four coordinates. Set $A_0' = A_0 \setminus (V \cup U)$. Then we have $A_0 \subset S_0$ if and only if $A_0' \subseteq S_0$. Moreover, A_0' is contained in $Z(f, g)$, and since f and g are linear in v and u respectively, we have $i^{-1}(A_0') \cong A_0'$. Set $A_1 = i^{-1}(A_0')$ and $S_1 = i^{-1}(S_0)$, then $A_0' \subseteq S_0$ is equivalent to $A_1 \subseteq S_1$.
Let L'' be the matrix denoted by L in Lemma 4.3.4 with

$$(R_1, \ldots, R_8) = (Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, P, Q_7).$$

Similarly to $C_{7,8}$, the fact that $C_{8,7}$ contains P is expressed by the vanishing of the determinant of L''. We compute this determinant and write it in terms of the coordinates of $\mathbb{A}^4$ using ψ. We find the expression

$$-2abm(m-1)^3(b-1)(a-1)(a+b-1)f_1 f_2 f_3, \tag{4.1}$$

with

$$f_1 = ac - a + bcm - bm^2 - c^2 + cm + c - m,$$

$$f_2 = abm^2 - 2abm + ab - ac^2 + 2ac - a - bc^2 + 2bcm - bm^2,$$

and

$$\begin{aligned} f_3 = abcm^2 &- 2abcm + abc - abm^3 + abm^2 + abm - ab - ac^2m + 2ac^2 \\ &+ acm^2 - 3ac - am^2 + am + a + 2bc^2m - bc^2 - 3bcm^2 + bc + bm^3 \\ &\quad + bm^2 - bm - 2c^3 + 3c^2m + 3c^2 - cm^2 - 4cm - c + m^2 + m. \end{aligned}$$

Expression (4.1) defines the set A_1 in $\mathbb{A}^4$. Since char $k \neq 2$, we have $(4.1) = 0$ if and only if at least one of the non-constant factors of (4.1) equals zero. We show that all non-constant factors of expression (4.1) define components of S_1. If $a = 0$, then Q_2, Q_3 and Q_5 are contained in the line given by $x - z = 0$. Similarly, $b = 0$ implies that Q_1, Q_4 and Q_5 are on the line given by $y - z = 0$, and $a + b - 1 = 0$ implies that Q_2, Q_4, and Q_5 are on the line given by $bx + ay - z = 0$. If $m = 0$ then $L_4 = L_2$, and $m = 1$ implies $L_4 = L_3$, so in both cases there are four points on a line. If $a = 1$ or $b = 1$, then two of the eight points would be the same. Set

$(R_1, \ldots, R_6) = (Q_3, \ldots, Q_8)$, and let N be the corresponding matrix from Lemma 4.3.4. We compute the determinant of N and find that $f_1 f_2 f_3$ divides $\det(N)$. This means that f_1, f_2, as well as f_3 define components of S_1, more specifically, they define configurations where $Q_3, \ldots, Q_8$ are on a conic. We conclude that all irreducible components of A_1 are contained in S_1, which finishes the proof. □

Remark 4.3.7. Note that, theoretically, we could have proved Proposition 4.3.6 with a computer, by checking that A_0 is contained in S_0 using Groebner bases. However, in practice, this turned out to be too big for `magma` to do.

We can now prove Theorem 4.1.1. We use the following notation.

Notation 4.3.8. Let $P_1, \ldots, P_8$ be eight points in general position in $\mathbb{P}^2$ such that X is isomorphic to $\mathbb{P}^2$ blown up these points. For $i \in \{1, \ldots, 8\}$, let E_i be the class in Pic X corresponding to the exceptional curve above P_i, and let L be the class in Pic X corresponding to the pullback of a line in $\mathbb{P}^2$ that does not contain any of the points $P_1, \ldots, P_8$.

Recall that a maximal set of exceptional curves that are concurrent in a point on the ramification curve consists of curves and their partners (Remark 4.3.2).

Proof of Theorem 4.1.1. First note that by Proposition 4.2.2, the number of exceptional curves through any point in X is at most sixteen in all characteristics; this proves the case char $k = 2$.
Now assume char $k \neq 2$. Consider the clique $K = \{e_1, \ldots, e_6, e_1', \ldots, e_6'\}$ in G, where

$$\begin{aligned}
e_1 &= L - E_1 - E_2;\\
e_2 &= L - E_3 - E_4;\\
e_3 &= L - E_5 - E_6;\\
e_4 &= L - E_7 - E_8;\\
e_5 &= 3L - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - 2E_8;\\
e_6 &= 3L - E_1 - E_2 - E_3 - E_4 - 2E_5 - E_7 - E_8,
\end{aligned}$$

and e_i' is the partner of e_i, for all $i \in \{1, \ldots, 6\}$. By Remark 1.2.7, the classes $e_1, \ldots, e_4$ correspond to the strict transforms of the four lines through P_i and P_{i+1} for $i \in \{1, 3, 5, 7\}$, and e_5, e_6, e_5' correspond to the strict transforms of the unique cubics through the points $P_1, \ldots, P_6, P_8$, and the points $P_1, \ldots, P_5, P_7, P_8$, and the points $P_1, \ldots, P_6, P_7$, respectively, that are singular in P_8, and P_5, and P_7, respectively.

Now let K' be a clique in G with only edges of weights 1 and 3, consisting of at least six sets of an exceptional class with its partner. Let $\{\{f_1, f_1'\}, \ldots, \{f_6, f_6'\}\}$ be a set of six such sets in K'. Since W acts transitively on the set of cliques of six exceptional classes and their partners by Proposition 4.3.3, after changing the indices and interchanging f_i's with their partner if necessary, there is an element $w \in W$ such that $f_i = w(e_i)$ and $f_i' = w(e_i')$ for $i \in \{1, \ldots, 6\}$. For $i \in \{1, \ldots, 8\}$, set $E_i' = w(E_i)$. Since the E_i' are pairwise disjoint, by Lemma 1.2.8 we can blow down $E_1', \ldots, E_8'$ to points $Q_1, \ldots, Q_8 \in \mathbb{P}^2$ that are in general position, such that X is isomorphic to the blow-up of $\mathbb{P}^2$ at $Q_1, \ldots, Q_8$, and E_i' is the class in Pic X corresponding to the exceptional curve above Q_i for all i. By Remark 1.2.9, the sequence $(E_1', \ldots, E_8')$ induces a bijection between the exceptional curves on X and the 240 vectors in Proposition 1.2.6, such that the element f_i corresponds to the class of the strict transform of the line through Q_{2i-1} and Q_i for $i \in \{1, \ldots, 4\}$, the elements f_5 and f_6 correspond to the classes of the strict transforms of the unique cubics through the points $Q_1, \ldots, Q_6, Q_8$ and $Q_1, \ldots, Q_5, Q_7, Q_8$, respectively, that are singular in Q_8 and Q_5 respectively, and f_i' is the unique class in I intersecting f_i with multiplicity three for all i. From Proposition 4.3.6 it follows that the curves on X corresponding to $f_1, \ldots, f_6, f_5'$ and f_6' are not concurrent.

We conclude that a set of at least six exceptional curves and their partners is never concurrent. Since any maximal set of exceptional curves going through the same point on the ramification curve forms a clique consisting of curves and their partners, hence of even size, we conclude that this maximum is at most ten. □

4.4 Proof of Theorem 4.1.2

In this section we prove Theorem 4.1.2. The structure of the proof is similar to that of Theorem 4.1.1; we first determine the cliques in G that possibly come from a set of exceptional curves that are concurrent outside the ramification curve of φ (Remark 4.4.1), and show that their maximal size is 12 (Proposition 4.4.2). Then we show that the group W acts transitively on these cliques of size 12 (Proposition 4.4.3) and 11 (Proposition 4.4.4), and finally we show that ten curves in $\mathbb{P}^2$ in a specific configuration are not concurrent in Proposition 4.4.6. This final proposition is again key to the proof of Theorem 4.1.2.

Remark 4.4.1. From Remark 4.2.5 we know that cliques in G corresponding to exceptional curves that intersect each other in a point outside the ramification curve have no edges of weight 3. We conclude that these cliques contain only edges of weights 1 and 2.

Proposition 4.4.2. *The maximal size of cliques in G with only edges of weights 1 and 2 is 12, and there are no maximal cliques with only edges of weights 1 and 2 of size 11.*

Proof. We use the correspondence with the graph Γ in Chapter 3, where the corresponding cliques have only edges of colors -1 and 0; the statement is Proposition 3.5.23. □

Proposition 4.4.3. *The group W acts transitively on the set of cliques of size 12 in G with only edges of weights 1 and 2.*

Proof. This is Proposition 3.5.24. □

Proposition 4.4.4. *The group W acts transitively on the set of cliques of size 11 in G with only edges of weights 1 and 2.*

Proof. By Proposition 4.4.2, any clique of size 11 with only edges of weights 1 and 2 is contained in a clique of size 12 with only edges of weights 1 and 2. By Corollary 3.5.25, for such a clique K of size 12, the stabilizer W_K acts transitively on K, which implies that W_K also acts transitively on the set of cliques of size 11 within K. Since W acts transitively on the set of all cliques of size 12 with only edges of weights 1 and 2 by Proposition 4.4.3, the statement follows. □

Now that we know which cliques in G to look at and what their maximal size is, we show that ten curves in $\mathbb{P}^2$ in a specific configuration are not concurrent in Proposition 4.4.6.

Remark 4.4.5. It is well known that two distinct points in $\mathbb{P}^2$ define a unique line, and five points in $\mathbb{P}^2$ in general position define a unique conic. Now let $R_1, \ldots, R_8$ be eight distinct points in $\mathbb{P}^2$ in general position. The linear system $\mathcal{Q}$ of quartics in $\mathbb{P}^2$ has dimension 14. For three distinct points $R_i, R_j, R_l \in \{R_1, \dots, R_8\}$, requiring a quartic to contain $R_1, \dots, R_8$ and be singular in in R_i, R_j, R_l gives $8+3\cdot 2 = 14$ linear relations. Since the eight points are in general position, the 14 linear conditions are linearly independent, so this gives a zero-dimensional linear subsystem of $\mathcal{Q}$. Hence there is a unique quartic containing all eight points that is singular in R_i, R_j, R_l.

Let $R_1, \dots, R_8$ be eight points in $\mathbb{P}^2$ in general position. Remark 4.4.5 allows us to define the following curves.

L_1 is the line through R_1 and R_2;

L_2 is the line through R_3 and R_4;

C_1 is the conic through R_1, R_3, R_5, R_6 and R_7;

C_2 is the conic through R_1, R_4, R_5, R_6 and R_8;

C_3 is the conic through R_2, R_3, R_5, R_7 and R_8;

C_4 is the conic through R_2, R_4, R_6, R_7 and R_8;

D_1 is the quartic through all eight points, singular in R_1, R_7 and R_8;

D_2 is the quartic through all eight points, singular in R_2, R_5 and R_6;

D_3 is the quartic through all eight points, singular in R_3, R_6 and R_8;

D_4 is the quartic through all eight points, singular in R_4, R_5 and R_7.

Proposition 4.4.6. *Assume that the characteristic of k is not 3. Then the ten curves L_1, L_2, $C_1, \dots C_4$, $D_1, \dots, D_4$ are not concurrent.*

Remark 4.4.7. As in the case of Proposition 4.3.6, in theory we could prove Proposition 4.4.6 with a computer by using Groebner bases, but in practice, this is undoable since the computations become too big (see also Remark 4.3.7). In the case of Proposition 4.4.6 the computations become even bigger, since we now have 10 curves to check, four of which are of degree 4, in contrast to the 7 curves of degrees at most 3 in Proposition 4.3.6.

Before we write down the proof of Proposition 4.4.6, we make some reductions. In $\mathbb{P}^2$, we can choose four points in general position. Fix these and call them Q_1, Q_5, Q_6, and R. We are interested in those configurations of five points Q_2, Q_3, Q_4, Q_7 and Q_8 in $\mathbb{P}^2$ such that the following 11 conditions hold.

0) The points $Q_1, \ldots, Q_8$ are in general position.
1) There is a line through R, Q_1, Q_2.
2) There is a line through R, Q_3, Q_4.
3) There is a conic through $R, Q_1, Q_3, Q_5, Q_6, Q_7$.
4) There is a conic through $R, Q_1, Q_4, Q_5, Q_6, Q_8$.
5) There is a conic through $R, Q_2, Q_3, Q_5, Q_7, Q_8$.
6) There is a conic through $R, Q_2, Q_4, Q_6, Q_7, Q_8$.
7) There is a quartic through all nine points, singular in Q_1, Q_7, Q_8.
8) There is a quartic through all nine points, singular in Q_2, Q_5, Q_6.
9) There is a quartic through all nine points, singular in Q_3, Q_6, Q_8.
10) There is a quartic through all nine points, singular in Q_4, Q_5, Q_7.

We will prove Proposition 4.4.6 by showing that there are no such configurations: all of the configurations satisfying 1–10 violate condition 0.

We consider the space $(\mathbb{P}^2)^5$. Within this space, we define the following two sets.

$$Y = \left\{ (Q_2, Q_3, Q_4, Q_7, Q_8) \in (\mathbb{P}^2)^5 \mid \text{conditions 1–5 are satisfied} \right\}.$$

$$S = \left\{ (Q_2, Q_3, Q_4, Q_7, Q_8) \in (\mathbb{P}^2)^5 \mid \text{three of } Q_1, \ldots, Q_8 \text{ are collinear} \right\}.$$

Note that for an element $(Q_2, Q_3, Q_4, Q_7, Q_8)$ in S, condition 0 is violated. Let F_1 be the linear system of conics through R, Q_1, Q_5, Q_6. Note that this is a one-dimensional linear system that is isomorphic to $\mathbb{P}^1$. Let F_2 be the linear system of lines through R, which is also isomorphic to $\mathbb{P}^1$. We will show that there is a bijection between $Y \setminus S$ and a subset of $F_1^2 \times F_2^3$ in Proposition 4.4.15. We start with two lemmas.

Lemma 4.4.8. *If $(Q_2, Q_3, Q_4, Q_7, Q_8)$ is a point in $Y \setminus S$, then we have $Q_i \neq R$ for $i = 2, 3, 4, 7, 8$.*

Proof. Take a point $Q = (Q_2, Q_3, Q_4, Q_7, Q_8)$ in $Y \setminus S$. Since Q is an element of Y, by condition 1 the points R, Q_1, Q_2 are on a line. That means that if $R = Q_i$ for $i = 3, 4, 7, 8$, the points Q_i, Q_1, Q_2 would be on a line, contradicting the fact that Q is not in S. Moreover, by condition 2, the points R, Q_3, Q_4 are on a line, so if $R = Q_2$ then Q_2, Q_3, Q_4 are on a line, again contradicting the fact that Q is not in S. $\square$

The following result is well known, but we include a proof, as we could not find a reference for this exact statement.

Lemma 4.4.9. *If $S_1, \ldots, S_5$ are five distinct points in $\mathbb{P}^2$, such that the four points $S_1, \ldots, S_4$ are in general position, then there is a unique conic containing $S_1, \ldots, S_5$, which is irreducible if all five points are in general position.*

Proof. The linear system of conics containing $S_1, \ldots, S_4$ is one-dimensional and has only these four points as base points. Requiring for a conic in this linear system to contain the point S_5 gives a linear condition, and since S_5 is different from $S_1, \ldots, S_4$, this condition defines a linear subspace of dimension at least zero. If there were two distinct conics in this subspace, they would intersect in 5 distinct points, so they would have a common component, which is a line. Since no 4 of the points $S_1, \ldots, S_5$ are collinear, there are at most 3 of the 5 points on this line. But then the other two points uniquely determine the second component of both conics, contradicting that they are distinct. We conclude that there is a unique conic containing $S_1, \ldots, S_5$. If, moreover, S_5 is such that all five points are in general position, then no three of them are collinear by definition, so the unique conic containing them cannot contain a line, hence it is irreducible. $\square$

Notation 4.4.10. Let $(Q_2, Q_3, Q_4, Q_7, Q_8)$ be a point in $Y \setminus S$. Note that by condition 3, there is a conic through the points R, Q_1, Q_3, Q_5, Q_6, and Q_7, and by Lemma 4.4.9 it is unique, since R, Q_1, Q_5, Q_6 are in general position. We call this conic A_1. By the same reasoning and condition 4, there is a unique conic containing the points $R, Q_1, Q_4, Q_5, Q_6, Q_8$. We call this conic A_2. By Lemma 4.4.8, the points Q_3, Q_7, Q_8 are all different from R, so we can define the line M_1 through X and Q_3, the line M_2 through R and Q_7, and the line M_3 through R and Q_8.

Recall that F_1 is the linear system of conics through R, Q_1, Q_5, Q_6, and F_2 the linear system of lines through R. We define a map

$$\varphi\colon Y \setminus S \longrightarrow F_1^2 \times F_2^3,$$
$$(Q_2, Q_3, Q_4, Q_7, Q_8) \longmapsto (A_1, A_2, M_1, M_2, M_3).$$

Note that φ is well defined by the definitions of A_1, A_2, M_1, M_2, M_3 in Notation 4.4.10. We want to describe its image. To this end, define the set

$$U = \left\{ (B_1, B_2, N_1, N_2, N_3) \in F_1^2 \times F_2^3 \;\middle|\; \begin{array}{c} B_1, B_2 \text{ irreducible} \\ B_1 \neq B_2 \\ N_1, N_2 \text{ not tangent to } B_1 \\ N_1, N_3 \text{ not tangent to } B_2 \\ N_1 \neq N_2, N_3 \\ Q_1, Q_5, Q_6 \notin N_1, N_2, N_3 \end{array} \right\}.$$

Lemma 4.4.11. *The image of φ is contained in U.*

Proof. Take a point $Q = (Q_2, Q_3, Q_4, Q_7, Q_8) \in Y \setminus S$ and consider its image under φ given by $\varphi(Q) = (A_1, A_2, M_1, M_2, M_3)$. Since Q is not in S, by Lemma 4.4.9, the conics A_1 and A_2 are unique and irreducible. Moreover, if they were equal to each other, then they would both contain the points R, Q_3, Q_4, which are collinear by condition 2, contradicting the fact that they are irreducible.
The line M_1 is tangent to A_1 only if R is equal to Q_3, the line M_2 is tangent to A_1 only if R is equal to Q_7, and the line M_3 is tangent to A_2 only if R is equal to Q_8, all of which are impossible by Lemma 4.4.8. Note that by condition 2, the line M_1 contains Q_4, so M_1 is tangent to A_2 only if $R = Q_4$, which is again impossible by Lemma 4.4.8. If M_2 or M_3 were equal to M_1, then either Q_7 or Q_8 is contained in M_1, which also contains the points R, Q_3, Q_4. But this can not be true since Q is not in S. If M_1 or M_2 contained any of the points Q_1, Q_5, Q_6, then this line would have three points in common with A_1, which implies that A_1 contains a line, contradicting the fact that A_1 is irreducible. Similarly, if M_3 contained Q_1, Q_5, or Q_6, then A_2 would contain M_3, contradicting the irreducibility of A_2. $\square$

We want to define an inverse to φ. We set up the following notation for a point in U.

Notation 4.4.12. Let $u = (B_1, B_2, N_1, N_2, N_3)$ be a point in U. Since the conics B_1 and B_2 are irreducible, they do not contain any of the lines N_1, N_2, N_3, and moreover, since N_1, N_2 are not tangent to B_1, and N_1, N_3

are not tangent to B_2, we can define the following five points in $\mathbb{P}^2$.

$S_3 =$ the point of intersection of B_1 with N_1 that is not X.

$S_4 =$ the point of intersection of B_2 with N_1 that is not X.

$S_7 =$ the point of intersection of B_1 with N_2 that is not X.

$S_8 =$ the point of intersection of B_2 with N_3 that is not X.

Lemma 4.4.13. *Let $u = (B_1, B_2, N_1, N_2, N_3)$ be a point in U. Define the points S_3, S_4, S_7, S_8 as in Notation 4.4.12. There is a unique conic through R, S_3, Q_5, S_7, and S_8, which does not contain the line through R and Q_1.*

Proof. Note that S_3 and S_7 are different from R by definition, and they are different from Q_1, Q_5, Q_6 since Q_1, Q_5, Q_6 are not contained in N_1, nor in N_2, by definition of U. If S_3 were equal to S_7, then N_1 and N_2 would both contain R and S_3, hence they would be equal, contradicting the fact that u is an element of U. So R, S_3, Q_5, S_7 are all distinct, and since they are all contained in B_1, they are in general position because B_1 is irreducible. We will show that S_8 is different from any of these four points. By definition, S_8 is different from R. If S_8 were equal to S_3, then B_1 and B_2 would both contain R, Q_1, Q_5, Q_6 and S_3. But since S_3 is different from R, Q_1, Q_5, Q_6, there is a unique conic through these five points by Lemma 4.4.9. So this would imply $B_1 = B_2$, contradicting the fact that u is in U. Hence S_8 is different from S_3, and similarly, S_8 is different from S_7. Finally, S_8 is different from Q_5, since the line N_3 does not contain Q_5. We conclude that by Lemma 4.4.9, there is a unique conic C through the points R, S_3, Q_5, S_7, and S_8. Note that R, S_3, Q_5, S_7 are all distinct from Q_1. If C contained the line L through R and Q_1, then C would be the union of two lines (one of them being L). This means that either L would contain one of the points S_3, Q_5, S_7, or the points S_3, Q_5, S_7 are all on the second line. But since R, Q_1, S_3, Q_5, S_7 are all in B_1, which is irreducible, both of these cases would be a contradiction. We conclude that C does not contain L. □

Notation 4.4.14. Let $u = (B_1, B_2, N_1, N_2, N_3)$ be a point in U, and let S_3, S_4, S_7, S_8 be the corresponding points as in Notation 4.4.12. We define a fifth point S_2 to be the point of intersection of the conic through R, S_3, Q_5, S_7, S_8 with the line through R and Q_1, that is not R. Note that S_2 is well defined by Lemma 4.4.13.

Using Notations 4.4.12 and 4.4.14, for any point u in U we have now defined an element $(S_2, S_3, S_4, S_7, S_8)$ of $(\mathbb{P}^2)^5$, and it is easy to see that for such a point conditions 1–5 are satisfied, hence it is an element of Y. This leads us to define the following map.

$$\begin{aligned} \psi\colon U &\longrightarrow Y, \\ (B_1, B_2, N_1, N_2, N_3) &\longmapsto (S_2, S_3, S_4, S_7, S_8). \end{aligned}$$

Let T be the set $\psi^{-1}(S)$.

Proposition 4.4.15. *The map $\psi|_{U\setminus T}\colon U \setminus T \longrightarrow Y \setminus S$ is a bijection, with inverse given by φ.*

Proof. Let $u = (B_1, B_2, N_1, N_2, N_3)$ be an element in $U \setminus T$. Write $\psi(u) = (S_2, S_3, S_4, S_7, S_8)$ and $\varphi(\psi(u)) = (B_1', B_2', N_1', N_2', N_3')$. Since $\psi(u)$ is not in S by definition of T, no three of the points $Q_1, Q_5, Q_6, S_2, S_3, S_4, S_7, S_8$ are collinear. Therefore, B_1' and B_2' are the unique and irreducible conics through Q_1, S_3, Q_5, Q_6, S_7 and through Q_1, S_4, Q_5, Q_6, S_8, respectively, by Lemma 4.4.9. Since B_1 and B_2 both contain Q_1, Q_5, Q_6, and B_1 contains S_3 and S_7 and B_2 contains S_4 and S_8 by definition of $\psi(u)$, we conclude that $B_1' = B_1$ and $B_2' = B_2$. The line N_1' is defined as the line containing R and S_3, which are both contained in N_1 as well by definition. We conclude that $N_1' = N_1$, and similarly $N_2' = N_2$, and $N_3' = N_3$. We conclude that $\varphi(\psi(u)) = u$. This proves injectivity of $\psi|_{U\setminus T}$. We now prove surjectivity. Take $Q = (Q_2, Q_3, Q_4, Q_7, Q_8) \in Y\setminus S$; write $\varphi(Q) = (A_1, A_2, M_1, M_2, M_3)$ and $\psi(A_1, A_2, M_1, M_2, M_3) = (Q_2', Q_3', Q_4', Q_7', Q_8')$. The point Q_3' is defined by taking the second point of intersection of A_1 with the line M_1 through R and Q_3. Since A_1 is irreducible ($\varphi(Q)$ is in U by Lemma 4.4.11), it does not contain M_1, so $Q_3' = Q_3$. Similarly, we have $Q_7' = Q_7$, $Q_4' = Q_4$, and $Q_8' = Q_8$. Therefore there is a unique conic C containing the points R, Q_3, Q_5, Q_7, Q_8 by Lemma 4.4.13. Since there is a conic through R, Q_3, Q_5, Q_7, Q_8 and Q_2 by condition 5, we conclude that C contains Q_2 by uniqueness. Since the line L through R and Q_1 is not contained in C by Lemma 4.4.13, and since L contains Q_2 by condition 1, it follows that Q_2 is the second point of intersection of L and C. Hence $Q_2' = Q_2$. We conclude that $\psi(\varphi(Q)) = Q$, and hence $\varphi(Q)$ is contained in $U \setminus T$, and $\psi|_{U\setminus T}$ is surjective.
Since $\psi_{U\setminus T}\colon U \setminus T \longrightarrow Y \setminus S$ is a bijection and we showed that for all elements $u \in U \setminus T$ we have $\varphi(\psi(u)) = u$, we conclude that φ is the inverse function. $\square$

We now prove Proposition 4.4.6. The computations are verified in `magma`; see [Codc] for the code. Recall that we fixed eight points $R_1, \ldots, R_8$ in general position $\mathbb{P}^2$ and ten curves $L_1, L_2, C_1, \ldots, C_4, D_1, \ldots, D_4$, above Proposition 4.4.6.

Proof of Proposition 4.4.6. We assume that these ten curves contain a common point P, and will show that this contradicts the fact that $R_1, \ldots, R_8$ are in general position. First note that if P were equal to one of the eight points $R_1, \ldots, R_8$, then one of the conics would contain six of the eight points, which would contradict the fact that $R_1, \ldots, R_8$ are in general position. Moreover, if P and any two of the three points R_1, R_5, R_6 lie on a line L, then the conic C_1 would intersect L in P and the two points. But this implies that C_1 is not irreducible, and since C_1 contains five of the points $R_1, \ldots, R_8$, this implies that at least three of them are collinear, contradicting the fact that $R_1, \ldots, R_8$ are in general position. We conclude that R_1, R_5, R_6 and P are in general position.
Let $(x : y : z)$ be the coordinates in $\mathbb{P}^2$. Without loss of generality, after applying an automorphism of $\mathbb{P}^2$ if necessary, we can choose R_1, R_5, R_6, and P to be any four points in general position in $\mathbb{P}^2$. We now distinguish between char $k \neq 2$ and char $k = 2$.

Assume char $k \neq 2$. Set

$$R_1 = (1 : 0 : 1); \qquad R_6 = (0 : -1 : 1);$$
$$R_5 = (0 : 1 : 1); \qquad P = (-1 : 0 : 1).$$

It follows that the line L_1, which contains R_1 and P, is given by $y = 0$. The linear system of quadrics through R_1, R_5, R_6 and P is generated by two linearly independent quadrics, and we take these to be $x^2 + y^2 - z^2$ and xy. Let $l, m \in k$ be such that

$$C_1 \text{ is given by } x^2 + y^2 - z^2 = 2lxy;$$
$$C_2 \text{ is given by } x^2 + y^2 - z^2 = 2mxy.$$

Since R_3, R_4, R_7, and R_8 are not contained in L_1, there are $s, t, u \in k$ such that

the line L_2 is given by $sy = x + z$;
the line L_3 through P and R_7 is given by $ty = x + z$;
the line L_4 through P and R_8 is given by $uy = x + z$.

We want to show that all possible configurations of the five points R_2, R_3, R_4, R_7, and R_8 in $\mathbb{P}^2$ such that all ten curves contain P, are such that

$R_1, \ldots, R_8$ are not in general position. By Proposition 4.4.15, all configurations of R_2, R_3, R_4, R_7, R_8 such that L_1, L_2, C_1, C_2, C_3 contain the point P and no three of the points $R_1, \ldots, R_8$ are collinear are given in terms of the conics C_1 and C_2 and the lines L_2, L_3, L_4. By computing the appropriate intersections we find

$$\begin{aligned}
R_3 &= \left(-s^2+1 : 2l-2s : 2ls-s^2-1\right);\\
R_4 &= \left(-s^2+1 : 2m-2s : 2ms-s^2-1\right);\\
R_7 &= \left(-t^2+1 : 2l-2t : 2lt-t^2-1\right);\\
R_8 &= \left(-u^2+1 : 2m-2u : 2mu-u^2-1\right).
\end{aligned}$$

By Lemma 4.4.13, there is a unique conic containing R_3, R_5, R_7, R_8, and P, and we compute a defining polynomial and find

$$\begin{gathered}
\Big(2l^2u+2l^2-2lmu-2lm-lsu-ls-ltu-lt+lu^2+2lu+l+mst\\
+ms+mt-2mu-m+st-su-tu+u^2\Big)\,x^2+\Big(2l^2u^2+2l^2u\\
+2lmst-2lmsu-2lmtu-2lmu-lstu+lst-lsu+ls-ltu+lt\\
+2lu^2+lu+l+mstu+mst-msu-ms-mtu-mt-mu-m\Big)\,xy\\
+2(u+1)(l+1)(l-m)xz+\Big(lstu+lst+lu^2+lu-mstu-msu-mtu\\
-mu+st-su-tu+u^2\Big)\,y^2+(u+1)(t+1)(s+1)(l-m)yz+\Big(lsu\\
+ls+ltu+lt-lu^2+l-mst-ms-mt-m-st+su+tu-u^2\Big)\,z^2.
\end{gathered}$$

Intersecting this conic with the line L_1 gives besides P the point R_2, and we find

$$\begin{aligned}
R_2 = (-(&lsu+ls+ltu+lt-lu^2+l-mst-ms-mt-m\\
&-st+su+tu-u^2) : 0 : (2l^2-2lm-ls-lt)(u+1)+lu^2\\
&+2lu+l+mst+ms+mt-2mu-m+st-su-tu+u^2).
\end{aligned}$$

We define $\mathbb{A}^5$ to be the affine space with coordinate ring $T_5 = k[l, m, s, t, u]$. Following all the above, points in $\mathbb{A}^5$ correspond to configurations of the points $R_1, \ldots, R_8$. The fact that the ten curves contain P gives polynomial equations in these five variables, and hence defines an algebraic set A_0 in $\mathbb{A}^5$. We define S_0 to be the algebraic set of all points in $\mathbb{A}^5$ that correspond to the configurations where the points $R_1, \ldots, R_8$ are not in general position. We want to show that A_0 is contained in S_0, which would prove the proposition. In what follows we will show that indeed every component of A_0 is contained in S_0.

Note that by construction of $R_1, \dots, R_8$, the curves L_1, L_2, C_1, C_2, C_3 contain P. We will add conditions for $C_4, D_1, \dots, D_4$ to contain P, too. We start with C_4. The equation expressing that P is contained in C_4, is given by $\det(N) = 0$, where N is the matrix in Lemma 4.3.4 corresponding to $(R_2, R_4, R_6, R_7, R_8, P)$. This determinant is given by

$$\det(N) = 16(u+1)(t+1)(s+1)(s-u)(m-u)(m-s)(l-t)(l-m)f_1 f_2,$$

where

$$\begin{aligned} f_1 = l^2u + l^2 - lmu - lm - lsu - ls - ltu - lt + lu^2 + lu + mst + ms \\ + mt - mu + st - su - tu + u^2, \end{aligned}$$

and

$$f_2 = at^2 + btu + cu^2 + dt + eu + f,$$

with

$$\begin{aligned} a &= (s+1)(m-1)(m+1), & b = d = -e &= 2s(m-1)(l+1), \\ c &= (s-1)(l-1)(l+1), & f &= (l-m)(ls - l - ms - m + 2s). \end{aligned}$$

Let $F_2 \subset \mathbb{A}^5$ be the affine variety given by $f_2 = 0$. Every component of A_0 is contained in one of the components of the algebraic set given by $\det(N) = 0$. With `magma` it is an easy check that apart from f_2, all non-constant factors of $\det(N)$ define configurations of $R_1, \dots, R_8$ where three of the points are collinear (see [Codc]; $f_1 = 0$ corresponds to R_2, R_3, R_4 being collinear), and hence they define components of S_0. Therefore, it suffices to prove that $A_0 \cap F_2$ is contained in S_0.

Since f_2 is quadratic in t and u, the projection π from F_2 to the affine space $\mathbb{A}^3$ with coordinates l, m, s has fibers that are (possibly non-integral) affine conics. Let Δ be the discriminant of the quadratic form that is the homogenisation of f_2 with respect to t and u, which is given by

$$\Delta = 4acf - ae^2 - b^2 f + bde - cd^2;$$

the singular fibers of π lie exactly above the points $(l, m, s) \in \mathbb{A}^3$ for which $\Delta = 0$. We compute the factorization of Δ in $\mathbb{Z}[l, m, s]$, and find

$$\Delta = 4(s-1)(s+1)(m-1)(m+1)(l-1)(l+1)(l-m)g,$$

with $g = ls - l - ms - m + 2s$. All non-constant factors of Δ except for g, when viewed as elements of T_5, define components of S_0 in $\mathbb{A}^5$. Therefore, the fibers under π above the zero sets of these factors in $\mathbb{A}^3$ are contained in S_0. We will show that the same holds for the inverse image under π of the zero set $Z(g) \subset \mathbb{A}^3$ of g, which is given by the zero set $Z(f_2, g)$ in $\mathbb{A}^5$. Note that we can write

$$f_2 = (s-1)(l+1)(u-t)a_1 + (t-1)ga_2,$$

with $a_1 = (l-1)(u+1)-(m+1)(t-1)$ and $a_2 = (l+1)(u+1)-(m+1)(t+1)$. Therefore, the set $Z(f_2, g)$ is given by $g = (s-1)(l+1)(u-t)a_1 = 0$, so $Z(f_2, g)$ is the union of four algebraic sets:

$$Z(f_2, g) = Z(g, s-1) \cup Z(g, l+1) \cup Z(g, u-t) \cup Z(g, a_1) \subset \mathbb{A}^5.$$

Note that $s-1, l+1$, and $u-t$ define components of S_0, so the first three terms in this union are contained in S_0. With `magma`, we check that the irreducible polynomial $\gamma = (m-u)(l-1)g + (l-s)(m-1)a_1$ corresponds to a configuration where the six points $R_3, \ldots, R_8$ are contained in a conic, and hence it defines a component of S_0. Since γ is contained in the ideal in $\mathbb{Z}[l, m, s, t, u]$ generated by g and a_1, it follows that $Z(g, a_1)$ is also contained in S_0. We conclude that all the singular fibers of π lie in S_0.

The generic fiber $F_{2,\eta}$ of π is a conic in the affine plane $\mathbb{A}^2$ with coordinates t and u over the function field $k(l, m, s)$, where l, m, s are transcendentals. This fiber contains the point $(t, u) = (l, m)$. We can parametrize $F_{2,\eta}$ with a parameter v by intersecting it with the line M given by $v(t-l) = (u-m)$, which intersects $F_{2,\eta}$ in the point (l, m) and a second intersection point that we associate to v. Consider the open subset $F_2' \subset F_2$ given by the complement in F_2 of the singular fibers of π and the hyperplane section defined by $t - l = 0$, so $F_2 \setminus F_2' \subset S_0$. In what follows, we use the idea of this parametrization to construct an isomorphism between F_2' and an open subset of the affine space $\mathbb{A}^4$ with coordinates l, m, s, v.

Consider the ring $T_5^v = k[l, m, s, t, v]$, and let φ be the map $\varphi\colon T_5 \longrightarrow T_5^v$ that sends u to $v(t-l) + m$ and l, m, s, t to themselves. Then we have $\varphi(f_2) = (t-l)(\alpha t + \beta)$, where

$$\alpha = l^2sv^2 - l^2v^2 - 2lmsv + 2lsv + m^2s + m^2 - 2msv - sv^2 + 2sv - s + v^2 - 1,$$

and

$$\begin{aligned}\beta = l^3sv^2 - l^3v^2 - 2l^2msv + 2l^2mv + lm^2s - lm^2 - 2lmsv - lsv^2 \\ + 2lsv - ls + lv^2 + l + 2m^2s - 2mv + 2sv - 2s.\end{aligned}$$

The map φ induces a birational morphism $\psi\colon \mathbb{A}^5_v \longrightarrow \mathbb{A}^5$, where $\mathbb{A}^5_v$ is the affine space with coordinate ring T_5^v. Moreover, ψ is an isomorphism on the complements of the zero sets of $t-l$ in its domain and codomain. Set

$$G = Z(\alpha t + \beta) \setminus Z(t-l) \subset \mathbb{A}^5_v,$$

then ψ induces an isomorphism $G \cong F_2 \setminus Z(t-l)$. In particular, ψ induces an isomorphism from $G \setminus Z(\Delta)$ to F_2'. We want to show that $G \setminus Z(\alpha\Delta)$ equals $G \setminus Z(\Delta)$; to do this it suffices to show that $\psi(G \cap Z(\alpha))$ is contained in a union of singular fibers of π. Note that we have $G \cap Z(\alpha) = G \cap Z(\alpha, \beta)$. Let $(l_0, m_0, s_0, t_0, v_0)$ be a point in $G \cap Z(\alpha, \beta)$, then, since α and β do not depend on t, the point (l_0, m_0, s_0, t, v_0) is contained in $Z(\alpha t + \beta)$ for all t. It follows that the fiber on F_2 in $\mathbb{A}^2(t, u)$ under π above the point $(l_0, m_0, s_0) \in \mathbb{A}^3$ contains the line $u = v_0(t - l_0) + m_0$, hence is singular. Moreover, this fiber contains the point $\psi((l_0, m_0, s_0, t_0, v_0))$. We conclude that $\psi(G \cap Z(\alpha))$ is contained in a union of singular fibers of F_2. It follows that

$$\psi(G \setminus Z(\alpha\Delta)) = \psi(G \setminus Z(\Delta)) = F_2'.$$

Consider the ring $T_4 = k[l, m, s, v]$, and let K_4 be its field of fractions. Consider the ring homomorphism $\rho\colon T_5^v \longrightarrow K_4$ that sends t to $\frac{-\beta}{\alpha}$, and l, m, s, v to themselves. This induces a birational map

$$i\colon \mathbb{A}^4 \longrightarrow Z(\alpha t + \beta) \subset \mathbb{A}^5_v,$$

where $\mathbb{A}^4$ is the affine space with coordinate ring T_4. The map i induces an isomorphism from $\mathbb{A}^4 \setminus Z(\alpha)$ to $Z(\alpha t + \beta) \setminus Z(\alpha)$; this isomorphism sends the zero set of Δ in $\mathbb{A}^4 \setminus Z(\alpha)$ to the zero set of Δ in $Z(\alpha t + \beta) \setminus Z(\alpha)$, and the zero set of $t - l$ in $Z(\alpha t + \beta) \setminus Z(\alpha)$ corresponds to the zero set of $\alpha l + \beta$ in $\mathbb{A}^4 \setminus Z(\alpha)$. Hence, we have an isomorphism

$$\mathbb{A}^4 \setminus Z(\alpha\Delta(\alpha l + \beta)) \cong G \setminus Z(\alpha\Delta).$$

We conclude that we have an isomorphism

$$\psi \circ i\colon \mathbb{A}^4 \setminus Z(\alpha\Delta(\alpha l + \beta)) \longrightarrow F_2'.$$

Recall that our aim is to show that $A_0 \cap F_2$ is contained in S_0. Since we showed that all components of $F_2 \setminus F_2'$ are contained in S_0, we have $A_0 \cap F_2 \subset S_0$ if and only if $A_0 \cap F_2' \subset S_0$. Moreover, after setting

$$A_1 = i^{-1}(\psi^{-1}(A_0 \cap F_2')) \text{ and } S_1 = i^{-1}(\psi^{-1}(S_0 \cap F_2')),$$

showing $A_0 \subseteq S_0$ is equivalent to showing $A_1 \subseteq S_1$.
For i in $\{1,2,3,4\}$, the expression stating that P is contained in D_i is given by $\det(H_i) = 0$, where H_i is the matrix denoted by $H_{\alpha_7,\alpha_8,\alpha_9}$ in Lemma 4.3.4 associated to

$$\begin{aligned}
&(R_2, R_3, R_4, R_5, R_6, R_1, R_7, R_8) \text{ for } i = 1;\\
&(R_1, R_3, R_4, R_7, R_8, R_2, R_5, R_6) \text{ for } i = 2;\\
&(R_1, R_2, R_4, R_5, R_7, R_3, R_6, R_8) \text{ for } i = 3;\\
&(R_1, R_2, R_3, R_6, R_8, R_4, R_5, R_7) \text{ for } i = 4,
\end{aligned}$$

where we set $\alpha_7 = x$, $\alpha_8 = \alpha_9 = y$ for $i \in \{1,2\}$, and $\alpha_7 = \alpha_8 = \alpha_9 = y$ for $i \in \{3,4\}$. For $i \in \{1,2,3,4\}$, let $B_i \subset F_2 \subset \mathbb{A}^5$ be the locus of points corresponding to configurations of $R_1, \ldots, R_8$ such that D_i contains P. Then we have $A_0 \cap F_2 = \bigcap_{i=1}^4 B_i$, so $A_0 \cap F_2' = \bigcap_{i=1}^4 (B_i \cap F_2')$, and hence $A_1 = \bigcap_{i=1}^4 i^{-1}(\psi^{-1}(B_i \cap F_2'))$. Note that B_i is defined by $f_2 = \det(H_i) = 0$. For $i \in \{1,2,3,4\}$, we compute the determinant of H_i and its factorization in $\mathbb{Z}[l,m,s,t,u]$ in `magma`. For all i, this factorization has a constant factor that is a power of 2, and there is exactly one irreducible factor h_i that does not define a component of S_0; it follows that $Z(f_2, h_i) \setminus S_0 = B_i \setminus S_0$. Note that for $i \in \{1,2,3,4\}$, the set $i^{-1}(\psi^{-1}(Z(f_2,h_i) \setminus Z(\alpha\Delta(t-l)))$ is defined in $\mathbb{A}^4 \setminus Z(\alpha\Delta(\alpha l + \beta))$ by the numerator of $\rho(\varphi(h_i))$; we compute the factorization of this numerator in $\mathbb{Z}[l,m,s,v]$. Again, for all i, this factorization has as constant factor a power of 2, and contains exactly one irreducible factor that does not define a component of S_1; we call this factor g_i. It follows that for $i \in \{1,2,3,4\}$, the set $i^{-1}(\psi^{-1}(B_i \setminus S_0))$ is contained in $Z(g_i)$, so $A_1 \setminus S_1$ is contained in $Z(g_1, g_2, g_3, g_4)$. Computing g_1, g_2, g_3, g_4 takes `magma` over an hour, and these polynomials are too big to write down here; you can find them in [Codd]. Set

$$\delta = (ls - l - ms - m + 2s)^2 (l-m)(l-s)(l+1)(m-1)(s+1) \cdot \\ (l-1)(m+1)(s-1)v^2.$$

We check that all factors of $\delta \in \mathbb{Z}[l,m,s,v]$ define components of S_1 (the first factor corresponds to both R_2, R_3, R_5 and R_2, R_4, R_6 being collinear). We will show that δ is contained in the ideal $\mathcal{I}$ of T_4 generated by g_1, g_2, g_3, and g_4. We use a Gröbner basis for $\mathcal{I}$ to check this. In `magma`, we define the ideal $\mathcal{I}$ in the ring T_4 with $k = \mathbb{Q}$ with the ordering $s > v > m > l$. With the function `G,b:=GroebnerBasis(I:ReturnDenominators)` we compute the reduced Gröbner basis G for $\mathcal{I}$; after using this function, `magma` uses G as a generator set for $\mathcal{I}$. We then use G to check that δ is contained in

$\mathcal{I}$, again over $\mathbb{Q}$. This finishes the proof for char $k = 0$; We continue the proof for char $k = p > 0$ with $p \neq 2, 3$.
The element δ can be written as a linear combination of the elements in G with coefficients in T_4. Let C be the set of these coefficients (obtained by the function `Coordinates(I,f)`). In the proces of computing G, `magma` makes divisions by integers, which are stored in the set b. Let $\mathcal{P}$ be the set containing the prime divisors of all elements in b, and all prime divisors of the denominators of the coefficients of the elements in G, and all prime divisors of the denominators of the coefficients of the elements in C. Then for a prime $p \notin \mathcal{P}$, the reductions modulo p of the elements in G are well defined. Moreover, since $\mathcal{P}$ contains all prime divisors of the elements in b, the reductions modulo p of the elements in G still form a Gröbner basis for the ideal $\mathcal{J}$ generated by the reductions modulo p of g_1, g_2, g_3, g_4. Finally, the reduction modulo p of δ is contained in $\mathcal{J}$, since the prime divisors of the denominators of the coefficients of the elements in C are in $\mathcal{P}$. This finishes the proof for char $k = p > 0$ with $p \neq 2, 3$, $p \notin \mathcal{P}$.
For all finitely many $p \in \mathcal{P} \setminus \{2, 3\}$, let $\overline{T_4}$ be the ring $\mathbb{F}_p[l, m, s, v]$, let $\overline{\delta}$ be the reduction of δ modulo p, and for $i \in \{1, 2, 3, 4\}$, let $\overline{g_i}$ be the reduction of g_i modulo p; then it is a quick check in `magma` that $\overline{\delta}$ is contained in the ideal $(\overline{g_1}, \overline{g_2}, \overline{g_3}, \overline{g_4})$ of $\overline{T_4}$. We conclude that for char $k \neq 2, 3$, the set $A_1 \setminus S_1$ is contained in the union of the varieties defined by the factors of δ, so $A_1 \setminus S_1$ is a subset of S_1. We conclude that A_1 is contained in S_1. This finishes the proof for char $k \neq 2$.

<u>Assume char $k = 2$.</u>
Since the points R_1, R_5, R_6, P as defined in the previous case are not in general position over a field of characteristic 2, we redefine these points here. The proof then goes completely analogous to the previous case; see [Codc] for the code in `magma` where we verify everything over the field $k = \mathbb{F}_2$ of two elements. Set

$$R_1 = (1 : 0 : 1); \qquad R_6 = (0 : 1 : 1);$$
$$R_5 = (0 : 1 : 0); \qquad P = (1 : 0 : 0).$$

These four points are in general position in $\mathbb{P}^2$. We take $z^2 + xz + yz$ and xy for the two generators of the linear system of quadrics through R_1, R_5, R_6 and P.
We now do all the steps as in the previous case, and everything works analogously. In fact, checking that all singular fibers of the analog of π from the previous case are contained in the analog of S_0 can be done even more directly in `magma` than as described in the previous case. We

obtain again an algebraic set $A_1 \subset \mathbb{A}^4$, where $\mathbb{A}^4$ is the affine space over $\mathbb{F}_2$ with coordinates l, m, s, v, and A_1 is the algebraic set corresponding to the configurations where the ten curves $L_1, L_2, C_1, \ldots, C_4, D_1, \ldots, D_4$ all contain the point P. Again, we want to show that A_1 is contained in S_1, where $S_1 \subset \mathbb{A}^4$ is the algebraic set defined by the polynomials that correspond to the eight points $R_1, \ldots, R_8$ not being in general position. Completely analogously to the case char $k \neq 2$, from the conditions that P is contained in D_1, D_2, D_3, D_4, we now obtain four polynomials g_1, g_2, g_3, g_4 in $\mathbb{F}_2[l, m, s, v]$ (see [Codd]). Again, we have $A_1 \setminus S_1 \subset Z(g_1, g_2, g_3, g_4)$. Set

$$\delta = (ls+ms+m+s)(lv+m+1)(l+m)(l+s)(m+s)(l+1)(m+1)m^3(s+1)lvs.$$

It is a quick check with `magma` that δ is contained in $\mathcal{I}$. Moreover, it is again a quick check that all factors of δ correspond to three points being collinear, and hence define a component of S_1. We conclude again that A_1 is contained in S_1. $\square$

We can now prove Theorem 4.1.2. Recall Notation 4.3.8.

Proof of Theorem 4.1.2. Recall that every set of exceptional curves without partners corresponds to a clique in G with only edges of weights 1 and 2, so by Lemma 4.4.2, the number of exceptional curves that are concurrent in a point outside the ramification curve of φ is at most twelve. This proves the case char $k = 3$.

Now assume that char $k \neq 3$. Consider the eleven classes in C given by

$$\begin{aligned}
e_1 &= L - E_1 - E_2;\\
e_2 &= L - E_3 - E_4;\\
e_3 &= 2L - E_1 - E_3 - E_5 - E_6 - E_7;\\
e_4 &= 2L - E_1 - E_4 - E_5 - E_6 - E_8;\\
e_5 &= 2L - E_2 - E_3 - E_5 - E_7 - E_8;\\
e_6 &= 2L - E_2 - E_4 - E_6 - E_7 - E_8;\\
e_7 &= 4L - 2E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - 2E_7 - 2E_8;\\
e_8 &= 4L - E_1 - 2E_2 - E_3 - E_4 - 2E_5 - 2E_6 - E_7 - E_8;\\
e_9 &= 4L - E_1 - E_2 - 2E_3 - E_4 - E_5 - 2E_6 - E_7 - 2E_8;\\
e_{10} &= 4L - E_1 - E_2 - E_3 - 2E_4 - 2E_5 - E_6 - 2E_7 - E_8;\\
e_{11} &= 5L - 2E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - E_6 - E_7 - 2E_8;
\end{aligned}$$

It is straightforward to check that they form a clique with only edges of weights 1 and 2 in G. By Remark 1.2.7, we know that $e_1, \ldots, e_{10}$ correspond to the classes in Pic X of the strict transforms of the curves $L_1, L_2, C_1, \ldots, C_4, D_1, \ldots, D_4$, defined as above Proposition 4.4.6 with respect to P_i instead of R_i for $i \in \{1, \ldots, 8\}$.
Let $K = \{c_1, \ldots, c_{11}\}$ be a clique of size eleven in G with only edges of weights 1 and 2. By Proposition 4.4.4, after changing the indices if necessary, there is an element $w \in W$ such that $c_i = w(e_i)$ for i in $\{1, \ldots, 11\}$. Set $E_i' = w(E_i)$. Then, since the E_i' are pairwise disjoint, by Lemma 1.2.8 we can blow down $E_1', \ldots, E_8'$ to points $Q_1, \ldots, Q_8$ in $\mathbb{P}^2$ that are in general position, such that X is isomorphic to the blow-up of $\mathbb{P}^2$ at $Q_1, \ldots, Q_8$, and E_i' is the class in Pic X that corresponds to the exceptional curve above Q_i for all i. By the bijection in Remark 1.2.7, the elements $c_1, \ldots, c_{10}$ are the classes that correspond to the strict transforms of $L_1, L_2, C_1, \ldots, C_4, D_1, \ldots, D_4$ defined as above Proposition 4.4.6 with respect to Q_i instead of R_i for $i \in \{1, \ldots, 8\}$. Since char $k \neq 3$, it follows from Proposition 4.4.6 that the curves corresponding to $c_1, \ldots, c_{10}$ are not concurrent. We conclude that the number of concurrent exceptional curves in a point outside the ramification curve of φ is less than eleven. $\square$

4.5 Examples

4.5.1 On the ramification curve

This section contains examples that show that the upper bounds in Theorem 4.1.1 are sharp. Example 4.5.1 is a del Pezzo surface over a field of characteristic 2 with 16 concurrent exceptional curves, Example 4.5.2 is a del Pezzo surface over any field of characteristic unequal to $2, 3, 5, 7, 11, 13, 17, 19$ with 10 concurrent exceptional curves, and Example 4.5.3 contains examples of ten concurrent exceptional curves on del Pezzo surfaces in the remaining 7 characteristics.

Example 4.5.1. Set $f = x^5 + x^2 + 1 \in \mathbb{F}_2[x]$, and let $F \cong \mathbb{F}_2[x]/(f)$ be the finite field of 32 elements defined by adjoining a root α of f to $\mathbb{F}_2$.

Define the following eight points in $\mathbb{P}^2_F$.

$$\begin{aligned} Q_1 &= (0:1:1); & Q_5 &= (1:1:1);\\ Q_2 &= (0:1:\alpha^{19}); & Q_6 &= (\alpha^{20}:\alpha^{20}:\alpha^{16});\\ Q_3 &= (1:0:1); & Q_7 &= (\alpha^{24}:\alpha^{25}:1);\\ Q_4 &= (1:0:\alpha^5); & Q_8 &= (\alpha^{30}:1:\alpha^5). \end{aligned}$$

With `magma` we check that the determinants of the appropriate matrices in Lemma 4.3.4 are all non-zero, so these eight points are in general position. Therefore, the blow-up of $\mathbb{P}^2$ in $\{Q_1,\dots,Q_8\}$ is a del Pezzo surface S. We have the following four lines in $\mathbb{P}^2$.

$$\begin{aligned} &\text{The line } L_1 \text{ through } Q_1 \text{ and } Q_2, \text{ which is given by } x=0;\\ &\text{the line } L_2 \text{ through } Q_3 \text{ and } Q_4, \text{ which is given by } y=0;\\ &\text{the line } L_3 \text{ through } Q_5 \text{ and } Q_6, \text{ which is given by } x=y;\\ &\text{the line } L_4 \text{ through } Q_7 \text{ and } Q_8, \text{ which is given by } y=\alpha x. \end{aligned}$$

Let $C_{i,j}$ be the unique cubic through $Q_1,\dots,Q_{i-1},Q_{i+1},\dots,Q_8$ that is singular in Q_j. Set $(R_1,\dots,R_8)=(Q_1,Q_3,Q_4,Q_5,Q_6,Q_7,Q_8,Q_2)$, and let L be the corresponding matrix from Lemma 4.3.4. Then the equation defining $C_{1,2}$ is the determinant of L', where L' is equal to L after replacing the first row by Mon_3. Similarly, we compute the defining equations of $C_{3,4}$, $C_{5,6}$, $C_{7,8}$ and $C_{8,7}$, and find the following.

$$C_{1,2}\colon x^3+\alpha^{24}x^2y+\alpha^{28}x^2z+\alpha^{30}xy^2+\alpha^9xyz+\alpha^{26}xz^2+\alpha^{13}y^3+\alpha^6yz^2=0$$

$$C_{3,4}\colon x^3+\alpha^{12}x^2y+\alpha^4xy^2+\alpha^{11}xyz+\alpha^{21}xz^2+y^3+\alpha^{23}y^2z+\alpha^{12}yz^2=0$$

$$\begin{aligned} C_{5,6}\colon x^3+\alpha^4x^2y+\alpha^{28}x^2z+\alpha^{25}xy^2+\alpha^{20}xyz+\alpha^{26}xz^2+\alpha^{17}y^3&\\ +\alpha^9y^2z+\alpha^{29}yz^2&=0 \end{aligned}$$

$$\begin{aligned} C_{7,8}\colon x^3+\alpha x^2y+\alpha^{28}x^2z+\alpha^{17}xy^2+\alpha^{10}xyz+\alpha^{26}xz^2+\alpha^{16}y^3&\\ +\alpha^8y^2z+\alpha^{28}yz^2&=0 \end{aligned}$$

$$\begin{aligned} C_{8,7}\colon x^3+\alpha^{26}x^2y+\alpha^{28}x^2z+\alpha^{19}xy^2+\alpha^{10}xyz+\alpha^{26}xz^2+\alpha^{16}y^3&\\ +\alpha^8y^2z+\alpha^{28}yz^2&=0 \end{aligned}$$

Let $e_1, \ldots, e_8$ be the strict transforms of the eight curves

$$L_1, \ldots, L_4, C_{1,2}, C_{3,4}, C_{5,6}, C_{7,8},$$

and let c_8 be the strict transform of $C_{8,7}$. Since these nine curves all contain the point $(0:0:1)$, the exceptional curves $e_1, \ldots, e_8, c_8$ are concurrent in a point P on S. Let ψ be the morphism associated to the linear system $|-2K_S|$. Since $e_8 \cdot c_8 = 3$, the point P lies on the ramification curve of ψ by Remark 4.2.5. Therefore, by the same remark, for $i \in \{1, \ldots, 7\}$, the partners of $e_1, \ldots, e_7$ contain P, too. We conclude that there are sixteen exceptional curves on S that are concurrent in P.

Example 4.5.2. Let k be a field of characteristic unequal to $2, 3, 5, 7, 11, 13, 17, 19$. Define the following eight points in $\mathbb{P}^2_k$.

$$\begin{aligned} Q_1 &= (0:1:1); & Q_5 &= (1:1:1); \\ Q_2 &= (0:5:3); & Q_6 &= (4:4:5); \\ Q_3 &= (1:0:1); & Q_7 &= (-2:2:1); \\ Q_4 &= (-1:0:1); & Q_8 &= (2:-2:1). \end{aligned}$$

With `magma` we compute the determinants of the matrices in Lemma 4.3.4 that determine whether three of the points are on a line, or six of the points are on a conic, or seven of them are on a cubic that is singular at one of them. These determinants are non-zero for char $k \neq 2, 3, 5, 7, 11, 13, 17, 19$, so the points are in general position. Therefore, the blow-up of $\mathbb{P}^2_k$ in $\{Q_1, \ldots, Q_8\}$ is a del Pezzo surface S. We define the lines L_1, L_2, L_3 as in Example 4.5.1. We define L_4 to be the line containing Q_7 and Q_8, which is given by $x = -y$.
Let $C_{7,8}$ be the unique cubic through $Q_1, \ldots, Q_6, Q_8$ that is singular in Q_8, and $C_{8,7}$ the unique cubic through $Q_1, \ldots, Q_7$ that is singular in Q_7. As in Example 4.5.1 we compute the defining equations for $C_{7,8}$ and $C_{8,7}$, and we find

$$\begin{aligned} &C_{7,8}\colon x^3 - \tfrac{3}{4}x^2y - \tfrac{31}{12}xy^2 + \tfrac{10}{3}xyz - xz^2 - y^3 + \tfrac{8}{3}y^2z - \tfrac{5}{3}yz^2 = 0, \\ &C_{8,7}\colon x^3 + \tfrac{13}{4}x^2y + \tfrac{43}{4}xy^2 - 14xyz - xz^2 + 15y^3 - 40y^2z + 25yz^2 = 0. \end{aligned}$$

On S, we define the four exceptional curves $e_1, \ldots, e_4$ to be the strict transforms of $L_1, \ldots, L_4$, and e_5, e_5' the strict transforms of $C_{7,8}$ and $C_{8,7}$, respectively. Since $L_1, \ldots, L_4, C_{7,8}, C_{8,7}$ all contain the point $(0:0:1)$, the six exceptional curves $e_1, \ldots, e_5, e_5'$ are concurrent in a point P in S.

Let ψ be the morphism associated to the linear system $|-2K_S|$. By Remark 4.2.5, since $e_5 \cdot e_5' = 3$, the point P lies on the ramification curve of ψ, and for $i \in \{1, \ldots, 4\}$, the partners of $e_1, \ldots, e_4$ contain P, too. We conclude that there are ten exceptional curves on S that are concurrent in P.

Example 4.5.3. For $p \in \{3, 5, 7, 11, 13, 17, 19\}$, we construct a del Pezzo surface over a field of characteristic p with ten exceptional curves that are concurrent in a completely analogous way to the one in Example 4.5.2.
Let p be a prime, and $\mathbb{F}_p$ be the finite field of p elements. Let $f_p \in \mathbb{F}_p[x]$ be an irreducible polynomial. Let α be a root of f_p, and $\mathbb{F} \cong \mathbb{F}_p[x]/(f_p)$ the field extension of $\mathbb{F}_p$ obtained by adjoining α to $\mathbb{F}_p$. For $a, b, c, m, u, v \in \mathbb{F}$, define the following eight points in $\mathbb{P}^2_{\mathbb{F}}$.

$$
\begin{aligned}
Q_1 &= (0:1:1); & Q_5 &= (1:1:1);\\
Q_2 &= (0:1:a); & Q_6 &= (1:1:c);\\
Q_3 &= (1:0:1); & Q_7 &= (m:1:u);\\
Q_4 &= (1:0:b); & Q_8 &= (m:1:v).
\end{aligned}
$$

Let x, y, z be the coordinates of $\mathbb{P}^2_{\mathbb{F}}$. We define again the lines L_1, L_2, L_3 as in Example 4.5.1, and the line L_4 by $x = my$. Note that $L_1, \ldots, L_4$ all contain the point $(0:0:1)$. Let $C_{7,8}$ be the unique cubic through $Q_1, \ldots, Q_6, Q_8$ that is singular in Q_8, and $C_{8,7}$ the unique cubic through $Q_1, \ldots, Q_7$ that is singular in Q_7. For all fixed $(p, f_p, a, b, c, m, u, v)$ that we describe below, we check as we did in Example 4.5.2 that the eight points are in general position, and compute the defining equations for $C_{7,8}$ and $C_{8,7}$. In all cases, the point $(0:0:1)$ is also contained in $C_{7,8}$ and $C_{8,7}$, and as in Example 4.5.2 this implies that there are 10 exceptional curves on the del Pezzo surface obtained by blowing up $\mathbb{P}^2_{\mathbb{F}}$ in $Q_1, \ldots, Q_8$, that are concurrent in a point on the ramification curve.

- For $p = 3$ we take

$$f_p = x^3 + 2x + 1, \quad (a, b, c, m, u, v) = (\alpha, \alpha^{20}, \alpha^{15}, \alpha^8, \alpha^2, \alpha^{12}).$$

- For $p = 5$ we take

$$f_p = x^2 + 4x + 2, \quad (a, b, c, m, u, v) = (\alpha^{19}, \alpha^{11}, \alpha^{10}, \alpha^{21}, \alpha^3, \alpha^{14}).$$

- For $p = 7$ we take

$$f_p = x^2 + 6x + 3, \quad (a, b, c, m, u, v) = (3, \alpha^{45}, \alpha^{35}, \alpha^4, \alpha^{46}, \alpha^9).$$

- For $p = 11$ we take

$$f_p = x^2 + 7x + 2, \quad (a, b, c, m, u, v) = (\alpha^{106}, \alpha^{94}, 4, \alpha^{62}, \alpha^{111}, \alpha^{6}).$$

- For $p = 13$ we take

$$f_p = x^2 + 12x + 2, \quad (a, b, c, m, u, v) = (\alpha^{161}, \alpha^{156}, \alpha^{83}, \alpha^{94}, \alpha^{132}, \alpha^{146}).$$

- For $p = 17$ we take

$$f_p = x^2 + 16x + 3, \quad (a, b, c, m, u, v) = (\alpha^{74}, \alpha^{166}, \alpha^{64}, \alpha^{24}, \alpha^{178}, \alpha^{250}).$$

- For $p = 19$, we take $\mathbb{F} = \mathbb{F}_{19}$, and $(a, b, c, m, u, v) = (2, 2, 14, 8, 7, 12)$.

All these examples are generated in `magma` by generating random values for the elements a, b, c, m, u, v in each case, until the points defined by the values are in general position.

4.5.2 Outside the ramification curve

In this section we give examples that show that the upper bound in Theorem 4.1.2 is sharp. Example 4.5.4 gives a del Pezzo surface of degree one over a field of characteristic 3 with twelve exceptional curves that are concurrent in a point outside the ramification curve. In Example 4.5.5 we give a del Pezzo surface over a field of characteristic unequal to 5 that contains ten exceptional curves that are concurrent in a point outside the ramification curve. This surface is isomorphic to the one in Example 4.1 in [SvL14] if the characteristic of k is unequal to 2 and 3. We do not give an example in characteristic 5, since we have not found one; it might very well be that the maximum in this case is less than ten.

Example 4.5.4. Let $f = x^3 + 2x + 1$ be a polynomial in $\mathbb{F}_3[x]$. Let α be a root of f, and let $\mathbb{F} \cong \mathbb{F}_3[x]/f$ be the field of 27 elements obtained by adjoining α to $\mathbb{F}_3$. Let $\mathbb{P}^2_{\mathbb{F}}$ be the projective plane over $\mathbb{F}$, and define the following eight points in this plane.

$$\begin{aligned}
Q_1 &= (1 : 0 : 1); & Q_5 &= (0 : 1 : 1);\\
Q_2 &= (\alpha^{20} : 0 : \alpha^{18}); & Q_6 &= (0 : 2 : 1);\\
Q_3 &= (\alpha^{6} : \alpha^{23} : \alpha^{2}); & Q_7 &= (\alpha^{9} : \alpha^{23} : 2);\\
Q_4 &= (\alpha^{15} : \alpha^{19} : \alpha^{18}); & Q_8 &= (\alpha^{24} : \alpha^{7} : \alpha^{5}).
\end{aligned}$$

With `magma` we check that no three of these points are on a line, no six of them are on a conic, and no seven of them are on a cubic that is singular

at one of them, by checking that the appropriate determinants of the matrices in Lemma 4.3.4 are non-zero. Therefore, the blow-up of $\mathbb{P}^2_{\mathbb{F}}$ in these eight points is a del Pezzo surface S of degree one.
Let L_1 be the line containing Q_1 and Q_2, which is given by $y = 0$. Let L_2 be the line containing Q_3 and Q_4, which is given by $\alpha^{23}y = x + z$. For five points $Q_{i_1}, \ldots, Q_{i_5}$ we find the equation of the conic containing these points by computing the determinant of the matrix N in Lemma 4.3.4, with $(R_2, \ldots, R_6) = (Q_{i_1}, \ldots, Q_{i_5})$, and where the first row is replaced by the list Mon_2. We obtain the following conics in $\mathbb{P}^2_{\mathbb{F}}$.

$C_1\colon x^2 + \alpha^7 xy + y^2 + 2z^2 = 0,$ containing Q_1, Q_3, Q_5, Q_6, Q_7.
$C_2\colon x^2 + \alpha^{16} xy + y^2 + 2z^2 = 0,$ containing Q_1, Q_4, Q_5, Q_6, Q_8.
$C_3\colon x^2+\alpha^{25}xz+\alpha^{16}y^2+\alpha^{11}yz+\alpha^{15}z^2 = 0,$ containing Q_2, Q_3, Q_5, Q_7, Q_8.
$C_4\colon x^2+\alpha^9xy+\alpha^{25}xz+\alpha^{20}y^2+\alpha^6yz+\alpha^{15}z^2 = 0,$ cont. Q_2, Q_4, Q_6, Q_7, Q_8.

Similarly, we compute defining equations for the quartics D_1, D_2, D_3, D_4 containing all the eight points with singularities in Q_1, Q_7, Q_8, and Q_2, Q_5, Q_6, and Q_3, Q_6, Q_8, and Q_4, Q_5, Q_7, respectively. We find

$$D_1\colon \alpha^4x^4 + \alpha^{11}x^3y + \alpha^{12}x^3z + \alpha^{24}x^2y^2 + \alpha^{10}x^2yz + \alpha^{16}x^2z^2 + \alpha^{16}xy^3 + \alpha^{21}xy^2z + \alpha^{17}xyz^2 + \alpha^{25}xz^3 + \alpha^6y^4 + \alpha^{12}y^3z + \alpha^{25}yz^3 + \alpha^{19}z^4 = 0,$$

$$D_2\colon \alpha^{14}x^4 + x^3y + \alpha^{16}x^3z + \alpha^4x^2y^2 + \alpha^4x^2yz + \alpha^{21}x^2z^2 + \alpha^{25}xy^3 + \alpha^{16}xy^2z + \alpha^{12}xyz^2 + \alpha^3xz^3 + \alpha^5y^4 + \alpha^5y^2z^2 + \alpha^5z^4 = 0,$$

$$D_3\colon \alpha^{21}x^4 + \alpha^4x^3y + \alpha^{20}x^3z + \alpha^9x^2y^2 + \alpha^{19}x^2yz + \alpha^3x^2z^2 + \alpha^{21}xy^3 + \alpha^{11}xy^2z+\alpha^2xyz^2+\alpha^7xz^3+\alpha^2y^4+\alpha^{17}y^3z+\alpha y^2z^2+\alpha^4yz^3+\alpha^{23}z^4 = 0,$$

$$D_4\colon \alpha^{19}x^4 + \alpha^{22}x^3y + \alpha^{18}x^3z + \alpha^{20}x^2y^2 + \alpha^{21}x^2yz + \alpha x^2z^2 + \alpha^2xy^3 + \alpha^{20}xy^2z+\alpha^{10}xyz^2+\alpha^5xz^3+\alpha^{23}y^4+\alpha^{20}y^3z+\alpha^3y^2z^2+\alpha^7yz^3+\alpha^{21}z^4 = 0.$$

Finally, in a similar way we compute the defining equations of the quintics G_1 and G_2, which contain all eight points and are singular in $Q_1, Q_2, Q_3, Q_4, Q_5, Q_8$, and $Q_1, Q_2, Q_3, Q_4, Q_6, Q_7$, respectively. We obtain

$$G_1\colon \alpha x^5 + \alpha^8x^4y + 2x^4z + \alpha^{21}x^3y^2 + \alpha^{20}x^3yz + \alpha^{23}x^3z^2 + \alpha^5x^2y^3 + \alpha^{25}x^2y^2z + \alpha^{22}x^2yz^2 + \alpha^7x^2z^3 + \alpha^{25}xy^4 + \alpha^{12}xy^3z + 2xy^2z^2 + \alpha^{25}xyz^3 + \alpha^2xz^4 + \alpha^{21}y^5 + \alpha^6y^4z + \alpha^8y^3z^2 + \alpha y^2z^3 + \alpha^5z^5 = 0,$$

$$G_2\colon \alpha^4x^5+\alpha^{11}x^4y+\alpha^{16}x^4z+\alpha^7x^3y^2+\alpha^{16}x^3yz+x^3z^2+\alpha x^2y^3 \\ +\alpha^{25}x^2y^2z+\alpha^2x^2yz^2+\alpha^{10}x^2z^3+\alpha^{17}xy^3z+\alpha^{15}xy^2z^2+\alpha^8xyz^3 \\ +\alpha^5xz^4+\alpha^{14}y^5+\alpha^{16}y^4z+\alpha^{11}y^3z^2+\alpha^{10}y^2z^3+\alpha^{25}yz^4+\alpha^8z^5=0.$$

Now consider the point $P=(2:0:1)$ in $\mathbb{P}^2_{\mathbb{F}}$. It is an easy check that P is contained in all twelve curves $L_1,L_2,C_1,\dots,C_4,D_1,\dots,D_4,G_1,G_2$. Therefore, the twelve exceptional curves on S that are the strict transforms of these twelve curves in $\mathbb{P}^2_{\mathbb{F}}$ are concurrent in a point Q on S. Let ψ be the morphism associated to the linear system $|-2K_S|$. Since none of the twelve exceptional curves intersect each other with multiplicity 3, the point Q is outside the ramification curve of ψ.

Example 4.5.5. Let k be a field of characteristic unequal to 5. For β an element in k^*, let S be the del Pezzo surface of degree one in $\mathbb{P}(2,3,1,1)$ with coordinates x,y,z,w over k given by

$$y^2+(\beta+1)xyw+\beta yw^3=x^3+\beta x^2w^2-z^5w.$$

For char $k\neq 2,3$, this surface is isomorphic to the surface in [SvL14, Example 4.1]. The blow-up of S in the point $(1:1:0:0)$ has the structure of an elliptic surface over $\mathbb{P}^1$ with coordinates z,w. The fiber above $z=0$ contains a point of order 5, which is given by $Q=(0:0:0:1)$; in fact, the cubic curve $E: y^2+(\beta+1)xy+\beta y=x^3+\beta x^2$ is the universal elliptic curve over the modular curve $Y_1(5)=\operatorname{Spec}(k[\beta,1/\Delta(E)])$ with $\Delta(E)=-\beta^5(\beta^2+11\beta-1)$ that parametrizes elliptic curves over extensions of k with a point of order 5 [CE11, Proposition 8.2.8].
Choose β such that S is smooth in all characteristics; for example, we can set $\beta=2$ in characteristic 11, and $\beta=1$ in all other characteristics. Let ρ,σ be elements of a field extension of k such that $\rho^2=\rho+1$, and $(\beta+\rho^5)\sigma^5=1$. Consider the curve $C_{\rho,\sigma}$ in $\mathbb{P}(2,3,1,1)$ defined by

$$x=\sigma^2z^2w^4+\rho\sigma zw^5,$$
$$y=-\sigma^3z^3w^3+(\rho+1)\sigma^2z^2w^4.$$

Then $C_{\rho,\sigma}$ is an exceptional curve in S, defined over $k(\rho,\sigma)$. It is easy to see that Q is contained in $C_{\rho,\sigma}$. There are ten pairs (ρ,σ), so we conclude that there are ten exceptional curves through Q over a field extension of k. Finally, let φ be the morphism associated to $|-2K_S|$. Since the points on the ramification curve of φ are exactly the points on S that are 2-torsion on their fiber, we conclude that Q is outside the ramification curve.