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Density of rational points on a family of del Pezzo surfaces of degree 1

In this chapter we study the Zariski density of the set of rational points on del Pezzo surfaces of degree 1. In Section 2.1 we give some background and known results. In Section 2.2 we state our main result (Theorem 2.2.1) and the main ingredient for its proof (Proposition 2.2.6). We prove the latter in Section 2.3, and prove our main theorem in Section 2.4. Finally, in Section 2.5 we give examples. This chapter is based on work with Julie Desjardins.

2.1 Rational points on del Pezzo surfaces

Let X be a variety defined over a number field k. In arithmetic geometry we are interested in the set of k-rational points $X(k)$ on X. For example, we can ask whether $X(k)$ is empty, and if so, if we can explain why. If $X(k)$ is not empty, we can further ask how big this set is: is it finite? Infinite? And if it is infinite, what does it look like? Is it dense with respect to the Zariski topology?

For del Pezzo surfaces, some (partial) answers to these questions are known. An overview can be found in [VA09, 1.4]; the following results are stated there. For example, del Pezzo surfaces of degrees 1, 5, and 7 over a field k always contain a k-rational point, and del Pezzo surfaces of degree at least 5 over a number field k satisfy the *Hasse principle*, meaning that if such a surface contains an element in $X(k_v)$ for the completion k_v at every place v of k, then it contains a k-rational point. There are also examples of del Pezzo surface of degrees 2, 3, and 4 over $\mathbb{Q}$ without a $\mathbb{Q}$-rational point even though they do have $\mathbb{R}$-, $\mathbb{C}$-, and $\mathbb{Q}_p$-rational points for all primes p [VA09, Examples 1.4.1–1.4.3].

Zariski density of rational points

In the rest of this chapter, by *density* we mean density with respect to the Zariski topology, unless stated otherwise. To give an overview of what is known for the Zariski density of the set of rational points on del Pezzo surfaces, we introduce another property of a variety.

Definition 2.1.1. A variety X over a field k is *k-unirational* if there is a dominant rational map $\mathbb{P}^n_k \dashrightarrow X$ for some n.

Remark 2.1.2. Note that if two varieties are birationally equivalent over a field k, one is k-unirational if and only if the other one is. Moreover, if k is infinite, then k-unirationality implies Zariski density of the set of k-rational points.

Theorem 2.1.3. *Let k be a field. The following hold.*

(i) Del Pezzo surfaces of degree at least 3 over k with a k-rational point are k-unirational.

(ii) Del Pezzo surfaces of degree 2 over k that contain a point that is neither in the ramification locus of the anticanonical map, nor in the intersection of four exceptional curves, are k-unirational.

(iii) Del Pezzo surfaces of degree 1 that admit a conic bundle structure are k-unirational.

Proof. (i) Segre proved this for degree 3 and $k = \mathbb{Q}$ in [Seg43] and [Seg51]. Manin proved it for $d \geq 5$, as well as for $d = 3, 4$ for large enough cardinality of k [Man86, Theorems 29.4, 30.1]. Kollár finished the case $d = 3$

[Kol02], and Pieropan the case $d = 4$ [Pie12, Proposition 5.19]. Part (ii) is in [STVA14]; part (iii) is in [KM17]. $\square$

Of course, if a del Pezzo surface S of degree 1 over a field k is not minimal, then we can blow down exceptional curves to obtain a del Pezzo surface S' of higher degree, and use Theorem 2.1.3 (i) or (ii) hold to determine whether S' is k-unirational. Since S and S' are birationally equivalent, S is unirational if and only S' is. The del Pezzo surfaces of degree 1 in Theorem 2.1.3 are those that are minimal with Picard rank 2; see Theorem 1.3.4. Outside this case the question of k-unirationality for minimal del Pezzo surfaces of degree 1 is wide open. Even though these surfaces always contain a k-rational point (the base point of the anticanonical linear system), we do not have any example of a minimal del Pezzo surface of degree 1 with Picard rank 1 that is known to be k-unirational, nor of one that is known not to be k-unirational.

If k is infinite, then k-unirationality implies density of the set of k-rational points. Therefore, for k infinite, Theorem 2.1.3 implies that for a del Pezzo surface X of degree at least 3, the set $X(k)$ of k-rational points is Zariski-dense if and only if it is not empty, and if X has degree 2, the set $X(k)$ is Zariski-dense if it contains a point outside the ramification locus of the anticanonical map and not contained in the intersection of four exceptional curves. While unirationality for del Pezzo surfaces of degree 1 is still out of reach, we can at least try to prove Zariski density of the set of k-rational points for these surfaces. A strong reason why we expect that the set of k-rational points on a del Pezzo surface of degree 1 is dense, at least when k is a number field, is the following conjecture by Colliot-Thélène and Sansuc.

Conjecture 2.1.4. *[CT92, Conjecture d] For every geometrically rationally connected variety over a number field, its set of rational points is dense in the Brauer–Manin set for the adelic topology.*

Since del Pezzo surfaces of degree 1 are geometrically rationally connected and have a rational point, this conjecture implies the density of their set of rational points over number fields [Wit18, Remark 2.4(iii)].

Known results

Let S be a del Pezzo surface of degree 1 over a field k with char $k \neq 2, 3$, and let $\mathcal{E}$ be the associated elliptic surface obtained by blowing up the base point of the linear system $|-K_S|$. We identify S with its anticanonical model in the weighted projective space $\mathbb{P}_k(2,3,1,1)$ with coordinates x, y, z, w, and since char $k \neq 2, 3$, we define S as the zero locus of

$$y^2 = x^3 + xf(z,w) + g(z,w),$$

where f and $g \in k[z,w]$ are homogeneous of degrees 4 and 6, respectively.

Previous results on Zariski density of $S(k)$ are obtained by proving that the set $\mathcal{E}(k)$ is dense in $\mathcal{E}$, which implies the result for $S(k)$. People have done this either by considering root numbers of fibers, or by constructing a multisection.

Remark 2.1.5. If $\mathcal{E}$ contains a section over k other than the exceptional curve above the base point of $|-K_S|$, then this section corresponds to a non-zero $k(t)$-rational point in the Mordell–Weil group of $\mathcal{E}$, which has no torsion (Remark 1.4.17). By Silverman's Specialization Theorem [Sil83, Theorem C], this gives a non-torsion k-rational point on all but finitely many fibers of $\mathcal{E}$, thus implying the density of the set of k-rational points on $\mathcal{E}$, hence on S.

We briefly state previous results here.

In [VA11], Várilly-Alvarado proves Zariski density of the set of $\mathbb{Q}$-rational points of S when $f = 0$ and $g = Az^6 + Bw^6$, with non-zero $A, B \in \mathbb{Z}$, such that either $3A/B$ is not a square, or $\gcd(A,B) = 1$ and $9 \nmid AB$. His results are conditional under the finiteness of the Tate–Shafarevich group of elliptic curves with j-invariant 0. Over $\mathbb{Q}$, the latter implies that the root number of such an elliptic curve E equals $(-1)^{\mathrm{rank}(E)}$. Várilly-Alvarado shows that his surfaces have infinitely many disjoint pairs of fibers of $\mathcal{E}$ with opposite root number, thus showing that there are infinitely many fibers with positive rank.

Ulas and Togbé, prove Zariski density of the set of $\mathbb{Q}$-rational points of S in the following cases.

- $g = 0$ and $\deg(f(z,1)) \leq 3$, or $g = 0$ and $\deg(f(z,1)) = 4$ with f not even, or $f = 0$ and $g(z,1)$ is monic of degree 6 and not even [Ula07,

Theorems 2.1 (1), 2.2, and 3.1].
• $g = 0$ and $\deg(f(z,1)) = 4$, or $f = 0$ and $g(z,1)$ is even and monic of degree 6, both cases under the condition that there is a fiber of $\mathcal{E}$ with infinitely many rational points [Ula07, Theorems 2.1 (2) and 3.2].
• S can be defined by $y^2 = x^3 - h(z,w)$, with $h(z,1) = z^5 + az^3 + bz^2 + cz + d \in \mathbb{Z}[z]$, and the set of rational points on the curve $Y^2 = X^3 + 135(2a-15)X - 1350(5a+2b-26)$ is infinite [Ula08, Theorem 2.1].
• $f(z,1)$ and $g(z,1)$ are both even of degree 4 and there is a fiber of $\mathcal{E}$ with infinitely many rational points [UT10, Theorem 2.1].

Jabara generalized the results from [Ula07] mentioned above in [Jab12, Theorems C and D]. Though the proofs of these two theorems are incomplete (see [SvL14, Remark 2.7]), they hold for sufficiently general cases.

In [SvL14], Salgado and van Luijk generalize some of the previous results, proving Zariski density of the set of k-rational points of S for any infinite field k with char $k \neq 2, 3$, assuming that there exists a point Q on a smooth fiber of $\mathcal{E}$ satisfying several conditions, among which that a multisection that they construct from Q has infinitely many k-rational points.

2.2 Main result

Our main theorem is the following; recall that this is joint work with Julie Desjardins.

Theorem 2.2.1. *Let k be a number field, let $A, B \in k$ be non-zero, and let S in $\mathbb{P}(2,3,1,1)$ be the del Pezzo surface of degree 1 over k given by*

$$y^2 = x^3 + Az^6 + Bw^6. \tag{2.1}$$

Let $\mathcal{E}$ be the elliptic surface obtained by blowing up the base point of the linear system $|-K_S|$. Then the set of k-rational points on S is dense in S with respect to the Zariski topology if and only if S contains a k-rational point P with non-zero z, w coordinates, such that the corresponding point on $\mathcal{E}$ lies on a smooth fiber and is non-torsion on that fiber.

Remark 2.2.2. Note that the family of surfaces we consider is the same as the one studied by Várilly-Alvarado in [VA11]. Moreover, the case $A = 1$ is proven by Ulas in [Ula07] for $k = \mathbb{Q}$ under the same condition that we have (the existence of a fiber of $\mathcal{E}$ with infinitely many rational points); we generalize his result to any non-zero A, and to any number field.

While Salgado and van Luijk prove their result over all infinite fields with characteristic unequal to $2, 3$ in [SvL14], their condition that there exists a point Q such that their multisection has infinitely many rational points is not easy to verify, nor is it clear to hold for every surface whose set of rational points is dense, that is, it might not be a necessary condition. For the family in Theorem 2.2.1, we give sufficient *and* necessary conditions for the set of rational points of S to be dense.

Let k be an infinite field with char $k \neq 2, 3$, let $A, B \in k$ non-zero, and let S be the del Pezzo surface of degree 1 over k given by (2.1), with canonical divisor K_S. Let $\mathcal{E}$ be the elliptic surface obtained by blowing up the base point of the linear system $|-K_S|$. The key ingredient of the proof of Theorem 2.2.1 is Proposition 2.2.6. We recall some notation from Section 1.4.3, which we will use throughout this chapter.

Notation 2.2.3. Let $\pi\colon \mathcal{E} \longrightarrow S$ be the blow-up of S in $\mathcal{O} = (1:1:0:0)$ with exceptional divisor $\tilde{\mathcal{O}}$. Since π gives an isomorphism between $\mathcal{E} \setminus \tilde{\mathcal{O}}$ and $S \setminus \{\mathcal{O}\}$, we denote a point $R \in \mathcal{E} \setminus \tilde{\mathcal{O}}$ by the coordinates of $\pi(R)$ in $\mathbb{P}_k(2, 3, 1, 1)$. Let $\nu\colon \mathcal{E} \longrightarrow \mathbb{P}^1$ be the elliptic fibration on $\mathcal{E}$, which is given on S by the projection onto $(z : w)$. For $R = (x_R : y_R : z_R : w_R) \in S\setminus\{\mathcal{O}\}$, we denote by $R_{\mathcal{E}}$ the inverse image $\pi^{-1}(R)$ on $\mathcal{E}$, which is a point on the fiber $\nu^{-1}((z_R : w_R))$.

Definition 2.2.4. For any point $R = (x_R : y_R : z_R : w_R)$ in $\mathcal{E}$ with $y_R, z_R \neq 0$, we define the curve $C_R \subset \mathcal{E}$ as the strict transform of the intersection of S with the surface given by

$$3x_R^2 z_R^2 xz - 2y_R z_R^3 y - (x_R^3 - 2Az_R^6)z^3 + 2Bz_R^3 w^3 = 0. \qquad (2.2)$$

Remark 2.2.5. For $R = (x_R : y_R : z_R : w_R)$ in $\mathcal{E}$ with $y_R, z_R \neq 0$, the curve $\pi(C_R)$ does not contain the point $\mathcal{O}$, so we identify the curve C_R with $\pi(C_R) \subset \mathbb{P}(2, 3, 1, 1)$; see Notation 2.2.3.

If R is a point on S with non-zero z-coordinate and such that $R_{\mathcal{E}}$ lies on a smooth fiber and is non-torsion, then its y-coordinate is non-zero as well, and every non-zero multiple $nR_{\mathcal{E}}$ of $R_{\mathcal{E}}$ on its fiber has non-zero z- and y-coordinate; therefore we can define $C_{nR_{\mathcal{E}}}$ for every non-zero integer n. We use this in the following proposition. Recall the definition of d-section (Definition 1.4.18).

Proposition 2.2.6. *Let P be a point in $S(k)$ with non-zero z, w coordinates, such that $P_{\mathcal{E}}$ lies on a smooth fiber and is non-torsion. If k is a* *number field, then there exists an integer n such that one of the following holds:*

(i) $C_{nP_{\mathcal{E}}}$ has a component that is a section of $\mathcal{E}$ that is defined over k;

(ii) $C_{nP_{\mathcal{E}}}$ is a 3-section of $\mathcal{E}$ of geometric genus 0;

(iii) $C_{nP_{\mathcal{E}}}$ is a 3-section of $\mathcal{E}$ whose normalization is an elliptic curve with positive rank over k.

Remark 2.2.7. Note that case (i) in the previous proposition immediately implies the density of the set of k-rational points on S, see Remark 2.1.5.

2.3 Creating a multisection

In this section we prove Proposition 2.2.6. We use Notation 2.2.3.

Remark 2.3.1. Let $R = (x_R : y_R : z_R : w_R)$ be a point in $\mathcal{E}$, with $y_R, z_R \neq 0$, and let C_R be the corresponding curve as in Definition 2.2.4. Let $\mathbb{A}^3$ be the affine open subset of $\mathbb{P}(2,3,1,1)$ given by $w \neq 0$, with coordinates $X = \frac{x}{w^2}$, $Y = \frac{y}{w^3}$, and $T = \frac{z}{w}$. We describe the intersection $C_R \cap \mathbb{A}^3$. Write

$$\begin{aligned} F &= Y^2 - X^3 - AT^6 - B, \\ G &= 3x_R^2 z_R^2 XT - 2y_R z_R^3 Y - (x_R^3 - 2Az_R^6)T^3 + 2Bz_R^3. \end{aligned} \tag{2.3}$$

We have $C_R \cap \mathbb{A}^3 = Z(F) \cap Z(G)$, where $Z(F)$ and $Z(G)$ are the zero loci of F and G, respectively. Since $y_R, z_R \neq 0$, the projection $p\colon \mathbb{A}^3 \longrightarrow \mathbb{A}^2$ to the X,T-coordinates has a section given by

$$r\colon (X,T) \longmapsto \left(X, \frac{3x_R^2 z_R^2 XT - (x_R^3 - 2Az_R^6)T^3 + 2Bz_R^3}{2y_R z_R^3}, T \right).$$

Note that p induces an isomorphism $Z(G) \longrightarrow \mathbb{A}^2$ with inverse r. It follows that $C_R \cap \mathbb{A}^3$ is isomorphic to $p(Z(F))$, and the latter is defined by $H_R = 0$, where

$$\begin{aligned} H_R = {} & 4y_R^2 z_R^6 X^3 - 9x_R^4 z_R^4 X^2 T^2 + (6x_R^5 z_R^2 - 12Ax_R^2 z_R^8)XT^4 \\ & - 12Bx_R^2 z_R^5 XT + (4Ax_R^3 z_R^6 + 4Ay_R^2 z_R^6 - 4A^2 z_R^{12} - x_R^6)T^6 \\ & + 4Bz_R^3(x_R^3 - 2Az_R^6)T^3 + 4Bz_R^6(y_R^2 - B). \end{aligned} \tag{2.4}$$

We denote by $K_{\mathcal{E}}$ the canonical divisor of $\mathcal{E}$. Let $\overline{k}$ be an algebraic closure of k, and write $\overline{C}_R$ for the base change $C_R \times_k \overline{k}$.

Lemma 2.3.2. *Let $R = (x_R : y_R : z_R : w_R)$ be a point in $\mathcal{E}$ with y_R, z_R non-zero, and let C_R be the curve in Definition 2.2.4. The following hold.*

(i) The curve C_R does not contain a fiber of $\mathcal{E}$.

(ii) The curve C_R is contained in the linear system $|-3K_{\mathcal{E}} + 3\tilde{\mathcal{O}}|$, and intersects every fiber of ν in three points counted with multiplicity.

Proof. (i). From equation (2.2) it is clear that C_R does not contain the fiber $w = 0$. Moreover, since the coefficient of X^3 of H_R (2.4) as a polynomial in $k[T]$ is constant and non-zero, C_R does not contain any fiber with $w \neq 0$, either.
(ii). The linear system $|-3K_S|$ induces the 3-uple embedding of S into $\mathbb{P}^6$ (Section 1.4.1). Under this embedding, the curve $\pi(C_R)$ is given by the intersection of S with a hyperplane, hence we have $\pi(C_R) \sim -3K_S$. Since $y_R, z_R \neq 0$, the image $\pi(C_R)$ does not contain the point $\mathcal{O}$, so this implies

$$C_R = \pi^*(\pi(C_R)) \in |\pi^*(-3K_S)| = |-3K_{\mathcal{E}} + 3\tilde{\mathcal{O}}|.$$

Since a fiber $\mathcal{F}$ of ν is linearly equivalent to $-K_{\mathcal{E}}$, it has self-intersection zero (Lemma 1.4.16), and $\tilde{\mathcal{O}}$ is a section of ν, we have

$$\mathcal{F} \cdot C_R = \mathcal{F} \cdot (-3K_{\mathcal{E}} + 3\tilde{\mathcal{O}}) = 0 + 3 = 3.$$

Since $\mathcal{F}$ is irreducible, it follows that, since $\mathcal{F}$ is not contained in C_R, the number of intersection points of $\mathcal{F}$ and C_R is 3, counted with multiplicity. □

Let $\zeta_3 \in \overline{k}$ be a primitive third root of unity. Note that, for a curve C_R as in Definition 2.2.4, the morphism of $\mathbb{P}(2,3,1,1)$ given by multiplying the w-coordinate with ζ_3^2 restricts to an automorphism of $\overline{C}_R = C_R \times_k \overline{k}$.

Definition 2.3.3. Let $R = (x_R : y_R : z_R : w_R)$ be a point in $\mathcal{E}$, with $y_R, z_R \neq 0$, and let C_R be the corresponding curve as in Definition 2.2.4. By σ we denote the automorphism of $\overline{C}_R$ given by

$$\sigma\colon (x : y : z : w) \longmapsto (x : y : z : \zeta_3^2 w) = (\zeta_3^2 x : y : \zeta_3 z : w) \tag{2.5}$$

Recall that $\pi\colon \mathcal{E} \longrightarrow S$ is the blow-up of S in $\mathcal{O}$, and $\nu\colon \mathcal{E} \longrightarrow \mathbb{P}^1$ is the elliptic fibration on $\mathcal{E}$.

Proposition 2.3.4. *Let $R = (x_R : y_R : z_R : 1)$ be a point in $\mathcal{E}$, with $x_R \in k$, $y_R, z_R \in k^*$, and let C_R be the curve in Definition 2.2.4. The following hold.*

(i) The curve C_R is singular in R, $\sigma(R)$, and $\sigma^2(R)$.

(ii) If $\pi(R)$ is not contained in an exceptional curve on $\overline{S} = S \times_k \overline{k}$, then C_R either contains a section that is defined over k, or it is geometrically integral and has geometric genus at most 1, in which case R, $\sigma(R)$, $\sigma^2(R)$ are all double points.

Proof. (i). It is an easy check that R is contained in C_R. Let m_R be the maximal ideal in the local ring of R on $\mathcal{E}$. The point R lies in the affine space $\mathbb{A}^3 \subset \mathbb{P}(2,3,1,1)$ defined by $w \neq 0$ as in Remark 2.3.1. The ideal m_R is generated by $X - x_R$, $Y - y_R$, and $T - z_R$. Let F, G be as in (2.3). We have $\mathcal{E} \cap \mathbb{A}^3 = Z(F)$, and using the identity $B = y_R^2 - x_R^3 - At_R^2$, we can write F as

$$\begin{aligned} F = 2y_R(Y - y_R) - 3x_R^2(X - x_R) - 6At_0^5(T - z_R) \\ + (Y - y_R)^2 - (X - x_R)^3 - 3x_R(X - x_R)^2 - A(T - z_R)^6 \\ - 6Az_R(T - z_R)^5 - 15At_0^2(T - z_R)^4 \\ - 20Az_R^3(T - z_R)^3 - 15Az_R^4(T - z_R)^2. \end{aligned}$$

Set

$$\alpha = 2y_R(Y - y_R) - 3x_R^2(X - x_R) - 6Az_R^5(T - z_R),$$

then it follows that α is contained in m_R^2, so the tangent line to $\mathcal{E}$ at R is given by $\alpha = 0$.

Similarly, we can rewrite G as

$$\begin{aligned} G = -z_R^3\alpha + 3x_R^2 z_R^2(X - x_R)(T - z_R) - (x_R^3 - 2Az_R^6)(T - z_R)^3 \\ - (3x_R^3 z_R - 6Az_R^7)(T - z_R)^2, \end{aligned}$$

so we conclude that G is contained in m_R^2, hence C_R is singular in R. Since σ is an automorphism of C_R, this implies that C_R is singular in $\sigma(R)$ and $\sigma^2(R)$, as well.

(ii). Assume that $\pi(R)$ is not contained in an exceptional curve on $\overline{S}$. We distinguish two cases. First assume that $\overline{C}_R$ is not irreducible or not reduced. Since $\overline{C}_R$ does not contain a fiber and intersects every fiber with multiplicity 3 (Lemma 2.3.2), this implies that there is a curve that intersects every fiber with multiplicity one (hence is a section), say H_1, such that $\overline{C}_R$ either contains H_1 as irreducible component, or $\overline{C}_R$ is a multiple of H_1. Since $\overline{C}_R$ is disjoint from the zero section, it follows that $\pi(H_1)$ is an exceptional curve on $\overline{S}$ (Proposition 1.4.21). Therefore, by our assumption, R is not contained in H_1, so $\overline{C}_R$ is not a multiple of H_1, and H_1 is an irreducible component of $\overline{C}_R$. Let H_2 be the other (not necessarily irreducible or reduced) component of $\overline{C}_R$, which contains R. If H_2 were not irreducible or not reduced, it would either be a double section or two sections intersecting in R. In both cases, $\pi(R)$ lies on an exceptional curve, contradicting our first assumption. We conclude that H_2 is irreducible and reduced. Since C_R is defined over k, it is fixed by the action of the absolute Galois group of k on Pic S. The exceptional curves of $\overline{S}$ are all defined over the separable closure k^{sep} of k by Theorem 1.1.8, so the Galois group $\text{Gal}(k^{\text{sep}}/k)$ acts on them. Since $\overline{C}_R$ contains only one exceptional curve of $\overline{S}$, which is H_1, it follows that this component is invariant under the Galois action, hence it is defined over k. This finishes the first case. Now assume that $\overline{C}_R$ is irreducible and reduced. Since C_R is contained in the linear system $|-3K_{\mathcal{E}} + 3\tilde{\mathcal{O}}|$ by Lemma 2.3.2, from the adjunction formula it follows that its arithmetic genus is $\frac{1}{2}\cdot(9-3)+1 = 4$. Since the three distinct points R, $\sigma(R)$, $\sigma^2(R)$ are all singular on C_R, we conclude that they all have multiplicity 2, and the geometric genus of C_R is at most 1. $\square$

Remark 2.3.5. In the last proof, we concluded that in the case where C_R is geometrically integral, the geometric genus of C_R is at most 1. If it were 0, then C_R would contain exactly one more singular point besides R, $\sigma(R)$, $\sigma^2(R)$, say Q. Then $\sigma(Q)$ and $\sigma^2(Q)$ would be singular points of C_R as well, so Q would be a fixed point of σ. We study this case further here. Note that the points on the intersection of C_R with the fiber above $(1:0)$ are fixed points of σ. Assume that σ has a fixed point $Q = (x_Q : y_Q : z_Q : w_Q)$ in $C_R \setminus (C_R \cap \mathcal{E}_{(1:0)})$. From (2.5) it follows that there is a $\lambda \in k$ such that $\lambda^3 y_Q = y_Q$, $\lambda^2 x_Q = x_Q$, $\lambda z_Q = z_Q$, and $\lambda\zeta_3^2 w_Q = w_Q$. Since $w_Q \neq 0$, the last equation implies $\lambda = \zeta_3^{3n-2}$ for some $n > 0$, and it follows that $x_Q = z_Q = 0$. From (2.2) and the fact that C_R lies in $\mathcal{E}$ we find

$$2y_R z_R^3 y_Q = 2B z_R^3 w_Q^3; \tag{2.6}$$

$$y_Q^2 = B w_Q^6. \tag{2.7}$$

Since $B, w_Q \neq 0$, it follows from (2.7) that $y_Q \neq 0$ and we can write

$B = \left(\frac{y_Q}{w_Q^3}\right)^2$. Substituting this in (2.6), we find

$$2y_R z_R^3 y_Q = 2\left(\frac{y_Q}{w_Q^3}\right)^2 z_R^3 w_Q^3.$$

Since $y_Q, z_R \neq 0$ this implies $y_R = \frac{y_Q}{w_Q^3}$, from which it follows that we have $B = y_R^2$. Since R is contained in $\mathcal{E}$, it follows that $y_R^2 = x_R^3 + Az_R^6 + y_R^2$, from which we get $A = \frac{-x_R^3}{z_R^6}$. So in this case, the surface S is of the form

$$y^2 = x^3 + \frac{-x_R^3}{z_R^6} z^6 + y_R^2 w^6,$$

and $Q = (0 : y_R : 0 : 1)$. But then C_R contains the section

$$D\colon \begin{cases} x = \frac{x_R}{z_R^2} z^2, \\ y = y_R w^3, \end{cases} \tag{2.8}$$

contradicting the fact that C_R is irreducible. We conclude that if C_R is geometrically integral, then it has genus 0 if and only if it has a singular point on the fiber above $(1 : 0)$.

Remark 2.3.6. Let R be as in Proposition 2.3.4. If C_R is a geometrically integral curve of geometric genus 1, then, since C_R intersects every fiber of ν in three points counted with multiplicity (Lemma 2.3.2), this implies that C_R is a 3-section. Moreover, since R is a double point on C_R, there is a unique third point of intersection of C_R with the fiber above $(z_R : 1)$ in $\mathcal{E}$, say Q. Since x_R, y_R, z_R are elements in k, the fiber above $(z_R : 1)$ is defined over k, and C_R and R are both defined over k. It follows that Q is defined over k. Hence $E_R = (\tilde{C}_R, Q)$ is an elliptic curve defined over k, where $\tilde{C}_R$ is the normalization of C_R. Let D_R be the sum on E_R of the points corresponding to $\sigma(Q)$ and $\sigma^2(Q)$ on C_R. Note that $\sigma(Q)$ and $\sigma^2(Q)$ are either both defined over k or conjugated, so D_R is defined over k.

Notation 2.3.7. If R is as in Proposition 2.3.4, and such that C_R is a geometrically integral curve of geometric genus 1, we denote by E_R the elliptic curve and by D_R the point on it, both as defined in Remark 2.3.6.

Let η be the generic point of S, that is, η is the point $(\tilde{x} : \tilde{y} : \tilde{z} : 1)$ over the function field $k(S) = k(\tilde{x}, \tilde{y}, \tilde{z}) = \mathrm{Frac}(k[x, y, z]/(y^2 - x^3 - Az^6 - B))$ of S.

Let $C_\eta \in \mathbb{P}_{k(S)}(2,3,1,1)$ be the corresponding curve given by (2.2). From Proposition 2.3.4 and Remark 2.3.5 it follows that C_η is geometrically integral of genus 1. Let E_η be the corresponding elliptic curve with point D_η as in Notation 2.3.7. In Lemma 2.3.8 we give a Weierstrass model for the curve E_η, which we will use in Proposition 2.3.10.

Recall that A, B are fixed non-zero elements in k. We define the polynomial

$$q = q_1 q_2 q_3 q_4 \tag{2.9}$$

in the polynomial ring $k[\tilde{x}, \tilde{z}]$ as follows.

$$\begin{aligned}
q_1 &= \tilde{x};\\
q_2 &= -\tilde{x}^6 + 8A\tilde{z}^6\tilde{x}^3 + 4AB\tilde{z}^6;\\
q_3 &= \tilde{x}^6 + 8(A\tilde{z}^6 - B)\tilde{x}^3 + 16(A^2\tilde{z}^{12} + AB\tilde{z}^6);\\
q_4 &= 29\tilde{x}^{12} + \left(40B + 24A\tilde{z}^6\right)\tilde{x}^9 + 16\left(9AB\tilde{z}^6 - B^2 + 6A^2\tilde{z}^{12}\right)\tilde{x}^6\\
&\qquad\qquad + 128\left(A^3\tilde{z}^{18} + 3A^2B\tilde{z}^{12} + 2AB^2\tilde{z}^6\right)\tilde{x}^3\\
&\qquad\qquad\qquad + 64(AB^3\tilde{z}^6 + 2A^2B^2\tilde{z}^{12} + A^3B\tilde{z}^{18}).
\end{aligned}$$

Lemma 2.3.8. *There exists a unique polynomial $\delta \in k[\tilde{x}, \tilde{z}]$, and unique rational functions*

$$\xi_D = \tfrac{\alpha}{(q_1q_3)^2}, \qquad \gamma_D = \tfrac{\beta}{(q_1q_3)^3},$$

where α and β are polynomials in $k[\tilde{x}, \tilde{z}]$, such that the leading terms of δ, α and β, as univariate polynomials in $\tilde{x}$, are given by

$$-27B\tilde{z}^{48}\tilde{x}^{81}, \qquad \tfrac{1}{4}\tilde{z}^{16}\tilde{x}^{42}, \qquad \tfrac{1}{8}\tilde{z}^{24}\tilde{x}^{63},$$

respectively, and such that the following holds. There is an isomorphism ω between the elliptic curve E_η and the curve with Weierstrass equation given by

$$\gamma^2 = \xi^3 + \delta, \tag{2.10}$$

such that the denominators in the defining equations of ω and ω^{-1} are all of the form $2^a3^b(q_2q_4)^c$ for positive integers a, b, c. Moreover, the point on (2.10) corresponding to the point D_η on E_η is given by

$$\omega(D_\eta) = (\xi_D, \gamma_D). \tag{2.11}$$

Proof. The `magma` code that is used in this proof can be found in [Coda]. Let Q be the third point of intersection of C_η with the fiber of η on the base change $\mathcal{E} \times_k k(S)$ over $\mathbb{P}^1 \times_k k(S)$. Write $Q = (x_Q : y_Q : z_Q : 1)$, with $x_Q, y_Q, z_Q \in k(S)$. Then Q lies in $C_\eta \cap (\mathbb{A}^3 \times_k k(S))$, which is isomorphic to the curve C^1_η in $\mathbb{A}^2 \times_k k(S)$ defined by $H_\eta = 0$, where H_η is given in (2.4) after substituting R by η. We find x_Q by substituting $T = \tilde{z}$, $B = \tilde{y}^2 - \tilde{x}^3 - A\tilde{z}^6$ in (2.4) and factorizing, which yields

$$x_Q = \frac{9\tilde{x}^4 - 8\tilde{x}\tilde{y}^2}{4\tilde{y}^2}.$$

We conclude that the elliptic curve E_η as defined in Remark 2.3.6 is isomorphic to the curve $\left(\tilde{C}^1_\eta, \left(\frac{9\tilde{x}^4 - 8\tilde{x}\tilde{y}^2}{4\tilde{y}^2}, \tilde{z}\right)\right)$, where $\tilde{C}^1_\eta$ is the normalization of C^1_η. With `magma` we compute a Weierstrass model for E_η, which is given by

$$\gamma'^2 = \xi'^3 + \frac{(3 \cdot 2^5)^6 \delta}{(q_2 q_4)^6}, \tag{2.12}$$

where δ is a polynomial in $k[\tilde{x}, \tilde{z}]$ with leading term $-27B\tilde{z}^{48}\tilde{x}^{81}$. We verify with `magma` that the denominators in the defining equations of the isomorphism ω_1 between E_η and the curve (2.12), as wel as those of ω_1^{-1}, are all of the form $2^{a'}(q_2q_4)^{b'}$ for positive integers a', b'. The change of coordinates

$$\xi' = \tfrac{(3 \cdot 2^5)^2}{(q_2 q_4)^2}\xi, \qquad \gamma' = \tfrac{(3 \cdot 2^5)^3}{(q_2 q_4)^3}\gamma,$$

induces an isomorphism ω_2 between the curve (2.12) and the curve defined by

$$\gamma^2 = \xi^3 + \delta. \tag{2.13}$$

We conclude that $\omega = \omega_2 \circ \omega_1$ is an isomorphism between E_η and the curve (2.13), and the denominators in the defining equations of ω and ω^{-1} are all of the form $2^a 3^b (q_2 q_4)^c$ for positive integers a, b, c.
If δ' was another polynomial in $k[\tilde{x}, \tilde{z}]$ such that E_η were isomorphic to the curve given by $\gamma^2 = \xi^3 + \delta'$, then we would have $\delta' = \upsilon^6 \delta$ for some $\upsilon \in k(S)$, hence δ' would not have leading term $-27B\tilde{z}^{48}\tilde{x}^{81}$ as univariate polynomial in $\tilde{x}$. We conclude that δ is the unique polynomial with leading term $-27B\tilde{z}^{48}\tilde{x}^{81}$ such that E_η is isomorphic to the curve with Weierstrass model (2.13). With `magma` we compute the sum D on the curve (2.13) of the points corresponding to $\left(\zeta_3^2 \frac{9\tilde{x}^4 - 8\tilde{x}\tilde{y}^2}{4\tilde{y}^2}, \zeta_3 \tilde{z}\right)$ and $\left(\zeta_3 \frac{9\tilde{x}^4 - 8\tilde{x}\tilde{y}^2}{4\tilde{y}^2}, \zeta_3^2 \tilde{z}\right)$ on C_η. We find $D = (\xi_D, \gamma_D)$ with $\xi_D = \frac{\alpha}{(q_1 q_3)^2}$, $\gamma_D = \frac{\beta}{(q_2 q_3)^3}$, where α, β are elements in $k[\tilde{x}, \tilde{z}]$ with leading terms as univariate polynomials in $\tilde{x}$ given by $\frac{1}{4}\tilde{z}^{16}\tilde{x}^{42}$ and $\frac{1}{8}\tilde{z}^{24}\tilde{x}^{63}$, respectively. $\square$

Remark 2.3.9. Let L be the hypersurface in $\mathbb{A}^2 \times S$ defined by

$$\begin{aligned} 4y^2z^6X^3 - 9x^4z^4X^2T^2 + (6x^5z^2 - 12Ax^2z^8)XT^4 - 12Bx^2z^5XT \\ + (4Ax^3z^6 + 4Ay^2z^6 - 4A^2z^{12} - x^6)T^6 + 4Bz^3(x^3 - 2Az^6)T^3 \\ + 4Bz^6(y^2 - B) = 0, \end{aligned}$$

and let $\lambda\colon L \longrightarrow S$ be the projection to S. Let $R = (x_R : y_R : z_R : 1)$ be a point in $\mathcal{E}$ with $x_R \in k$, $y_R, z_R \in k^*$, $q(x_R, z_R) \neq 0$, and such that C_R is geometrically integral of genus 1. We identify R with $\pi(R) \in S$; the fiber of λ above R is the curve in $\mathbb{A}^2$ given by $H_R = 0$, where H_R is in (2.4), hence it is isomorphic to $C_R \cap \mathbb{A}^3$, where $\mathbb{A}^3$ is defined by $w \neq 0$ in $\mathbb{P}(2,3,1,1)$. Moreover, the curve $C_\eta \cap (\mathbb{A}^3 \times_k k(S))$ is isomorphic to the generic fiber of λ.
Let δ and ω be as in Lemma 2.3.8, and E_R, D_R as in Notation 2.3.7. Since $q(x_R, z_R)$ is non-zero, the isomorphism ω specializes to the fiber $\lambda^{-1}(R)$, and we obtain an isomorphism between E_R and the curve given by

$$\gamma^3 = \xi^2 + \delta(x_R, z_R), \tag{2.14}$$

that sends the point D_R to the point

$$(\xi_D(x_R, z_R), \gamma_D(x_R, z_R)).$$

Let $P = (x_0 : y_0 : z_0 : 1)$ be a point on S with $z_0 \neq 0$. Let V be the set of points $R = (x_R : y_R : z_0 : 1)$ on the fiber of $P_{\mathcal{E}}$ with $x_R \in k$, $y_R \in k^*$, such that $q(x_R, z_0) \neq 0$ (where q is given in (2.9)), and such that C_R is geometrically integral of genus 1.

Proposition 2.3.10. *If k is a number field, then for all but a finite number of points R in the set V, the curve E_R has positive rank over k.*

Proof. Let δ be as in Lemma 2.3.8. We define the following polynomials in $k[\tilde{x}, \xi, \gamma]$.

$$\psi_1 = 1, \qquad \psi_2 = 2\gamma, \qquad \psi_3 = 3\xi^4 + 12\delta(\tilde{x}, z_0)\xi,$$
$$\psi_4 = 2\psi_2\left(\xi^6 + 20\delta(\tilde{x}, z_0)\xi^3 - 8\delta(\tilde{x}, z_0)^2\right),$$

and recursively,

$$\psi_{2m+1} = \psi_{m+2}\psi_m^3 - \psi_{m-1}\psi_{m+1}^3 \quad \text{for } m \geq 2, \tag{2.15}$$
$$\psi_2\psi_{2m} = \psi_{m-1}^2\psi_m\psi_{m+2} - \psi_{m-2}\psi_m\psi_{m+1}^2 \quad \text{for } m \geq 3. \tag{2.16}$$

Let ξ_D, γ_D be as in Lemma 2.3.8. For $m \geq 1$, we define $\psi_{m,\tilde{x}}$ to be the rational function

$$\psi_m(\tilde{x}, \xi_D(\tilde{x}, z_0), \gamma_D(\tilde{x}, z_0)) \in k(\tilde{x}).$$

Write $d = q_1(\tilde{x}, z_0)q_3(\tilde{x}, z_0) \in k[\tilde{x}]$. From Remark 2.3.9, we find

$$\psi_{2,\tilde{x}} = \frac{N_2}{d^3}, \quad \psi_{3,\tilde{x}} = \frac{N_3}{d^8}, \quad \psi_{4,\tilde{x}} = \frac{N_4}{d^{15}},$$

where N_2, N_3, N_4 are polynomials in $k[\tilde{x}]$. Let c_i be the leading coefficient of N_i for $i \in \{2, 3, 4\}$, then we have

$$\deg(N_2) = 63, \quad \deg(N_3) = 168, \quad \deg(N_4) = 315,$$

$$c_2 = \frac{1}{4}z_0^{24}, \quad c_3 = \frac{3}{2^8}z_0^{64}, \quad c_4 = \frac{1}{2^{13}}t_0^{120}.$$

We claim that for all $m \geq 1$ we have

$$\psi_{m,\tilde{x}} = \frac{N_m}{d^{m^2-1}},$$

Where N_m is a polynomial in $k[\tilde{x}]$ with leading coefficient c_m such that

$$\deg(N_m) = 21(m^2 - 1) \qquad \text{and} \qquad c_m = m\left(\frac{1}{2}z_0^8\right)^{m^2-1}.$$

Assume that this claim is true (we prove this below). Since k is a number field, there is an upper bound $B = B(k)$ such that the torsion points on the fiber of $P_{\mathcal{E}}$ have order at most B [Mer96]. Let $R = (x_R : y_R : z_0 : 1)$ be a point in V such that $\psi_{m,\tilde{x}}(x_R)$ is non-zero for all $m \leq B$. Note that, since $z_0 \neq 0$, this holds for all but finitely many points in V by our claim. By Remark 2.3.9, the curve E_R is isomorphic to the elliptic curve in $\mathbb{A}^2$ given by equation (2.14), where $z_R = z_0$. We identify E_R with this model. Let D_R be the point on E_R as in Notation 2.3.7, and note that D_R is defined over k because R is. Write

$$\xi_R = \xi_D(x_R, z_0), \quad \gamma_R = \gamma_D(x_R, z_0),$$

then we have $D_R = (\xi_R, \gamma_R)$ by Remark 2.3.9, and since $q(x_R, z_0) \neq 0$, the point D_R is non-zero on E_R.

Note that for $m \geq 1$, the polynomial $\psi_m(x_R, \xi, \gamma) \in k[\xi, \gamma]$ is the m-th division polynomial of E_R, as defined in [Sil09, Exercise 3.7], and from the same reference we know that D_R is m-torsion for $m \geq 2$ if and only if

$\psi_m(x_R, \xi_R, \gamma_R) = \psi_{m,\tilde{x}}(x_R) = 0$. Since we chose R such that $\psi_{m,\tilde{x}}(x_R) \neq 0$ for all $m \leq B$, we conclude that D_R is non-torsion on E_R. This, together with the proof of the claim below, proves the propostion.
Proof claim.
We prove this by induction. Set $k \geq 2$, and assume that the claim holds for $m < 2k+1$ (note that this is indeed the case for $k = 2$). Then we have

$$\begin{aligned} \deg\left(N_{k+2}N_k^3\right) &= 21((k+2)^2-1) + 63(k^2-1) = 21(4k^2+4k); \\ \deg\left(N_{k-1}N_{k+1}^3\right) &= 21((k-1)^2-1) + 63((k+1)^2-1) = 21(4k^2+4k), \end{aligned}$$

so we find

$$\deg(N_{k+1}N_k^3) = \deg\left(N_{k-1}N_{k+1}^3\right) = 21((2k+1)^2-1). \tag{2.17}$$

Completely analogously, we find that the denominators of $\psi_{k+2,\tilde{x}}\psi_{k,\tilde{x}}^3$ and $\psi_{k-1,\tilde{x}}\psi_{k+3,\tilde{x}}^3$ are both equal to $d^{(2k+1)^2-1}$. Combining this with (2.17), we find from the recursion in (2.15) that the denominator of $\psi_{2k+1,\tilde{x}}$ is equal to $d^{(2k+1)^2-1}$, that the degree of N_{2k+1} is at most $21((2k+1)^2-1)$, and that the coefficient of the monomial $\tilde{x}^{21((2k+1)^2-1)}$ in N_{2k+1} is given by $c_{k+2}c_k^3 - c_{k-1}c_{k+1}^3$, which by induction is equal to

$$\begin{aligned} &(k+2)(\frac{1}{2}z_0^8)^{(k+2)^2-1}k^3(\frac{1}{2}z_0^8)^{3(k^2-1)} \\ &-(k-1)(\frac{1}{2}z_0^8)^{(k-1)^2-1}(k+1)^3(\frac{1}{2}z_0^8)^{3((k+1)^2-1)} \\ &\quad = (k^3(k+2)-(k-1)(k+1)^3)(\frac{1}{2}z_0^8)^{4k^2+4k} \\ &\quad = (2k+1)(\frac{1}{2}z_0^8)^{(2k+1)^2-1}. \end{aligned}$$

Since the latter is non-zero we conclude that it is the leading coefficient of N_{2k+1}, so we find $c_{2k+1} = (2k+1)(\frac{1}{2}z_0^8)^{(2k+1)^2-1}$, and we conclude $\deg(N_{2k+1}) = 21((2k+1)^2-1)$. This finishes the proof of the claim for $m = 2k+1$; we will now prove it for $m = 2k+2$ in a similar way. By induction, we have

$$\begin{aligned}
\deg\left(N_k^2 N_{k+1} N_{k+3}\right) &= 42(k^2-1) + 21((k+1)^2-1) \\
&\quad + 21((k+3)^2-1) \\
&= 21((2k+2)^2-1+3) \\
\deg\left(N_{k-1} N_{k+1} N_{k+2}^2\right) &= 21((k-1)^2-1) + 21((k+1)^2-1) \\
&\quad + 42((k+2)^2-1) \\
&= 21((2k+2)^2-1+3),
\end{aligned}$$

so we find

$$\begin{aligned}
\deg\left(N_k^2 N_{k+1} N_{k+3}\right) = \deg\left(N_{k-1} N_{k+1} N_{k+2}^2\right) \\
= 21((2k+2)^2-1) + \deg\left(N_2\right). \quad (2.18)
\end{aligned}$$

Analogously we find that the denominators of both $\psi_{k,\tilde{x}}^2 \psi_{k+1,\tilde{x}} \psi_{k+3,\tilde{x}}$ and $\psi_{k-1,\tilde{x}} \psi_{k+1,\tilde{x}} \psi_{k+2,\tilde{x}}^2$ are equal to $d^{(2k+2)^2-1} d^{2^2-1}$. Combining this with (2.18), we find from the recursion in (2.16) that the denominator of ψ_{2k+2} is equal to $d^{(2k+2)^2-1}$, that the degree of N_{2k+2} is at most $21((2k+2)^2-1)$, and that the coefficient of the monomial $\tilde{x}^{21((2k+2)^2-1)}$ in N_{2k+2} is given by $\frac{1}{c_2}(c_k^2 c_{k+1} c_{k+3} - c_{k-1} c_{k+1} c_{k+2}^2)$, which by induction is equal to

$$\begin{aligned}
&\frac{1}{c_2}\left(k^2(\frac{1}{2}z_0^8)^{2(k^2-1)}(k+1)(\frac{1}{2}z_0^8)^{(k+1)^2-1}(k+3)(\frac{1}{2}z_0^8)^{(k+3)^2-1}\right. \\
&\qquad \left. -(k-1)(\frac{1}{2}z_0^8)^{(k-1)^2-1}(k+1)(\frac{1}{2}z_0^8)^{(k+1)^2-1}(k+2)^2(\frac{1}{2}z_0^8)^{2(k+2)^2-2}\right) \\
&= \frac{1}{c_2}\left((k^2(k+1)(k+3)-(k-1)(k+1)(k+2)^2)(\frac{1}{2}z_0^8)^{4k^2+8k+6}\right) \\
&= \frac{1}{c_2}\left(2(\frac{1}{2}z_0^8)^3(2k+2)(\frac{1}{2}z_0^8)^{(2k+2)^2-2}\right) \\
&= \frac{1}{c_2}\left(c_2(2k+2)(\frac{1}{2}z_0^8)^{(2k+2)^2-1}\right) = (2k+2)(\frac{1}{2}z_0^8)^{(2k+2)^2-1}.
\end{aligned}$$

Since the latter is non-zero we conclude that it is the leading coefficient of N_{2k+2}, so we find $c_{2k+2} = (2k+2)(\frac{1}{2}z_0^8)^{(2k+2)^2-1}$, and we conclude $\deg(N_{2k+2}) = 21((2k+2)^2-1)$. This finishes the proof of the claim for $m = 2k+2$. The rest of the claim now follows from induction. □

We are now ready to prove Proposition 2.2.6. Recall that for a point $P \in S \setminus \{(1:1:0:0)\}$, we denote by $P_{\mathcal{E}}$ the corresponding point on $\mathcal{E}$.

By the fiber of $P_{\mathcal{E}}$ we mean the fiber of the elliptic fibration $\nu\colon \mathcal{E} \longrightarrow \mathbb{P}^1$ that contains $P_{\mathcal{E}}$; see also Notation 2.2.3.

Proof of Proposition 2.2.6. Let k be a number field, and let P be a point $P = (x_0 : y_0 : z_0 : 1)$ as in Proposition 2.2.6. Since P is defined over k and $P_{\mathcal{E}}$ has infinite order on its fiber, the set $\mathcal{P} = \{nP_{\mathcal{E}} : n \in \mathbb{Z} \setminus 0\}$ contains infinitely many points on the fiber of $P_{\mathcal{E}}$ that are all defined over k and have non-zero y, z-coordinates. Since the strict transform of an exceptional curve on S is a section of $\mathcal{E}$ (Remark 1.4.20), there are at most 240 points in $\mathcal{P}$ that are contained in the strict transform of an exceptional curve on S (Table 1.1). Let V_1 be the set of these points. Let V_2 be the set of points $(x_R : y_R : z_0 : 1) \in \mathcal{P}$ such that x_R is a root of the polynomial $q(\tilde{x}, z_0) \in k[\tilde{x}]$ defined in (2.9); there are at most 25 points in V_2. For all points R in $\mathcal{P} \setminus V_1$, the curve C_R is defined over k, and it either contains a section defined over k, or is geometrically integral of genus at most 1, by Lemma 2.3.4. Let V_3 be the set in $\mathcal{P} \setminus (V_1 \cup V_2)$ for which C_R is geometrically integral of genus 1, and for which the elliptic curve E_R has rank 0 over k; the set V_3 is finite by Proposition 2.3.10. We conlude that the set $\mathcal{P} \setminus (V_1 \cup V_2 \cup V_3)$ contains infinitely many points, and all integers n for which $nP_{\mathcal{E}}$ is in this set satisfy the statement in Proposition 2.2.6.

2.4 Proof of the main result

In this section we prove Theorem 2.2.1. Let A, B, k, S, and $\mathcal{E}$ be as in the theorem (in particular, k is now a number field), and recall Notation 2.2.3.

Proof of Theorem 2.2.1. Let P be a point satisfying the conditions in Theorem 2.2.1. By Proposition 2.2.6, there exists an integer n such that one of the following holds.

(i) $C_{nP_{\mathcal{E}}}$ has a component that is a section defined over k,

(ii) $C_{nP_{\mathcal{E}}}$ is a 3-section of $\mathcal{E}$ of geometric genus 0, or

(iii) $C_{nP_{\mathcal{E}}}$ is a 3-section of $\mathcal{E}$ whose normalization is an elliptic curve with positive rank over k.

Choose such an n and set $R = nP_{\mathcal{E}}$. Note that in case (i) we are done by Remark 2.1.5. In case (ii), the desingularization of C_R is a smooth curve of genus 0. Since R is not a triple point on C_R, the latter contains a rational point given by the unique other point in the intersection of

C_R with the fiber of R, hence C_R has infinitely many k-rational points. In case (iii), C_R contains infinitely many k-rational points as well. Now assume we are in case (ii) or (iii). Then C_R contains infinitely many k-rational points, and since C_R intersects each fiber of $\mathcal{E}$ in 3 points counted with multiplicity, this implies that C_R intersects infinitely many fibers in a k-rational point. We show that infinitely many of these points are non-torsion on their fiber. Note that every smooth fiber is an elliptic curve over k, hence there is an upper bound $B = B(k)$ such that on all the fibers, all the torsion points have order at most B [Mer96]. Let $m \leq B$ be an integer, and let T_m be the zero locus of the m-th division polynomial $\psi_m \in k[x, y, t]$ of the generic fiber E over the function field $k(t)$. We have $\psi_m \in k[x, t]$, and for any $\tau \in k$, the polynomial $\psi_m(x, \tau) \in k[x]$ has degree m^2 [Sil09, Exercise III.3.7]. So T_m is an m^2-section of $\mathcal{E}$. Moreover, for every smooth fiber $\mathcal{E}_t$, the intersection of T_m with $\mathcal{E}_t$ is exactly the set of m-torsion points on $\mathcal{E}_t$, which has size m^2 [Sil09, Exercise III.3.7 and Corollary III.6.4]. It follows that T_m intersects every smooth fiber of $\mathcal{E}$ in m^2 points, all with multiplicity 1. In particular, the curve C_R is not a component of T_m, since in all three cases above, C_R intersects the smooth fiber of P in a point with multiplicity 2. Therefore, the curve C_R intersects T_m only in finitely many points. Since all the torsion points on the fibers of $\mathcal{E}$ are contained in the finite union $\cup_{m \leq B} T_m$, we conclude that C_R intersects only finitely many fibers in a torsion point. Since we already showed that C_R intersects infinitely many fibers in a k-rational point, this implies that C_R intersects infinitely many fibers in a k-rational point that is non-torsion on its fiber. We conclude that infinitely many smooth fibers of $\mathcal{E}$ have infinitely many k-rational points. Since a smooth fiber $\mathcal{E}_t$ is closed and irreducible in $\mathcal{E}$, it follows that $\mathcal{E}_t \cap \overline{\mathcal{E}(k)} = \mathcal{E}_t$. So $\overline{\mathcal{E}(k)}$ contains infinitely many one-dimensional irreducible subsets, which implies that it is of dimension 2, and since $\mathcal{E}$ is irreducible we conclude that $\overline{\mathcal{E}(k)} = \mathcal{E}$, i.e., the set of k-rational points of $\mathcal{E}$ is dense in $\mathcal{E}$. Since $\mathcal{E}$ and S are birationally equivalent, it follows that $S(k)$ is dense in S as well. Conversely, if S did not contain a point P as in the theorem, then $S(k)$ would be contained in the union of the torsion locus $\cup_{m \leq B} T_m$ with the two fibers $(1 : 0)$ and $(0 : 1)$ and the singular fibers, which is a strict closed subset of S, hence $S(k)$ would not be dense in S.

2.5 Examples

We conclude this chapter by giving two examples where we prove the density of the set of rational points on specific del Pezzo surfaces of degree 1. The rank of the Mordell–Weil group over $\mathbb{Q}$ of the surfaces in Examples 2.5.1 and 2.5.2 is 0 by [DN, Corollary 2.4 and Figure 5], so in these cases the density of the set of $\mathbb{Q}$-rational points can not be proven by the existence of a section over $\mathbb{Q}$ (see also Remark 2.1.5).

Example 2.5.1. Let k be a number field and let S be the del Pezzo surface of degree 1 in $\mathbb{P}(2,3,1,1)$ given by

$$y^2 = x^3 + 6(27z^6 + w^6).$$

Note that S does not satisfy the conditions of [VA11, Theorem 1.1] since $3 \cdot 27$ is a square and $\gcd(6 \cdot 27, 6) \neq 1$, hence the density of the set of $\mathbb{Q}$-rational points could not be proven by Várilly-Alvarado [VA11, Example 7.2]. However, the fiber $\mathcal{E}_{(1:1)}$ of the anticanonical elliptic surface $\mathcal{E}$ above $(1:1)$ is smooth, and with `magma` we find that this fiber has rank 2. So S contains a point that lies on a smooth fiber of $\mathcal{E}$ and has infinite order, hence $S(k)$ is dense in S by Theorem 2.2.1.
We illustrate this by constructing a 3-section as in (2.2). With `magma` we find two generators for $\mathcal{E}_{(1:1)}(\mathbb{Q})$, given by $P_1 = (1:13:1:1)$ and $P_2 = (22:104:1:1)$. The curve C_{P_1} is cut out from S by

$$3xz - 26y + 323z^3 + 12w^3,$$

and it has geometric genus 1. We find $C_{P_1} \cap \mathcal{E}_{(1:1)} = \{P_1, Q_1\}$ with $Q_1 = \left(-\frac{1343}{676} : \frac{222431}{17576} : 1 : 1\right)$. The elliptic curve $E = (\tilde{C}_{P_1}, Q_1)$ is given by Weierstrass equation

$$\gamma^2 = \xi^3 - 2 \cdot 3^4 \cdot 5^2 \cdot 28368481,$$

and the point $D = \sigma(Q_1) + \sigma^2(Q_1)$ has infinite order on E; its ξ-coordinate is given by

$$\xi_D = \frac{11 \cdot 33487 \cdot 580020724757}{(2 \cdot 12 \cdot \cdot 167 \cdot 523)^2},$$

so D has infinite order on E by a result of Lutz and Nagel ([Corollary VIII.7.2][Sil09]). We conclude that the 3-section C_{P_1} has infinitely many k-rational points. Equivalently, we could have used the point P_2 to create a 3-section with infinitely many k-rational points: the curve C_{P_2} is cut out from S by $1452xz - 208y - 10324z^3 + 12w^3$; it has geometric genus 1, the third point of intersection of C_{P_2} with the fiber $\mathcal{E}_{(1:1)}$ is given by $Q_2 = \left(\frac{12793}{2704} : -\frac{2327053}{140608} : 1 : 1\right)$, and the point $\sigma(Q_2) + \sigma^2(Q_2)$ again has infinite order on the elliptic curve $(\tilde{C}_{P_2}, Q_2)$. We conclude that also C_{P_2} has infinitely many k-rational points.

Example 2.5.2. Let k be a number field and consider the del Pezzo surface S of degree 1 in $\mathbb{P}(2,3,1,1)$ given by

$$y^2 = x^3 + 243z^6 + 16w^6.$$

Note that this surface does not satisfy the conditions of [VA11, Theorem 1.1], so the method there failed in this case [VA11, Remark 7.4]. Salgado and van Luijk made the observation that this surface contains the point $P = (0 : 4 : 0 : 1)$, which is 3-torsion on its fiber on $\mathcal{E}$ (more generally, a surface of the form $y^2 = x^3 + \beta^2 w^6$ has the 3-torsion point $(0 : \beta : 0 : 1)$). However, this point is contained in 9 exceptional curves, so their method does not work with P. They did not find another point for which the computations were doable to show density of $S(k)$ [SvL14, Examples 7.3 and 4.4 (iii)]. Finally, Elkies showed that the set $S(\mathbb{Q})$ is Zariski-dense in S, by constructing a multisection with infinitely many rational points in the linear system $|-3K_S|$ that contains P as a point of multiplicity 3 (this idea was generalized to any surface with a torsion point in the master thesis [Bul18], though under the assumption that at least one of the infinitely many multisections constructed there has infinitely many rational points).

We prove the density of $S(k)$ in S using Theorem 2.2.1: with `magma` we find that the fiber $\mathcal{E}_{(1:5)}$ above $(1 : 5)$ is smooth and has rank 2, so S contains a point that lies on a smooth fiber of $\mathcal{E}$ and has infinite order (for example $P = (-63 : -14 : 1 : 5)$), hence $S(k)$ is dense in S.