Geometry and arithmetic of del Pezzo surfaces of degree 1

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Background

This chapter contains the background for the rest of this thesis. We assume that the reader is familiar with basic algebraic geometry, and more specifically with schemes, divisors, Picard groups, and the process of blowing up a scheme in a point. A classic reference for this is [Har77]. We introduce *del Pezzo surfaces*, and we focus especially on del Pezzo surfaces of degree 1 in Section 1.4. In Sections 1.1, 1.3, 1.4.1, and 1.4.3, we work with del Pezzo surfaces over any field; most results in Sections 1.2 and 1.4.2, however, only hold over algebraically closed fields.

1.1 Del Pezzo surfaces

Definition 1.1.1. A *variety* is a separated scheme of finite type over a field. A variety is *nice* if it is projective, smooth, and geometrically integral.

Definition 1.1.2. A *curve* is a variety of pure dimension 1, and a *surface* is a variety of pure dimension 2.

Notation 1.1.3. For a field k, we denote by $\overline{k}$ a fixed algebraic closure and by k^{sep} the separable closure of k in $\overline{k}$. For a ring A, an A-algebra B, and a scheme X over Spec A, we denote by $X \times_A B$ the base change $X \times_{\text{Spec } A} \text{Spec } B$.

Definition 1.1.4. A *del Pezzo surface* is a nice surface with ample anticanonical divisor. The *degree* of a del Pezzo surface is the self-intersection number of the anticanonical divisor.

If X is a del Pezzo surface of degree d, then, since $-K_X$ is ample, there is an integer $m > 0$ such that $-mK_X$ determines an embedding of X into some projective space. The degree of X under this embedding is $(-mK_X)^2 = m^2K_X^2$, so we have $d = K_X^2 > 0$. Moreover, d is an integer between 1 and 9 [Man86, 24.3 (i)]. A well-known class of del Pezzo surfaces consists of those of degree 3, which are exactly the smooth cubic surfaces in $\mathbb{P}^3$.

Remark 1.1.5. For $d \geq 3$, the anticanonical divisor of a del Pezzo surface of degree d is very ample, and defines an embedding of the surface into a projective space of dimension d [Kol96, III.3.4.3, III.3.5.2]; the image is a surface of degree d. Del Pezzo surfaces of degree 2 are exactly the smooth hypersurfaces of degree 4 in the weighted projective space $\mathbb{P}(2,1,1,1)$, and del Pezzo surfaces of degree 1 are exactly the smooth hypersurfaces of degree 6 in the weighted projective space $\mathbb{P}(2,3,1,1)$ (see [Kol96, III.3.5]; we will show this for the latter in Section 1.4.1).

Remark 1.1.6. If X is a del Pezzo surface over a perfect field k, then the base change $\overline{X} = X \times_k \overline{k}$ is a del Pezzo surface too: assume that $-K_X$ is ample, then we have $-K_X \cdot C > 0$ for every irreducible curve C on X. Now let D be an integral curve on $\overline{X}$, and let C be the image of D on X under the map $\overline{X} \longrightarrow X$. The pullback of C to $\overline{X}$ consists of the Galois conjugates $D_1, \ldots, D_n$ of D under the action of the Galois group $G =\mathrm{Gal}(\overline{k}/k)$. Since $-K_X \cdot C > 0$, and the intersection pairing is preserved under base change, we have $\sum_{i=1}^n -K_{\overline{X}} \cdot D_i = -K_X \cdot C > 0$. Since G acts transitively on the set $\{D_1, \ldots, D_n\}$ [Sta20, Tag 04KY], it follows that $-K_{\overline{X}} \cdot D > 0$. Finally, from $(-K_X)^2 > 0$ it follows that $(-K_{\overline{X}})^2 > 0$, and therefore $-K_{\overline{X}}$ is ample by Nakai-Moishezon [Har77, Theorem V.1.10].

Del Pezzo surfaces over a separably closed field are birationally equivalent to the projective plane. To state a more precise version of this we introduce the notion of *general position*.

Definition 1.1.7. For $r \leq 8$, points $P_1, \ldots, P_r$ in $\mathbb{P}^2$ are in *general position* if no three of them lie on a line, no six of them lie on a conic, and no eight of them lie on a singular cubic with one of these eight points at the singularity.

Theorem 1.1.8. *A del Pezzo surface of degree d over a separably closed field k is isomorphic to either $\mathbb{P}^1_k \times \mathbb{P}^1_k$, in which case $d = 8$, or to $\mathbb{P}^2_k$ blown up at $r \leq 8$ k-rational points in general position, in which case $d = 9 - r$.*

Proof. Manin proved this for k algebraically closed in [Man86, 24.4]; the result for k separably closed followes from [Coo88, Propositions 5 and 7], see for example [VA09, Theorem 2.1.1]. □

The previous theorem and Remark 1.1.6 show that a del Pezzo surface over a perfect field k becomes birationally equivalent to $\mathbb{P}^2$ after a base change to the algebraic closure of k; varieties with this property are called *geometrically rational*. In Theorem 1.3.6 we state Iskoviskih's classificaton of all geometrically rational surfaces.

1.2 The geometric Picard group

Since del Pezzo surfaces are nice, we can identify their Picard group with their Weil divisor class group [Har77, II.6.16]. In this section we state some results about the Picard group of a del Pezzo surface over an algebraically closed field; in this case we can easily describe the Picard group as a result of Theorem 1.1.8. We spend particular attention to the *exceptional classes* in the Picard group. Our main reference for this theory is [Man86].

Let k be an algebraically closed field. Let X be a del Pezzo surface of degree d over k, and assume that X is isomorphic to $\mathbb{P}^2$ blown up in $r = 9 - d$ points $P_1, \ldots, P_r$ in general position. Let K_X be the class in Pic X of a canonical divisor of X, and for $i \in \{1, \ldots, r\}$, let E_i be the class in Pic X corresponding to the exceptional curve above P_i. Finally, let L be the class in Pic X corresponding to the pullback of a line in $\mathbb{P}^2$ that does not contain any of the points $P_1, \ldots, P_r$.

Theorem 1.2.1. *The Picard group Pic X is ismorphic to $\mathbb{Z}^{10-d}$. Moreover, the set $\{L, E_1, \ldots, E_r\}$ forms a basis for Pic X, and we have $-K_X = 3L - \sum_{i=1}^{r} E_i$.*

Proof. This follows from Theorem 1.1.8 and [Man86, 20.9.1 and 20.10]. □

Remark 1.2.2. By Theorem 1.1.8, our assumptions on X are satisfied by all del Pezzo surfaces except for a del Pezzo surface of degree 8 that is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The Picard group of such a surface is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

For $i, j \in \{1, \ldots, r\}$, $i \neq j$, we have $E_i \cdot E_j = 0$, $L \cdot E_i = 0$, $L^2 = 1$, and

$$E_i^2 = -1, \qquad -K_X \cdot E_i = 1. \tag{1.1}$$

Besides $E_1, \ldots, E_r$, there are more classes in Pic X satisfying (1.1). In the rest of this section we will list results about these so-called *exceptional classes*.

Definition 1.2.3. Let Y be a nice surface with canonical class K_Y. An *exceptional class* in Pic Y is a class D with $D^2 = D \cdot K_Y = -1$. An *exceptional curve* on Y is an irreducible curve on Y whose class in Pic Y is an exceptional class.

Every exceptional class in Pic X contains exactly one exceptional curve on X [Man86, 26.2 (i)].

For $d \geq 3$, the anticanonical divisor $-K_X$ determines an embedding φ of X in $\mathbb{P}^d$ (see Remark 1.1.5). If this is the case, and if C is an exceptional curve on X, then its image $\varphi(C)$ has degree $-K_X \cdot C = 1$, hence $\varphi(C)$ is a line on $\varphi(X)$. Therefore one often refers to exceptional curves on del Pezzo surfaces as *lines*.

Remark 1.2.4. By Castelnuovo's contraction theorem, an exceptional curve C on a nice surface Y can be 'blown down' in the sense that there exists a nonsingular projective surface Y_0 with a point P, and a morphism $f\colon Y \longrightarrow Y_0$, such that f is the blow-up of Y_0 in P, and $C = f^{-1}(P)$ [Har77, Theorem V.5.7]. If Y is a del Pezzo surface, then Y_0 is a del Pezzo surface too [Man86, 24.5.2 (i)], of degree one higher than Y.

Proposition 1.2.5. *Every geometrically integral curve on X with negative self-intersection is an exceptional curve, and isomorphic to $\mathbb{P}^1$.*

Proof. This is in [Man86, 24.3 (ii)]; it follows from adjunction. □

The following proposition tells us exactly what the exceptional classes in Pic X look like. Recall that d is the degree of X, and $r = 9 - d$.

Proposition 1.2.6. *For $d \leq 8$, the exceptional classes in Pic X are those of the form $aL - \sum_{i=1}^{r} b_i E_i$ where $r = 9 - d$, and $(a, b_1, \ldots, b_r)$ is given by taking the first $r + 1$ entries of any of the rows of the following table for which the remaining $d - 1$ entries are zero, and permuting $b_1, \ldots, b_r$. So for $d = 1$ all rows are used, for $d = 2$ only rows 1–4, for $d = 3, 4$ rows 1–3, for $d = 5, 6, 7$ rows 1–2, and for $d = 8$ row 1.*

a	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
0	−1	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0
2	1	1	1	1	1	0	0	0
3	2	1	1	1	1	1	1	0
4	2	2	2	1	1	1	1	1
5	2	2	2	2	2	2	1	1
6	3	2	2	2	2	2	2	2

Proof. [Man86, 26.1] □

From this table we find the number of exceptional classes in Pic X, depending on d. Since every exceptional class in Pic X contains exactly one exceptional curve on X, this equals the number of exceptional curves.

d	1	2	3	4	5	6	7	8
# exceptional classes	240	56	27	16	10	6	3	1

Table 1.1: Number of exceptional classes in Pic X, depending on the degree of X.

Remark 1.2.7. We give a geometric description of the table in Proposition 1.2.6 [Man86, 26.2]: an exceptional class of the form $D = aL - \sum_{i=1}^{r} b_i E_i$, with $(a, b_1, \ldots, b_r)$ a vector given by Proposition 1.2.6, is either one of the E_i, where $i \in \{1, \ldots, r\}$ (which is the case if $b_i = -1$), or it is the class corresponding to the strict transform of a curve in $\mathbb{P}^2$ of degree a, going through P_i with multiplicity b_i for each i.

Let I be the set of exceptional classes in Pic X, and let I_0 be the set

$$\{(e_1, \ldots, e_r) \in I^r \mid \forall i \neq j : e_i \cdot e_j = 0\}.$$

Note that $(E_1, \ldots, E_r)$ is an element in I_0. We will show that every element in I_0 gives rise to a basis for Pic X.

Lemma 1.2.8. *For $(e_1, \ldots, e_r) \in I_0$, there is a morphism $f\colon X \longrightarrow \mathbb{P}^2$, and points $Q_1, \ldots, Q_r \in \mathbb{P}^2$ that are in general position, such that f is the blow-up of $\mathbb{P}^2$ at $Q_1, \ldots, Q_r$, and, for all i, the element e_i is the class in Pic X of the exceptional curve above Q_i.*

Proof. Recall that we can blow down an exceptional curve on X and obtain a del Pezzo surface of degree $d+1$ (Remark 1.2.4). Since the exceptional curves in the classes $e_1, \ldots, e_r$ are pairwise disjoint, after blowing down one of them the remaining ones are exceptional curves on the resulting surface. Therefore we can repeatedly blow down the exceptional curves in all the classes $e_1, \ldots, e_r$. It follows that we obtain a morphism $f\colon X \longrightarrow \mathbb{P}^2$, which is the blow-up in r points $Q_1, \ldots, Q_r$. If $Q_1, \ldots, Q_r$ were not in general position, then X would contain curves with self-intersection ≤ -2, contradicting Proposition 1.2.5. $\square$

Let $\iota = (e_1, \ldots, e_r)$ be an element in I_0, and $Q_1, \ldots, Q_r \in \mathbb{P}^2$ as in the previous lemma. Then we have $K_X = -3l + \sum_{i=1}^{r} e_i$, where l is the class of the strict transform of a line in $\mathbb{P}^2$ not containing any of the Q_i, and it follows that $\{l, e_1, \ldots, e_r\}$ forms a basis for Pic X (Theorem 1.2.1).

Remark 1.2.9. Let V be the set of 240 vectors $(a, b_1, \ldots, b_r)$ that are in the table in Proposition 1.2.6 (where the b_i can be permuted). We have a map

$$f\colon I_0 \longrightarrow \mathrm{Hom}_{\mathrm{Set}}(I, V)$$

as follows. For $\iota = (e_1, \ldots, e_r) \in I_0$, let l be the unique class in Pic X such that $K_X = -3l + \sum_{i=1}^{r} e_i$. Then we define $f(\iota)$ as follows.

$$f(\iota)\colon I \longrightarrow V,\ e \longmapsto (e \cdot l, e \cdot e_1, \ldots, e \cdot e_r).$$

The map $f(\iota)$ is a bijection with inverse $f(\iota)^{-1}((a, b_1, \ldots, b_r)) = al - \sum_{i=1}^{r} b_i e_i \in I$. We conclude that every element of I_0 gives rise to a bijection between I and V.

1.3 Minimality

In this section we consider del Pezzo surfaces over non-algebraically closed fields. We state a useful classification of *minimal* del Pezzo surfaces (Theorem 1.3.4). Recall that for a field k we denote by k^{sep} its separable closure. From [Coo88, Proposition 5] it follows that the exceptional curves on a nice surface over a field k are all defined over k^{sep}.

Definition 1.3.1. A nice surface X over a field k is *minimal* if there is no set of pairwise disjoint exceptional curves on X that form an orbit under the action of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ on Pic $(X \times_k k^{\mathrm{sep}})$.

Note that this definition makes sense when we consider Remark 1.2.4; disjoint exceptional curves that are conjugate under the action of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ can be blown down simultaneously. Since after blowing down one obtains a surface that has smaller Picard number, this is a finite process that results in a minimal surface.

If $k = k^{\mathrm{sep}}$, then a minimal del Pezzo surface over k is isomorphic to either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$; this follows from the definition of minimality and from Theorem 1.1.8. For general k, the minimal del Pezzo surfaces are classified in Theorem 1.3.4. We first introduce the following definition.

Definition 1.3.2. A *rational conic bundle* is a minimal nice geometrically rational surface X with a morphism $f\colon X \longrightarrow C$ to a nice curve C of genus 0, such that the generic fiber of f is a smooth curve of genus 0.

The following theorem describes the geometric fibers of a rational conic bundle.

Theorem 1.3.3. *If X is a rational conic bundle over a perfect field k with morphism $f\colon X \longrightarrow C$, then any reducible fiber of the base change $f_{\overline{k}}\colon X \times_k \overline{k} \longrightarrow C \times_k \overline{k}$ consists of two exceptional curves on $X \times_k \overline{k}$ that intersect in a point and are conjugate under the action of* $\mathrm{Gal}(\overline{k}/k)$.

Proof. [Has09, Theorem 3.6] □

We can now classify all minimal del Pezzo surfaces.

Theorem 1.3.4. *Let X be a del Pezzo surface of degree d over a field k.*

(i) If $d = 3, 5, 6, 9$, then X is minimal if and only if Pic $X \simeq \mathbb{Z}$.

(ii) If $d = 1, 2, 4$, then X is minimal if and only if Pic $X \simeq \mathbb{Z}$, or Pic $X \simeq \mathbb{Z} \oplus \mathbb{Z}$ and X is a rational conic bundle.

(iii) If $d = 8$ then X is minimal if and only if Pic $X \simeq \mathbb{Z}$, or $X \simeq C \times C'$, where C, C' are smooth curves of genus 0.

(iv) If $d = 7$ then X is not minimal.

Proof. [Isk80, Corollary of Theorems 5, 4, and 1, before paragraph 4] □

Remark 1.3.5. In case (ii) of Theorem 1.3.4, the surface X admits two representations as a conic bundle; see [Isk80, Theorem 5].

Theorem 1.3.6 classifies all geometrically rational surfaces.

Theorem 1.3.6. *Let X be a smooth projective geometrically rational surface over a field k. Then X is birationally equivalent (over k) to one of the following surfaces.*

(i) A quadric in $\mathbb{P}^3_k$;

(ii) a del Pezzo surface;

(iii) a rational conic bundle.

Proof. [Isk80, Theorem 1] □

1.4 Del Pezzo surfaces of degree 1

In this section we focus on del Pezzo surfaces of degree 1, which are the main objects of study in this thesis. In Section 1.4.1 we show that a del Pezzo surface X of degree 1 with canonical divisor K_X can be embedded as a smooth sextic in the weighted space $\mathbb{P}(2,3,1,1)$, and we describe the different maps induced by the *linear systems* $|-3K_X|$, $|-2K_X|$, and $|-K_X|$. In Section 1.4.2 we describe how the exceptional curves on a del Pezzo surface of degree 1 over an algebraically closed field can be identified with the classical *root system* $\mathbf{E}_8$. Finally, in Section 1.4.3 we study the *elliptic surface* that arises from a del Pezzo surface of degree 1 by blowing up the base point of the anticanonical linear system.

1.4.1 The anticanonical model and linear systems

Let X be a del Pezzo surface of degree 1 over a field k with anticanonical divisor $-K_X$. We start this section by recalling some concepts associated to divisors on X.

Definition 1.4.1. For a divisor D on X, we define $\mathcal{L}(D)$ to be the k-vector space consisting of all the rational functions over k on X with poles at most at D. We denote its dimension by $l(D)$. The *complete linear* *system* $|D|$ associated to D consists of all effective divisors on X that are linearly equivalent to D.

For a divisor D on X, the map $f \longmapsto \mathrm{div}(f) + D$ induces a bijection between the space $(\mathcal{L}(D) - 0)/k^*$ and the complete linear system $|D|$.

Definition 1.4.2. A *linear system* on X is a subset L of a complete linear system $|D|$ for some divisor D on X, such that the image of L under the bijection $\alpha\colon |D| \longrightarrow (\mathcal{L}(D)-0)/k^*$, together with 0, is a sub-vector space, say V, of $\mathcal{L}(D)$. The *dimension* of L is $\dim_k V - 1$.

Definition 1.4.3. A *base point* of a linear system L on X is a point $P \in X$ such that $P \in C$ for all divisors $C \in L$.

Let L be a non-empty linear system on X, such that L corresponds to the sub-vector space $V \subset \mathcal{L}(D)$ for some divisor D on X. Then L determines a rational map $\varphi_L\colon X \dashrightarrow \mathbb{P}^n_k$, where n is the dimension of L. If L is base-point-free, then φ_L is a morphism.

We describe the *anticanonical model* of X.

Definition 1.4.4. The *anticanonical ring* of X is the graded ring

$$R(X, -K_X) = \bigoplus_{m\geq 0} \mathcal{L}(-mK_X),$$

and the *anticanonical model* of X is the scheme Proj $R(X, -K_X)$.

Since $-K_X$ is ample, the ring $R(X, -K_X)$ is non-empty and non-zero, so the anticanonical model of X is well defined. Moreover, X is isomorphic to Proj $R(X, -K_X)$ [Kol96, III.3.5]. We construct the anticanonical model for X, following [CO99].

Lemma 1.4.5. *For all positive integers m we have*

$$l(-mK_X) = 1 + \frac{1}{2}m(m+1)d.$$

Proof. [Kol96, III.3.2.5.2]. □

By the previous lemma, we have $l(-K_X) = 2$. Let $\{z, w\}$ be a basis for $\mathcal{L}(-K_X)$. For all $m \geq 1$, the elements $z^m, z^{m-1}w, \ldots, zw^{m-1}, w^m$ are linearly independent in $\mathcal{L}(-mK_X)$ by [CO99, 2.3]. Therefore, since

$l(-2K_X) = 4$, we can choose an element $x \in \mathcal{L}(-2K_X)$ such that the set $\{z^2, zw, w^2, x\}$ forms a basis for $\mathcal{L}(-2K_X)$. The elements $z^3, z^2w, zw^2, w^3, zx, wx$ in $\mathcal{L}(-3K_X)$ are linearly independent [CO99, p.1200]. Since we have $l(-3K_X) = 7$ we can therefore choose an element $y \in \mathcal{L}(-3K_X)$ to obtain a basis $\{z^3, z^2w, zw^2, w^3, zx, wx, y\}$ of $\mathcal{L}(-3K_X)$. Finally, since $l(-6K_X) = 22$, the 23 elements

$$z^6, z^5w, z^4w^2, z^3w^3, z^2w^4, zw^5, w^6, x^3, x^2z^2, x^2w^2, x^2zw, xz^4, xz^3w,$$
$$xz^2w^2, xzw^3, xw^4, xyz, xyw, y^2, yz^3, yz^2w, yzw^2, yw^3$$

of $\mathcal{L}(-6K_X)$ are linearly dependent. Let $h(x, y, z, w) = 0$ be a dependence relation between them. We can rescale x and y such that the coefficients of the monomials x^3 and y^2 are ± 1, and write

$$h = y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6, \tag{1.2}$$

where $a_i \in k[z, w]$ is homogeneous of degree i for each i in $\{1, \ldots, 6\}$. Let $k[x, y, z, w]$ be the graded k-algebra where x has degree 2, y has degree 3, and z, w have degree 1. Then the anticanonical model of X is Proj $k[x, y, z, w]/(h)$.

The linear system $|-3K_X|$

The linear system $|-3K_X|$ induces an embedding of X into $\mathbb{P}^6$, with coordinates $\{z^3, z^2w, zw^2, w^3, zx, wx, y\}$. This embedding factors through the anticanonical model of X.

For the rest of this section we identify X with its anticanonical model, that is, the zero locus of h in $\mathbb{P}_k(2, 3, 1, 1)$, where h is given by (1.2).

The linear system $|-2K_X|$

Let $p : \mathbb{P}_k(2, 3, 1, 1) \dashrightarrow \mathbb{P}_k(2, 1, 1)$ be the projection to $(x : z : w)$; its restriction to X is a morphism of degree 2. Let $i : \mathbb{P}_k(2, 1, 1) \hookrightarrow \mathbb{P}^3_k$ be the 2-uple embedding, sending $(x : z : w)$ to $(x : z^2 : zw : w^2)$. Write $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ for the coordinates of $\mathbb{P}^3_k$, then $i(\mathbb{P}_k(2, 1, 1))$ is a cone Q given by $\alpha_2^2 = \alpha_1\alpha_3$, with vertex $v = (1 : 0 : 0 : 0)$. The composition $\varphi = i \circ p \colon X \longrightarrow \mathbb{P}^3_k$ is a double cover of Q, and this is the morphism defined by the linear system $|-2K_X|$. If char $k \neq 2$ then we can do a coordinate change such that h is given by $y^2 - x^3 - a_2'x^2 - a_4'x - a_6'$, and the morphism φ is ramified at the points $(x : y : z : w) \in X$ for which

$$x^3 + a_2'x^2 + a_4'x + a_6' = 0. \tag{1.3}$$

In that case, the branch locus of φ is the union of v and the curve B that is the intersection of the cubic surface in $\mathbb{P}^3_k$ defined by (1.3) with Q, and B is a smooth integral curve of degree six and genus four [CO99, Proposition 3.1]. In the case char $k = 2$, the morphism φ is ramified at the points $(x : y : z : w) \in X$ for which $a_1 x + a_3 = 0$, and the branch curve of φ is smooth if and only if the intersection of the zero loci of a_1 and a_3 in $\mathbb{P}^1$ is empty [CO00, Remark 2.5].

The linear system $|-K_X|$

The linear system $|-K_X|$ defines a rational map $\mu\colon X \dashrightarrow \mathbb{P}^1_k$, projecting to the coordinates z, w. This is not defined in the point $\mathcal{O} = (1 : 1 : 0 : 0)$, which is the unique base point of $|-K_X|$. Let $\mathcal{E}$ be the blow-up of X in $\mathcal{O}$, then the rational map μ induces a morphism $\nu : \mathcal{E} \longrightarrow \mathbb{P}^1_k$. This gives $\mathcal{E}$ the structure of an *elliptic surface*; see Section 1.4.3.

Some of the rational maps and morphisms described above are shown in the following commutative diagram.

$$
\begin{tikzcd}
 & & & \mathbb{P}^1_k \\
\mathcal{E} \arrow[r, "\pi"] \arrow[urrr, bend left, "\nu"] & X \arrow[r, hook] \arrow[urr, dashed, "\mu", "|-K_X|"'] \arrow[ddrr, "\varphi", "|-2K_X|"'] & \mathbb{P}_k(2,3,1,1) \arrow[r, dashed, "p"] & \mathbb{P}_k(2,1,1) \arrow[u] \arrow[d, "\simeq"] \arrow[dd, hook, bend left, "i"] \\
 & & & Q \arrow[d, hook] \\
 & & & \mathbb{P}^3_k
\end{tikzcd}
$$

1.4.2 Exceptional curves and the $\mathbf{E}_8$ root system

Let X be a del Pezzo surface of degree 1 over an algebraically closed field. Recall that Pic X contains exactly 240 exceptional classes (Table 1.1); let I be the set of these classes. In this section we describe the relation between I and the *root system* $\mathbf{E}_8$. In particular, we show that the group of permutations of I that preserve the intersection pairing is isomorphic to the automorphism group of $\mathbf{E}_8$ (Corollary 1.4.10), which gives us a very useful tool when studying configurations of exceptional curves. Root systems arise in the study of many different objects, such as Lie groups and the classification of singularities on varieties. We will only treat a very small fraction of the theory of root systems here and in Chapter 3. Useful references for more on root systems are [Bou68] and [Hum72].

We start by recalling the definition of a root system.

Definition 1.4.6. Let V be a finite-dimensional vector space over $\mathbb{R}$ with a positive-definite inner product $\langle \cdot, \cdot \rangle$. A *root system* in V is a finite set R of non-zero vectors, called *roots*, that satisfy the following conditions:

(i) the roots span V;

(ii) for all $r \in R$, we have $\lambda r \in R \Longrightarrow \lambda = \pm 1$;

(iii) for all $r, s \in R$, we have $s - 2r\frac{\langle r,s \rangle}{\langle r,r \rangle} \in R$;

(iv) for all $r, s \in R$, the number $2\frac{\langle r,s \rangle}{\langle r,r \rangle}$ is an integer.

The *rank* of R is the dimension of V.

Definition 1.4.7. If R is a root system in a vector space V with inner product $\langle \cdot, \cdot \rangle$, and S is a root system in a vector space W with inner product $[\cdot, \cdot]$, then R and S are *isomorphic* if there is an isomorphism of vector spaces $\varphi \colon V \longrightarrow W$, which sends R to S, and such that $[\varphi(r_1), \varphi(r_2)] = \langle r_1, r_2 \rangle$ for all $r_1, r_2 \in R$.

Let Λ be the $\boldsymbol{E}_8$ *lattice*, given by

$$\Lambda = \left\{ a \in \mathbb{Z}^8 + \left\langle \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right) \right\rangle \;\middle|\; \sum_{i=1}^{8} a_i \in 2\mathbb{Z} \right\} \subset \mathbb{R}^8.$$

This is the unique positive-definite, even, unimodular lattice of dimension 8 [MH73, II.§6]. The set

$$\mathbf{E}_8 = \{a \in \Lambda \mid \|a\| = \sqrt{2}\}$$

forms a root system in $\mathbb{R}^8$, known as the $\boldsymbol{E}_8$ *root system*. [Hum72, 12.1]. We will show that Pic X contains a subset R that forms a root system isomorphic to $\mathbf{E}_8$ (Proposition 1.4.8), and we will give a bijection between R and I (Remark 1.4.9).

Recall that X is isomorphic to $\mathbb{P}^2$ blown up in 8 points $P_1, \ldots, P_8$ in general position (Theorem 1.1.8). Let K_X be the class in Pic X of a canonical divisor of X. For $i \in \{1, \ldots, 8\}$, let E_i be the class in Pic X corresponding to the exceptional curve above P_i, and let L be the class in Pic X corresponding to the pullback of a line in $\mathbb{P}^2$ that does not contain any of the points $P_1, \ldots, P_8$. Consider the subgroup

$$K_X^{\perp} = \{D \in \text{Pic } X \mid D \cdot K_X = 0\} \subset \text{Pic } X,$$

and its subset

$$R = \{D \in K_X^{\perp} \mid D^2 = -2\}.$$

Let $\left(K_X^{\perp}, \langle \cdot, \cdot \rangle\right)$ be the vector space $\mathbb{R} \otimes_{\mathbb{Z}} K_X^{\perp}$ with inner product $\langle \cdot, \cdot \rangle$ defined as the negative of the intersection pairing in Pic X.

Proposition 1.4.8. *The set R is a root system of rank 8 in $\left(K_X^{\perp}, \langle \cdot, \cdot \rangle\right)$. Moreover, it is isomorphic to $\mathbf{E}_8$, and every element in R can be given as a linear combination with integer coefficients of the elements $r_1, \ldots, r_8 \in R$, given by*

$$E_1 - E_2,\ E_2 - E_3, \ldots, E_7 - E_8,\ L - E_1 - E_2 - E_3.$$

Proof. In [Man86, Propositions 25.1.1 and 25.2] it is shown that R is a root system of rank 8; in [Man86, Theorem 25.4 and Proposition 25.5.6] it is shown that this root system is isomorphic to $\mathbf{E}_8$, and the basis is given. $\square$

Remark 1.4.9. For $e \in I$ we have $e+K_X \in K_X^{\perp}$ and $\langle e+K_X, e+K_X \rangle = 2$, and this gives a bijection

$$I \longrightarrow R, \qquad e \longmapsto e + K_X.$$

For $e_1, e_2 \in I$ we have $\langle e_1 + K_X, e_2 + K_X \rangle = 1 - e_1 \cdot e_2$, where $\cdot$ is the intersection pairing in Pic X.

As a consequence of Proposition 1.4.8 and the bijection in Remark 1.4.9 we have the following result.

Corollary 1.4.10. *The group of permutations of I that preserve the intersection pairing is isomorphic to the Weyl group W_8, which is the group* *of permutations of $\boldsymbol{E}_8$ generated by the reflections in the hyperplanes orthogonal to the roots.*

Proof. [Man86, 25.1.1 and 23.9] $\square$

Another way of phrasing Corollary 1.4.10 is that the *weighted graphs* on I and $\mathbf{E}_8$ and their automorphism groups are isomorphic (Corollary 1.4.14).

Definition 1.4.11. By a *graph* we mean a pair (V, D), where V is a set of elements called *vertices*, and D a subset of the power set of V such that every element in D has cardinality 2; elements in D are called *edges*, and the *size* of the graph is the cardinality of V. A graph (V, D) is *complete* if for every two distinct vertices $v_1, v_2 \in V$, the pair $\{v_1, v_2\}$ is in D.
By a *weighted graph* we mean a graph (V, D) with a map $\psi\colon D \longrightarrow A$, where A is any set, whose elements we call *weights*; for any element d in D we call $\psi(d)$ its weight. If (V, D) is a weighted graph with weight function ψ, then we define a *weighted subgraph* of (V, D) to be a graph (V', D') with map ψ', where V' is a subset of V, while D' is a subset of the intersection of D with the power set of V', and ψ' is the restriction of ψ to D'. A *clique* of a weighted graph is a complete weighted subgraph.
An *isomorphism* between two weighted graphs (V, D) and (V', D') with weight functions $\psi\colon D \longrightarrow A$ and $\psi'\colon D' \longrightarrow A'$, respectively, consists of a bijection f between the sets V and V' and a bijection g between the sets A and A', such that for any two vertices $v_1, v_2 \in V$, we have $\{v_1, v_2\} \in D$ with weight w if and only if $\{f(v_1), f(v_2)\} \in D'$ with weight $g(w)$. We call the map f an *automorphism* of (V, D) if $(V, D) = (V', D')$, $\psi = \psi'$, and g is the identity on A.

Definition 1.4.12. By Γ we denote the complete weighted graph whose vertex set is the set of roots in $\mathbf{E}_8$, and where the weight function is induced by the dot product. Similarly, by G we denote the complete weighted graph whose vertex set is I, and where the weight function is the intersection pairing in Pic X.

We can rephrase Remark 1.4.9 and Corollary 1.4.10 in terms of Γ and G as follows.

Remark 1.4.13. There is an isomorphsim of weighted graphs between G and Γ, that sends a vertex e in G to the corresponding vertex $e + K_X$ in Γ, and an edge $d = \{e_1, e_2\}$ in G with weight w to the edge $\delta = \{e_1 + K_X, e_2 + K_X\}$ in Γ with weight $1 - w$. The different weights that occur in G are $0, 1, 2$, and 3, and they correspond to weights $1, 0, -1$, and -2, respectively, in Γ.

Corollary 1.4.14. *The weighted graphs G and Γ have isomorphic automorphism groups, given by the Weyl group W_8.*

1.4.3 The anticanonical elliptic surface

Let k be a field, and S a del Pezzo surface of degree 1 over k. In this section we give more details about the surface $\mathcal{E}$ that was introduced in Section 1.4.1: it is obtained from S by blowing up the base point $\mathcal{O}$ of the anticanonical linear system $|-K_S|$. We show that it is an *elliptic surface*, and we study the *sections* of this surface and relate these to the exceptional curves on S (Proposition 1.4.21). For more theory on elliptic surfaces, see [Shi90] and [SS10].

Definition 1.4.15. An *elliptic surface* Y is a nice surface with a surjective morphism $f\colon Y \longrightarrow C$, where C is a nice curve, such that the following holds.

- The morphism f admits a *section*, that is, a morphism $s\colon C \longrightarrow Y$ such that
$$f \circ s = id_C.$$
- Almost all fibers of f are elliptic curves.
- No fibers of f contain an exceptional curve of Y.

We call the morphism f an *elliptic fibration*.

We will now describe the surface $\mathcal{E}$, and show that it is an elliptic surface over $\mathbb{P}^1$ (Lemma 1.4.16). We use the same notation as in Section 1.4.1; specifically, we identify the surface S with the smooth sextic in $\mathbb{P}_k(2, 3, 1, 1)$ with coordinates $(x : y : z : w)$ given by $h = 0$, where

$$h = y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6,$$

with $a_i \in k[z, w]$ homogeneous of degree i for each i. The point $\mathcal{O}$ is then given by $(1 : 1 : 0 : 0)$, and the blow-up of S in $\mathcal{O}$ is denoted by $\pi\colon \mathcal{E} \longrightarrow S$. We follow [VAZ09, 7.3] to describe $\mathcal{E}$: it is the subscheme of $\mathbb{P}_k(2, 3, 1, 1) \times \mathbb{P}^1_k$ given by

$$\mathcal{E}\colon \begin{cases} y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6 = 0; \\ vz - uw = 0, \end{cases}$$

where u, v are the coordinates of $\mathbb{P}^1_k$. The projection to $\mathbb{P}^1_k$ is the morphism $\nu\colon \mathcal{E} \longrightarrow \mathbb{P}^1_k$, which was also introduced in Section 1.4.1. Outside the exceptional divisor of π, which is given by $\tilde{\mathcal{O}} = \{(1:1:0:0)\} \times \mathbb{P}^1_k \subset \mathcal{E}$, we have $(u:v) = (z:w)$. Set $t = \frac{u}{v}$, which gives $z = tw$ on $\mathcal{E}$. The *generic fiber* of ν is a cubic curve over the function field $k(t)$ of $\mathbb{P}^1$, and it is the subset of $\mathbb{P}_{k(t)}(2,3,1)$ given by

$$\begin{aligned} E\colon y^2 + wa_1(t,1)xy + w^3a_3(t,1)y - x^3 - w^2a_2(t,1)x^2 \\ - w^4a_4(t,1)x - w^6a_6(t,1) = 0. \end{aligned}$$

Let $\mathbb{A}^2_{k(t)}$ be the affine open subset $w \neq 0$ of $\mathbb{P}_{k(t)}(2,3,1)$ with coordinates $X = \frac{x}{w^2}$, $Y = \frac{y}{w^3}$. The intersection of E with $\mathbb{A}^2_{k(t)}$ is given by

$$Y^2 + a_1\,(t,1)\,XY + a_3\,(t,1)\,Y = X^3 + a_2(t,1)X^2 + a_4(t,1)X + a_6(t,1).$$

Since S is smooth and geometrically rational, the discriminant Δ of E is a polynomial in $k[t]$ of degree between 10 and 12 [SS10, 4.3, 4.4, 8.2, 8.3]. In particular, Δ is not identically 0, so E is an elliptic curve over $k(t)$. Similarly, for $(u_0 : v_0) \in \mathbb{P}^1_k$, the fiber $\nu^{-1}((u_0 : v_0))$ is isomorphic to the cubic curve in $\mathbb{P}^2_k$ with affine Weierstrass equation

$$\begin{aligned} Y^2 + a_1\,(u_0,v_0)\,XY + a_3\,(u_0,v_0)\,Y = X^3 + a_2(u_0,v_0)X^2 \\ + a_4(u_0,v_0)X + a_6(u_0,v_0). \quad (1.4) \end{aligned}$$

This is an elliptic curve for all $(u_0 : v_0) \in \mathbb{P}^1_k$ such that $v_0 \neq 0$ and $\Delta(\frac{u_0}{v_0}) \neq 0$. Therefore, all but finitely many fibers of ν are elliptic curves, with zero-point given by the intersection with the exceptional divisor $\tilde{\mathcal{O}}$.

Let $K_{\mathcal{E}}$ be the canonical divisor on $\mathcal{E}$.

Lemma 1.4.16. *The surface $\mathcal{E}$ is an elliptic surface with elliptic fibration ν. Moreover, every fiber of ν is linearly equivalent to $-K_{\mathcal{E}}$ and has self-intersection 0.*

Proof. We already showed that almost every fiber of ν is an elliptic curve, so we only have to show that no fibers of ν contain an exceptional curve on $\mathcal{E}$. Since all fibers of ν are given by (1.4), they are integral, so the only way they could contain an exceptional curve is if they are one. Since ν restricted to $\mathcal{E} \setminus \tilde{O}$ is the map μ induced by the anticanonical linear system $|-K_S|$ (see Section 1.4.1), the fibers of ν are linearly equivalent to $-K_{\mathcal{E}} = \pi^*(-K_S) + \tilde{\mathcal{O}}$. Since all fibers of ν are linearly equivalent and pairwise disjoint, they have self-intersection 0. Therefore no fiber is equal to an exceptional curve. We conclude that $\mathcal{E}$ is an elliptic surface with elliptic fibration ν. $\square$

Remark 1.4.17. The set of $k(t)$-rational points on E forms a group, the *Mordell–Weil group* of E over $k(t)$ or of $\mathcal{E}$ [Shi90, Theorem 1.1]. This group is torsion-free and has rank at most 8 over k [Shi90, Theorem 10.4]. The set of sections of ν form a group as well, and the map

$$P = (X_P, Y_P) \longmapsto \Big(s\colon \mathbb{P}^1 \setminus \{(1:0)\} \longrightarrow \mathcal{E},\ (t:1) \longmapsto (X_P(t), Y_P(t), t)\Big)$$

induces an isomorphism between the group of $k(t)$-rational points on E and the group of sections of ν that are defined over k [Sil94, Proposition 3.10]. As a consequence of this correspondence, we sometimes talk about a k-section as a morphism $\mathbb{P}^1_k \longrightarrow \mathcal{E}$, and sometimes as a curve on $\mathcal{E}$, whose generic point is the corresponding $k(t)$-rational point on E.

The following definition generalizes the notion of section.

Definition 1.4.18. A *multisection of degree d* or *d-section* of $\mathcal{E}$ is an irreducible curve C contained in $\mathcal{E}$ such that the projection $\varphi|_C\colon C \longrightarrow \mathbb{P}^1_k$ is non-constant and of degree d.

Remark 1.4.19. Note that a section is a multisection of degree 1, and in a similar way as with sections, the d-sections of $\mathcal{E}$ correspond to points on the generic fiber E of $\mathcal{E}$ that are defined over a degree d extension of $k(t)$.

We end this chapter by showing that the exceptional curves on S induce sections of ν, and by giving a characterization of these sections on $\mathcal{E}$.

Remark 1.4.20. Since exceptional curves on S are defined over a separable closure of k (Theorem 1.1.8), from [VA08, Theorem 1.2] it follows that the exceptional curves on S are exactly the curves given by

$$x = p(z,w), \quad y = q(z,w),$$

where $p, q \in k[z,w]$ are homogeneous of degrees 2 and 3. Note that this implies that an exceptional curve never contains $\mathcal{O} = (1:1:0:0)$. Therefore, for an exceptional curve C on S, its strict transform $\pi^*(C)$ on $\mathcal{E}$ satisfies

$$\pi^*(C)^2 = -1, \qquad \pi^*(C) \cdot -K_{\mathcal{E}} = \pi^*(C) \cdot (\pi^*(-K_S) + \tilde{\mathcal{O}}) = 1 + 0 = 1,$$

so $\pi^*(C)$ is an exceptional curve on $\mathcal{E}$ as well. Moreover, since a fiber of ν is linearly equivalent to $-K_{\mathcal{E}}$, the curve $\pi^*(C)$ intersects every fiber once. This gives a section of ν.

Proposition 1.4.21. *Let C be a section of ν on $\mathcal{E}$. The following are equivalent.*

(i) C is the strict transform of an exceptional curve on S.

(ii) C is of the form

$$x = p(z,w), \quad y = q(z,w),$$

where $p, q \in k[z,w]$ are homogeneous of degree 2 and 3.

(iii) C is disjoint from $\tilde{\mathcal{O}}$.

Proof. (i) is equivalent to (ii) by Remark 1.4.20, and (ii) and (iii) are equivalent by [Shi90, Lemma 10.9]. □