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Introduction

Del Pezzo surfaces are surfaces that can be classified by their degree, which is an integer between 1 and 9. They are named after Pasquale del Pezzo, who studied surfaces of degree d in $\mathbb{P}^d$, corresponding to del Pezzo surfaces of degree at least 3; well-known examples are smooth cubic surfaces in $\mathbb{P}^3$. Over an algebraically closed field, del Pezzo surfaces are birationally equivalent to the projective plane, and therefore they have a geometric structure that is easy to describe. However, for lower degree del Pezzo surfaces, this structure is rich enough to provide interesting questions. Moreover, over a non-algebraically closed field k, del Pezzo surfaces are in general not birationally equivalent to the projective plane, and therefore their set of k-rational points can a priori take many forms.

This thesis contains results on both the arithmetic (Chapter 2) and the geometry (Chapters 3–5) of del Pezzo surfaces of degree 1.

Chapter 1 covers the necessary background, assuming the reader is already familiar with basic algebraic geometry. Del Pezzo surfaces are defined there, and it is shown that they contain a finite number of exceptional curves (also called *lines*), based on the degree of the surface. A well-known example of this is the fact that smooth cubic surfaces over $\mathbb{C}$ contain exactly 27 lines. From Section 1.4 on, the focus is on del Pezzo surfaces of degree 1. Such a surface, over a field k, can be defined as the set of solutions in the weighted projective space $\mathbb{P}(2,3,1,1)$ with coordinates x, y, z, w to an equation of the form

$$y^2 + a_1(z,w)xy + a_3(z,w)y = x^3 + a_2(z,w)x^2 + a_4(z,w)x + a_6(z,w), \quad (1)$$

where $a_i \in k[z,w]$ is homogeneous of degree i for each i. The two main features of del Pezzo surfaces of degree 1 that are covered are their associated elliptic surface, which is obtained by blowing up the base point of the anticanonical linear system, and the connection between their exceptional curves and the root system $\mathbf{E}_8$.

In Chapter 2, which is joint work with Julie Desjardins, we study the k-rational points on a family del Pezzo surfaces of degree 1, where k is a number field. These correspond to the solutions to (1) for which all coordinates are elements in k. Our main result is the following.

Theorem 1. *Let k be a number field, let $A, B \in k$ be non-zero, and let S in $\mathbb{P}(2,3,1,1)$ be the del Pezzo surface of degree 1 over k given by*

$$y^2 = x^3 + Az^6 + Bw^6. \tag{2}$$

Let $\mathcal{E}$ be the elliptic surface obtained by blowing up the base point of the linear system $|-K_S|$. Then the set of k-rational points on S is dense in S with respect to the Zariski topology if and only if S contains a k-rational point P with non-zero z, w coordinates, such that the corresponding point on $\mathcal{E}$ lies on a smooth fiber and is non-torsion on that fiber.

As mentioned before, if k is a non-algebraically closed field, it is in general not true that a del Pezzo surface over k is birationally equivalent to the projective plane. One measure of 'how close' a variety is to being birational to projective space is *unirationality*: a variety X over a field k is k-unirational if there is a dominant map $\mathbb{P}^n_k \dashrightarrow X$ for some n. Del Pezzo surfaces of degree at least 2 have been known to be k-unirational for any field k, with an extra condition for degree 2 (summarized in Theorem 2.1.3). For minimal del Pezzo surfaces of degree 1, for a long time no results on unirationality were known, and only recently Kollár and Mella proved that those with Picard rank 2 are unirational [KM17]. Outside this case, the question of k-unirationality for minimal del Pezzo surfaces of degree 1 is wide open. Even though these surfaces always contain a k-rational point, we do not have any example of a minimal del Pezzo surface of degree 1 with Picard rank 1 that is known to be unirational, nor of one that is known not to be unirational.

For an infinite field k, the k-unirationality of a variety X implies that the set $X(k)$ of k-rational points is Zariski-dense in X. Partial results on the Zariski density of the set of rational points on del Pezzo surfaces of degree 1 are known, though most results either depend on conjectures, or

give sufficient conditions that might not be necessary (for an overview, see Section 2.1). Theorem 1 is the first result that gives necessary and sufficient conditions for the set of k-rational points of the family given by (2) to be Zariski-dense, where k is any number field.
After finishing this thesis, Jean-Louis Colliot-Thélène showed us that we can generalize the part of the proof where we show that the conditions are sufficient to any field of characteristic 0 (these conditions are in general not necessary if k is not a number field). We will include this result in the paper that is based on Chapter 2.

Chapters 3 and 4 are adaptations of the preprints [vLWa] and [vLWb], respectively, which have been submitted to journals. As mentioned before, a del Pezzo surface of degree d over an algebraically closed field contains a finite number of exceptional curves, which depends on the degree d. When studying arithmetic questions, the configuration of these curves can be important. For example, one of the conditions that the authors of [SvL14] impose on a del Pezzo surface of degree 1 for the set of rational points to be dense, is for the existence of a point that is not contained in six exceptional curves. Moreover, from [STVA14, Corollary 18], it follows that the set of rational points on a del Pezzo surface of degree 2 is dense if it contains a point that is not contained in four exceptional curves, and lies outside a specific closed subset of the surface. A natural question is therefore the following.

Question 1. Let P be a point on a del Pezzo surface S of degree 1 over an algebraically closed field. How many exceptional curves of S can go through P?

The analogue to Question 1 has been known classically for del Pezzo surfaces of degree at least 2. As an example, del Pezzo surfaces of degree 3 contain 27 exceptional curves, and the maximal number of intersecting exceptional curves is 3. The intersection graph of these curves, where each vertex represents a curve and two vertices are connected if the corresponding curves intersect, contains no fully connected subgraph of size bigger than 3, so the graph immediately gives a sharp upper bound for the number of intersecting exceptional curves. This is also the case for del Pezzo surfaces of degree 2, which contain 56 exceptional curves, of which at most 4 go through a single point. For del Pezzo surfaces of degree 1, which contain 240 exceptional curves, we do not get a sharp upper bound outside characteristic 2 just by looking at the intersection graph. The latter contains fully connected subgraphs of size 16, but we prove that the answer to Question 1 is 10 outside characteristics 2 and 3. More precicely, if S is a del Pezzo surface of degree 1 given by (1), then S is a double cover $\varphi\colon S \longrightarrow Q$ of a cone Q in $\mathbb{P}^3$, ramified over a sextic curve, and in Chapter 4 we prove the following two theorems.

Theorem 2. *Let S be a del Pezzo surface of degree 1, and let P be a point on the ramification curve of φ. The number of exceptional curves that go through P is at most ten if char $k \neq 2$, and at most sixteen if char $k = 2$.*

Theorem 3. *Let S be a del Pezzo surface of degree 1, and let P be a point on S outside the ramification curve of φ. The number of exceptional curves that go through P is at most ten if char $k \neq 3$, and at most twelve if char $k = 3$.*

Chapter 4 is based on work by the same author in [Win14]; Theorem 2 is stated there, as well as the weaker version of Theorem 3 that for a point P ouside the ramification curve, there are at most 12 exceptional curves going through P and at most 10 in characteristic 0. In Chapter 4 we extend these results to all characteristics, and we give examples that show that the upper bounds are sharp in all characteristics except possibly characteristic 5. Moreover, we heavily reduce the use of computer computations in the proof of [Win14, Proposition 4.29]; this is done in Section 4.4.

The 240 exceptional curves on a del Pezzo surface of degree 1 are in one-to-one correspondence with 240 vectors in $\mathbb{R}^8$ that form the $\mathbf{E}_8$ root system. As a consequence of this correspondence, the intersection graph on the exceptional curves, where edges have weight w if the corresponding exceptional curves have intersection multiplicity w, is isomorphic to the graph Γ where vertices represent the vectors in $\mathbf{E}_8$, and where two vertices are connected by an edge of weight a if the two corresponding vectors have dot product a in $\mathbb{R}^8$. In order to prove Theorems 2 and 3 we use results on Γ that were already in [Win14], and are now part of Chapter 3. The graph Γ is too big to let a computer find all the information we needed there in a reasonable time. However, Γ has 696,729,600 symmetries (the automorphism group is the Weyl group W_8), which can be used to reduce computations.

In Chapter 3 we extend the results on Γ that were in [Win14] to a thorough study of the action of W_8 on Γ. The root system $\mathbf{E}_8$ pops up in many branches of mathematics and physics (for example Lie groups, sphere packings, string theory). This chapter can be read separately from the rest of the thesis, and is also interesting for the reader that wants to know about the $\mathbf{E}_8$ root system without any interest in del Pezzo surfaces. However, using the relation with del Pezzo surfaces of degree 1, this chapter also gives a list of all potentially possible configurations of a maximal set of exceptional curves that all intersect in a point. In Theorem 3.1.3 we prove that for a large class of subgraphs of Γ, any two subgraphs from this class are isomorphic if and only if there is a symmetry of Γ that maps one to the other. We also give invariants that determine the isomorphism type of a subgraph. Moreover, in Theorem 3.1.4 we show that for two isomorphic subgraphs G_1, G_2 from this class that do not contain one of 7 specific subgraphs, any isomorphism between G_1 and G_2 extends to a symmetry of the whole graph Γ. These results reduce computations on the graph Γ significantly, since they allow us to study many subgraphs by choosing one representative from their isomorphism class.

In Theorem 1 we require the existence of a point on a del Pezzo surface of degree 1 that is non-torsion on its fiber in the corresponding elliptic fibration. This requirement seems to be related to the existence of a point not being contained in too many exceptional curves: in Section 4 of [SvL14], where many examples of del Pezzo surfaces of degree 1 are given, every point on such a surface that is contained in at least 6 exceptional curves corresponds to a point which is torsion on its fiber. It is therefore a natural question to ask whether there is a relation between these two situations.

Question 2. Let S be a del Pezzo surface of degree 1, and let P be a point on S. If 'many' exceptional curves on S intersect in P, is the corresponding point on the elliptic surface constructed from S then a torsion point on its fiber?

Of course, the word 'many' has to be specified in the above question. Kuwata proved that for del Pezzo surfaces of degree 2, if we take 'many' to be 4, the analogous question has a positive answer; see [Kuw05]. This number is also the maximal number of exceptional curves that can intersect in a point on the surface. In the case of del Pezzo surfaces of degree 1, the question is more complicated, since more exceptional curves can intersect in a single point, and in many different configurations.

In Chapter 5 we prove the following theorem.

Theorem 4. *Let S be a del Pezzo surface of degree 1, and let P be a point on S. If at least 9 exceptional curves on S intersect in P, then the corresponding point on the elliptic surface constructed from S is torsion on its fiber.*

To prove Theorem 4, we use results on the configurations of the 240 lines on a del Pezzo surface of degree 1 from Chapter 3. Moreover, using results from Chapter 4, we give examples of surfaces with 6 exceptional curves that pass through a point P that does not correspond to a torsion point, proving that in general the answer to Question 2 is negative if we take 'many' to be 6 or less. What still remains to be done are the cases of 7 and 8 exceptional curves that intersect in a point.