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## Geometry and arithmetic of del Pezzo surfaces of degree 1

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# **Geometry and arithmetic of del Pezzo surfaces of degree 1**

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