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Geometry and arithmetic of del Pezzo surfaces of degree 1

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Contents

Introduction

Del Pezzo surfaces are surfaces that can be classified by their degree, which is an integer between 1 and 9. They are named after Pasquale del Pezzo, who studied surfaces of degree d in $\mathbb{P}^d$, corresponding to del Pezzo surfaces of degree at least 3; well-known examples are smooth cubic surfaces in $\mathbb{P}^3$. Over an algebraically closed field, del Pezzo surfaces are birationally equivalent to the projective plane, and therefore they have a geometric structure that is easy to describe. However, for lower degree del Pezzo surfaces, this structure is rich enough to provide interesting questions. Moreover, over a non-algebraically closed field k, del Pezzo surfaces are in general not birationally equivalent to the projective plane, and therefore their set of k-rational points can a priori take many forms.

This thesis contains results on both the arithmetic (Chapter 2) and the geometry (Chapters 3–5) of del Pezzo surfaces of degree 1.

Chapter 1 covers the necessary background, assuming the reader is already familiar with basic algebraic geometry. Del Pezzo surfaces are defined there, and it is shown that they contain a finite number of exceptional curves (also called *lines*), based on the degree of the surface. A well-known example of this is the fact that smooth cubic surfaces over $\mathbb{C}$ contain exactly 27 lines. From Section 1.4 on, the focus is on del Pezzo surfaces of degree 1. Such a surface, over a field k, can be defined as the set of solutions in the weighted projective space $\mathbb{P}(2,3,1,1)$ with coordinates x, y, z, w to an equation of the form

$$y^2 + a_1(z,w)xy + a_3(z,w)y = x^3 + a_2(z,w)x^2 + a_4(z,w)x + a_6(z,w), \quad (1)$$

where $a_i \in k[z,w]$ is homogeneous of degree i for each i. The two main features of del Pezzo surfaces of degree 1 that are covered are their associated elliptic surface, which is obtained by blowing up the base point of the anticanonical linear system, and the connection between their exceptional curves and the root system $\mathbf{E}_8$.

In Chapter 2, which is joint work with Julie Desjardins, we study the k-rational points on a family del Pezzo surfaces of degree 1, where k is a number field. These correspond to the solutions to (1) for which all coordinates are elements in k. Our main result is the following.

Theorem 1. *Let k be a number field, let $A, B \in k$ be non-zero, and let S in $\mathbb{P}(2,3,1,1)$ be the del Pezzo surface of degree 1 over k given by*

$$y^2 = x^3 + Az^6 + Bw^6. \tag{2}$$

Let $\mathcal{E}$ be the elliptic surface obtained by blowing up the base point of the linear system $|-K_S|$. Then the set of k-rational points on S is dense in S with respect to the Zariski topology if and only if S contains a k-rational point P with non-zero z, w coordinates, such that the corresponding point on $\mathcal{E}$ lies on a smooth fiber and is non-torsion on that fiber.

As mentioned before, if k is a non-algebraically closed field, it is in general not true that a del Pezzo surface over k is birationally equivalent to the projective plane. One measure of 'how close' a variety is to being birational to projective space is *unirationality*: a variety X over a field k is k-unirational if there is a dominant map $\mathbb{P}^n_k \dashrightarrow X$ for some n. Del Pezzo surfaces of degree at least 2 have been known to be k-unirational for any field k, with an extra condition for degree 2 (summarized in Theorem 2.1.3). For minimal del Pezzo surfaces of degree 1, for a long time no results on unirationality were known, and only recently Kollár and Mella proved that those with Picard rank 2 are unirational [KM17]. Outside this case, the question of k-unirationality for minimal del Pezzo surfaces of degree 1 is wide open. Even though these surfaces always contain a k-rational point, we do not have any example of a minimal del Pezzo surface of degree 1 with Picard rank 1 that is known to be unirational, nor of one that is known not to be unirational.
For an infinite field k, the k-unirationality of a variety X implies that the set $X(k)$ of k-rational points is Zariski-dense in X. Partial results on the Zariski density of the set of rational points on del Pezzo surfaces of degree 1 are known, though most results either depend on conjectures, or

give sufficient conditions that might not be necessary (for an overview, see Section 2.1). Theorem 1 is the first result that gives necessary and sufficient conditions for the set of k-rational points of the family given by (2) to be Zariski-dense, where k is any number field.
After finishing this thesis, Jean-Louis Colliot-Thélène showed us that we can generalize the part of the proof where we show that the conditions are sufficient to any field of characteristic 0 (these conditions are in general not necessary if k is not a number field). We will include this result in the paper that is based on Chapter 2.

Chapters 3 and 4 are adaptations of the preprints [vLWa] and [vLWb], respectively, which have been submitted to journals. As mentioned before, a del Pezzo surface of degree d over an algebraically closed field contains a finite number of exceptional curves, which depends on the degree d. When studying arithmetic questions, the configuration of these curves can be important. For example, one of the conditions that the authors of [SvL14] impose on a del Pezzo surface of degree 1 for the set of rational points to be dense, is for the existence of a point that is not contained in six exceptional curves. Moreover, from [STVA14, Corollary 18], it follows that the set of rational points on a del Pezzo surface of degree 2 is dense if it contains a point that is not contained in four exceptional curves, and lies outside a specific closed subset of the surface. A natural question is therefore the following.

Question 1. Let P be a point on a del Pezzo surface S of degree 1 over an algebraically closed field. How many exceptional curves of S can go through P?

The analogue to Question 1 has been known classically for del Pezzo surfaces of degree at least 2. As an example, del Pezzo surfaces of degree 3 contain 27 exceptional curves, and the maximal number of intersecting exceptional curves is 3. The intersection graph of these curves, where each vertex represents a curve and two vertices are connected if the corresponding curves intersect, contains no fully connected subgraph of size bigger than 3, so the graph immediately gives a sharp upper bound for the number of intersecting exceptional curves. This is also the case for del Pezzo surfaces of degree 2, which contain 56 exceptional curves, of which at most 4 go through a single point. For del Pezzo surfaces of degree 1, which contain 240 exceptional curves, we do not get a sharp upper bound outside characteristic 2 just by looking at the intersection graph. The latter contains fully connected subgraphs of size 16, but we prove that the answer to Question 1 is 10 outside characteristics 2 and 3. More precicely, if S is a del Pezzo surface of degree 1 given by (1), then S is a double cover $\varphi\colon S \longrightarrow Q$ of a cone Q in $\mathbb{P}^3$, ramified over a sextic curve, and in Chapter 4 we prove the following two theorems.

Theorem 2. *Let S be a del Pezzo surface of degree 1, and let P be a point on the ramification curve of φ. The number of exceptional curves that go through P is at most ten if char $k \neq 2$, and at most sixteen if char $k = 2$.*

Theorem 3. *Let S be a del Pezzo surface of degree 1, and let P be a point on S outside the ramification curve of φ. The number of exceptional curves that go through P is at most ten if char $k \neq 3$, and at most twelve if char $k = 3$.*

Chapter 4 is based on work by the same author in [Win14]; Theorem 2 is stated there, as well as the weaker version of Theorem 3 that for a point P ouside the ramification curve, there are at most 12 exceptional curves going through P and at most 10 in characteristic 0. In Chapter 4 we extend these results to all characteristics, and we give examples that show that the upper bounds are sharp in all characteristics except possibly characteristic 5. Moreover, we heavily reduce the use of computer computations in the proof of [Win14, Proposition 4.29]; this is done in Section 4.4.

The 240 exceptional curves on a del Pezzo surface of degree 1 are in one-to-one correspondence with 240 vectors in $\mathbb{R}^8$ that form the $\mathbf{E}_8$ root system. As a consequence of this correspondence, the intersection graph on the exceptional curves, where edges have weight w if the corresponding exceptional curves have intersection multiplicity w, is isomorphic to the graph Γ where vertices represent the vectors in $\mathbf{E}_8$, and where two vertices are connected by an edge of weight a if the two corresponding vectors have dot product a in $\mathbb{R}^8$. In order to prove Theorems 2 and 3 we use results on Γ that were already in [Win14], and are now part of Chapter 3. The graph Γ is too big to let a computer find all the information we needed there in a reasonable time. However, Γ has 696,729,600 symmetries (the automorphism group is the Weyl group W_8), which can be used to reduce computations.

In Chapter 3 we extend the results on Γ that were in [Win14] to a thorough study of the action of W_8 on Γ. The root system $\mathbf{E}_8$ pops up in many branches of mathematics and physics (for example Lie groups, sphere packings, string theory). This chapter can be read separately from the rest of the thesis, and is also interesting for the reader that wants to know about the $\mathbf{E}_8$ root system without any interest in del Pezzo surfaces. However, using the relation with del Pezzo surfaces of degree 1, this chapter also gives a list of all potentially possible configurations of a maximal set of exceptional curves that all intersect in a point. In Theorem 3.1.3 we prove that for a large class of subgraphs of Γ, any two subgraphs from this class are isomorphic if and only if there is a symmetry of Γ that maps one to the other. We also give invariants that determine the isomorphism type of a subgraph. Moreover, in Theorem 3.1.4 we show that for two isomorphic subgraphs G_1, G_2 from this class that do not contain one of 7 specific subgraphs, any isomorphism between G_1 and G_2 extends to a symmetry of the whole graph Γ. These results reduce computations on the graph Γ significantly, since they allow us to study many subgraphs by choosing one representative from their isomorphism class.

In Theorem 1 we require the existence of a point on a del Pezzo surface of degree 1 that is non-torsion on its fiber in the corresponding elliptic fibration. This requirement seems to be related to the existence of a point not being contained in too many exceptional curves: in Section 4 of [SvL14], where many examples of del Pezzo surfaces of degree 1 are given, every point on such a surface that is contained in at least 6 exceptional curves corresponds to a point which is torsion on its fiber. It is therefore a natural question to ask whether there is a relation between these two situations.

Question 2. Let S be a del Pezzo surface of degree 1, and let P be a point on S. If 'many' exceptional curves on S intersect in P, is the corresponding point on the elliptic surface constructed from S then a torsion point on its fiber?

Of course, the word 'many' has to be specified in the above question. Kuwata proved that for del Pezzo surfaces of degree 2, if we take 'many' to be 4, the analogous question has a positive answer; see [Kuw05]. This number is also the maximal number of exceptional curves that can intersect in a point on the surface. In the case of del Pezzo surfaces of degree 1, the question is more complicated, since more exceptional curves can intersect in a single point, and in many different configurations.

In Chapter 5 we prove the following theorem.

Theorem 4. *Let S be a del Pezzo surface of degree 1, and let P be a point on S. If at least 9 exceptional curves on S intersect in P, then the corresponding point on the elliptic surface constructed from S is torsion on its fiber.*

To prove Theorem 4, we use results on the configurations of the 240 lines on a del Pezzo surface of degree 1 from Chapter 3. Moreover, using results from Chapter 4, we give examples of surfaces with 6 exceptional curves that pass through a point P that does not correspond to a torsion point, proving that in general the answer to Question 2 is negative if we take 'many' to be 6 or less. What still remains to be done are the cases of 7 and 8 exceptional curves that intersect in a point.

1

Background

This chapter contains the background for the rest of this thesis. We assume that the reader is familiar with basic algebraic geometry, and more specifically with schemes, divisors, Picard groups, and the process of blowing up a scheme in a point. A classic reference for this is [Har77]. We introduce *del Pezzo surfaces*, and we focus especially on del Pezzo surfaces of degree 1 in Section 1.4. In Sections 1.1, 1.3, 1.4.1, and 1.4.3, we work with del Pezzo surfaces over any field; most results in Sections 1.2 and 1.4.2, however, only hold over algebraically closed fields.

1.1 Del Pezzo surfaces

Definition 1.1.1. A *variety* is a separated scheme of finite type over a field. A variety is *nice* if it is projective, smooth, and geometrically integral.

Definition 1.1.2. A *curve* is a variety of pure dimension 1, and a *surface* is a variety of pure dimension 2.

Notation 1.1.3. For a field k, we denote by $\overline{k}$ a fixed algebraic closure and by k^{sep} the separable closure of k in $\overline{k}$. For a ring A, an A-algebra B, and a scheme X over Spec A, we denote by $X \times_A B$ the base change $X \times_{\mathrm{Spec}\ A} \mathrm{Spec}\ B$.

Definition 1.1.4. A *del Pezzo surface* is a nice surface with ample anticanonical divisor. The *degree* of a del Pezzo surface is the self-intersection number of the anticanonical divisor.

If X is a del Pezzo surface of degree d, then, since $-K_X$ is ample, there is an integer $m > 0$ such that $-mK_X$ determines an embedding of X into some projective space. The degree of X under this embedding is $(-mK_X)^2 = m^2K_X^2$, so we have $d = K_X^2 > 0$. Moreover, d is an integer between 1 and 9 [Man86, 24.3 (i)]. A well-known class of del Pezzo surfaces consists of those of degree 3, which are exactly the smooth cubic surfaces in $\mathbb{P}^3$.

Remark 1.1.5. For $d \geq 3$, the anticanonical divisor of a del Pezzo surface of degree d is very ample, and defines an embedding of the surface into a projective space of dimension d [Kol96, III.3.4.3, III.3.5.2]; the image is a surface of degree d. Del Pezzo surfaces of degree 2 are exactly the smooth hypersurfaces of degree 4 in the weighted projective space $\mathbb{P}(2,1,1,1)$, and del Pezzo surfaces of degree 1 are exactly the smooth hypersurfaces of degree 6 in the weighted projective space $\mathbb{P}(2,3,1,1)$ (see [Kol96, III.3.5]; we will show this for the latter in Section 1.4.1).

Remark 1.1.6. If X is a del Pezzo surface over a perfect field k, then the base change $\overline{X} = X \times_k \overline{k}$ is a del Pezzo surface too: assume that $-K_X$ is ample, then we have $-K_X \cdot C > 0$ for every irreducible curve C on X. Now let D be an integral curve on $\overline{X}$, and let C be the image of D on X under the map $\overline{X} \longrightarrow X$. The pullback of C to $\overline{X}$ consists of the Galois conjugates $D_1, \ldots, D_n$ of D under the action of the Galois group $G = \mathrm{Gal}(\overline{k}/k)$. Since $-K_X \cdot C > 0$, and the intersection pairing is preserved under base change, we have $\sum_{i=1}^n -K_{\overline{X}} \cdot D_i = -K_X \cdot C > 0$. Since G acts transitively on the set $\{D_1, \ldots, D_n\}$ [Sta20, Tag 04KY], it follows that $-K_{\overline{X}} \cdot D > 0$. Finally, from $(-K_X)^2 > 0$ it follows that $(-K_{\overline{X}})^2 > 0$, and therefore $-K_{\overline{X}}$ is ample by Nakai-Moishezon [Har77, Theorem V.1.10].

Del Pezzo surfaces over a separably closed field are birationally equivalent to the projective plane. To state a more precise version of this we introduce the notion of *general position*.

Definition 1.1.7. For $r \leq 8$, points $P_1, \ldots, P_r$ in $\mathbb{P}^2$ are in *general position* if no three of them lie on a line, no six of them lie on a conic, and no eight of them lie on a singular cubic with one of these eight points at the singularity.

Theorem 1.1.8. *A del Pezzo surface of degree d over a separably closed field k is isomorphic to either $\mathbb{P}^1_k \times \mathbb{P}^1_k$, in which case $d = 8$, or to $\mathbb{P}^2_k$ blown up at $r \leq 8$ k-rational points in general position, in which case $d = 9 - r$.*

Proof. Manin proved this for k algebraically closed in [Man86, 24.4]; the result for k separably closed followes from [Coo88, Propositions 5 and 7], see for example [VA09, Theorem 2.1.1]. □

The previous theorem and Remark 1.1.6 show that a del Pezzo surface over a perfect field k becomes birationally equivalent to $\mathbb{P}^2$ after a base change to the algebraic closure of k; varieties with this property are called *geometrically rational*. In Theorem 1.3.6 we state Iskoviskih's classificaton of all geometrically rational surfaces.

1.2 The geometric Picard group

Since del Pezzo surfaces are nice, we can identify their Picard group with their Weil divisor class group [Har77, II.6.16]. In this section we state some results about the Picard group of a del Pezzo surface over an algebraically closed field; in this case we can easily describe the Picard group as a result of Theorem 1.1.8. We spend particular attention to the *exceptional classes* in the Picard group. Our main reference for this theory is [Man86].

Let k be an algebraically closed field. Let X be a del Pezzo surface of degree d over k, and assume that X is isomorphic to $\mathbb{P}^2$ blown up in $r = 9 - d$ points $P_1, \ldots, P_r$ in general position. Let K_X be the class in Pic X of a canonical divisor of X, and for $i \in \{1, \ldots, r\}$, let E_i be the class in Pic X corresponding to the exceptional curve above P_i. Finally, let L be the class in Pic X corresponding to the pullback of a line in $\mathbb{P}^2$ that does not contain any of the points $P_1, \ldots, P_r$.

Theorem 1.2.1. *The Picard group Pic X is ismorphic to $\mathbb{Z}^{10-d}$. Moreover, the set $\{L, E_1, \ldots, E_r\}$ forms a basis for Pic X, and we have $-K_X = 3L - \sum_{i=1}^r E_i$.*

Proof. This follows from Theorem 1.1.8 and [Man86, 20.9.1 and 20.10]. □

Remark 1.2.2. By Theorem 1.1.8, our assumptions on X are satisfied by all del Pezzo surfaces except for a del Pezzo surface of degree 8 that is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The Picard group of such a surface is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

For $i, j \in \{1, \ldots, r\},\ i \neq j$, we have $E_i \cdot E_j = 0,\ \ L \cdot E_i = 0,\ \ L^2 = 1$, and

$$E_i^2 = -1, \qquad -K_X \cdot E_i = 1. \tag{1.1}$$

Besides $E_1, \ldots, E_r$, there are more classes in Pic X satisfying (1.1). In the rest of this section we will list results about these so-called *exceptional classes*.

Definition 1.2.3. Let Y be a nice surface with canonical class K_Y. An *exceptional class* in Pic Y is a class D with $D^2 = D \cdot K_Y = -1$. An *exceptional curve* on Y is an irreducible curve on Y whose class in Pic Y is an exceptional class.

Every exceptional class in Pic X contains exactly one exceptional curve on X [Man86, 26.2 (i)].

For $d \geq 3$, the anticanonical divisor $-K_X$ determines an embedding φ of X in $\mathbb{P}^d$ (see Remark 1.1.5). If this is the case, and if C is an exceptional curve on X, then its image $\varphi(C)$ has degree $-K_X \cdot C = 1$, hence $\varphi(C)$ is a line on $\varphi(X)$. Therefore one often refers to exceptional curves on del Pezzo surfaces as *lines*.

Remark 1.2.4. By Castelnuovo's contraction theorem, an exceptional curve C on a nice surface Y can be 'blown down' in the sense that there exists a nonsingular projective surface Y_0 with a point P, and a morphism $f\colon Y \longrightarrow Y_0$, such that f is the blow-up of Y_0 in P, and $C = f^{-1}(P)$ [Har77, Theorem V.5.7]. If Y is a del Pezzo surface, then Y_0 is a del Pezzo surface too [Man86, 24.5.2 (i)], of degree one higher than Y.

Proposition 1.2.5. *Every geometrically integral curve on X with negative self-intersection is an exceptional curve, and isomorphic to $\mathbb{P}^1$.*

Proof. This is in [Man86, 24.3 (ii)]; it follows from adjunction. □

The following proposition tells us exactly what the exceptional classes in Pic X look like. Recall that d is the degree of X, and $r = 9 - d$.

Proposition 1.2.6. *For $d \leq 8$, the exceptional classes in Pic X are those of the form $aL - \sum_{i=1}^{r} b_i E_i$ where $r = 9 - d$, and $(a, b_1, \ldots, b_r)$ is given by taking the first $r+1$ entries of any of the rows of the following table for which the remaining $d-1$ entries are zero, and permuting $b_1, \ldots, b_r$. So for $d = 1$ all rows are used, for $d = 2$ only rows 1–4, for $d = 3, 4$ rows 1–3, for $d = 5, 6, 7$ rows 1–2, and for $d = 8$ row 1.*

a	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
0	−1	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0
2	1	1	1	1	1	0	0	0
3	2	1	1	1	1	1	1	0
4	2	2	2	1	1	1	1	1
5	2	2	2	2	2	2	1	1
6	3	2	2	2	2	2	2	2

Proof. [Man86, 26.1] □

From this table we find the number of exceptional classes in Pic X, depending on d. Since every exceptional class in Pic X contains exactly one exceptional curve on X, this equals the number of exceptional curves.

d	1	2	3	4	5	6	7	8
# exceptional classes	240	56	27	16	10	6	3	1

Table 1.1: Number of exceptional classes in Pic X, depending on the degree of X.

Remark 1.2.7. We give a geometric description of the table in Proposition 1.2.6 [Man86, 26.2]: an exceptional class of the form $D = aL - \sum_{i=1}^{r} b_i E_i$, with $(a, b_1, \ldots, b_r)$ a vector given by Proposition 1.2.6, is either one of the E_i, where $i \in \{1, \ldots, r\}$ (which is the case if $b_i = -1$), or it is the class corresponding to the strict transform of a curve in $\mathbb{P}^2$ of degree a, going through P_i with multiplicity b_i for each i.

Let I be the set of exceptional classes in Pic X, and let I_0 be the set

$$\{(e_1, \ldots, e_r) \in I^r \mid \forall i \neq j : e_i \cdot e_j = 0\}.$$

Note that $(E_1, \ldots, E_r)$ is an element in I_0. We will show that every element in I_0 gives rise to a basis for Pic X.

Lemma 1.2.8. *For $(e_1, \ldots, e_r) \in I_0$, there is a morphism $f\colon X \longrightarrow \mathbb{P}^2$, and points $Q_1, \ldots, Q_r \in \mathbb{P}^2$ that are in general position, such that f is the blow-up of $\mathbb{P}^2$ at $Q_1, \ldots, Q_r$, and, for all i, the element e_i is the class in Pic X of the exceptional curve above Q_i.*

Proof. Recall that we can blow down an exceptional curve on X and obtain a del Pezzo surface of degree $d+1$ (Remark 1.2.4). Since the exceptional curves in the classes $e_1, \ldots, e_r$ are pairwise disjoint, after blowing down one of them the remaining ones are exceptional curves on the resulting surface. Therefore we can repeatedly blow down the exceptional curves in all the classes $e_1, \ldots, e_r$. It follows that we obtain a morphism $f\colon X \longrightarrow \mathbb{P}^2$, which is the blow-up in r points $Q_1, \ldots, Q_r$. If $Q_1, \ldots, Q_r$ were not in general position, then X would contain curves with self-intersection ≤ -2, contradicting Proposition 1.2.5. $\square$

Let $\iota = (e_1, \ldots, e_r)$ be an element in I_0, and $Q_1, \ldots, Q_r \in \mathbb{P}^2$ as in the previous lemma. Then we have $K_X = -3l + \sum_{i=1}^r e_i$, where l is the class of the strict transform of a line in $\mathbb{P}^2$ not containing any of the Q_i, and it follows that $\{l, e_1, \ldots, e_r\}$ forms a basis for Pic X (Theorem 1.2.1).

Remark 1.2.9. Let V be the set of 240 vectors $(a, b_1, \ldots, b_r)$ that are in the table in Proposition 1.2.6 (where the b_i can be permuted). We have a map

$$f\colon I_0 \longrightarrow \mathrm{Hom}_{\mathrm{Set}}(I, V)$$

as follows. For $\iota = (e_1, \ldots, e_r) \in I_0$, let l be the unique class in Pic X such that $K_X = -3l + \sum_{i=1}^r e_i$. Then we define $f(\iota)$ as follows.

$$f(\iota)\colon I \longrightarrow V,\ e \longmapsto (e \cdot l, e \cdot e_1, \ldots, e \cdot e_r).$$

The map $f(\iota)$ is a bijection with inverse $f(\iota)^{-1}((a, b_1, \ldots, b_r)) = al - \sum_{i=1}^r b_i e_i \in I$. We conclude that every element of I_0 gives rise to a bijection between I and V.

1.3 Minimality

In this section we consider del Pezzo surfaces over non-algebraically closed fields. We state a useful classification of *minimal* del Pezzo surfaces (Theorem 1.3.4). Recall that for a field k we denote by k^{sep} its separable closure. From [Coo88, Proposition 5] it follows that the exceptional curves on a nice surface over a field k are all defined over k^{sep}.

Definition 1.3.1. A nice surface X over a field k is *minimal* if there is no set of pairwise disjoint exceptional curves on X that form an orbit under the action of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ on $\mathrm{Pic}\,(X \times_k k^{\mathrm{sep}})$.

Note that this definition makes sense when we consider Remark 1.2.4; disjoint exceptional curves that are conjugate under the action of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ can be blown down simultaneously. Since after blowing down one obtains a surface that has smaller Picard number, this is a finite process that results in a minimal surface.

If $k = k^{\mathrm{sep}}$, then a minimal del Pezzo surface over k is isomorphic to either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$; this follows from the definition of minimality and from Theorem 1.1.8. For general k, the minimal del Pezzo surfaces are classified in Theorem 1.3.4. We first introduce the following definition.

Definition 1.3.2. A *rational conic bundle* is a minimal nice geometrically rational surface X with a morphism $f\colon X \longrightarrow C$ to a nice curve C of genus 0, such that the generic fiber of f is a smooth curve of genus 0.

The following theorem describes the geometric fibers of a rational conic bundle.

Theorem 1.3.3. *If X is a rational conic bundle over a perfect field k with morphism $f\colon X \longrightarrow C$, then any reducible fiber of the base change $f_{\overline{k}}\colon X \times_k \overline{k} \longrightarrow C \times_k \overline{k}$ consists of two exceptional curves on $X \times_k \overline{k}$ that intersect in a point and are conjugate under the action of* $\mathrm{Gal}(\overline{k}/k)$.

Proof. [Has09, Theorem 3.6] □

We can now classify all minimal del Pezzo surfaces.

Theorem 1.3.4. *Let X be a del Pezzo surface of degree d over a field k.*

(i) If $d = 3, 5, 6, 9$, then X is minimal if and only if Pic $X \simeq \mathbb{Z}$.

(ii) If $d = 1, 2, 4$, then X is minimal if and only if Pic $X \simeq \mathbb{Z}$, or Pic $X \simeq \mathbb{Z} \oplus \mathbb{Z}$ and X is a rational conic bundle.

(iii) If $d = 8$ then X is minimal if and only if Pic $X \simeq \mathbb{Z}$, or $X \simeq C \times C'$, where C, C' are smooth curves of genus 0.

(iv) If $d = 7$ then X is not minimal.

Proof. [Isk80, Corollary of Theorems 5, 4, and 1, before paragraph 4] □

Remark 1.3.5. In case (ii) of Theorem 1.3.4, the surface X admits two representations as a conic bundle; see [Isk80, Theorem 5].

Theorem 1.3.6 classifies all geometrically rational surfaces.

Theorem 1.3.6. *Let X be a smooth projective geometrically rational surface over a field k. Then X is birationally equivalent (over k) to one of the following surfaces.*

(i) A quadric in $\mathbb{P}^3_k$;

(ii) a del Pezzo surface;

(iii) a rational conic bundle.

Proof. [Isk80, Theorem 1] □

1.4 Del Pezzo surfaces of degree 1

In this section we focus on del Pezzo surfaces of degree 1, which are the main objects of study in this thesis. In Section 1.4.1 we show that a del Pezzo surface X of degree 1 with canonical divisor K_X can be embedded as a smooth sextic in the weighted space $\mathbb{P}(2,3,1,1)$, and we describe the different maps induced by the *linear systems* $|-3K_X|$, $|-2K_X|$, and $|-K_X|$. In Section 1.4.2 we describe how the exceptional curves on a del Pezzo surface of degree 1 over an algebraically closed field can be identified with the classical *root system* $\mathbf{E}_8$. Finally, in Section 1.4.3 we study the *elliptic surface* that arises from a del Pezzo surface of degree 1 by blowing up the base point of the anticanonical linear system.

1.4.1 The anticanonical model and linear systems

Let X be a del Pezzo surface of degree 1 over a field k with anticanonical divisor $-K_X$. We start this section by recalling some concepts associated to divisors on X.

Definition 1.4.1. For a divisor D on X, we define $\mathcal{L}(D)$ to be the k-vector space consisting of all the rational functions over k on X with poles at most at D. We denote its dimension by $l(D)$. The *complete linear* *system* $|D|$ associated to D consists of all effective divisors on X that are linearly equivalent to D.

For a divisor D on X, the map $f \longmapsto \mathrm{div}(f) + D$ induces a bijection between the space $(\mathcal{L}(D) - 0)/k^*$ and the complete linear system $|D|$.

Definition 1.4.2. A *linear system* on X is a subset L of a complete linear system $|D|$ for some divisor D on X, such that the image of L under the bijection $\alpha\colon |D| \longrightarrow (\mathcal{L}(D)-0)/k^*$, together with 0, is a sub-vector space, say V, of $\mathcal{L}(D)$. The *dimension* of L is $\dim_k V - 1$.

Definition 1.4.3. A *base point* of a linear system L on X is a point $P \in X$ such that $P \in C$ for all divisors $C \in L$.

Let L be a non-empty linear system on X, such that L corresponds to the sub-vector space $V \subset \mathcal{L}(D)$ for some divisor D on X. Then L determines a rational map $\varphi_L\colon X \dashrightarrow \mathbb{P}^n_k$, where n is the dimension of L. If L is base-point-free, then φ_L is a morphism.

We describe the *anticanonical model* of X.

Definition 1.4.4. The *anticanonical ring* of X is the graded ring

$$R(X, -K_X) = \bigoplus_{m \geq 0} \mathcal{L}(-mK_X),$$

and the *anticanonical model* of X is the scheme Proj $R(X, -K_X)$.

Since $-K_X$ is ample, the ring $R(X, -K_X)$ is non-empty and non-zero, so the anticanonical model of X is well defined. Moreover, X is isomorphic to Proj $R(X, -K_X)$ [Kol96, III.3.5]. We construct the anticanonical model for X, following [CO99].

Lemma 1.4.5. *For all positive integers m we have*

$$l(-mK_X) = 1 + \frac{1}{2}m(m+1)d.$$

Proof. [Kol96, III.3.2.5.2]. □

By the previous lemma, we have $l(-K_X) = 2$. Let $\{z, w\}$ be a basis for $\mathcal{L}(-K_X)$. For all $m \geq 1$, the elements $z^m, z^{m-1}w, \ldots, zw^{m-1}, w^m$ are linearly independent in $\mathcal{L}(-mK_X)$ by [CO99, 2.3]. Therefore, since

$l(-2K_X) = 4$, we can choose an element $x \in \mathcal{L}(-2K_X)$ such that the set $\{z^2, zw, w^2, x\}$ forms a basis for $\mathcal{L}(-2K_X)$. The elements z^3, z^2w, zw^2, w^3, zx, wx in $\mathcal{L}(-3K_X)$ are linearly independent [CO99, p.1200]. Since we have $l(-3K_X) = 7$ we can therefore choose an element $y \in \mathcal{L}(-3K_X)$ to obtain a basis $\{z^3, z^2w, zw^2, w^3, zx, wx, y\}$ of $\mathcal{L}(-3K_X)$. Finally, since $l(-6K_X) = 22$, the 23 elements

$$z^6, z^5w, z^4w^2, z^3w^3, z^2w^4, zw^5, w^6, x^3, x^2z^2, x^2w^2, x^2zw, xz^4, xz^3w,\\ xz^2w^2, xzw^3, xw^4, xyz, xyw, y^2, yz^3, yz^2w, yzw^2, yw^3$$

of $\mathcal{L}(-6K_X)$ are linearly dependent. Let $h(x, y, z, w) = 0$ be a dependence relation between them. We can rescale x and y such that the coefficients of the monomials x^3 and y^2 are ± 1, and write

$$h = y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6, \tag{1.2}$$

where $a_i \in k[z, w]$ is homogeneous of degree i for each i in $\{1, \ldots, 6\}$. Let $k[x, y, z, w]$ be the graded k-algebra where x has degree 2, y has degree 3, and z, w have degree 1. Then the anticanonical model of X is Proj $k[x, y, z, w]/(h)$.

The linear system $|-3K_X|$

The linear system $|-3K_X|$ induces an embedding of X into $\mathbb{P}^6$, with coordinates $\{z^3, z^2w, zw^2, w^3, zx, wx, y\}$. This embedding factors through the anticanonical model of X.

For the rest of this section we identify X with its anticanonical model, that is, the zero locus of h in $\mathbb{P}_k(2, 3, 1, 1)$, where h is given by (1.2).

The linear system $|-2K_X|$

Let $p : \mathbb{P}_k(2, 3, 1, 1) \dashrightarrow \mathbb{P}_k(2, 1, 1)$ be the projection to $(x : z : w)$; its restriction to X is a morphism of degree 2. Let $i : \mathbb{P}_k(2, 1, 1) \hookrightarrow \mathbb{P}^3_k$ be the 2-uple embedding, sending $(x : z : w)$ to $(x : z^2 : zw : w^2)$. Write $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ for the coordinates of $\mathbb{P}^3_k$, then $i(\mathbb{P}_k(2, 1, 1))$ is a cone Q given by $\alpha_2^2 = \alpha_1\alpha_3$, with vertex $v = (1 : 0 : 0 : 0)$. The composition $\varphi = i \circ p \colon X \longrightarrow \mathbb{P}^3_k$ is a double cover of Q, and this is the morphism defined by the linear system $|-2K_X|$. If char $k \neq 2$ then we can do a coordinate change such that h is given by $y^2 - x^3 - a_2'x^2 - a_4'x - a_6'$, and the morphism φ is ramified at the points $(x : y : z : w) \in X$ for which

$$x^3 + a_2'x^2 + a_4'x + a_6' = 0. \tag{1.3}$$

In that case, the branch locus of φ is the union of v and the curve B that is the intersection of the cubic surface in $\mathbb{P}^3_k$ defined by (1.3) with Q, and B is a smooth integral curve of degree six and genus four [CO99, Proposition 3.1]. In the case char $k = 2$, the morphism φ is ramified at the points $(x : y : z : w) \in X$ for which $a_1 x + a_3 = 0$, and the branch curve of φ is smooth if and only if the intersection of the zero loci of a_1 and a_3 in $\mathbb{P}^1$ is empty [CO00, Remark 2.5].

The linear system $|-K_X|$

The linear system $|-K_X|$ defines a rational map $\mu\colon X \dashrightarrow \mathbb{P}^1_k$, projecting to the coordinates z, w. This is not defined in the point $\mathcal{O} = (1 : 1 : 0 : 0)$, which is the unique base point of $|-K_X|$. Let $\mathcal{E}$ be the blow-up of X in $\mathcal{O}$, then the rational map μ induces a morphism $\nu : \mathcal{E} \longrightarrow \mathbb{P}^1_k$. This gives $\mathcal{E}$ the structure of an *elliptic surface*; see Section 1.4.3.

Some of the rational maps and morphisms described above are shown in the following commutative diagram.

$$\begin{tikzcd}
 & & & \mathbb{P}^1_k \\
\mathcal{E} \arrow[r, "\pi"] \arrow[rrru, bend left, "\nu"] & X \arrow[r, hook] \arrow[rru, dashed, "\mu", "|-K_X|"'] \arrow[rrdd, "\varphi", "|-2K_X|"'] & \mathbb{P}_k(2,3,1,1) \arrow[r, dashed, "p"] & \mathbb{P}_k(2,1,1) \arrow[u] \arrow[d, "\simeq"] \arrow[dd, hook, bend left, "i"] \\
 & & & Q \arrow[d, hook] \\
 & & & \mathbb{P}^3_k
\end{tikzcd}$$

1.4.2 Exceptional curves and the $\mathbf{E}_8$ root system

Let X be a del Pezzo surface of degree 1 over an algebraically closed field. Recall that Pic X contains exactly 240 exceptional classes (Table 1.1); let I be the set of these classes. In this section we describe the relation between I and the *root system* $\mathbf{E}_8$. In particular, we show that the group of permutations of I that preserve the intersection pairing is isomorphic to the automorphism group of $\mathbf{E}_8$ (Corollary 1.4.10), which gives us a very useful tool when studying configurations of exceptional curves. Root systems arise in the study of many different objects, such as Lie groups and the classification of singularities on varieties. We will only treat a very small fraction of the theory of root systems here and in Chapter 3. Useful references for more on root systems are [Bou68] and [Hum72].

We start by recalling the definition of a root system.

Definition 1.4.6. Let V be a finite-dimensional vector space over $\mathbb{R}$ with a positive-definite inner product $\langle\cdot,\cdot\rangle$. A *root system* in V is a finite set R of non-zero vectors, called *roots*, that satisfy the following conditions:

(i) the roots span V;

(ii) for all $r \in R$, we have $\lambda r \in R \Longrightarrow \lambda = \pm 1$;

(iii) for all $r, s \in R$, we have $s - 2r\frac{\langle r,s\rangle}{\langle r,r\rangle} \in R$;

(iv) for all $r, s \in R$, the number $2\frac{\langle r,s\rangle}{\langle r,r\rangle}$ is an integer.

The *rank* of R is the dimension of V.

Definition 1.4.7. If R is a root system in a vector space V with inner product $\langle\cdot,\cdot\rangle$, and S is a root system in a vector space W with inner product $[\cdot,\cdot]$, then R and S are *isomorphic* if there is an isomorphism of vector spaces $\varphi\colon V \longrightarrow W$, which sends R to S, and such that $[\varphi(r_1),\varphi(r_2)] = \langle r_1, r_2\rangle$ for all $r_1, r_2 \in R$.

Let Λ be the $\boldsymbol{E}_8$ *lattice*, given by

$$\Lambda = \left\{ a \in \mathbb{Z}^8 + \left\langle \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right)\right\rangle \;\middle|\; \sum_{i=1}^{8} a_i \in 2\mathbb{Z} \right\} \subset \mathbb{R}^8.$$

This is the unique positive-definite, even, unimodular lattice of dimension 8 [MH73, II.§6]. The set

$$\mathbf{E}_8 = \{a \in \Lambda \mid \|a\| = \sqrt{2}\}$$

forms a root system in $\mathbb{R}^8$, known as the $\boldsymbol{E}_8$ *root system*. [Hum72, 12.1]. We will show that Pic X contains a subset R that forms a root system isomorphic to $\mathbf{E}_8$ (Proposition 1.4.8), and we will give a bijection between R and I (Remark 1.4.9).

Recall that X is isomorphic to $\mathbb{P}^2$ blown up in 8 points $P_1, \ldots, P_8$ in general position (Theorem 1.1.8). Let K_X be the class in Pic X of a canonical divisor of X. For $i \in \{1, \ldots, 8\}$, let E_i be the class in Pic X corresponding to the exceptional curve above P_i, and let L be the class in Pic X corresponding to the pullback of a line in $\mathbb{P}^2$ that does not contain any of the points $P_1, \ldots, P_8$. Consider the subgroup

$$K_X^{\perp} = \{D \in \text{Pic } X \mid D \cdot K_X = 0\} \subset \text{Pic } X,$$

and its subset

$$R = \{D \in K_X^{\perp} \mid D^2 = -2\}.$$

Let $\left(K_X^{\perp}, \langle \cdot, \cdot \rangle\right)$ be the vector space $\mathbb{R} \otimes_{\mathbb{Z}} K_X^{\perp}$ with inner product $\langle \cdot, \cdot \rangle$ defined as the negative of the intersection pairing in Pic X.

Proposition 1.4.8. *The set R is a root system of rank 8 in $\left(K_X^{\perp}, \langle \cdot, \cdot \rangle\right)$. Moreover, it is isomorphic to $\mathbf{E}_8$, and every element in R can be given as a linear combination with integer coefficients of the elements $r_1, \ldots, r_8 \in R$, given by*

$$E_1 - E_2,\ E_2 - E_3, \ldots, E_7 - E_8,\ L - E_1 - E_2 - E_3.$$

Proof. In [Man86, Propositions 25.1.1 and 25.2] it is shown that R is a root system of rank 8; in [Man86, Theorem 25.4 and Proposition 25.5.6] it is shown that this root system is isomorphic to $\mathbf{E}_8$, and the basis is given. $\square$

Remark 1.4.9. For $e \in I$ we have $e + K_X \in K_X^{\perp}$ and $\langle e+K_X, e+K_X \rangle = 2$, and this gives a bijection

$$I \longrightarrow R, \qquad e \longmapsto e + K_X.$$

For $e_1, e_2 \in I$ we have $\langle e_1 + K_X, e_2 + K_X \rangle = 1 - e_1 \cdot e_2$, where $\cdot$ is the intersection pairing in Pic X.

As a consequence of Proposition 1.4.8 and the bijection in Remark 1.4.9 we have the following result.

Corollary 1.4.10. *The group of permutations of I that preserve the intersection pairing is isomorphic to the Weyl group W_8, which is the group* *of permutations of* $\mathbf{E}_8$ *generated by the reflections in the hyperplanes orthogonal to the roots.*

Proof. [Man86, 25.1.1 and 23.9] □

Another way of phrasing Corollary 1.4.10 is that the *weighted graphs* on I and $\mathbf{E}_8$ and their automorphism groups are isomorphic (Corollary 1.4.14).

Definition 1.4.11. By a *graph* we mean a pair (V, D), where V is a set of elements called *vertices*, and D a subset of the power set of V such that every element in D has cardinality 2; elements in D are called *edges*, and the *size* of the graph is the cardinality of V. A graph (V, D) is *complete* if for every two distinct vertices $v_1, v_2 \in V$, the pair $\{v_1, v_2\}$ is in D.
By a *weighted graph* we mean a graph (V, D) with a map $\psi\colon D \longrightarrow A$, where A is any set, whose elements we call *weights*; for any element d in D we call $\psi(d)$ its weight. If (V, D) is a weighted graph with weight function ψ, then we define a *weighted subgraph* of (V, D) to be a graph (V', D') with map ψ', where V' is a subset of V, while D' is a subset of the intersection of D with the power set of V', and ψ' is the restriction of ψ to D'. A *clique* of a weighted graph is a complete weighted subgraph.
An *isomorphism* between two weighted graphs (V, D) and (V', D') with weight functions $\psi\colon D \longrightarrow A$ and $\psi'\colon D' \longrightarrow A'$, respectively, consists of a bijection f between the sets V and V' and a bijection g between the sets A and A', such that for any two vertices $v_1, v_2 \in V$, we have $\{v_1, v_2\} \in D$ with weight w if and only if $\{f(v_1), f(v_2)\} \in D'$ with weight $g(w)$. We call the map f an *automorphism* of (V, D) if $(V, D) = (V', D')$, $\psi = \psi'$, and g is the identity on A.

Definition 1.4.12. By Γ we denote the complete weighted graph whose vertex set is the set of roots in $\mathbf{E}_8$, and where the weight function is induced by the dot product. Similarly, by G we denote the complete weighted graph whose vertex set is I, and where the weight function is the intersection pairing in Pic X.

We can rephrase Remark 1.4.9 and Corollary 1.4.10 in terms of Γ and G as follows.

Remark 1.4.13. There is an isomorphsim of weighted graphs between G and Γ, that sends a vertex e in G to the corresponding vertex $e + K_X$ in Γ, and an edge $d = \{e_1, e_2\}$ in G with weight w to the edge $\delta = \{e_1 + K_X, e_2 + K_X\}$ in Γ with weight $1 - w$. The different weights that occur in G are $0, 1, 2$, and 3, and they correspond to weights $1, 0, -1$, and -2, respectively, in Γ.

Corollary 1.4.14. *The weighted graphs G and Γ have isomorphic automorphism groups, given by the Weyl group W_8.*

1.4.3 The anticanonical elliptic surface

Let k be a field, and S a del Pezzo surface of degree 1 over k. In this section we give more details about the surface $\mathcal{E}$ that was introduced in Section 1.4.1: it is obtained from S by blowing up the base point $\mathcal{O}$ of the anticanonical linear system $|-K_S|$. We show that it is an *elliptic surface*, and we study the *sections* of this surface and relate these to the exceptional curves on S (Proposition 1.4.21). For more theory on elliptic surfaces, see [Shi90] and [SS10].

Definition 1.4.15. An *elliptic surface* Y is a nice surface with a surjective morphism $f\colon Y \longrightarrow C$, where C is a nice curve, such that the following holds.

- The morphism f admits a *section*, that is, a morphism $s\colon C \longrightarrow Y$ such that
$$f \circ s = id_C.$$
- Almost all fibers of f are elliptic curves.
- No fibers of f contain an exceptional curve of Y.

We call the morphism f an *elliptic fibration*.

We will now describe the surface $\mathcal{E}$, and show that it is an elliptic surface over $\mathbb{P}^1$ (Lemma 1.4.16). We use the same notation as in Section 1.4.1; specifically, we identify the surface S with the smooth sextic in $\mathbb{P}_k(2, 3, 1, 1)$ with coordinates $(x : y : z : w)$ given by $h = 0$, where

$$h = y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6,$$

with $a_i \in k[z, w]$ homogeneous of degree i for each i. The point $\mathcal{O}$ is then given by $(1 : 1 : 0 : 0)$, and the blow-up of S in $\mathcal{O}$ is denoted by $\pi\colon \mathcal{E} \longrightarrow S$. We follow [VAZ09, 7.3] to describe $\mathcal{E}$: it is the subscheme of $\mathbb{P}_k(2, 3, 1, 1) \times \mathbb{P}^1_k$ given by

$$\mathcal{E}\colon \begin{cases} y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6 = 0; \\ vz - uw = 0, \end{cases}$$

where u, v are the coordinates of $\mathbb{P}^1_k$. The projection to $\mathbb{P}^1_k$ is the morphism $\nu\colon \mathcal{E} \longrightarrow \mathbb{P}^1_k$, which was also introduced in Section 1.4.1. Outside the exceptional divisor of π, which is given by $\tilde{\mathcal{O}} = \{(1:1:0:0)\} \times \mathbb{P}^1_k \subset \mathcal{E}$, we have $(u:v) = (z:w)$. Set $t = \frac{u}{v}$, which gives $z = tw$ on $\mathcal{E}$. The *generic fiber* of ν is a cubic curve over the function field $k(t)$ of $\mathbb{P}^1$, and it is the subset of $\mathbb{P}_{k(t)}(2,3,1)$ given by

$$\begin{aligned} E\colon y^2 + wa_1(t,1)xy + w^3a_3(t,1)y - x^3 - w^2a_2(t,1)x^2 \\ - w^4a_4(t,1)x - w^6a_6(t,1) = 0. \end{aligned}$$

Let $\mathbb{A}^2_{k(t)}$ be the affine open subset $w \neq 0$ of $\mathbb{P}_{k(t)}(2,3,1)$ with coordinates $X = \frac{x}{w^2}$, $Y = \frac{y}{w^3}$. The intersection of E with $\mathbb{A}^2_{k(t)}$ is given by

$$Y^2 + a_1(t,1)\,XY + a_3(t,1)\,Y = X^3 + a_2(t,1)X^2 + a_4(t,1)X + a_6(t,1).$$

Since S is smooth and geometrically rational, the discriminant Δ of E is a polynomial in $k[t]$ of degree between 10 and 12 [SS10, 4.3, 4.4, 8.2, 8.3]. In particular, Δ is not identically 0, so E is an elliptic curve over $k(t)$. Similarly, for $(u_0 : v_0) \in \mathbb{P}^1_k$, the fiber $\nu^{-1}((u_0 : v_0))$ is isomorphic to the cubic curve in $\mathbb{P}^2_k$ with affine Weierstrass equation

$$\begin{aligned} Y^2 + a_1(u_0,v_0)\,XY + a_3(u_0,v_0)\,Y = X^3 + a_2(u_0,v_0)X^2 \\ + a_4(u_0,v_0)X + a_6(u_0,v_0). \quad (1.4) \end{aligned}$$

This is an elliptic curve for all $(u_0 : v_0) \in \mathbb{P}^1_k$ such that $v_0 \neq 0$ and $\Delta(\frac{u_0}{v_0}) \neq 0$. Therefore, all but finitely many fibers of ν are elliptic curves, with zero-point given by the intersection with the exceptional divisor $\tilde{\mathcal{O}}$.

Let $K_{\mathcal{E}}$ be the canonical divisor on $\mathcal{E}$.

Lemma 1.4.16. *The surface $\mathcal{E}$ is an elliptic surface with elliptic fibration ν. Moreover, every fiber of ν is linearly equivalent to $-K_{\mathcal{E}}$ and has self-intersection 0.*

Proof. We already showed that almost every fiber of ν is an elliptic curve, so we only have to show that no fibers of ν contain an exceptional curve on $\mathcal{E}$. Since all fibers of ν are given by (1.4), they are integral, so the only way they could contain an exceptional curve is if they are one. Since ν restricted to $\mathcal{E} \setminus \tilde{O}$ is the map μ induced by the anticanonical linear system $|-K_S|$ (see Section 1.4.1), the fibers of ν are linearly equivalent to $-K_{\mathcal{E}} = \pi^*(-K_S) + \tilde{\mathcal{O}}$. Since all fibers of ν are linearly equivalent and pairwise disjoint, they have self-intersection 0. Therefore no fiber is equal to an exceptional curve. We conclude that $\mathcal{E}$ is an elliptic surface with elliptic fibration ν. □

Remark 1.4.17. The set of $k(t)$-rational points on E forms a group, the *Mordell–Weil group* of E over $k(t)$ or of $\mathcal{E}$ [Shi90, Theorem 1.1]. This group is torsion-free and has rank at most 8 over k [Shi90, Theorem 10.4]. The set of sections of ν form a group as well, and the map

$$P = (X_P, Y_P) \longmapsto \Big(s\colon \mathbb{P}^1 \setminus \{(1:0)\} \longrightarrow \mathcal{E},\ (t:1) \longmapsto (X_P(t), Y_P(t), t)\Big)$$

induces an isomorphism between the group of $k(t)$-rational points on E and the group of sections of ν that are defined over k [Sil94, Proposition 3.10]. As a consequence of this correspondence, we sometimes talk about a k-section as a morphism $\mathbb{P}^1_k \longrightarrow \mathcal{E}$, and sometimes as a curve on $\mathcal{E}$, whose generic point is the corresponding $k(t)$-rational point on E.

The following definition generalizes the notion of section.

Definition 1.4.18. A *multisection of degree* d or d*-section* of $\mathcal{E}$ is an irreducible curve C contained in $\mathcal{E}$ such that the projection $\varphi|_C\colon C \longrightarrow \mathbb{P}^1_k$ is non-constant and of degree d.

Remark 1.4.19. Note that a section is a multisection of degree 1, and in a similar way as with sections, the d-sections of $\mathcal{E}$ correspond to points on the generic fiber E of $\mathcal{E}$ that are defined over a degree d extension of $k(t)$.

We end this chapter by showing that the exceptional curves on S induce sections of ν, and by giving a characterization of these sections on $\mathcal{E}$.

Remark 1.4.20. Since exceptional curves on S are defined over a separable closure of k (Theorem 1.1.8), from [VA08, Theorem 1.2] it follows that the exceptional curves on S are exactly the curves given by

$$x = p(z, w), \quad y = q(z, w),$$

where $p, q \in k[z, w]$ are homogeneous of degrees 2 and 3. Note that this implies that an exceptional curve never contains $\mathcal{O} = (1:1:0:0)$. Therefore, for an exceptional curve C on S, its strict transform $\pi^*(C)$ on $\mathcal{E}$ satisfies

$$\pi^*(C)^2 = -1, \qquad \pi^*(C) \cdot -K_{\mathcal{E}} = \pi^*(C) \cdot (\pi^*(-K_S) + \tilde{\mathcal{O}}) = 1 + 0 = 1,$$

so $\pi^*(C)$ is an exceptional curve on $\mathcal{E}$ as well. Moreover, since a fiber of ν is linearly equivalent to $-K_{\mathcal{E}}$, the curve $\pi^*(C)$ intersects every fiber once. This gives a section of ν.

Proposition 1.4.21. *Let C be a section of ν on $\mathcal{E}$. The following are equivalent.*

(i) C is the strict transform of an exceptional curve on S.

(ii) C is of the form

$$x = p(z, w), \quad y = q(z, w),$$

where $p, q \in k[z, w]$ are homogeneous of degree 2 and 3.

(iii) C is disjoint from $\tilde{\mathcal{O}}$.

Proof. (i) is equivalent to (ii) by Remark 1.4.20, and (ii) and (iii) are equivalent by [Shi90, Lemma 10.9]. $\square$

2

Density of rational points on a family of del Pezzo surfaces of degree 1

In this chapter we study the Zariski density of the set of rational points on del Pezzo surfaces of degree 1. In Section 2.1 we give some background and known results. In Section 2.2 we state our main result (Theorem 2.2.1) and the main ingredient for its proof (Proposition 2.2.6). We prove the latter in Section 2.3, and prove our main theorem in Section 2.4. Finally, in Section 2.5 we give examples. This chapter is based on work with Julie Desjardins.

2.1 Rational points on del Pezzo surfaces

Let X be a variety defined over a number field k. In arithmetic geometry we are interested in the set of k-rational points $X(k)$ on X. For example, we can ask whether $X(k)$ is empty, and if so, if we can explain why. If $X(k)$ is not empty, we can further ask how big this set is: is it finite? Infinite? And if it is infinite, what does it look like? Is it dense with respect to the Zariski topology?

For del Pezzo surfaces, some (partial) answers to these questions are known. An overview can be found in [VA09, 1.4]; the following results are stated there. For example, del Pezzo surfaces of degrees 1, 5, and 7 over a field k always contain a k-rational point, and del Pezzo surfaces of degree at least 5 over a number field k satisfy the *Hasse principle*, meaning that if such a surface contains an element in $X(k_v)$ for the completion k_v at every place v of k, then it contains a k-rational point. There are also examples of del Pezzo surface of degrees 2, 3, and 4 over $\mathbb{Q}$ without a $\mathbb{Q}$-rational point even though they do have $\mathbb{R}$-, $\mathbb{C}$-, and $\mathbb{Q}_p$-rational points for all primes p [VA09, Examples 1.4.1–1.4.3].

Zariski density of rational points

In the rest of this chapter, by *density* we mean density with respect to the Zariski topology, unless stated otherwise. To give an overview of what is known for the Zariski density of the set of rational points on del Pezzo surfaces, we introduce another property of a variety.

Definition 2.1.1. A variety X over a field k is *k-unirational* if there is a dominant rational map $\mathbb{P}^n_k \dashrightarrow X$ for some n.

Remark 2.1.2. Note that if two varieties are birationally equivalent over a field k, one is k-unirational if and only if the other one is. Moreover, if k is infinite, then k-unirationality implies Zariski density of the set of k-rational points.

Theorem 2.1.3. *Let k be a field. The following hold.*

(i) Del Pezzo surfaces of degree at least 3 over k with a k-rational point are k-unirational.

(ii) Del Pezzo surfaces of degree 2 over k that contain a point that is neither in the ramification locus of the anticanonical map, nor in the intersection of four exceptional curves, are k-unirational.

(iii) Del Pezzo surfaces of degree 1 that admit a conic bundle structure are k-unirational.

Proof. (i) Segre proved this for degree 3 and $k = \mathbb{Q}$ in [Seg43] and [Seg51]. Manin proved it for $d \geq 5$, as well as for $d = 3, 4$ for large enough cardinality of k [Man86, Theorems 29.4, 30.1]. Kollár finished the case $d = 3$

[Kol02], and Pieropan the case $d = 4$ [Pie12, Proposition 5.19]. Part (ii) is in [STVA14]; part (iii) is in [KM17]. $\square$

Of course, if a del Pezzo surface S of degree 1 over a field k is not minimal, then we can blow down exceptional curves to obtain a del Pezzo surface S' of higher degree, and use Theorem 2.1.3 (i) or (ii) hold to determine whether S' is k-unirational. Since S and S' are birationally equivalent, S is unirational if and only S' is. The del Pezzo surfaces of degree 1 in Theorem 2.1.3 are those that are minimal with Picard rank 2; see Theorem 1.3.4. Outside this case the question of k-unirationality for minimal del Pezzo surfaces of degree 1 is wide open. Even though these surfaces always contain a k-rational point (the base point of the anticanonical linear system), we do not have any example of a minimal del Pezzo surface of degree 1 with Picard rank 1 that is known to be k-unirational, nor of one that is known not to be k-unirational.

If k is infinite, then k-unirationality implies density of the set of k-rational points. Therefore, for k infinite, Theorem 2.1.3 implies that for a del Pezzo surface X of degree at least 3, the set $X(k)$ of k-rational points is Zariski-dense if and only if it is not empty, and if X has degree 2, the set $X(k)$ is Zariski-dense if it contains a point outside the ramification locus of the anticanonical map and not contained in the intersection of four exceptional curves. While unirationality for del Pezzo surfaces of degree 1 is still out of reach, we can at least try to prove Zariski density of the set of k-rational points for these surfaces. A strong reason why we expect that the set of k-rational points on a del Pezzo surface of degree 1 is dense, at least when k is a number field, is the following conjecture by Colliot-Thélène and Sansuc.

Conjecture 2.1.4. *[CT92, Conjecture d] For every geometrically rationally connected variety over a number field, its set of rational points is dense in the Brauer–Manin set for the adelic topology.*

Since del Pezzo surfaces of degree 1 are geometrically rationally connected and have a rational point, this conjecture implies the density of their set of rational points over number fields [Wit18, Remark 2.4(iii)].

Known results

Let S be a del Pezzo surface of degree 1 over a field k with char $k \neq 2, 3$, and let $\mathcal{E}$ be the associated elliptic surface obtained by blowing up the base point of the linear system $|-K_S|$. We identify S with its anticanonical model in the weighted projective space $\mathbb{P}_k(2, 3, 1, 1)$ with coordinates x, y, z, w, and since char $k \neq 2, 3$, we define S as the zero locus of

$$y^2 = x^3 + xf(z, w) + g(z, w),$$

where f and $g \in k[z, w]$ are homogeneous of degrees 4 and 6, respectively.

Previous results on Zariski density of $S(k)$ are obtained by proving that the set $\mathcal{E}(k)$ is dense in $\mathcal{E}$, which implies the result for $S(k)$. People have done this either by considering root numbers of fibers, or by constructing a multisection.

Remark 2.1.5. If $\mathcal{E}$ contains a section over k other than the exceptional curve above the base point of $|-K_S|$, then this section corresponds to a non-zero $k(t)$-rational point in the Mordell–Weil group of $\mathcal{E}$, which has no torsion (Remark 1.4.17). By Silverman's Specialization Theorem [Sil83, Theorem C], this gives a non-torsion k-rational point on all but finitely many fibers of $\mathcal{E}$, thus implying the density of the set of k-rational points on $\mathcal{E}$, hence on S.

We briefly state previous results here.

In [VA11], Várilly-Alvarado proves Zariski density of the set of $\mathbb{Q}$-rational points of S when $f = 0$ and $g = Az^6 + Bw^6$, with non-zero $A, B \in \mathbb{Z}$, such that either $3A/B$ is not a square, or $\gcd(A, B) = 1$ and $9 \nmid AB$. His results are conditional under the finiteness of the Tate–Shafarevich group of elliptic curves with j-invariant 0. Over $\mathbb{Q}$, the latter implies that the root number of such an elliptic curve E equals $(-1)^{\mathrm{rank}(E)}$. Várilly-Alvarado shows that his surfaces have infinitely many disjoint pairs of fibers of $\mathcal{E}$ with opposite root number, thus showing that there are infinitely many fibers with positive rank.

Ulas and Togbé, prove Zariski density of the set of $\mathbb{Q}$-rational points of S in the following cases.
- $g = 0$ and $\deg(f(z, 1)) \leq 3$, or $g = 0$ and $\deg(f(z, 1)) = 4$ with f not even, or $f = 0$ and $g(z, 1)$ is monic of degree 6 and not even [Ula07,

Theorems 2.1 (1), 2.2, and 3.1].
• $g = 0$ and $\deg(f(z,1)) = 4$, or $f = 0$ and $g(z,1)$ is even and monic of degree 6, both cases under the condition that there is a fiber of $\mathcal{E}$ with infinitely many rational points [Ula07, Theorems 2.1 (2) and 3.2].
• S can be defined by $y^2 = x^3 - h(z,w)$, with $h(z,1) = z^5 + az^3 + bz^2 + cz + d \in \mathbb{Z}[z]$, and the set of rational points on the curve $Y^2 = X^3 + 135(2a-15)X - 1350(5a+2b-26)$ is infinite [Ula08, Theorem 2.1].
• $f(z,1)$ and $g(z,1)$ are both even of degree 4 and there is a fiber of $\mathcal{E}$ with infinitely many rational points [UT10, Theorem 2.1].

Jabara generalized the results from [Ula07] mentioned above in [Jab12, Theorems C and D]. Though the proofs of these two theorems are incomplete (see [SvL14, Remark 2.7]), they hold for sufficiently general cases.

In [SvL14], Salgado and van Luijk generalize some of the previous results, proving Zariski density of the set of k-rational points of S for any infinite field k with char $k \neq 2,3$, assuming that there exists a point Q on a smooth fiber of $\mathcal{E}$ satisfying several conditions, among which that a multisection that they construct from Q has infinitely many k-rational points.

2.2 Main result

Our main theorem is the following; recall that this is joint work with Julie Desjardins.

Theorem 2.2.1. *Let k be a number field, let $A, B \in k$ be non-zero, and let S in $\mathbb{P}(2,3,1,1)$ be the del Pezzo surface of degree 1 over k given by*

$$y^2 = x^3 + Az^6 + Bw^6. \tag{2.1}$$

Let $\mathcal{E}$ be the elliptic surface obtained by blowing up the base point of the linear system $|-K_S|$. Then the set of k-rational points on S is dense in S with respect to the Zariski topology if and only if S contains a k-rational point P with non-zero z, w coordinates, such that the corresponding point on $\mathcal{E}$ lies on a smooth fiber and is non-torsion on that fiber.

Remark 2.2.2. Note that the family of surfaces we consider is the same as the one studied by Várilly-Alvarado in [VA11]. Moreover, the case $A = 1$ is proven by Ulas in [Ula07] for $k = \mathbb{Q}$ under the same condition that we have (the existence of a fiber of $\mathcal{E}$ with infinitely many rational points); we generalize his result to any non-zero A, and to any number field.

While Salgado and van Luijk prove their result over all infinite fields with characteristic unequal to $2, 3$ in [SvL14], their condition that there exists a point Q such that their multisection has infinitely many rational points is not easy to verify, nor is it clear to hold for every surface whose set of rational points is dense, that is, it might not be a necessary condition. For the family in Theorem 2.2.1, we give sufficient *and* necessary conditions for the set of rational points of S to be dense.

Let k be an infinite field with char $k \neq 2, 3$, let $A, B \in k$ non-zero, and let S be the del Pezzo surface of degree 1 over k given by (2.1), with canonical divisor K_S. Let $\mathcal{E}$ be the elliptic surface obtained by blowing up the base point of the linear system $|-K_S|$. The key ingredient of the proof of Theorem 2.2.1 is Proposition 2.2.6. We recall some notation from Section 1.4.3, which we will use throughout this chapter.

Notation 2.2.3. Let $\pi\colon \mathcal{E} \longrightarrow S$ be the blow-up of S in $\mathcal{O} = (1:1:0:0)$ with exceptional divisor $\tilde{\mathcal{O}}$. Since π gives an isomorphism between $\mathcal{E} \setminus \tilde{\mathcal{O}}$ and $S \setminus \{\mathcal{O}\}$, we denote a point $R \in \mathcal{E} \setminus \tilde{\mathcal{O}}$ by the coordinates of $\pi(R)$ in $\mathbb{P}_k(2, 3, 1, 1)$. Let $\nu\colon \mathcal{E} \longrightarrow \mathbb{P}^1$ be the elliptic fibration on $\mathcal{E}$, which is given on S by the projection onto $(z : w)$. For $R = (x_R : y_R : z_R : w_R) \in S \setminus \{\mathcal{O}\}$, we denote by $R_{\mathcal{E}}$ the inverse image $\pi^{-1}(R)$ on $\mathcal{E}$, which is a point on the fiber $\nu^{-1}((z_R : w_R))$.

Definition 2.2.4. For any point $R = (x_R : y_R : z_R : w_R)$ in $\mathcal{E}$ with $y_R, z_R \neq 0$, we define the curve $C_R \subset \mathcal{E}$ as the strict transform of the intersection of S with the surface given by

$$3x_R^2 z_R^2 xz - 2y_R z_R^3 y - (x_R^3 - 2Az_R^6)z^3 + 2Bz_R^3 w^3 = 0. \tag{2.2}$$

Remark 2.2.5. For $R = (x_R : y_R : z_R : w_R)$ in $\mathcal{E}$ with $y_R, z_R \neq 0$, the curve $\pi(C_R)$ does not contain the point $\mathcal{O}$, so we identify the curve C_R with $\pi(C_R) \subset \mathbb{P}(2, 3, 1, 1)$; see Notation 2.2.3.

If R is a point on S with non-zero z-coordinate and such that $R_{\mathcal{E}}$ lies on a smooth fiber and is non-torsion, then its y-coordinate is non-zero as well, and every non-zero multiple $nR_{\mathcal{E}}$ of $R_{\mathcal{E}}$ on its fiber has non-zero z- and y-coordinate; therefore we can define $C_{nR_{\mathcal{E}}}$ for every non-zero integer n. We use this in the following proposition. Recall the definition of d-section (Definition 1.4.18).

Proposition 2.2.6. *Let P be a point in $S(k)$ with non-zero z, w coordinates, such that $P_{\mathcal{E}}$ lies on a smooth fiber and is non-torsion. If k is a* *number field, then there exists an integer n such that one of the following holds:*

(i) $C_{nP_{\mathcal{E}}}$ has a component that is a section of $\mathcal{E}$ that is defined over k;

(ii) $C_{nP_{\mathcal{E}}}$ is a 3-section of $\mathcal{E}$ of geometric genus 0;

(iii) $C_{nP_{\mathcal{E}}}$ is a 3-section of $\mathcal{E}$ whose normalization is an elliptic curve with positive rank over k.

Remark 2.2.7. Note that case (i) in the previous proposition immediately implies the density of the set of k-rational points on S, see Remark 2.1.5.

2.3 Creating a multisection

In this section we prove Proposition 2.2.6. We use Notation 2.2.3.

Remark 2.3.1. Let $R = (x_R : y_R : z_R : w_R)$ be a point in $\mathcal{E}$, with $y_R, z_R \neq 0$, and let C_R be the corresponding curve as in Definition 2.2.4. Let $\mathbb{A}^3$ be the affine open subset of $\mathbb{P}(2,3,1,1)$ given by $w \neq 0$, with coordinates $X = \frac{x}{w^2}$, $Y = \frac{y}{w^3}$, and $T = \frac{z}{w}$. We describe the intersection $C_R \cap \mathbb{A}^3$. Write

$$\begin{aligned} F &= Y^2 - X^3 - AT^6 - B, \\ G &= 3x_R^2 z_R^2 XT - 2y_R z_R^3 Y - (x_R^3 - 2Az_R^6)T^3 + 2Bz_R^3. \end{aligned} \tag{2.3}$$

We have $C_R \cap \mathbb{A}^3 = Z(F) \cap Z(G)$, where $Z(F)$ and $Z(G)$ are the zero loci of F and G, respectively. Since $y_R, z_R \neq 0$, the projection $p\colon \mathbb{A}^3 \longrightarrow \mathbb{A}^2$ to the X, T-coordinates has a section given by

$$r\colon (X,T) \longmapsto \left(X, \frac{3x_R^2 z_R^2 XT - (x_R^3 - 2Az_R^6)T^3 + 2Bz_R^3}{2y_R z_R^3}, T \right).$$

Note that p induces an isomorphism $Z(G) \longrightarrow \mathbb{A}^2$ with inverse r. It follows that $C_R \cap \mathbb{A}^3$ is isomorphic to $p(Z(F))$, and the latter is defined by $H_R = 0$, where

$$\begin{aligned} H_R = {} & 4y_R^2 z_R^6 X^3 - 9x_R^4 z_R^4 X^2 T^2 + (6x_R^5 z_R^2 - 12Ax_R^2 z_R^8)XT^4 \\ & - 12Bx_R^2 z_R^5 XT + (4Ax_R^3 z_R^6 + 4Ay_R^2 z_R^6 - 4A^2 z_R^{12} - x_R^6)T^6 \\ & + 4Bz_R^3(x_R^3 - 2Az_R^6)T^3 + 4Bz_R^6(y_R^2 - B). \end{aligned} \tag{2.4}$$

We denote by $K_{\mathcal{E}}$ the canonical divisor of $\mathcal{E}$. Let $\overline{k}$ be an algebraic closure of k, and write $\overline{C}_R$ for the base change $C_R \times_k \overline{k}$.

Lemma 2.3.2. *Let $R = (x_R : y_R : z_R : w_R)$ be a point in $\mathcal{E}$ with y_R, z_R non-zero, and let C_R be the curve in Definition 2.2.4. The following hold.*

(i) The curve C_R does not contain a fiber of $\mathcal{E}$.

(ii) The curve C_R is contained in the linear system $|-3K_{\mathcal{E}} + 3\tilde{\mathcal{O}}|$, and intersects every fiber of ν in three points counted with multiplicity.

Proof. (i). From equation (2.2) it is clear that C_R does not contain the fiber $w = 0$. Moreover, since the coefficient of X^3 of H_R (2.4) as a polynomial in $k[T]$ is constant and non-zero, C_R does not contain any fiber with $w \neq 0$, either.
(ii). The linear system $|-3K_S|$ induces the 3-uple embedding of S into $\mathbb{P}^6$ (Section 1.4.1). Under this embedding, the curve $\pi(C_R)$ is given by the intersection of S with a hyperplane, hence we have $\pi(C_R) \sim -3K_S$. Since $y_R, z_R \neq 0$, the image $\pi(C_R)$ does not contain the point $\mathcal{O}$, so this implies

$$C_R = \pi^*(\pi(C_R)) \in |\pi^*(-3K_S)| = |-3K_{\mathcal{E}} + 3\tilde{\mathcal{O}}|.$$

Since a fiber $\mathcal{F}$ of ν is linearly equivalent to $-K_{\mathcal{E}}$, it has self-intersection zero (Lemma 1.4.16), and $\tilde{\mathcal{O}}$ is a section of ν, we have

$$\mathcal{F} \cdot C_R = \mathcal{F} \cdot (-3K_{\mathcal{E}} + 3\tilde{\mathcal{O}}) = 0 + 3 = 3.$$

Since $\mathcal{F}$ is irreducible, it follows that, since $\mathcal{F}$ is not contained in C_R, the number of intersection points of $\mathcal{F}$ and C_R is 3, counted with multiplicity. □

Let $\zeta_3 \in \overline{k}$ be a primitive third root of unity. Note that, for a curve C_R as in Definition 2.2.4, the morphism of $\mathbb{P}(2,3,1,1)$ given by multiplying the w-coordinate with ζ_3^2 restricts to an automorphism of $\overline{C}_R = C_R \times_k \overline{k}$.

Definition 2.3.3. Let $R = (x_R : y_R : z_R : w_R)$ be a point in $\mathcal{E}$, with $y_R, z_R \neq 0$, and let C_R be the corresponding curve as in Definition 2.2.4. By σ we denote the automorphism of $\overline{C}_R$ given by

$$\sigma\colon (x : y : z : w) \longmapsto (x : y : z : \zeta_3^2 w) = (\zeta_3^2 x : y : \zeta_3 z : w) \qquad (2.5)$$

Recall that $\pi\colon \mathcal{E} \longrightarrow S$ is the blow-up of S in $\mathcal{O}$, and $\nu\colon \mathcal{E} \longrightarrow \mathbb{P}^1$ is the elliptic fibration on $\mathcal{E}$.

PROPOSITION 2.3.4. *Let $R = (x_R : y_R : z_R : 1)$ be a point in $\mathcal{E}$, with $x_R \in k$, $y_R, z_R \in k^*$, and let C_R be the curve in Definition 2.2.4. The following hold.*

(i) The curve C_R is singular in R, $\sigma(R)$, and $\sigma^2(R)$.

(ii) If $\pi(R)$ is not contained in an exceptional curve on $\overline{S} = S \times_k \overline{k}$, then C_R either contains a section that is defined over k, or it is geometrically integral and has geometric genus at most 1, in which case R, $\sigma(R)$, $\sigma^2(R)$ are all double points.

Proof. (i). It is an easy check that R is contained in C_R. Let m_R be the maximal ideal in the local ring of R on $\mathcal{E}$. The point R lies in the affine space $\mathbb{A}^3 \subset \mathbb{P}(2,3,1,1)$ defined by $w \neq 0$ as in Remark 2.3.1. The ideal m_R is generated by $X - x_R$, $Y - y_R$, and $T - z_R$. Let F, G be as in (2.3). We have $\mathcal{E} \cap \mathbb{A}^3 = Z(F)$, and using the identity $B = y_R^2 - x_R^3 - At_R^2$, we can write F as

$$\begin{aligned} F = 2y_R(Y - y_R) - 3x_R^2(X - x_R) - 6At_0^5(T - z_R) \\ + (Y - y_R)^2 - (X - x_R)^3 - 3x_R(X - x_R)^2 - A(T - z_R)^6 \\ - 6Az_R(T - z_R)^5 - 15At_0^2(T - z_R)^4 \\ - 20Az_R^3(T - z_R)^3 - 15Az_R^4(T - z_R)^2. \end{aligned}$$

Set

$$\alpha = 2y_R(Y - y_R) - 3x_R^2(X - x_R) - 6Az_R^5(T - z_R),$$

then it follows that α is contained in m_R^2, so the tangent line to $\mathcal{E}$ at R is given by $\alpha = 0$.

Similarly, we can rewrite G as

$$\begin{aligned} G = -z_R^3\alpha + 3x_R^2 z_R^2(X - x_R)(T - z_R) - (x_R^3 - 2Az_R^6)(T - z_R)^3 \\ - (3x_R^3 z_R - 6Az_R^7)(T - z_R)^2, \end{aligned}$$

so we conclude that G is contained in m_R^2, hence C_R is singular in R. Since σ is an automorphism of C_R, this implies that C_R is singular in $\sigma(R)$ and $\sigma^2(R)$, as well.

(ii). Assume that $\pi(R)$ is not contained in an exceptional curve on $\overline{S}$. We distinguish two cases. First assume that $\overline{C}_R$ is not irreducible or not reduced. Since $\overline{C}_R$ does not contain a fiber and intersects every fiber with multiplicity 3 (Lemma 2.3.2), this implies that there is a curve that intersects every fiber with multiplicity one (hence is a section), say H_1, such that $\overline{C}_R$ either contains H_1 as irreducible component, or $\overline{C}_R$ is a multiple of H_1. Since $\overline{C}_R$ is disjoint from the zero section, it follows that $\pi(H_1)$ is an exceptional curve on $\overline{S}$ (Proposition 1.4.21). Therefore, by our assumption, R is not contained in H_1, so $\overline{C}_R$ is not a multiple of H_1, and H_1 is an irreducible component of $\overline{C}_R$. Let H_2 be the other (not necessarily irreducible or reduced) component of $\overline{C}_R$, which contains R. If H_2 were not irreducible or not reduced, it would either be a double section or two sections intersecting in R. In both cases, $\pi(R)$ lies on an exceptional curve, contradicting our first assumption. We conclude that H_2 is irreducible and reduced. Since C_R is defined over k, it is fixed by the action of the absolute Galois group of k on Pic S. The exceptional curves of $\overline{S}$ are all defined over the separable closure k^{sep} of k by Theorem 1.1.8, so the Galois group $\text{Gal}(k^{\text{sep}}/k)$ acts on them. Since $\overline{C}_R$ contains only one exceptional curve of $\overline{S}$, which is H_1, it follows that this component is invariant under the Galois action, hence it is defined over k. This finishes the first case. Now assume that $\overline{C}_R$ is irreducible and reduced. Since C_R is contained in the linear system $|-3K_{\mathcal{E}}+3\tilde{\mathcal{O}}|$ by Lemma 2.3.2, from the adjunction formula it follows that its arithmetic genus is $\frac{1}{2}\cdot(9-3)+1=4$. Since the three distinct points R, $\sigma(R)$, $\sigma^2(R)$ are all singular on C_R, we conclude that they all have multiplicity 2, and the geometric genus of C_R is at most 1. □

Remark 2.3.5. In the last proof, we concluded that in the case where C_R is geometrically integral, the geometric genus of C_R is at most 1. If it were 0, then C_R would contain exactly one more singular point besides R, $\sigma(R)$, $\sigma^2(R)$, say Q. Then $\sigma(Q)$ and $\sigma^2(Q)$ would be singular points of C_R as well, so Q would be a fixed point of σ. We study this case further here. Note that the points on the intersection of C_R with the fiber above $(1:0)$ are fixed points of σ. Assume that σ has a fixed point $Q=(x_Q:y_Q:z_Q:w_Q)$ in $C_R\setminus(C_R\cap\mathcal{E}_{(1:0)})$. From (2.5) it follows that there is a $\lambda\in k$ such that $\lambda^3y_Q=y_Q$, $\lambda^2x_Q=x_Q$, $\lambda z_Q=z_Q$, and $\lambda\zeta_3^2w_Q=w_Q$. Since $w_Q\neq 0$, the last equation implies $\lambda=\zeta_3^{3n-2}$ for some $n>0$, and it follows that $x_Q=z_Q=0$. From (2.2) and the fact that C_R lies in $\mathcal{E}$ we find

$$2y_Rz_R^3y_Q=2Bz_R^3w_Q^3; \tag{2.6}$$

$$y_Q^2=Bw_Q^6. \tag{2.7}$$

Since $B,w_Q\neq 0$, it follows from (2.7) that $y_Q\neq 0$ and we can write

$B = \left(\frac{y_Q}{w_Q^3}\right)^2$. Substituting this in (2.6), we find

$$2y_R z_R^3 y_Q = 2\left(\frac{y_Q}{w_Q^3}\right)^2 z_R^3 w_Q^3.$$

Since $y_Q, z_R \neq 0$ this implies $y_R = \frac{y_Q}{w_Q^3}$, from which it follows that we have $B = y_R^2$. Since R is contained in $\mathcal{E}$, it follows that $y_R^2 = x_R^3 + Az_R^6 + y_R^2$, from which we get $A = \frac{-x_R^3}{z_R^6}$. So in this case, the surface S is of the form

$$y^2 = x^3 + \frac{-x_R^3}{z_R^6} z^6 + y_R^2 w^6,$$

and $Q = (0 : y_R : 0 : 1)$. But then C_R contains the section

$$D\colon \begin{cases} x = \frac{x_R}{z_R^2} z^2, \\ y = y_R w^3, \end{cases} \tag{2.8}$$

contradicting the fact that C_R is irreducible. We conclude that if C_R is geometrically integral, then it has genus 0 if and only if it has a singular point on the fiber above $(1 : 0)$.

Remark 2.3.6. Let R be as in Proposition 2.3.4. If C_R is a geometrically integral curve of geometric genus 1, then, since C_R intersects every fiber of ν in three points counted with multiplicity (Lemma 2.3.2), this implies that C_R is a 3-section. Moreover, since R is a double point on C_R, there is a unique third point of intersection of C_R with the fiber above $(z_R : 1)$ in $\mathcal{E}$, say Q. Since x_R, y_R, z_R are elements in k, the fiber above $(z_R : 1)$ is defined over k, and C_R and R are both defined over k. It follows that Q is defined over k. Hence $E_R = (\tilde{C}_R, Q)$ is an elliptic curve defined over k, where $\tilde{C}_R$ is the normalization of C_R. Let D_R be the sum on E_R of the points corresponding to $\sigma(Q)$ and $\sigma^2(Q)$ on C_R. Note that $\sigma(Q)$ and $\sigma^2(Q)$ are either both defined over k or conjugated, so D_R is defined over k.

Notation 2.3.7. If R is as in Proposition 2.3.4, and such that C_R is a geometrically integral curve of geometric genus 1, we denote by E_R the elliptic curve and by D_R the point on it, both as defined in Remark 2.3.6.

Let η be the generic point of S, that is, η is the point $(\tilde{x} : \tilde{y} : \tilde{z} : 1)$ over the function field $k(S) = k(\tilde{x}, \tilde{y}, \tilde{z}) = \mathrm{Frac}(k[x, y, z]/(y^2 - x^3 - Az^6 - B))$ of S.

Let $C_\eta \in \mathbb{P}_{k(S)}(2,3,1,1)$ be the corresponding curve given by (2.2). From Proposition 2.3.4 and Remark 2.3.5 it follows that C_η is geometrically integral of genus 1. Let E_η be the corresponding elliptic curve with point D_η as in Notation 2.3.7. In Lemma 2.3.8 we give a Weierstrass model for the curve E_η, which we will use in Proposition 2.3.10.

Recall that A, B are fixed non-zero elements in k. We define the polynomial

$$q = q_1 q_2 q_3 q_4 \tag{2.9}$$

in the polynomial ring $k[\tilde{x}, \tilde{z}]$ as follows.

$$\begin{aligned}
q_1 &= \tilde{x};\\
q_2 &= -\tilde{x}^6 + 8A\tilde{z}^6\tilde{x}^3 + 4AB\tilde{z}^6;\\
q_3 &= \tilde{x}^6 + 8(A\tilde{z}^6 - B)\tilde{x}^3 + 16(A^2\tilde{z}^{12} + AB\tilde{z}^6);\\
q_4 &= 29\tilde{x}^{12} + \left(40B + 24A\tilde{z}^6\right)\tilde{x}^9 + 16\left(9AB\tilde{z}^6 - B^2 + 6A^2\tilde{z}^{12}\right)\tilde{x}^6\\
&\qquad + 128\left(A^3\tilde{z}^{18} + 3A^2B\tilde{z}^{12} + 2AB^2\tilde{z}^6\right)\tilde{x}^3\\
&\qquad\qquad + 64(AB^3\tilde{z}^6 + 2A^2B^2\tilde{z}^{12} + A^3B\tilde{z}^{18}).
\end{aligned}$$

Lemma 2.3.8. *There exists a unique polynomial $\delta \in k[\tilde{x}, \tilde{z}]$, and unique rational functions*

$$\xi_D = \tfrac{\alpha}{(q_1q_3)^2}, \qquad \gamma_D = \tfrac{\beta}{(q_1q_3)^3},$$

where α and β are polynomials in $k[\tilde{x}, \tilde{z}]$, such that the leading terms of δ, α and β, as univariate polynomials in $\tilde{x}$, are given by

$$-27B\tilde{z}^{48}\tilde{x}^{81}, \qquad \tfrac{1}{4}\tilde{z}^{16}\tilde{x}^{42}, \qquad \tfrac{1}{8}\tilde{z}^{24}\tilde{x}^{63},$$

respectively, and such that the following holds. There is an isomorphism ω between the elliptic curve E_η and the curve with Weierstrass equation given by

$$\gamma^2 = \xi^3 + \delta, \tag{2.10}$$

such that the denominators in the defining equations of ω and ω^{-1} are all of the form $2^a3^b(q_2q_4)^c$ for positive integers a, b, c. Moreover, the point on (2.10) corresponding to the point D_η on E_η is given by

$$\omega(D_\eta) = (\xi_D, \gamma_D). \tag{2.11}$$

Proof. The `magma` code that is used in this proof can be found in [Coda]. Let Q be the third point of intersection of C_η with the fiber of η on the base change $\mathcal{E} \times_k k(S)$ over $\mathbb{P}^1 \times_k k(S)$. Write $Q = (x_Q : y_Q : z_Q : 1)$, with $x_Q, y_Q, z_Q \in k(S)$. Then Q lies in $C_\eta \cap (\mathbb{A}^3 \times_k k(S))$, which is isomorphic to the curve C_η^1 in $\mathbb{A}^2 \times_k k(S)$ defined by $H_\eta = 0$, where H_η is given in (2.4) after substituting R by η. We find x_Q by substituting $T = \tilde{z}$, $B = \tilde{y}^2 - \tilde{x}^3 - A\tilde{z}^6$ in (2.4) and factorizing, which yields

$$x_Q = \frac{9\tilde{x}^4 - 8\tilde{x}\tilde{y}^2}{4\tilde{y}^2}.$$

We conclude that the elliptic curve E_η as defined in Remark 2.3.6 is isomorphic to the curve $\left(\tilde{C}_\eta^1, \left(\frac{9\tilde{x}^4-8\tilde{x}\tilde{y}^2}{4\tilde{y}^2}, \tilde{z}\right)\right)$, where $\tilde{C}_\eta^1$ is the normalization of C_η^1. With `magma` we compute a Weierstrass model for E_η, which is given by

$$\gamma'^2 = \xi'^3 + \frac{(3 \cdot 2^5)^6 \delta}{(q_2 q_4)^6}, \tag{2.12}$$

where δ is a polynomial in $k[\tilde{x}, \tilde{z}]$ with leading term $-27B\tilde{z}^{48}\tilde{x}^{81}$. We verify with `magma` that the denominators in the defining equations of the isomorphism ω_1 between E_η and the curve (2.12), as wel as those of ω_1^{-1}, are all of the form $2^{a'}(q_2q_4)^{b'}$ for positive integers a', b'. The change of coordinates

$$\xi' = \tfrac{(3\cdot 2^5)^2}{(q_2q_4)^2}\xi, \qquad \gamma' = \tfrac{(3\cdot 2^5)^3}{(q_2q_4)^3}\gamma,$$

induces an isomorphism ω_2 between the curve (2.12) and the curve defined by

$$\gamma^2 = \xi^3 + \delta. \tag{2.13}$$

We conclude that $\omega = \omega_2 \circ \omega_1$ is an isomorphism between E_η and the curve (2.13), and the denominators in the defining equations of ω and ω^{-1} are all of the form $2^a 3^b (q_2 q_4)^c$ for positive integers a, b, c.
If δ' was another polynomial in $k[\tilde{x}, \tilde{z}]$ such that E_η were isomorphic to the curve given by $\gamma^2 = \xi^3 + \delta'$, then we would have $\delta' = \upsilon^6\delta$ for some $\upsilon \in k(S)$, hence δ' would not have leading term $-27B\tilde{z}^{48}\tilde{x}^{81}$ as univariate polynomial in $\tilde{x}$. We conclude that δ is the unique polynomial with leading term $-27B\tilde{z}^{48}\tilde{x}^{81}$ such that E_η is isomorphic to the curve with Weierstrass model (2.13). With `magma` we compute the sum D on the curve (2.13) of the points corresponding to $\left(\zeta_3^2 \frac{9\tilde{x}^4-8\tilde{x}\tilde{y}^2}{4\tilde{y}^2}, \zeta_3\tilde{z}\right)$ and $\left(\zeta_3 \frac{9\tilde{x}^4-8\tilde{x}\tilde{y}^2}{4\tilde{y}^2}, \zeta_3^2\tilde{z}\right)$ on C_η. We find $D = (\xi_D, \gamma_D)$ with $\xi_D = \frac{\alpha}{(q_1q_3)^2}$, $\gamma_D = \frac{\beta}{(q_2q_3)^3}$, where α, β are elements in $k[\tilde{x}, \tilde{z}]$ with leading terms as univariate polynomials in $\tilde{x}$ given by $\frac{1}{4}\tilde{z}^{16}\tilde{x}^{42}$ and $\frac{1}{8}\tilde{z}^{24}\tilde{x}^{63}$, respectively. $\square$

Remark 2.3.9. Let L be the hypersurface in $\mathbb{A}^2 \times S$ defined by

$$\begin{aligned}4y^2z^6X^3 - 9x^4z^4X^2T^2 + (6x^5z^2 - 12Ax^2z^8)XT^4 - 12Bx^2z^5XT \\ + (4Ax^3z^6 + 4Ay^2z^6 - 4A^2z^{12} - x^6)T^6 + 4Bz^3(x^3 - 2Az^6)T^3 \\ + 4Bz^6(y^2 - B) = 0,\end{aligned}$$

and let $\lambda\colon L \longrightarrow S$ be the projection to S. Let $R = (x_R : y_R : z_R : 1)$ be a point in $\mathcal{E}$ with $x_R \in k$, $y_R, z_R \in k^*$, $q(x_R, z_R) \neq 0$, and such that C_R is geometrically integral of genus 1. We identify R with $\pi(R) \in S$; the fiber of λ above R is the curve in $\mathbb{A}^2$ given by $H_R = 0$, where H_R is in (2.4), hence it is isomorphic to $C_R \cap \mathbb{A}^3$, where $\mathbb{A}^3$ is defined by $w \neq 0$ in $\mathbb{P}(2,3,1,1)$. Moreover, the curve $C_\eta \cap (\mathbb{A}^3 \times_k k(S))$ is isomorphic to the generic fiber of λ.
Let δ and ω be as in Lemma 2.3.8, and E_R, D_R as in Notation 2.3.7. Since $q(x_R, z_R)$ is non-zero, the isomorphism ω specializes to the fiber $\lambda^{-1}(R)$, and we obtain an isomorphism between E_R and the curve given by

$$\gamma^3 = \xi^2 + \delta(x_R, z_R), \tag{2.14}$$

that sends the point D_R to the point

$$(\xi_D(x_R, z_R), \gamma_D(x_R, z_R)).$$

Let $P = (x_0 : y_0 : z_0 : 1)$ be a point on S with $z_0 \neq 0$. Let V be the set of points $R = (x_R : y_R : z_0 : 1)$ on the fiber of $P_{\mathcal{E}}$ with $x_R \in k$, $y_R \in k^*$, such that $q(x_R, z_0) \neq 0$ (where q is given in (2.9)), and such that C_R is geometrically integral of genus 1.

Proposition 2.3.10. *If k is a number field, then for all but a finite number of points R in the set V, the curve E_R has positive rank over k.*

Proof. Let δ be as in Lemma 2.3.8. We define the following polynomials in $k[\tilde{x}, \xi, \gamma]$.

$$\psi_1 = 1, \qquad \psi_2 = 2\gamma, \qquad \psi_3 = 3\xi^4 + 12\delta(\tilde{x}, z_0)\xi,$$
$$\psi_4 = 2\psi_2\left(\xi^6 + 20\delta(\tilde{x}, z_0)\xi^3 - 8\delta(\tilde{x}, z_0)^2\right),$$

and recursively,

$$\psi_{2m+1} = \psi_{m+2}\psi_m^3 - \psi_{m-1}\psi_{m+1}^3 \quad \text{for } m \geq 2, \tag{2.15}$$
$$\psi_2\psi_{2m} = \psi_{m-1}^2\psi_m\psi_{m+2} - \psi_{m-2}\psi_m\psi_{m+1}^2 \quad \text{for } m \geq 3. \tag{2.16}$$

Let ξ_D, γ_D be as in Lemma 2.3.8. For $m \geq 1$, we define $\psi_{m,\tilde{x}}$ to be the rational function

$$\psi_m(\tilde{x}, \xi_D(\tilde{x}, z_0), \gamma_D(\tilde{x}, z_0)) \in k(\tilde{x}).$$

Write $d = q_1(\tilde{x}, z_0)q_3(\tilde{x}, z_0) \in k[\tilde{x}]$. From Remark 2.3.9, we find

$$\psi_{2,\tilde{x}} = \frac{N_2}{d^3}, \quad \psi_{3,\tilde{x}} = \frac{N_3}{d^8}, \quad \psi_{4,\tilde{x}} = \frac{N_4}{d^{15}},$$

where N_2, N_3, N_4 are polynomials in $k[\tilde{x}]$. Let c_i be the leading coefficient of N_i for $i \in \{2, 3, 4\}$, then we have

$$\deg(N_2) = 63, \quad \deg(N_3) = 168, \quad \deg(N_4) = 315,$$

$$c_2 = \frac{1}{4}z_0^{24}, \quad c_3 = \frac{3}{2^8}z_0^{64}, \quad c_4 = \frac{1}{2^{13}}t_0^{120}.$$

We claim that for all $m \geq 1$ we have

$$\psi_{m,\tilde{x}} = \frac{N_m}{d^{m^2-1}},$$

Where N_m is a polynomial in $k[\tilde{x}]$ with leading coefficient c_m such that

$$\deg(N_m) = 21(m^2 - 1) \qquad \text{and} \qquad c_m = m\left(\frac{1}{2}z_0^8\right)^{m^2-1}.$$

Assume that this claim is true (we prove this below). Since k is a number field, there is an upper bound $B = B(k)$ such that the torsion points on the fiber of $P_{\mathcal{E}}$ have order at most B [Mer96]. Let $R = (x_R : y_R : z_0 : 1)$ be a point in V such that $\psi_{m,\tilde{x}}(x_R)$ is non-zero for all $m \leq B$. Note that, since $z_0 \neq 0$, this holds for all but finitely many points in V by our claim. By Remark 2.3.9, the curve E_R is isomorphic to the elliptic curve in $\mathbb{A}^2$ given by equation (2.14), where $z_R = z_0$. We identify E_R with this model. Let D_R be the point on E_R as in Notation 2.3.7, and note that D_R is defined over k because R is. Write

$$\xi_R = \xi_D(x_R, z_0), \quad \gamma_R = \gamma_D(x_R, z_0),$$

then we have $D_R = (\xi_R, \gamma_R)$ by Remark 2.3.9, and since $q(x_R, z_0) \neq 0$, the point D_R is non-zero on E_R.

Note that for $m \geq 1$, the polynomial $\psi_m(x_R, \xi, \gamma) \in k[\xi, \gamma]$ is the m-th division polynomial of E_R, as defined in [Sil09, Exercise 3.7], and from the same reference we know that D_R is m-torsion for $m \geq 2$ if and only if

$\psi_m(x_R, \xi_R, \gamma_R) = \psi_{m,\tilde{x}}(x_R) = 0$. Since we chose R such that $\psi_{m,\tilde{x}}(x_R) \neq 0$ for all $m \leq B$, we conclude that D_R is non-torsion on E_R. This, together with the proof of the claim below, proves the propostion.
Proof claim.
We prove this by induction. Set $k \geq 2$, and assume that the claim holds for $m < 2k+1$ (note that this is indeed the case for $k = 2$). Then we have

$$\begin{aligned}\deg\left(N_{k+2}N_k^3\right) &= 21((k+2)^2-1) + 63(k^2-1) = 21(4k^2+4k);\\ \deg\left(N_{k-1}N_{k+1}^3\right) &= 21((k-1)^2-1) + 63((k+1)^2-1) = 21(4k^2+4k),\end{aligned}$$

so we find

$$\deg(N_{k+1}N_k^3) = \deg\left(N_{k-1}N_{k+1}^3\right) = 21((2k+1)^2-1). \tag{2.17}$$

Completely analogously, we find that the denominators of $\psi_{k+2,\tilde{x}}\psi_{k,\tilde{x}}^3$ and $\psi_{k-1,\tilde{x}}\psi_{k+3,\tilde{x}}^3$ are both equal to $d^{(2k+1)^2-1}$. Combining this with (2.17), we find from the recursion in (2.15) that the denominator of $\psi_{2k+1,\tilde{x}}$ is equal to $d^{(2k+1)^2-1}$, that the degree of N_{2k+1} is at most $21((2k+1)^2-1)$, and that the coefficient of the monomial $\tilde{x}^{21((2k+1)^2-1)}$ in N_{2k+1} is given by $c_{k+2}c_k^3 - c_{k-1}c_{k+1}^3$, which by induction is equal to

$$\begin{aligned}&(k+2)(\frac{1}{2}z_0^8)^{(k+2)^2-1}k^3(\frac{1}{2}z_0^8)^{3(k^2-1)}\\ &-(k-1)(\frac{1}{2}z_0^8)^{(k-1)^2-1}(k+1)^3(\frac{1}{2}z_0^8)^{3((k+1)^2-1)}\\ &\quad=(k^3(k+2)-(k-1)(k+1)^3)(\frac{1}{2}z_0^8)^{4k^2+4k}\\ &\quad=(2k+1)(\frac{1}{2}z_0^8)^{(2k+1)^2-1}.\end{aligned}$$

Since the latter is non-zero we conclude that it is the leading coefficient of N_{2k+1}, so we find $c_{2k+1} = (2k+1)(\frac{1}{2}z_0^8)^{(2k+1)^2-1}$, and we conclude $\deg(N_{2k+1}) = 21((2k+1)^2-1)$. This finishes the proof of the claim for $m = 2k+1$; we will now prove it for $m = 2k+2$ in a similar way. By induction, we have

$$\begin{aligned}
\deg\left(N_k^2 N_{k+1} N_{k+3}\right) &= 42(k^2-1) + 21((k+1)^2-1) \\
&\quad + 21((k+3)^2-1) \\
&= 21((2k+2)^2 - 1 + 3) \\
\deg\left(N_{k-1} N_{k+1} N_{k+2}^2\right) &= 21((k-1)^2-1) + 21((k+1)^2-1) \\
&\quad + 42((k+2)^2-1) \\
&= 21((2k+2)^2 - 1 + 3),
\end{aligned}$$

so we find

$$\begin{aligned}
\deg\left(N_k^2 N_{k+1} N_{k+3}\right) &= \deg\left(N_{k-1} N_{k+1} N_{k+2}^2\right) \\
&= 21((2k+2)^2-1) + \deg\left(N_2\right). \quad (2.18)
\end{aligned}$$

Analogously we find that the denominators of both $\psi_{k,\tilde{x}}^2 \psi_{k+1,\tilde{x}} \psi_{k+3,\tilde{x}}$ and $\psi_{k-1,\tilde{x}} \psi_{k+1,\tilde{x}} \psi_{k+2,\tilde{x}}^2$ are equal to $d^{(2k+2)^2-1} d^{2^2-1}$. Combining this with (2.18), we find from the recursion in (2.16) that the denominator of ψ_{2k+2} is equal to $d^{(2k+2)^2-1}$, that the degree of N_{2k+2} is at most $21((2k+2)^2-1)$, and that the coefficient of the monomial $\tilde{x}^{21((2k+2)^2-1)}$ in N_{2k+2} is given by $\frac{1}{c_2}(c_k^2 c_{k+1} c_{k+3} - c_{k-1} c_{k+1} c_{k+2}^2)$, which by induction is equal to

$$\begin{aligned}
&\frac{1}{c_2}\Big(k^2(\frac{1}{2}z_0^8)^{2(k^2-1)}(k+1)(\frac{1}{2}z_0^8)^{(k+1)^2-1}(k+3)(\frac{1}{2}z_0^8)^{(k+3)^2-1} \\
&\qquad -(k-1)(\frac{1}{2}z_0^8)^{(k-1)^2-1}(k+1)(\frac{1}{2}z_0^8)^{(k+1)^2-1}(k+2)^2(\frac{1}{2}z_0^8)^{2(k+2)^2-2}\Big) \\
&= \frac{1}{c_2}\left((k^2(k+1)(k+3) - (k-1)(k+1)(k+2)^2)(\frac{1}{2}z_0^8)^{4k^2+8k+6}\right) \\
&= \frac{1}{c_2}\left(2(\frac{1}{2}z_0^8)^3(2k+2)(\frac{1}{2}z_0^8)^{(2k+2)^2-2}\right) \\
&= \frac{1}{c_2}\left(c_2(2k+2)(\frac{1}{2}z_0^8)^{(2k+2)^2-1}\right) = (2k+2)(\frac{1}{2}z_0^8)^{(2k+2)^2-1}.
\end{aligned}$$

Since the latter is non-zero we conclude that it is the leading coefficient of N_{2k+2}, so we find $c_{2k+2} = (2k+2)(\frac{1}{2}z_0^8)^{(2k+2)^2-1}$, and we conclude $\deg(N_{2k+2}) = 21((2k+2)^2-1)$. This finishes the proof of the claim for $m = 2k+2$. The rest of the claim now follows from induction. □

We are now ready to prove Proposition 2.2.6. Recall that for a point $P \in S \setminus \{(1:1:0:0)\}$, we denote by $P_{\mathcal{E}}$ the corresponding point on $\mathcal{E}$.

By the fiber of $P_{\mathcal{E}}$ we mean the fiber of the elliptic fibration $\nu\colon \mathcal{E} \longrightarrow \mathbb{P}^1$ that contains $P_{\mathcal{E}}$; see also Notation 2.2.3.

Proof of Proposition 2.2.6. Let k be a number field, and let P be a point $P = (x_0 : y_0 : z_0 : 1)$ as in Proposition 2.2.6. Since P is defined over k and $P_{\mathcal{E}}$ has infinite order on its fiber, the set $\mathcal{P} = \{nP_{\mathcal{E}} : n \in \mathbb{Z} \setminus 0\}$ contains infinitely many points on the fiber of $P_{\mathcal{E}}$ that are all defined over k and have non-zero y, z-coordinates. Since the strict transform of an exceptional curve on S is a section of $\mathcal{E}$ (Remark 1.4.20), there are at most 240 points in $\mathcal{P}$ that are contained in the strict transform of an exceptional curve on S (Table 1.1). Let V_1 be the set of these points. Let V_2 be the set of points $(x_R : y_R : z_0 : 1) \in \mathcal{P}$ such that x_R is a root of the polynomial $q(\tilde{x}, z_0) \in k[\tilde{x}]$ defined in (2.9); there are at most 25 points in V_2. For all points R in $\mathcal{P} \setminus V_1$, the curve C_R is defined over k, and it either contains a section defined over k, or is geometrically integral of genus at most 1, by Lemma 2.3.4. Let V_3 be the set in $\mathcal{P} \setminus (V_1 \cup V_2)$ for which C_R is geometrically integral of genus 1, and for which the elliptic curve E_R has rank 0 over k; the set V_3 is finite by Proposition 2.3.10. We conlude that the set $\mathcal{P} \setminus (V_1 \cup V_2 \cup V_3)$ contains infinitely many points, and all integers n for which $nP_{\mathcal{E}}$ is in this set satisfy the statement in Proposition 2.2.6.

2.4 Proof of the main result

In this section we prove Theorem 2.2.1. Let A, B, k, S, and $\mathcal{E}$ be as in the theorem (in particular, k is now a number field), and recall Notation 2.2.3.

Proof of Theorem 2.2.1. Let P be a point satisfying the conditions in Theorem 2.2.1. By Proposition 2.2.6, there exists an integer n such that one of the following holds.

(i) $C_{nP_{\mathcal{E}}}$ has a component that is a section defined over k,

(ii) $C_{nP_{\mathcal{E}}}$ is a 3-section of $\mathcal{E}$ of geometric genus 0, or

(iii) $C_{nP_{\mathcal{E}}}$ is a 3-section of $\mathcal{E}$ whose normalization is an elliptic curve with positive rank over k.

Choose such an n and set $R = nP_{\mathcal{E}}$. Note that in case (i) we are done by Remark 2.1.5. In case (ii), the desingularization of C_R is a smooth curve of genus 0. Since R is not a triple point on C_R, the latter contains a rational point given by the unique other point in the intersection of

C_R with the fiber of R, hence C_R has infinitely many k-rational points. In case (iii), C_R contains infinitely many k-rational points as well. Now assume we are in case (ii) or (iii). Then C_R contains infinitely many k-rational points, and since C_R intersects each fiber of $\mathcal{E}$ in 3 points counted with multiplicity, this implies that C_R intersects infinitely many fibers in a k-rational point. We show that infinitely many of these points are non-torsion on their fiber. Note that every smooth fiber is an elliptic curve over k, hence there is an upper bound $B = B(k)$ such that on all the fibers, all the torsion points have order at most B [Mer96]. Let $m \leq B$ be an integer, and let T_m be the zero locus of the m-th division polynomial $\psi_m \in k[x, y, t]$ of the generic fiber E over the function field $k(t)$. We have $\psi_m \in k[x, t]$, and for any $\tau \in k$, the polynomial $\psi_m(x, \tau) \in k[x]$ has degree m^2 [Sil09, Exercise III.3.7]. So T_m is an m^2-section of $\mathcal{E}$. Moreover, for every smooth fiber $\mathcal{E}_t$, the intersection of T_m with $\mathcal{E}_t$ is exactly the set of m-torsion points on $\mathcal{E}_t$, which has size m^2 [Sil09, Exercise III.3.7 and Corollary III.6.4]. It follows that T_m intersects every smooth fiber of $\mathcal{E}$ in m^2 points, all with multiplicity 1. In particular, the curve C_R is not a component of T_m, since in all three cases above, C_R intersects the smooth fiber of P in a point with multiplicity 2. Therefore, the curve C_R intersects T_m only in finitely many points. Since all the torsion points on the fibers of $\mathcal{E}$ are contained in the finite union $\cup_{m \leq B} T_m$, we conclude that C_R intersects only finitely many fibers in a torsion point. Since we already showed that C_R intersects infinitely many fibers in a k-rational point, this implies that C_R intersects infinitely many fibers in a k-rational point that is non-torsion on its fiber. We conclude that infinitely many smooth fibers of $\mathcal{E}$ have infinitely many k-rational points. Since a smooth fiber $\mathcal{E}_t$ is closed and irreducible in $\mathcal{E}$, it follows that $\mathcal{E}_t \cap \overline{\mathcal{E}(k)} = \mathcal{E}_t$. So $\overline{\mathcal{E}(k)}$ contains infinitely many one-dimensional irreducible subsets, which implies that it is of dimension 2, and since $\mathcal{E}$ is irreducible we conclude that $\overline{\mathcal{E}(k)} = \mathcal{E}$, i.e., the set of k-rational points of $\mathcal{E}$ is dense in $\mathcal{E}$. Since $\mathcal{E}$ and S are birationally equivalent, it follows that $S(k)$ is dense in S as well. Conversely, if S did not contain a point P as in the theorem, then $S(k)$ would be contained in the union of the torsion locus $\cup_{m \leq B} T_m$ with the two fibers $(1:0)$ and $(0:1)$ and the singular fibers, which is a strict closed subset of S, hence $S(k)$ would not be dense in S.

2.5 Examples

We conclude this chapter by giving two examples where we prove the density of the set of rational points on specific del Pezzo surfaces of degree 1. The rank of the Mordell–Weil group over $\mathbb{Q}$ of the surfaces in Examples 2.5.1 and 2.5.2 is 0 by [DN, Corollary 2.4 and Figure 5], so in these cases the density of the set of $\mathbb{Q}$-rational points can not be proven by the existence of a section over $\mathbb{Q}$ (see also Remark 2.1.5).

Example 2.5.1. Let k be a number field and let S be the del Pezzo surface of degree 1 in $\mathbb{P}(2,3,1,1)$ given by

$$y^2 = x^3 + 6(27z^6 + w^6).$$

Note that S does not satisfy the conditions of [VA11, Theorem 1.1] since $3 \cdot 27$ is a square and $\gcd(6 \cdot 27, 6) \neq 1$, hence the density of the set of $\mathbb{Q}$-rational points could not be proven by Várilly-Alvarado [VA11, Example 7.2]. However, the fiber $\mathcal{E}_{(1:1)}$ of the anticanonical elliptic surface $\mathcal{E}$ above $(1:1)$ is smooth, and with `magma` we find that this fiber has rank 2. So S contains a point that lies on a smooth fiber of $\mathcal{E}$ and has infinite order, hence $S(k)$ is dense in S by Theorem 2.2.1.
We illustrate this by constructing a 3-section as in (2.2). With `magma` we find two generators for $\mathcal{E}_{(1:1)}(\mathbb{Q})$, given by $P_1 = (1 : 13 : 1 : 1)$ and $P_2 = (22 : 104 : 1 : 1)$. The curve C_{P_1} is cut out from S by

$$3xz - 26y + 323z^3 + 12w^3,$$

and it has geometric genus 1. We find $C_{P_1} \cap \mathcal{E}_{(1:1)} = \{P_1, Q_1\}$ with $Q_1 = \left(-\frac{1343}{676} : \frac{222431}{17576} : 1 : 1\right)$. The elliptic curve $E = (\tilde{C}_{P_1}, Q_1)$ is given by Weierstrass equation

$$\gamma^2 = \xi^3 - 2 \cdot 3^4 \cdot 5^2 \cdot 28368481,$$

and the point $D = \sigma(Q_1) + \sigma^2(Q_1)$ has infinite order on E; its ξ-coordinate is given by

$$\xi_D = \frac{11 \cdot 33487 \cdot 580020724757}{(2 \cdot 12 \cdot \cdot 167 \cdot 523)^2},$$

so D has infinite order on E by a result of Lutz and Nagel ([Corollary VIII.7.2][Sil09]). We conclude that the 3-section C_{P_1} has infinitely many k-rational points. Equivalently, we could have used the point P_2 to create a 3-section with infinitely many k-rational points: the curve C_{P_2} is cut out from S by $1452xz - 208y - 10324z^3 + 12w^3$; it has geometric genus 1, the third point of intersection of C_{P_2} with the fiber $\mathcal{E}_{(1:1)}$ is given by $Q_2 = \left(\frac{12793}{2704} : -\frac{2327053}{140608} : 1 : 1\right)$, and the point $\sigma(Q_2) + \sigma^2(Q_2)$ again has infinite order on the elliptic curve $(\tilde{C}_{P_2}, Q_2)$. We conclude that also C_{P_2} has infinitely many k-rational points.

Example 2.5.2. Let k be a number field and consider the del Pezzo surface S of degree 1 in $\mathbb{P}(2,3,1,1)$ given by

$$y^2 = x^3 + 243z^6 + 16w^6.$$

Note that this surface does not satisfy the conditions of [VA11, Theorem 1.1], so the method there failed in this case [VA11, Remark 7.4]. Salgado and van Luijk made the observation that this surface contains the point $P = (0 : 4 : 0 : 1)$, which is 3-torsion on its fiber on $\mathcal{E}$ (more generally, a surface of the form $y^2 = x^3 + \beta^2 w^6$ has the 3-torsion point $(0 : \beta : 0 : 1)$). However, this point is contained in 9 exceptional curves, so their method does not work with P. They did not find another point for which the computations were doable to show density of $S(k)$ [SvL14, Examples 7.3 and 4.4 (iii)]. Finally, Elkies showed that the set $S(\mathbb{Q})$ is Zariski-dense in S, by constructing a multisection with infinitely many rational points in the linear system $|-3K_S|$ that contains P as a point of multiplicity 3 (this idea was generalized to any surface with a torsion point in the master thesis [Bul18], though under the assumption that at least one of the infinitely many multisections constructed there has infinitely many rational points).
We prove the density of $S(k)$ in S using Theorem 2.2.1: with `magma` we find that the fiber $\mathcal{E}_{(1:5)}$ above $(1 : 5)$ is smooth and has rank 2, so S contains a point that lies on a smooth fiber of $\mathcal{E}$ and has infinite order (for example $P = (-63 : -14 : 1 : 5)$), hence $S(k)$ is dense in S.

3

The action of the Weyl group on the E_8 root system

This chapter is an adaptation of the preprint [vLWa], which is at the time of writing submitted for publication. Some of the results here were already proved by the same author in the master thesis [Win14]; we state which results coincide in the relevant places (at Proposition 3.2.2 and Lemma 3.2.14, and in Remarks 3.3.20 and 3.5.1).

Let X be a del Pezzo surface of degree 1 over an algebraically closed field k. Recall that the 240 exceptional curves on X are in one-to-one correspondence with the exceptional classes in $\operatorname{Pic} X$, and as we have seen in Section 1.4.2, these are in one-to-one correspondence with the 240 roots in the $\mathbf{E}_8$ root system. In this chapter we study this root system and, more specifically, the action of its automorphism group on the roots. The reason we originally did this is because we wanted to study configurations of intersecting exceptional curves on X; the results on this are in Chapter 4. However, since $\mathbf{E}_8$ arises in many more areas of mathematics, we thought it useful to do a more thorough study of this root system. Therefore, while this chapter contains results that are used in Chapters 4 and 5, it is also self-contained, and the reader does not need to have any knowledge of or interest in del Pezzo surfaces or algebraic geometry to be able to appreciate it. In Remarks 3.2.8, 3.3.6, 3.3.23, 3.4.11, and 3.5.4, we explain how some of the results in this chapter translate to the 240 exceptional classes in Pic X. Everything in these remarks is over the algebraically closed field k.

3.1 Main results

Notation 3.1.1. Recall the definitions graph, weighted graph, weighted subgraph, clique, and isomorsphism between two weighted graphs in Definition 1.4.11. In this chapter we use the same definition for graph, and we use the term *colored graph* for weighted graph; we will talk about *colors* instead of weights, and define *colored subgraph*, *clique*, and *isomorphism* between colored graphs analogously. The reason for this terminology is that it allows us to talk about a *monochromatic graph*, i.e., a colored graph where all edges have the same color. Whenever we talk about an isomorphism of two cliques, we mean an isomorphism of colored graphs.

Let E be the set of roots in $\mathbf{E}_8$. The following definition is analogous to Definition 1.4.12.

Definition 3.1.2. Let Γ be the complete colored graph whose vertex set is E, of which the color function on the edge set is induced by the dot product in $\mathbf{E}_8$. The different colors of the edges in Γ are $-2, -1, 0, 1$. For a subset $c \subseteq \{-2, -1, 0, 1\}$, we denote by Γ_c the colored subgraph of Γ with vertex set E and including all edges whose color is an element in c.

Let W be the automorphism group of Γ as colored graph; recall that W is isomorphic to the Weyl group W_8 (Corollary 1.4.14). It is clear that if two cliques in Γ are conjugate under the action of W, they must be isomorphic as colored graphs. The converse is not always true, and in general it can be hard to determine whether two cliques in Γ are conjugate under the action of W. Dynkin and Minchenko studied in [DM10] the bases of subsystems of $\mathbf{E}_8$, and classified for which isomorphism classes of these bases being isomorphic implies being conjugate. They call these bases *normal*. In this chapter, we extend this classification to a large set of cliques in Γ (more specifically, cliques of type I, II, III, or IV, as defined below). In Theorem 3.1.3 we show that with two exceptions, two such cliques are isomorphic if and only if they are conjugate. One of the exceptions, which is the clique described in Theorem 3.1.3 (i), is one of the bases (of the system $4A_1$) that was also found as not being normal in [DM10, Theorem 4.7]. Additionally, in [DM10] the authors determine when a homomorphism of

two bases of subsytems extends to a homomorphism of the whole root system. We answer the same question for cliques of type I, II, III, or IV in Theorem 3.1.4.
Although the classification of different types of cliques and their orbits is a finite problem, because of the size of Γ it is practically impossible to naively let a computer find and classify the cliques according to their W-orbit. In fact, we avoid using a computer for our computations as much as possible.

The $\mathbf{E}_8$ root polytope is the convex polytope in $\mathbb{R}^8$ whose vertices are the roots in E. By a *face* of the root polytope we mean a non-empty intersection of a hyperplane in $\mathbb{R}^8$ and the root polytope, such that the root polytope lies entirely on one side of the hyperplane. If the dimension of this intersection is n then we call this an n-face, and a 7-face is called a *facet*. We study the following cliques in Γ, and their orbits under the action of W.

(I) Monochromatic cliques

(II) Cliques whose vertices are the vertices of a face of the $\mathbf{E}_8$ root polytope

(III) Cliques of size at most three

(IV) For all $c \neq \{-1, 0, 1\}$, the maximal cliques in Γ_c

More specifically, we prove the following theorem.

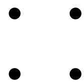
Theorem 3.1.3. *Let K_1, K_2 be two cliques in Γ, each of type I, II, III, or IV. Then the following hold.*

(i) If both K_1 and K_2 are of type I with color 0 and of size 4, then K_1 and K_2 are conjugate under the action of W if and only if the vertices sum to an element in 2Λ for both K_1 and K_2, or for neither.

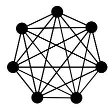
(ii) If both K_1 and K_2 are of type I with color 1 and of size 7, then K_1 and K_2 are conjugate under the action of W if and only if the vertices sum to an element in 2Λ for both K_1 and K_2, or for neither; this is equivalent to K_1 and K_2 both being maximal cliques or both being non-maximal cliques, respectively, under inclusion in $\Gamma_{\{1\}}$.

(iii) In all other cases, K_1 and K_2 are conjugate under the action of W if and only if they are isomorphic as colored graphs.

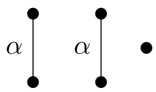
Furthermore, we give conditions for an isomorphism of two cliques of types I, II, III or IV to extend to an automorphism of the $\mathbf{E}_8$ lattice Λ (defined in Section 1.4.2). To this end we introduce the following complete colored graphs.



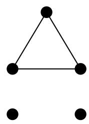
A



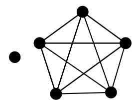
B



C_α



D



F

Here α is either -1 or 1, two disjoint vertices have an edge of color 0 between them, and all other edges have color 1.

Theorem 3.1.4. *Let K_1, K_2 be two cliques in Γ of types I, II, III, or IV, and let $f\colon K_1 \longrightarrow K_2$ be an isomorphism between them. The following hold.*

(i) The map f extends to an automorphism of Λ if and only if for every ordered sequence $S = (e_1, \ldots, e_r)$ of distinct roots in K_1 such that the colored graph on them induced by Γ is isomorphic to A, B, C_α, D, or F, its image $f(S) = (f(e_1), \ldots, f(e_r))$ is conjugate to S under the action of W.

(ii) If $S = (e_1, \ldots, e_r)$ is a sequence of distinct roots in K_1 such that the colored graph on them induced by Γ is isomorphic to either A or B, then S and $f(S)$ are conjugate under the action of W if and only if the sets $\{e_1, \ldots, e_r\}$ and $\{f(e_1), \ldots, f(e_r)\}$ are.

(iii) If K_1 and K_2 are maximal cliques, both in $\Gamma_{\{-1,0\}}$ or both in $\Gamma_{\{-2,-1,0\}}$, and $S = (e_1, \ldots, e_5)$ is a sequence of roots in K_1 such that the colored graph on them induced by Γ is isomorphic to C_{-1} with $e_1 \cdot e_4 = e_2 \cdot e_5 = -1$, then S and $f(S)$ are conjugate under the action

of W if and only if both $e = e_1 + e_2 + e_3 - e_4 - e_5$ and $f(e)$ are in the set $\{2f_1 + f_2 \mid f_1, f_2 \in E\}$, or neither are.

(iv) If K_1 and K_2 are maximal cliques in $\Gamma_{\{-2,0,1\}}$, and $S = (e_1, \ldots, e_r)$ is a sequence of distinct roots in K_1 such that the colored graph G on them induced by Γ is isomorphic to C_1, D, or F, then S and $f(S)$ are conjugate under the action of W if and only if the sets $\{e_1, \ldots, e_r\}$ and $\{f(e_1), \ldots, f(e_r)\}$ are, or equivalently, if and only if the following hold.
- *If $G \cong C_1$, both $\sum_{i=1}^{5} e_i$ and $\sum_{i=1}^{5} f(e_i)$ are in $\{2f_1 + f_2 \mid f_1, f_2 \in E\}$, or neither are.*
- *If $G \cong D$, both $\sum_{i=1}^{5} e_i$ and $\sum_{i=1}^{5} f(e_i)$ are in $\{2f_1 + 2f_2 \mid f_1, f_2 \in E\}$, or neither are.*
- *If $G \cong F$, then both $\sum_{i=1}^{6} e_i$ and $\sum_{i=1}^{6} f(e_i)$ are in 2Λ, or neither are.*

Remark 3.1.5. Note that to apply Theorem 3.1.4 (i) to an isomorphism f, we have to know whether certain ordered sequences of roots are conjugate. Theorem 3.1.4 (ii), in combination with Theorem 3.1.3 (i) and (ii), tells us how to verify this when the colored graph on the roots in an ordered sequence is isomorphic to A or B. Theorem 3.1.4 (iii) and (iv) tells us how to verify this when the colored graph on the roots in an ordered sequence is isomorphic to C_α, D, or F.

Remark 3.1.6. In the proof of Theorem 3.1.4, we specify for each type of K_1 and K_2 which of the graphs A, B, C_α, D, and F are needed to check whether an isomorphism f extends. Of course one can see this partially from the size and the colors, but it turns out that we can make stronger statements. For example, surprisingly, an isomorphism between two maximal graphs in $\Gamma_{\{0,1\}}$ always extends, and even uniquely (Corollary 3.5.37). In the table in Remark 3.6.1 we show the requirements for each type of K_1 and K_2.

As we mentioned before, because of the size of Γ it is practically impossible to naively let a computer find and classify all cliques of the above types according to their W-orbit. This holds mainly for the results in Section 3.5, where we study cliques of type IV. This is the only section where we use a computer program, but without using results from the previous sections to minimize the computations it would have been practically undoable. Checking that two cliques are isomorphic is easily done by hand for types I, II, and III, since with one exception of size fourteen, they are all of size at most eight (see Sections 3.3 and 3.4). For type IV we give necessary and sufficient invariants to check if two large cliques are isomorphic in Section 3.5.

Remark 3.1.7. Apart from the work in [DM10] on bases of subsystems of $\mathbf{E}_8$, some partial results of Theorems 3.1.3 and 3.1.4 were known before. We list them here and compare them to our results.
The orbits of the faces of the $\mathbf{E}_8$ root polytope under the action of W are described in [Cox30, Section 7.5]. These include all monochromatic cliques of color 1 (see Proposition 3.2.4). For one of the types of facets, we give a different, more group-theoretical proof of the fact that they form one orbit under the action of W, see Corollary 3.3.17.
The orbits of *ordered sequences* of the vertices in the faces (except for one type of facets) have been described in [Man86, Corollary 26.8]. We summarize his results in Proposition 3.2.12.
Monochromatic cliques of color 0 are orthogonal sets, and their orbits under the action of W are described in [DM10, Corollary 3.3]. We describe the action of W on the *ordered sequences* of orthogonal roots in Proposition 3.4.4.
Finally, in [CRS04] the authors give a classification of isomorphism types of all maximal *exceptional* graphs (i.e., connected graphs with least eigenvalue greater or equal to -2 that are not generalized line graphs [CRS04, Section 1.1]). From [CRS04, Corollary 3.6.4] it follows that these graphs correspond exactly to the maximal cliques in $\Gamma_{\{0,1\}}$. Therefore our classification of isomorphism types of cliques of Type IV for $c = \{0, 1\}$ (see Appendices A and B) coincides with the classification of isomorphism types of maximal exceptional graphs in [CRS04, Appendix A6]; see Remark 3.5.32 for a comparison between our method and the one in [CRS04]. However, the classification of the isomorphism types is only part of our results on the maximal cliques in $\Gamma_{\{0,1\}}$. We also give invariants for such a clique that determine its isomorphism type, and we show that each isomorphism class is a full orbit under the action of W (Propositions 3.5.35 and 3.5.36). Moreover, in Corollary 3.5.37 we show that every isomorphism between two representations of exceptional graphs in $\mathbf{E}_8$ extends to an automorphism of $\mathbf{E}_8$.

We split the chapter into sections that deal with one or more of the types I, II, III, or IV. Note that these four types do not exclude each other, and some results in one section may be part of a result in another section. We ordered the sections such that each section builds as much on the previous ones as possible.

Section 3.2 states all the needed definitions as well as many known results about $\mathbf{E}_8$ and the action of the Weyl group. We also set up the notation for the rest of this chapter. The reader who is familiar with root systems, and with $\mathbf{E}_8$ in particular, can skip this section. Section 3.3 contains all results on the facets of the $\mathbf{E}_8$ root polytope, and cliques of type III. Section 3.4 deals with cliques of type I. Section 3.5 classifies all cliques of type IV. This is the biggest section, and the only section where we use a computer for some of the results (from Section 3.5.3 onwards). The results from this section are summarized in the tables in the appendices. Finally, we prove Theorems 3.1.3 and 3.1.4 in Section 3.6.

All computations are done in `magma` [BCP97]. The code that we used can be found in [Codb]. We want to thank David Madore, who gave us useful references for results on $\mathbf{E}_8$ and the action of W. Moreover, there is a great interactive view of $\mathbf{E}_8$ on his website `http://www.madore.org/~david/math/e8w.html`, which has been very insightful.

3.2 The Weyl group and the $\mathbf{E}_8$ root polytope

Let Λ be the $\mathbf{E}_8$ lattice as defined in Section 1.4.2, let Γ be the graph defined in Definition 3.1.2, with automorphism group W, and let E be the set of roots in $\mathbf{E}_8$. In this section we recall some well-known results about these objects and the $\mathbf{E}_8$ root polytope. We also make a first step in proving Theorems 3.1.3 and 3.1.4, by showing that for two cliques of type I, II, III, or IV in Γ that are isomorphic as colored graphs, there is a type that they both belong to (Lemma 3.2.13).

Useful references for root systems and the Weyl group are [Bou68, Chapter 6], and [Hum72, Chapter III].

The subgroup of the isometry group of $\mathbb{R}^8$ that is generated by the reflections in the hyperplanes orthogonal to the roots in E is called the *Weyl group*, and denoted by W_8. This group permutes the elements in E, and since these roots span $\mathbb{R}^8$, the action of W_8 on E is faithful. The Weyl group is therefore finite: it has order $696729600 = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$. It is equal to the automorphism group of the $\mathbf{E}_8$ root system [Hum72, Section 12.2], hence also to the automorphism group of the root lattice Λ, and to the group W.

Lemma 3.2.1. *The Weyl group acts transitively on the $\mathbf{E}_8$ root system.*

Proof. [Hum72, Section 10.4, Lemma C]. $\square$

From the description of Λ and $\mathbf{E}_8$ we see that the roots in E are of two types. Either they are of the form $\left(\pm\frac{1}{2}, \ldots, \pm\frac{1}{2}\right)$, where an even number of entries is negative (giving $2^7 = 128$ roots), or exactly two entries are non-zero, and they can independently be chosen to be -1 or 1 (giving $4 \cdot \binom{8}{2} = 112$ roots).

The following proposition contains Proposition 3.17 in [Win14], where the results are written in terms of exceptional curves on a del Pezzo surface of degree 1.

Proposition 3.2.2. *The absolute value of the dot product of any two elements in E is at most 2. Let $e \in E$ be a root. Then e has dot product 2 only with itself, and dot product -2 only with its inverse $-e$. There are exactly 56 roots $f \in E$ with $e \cdot f = 1$, there are exactly 56 roots $g \in E$ with $e \cdot g = -1$, and there are exactly 126 roots in E that are orthogonal to e.*

Proof. From Cauchy–Schwarz it follows that for $e, e' \in E$ we have

$$|e \cdot e'| \leq \|e\| \cdot \|e'\| = 2,$$

and equality holds if and only if e, e' are scalar multiples of each other. Since all roots are primitive, it follows that $e \cdot e' = 2$ if and only if $e = e'$, and $e \cdot e' = -2$ if and only if $e = -e'$. Since W acts transitively on E (Lemma 3.2.1), to count the other cases it suffices to prove this for one element in E. Take $e = (1,1,0,0,0,0,0,0) \in E$.
The roots $f \in E$ with $e \cdot f = 1$ are of the form $f = (a_1, \ldots, a_8)$ with $a_1 + a_2 = 1$. So for these roots we either have $a_1 = a_2 = \frac{1}{2}$, which gives 32 different roots, or $\{a_1, a_2\} = \{0, 1\}$, which gives 24 different roots. This gives a total of 56 roots.
For $f \in E$, we have $e \cdot f = 1$ if and only if $e \cdot -f = -1$, so this gives also 56 roots $g \in E$ with $e \cdot g = -1$.
The roots in E that are orthogonal to e are of the form $f = (a_1, \ldots, a_8)$ with $a_1 + a_2 = 0$. So for these roots we have $a_1 = a_2 = 0$, which gives 60 roots, or $\{a_1, a_2\} = \{-1, 1\}$, which gives 2 roots, or $\{a_1, a_2\} = \left\{-\frac{1}{2}, \frac{1}{2}\right\}$, which gives 64 roots. This gives a total of 126 roots. $\square$

We continue with results on the $\mathbf{E}_8$ root polytope. Coxeter described all faces of the $\mathbf{E}_8$ root polytope, which he called the 4_{21} polytope, in

[Cox30]. The faces come in two types: *n-simplices* (for $n \leq 7$), given by $n+1$ vertices with angle $\frac{\pi}{3}$ and distance $\sqrt{2}$ between any two of them, and *n-crosspolytopes* (for $n = 7$), given by $2n$ vertices where every vertex is orthogonal to exactly one other vertex, and has angle $\frac{\pi}{3}$ and distance $\sqrt{2}$ with all the other vertices. We summarize his results in Propositions 3.2.4 and 3.2.5.

Lemma 3.2.3. *Two vertices in the $\boldsymbol{E}_8$ root polytope have distance $\sqrt{2}$ between them if and only if their dot product is one.*

Proof. For $e, f \in E$ we have $\|e - f\|^2 = e^2 - 2 \cdot e \cdot f + f^2 = 4 - 2 \cdot e \cdot f$. □

Proposition 3.2.4. *For $n \leq 7$, the set of n-simplices in the $\boldsymbol{E}_8$ root polytope is given by*

$$\{\{e_1, \ldots, e_{n+1}\} \mid \forall i : e_i \in E;\ \forall j \neq i : e_i \cdot e_j = 1\},$$

where an n-simplex is identified with the set of its vertices. For $n \leq 6$, the n-simplices in the $\boldsymbol{E}_8$ root polytope are exactly its n-faces.

Proof. The vertices in an n-simplex have dot product 1 by the previous lemma. The fact that the n-faces are exactly the n-simplices for $n \leq 6$ is in [Cox30, Section 7.5], or the table on page 414. □

Proposition 3.2.5. *The facets of the $\boldsymbol{E}_8$ root polytope are exactly the 7-simplices and the 7-crosspolytopes contained in it. The set of 7-crosspolytopes is given by*

$$\left\{ \{\{e_1, f_1\}, \ldots, \{e_7, f_7\}\} \;\middle|\; \begin{array}{l} \forall i \in \{1, \ldots, 7\} : e_i, f_i \in E;\ e_i \cdot f_i = 0; \\ \forall j \neq i : e_i \cdot e_j = e_i \cdot f_j = f_i \cdot f_j = 1. \end{array} \right\},$$

where a 7-crosspolytope is identified by the set of its 7 pairs of orthogonal roots.

Proof. The facets are the 7-simplices and the 7-crosspolytopes by [Cox30, Section 7.5], or see the table on page 414. The dot products follow from Lemma 3.2.3. □

Remark 3.2.6. We also show that the 7-simplices and the 7-crosspolytopes in the $\mathbf{E}_8$ root polytope are facets in Remarks 3.3.7 and 3.3.19.

Corollary 3.2.7. *The $\mathbf{E}_8$ root polytope has 6720 1-faces, 60480 2-faces, 241920 3-faces, 483840 4-faces, 483840 5-faces, 207360 6-faces, 17280 7-simplices, and 2160 7-crosspolytopes.*

Proof. See [Cox30, p.414]. □

Remark – analogy with geometry 3.2.8. Let X be a del Pezzo surface of degree 1 over an algebraically closed field k, with canonical divisor K_X. Recall from Remark 1.4.9 that there is a bijection φ between the set I of exceptional curves on X and the set E, such that for $c_1, c_2 \in I$, we have $\varphi(c_1) \cdot \varphi(c_2) = 1 - c_1 \cdot c_2$ (where the dot on the left-hand side is the dot product in $\mathbf{E}_8$, and the dot on the right-hand side is the intersection pairing in Pic X). As a consequence, the group of permutations of I that preserve the intersection multiplicity in Pic X is isomorphic to the Weyl group W_8 (Corollary 1.4.10). Moreover, φ gives an isomorphism of the weighted graph G on I as defined in Definition 1.4.12 with Γ.
It follows that the vertices of a n-simplex in the $\mathbf{E}_8$ root polytope correspond to a sequence of $n+1$ exceptional classes in I that have pairwise intersection pairing 0. Moreover, for r pairwise disjoint exceptional curves $e_1, \dots, e_r$, where $1 \leq r \leq 7$, the exceptional curves that are disjoint from $e_1, \dots, e_r$ correspond to the exceptional curves of the del Pezzo surface of degree $r+1$ that is obtained by blowing down $e_1, \dots, e_r$. We know the number of exceptional curves on del Pezzo surfaces (1.1), and we can use this to compute the number of n-faces of the $\mathbf{E}_8$ root polytope for $n \leq 5$.

Remark 3.2.9. For $n \leq 5$, the statement in Corollary 3.2.7 also follows from the last part of Remark 3.2.8 and Table 1.1: we have

$$\frac{240 \cdot 56}{2} = 6720, \quad \frac{240 \cdot 56 \cdot 27}{3!} = 60480, \quad \frac{240 \cdot 56 \cdot 27 \cdot 16}{4!} = 241920,$$

and so on. For n equal to 6 and for the 7-simplices, the statement is in Proposition 3.4.7. For the 7-crosspolytopes it follows from Lemma 3.3.15, see Remark 3.3.16.

The following propositions state results about the action of the Weyl group on the faces of the $\mathbf{E}_8$ root polytope.

Proposition 3.2.10. *For $n \leq 5$, the group W acts transitively on the set of n-faces. There are two orbits of facets.*

Proof. In [Cox30, Section 7.5] it is shown that all n-simplices are conjugate for $n \leq 5$, and that any two facets of the same type are conjugate as well. We know that there are two types of facets from Proposition 3.2.5. $\square$

Remark 3.2.11. There are two orbits of 6-faces. We describe them in Proposition 3.4.7; see also Remark 3.4.10.

We know something even stronger, namely, the action of W on the *ordered sequences* of roots in faces of the $\mathbf{E}_8$ root polytope.

Proposition 3.2.12. *For all $r \leq 8$ such that $r \neq 7$, the group W acts transitively on the set*

$$R_r = \{(e_1, \dots, e_r) \in E^s \mid \forall i \neq j : e_i \cdot e_j = 1\}.$$

For $r = 7$, there are two orbits under the action of W.

Proof. In Remark 3.2.8 we describe a bijection between E and the set I of 240 exceptional classes on a del Pezzo surface of degree 1, where two elements in E have dot product a if and only if the two corresponding elements in I have intersection product $1 - a$. This bijection respects the action of W, and under this bijection the set R_r corresponds to the set of sequences of length r of disjoint exceptional classes. The statement now follows from [Man86, Corollary 26.8]. $\square$

The following lemma is the first step in proving Theorems 3.1.3 and 3.1.4.

Lemma 3.2.13. *Let K_1, K_2 be two cliques in Γ of type I, II, III, or IV that are isomorphic. Then there is a type I, II, III, or IV that they both belong to.*

Proof. If a clique is of type I or III, then any clique that is isomorphic to it is of the same type. If K_1 is of type II, then its vertices form a n-simplex (for $n \leq 7$) or an n-crosspolytope (for $n = 7$) by Proposition 3.2.4 and Proposition 3.2.5. In both cases, K_2 is of the same type, again by Proposition 3.2.4 and Proposition 3.2.5. Analogously, if K_2 is of type II then so is K_1. Finally, if K_1 and K_2 are both not of types I, II, or III, then they are automatically both of type IV. $\square$

We conclude this section by stating a lemma that will be used throughout this chapter. Parts (i)–(iii) are Lemma 20 in [Win14].

Lemma 3.2.14. *Let H be a group, let A, B be H-sets, and $f\colon A \longrightarrow B$ a morphism of H-sets. Then the following hold.*

(i) If H acts transitively on A, then H acts transitively on $f(A)$.

(ii) If H acts transitively on B, then all fibers of f have the same cardinality.

(iii) If H acts transitively on A and A is finite, then all non-empty fibers of f have the same cardinality, say n, and $|f(A)| = \frac{|A|}{n}$.

(iv) If H acts transitively on $f(A)$, and there is an element $b \in f(A)$ such that its stabilizer H_b in H acts transitively on $f^{-1}(b)$, then f acts transitively on A.

Proof.

(i) Take $f(a), f(a') \in f(A)$ with $a, a' \in A$. Assume that H acts transitively on A, then there is an $h \in H$ such that $ha = a'$. Since f is a morphism of H-sets, we have $hf(a) = f(ha) = f(a')$, so H acts transitively on $f(A)$.

(ii) Take $b, b' \in B$. Since H acts transitively on B, there is an $h \in H$ such that $hb = b'$, so $|f^{-1}(b')| = |f^{-1}(hb)| = |hf^{-1}(b)| = |f^{-1}(b)|$.

(iii) Take $b, b' \in B$ such that $f^{-1}(b)$ and $f^{-1}(b')$ are non-empty. Then we have $b, b' \in f(A)$. Since H acts transitively on $f(A)$ by (i), it follows from (ii) that $f^{-1}(b)$ and $f^{-1}(b')$ have the same cardinality, say n. It is now immediate that $|A| = |f^{-1}(B)| = \sum_{b \in f(A)} n = n|f(A)|$, so we find $|f(A)| = \frac{|A|}{n}$.

(iv) Take $b \in f(A)$ such that H_b acts transitively on $f^{-1}(b)$. Take $a, a' \in A$. Since H acts transitively on $f(A)$, there are $h, h' \in H$ such that $hf(a) = b$ and $h'f(a') = b$. Then ha and $h'a'$ are contained in $f^{-1}(b)$. Since H_b acts transitively on $f^{-1}(b)$, there is an element $g \in H_b$ with $gha = h'a'$. So we have $h'^{-1}gha = a'$ and H acts transitively on A. $\square$

3.3 Facets and cliques of size at most three

In this section we study the cliques in Γ of type III, and the facets of the $\mathbf{E}_8$ root polytope. We give an alternative proof for the fact that W acts transitively on the set of facets that are 7-crosspolytopes (Corollary 3.3.17), and we prove the following propositions.

Proposition 3.3.1. *For $a \in \{\pm 1, -2, 0\}$, the group W acts transitively on the set*

$$\{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = a\}.$$

Proposition 3.3.2. *For $a, b, c \in \{-2, -1, 0, 1\}$, the group W acts transitively on the set*

$$\{(e_1, e_2, e_3) \in E^3 \mid e_1 \cdot e_2 = a,\ e_2 \cdot e_3 = b,\ e_1 \cdot e_3 = c\},$$

in all cases where it is not empty.

Remark 3.3.3. Proposition 3.3.1, as well as the cases $\{a, b, c\} = \{0, 0, 0\}$, $\{a, b, c\} = \{0, 0, 1\}$ of Proposition 3.3.2, were proved by the same author in [Win14]. In particular, the results 3.3.11 – 3.3.14 are the same as [Win14, results 3.18, 3.19, 3.21, 3.22], and the results 3.3.24 – 3.3.28 and the first statement in Proposition 3.3.29 are the same as [Win14, results 3.23 – 3.28]. We decided to restate the results here for completeness, as well as for the fact that everything in [Win14] is stated in terms of exceptional curves on del Pezzo surfaces of degree 1, and this chapter is also meant for the reader that wants to use the results in terms of the roots of $\mathbf{E}_8$.

Note that these two propositions describe the orbits under the action of W of *sequences* of the vertices of cliques in Γ, hence they also prove Theorem 3.1.4 for cliques of type III; see Corollary 3.3.34. The proof of Proposition 3.3.1 can be found below Proposition 3.3.14, and the proof of Proposition 3.3.2 below Lemma 3.3.33. Throughout this section we do not use any computer programs. More background on the $\mathbf{E}_8$ root polytope can be found in [Cox30] and [Cox49].

We start with some results on the facets of the $\mathbf{E}_8$ root polytope that are 7-simplices. The results on the facets that are 7-crosspolytopes are in Lemmas 3.3.17 and 3.3.18. Consider the set

$$U = \{(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8) \in E^8 \mid \forall i \neq j : e_i \cdot e_j = 1\}.$$

Note that an element in U is a sequence of eight roots that form a 7-simplex. Define the following roots, and note that $(u_1, \ldots, u_8)$ is an element in U.

$$\begin{aligned} u_1 &= (1,1,0,0,0,0,0,0); & u_5 &= (1,0,0,0,0,1,0,0);\\ u_2 &= (1,0,1,0,0,0,0,0); & u_6 &= (1,0,0,0,0,0,1,0);\\ u_3 &= (1,0,0,1,0,0,0,0); & u_7 &= (1,0,0,0,0,0,0,1);\\ u_4 &= (1,0,0,0,1,0,0,0); & u_8 &= \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right). \end{aligned}$$

Lemma 3.3.4. *Every element in U generates a sublattice of index 3 of the root lattice Λ, and the group W acts freely on U.*

Proof. By Proposition 3.2.12, it is enough to check the first statement for one element in U. The matrix whose i-th row is u_i for $i \in \{1,\dots,8\}$ has determinant 3, so $u_1,\dots,u_8$ are linearly independent and generate a sublattice of rank 8 and index 3 in Λ. Take $w \in W$ such that there is an element $u \in U$ with $w(u) = u$. Then w fixes the sublattice generated by u, so for all $x \in \Lambda$ we have $3w(x) = w(3x) = 3x$. Since Λ is torsion free, this implies that w fixes all of Λ. It follows that w is the identity. We conclude that the action of W on U is free. $\square$

Corollary 3.3.5. *Let $u = (e_1,\dots,e_8)$ be an element in U. Then $\frac{1}{3}\sum_{i=1}^{8} e_i$ is contained in Λ.*

Proof. By Lemma 3.3.4, we know that the roots $e_1,\dots,e_8$ generate a lattice M of index 3 in Λ. Set $v = \frac{1}{3}\sum_{i=1}^{8} e_i$. Since $v \cdot e_i = 3$ for $i \in \{1,\dots,8\}$, we have $\frac{1}{3}v \in M^\vee$, where $M^\vee$ is the dual lattice of M. But the dual lattice $\Lambda^\vee$ has index 3 in $M^\vee$, so it follows that $3 \cdot \frac{1}{3}v = v$ is contained in $\Lambda^\vee$. Since Λ is unimodular, it is self dual, so v is contained in Λ. $\square$

Remark – analogy with geometry 3.3.6. Let X be a del Pezzo surface of degree 1 over an algebraically closed field and K_X its canonical divisor. Lemma 3.3.4 can be stated in terms of X as follows. For every set of eight exceptional classes $c_1,\dots,c_8$ that have pairwise intersection pairing 0 there exists a unique class l such that we have $K_X = -3l + \sum_{i=1}^{8} c_i$ and $(l, c_1,\dots,c_8)$ is a basis for Pic X; one can blow down the exceptional curves corresponding to $c_1,\dots,c_8$ to eight points in $\mathbb{P}^2$, such that l is the class of the pullback of a line in $\mathbb{P}^2$ that does not contain any of these eight points.

Remark 3.3.7. Let $u = (e_1,\dots,e_8)$ be an element in U. We know that $e_1,\dots,e_8$ define a facet of the $\mathbf{E}_8$ root polytope. This also follows from

from Corollary 3.3.5. Indeed, for $v = \frac{1}{3}\sum_{i=1}^{8} e_i$ we have $v \cdot e_i = 3$ for $i \in \{1, \ldots, 8\}$, and we have

$$v \cdot e = \frac{1}{3}\sum_{i=1}^{8} e_i \cdot e \leq \frac{1}{3}\sum_{i=1}^{8} 1 = \frac{8}{3} < 3$$

for $e \in E \setminus \{e_1, \ldots, e_8\}$. This implies that the whole $\mathbf{E}_8$ root polytope lies on one side of the hyperplane given by $v \cdot x = 3$, and the intersection of the polytope with this hyperplane, which is exactly given by the convex combinations of $e_1, \ldots, e_8$, lies in the boundary of the polytope. Hence $e_1, \ldots, e_8$ generate a facet of the $\mathbf{E}_8$ root polytope, and v is the normal vector to this facet.

We now prove part of Proposition 3.3.1.

Lemma 3.3.8. *For any $a \in \{-2, \pm 1\}$, the group W acts transitively on the set*

$$A_a = \{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = a\}.$$

Proof. The group W acts transitively on A_1 by Proposition 3.2.12. There is a bijection between the W-sets A_1 and A_{-1} given by

$$f\colon A_1 \longrightarrow A_{-1},\ (e_1, e_2) \longmapsto (e_1, -e_2).$$

It follows from Lemma 3.2.14 that W acts transitively on A_{-1}, too. Finally, we have a bijection

$$E \longrightarrow A_{-2},\ e \longmapsto (e, -e),$$

so W acts transitively on A_{-2} by Proposition 3.2.12 and by Lemma 3.2.14. □

Before we prove the rest of Proposition 3.3.1, we prove Proposition 3.3.2 for the cases $(a, b, c) = (-1, -1, -1)$ (Corollary 3.3.10) and $(a, b, c) = (0, 0, 1)$ (Lemma 3.3.12), which we will use later.

Lemma 3.3.9. *For $e_1, e_2 \in E$ with $e_1 \cdot e_2 = -1$ there is a unique element $e \in E$ with $e \cdot e_1 = e \cdot e_2 = -1$, which is given by $e = -e_1 - e_2$.*

Proof. Take $e_1, e_2, e \in E$ with $e_1 \cdot e_2 = -1$ and $e \cdot e_1 = e \cdot e_2 = -1$. Set $f = e_1 + e_2 + e$. Then we have $\|f\| = 0$, hence $f = 0$, so $e = -e_1 - e_2$. Therefore e is unique if it exists. Moreover, we have $\| - e_1 - e_2\| = \sqrt{2}$, so $-e_1 - e_2$ is an element in E that satisfies the conditions. □

Corollary 3.3.10. *The group W acts transitively on the W-set*

$$\{(e_1, e_2, e_3) \in E^3 \mid e_1 \cdot e_2 = e_2 \cdot e_3 = e_1 \cdot e_3 = -1\}.$$

Proof. By Lemma 3.3.9 there is a bijection between the sets

$$\{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = -1\}$$

and

$$\{(e_1, e_2, e_3) \in E^3 \mid e_1 \cdot e_2 = e_2 \cdot e_3 = e_1 \cdot e_3 = -1\},$$

given by $(e_1, e_2) \longmapsto (e_1, e_2, -e_1 - e_2)$. The statement now follows from Lemma 3.3.8 and Lemma 3.2.14. □

Lemma 3.3.11. *Take $e_1,\ e_2 \in E$ such that $e_1 \cdot e_2 = 1$. Then there are exactly 72 elements of E orthogonal to e_1 and e_2.*

Proof. By Lemma 3.3.8 it is enough to check this for fixed $e_1, e_2 \in E$ with $e_1 \cdot e_2 = 1$. Set $e_1 = (1, 1, 0, 0, 0, 0, 0, 0)$, $e_2 = (1, 0, 1, 0, 0, 0, 0, 0)$. Then $e_1 \cdot e_2 = 1$. An element $f \in E$ with $f \cdot e_1 = f \cdot e_2 = 0$ is of the form $f = (a_1, \ldots, a_8)$ with $a_1 + a_2 = 0$ and $a_1 + a_3 = 0$, hence $a_1 = -a_2$ and $a_2 = a_3$. If f is of the form $\left(\pm\frac{1}{2}, \ldots, \pm\frac{1}{2}\right)$, then there are 32 such possibilities. If f has two non-zero entries, given by ± 1, then a_1, a_2, a_3 should all be zero, which gives 40 possibilities. We find a total of 72 possibilities for f. □

Lemma 3.3.12. *Consider the set*

$$B = \{(e_1, e_2, e_3) \in E^3 \mid e_1 \cdot e_2 = e_2 \cdot e_3 = 0;\ e_1 \cdot e_3 = 1\}.$$

We have $|B| = 967680$, and the following hold.

(i) The group W acts transitively on B.

(ii) For every element $b = (e_1, e_2, e_3) \in B$, there are exactly 6 roots that have dot product 1 with e_1, e_2 and e_3. These 6 roots, together with e_1 and e_3, form a facet in the set U.

Proof. We have $|B| = 240 \cdot 56 \cdot 72 = 967680$ by Proposition 3.2.2 and Lemma 3.3.11. Set $e_1 = (1, 1, 0, 0, 0, 0, 0, 0)$, $e_2 = (0, 0, 1, 1, 0, 0, 0, 0)$, and $e_3 = (1, 0, 0, 0, 1, 0, 0, 0)$. Then $b = (e_1, e_2, e_3)$ is an element in B. Let W_b be its stabilizer in W and Wb its orbit in B. Let U_b be the set

$$U_b = \{e \in E \mid e \cdot e_1 = e \cdot e_2 = e \cdot e_3 = 1\}.$$

For $e = (a_1, \dots, a_8) \in U_b$, we have $a_1 + a_2 = a_3 + a_4 = a_1 + a_5 = 1$. From this we find

$$U_b = \left\{ \begin{array}{c} (1,0,0,1,0,0,0,0) \\ (1,0,1,0,0,0,0,0) \\ \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\frac{1}{2}\right) \\ \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right) \\ \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\frac{1}{2},-\frac{1}{2}\right) \\ \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right) \end{array} \right\}.$$

We conclude that there are 6 roots that have dot product 1 with e_1, e_2, and e_3. It is obvious that these 6 elements, together with e_1 and e_3, form an element of the set U that is defined above Lemma 3.3.4.
We have $\frac{|W|}{|W_b|} = |Wb| \le |B|$. We want to show that the latter is an equality. The group W_b acts on U_b. Let w be an element of W_b that fixes all the roots in U_b. Since the roots in $\{e_1, e_3\} \cup U_b$ form an element in U, by Lemma 3.3.4 this implies that w is the identity. Therefore the action of W_b on U_b is faithful. This implies that W_b injects into S_6, so $|W_b| \le 720$. We now have

$$967680 = \frac{|W|}{720} \le \frac{|W|}{|W_b|} = |Wb| \le |B| = 967680,$$

so we have equality everywhere and therefore we have $Wb = B$. We conclude that W acts transitively on B, proving (i). Part (ii) clearly holds for the element b, and from part (i) it follows that it holds for all elements in B. ☐

We proceed to prove the rest of Proposition 3.3.1.

Lemma 3.3.13. *For $e_1 = (1,1,0,0,0,0,0,0)$, $e_2 = (0,0,1,1,0,0,0,0) \in E$, there are 32 elements e in E such that $e \cdot e_1 = 0$ and $e \cdot e_2 = 1$.*

Proof. Take $e \in E$ with $e \cdot e_1 = 0$ and $e \cdot e_2 = 1$. Then e is of the form $e = (a_1, a_2, a_3, a_4, \dots, a_8)$ with $a_1 + a_2 = 0$ and $a_3 + a_4 = 1$. If e is of the form $\left(\pm\frac{1}{2}, \dots, \pm\frac{1}{2}\right)$, then $a_1 = -a_2$ and $a_3 = a_4 = \frac{1}{2}$. There are 16 such possibilities. If e has two non-zero entries given by ± 1, then either $a_3 = 1,\ a_1 = a_2 = a_4 = 0$, or $a_4 = 1,\ a_1 = a_2 = a_3 = 0$. This gives 16 possibilities. We find a total of 32 possibilities for e. ☐

Proposition 3.3.14. *The group W acts transitively on the set*

$$A_0 = \{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = 0\}.$$

Proof. Consider the set

$$B' = \{(e_1, e_2, e_3) \in E^3 \mid e_1 \cdot e_2 = e_1 \cdot e_3 = 0;\ e_2 \cdot e_3 = 1\}.$$

Note that there is a bijection between the W-set B' and the W-set B in Lemma 3.3.12, given by $(e, f, g) \longmapsto (f, e, g)$. Therefore, the group W acts transitively on B' and we have $|B'| = 967680$ by Lemma 3.3.12. We have a projection $\lambda\colon B' \longrightarrow A_0$ to the first two coordinates. We show that λ is surjective. Fix the two roots $e_1 = (1, 1, 0, 0, 0, 0, 0, 0)$ and $e_2 = (0, 0, 1, 1, 0, 0, 0, 0)$ in E. Then (e_1, e_2) is an element of A_0. Take $e \in E$, then (e_1, e_2, e) is in B' if and only if $e \cdot e_1 = 0$ and $e \cdot e_2 = 1$. By Lemma 3.3.13 this gives 32 possibilities for e, so $|\lambda^{-1}((e_1, e_2))| = 32$. Since W acts transitively on B', it follows from Lemma 3.2.14 that all non-empty fibers of λ have cardinality 32, and $|\lambda(B')| = \frac{|B'|}{32} = 30240$. By Proposition 3.2.2 we have $|A_0| = 240 \cdot 126 = 30240$. We conclude that $\lambda(B') = A_0$. Hence λ is surjective. Therefore, the group W acts transitively on A_0 by Lemma 3.2.14. $\square$

Proof of Proposition 3.3.1. This follows from the previous proposition together with Lemma 3.3.8.

Before we continue proving Proposition 3.3.2, we complete our study of the facets of the $\mathbf{E}_8$ root polytope. Define the set

$$C = \left\{ \{\{e_1, f_1\}, \ldots, \{e_7, f_7\}\} \;\middle|\; \begin{array}{l} \forall i \in \{1, \ldots, 7\} : e_i, f_i \in E;\ e_i \cdot f_i = 0 \\ \forall j \neq i : e_i \cdot e_j = e_i \cdot f_j = f_i \cdot f_j = 1 \end{array} \right\}.$$

Elements in C are facets that are 7-crosspolytopes by Proposition 3.2.4. We define elements $c_1, \ldots, c_7, d_1, \ldots, d_7$; note that $\{\{c_1, d_1\}, \ldots, \{c_7, d_7\}\}$ is an element in C.

$$
\begin{aligned}
c_1 &= (1,1,0,0,0,0,0,0)\,, & d_1 &= (0,0,1,1,0,0,0,0),\\
c_2 &= (1,0,1,0,0,0,0,0)\,, & d_2 &= (0,1,0,1,0,0,0,0),\\
c_3 &= (1,0,0,1,0,0,0,0), & d_3 &= (0,1,1,0,0,0,0,0),\\
c_4 &= \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right), & d_4 &= \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}\right),\\
c_5 &= \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right), & d_5 &= \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}\right),\\
c_6 &= \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2}\right), & d_6 &= \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}\right),\\
c_7 &= \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}\right), & d_7 &= \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2}\right).
\end{aligned}
$$

Lemma 3.3.15. *For $e_1, e_2 \in E$ with $e_1 \cdot e_2 = 0$, there are exactly 12 elements $e \in E$ with $e \cdot e_1 = e \cdot e_2 = 1$. These 12 elements, together with e_1 and e_2, form an element in C, and this is the unique element in C containing e_1, e_2.*

Proof. By Proposition 3.3.14, it is enough to check this for fixed e_1, e_2 with $e_1 \cdot e_2 = 0$. Take $e_1 = c_1$, and $e_2 = d_1$. For a root $e = (a_1, \dots, a_8)$ in E with $e \cdot c_1 = e \cdot d_1 = 1$, we have either $a_1 = a_2 = a_3 = a_4 = \frac{1}{2}$, in which case e is contained in $\{c_4, \dots, c_7, d_4, \dots, d_7\}$, or $\{a_1, a_2\} = \{a_3, a_4\} = \{0, 1\}$, which implies $e \in \{c_2, c_3, d_2, d_3\}$. Therefore there are 12 possibilities $\{c_2, \dots, c_7, d_2, \dots, d_7\}$ for e, and we conclude that $\{\{c_1, d_1\}, \dots, \{c_7, d_7\}\}$ is the unique element in C containing c_1, d_1. $\square$

Remark 3.3.16. Since elements in C correspond to 7-crosspolytopes, we know that $|C| = 2160$ from Corollary 3.2.7. This also follows from the previous lemma. Recall the set $A_0 = \{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = 0\}$. By Lemma 3.3.15, for every element (e_1, e_2) in A_0 there is a unique element in C containing e_1, e_2. But every element in C contains seven pairs f_1, f_2 such that (f_1, f_2) and (f_2, f_1) are in A_0, so the map $A_0 \longrightarrow C$ is fourteen to one. Hence we have $|C| = \frac{|A_0|}{14} = \frac{240 \cdot 126}{14} = 2160$.

Corollary 3.3.17. *The group W acts transitively on C.*

Proof. Consider the set $A_0 = \{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = 0\}$. The group W acts transitively on A_0 by Proposition 3.3.14. By Lemma 3.3.15 there is a map $A_0 \longrightarrow C$, sending (e_1, e_2) to the unique element in C that contains

e_1 and e_2. This map is clearly surjective. It follows from Lemma 3.2.14 that W acts transitively on C. $\square$

Lemma 3.3.18. *Every element in C generates a sublattice of finite index in Λ.*

Proof. By Corollary 3.3.17, it is enough to check this for one element in C. Take the element $\{\{c_1, d_1\}, \ldots, \{c_7, d_7\}\}$ in C, where the c_i, d_i are defined above Lemma 3.3.15. The matrix whose rows are the vectors $c_1, \ldots, c_7, d_1, \ldots, d_7$ has rank 8, so these 14 elements generate a sublattice L of finite index in Λ. $\square$

Remark 3.3.19. Let $\{\{e_1, f_1\}, \ldots, \{e_7, f_7\}\}$ be an element in C, and let c be the set $c = \{e_1, \ldots, e_7, f_1, \ldots, f_7\}$. We know that the elements in c are the vertices of a facet of the $\mathbf{E}_8$ root polytope. We show how this also follows from the previous lemma. Take $i \in \{1, \ldots, 7\}$, then we have $(e_i + f_i) \cdot e = 2$ for all $e \in c$. Since the elements in c generate a full rank sublattice, this implies that $e_i + f_i = e_j + f_j$ for all $i, j \in \{1, \ldots, 7\}$. So the vector $n = \frac{1}{7} \sum_{i=1}^{7} (e_i + f_i) = e_1 + f_1$ is an element in Λ with $n \cdot e = 2$ for $e \in s$. Take $e \in E \setminus s$, and note that e cannot have dot product 1 with both e_1 and f_1 by Lemma 3.3.15. It follows that we have $n \cdot e < 2$, so the entire $\mathbf{E}_8$ root polytope lies on one side of the affine hyperplane given by $n \cdot x = 2$. Moreover, this hyperplane intersects the $\mathbf{E}_8$ root polytope in its boundary, and exactly in the convex combinations of the roots $e_1, \ldots, e_7, f_1, \ldots, f_7$. Therefore these roots are the vertices of a facet of the $\mathbf{E}_8$ root polytope with normal vector n.

We continue with Proposition 3.3.2, and prove it for $(a, b, c) = (0, 0, 0)$. Consider the sets

$$V_3 = \{(e_1, e_2, e_3) \in E^3 \mid \forall i \neq j : e_i \cdot e_j = 0\}$$

and

$$V_4 = \{(e_1, e_2, e_3, e_4) \in E^4 \mid \forall i \neq j : e_i \cdot e_j = 0\}.$$

We begin by studying V_4. To this end, recall the set U defined above Lemma 3.3.4, and define the set

$$Z = \{(\{e_1, e_2\}, \{e_3, e_4\}, \{e_5, e_6\}, \{e_7, e_8\}) \mid \forall i : e_i \in E;\ \forall j \neq i : e_i \cdot e_j = 1\}.$$

Remark 3.3.20. We have a surjective map $U \longrightarrow Z$ by simply forgetting the order of e_i and e_{i+1} for $i \in \{1, 3, 5, 7\}$. Since W acts transitively on U

(Proposition 3.2.12), it follows from Lemma 3.2.14 that W acts transitively on Z. By Lemma 3.3.4, the action of W on U is free, so we have $|U| = |W|$, and $|Z| = \frac{|U|}{2^4} = \frac{|W|}{2^4} = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7$.

We want to define a map $\alpha\colon Z \longrightarrow V_4$. To do this we use the following lemma.

Lemma 3.3.21. *For an element* $z = (\{e_1, e_2\}, \{e_3, e_4\}, \{e_5, e_6\}, \{e_7, e_8\})$ *in* Z*, there are unique roots* $f_1, f_2, f_3, f_4 \in E$ *with*

$$\begin{aligned}
f_1 \cdot e_i &= 0,\ \ f_1 \cdot e_j = 1 \ \ \text{for } i \in \{1,2\},\ j \notin \{1,2\};\\
f_2 \cdot e_i &= 0,\ \ f_2 \cdot e_j = 1 \ \ \text{for } i \in \{3,4\},\ j \notin \{3,4\};\\
f_3 \cdot e_i &= 0,\ \ f_3 \cdot e_j = 1 \ \ \text{for } i \in \{5,6\},\ j \notin \{5,6\};\\
f_4 \cdot e_i &= 0,\ \ f_4 \cdot e_j = 1 \ \ \text{for } i \in \{7,8\},\ j \notin \{7,8\}.
\end{aligned}$$

For these f_1, f_2, f_3, f_4 *we have* $f_i \cdot f_j = 0$ *for* $i \neq j$*, and* $3\sum_{i=1}^4 f_i = \sum_{i=1}^8 e_i$*.*

Proof. By Lemma 3.3.4, the elements $e_1, \dots, e_8$ generate a full rank sublattice of Λ, so an element $f \in E$ is uniquely determined by the intersection numbers $f \cdot e_i$ for $i \in \{1, \dots, 8\}$. We will show existence. Set $v = \frac{1}{3}\sum_{i=1}^8 e_i$. By Corollary 3.3.5, the vector v is an element in Λ. We have $\|v\| = \sqrt{8}$, and $v \cdot e_i = 3$ for $i \in \{1, \dots, 8\}$. For $i \in \{1,2,3,4\}$, set $f_i = v - e_{2i-1} - e_{2i}$. Then $\|f_i\| = \sqrt{2}$, so $f_i \in E$. Moreover, f_1, f_2, f_3, f_4 satisfy the conditions in the lemma. □

We now define a map

$$\alpha\colon Z \longrightarrow V_4, \quad (\{e_1, e_2\}, \dots, \{e_7, e_8\}) \longmapsto (f_1, f_2, f_3, f_4),$$

where f_1, f_2, f_3, f_4 are the unique elements found in Lemma 3.3.21.

Corollary 3.3.22. *If* (f_1, f_2, f_3, f_4) *is an element in the image of* α*, then* $x = \sum_{i=1}^4 f_i$ *is a primitive element of* Λ *with norm* $\sqrt{8}$*.*

Proof. Take (f_1, f_2, f_3, f_4) in the image of α, and let $(\{e_1, e_2\}, \dots, \{e_7, e_8\})$ be an element Z such that $(f_1, f_2, f_3, f_4) = \alpha((\{e_1, e_2\}, \dots, \{e_7, e_8\}))$. Set $x = \sum_{i=1}^4 f_i$. Then we have $3x = \sum_{i=1}^8 e_i$ by Lemma 3.3.21. It follows that $\|3x\|^2 = 72$, hence $\|x\|^2 = 8$. Moreover, for any $i \in \{1, \dots, 8\}$ we have $3x \cdot e_i = 9$, hence $x \cdot e_i = 3$. This implies that if we have $x = m \cdot x'$ for some $m \in \mathbb{Z}$, $x' \in \Lambda$, then $m|2$ and $m|3$, so $m = 1$ and x is primitive. □

Remark – analogy with geometry 3.3.23. Let X be a del Pezzo surface of degree 1 over an algebraically closed field, and I the set of exceptional classes in Pic X. The map α has a nice description in the geometric setting, through the bijection $I \longrightarrow E$, $c \longmapsto c + K_X$. Take an element $z = (\{e_1, e_2\}, \{e_3, e_4\}, \{e_5, e_6\}, \{e_7, e_8\})$ in Z. The roots $e_1, \ldots, e_8$ correspond to classes $c_1, \ldots, c_8$ in I with $c_i \cdot c_j = 0$ for all $i \neq j \in \{1, \ldots, 8\}$. These classes correspond to pairwise disjoint curves on X that can be blown down to points $P_1, \ldots, P_8$ in $\mathbb{P}^2$ such that c_i is the class of the exceptional curve above P_i for $i \in \{1, \ldots, 8\}$ (Lemma 1.2.8). The conditions for f_i in Lemma 3.3.21 are equivalent to f_i being the strict transform on X of the line in $\mathbb{P}^2$ through P_{2i-1} and P_{2i} for $i \in \{1, 2, 3, 4\}$. This geometrical argument immediately proves the uniqueness of f_i.

Let $\pi \colon V_4 \longrightarrow V_3$ be the projection to the first three coordinates. From the maps π and α, transitivity on V_3 will follow (Proposition 3.3.28). Let Y be the image of α. We will show that V_4 has two orbits under the action of W, given by Y and $V_4 \setminus Y$ (Proposition 3.3.29). The following commutative diagram shows the maps and sets that are defined.

$$\begin{tikzcd}
U \arrow[d, two heads] & & \\
Z \arrow[r, "\alpha"] \arrow[dr, two heads] & V_4 \arrow[r, "\pi"] & V_3 \\
& Y \arrow[u, hook] \arrow[ur] &
\end{tikzcd}$$

Lemma 3.3.24. *The map α is injective.*

Proof. Consider the roots in E given by

$$\begin{aligned} f_1 &= (1,1,0,0,0,0,0,0)\,, & f_3 &= (0,0,0,0,1,1,0,0)\,, \\ f_2 &= (0,0,1,1,0,0,0,0)\,, & f_4 &= (1,-1,0,0,0,0,0,0)\,. \end{aligned}$$

Note that $v = (f_1, f_2, f_3, f_4)$ is an element in V_4. Now take an element $(\{e_1, e_2\}, \{e_3, e_4\}, \{e_5, e_6\}, \{e_7, e_8\})$ in the fiber of α above v, then we have

$$\begin{aligned} &e_1 \cdot f_1 = e_2 \cdot f_1 = 0 \text{ and } e_1 \cdot f_i = e_2 \cdot f_i = 1 \text{ for all } i \neq 1; \\ &e_3 \cdot f_2 = e_4 \cdot f_2 = 0 \text{ and } e_3 \cdot f_i = e_4 \cdot f_i = 1 \text{ for all } i \neq 2; \\ &e_5 \cdot f_3 = e_6 \cdot f_3 = 0 \text{ and } e_5 \cdot f_i = e_6 \cdot f_i = 1 \text{ for all } i \neq 3; \\ &e_7 \cdot f_4 = e_8 \cdot f_4 = 0 \text{ and } e_7 \cdot f_i = e_8 \cdot f_i = 1 \text{ for all } i \neq 4. \end{aligned} \tag{3.1}$$

Write $e_1 = (a_1, \ldots, a_8)$. From (3.1) it follows that we have $a_1 + a_2 = 0$ and $a_1 - a_2 = a_3 + a_4 = a_5 + a_6 = 1$. So e_1 is $\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$ or $\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$, and e_2 is the other. Analogously we find:

$$\begin{aligned}
\{e_3, e_4\} &= \{(1,0,0,0,0,1,0,0), (1,0,0,0,1,0,0,0)\}, \\
\{e_5, e_6\} &= \{(1,0,0,1,0,0,0,0), (1,0,1,0,0,0,0,0)\}, \\
\{e_7, e_8\} &= \left\{\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}\right), \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right)\right\}.
\end{aligned}$$

Hence the fiber above v has cardinality one. Since W acts transitively on Z, we conclude from Lemma 3.2.14 that all non-empty fibers of α have cardinality one, so α is injective. □

Remark 3.3.25. By the previous lemma, there is a bijection between the sets Z and $\alpha(Z) = Y$. Since α is a W-map, it follows that Y is a W-set, and that W acts transitively on Y by Lemma 3.2.14.

We state two more lemmas before we prove that W acts transitively on V_3.

Lemma 3.3.26. *Consider the elements in E given by*

$$\begin{aligned}
e_1 &= (1,1,0,0,0,0,0,0); & f_1 &= (0,0,0,0,0,0,1,1) \\
e_2 &= (0,0,1,1,0,0,0,0); & f_2 &= (0,0,0,0,0,0,-1,-1). \\
e_3 &= (0,0,0,0,1,1,0,0); & &
\end{aligned}$$

Then $v = (e_1, e_2, e_3, f_1)$ and $v' = (e_1, e_2, e_3, f_2)$ are elements in V_4 that are not in Y.

Proof. It is easy to check that v and v' are in V_4. We have

$$e_1 + e_2 + e_3 + f_1 = 2 \cdot \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right)$$

and

$$e_1 + e_2 + e_3 + f_2 = 2 \cdot \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}\right),$$

hence both $e_1 + e_2 + e_3 + f_1$ and $e_1 + e_2 + e_3 + f_2$ are not primitive elements in Λ and therefore not contained in Y by Corollary 3.3.22. □

Lemma 3.3.27. *For two elements $e_1, e_2 \in E$ with $e_1 \cdot e_2 = 0$, there are exactly 60 roots $e \in E$ such that $e_1 \cdot e = e_2 \cdot e = 0$.*

Proof. By Proposition 3.3.14, it is enough to check this for two orthogonal roots e_1, e_2 in E. Set $e_1 = (1,1,0,0,0,0,0,0)$, $e_2 = (0,0,1,1,0,0,0,0)$. An element $f \in E$ with $f \cdot e_1 = f \cdot e_2 = 0$ is of the form $f = (a_1, a_2, a_3, \ldots, a_8)$ with $a_1 = -a_2$ and $a_3 = -a_4$. If f is of the form $\left(\pm\frac{1}{2}, \ldots, \pm\frac{1}{2}\right)$, then there are 32 such possibilities. If f has two non-zero entries, given by ± 1, then there are 28 possibilities. We find a total of 60 possibilities for f. $\square$

Figure 3.1 summarizes the results in Proposition 3.2.2 and Lemmas 3.3.9, 3.3.13 and 3.3.27. Vertices are roots, and the number in a subset is its cardinality. The number on an edge between two subsets is the dot product of two roots, one from each subset.

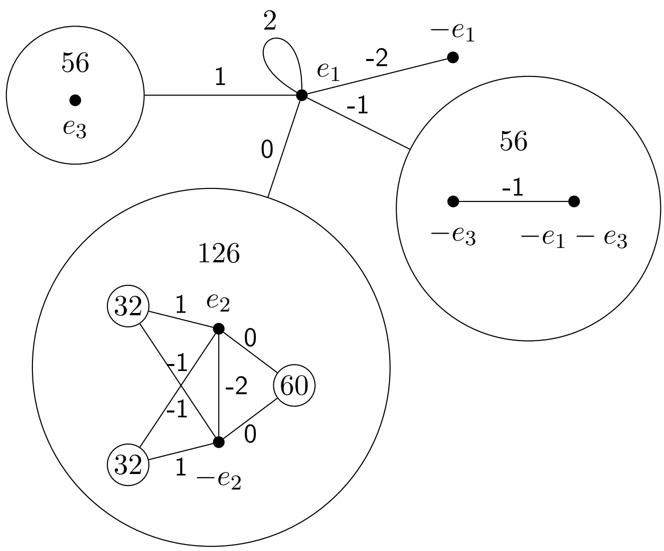


Figure 3.1: Graph Γ

Proposition 3.3.28. *Let $v = (f_1, f_2, f_3)$ be an element of V_3. The following hold.*

(i) We have $|V_3| = 1814400$, and the group W acts transitively on V_3.

(ii) We have $|\pi^{-1}(v)| = 26$, and $|\pi^{-1}(v) \cap Y| = 24$.

(iii) For $\{(f_1, f_2, f_3, u), (f_1, f_2, f_3, u')\} = \pi^{-1}(v) \setminus Y$, we have $u = -u'$, and for $(f_1, f_2, f_3, e) \in \pi^{-1}(v) \cap Y$, we have $e \cdot u = e \cdot u' = 0$.

Proof. From Proposition 3.2.2 and Lemma 3.3.27 it follows that

$$|V_3| = 240 \cdot 126 \cdot 60 = 1814400.$$

Consider the map $\lambda = \pi \circ \alpha \colon Z \to V_3$. Note that λ is a W-map, since both π and α are. We want to show that λ is surjective. Set

$$f_1 = (1,1,0,0,0,0,0,0), f_2 = (0,0,1,1,0,0,0,0), f_3 = (0,0,0,0,1,1,0,0).$$

Then we have $v = (f_1, f_2, f_3) \in V_3$. Define the roots

$$\begin{aligned}
e_1 &= \left(\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}\right), & e_5 &= (1,0,0,1,0,0,0,0)\,, \\
e_2 &= \left(\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}\right), & e_6 &= (1,0,1,0,0,0,0,0)\,, \\
e_3 &= (1,0,0,0,0,1,0,0)\,, & e_7 &= \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}\right) \\
e_4 &= (1,0,0,0,1,0,0,0)\,, & e_8 &= \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right).
\end{aligned}$$

Note that $e_i \cdot e_j = 1$ for $i \neq j$, so $(\{e_1, e_2\}, \{e_3, e_4\}, \{e_5, e_6\}, \{e_7, e_8\})$ is an element in Z. We have

$$\begin{aligned}
f_1 \cdot e_1 = f_1 \cdot e_2 = 0 &\text{ and } f_1 \cdot e_i = 1 \text{ for all } i \notin \{1,2\}; \\
f_2 \cdot e_3 = f_2 \cdot e_4 = 0 &\text{ and } f_2 \cdot e_i = 1 \text{ for all } i \notin \{3,4\}; \\
f_3 \cdot e_5 = f_3 \cdot e_6 = 0 &\text{ and } f_3 \cdot e_i = 1 \text{ for all } i \notin \{5,6\},
\end{aligned}$$

so $\lambda\left((\{e_1, e_2\}, \{e_3, e_4\}, \{e_5, e_6\}, \{e_7, e_8\})\right) = v$. Hence the fiber of λ above v is not empty; we compute its cardinality. We first compute the cardinality of the fiber of π above v. For an element $f = (a_1, \ldots, a_8) \in E$, we have $(f_1, f_2, f_3, f) \in V_4$ if and only if $a_1 + a_2 = a_3 + a_4 = a_5 + a_6 = 0$. This gives 16 possibilities for f with $a_i \in \left\{\pm\frac{1}{2}\right\}$ for $i \in \{1, \ldots, 8\}$, and 10 possibilities for f where the two non-zero entries are ± 1. We conclude $|\pi^{-1}(v)| = 26$. Set $g_1 = (0,0,0,0,0,0,1,1)$, $g_2 = (0,0,0,0,0,0,-1,-1)$, then $u = (f_1, f_2, f_3, g_1)$ and $u' = (f_1, f_2, f_3, g_2)$ are both elements in $\pi^{-1}(v)$. By Lemma 3.3.26, we know that the fibers of α above u and u' are empty. Since α is injective, this implies $|\lambda^{-1}(v)| \leq 24$. Since $\lambda^{-1}(v)$ is not empty, by Lemma 3.2.14, we have $|\lambda(Z)| = \frac{|Z|}{|\lambda^{-1}(v)|}$. Combining this, we find

$$\frac{|Z|}{24} \leq \frac{|Z|}{|\lambda^{-1}(v)|} = |\lambda(Z)| \leq |V_3| = 1814400 = \frac{|Z|}{24}.$$

So we have equality everywhere, hence $|\lambda^{-1}(v)| = 24$, and $|\lambda(Z)| = |V_3|$, so λ is surjective. Since W acts transitively on Z, we conclude from

Lemma 3.2.14 that W acts transitively on V_3, too. This proves (i). To prove (ii), note that we showed that $|\pi^{-1}(v)| = 26$ and $|\lambda^{-1}(v)| = 24$, and since α is injective, we have the equality $|\pi^{-1}(v) \cap Y| = |\lambda^{-1}(v)| = 24$. Since π is a W-map, and W acts transitively on V_3, the result holds for all elements in V_3. Finally, (iii) is an easy check for the element v, after writing down the 26 elements in $\pi^{-1}(v)$. Since W acts transitively on V_3, this holds for all elements in V_3. □

Proposition 3.3.29. *The set V_4 has two orbits under the action of W, which are Y and $V_4 \setminus Y$. We have $|Y| = 43545600$ and $|V_4 \setminus Y| = 3628800$. An element $(e_1, \ldots, e_4)$ is in $V_4 \setminus Y$ if and only if $\sum_{i=1}^{4} e_i \in 2\Lambda$.*

Proof. From Remark 3.3.25 it follows that Y is an orbit under the action of W on V_4. Therefore $O = V_4 \setminus Y$ is also a W-set. Consider the restriction $\pi|_O$ of π to O. Let e_1, e_2, e_3, f_1, f_2 be as in Lemma 3.3.26, and set $v = (e_1, e_2, e_3)$, $u = (e_1, e_2, e_3, f_1)$, and $u' = (e_1, e_2, e_3, f_2)$. Then we have $v \in V_3$, and $u, u' \in \pi|_O^{-1}(v)$ by Lemma 3.3.26. From Proposition 3.3.28 we know that $|\pi^{-1}(v) \cap Y| = 24$, so $\left|\pi|_O^{-1}(v)\right| = 2$. This implies $\pi|_O^{-1}(v) = \{u, u'\}$. Consider the element r in W given by the reflection in the hyperplane that is orthogonal to f_1. Since e_1, e_1, e_3 are contained in this hyperplane, the reflection r is contained in the stabilizer W_v in W of v. Moreover, since $f_2 = -f_1$, the reflection r interchanges f_1 and f_2, hence W_v acts transitively on $\pi|_O^{-1}(v)$. Since W acts transitively on V_3 by Proposition 3.3.28, we conclude that W acts transitively on O from Lemma 3.2.14. From Proposition 3.3.28 it follows that $|Y| = |V_3| \cdot 24 = 43545600$, and $|O| = |V_3| \cdot 2 = 3628800$. It follows from Corollary 3.3.22 that for every element $(g_1, g_2, g_3, g_4) \in Y$ the sum $\sum_{i=1}^{4} g_i$ is primitive. On the other hand, $u = (e_1, e_2, e_3, f_1)$ is an element in O, and we have $e_1 + e_2 + e_3 + f_1 = 2 \cdot \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \in 2\Lambda$. Since W acts transitively on O, this finishes the proof. □

Now that we proved that W acts transitively on V_3, there is one last case of Proposition 3.3.2 that we prove separately (Lemma 3.3.33). We state two auxiliary lemmas first.

Lemma 3.3.30. *Let r be a positive integer, and let G be a graph with vertex set $\{v_1, \ldots, v_r, w_1 \ldots, w_r\}$, and edge set $\{\{v_i, w_i\} \mid i \in \{1, \ldots, r\}\}$. Let A be the automorphism group of G. For an element $a \in A$ and for $i \in \{1, \ldots, r\}$, define an integer a_i by $a_i = 1$ if $a(v_i) \in \{v_1, \ldots, v_r\}$, and $a_i = -1$ otherwise. There exists an isomorphism $\varphi\colon A \xrightarrow{\sim} \mu_2^r \rtimes S_r$, where*

μ_2 is the multiplicative group with two elements and S_r the symmetric group on r elements, acting on μ_2^r by permuting the coordinates, given by

$$\varphi(a) = ((a_1, \ldots, a_r), (i \mapsto j \text{ for } a(v_i) \in \{v_j, w_j\})).$$

Proof. Let a be an element in A. Note that for all i, the image $a(v_i)$ of v_i is only connected to $a(w_i)$, so there is a j such that $\{a(v_i), a(w_i)\} = \{v_j, w_j\}$. Therefore we have a group homomorphism $\gamma\colon A \longrightarrow S_r$, given by

$$a \longmapsto (i \mapsto j \text{ for } a(v_i) \in \{v_j, w_j\}).$$

Note that γ is surjective, and its kernel consists of all elements $a \in A$ such that, for all $i \in \{1, \ldots, r\}$, either $a(v_i) = v_i$, or $a(v_i) = w_i$. We conclude that the kernel of γ is isomorphic to the group μ_2^r. So we have a short exact sequence

$$1 \longrightarrow \mu_2^r \longrightarrow A \xrightarrow{\gamma} S_r \longrightarrow 1.$$

Moreover, we have a section $S_r \longrightarrow A$, $g \longmapsto \{v_i \mapsto v_{g(i)},\ w_i \mapsto w_{g(i)}\}$, so the statement follows. $\square$

Lemma 3.3.31. *Let $c = \{\{e_1, f_1,\}, \ldots, \{e_7, f_7\}\}$ be an element in the set C that is defined above Lemma 3.3.15, and denote by s the set of roots $\{e_1, \ldots, e_7, f_1, \ldots, f_7\}$. Let A the automophism group of the colored graph associated to s, and let $\varphi\colon A \xrightarrow{\sim} \mu_2^7 \rtimes S_7$ be the isomorphism given in Lemma 3.3.30. Let W_s be the stabilizer in W of s. Then there is an injective map $W_s \longrightarrow A$, whose image has index 2 in A, and its image after composing with φ is given by*

$$\left\{ ((m_1, \ldots, m_7), g) \in \mu_2^7 \rtimes S_7 \mid \prod_{i=1}^{7} m_i = 1 \right\}.$$

Proof. Elements in W_s respect the dot product between roots, so we have a map $\beta\colon W_s \longrightarrow A$. If an element $w \in W_s$ fixes every element in s, then it fixes a sublattice of Λ of finite index by Lemma 3.3.18, and since Λ is torsion free this implies that w is the identity. So the action of W_s on s is faithful, hence β is injective, and $|\beta(W_s)| = |W_s|$. Since W acts transitively on C by Corollary 3.3.17, and $|C| = 2160$ by Remark 3.3.16, we have $|W_s| = |W_c| = \frac{|W|}{|C|} = \frac{|W|}{2160} = 322560$. Moreover, we have $|A| = 2^7 \cdot 7! = 645120$, so $|\beta(W_s)| = |W_s| = 322560 = \frac{1}{2} \cdot |A|$. Hence $\beta(W_s)$ is a subgroup of index two in A. We will now determine which subgroup. Note that $\|e_1 - e_2\| = \sqrt{2}$, so $e_1 - e_2$ is an element $e \in E$, and the reflection in the hyperplane orthogonal to e gives an element in W, say r_{12}. Note that $e_1 + f_1 = e_2 + f_2$ by Remark 3.3.19, so $e_1 - e_2 = f_2 - f_1$. Therefore r_{12} interchanges e_1 with e_2 and f_1 with f_2. Moreover, since all roots in $\{e_3, \ldots, e_7, f_3, \ldots, f_7\}$ are orthogonal to e, the element r_{12} acts trivially on them. Analogously, for $i, j \in \{1, \ldots, 7\}$, $i \neq j$, the reflection r_{ij} is an element in W_s that interchanges e_i and e_j, and f_i with f_j. Let $\gamma\colon A \longrightarrow S_7$ be the projection of $\varphi(A)$ to S_7, then it follows that $\gamma(\beta(W_s)) = S_7$. Now consider for $i, j \in \{1, \ldots, 7\}, i \neq j$, the element $e_i - f_j$. Again, this is an element in E, and the reflection t_{ij} in the hyperplane orthogonal to it is an element in W_s interchanging e_i with f_j, and e_j with f_i, and leaving all other roots in s fixed. It follows that the composition $t_{ij} \circ r_{ij}$ is an element in W_s with $\varphi(\beta(t_{ij} \circ r_{ij})) = ((-1, -1, 1, 1, 1, 1, 1), \mathrm{id}) \in \mu_2^7 \rtimes S_7$. By composing the automorphisms $t_{ij} \circ r_{ij}$ for different i, j, we see that $\varphi(\beta(W_c))$ contains all elements $((m_1, \ldots, m_7), g) \in \mu_2^7 \rtimes S_7$ with $\prod_{i=1}^7 m_i = 1$. Therefore, the reflections r_{ij}, t_{ij} generate a subgroup of A of order $7! \cdot 2^6 = \frac{1}{2}A$, and we conclude that this is all of W_s. $\square$

Corollary 3.3.32. *Let K_1 and K_2 be cliques in Γ whose vertices correspond to a 7-crosspolytope in the $\boldsymbol{E}_8$ root polytope, and let $f\colon K_1 \longrightarrow K_2$ be an isomorphism between them. Then f extends to an automorphism of Λ if and only if for every subclique $S = \{e_1, \ldots, e_7\}$ of K_1 of 7 vertices that are pairwise connected with edges of color 1, the vectors $\sum_{i=1}^7 e_i$ and $\sum_{i=1}^7 f(e_i)$ are either both in 2Λ, or neither are.*

Proof. Consider the set $H = \{c_1 \ldots, c_7, d_1, \ldots, d_7\}$, where the elements are defined above Lemma 3.3.15. Note that the vertices in H correspond to a 7-crosspolytope, and since W acts transitively on the set of cliques corresponding to 7-crosspolytopes (Corollary 3.3.17), there are elements α, β in W such that $\alpha(K_1) = \beta(K_2) = H$. So $\beta \circ f \circ \alpha^{-1}$ is an element in the automorphism group $\mathrm{Aut}(H)$ of H. Of course, f extends to an element in W if and only if $\beta \circ f \circ \alpha^{-1}$ does. Moreover, since α and β are automorphisms of Λ, the two sums $\sum_{i=1}^7 f(e_i)$ and $\sum_{i=1}^7 (\beta \circ f \circ \alpha^{-1})(e_i)$ are either both in or both not in 2Λ. We conclude that we can reduce to the case where $K_1 = K_2 = H$, and f is an element in $\mathrm{Aut}(H)$.
Let W_H be the stabilizer of H in W. By Lemma 3.3.31, there is an injective map $\psi\colon W_H \longrightarrow \mathrm{Aut}(H)$, whose image has index 2 in $\mathrm{Aut}(H)$. Of course, for all elements w in the image of ψ, and for all cliques $S = \{s_1, \ldots, s_7\}$ as in the statement, the sums $\sum_{i=1}^7 s_i$ and $\sum_{i=1}^7 w(s_i)$ are either both in, or both not in 2Λ. We will show that this completely determines the image of ψ, that is, we will show that every element in $\mathrm{Aut}(H) \setminus \psi(W_H)$ does

not have this property for all cliques S as in the statement. To this end, consider the element h in $\mathrm{Aut}(H)$ that exchanges c_1 and d_1, and fixes all other vertices. Since h exchanges an odd number of c_i with d_i, it is not in the image of ψ. Note that $S = \{c_1, \dots, c_7\}$ is a clique as in the statement. The sum $\sum_{i=1}^{7} c_i = (5, 3, 3, 3, -1, 1, 1, 1)$ is an element in 2Λ, and its image under h, which is $\sum_{i=1}^{7} h(c_i) = d_1 + \sum_{i=2}^{7} c_i = (4, 2, 4, 4, -1, 1, 1, 1)$, is not. Since all elements in $\mathrm{Aut}(H) \setminus \psi(W_H)$ are compositions of h with elements in W_H, we conclude that for all elements a in $\mathrm{Aut}(H) \setminus \psi(W_H)$, the sum $\sum_{i=1}^{7} a(c_i)$ is not an element in 2Λ. Since the image of ψ consists exactly of those elements in $\mathrm{Aut}(H)$ extending to an element in W, this finishes the proof. $\square$

Lemma 3.3.33. *The group W acts transitively on the set*

$$B = \{(e_1, e_2, e_3) \in E^3 \mid e_1 \cdot e_2 = 0,\ e_2 \cdot e_3 = e_1 \cdot e_3 = 1\}.$$

Proof. By Proposition 3.2.2 and Lemma 3.3.15, we have

$$|B| = 240 \cdot 126 \cdot 12 = 362880.$$

Let c, s, A be as defined in Lemma 3.3.31, and note that $b = (e_1, f_1, e_2)$ is an element in B. Let W_b be the stabilizer in W of b. Then we have

$$|W_b| = \frac{|W|}{|Wb|} \geq \frac{|W|}{|B|} = 1920.$$

We want to show that this is an equality.
Since c is the unique element in C containing e_1, f_1 by Lemma 3.3.15, the stabilizer W_b of b acts on the set s. If an element $w \in W_b$ fixes all the roots in s, then it fixes a full rank sublattice of finite index in Λ, and since Λ is torsion free this implies that w is the identity. Therefore the action of W_b on s is faithful, so there is an injective map $W_b \longrightarrow W_s$. Note that f_2 is uniquely determined in s as the root that is orthogonal to e_2, so every element in W_b fixes e_1, e_2, f_1, f_2, hence W_b acts faithfully on $s' = \{e_3, \dots, e_7, f_3, \dots, f_7\}$. Let A' be the automorphism group of the colored graph associated to s'. We know there is an isomorphism $\varphi' : A' \longrightarrow \mu_2^5 \rtimes S_5$ by Lemma 3.3.30. Since elements in W_b respect the dot product, we have an injective map $\beta' \colon W_b \longrightarrow A'$. Let $\beta \colon W_s \longrightarrow A$ be the injective map from Lemma 3.3.31. together with the injective maps $W_b \longrightarrow W_s$ and $A' \longrightarrow A$, we have the following commutative diagram.

$$\begin{tikzcd} W_b \arrow[r, hook, "\beta'"] \arrow[d, hook] & A' \arrow[r, "\varphi'", "\sim"'] \arrow[d, hook] & \mu_2^5 \rtimes S_5 \arrow[d, hook] \\ W_s \arrow[r, hook, "\beta"] & A \arrow[r, "\varphi", "\sim"'] & \mu_2^7 \rtimes S_7 \end{tikzcd}$$

By Lemma 3.3.31, the image $\varphi(\beta(W_s))$ is a subset of index 2 in $\mu_2^7 \rtimes S_7$, given by subset $\left\{((m_1,\ldots,m_7),g) \in \mu_2^7 \rtimes S_7 \mid \prod_{i=1}^7 m_i = 1\right\}$. Intersecting this subset with $\mu_2^5 \rtimes S_5$ gives a subset of index 2 in $\mu_2^5 \rtimes S_5$, so by the diagram above, the image $\varphi'(\beta'(W_b))$ has index at least 2 in $\mu_2^5 \rtimes S_5$. We find $|W_b| \leq \frac{1}{2} \cdot 2^5 \cdot 5! = 1920$, so together with the inequality above we conclude that $|W_b| = 1920$. We find $|Wb| = \frac{|W|}{|W_b|} = 362880 = |B|$, and W acts transitively on B. $\square$

We can now prove Proposition 3.3.2.

Proof of Proposition 3.3.2. Note that for a, b, c fixed and σ any permutation of them, there is a bijection between the sets $V_{a,b,c}$ and $V_{\sigma(a),\sigma(b),\sigma(c)}$, so if we prove that W acts transitively on one of them, then W also acts transitively on the other by Lemma 3.2.14. Therefore, we only consider the sets $V_{a,b,c}$ where $a \leq b \leq c$.
There are 4 different sets with $a = b = c$. There are 12 different sets where two of a, b, c are equal to each other and unequal to the third, and 4 different sets with a, b, c all distinct. So there are 20 different sets $V_{a,b,c}$ with $a \leq b \leq c$.

- If $V_{a,b,c}$ is a non-empty set with $a = -2$, then every element (e_1, e_2, e_3) in $V_{a,b,c}$ has $e_1 = -e_2$, so $b = -c$. Therefore the set $V_{a,b,c}$ is empty for (a, b, c) in

$$\begin{aligned}\{(-2,-2,-2),&(-2,-2,-1),(-2,-2,0),(-2,-2,1),\\ &(-2,-1,-1),(-2,-1,0),(-2,0,1),(-2,1,1)\}.\end{aligned}$$

- We have proved that W acts transitively on the sets $V_{-1,-1,-1}$ (Corollary 3.3.10), $V_{0,0,0}$ (Proposition 3.3.28), $V_{0,0,1}$ (Lemma 3.3.12), $V_{0,1,1}$ (Lemma 3.3.33), and $V_{1,1,1}$ (Proposition 3.2.12).

- We have the following bijections.

$$
\begin{aligned}
&\{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = -1\} \longrightarrow V_{-2,-1,1}, && (e_1, e_2) \longmapsto (-e_1, e_1, e_2);\\
&\{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = 0\} \longrightarrow V_{-2,0,0}, && (e_1, e_2) \longmapsto (-e_1, e_1, e_2);\\
&V_{0,1,1} \longrightarrow V_{-1,-1,0}, && (e_1, e_2, e_3) \longmapsto (e_1, -e_3, e_2);\\
&V_{1,1,1} \longrightarrow V_{-1,-1,1}, && (e_1, e_2, e_3) \longmapsto (e_1, -e_2, e_3);\\
&V_{0,0,1} \longrightarrow V_{-1,0,0}, && (e_1, e_2, e_3) \longmapsto (-e_1, e_3, e_2);\\
&V_{0,1,1} \longrightarrow V_{-1,0,1}, && (e_1, e_2, e_3) \longmapsto (-e_3, e_2, -e_1);\\
&V_{-1,-1,-1} \longrightarrow V_{-1,1,1}, && (e_1, e_2, e_3) \longmapsto (e_1, e_2, -e_3).
\end{aligned}
$$

  We proved that W acts transitively on the six different sets on the left-hand sides. From Lemma 3.2.14 it follows that W acts transitively on $V_{-2,-1,1}$, $V_{-2,0,0}$, $V_{-1,-1,0}$, $V_{-1,-1,1}$, $V_{-1,0,0}$, $V_{-1,0,1}$, and $V_{-1,1,1}$, too.

Since we proved that $V_{a,b,c}$ is either empty or W acts transitively on it for 20 different sets, we conclude that we proved the proposition.

The following corollary proves Theorem 3.1.4 for cliques of Type III.

Corollary 3.3.34. *Let K_1 and K_2 be two cliques of type III, and let $f\colon K_1 \longrightarrow K_2$ be an isomorphism between them. Then f extends to an automorphism of Λ.*

Proof. Since W acts transitively on the set of ordered sequences of n roots for $1 \leq n \leq 3$ by Propositions 3.3.1 and 3.3.2, there exists an automorphism $w \in W$ of Λ such that w restricted to K_1 equals f. □

3.4 Monochromatic cliques

In this section we study the cliques of type I, that is, cliques in $\Gamma_{\{-2\}}$, $\Gamma_{\{-1\}}$, $\Gamma_{\{0\}}$, and $\Gamma_{\{1\}}$. We describe the orbits under the action of W of *sequences* of roots that form a clique, thus obtaining the results in Theorem 3.1.4 for cliques of type I (see Corollaries 3.4.5 and 3.4.9). We also describe all maximal cliques per color. For $\Gamma_{\{-2\}}$ and $\Gamma_{\{-1\}}$, everything follows from the previous sections. For $\Gamma_{\{1\}}$ we have Proposition 3.2.12 already; we show moreover that there are no cliques of size bigger than eight, and describe the maximal cliques in Proposition 3.4.7. Finally, in this section we prove that W acts transitively on ordered sequences of length r of orthogonal roots for $r \geq 5$. The result is in Proposition 3.4.4. Throughout this section we do not use any computer.

Cliques in $\Gamma_{\{-2\}}$
The maximal size of a clique in $\Gamma_{\{-2\}}$ is two, since such a maximal clique consists of an element in E and its inverse (see Proposition 3.2.2). There are therefore 120 such cliques. In Lemma 3.3.8 we showed that W acts transitively on the set of ordered pairs $\{(e_1, e_2) \in E^2 \mid e_1 = -e_2\}$, so W acts transitively on the set of maximal cliques in $\Gamma_{\{-2\}}$.

Cliques in $\Gamma_{\{-1\}}$
In $\Gamma_{\{-1\}}$, the maximal size of a clique is three, and there are no maximal cliques of smaller size, by Lemma 3.3.9. From Proposition 3.2.2 and Lemma 3.3.9 it follows that there are $\frac{240 \cdot 56}{3!} = 2240$ maximal cliques. By Corollary 3.3.10, the group W acts transitively on the set of sequences $\{(e_1, e_2, e_3) \in E^3 \mid e_1 \cdot e_2 = e_2 \cdot e_3 = e_1 \cdot e_3 = -1\}$, so W acts transitively on the set of maximal cliques in $\Gamma_{\{-1\}}$. By Lemma 3.3.8, the group W acts transitively on the set $\{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = -1\}$, so W acts also transitively on the set of cliques of size two in $\Gamma_{\{-1\}}$, of which there are $\frac{240 \cdot 56}{2} = 6720$ (Proposition 3.2.2).

Cliques in $\Gamma_{\{0\}}$
Cliques in $\Gamma_{\{0\}}$ are studied in [DM10], where they are called *orthogonal subsets*. In their paper, the authors show that the maximal size of cliques in $\Gamma_{\{0\}}$ is eight [DM10, Table 1], that two cliques of the same size r are conjugate if $r \neq 4$, and that there are two orbits of cliques of size 4 [DM10, Corollary 2.3]. In the previous section we showed that W acts transitively on the set of *ordered sequences* of length at most 3 of orthogonal roots, and that there are two orbits of sequences of length 4. In this section we use this to conclude the same results as in [DM10] for cliques of size $r \leq 4$, and we compute the number of these cliques. Moreover, we study the action of W on ordered sequences of length ≥ 5 of orthogonal roots (Proposition 3.4.4), and compute the number of cliques of size ≥ 5 (Proposition 3.4.6).

The following proposition deals with the cliques of size at most 4.

Proposition 3.4.1.

(i) There are 15120 *cliques of size two in* $\Gamma_{\{0\}}$*, and the group* W *acts transitively on the set of all of them.*

(ii) There are 302400 *cliques of size three in* $\Gamma_{\{0\}}$*, and the group* W *acts transitively on the set of all of them.*

(iii) There are 1965600 *cliques of size four in* $\Gamma_{\{0\}}$*, and they form two orbits under the action of* W*: one of size* 151200*, in which all cliques have vertices whose roots sum up to a vector in* 2Λ*, and one of size* 1814400*, in which all cliques have vertices whose roots sum op to a vector that is not in* 2Λ*.*

Proof.

(i) We have shown that the group W acts transitively on the set

$$A_0 = \{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = 0\}$$

(Proposition 3.3.14), and $|A_0| = 240 \cdot 126 = 30240$ (Proposition 3.2.2). It follows that there are $\frac{30240}{2} = 15120$ cliques of size two in $\Gamma_{\{0\}}$, and the group W acts transitively on the set of all of them.

(ii) The group W acts transitively on the set

$$V_3 = \{(e_1, e_2, e_3) \in E^3 \mid \forall i \neq j : e_i \cdot e_j = 0\},$$

and we have $|V_3| = 1814400$ (Proposition 3.3.28 (i)). It follows that there are $\frac{1814400}{6} = 302400$ cliques of size three in $\Gamma_{\{0\}}$, and the group W acts transitively on the set of all of them.

(iii) By Proposition 3.3.29 there are two orbits under the action of W on the set

$$V_4 = \{(e_1, e_2, e_3, e_4) \in E^4 \mid \forall i \neq j : e_i \cdot e_j = 0\};$$

one of size 3628800 where all elements have coordinates that sum up to a vector that is in 2Λ, and one orbit of size 43545600 where all elements have coordinates that sum up to a vector that is not in 2Λ. Since the orbit in which an element is contained does not depend on the order of its coordinates, we conclude that this also gives two orbits with the same properties under the action of W on the set of all cliques of size four in $\Gamma_{\{0\}}$, of sizes $\frac{3628800}{4!} = 151200$ and $\frac{43545600}{4!} = 1814400$, respectively. □

We continue by studying the sequences of orthogonal roots of length greater than four. Recall the set V_4 and its orbits under the action of W, given by Y of size 43545600 and $O = V_4 \setminus Y$ of size 3628800 (shown in Proposition 3.3.29).

Lemma 3.4.2. *For an element $y = (e_1, \ldots, e_4) \in Y$, define the set*

$$C_y = \{e \in E \mid e \cdot e_i = 0 \text{ for } i \in \{1,2,3,4\}\}.$$

The following hold.

(i) The set C_y is the union of four sets $\{f_1, -f_1\}, \{f_2, -f_2\}, \{f_3, -f_3\}$, and $\{f_4, -f_4\}$, with $f_i \cdot f_j = 0$ for $i \neq j$. For such a set $\{f_i, -f_i\}$, there is exactly one triple $\{e_{i_1}, e_{i_2}, e_{i_3}\}$ of elements in y such that the permutations of $(e_{i_1}, e_{i_2}, e_{i_3}, f_i)$ (or equivalently of $(e_{i_1}, e_{i_2}, e_{i_3}, -f_i)$) form elements in O. Moreover, for $j \neq i$, and j_1, j_2, j_3 such that the permutations of $(e_{j_1}, e_{j_2}, e_{j_3}, f_j)$ form elements in O, the sets $\{e_{i_1}, e_{i_2}, e_{i_3}\}$ and $\{e_{j_1}, e_{j_2}, e_{j_3}\}$ are different.

(ii) The stabilizer of y is generated by the reflections in the hyperplanes orthogonal to f_i for $i \in \{1,2,3,4\}$.

Proof. Since W acts transtively on Y, it suffices to show this for a fixed element $y \in Y$. Set

$$e_1 = (1,1,0,0,0,0,0,0)\,, \qquad e_3 = (0,0,0,0,1,1,0,0)\,,$$
$$e_2 = (0,0,1,1,0,0,0,0)\,, \qquad e_4 = (1,-1,0,0,0,0,0,0)\,.$$

Then (e_1, e_2, e_3, e_4) is an element in V_4 and since $\sum_{i=1}^4 e_i \notin 2\Lambda$, it is an element in Y as well by Proposition 3.3.29. Take $e = (a_1, \ldots, a_8) \in E$ such that $e \cdot e_i = 0$ for $i \in \{1,2,3,4\}$. Then we have $a_1 + a_2 = a_1 - a_2 = a_3 + a_4 = a_5 + a_6 = 0$. We find the following possibilities.

$$\pm f_1 = \pm(0,0,0,0,0,0,1,-1)\,, \qquad \pm f_3 = \pm(0,0,1,-1,0,0,0,0)\,,$$
$$\pm f_2 = \pm(0,0,0,0,1,-1,0,0)\,, \qquad \pm f_4 = \pm(0,0,0,0,0,0,1,1)\,.$$

It is an easy check that $f_i \cdot f_j = 0$ for $i \neq j$, and for $i, k \in 1,2,3,4$, the sum $\left(\sum_{j \neq i} e_j\right) \pm f_k$ is contained in 2Λ if and only if $i = k$. This proves (i). We continue with (ii). Take $i \in \{1,2,3,4\}$. Since f_i is orthogonal to the elements in y the reflection r_i in the hyperplane orthogonal to f_i is an element of W_y. For $i \neq j$, the reflections r_i and r_j commute, since f_i and f_j are orthogonal. Therefore the elements r_1, r_2, r_3, r_4 generate a subgroup of W_y of order 16. Since we have

$$|W_y| = \frac{|W|}{|Y|} = \frac{696729600}{43545600} = 16,$$

they generate the whole group W_y. $\square$

Corollary 3.4.3. *Set $n_5 = 1$, $n_6 = 3$, $n_7 = 7$, and $n_8 = 14$. Let K be a clique of size $r \in \{5, 6, 7, 8\}$ in $\Gamma_{\{0\}}$. Then the number of sets of four vertices e_1, e_2, e_3, e_4 in K such that the permutations of (e_1, e_2, e_3, e_4) are elements in O is equal to n_r.*

Proof. First let K be a clique of size 5 in $\Gamma_{\{0\}}$. Assume in contradiction that there are two distinct subsets, say y_1, y_2, of four vertices in K that form an element in O. Then there are three vertices of K, say e_1, e_2, e_3, that are contained both in y_1 and y_2. Write $y_1 = \{e_1, e_2, e_3, f_1\}$ and $y_2 = \{e_1, e_2, e_3, f_2\}$. By applying Proposition 3.3.28 (iii) to the triple (e_1, e_2, e_3), it follows that $f_1 = -f_2$, so $f_1 \cdot f_2 = -2$. But this gives a contradiction, since f_1, f_2 are both in K. So the number of sets of four vertices in K that form an element in O is at most 1, which means that there is at least one subset $\{g_1, g_2, g_3, g_4\}$ of K of four roots such that (g_1, g_2, g_3, g_4) is an element in Y. For the fifth element in K, say g_5, it follows from the previous lemma that there is exactly one triple $\{g_\alpha, g_\beta, g_\gamma\}$ of elements in $\{g_1, \ldots, g_4\}$ that it forms an element in O with. We conclude that there is exactly 1 set of four vertices in K that form an element in O; this proves the statement for $r = 5$.
We proceed by induction. Take $s \in \{6, 7, 8\}$. Assume that the statement holds for $5 \leq r < s$, and let $K = \{e_1, \ldots, e_s\}$ be a clique of size s in $\Gamma_{\{0\}}$. By induction we know that $\{e_1, \ldots, e_{s-1}\}$ contains n_{s-1} subsets of size four that form an element in O. That means that there are $\binom{s-1}{4} - n_{s-1}$ subsets of size four in $\{e_1, \ldots, e_{s-1}\}$ that form an element in Y. By Lemma 3.4.2, each of these $\binom{s-1}{4} - n_{s-1}$ subsets contains exacty three elements that, together with e_s, form an element in O. Let d_1, d_2, d_3 be three elements in $\{e_1, \ldots, e_{s-1}\}$ such that (d_1, d_2, d_3, e_s) is an element in O. Then for every element $d \in \{e_1, \ldots, e_{s-1}\} \setminus \{d_1, d_2, d_3\}$, the set $\{d_1, d_2, d_3, e_s, d\}$ forms a clique of size 5 in $\Gamma_{\{0\}}$, and since $n_5 = 1$, it follows that (d_1, d_2, d_3, d) is an element in Y. This means that every set of three roots in $\{e_1, \ldots, e_{s-1}\}$ that forms an element in O with e_s forms an element in Y with all other roots in $\{e_1, \ldots, e_{s-1}\}$. Since every set of three roots in $\{e_1, \ldots, e_{s-1}\}$ is contained in $(s-1) - 3$ subsets of size four of $\{e_1, \ldots, e_{s-1}\}$, this gives $\frac{\binom{s-1}{4} - n_{s-1}}{s-4}$ distinct sets of three that form an element in O with e_s. In total this gives $n_{s-1} + \frac{\binom{s-1}{4} - n_{s-1}}{s-4}$ sets of four vertices in K that form an element in O. This is exactly equal to n_s for $s = 6, 7, 8$. $\square$

For $1 \leq r \leq 8$, let V_r be the set

$$V_r = \{(e_1, \ldots, e_r) \in E^r \mid \forall i \neq j : e_i \cdot e_j = 0\}.$$

Proposition 3.4.4. *For $5 \leq r \leq 8$, two elements $(e_1, \dots, e_r)$, $(f_1, \dots, f_r)$ in V_r are in the same orbit under the action of W if and only if for all $1 \leq i < j < k < l \leq r$, the elements (e_i, e_j, e_k, e_l) and (f_i, f_j, f_k, f_l) are conjugate in V_4 under the action of W.*

Proof. For $5 \leq r \leq 8$, define the relation $\sim$ on V_r by $(e_1, \dots, e_r) \sim (f_1, \dots, f_r)$ if and only if for all $1 \leq i < j < k < l \leq r$, the elements (e_i, e_j, e_k, e_l) and (f_i, f_j, f_k, f_l) are conjugate in V_4. Note that $\sim$ is an equivalence relation on V_r, and the group W acts on the equivalence classes. Our goal is to show that each equivalence class is an orbit in V_r under the action of W. We do this by induction on r.
For $r = 5$, let $X_5 \subset V_5$ be an equivalence class with respect to $\sim$. We distinguish two cases. If for every element in X_5 the first four coordinates form an element in Y, we let $p \colon X_5 \longrightarrow Y$ be the projection to the first four coordinates. Note that this is surjective by Lemma 3.4.2. Set $y = (y_1, \dots, y_4) \in Y$. Since the elements in the fiber $p^{-1}(y)$ are equivalent under $\sim$, there are exactly two elements $(y_1, \dots, y_4, f), (y_1, \dots, y_4, -f)$ in $p^{-1}(y)$ by Lemma 3.4.2 (i). Moreover, the stabilizer W_y acts transitively on these two elements by Lemma 3.4.2 (ii). From Lemma 3.2.14 it follows that W acts transitively on X_5. If, on the other hand, for every element in X_5 the first four coordinates form an element in O, then it follows from Corollary 3.4.3 that the last four coordinates of every element in X_5 form an element in Y. We now let $p \colon X_5 \longrightarrow Y$ be the projection to the last four coordinates, and the proof is the same.
Now assume that $r > 5$, and that each equivalence class in V_{r-1} is an orbit under the action of W. Let X_r be an equivalence class in V_r, and $p_r \colon X_r \longrightarrow V_{r-1}$ the projection to the first $r-1$ coordinates. Then W acts on $p_r(X_r)$, and $p_r(X_r)$ is contained in an equivalence class X_{r-1} with respect to $\sim$ in V_{r-1}. Since W acts transitively on X_{r-1} by hypothesis, it follows that $p_r(X_r) = X_{r-1}$, and W acts transitively on $p_r(X_r)$. Since $r > 5$, by Corollary 3.4.3 there exist $i, j, k, l \in \{1, \dots, r-1\}$ such that for all elements $(e_1, \dots, e_r) \in X_r$ we have $(e_i, e_j, e_k, e_l) \in Y$. Fix such i, j, k, l, and let $v = (v_1, \dots, v_{r-1})$ be an element in $p_r(X_r)$. Then (v_i, v_j, v_k, v_l) is an element in Y. Let $(v_1, \dots, v_{r-1}, f)$, $(v_1, \dots, v_{r-1}, g)$ be elements in the fiber $p_r^{-1}(v)$. Since $(v_1, \dots, v_{r-1}, f)$ is equivalent to $(v_1, \dots, v_{r-1}, g)$ with respect to $\sim$, by applying Lemma 3.4.2 to (v_i, v_j, v_k, v_l) we see that $f = -g$, and the fiber $p_r^{-1}(v)$ consists of the two elements $(v_1, \dots, v_{r-1}, f)$ and $(v_1, \dots, v_{r-1}, -f)$. Moreover, the reflection in the hyperplane orthogonal to f fixes $v_1, \dots, v_{r-1}$, hence is an element in the stabilizer of v that switches f and $-f$. So the stabilizer of v acts transitively on $p_r^{-1}(v)$, and

again from Lemma 3.2.14 we conclude that W acts transitively on X_r. $\square$

Corollary 3.4.5. *Let K_1 and K_2 be cliques in $\Gamma_{\{0\}}$, and $f\colon K_1 \longrightarrow K_2$ an isomorphism between them. Then f extends to an automorphism of Λ if and only if for every subclique S of size 4 in K_1, the image $f(S)$ in K_2 is conjugate to S under the action of W.*

Proof. If K_1 and K_2 have size ≤ 3, then it follows from Corollary 3.3.34 that f always extends. From Proposition 3.4.4 it follows that if K_1 and K_2 have size at least four, the isomorphism f extends to an element in W exactly when f sends every sequence of four roots that form an element in V_4 to a conjugate element in V_4. By Proposition 3.3.29, there are two orbits of ordered sequences of four pairwise orthogonal roots, that do not depend on the order of the roots. We conclude that if S and $f(S)$ are conjugate under the action of W for every set S of four vertices in K_1, there exists an automorphism $w \in W$ of Λ such that w restricted to K_1 equals f. $\square$

Theorem 3.4.6. *In $\Gamma_{\{0\}}$, the following hold.*

(i) There are no maximal cliques of size smaller than eight.

(ii) There are 3628800 *cliques of size five,* 3628800 *cliques of size six,* 2073600 *cliques of size seven, and* 518400 *cliques of size eight.*

(iii) The group W acts transitively on the cliques of size 5.

Proof.

(i) We know that every root in E is orthogonal to 126 other roots (Proposition 3.2.2). Moreover, we know that in $\Gamma_{\{0\}}$ every clique of size 2 extends to a clique of size 3 (Lemma 3.3.27), and every clique of size 3 extends to a clique of size 4 (Proposition 3.3.28 (ii)). Since $n_5 = 1 < \binom{5}{4}$ by Corollary 3.4.3, every clique of size 5 in $\Gamma_{\{0\}}$ contains both a subclique whose vertices form an element in O, and a subclique whose vertices form an element in Y. Since W acts transitively on O and on Y, and $V_4 = O \cup Y$, this means that every clique of size 4 in $\Gamma_{\{0\}}$ extends to a clique of size 5. Moreover, by Lemma 3.4.2 (i), every extension of a clique of size 4 whose vertices form an element in Y is contained in a clique of size 8. Since every clique of size at least 5 contains a clique of size 4 whose vertices form an element in Y, there are no maximal cliques of size smaller than 8.

(ii) By Lemma 3.4.2, if we fix an element $y = (e_1, e_2, e_3, e_4) \in Y$, there are exactly 8 elements in V_5, and $8\cdot 6$ elements in V_6, and $8\cdot 6\cdot 4$ elements in V_7, and $8\cdot 6\cdot 4\cdot 2$ elements in V_8, that have e_i as the i^{th} coordinate for $i \in \{1,2,3,4\}$. We call this number m_r for $r = 5,6,7,8$. For all $5 \le r \le 8$, for S a clique of size r, it follows from Corollary 3.4.3 that S contains $\binom{r}{4} - n_r$ cliques of size 4 that, together, form $4! \cdot (\binom{r}{4} - n_r)$ different elements in Y; for such a subclique of size 4 in S, the other $r-4$ elements can be permuted in $(r-4)!$ ways. For all $5 \le r \le 8$, let D_r be the set of cliques of size r in $\Gamma_{\{0\}}$. It follows that we have

$$|D_r| = \frac{|Y| \cdot m_r}{4! \cdot (\binom{r}{4} - n_r) \cdot (r-4)!}.$$

We find the following results.

$$|D_5| = \frac{|Y|\cdot 8}{4!\cdot 4} = 3628800,\ |D_6| = \frac{|Y|\cdot 8\cdot 6}{4!\cdot 12\cdot 2} = 3628800,$$

$$|D_7| = \frac{|Y|\cdot 8\cdot 6\cdot 4}{4!\cdot 28\cdot 3!} = 2073600,\ |D_8| = \frac{|Y|\cdot 8\cdot 6\cdot 4\cdot 2}{4!\cdot 56\cdot 4!} = 518400.$$

(iii) Let $K_1 = \{e_1, \ldots, e_5\}$, $K_2 = \{f_1, \ldots, f_5\}$ be two cliques in $\Gamma_{\{0\}}$. We have $n_5 = 1$ by Corollary 3.4.3, so without loss of generality we can assume that e_1, e_2, e_3, e_4 and f_1, f_2, f_3, f_4 are the unique four elements in K_1 and K_2, respectively, that form an element in O. Then $(e_1, e_2, e_3, e_4, e_5)$ and $(f_1, f_2, f_3, f_4, f_5)$ are conjugate under the action of W by Proposition 3.4.4, hence so are K_1 and K_2. □

Cliques in $\Gamma_{\{1\}}$

We know that cliques in $\Gamma_{\{1\}}$ form k-simplices that are k-faces of the $\mathbf{E}_8$ root polytope (Proposition 3.2.4), hence Corollary 3.2.7 states how many cliques of size n there are in $\Gamma_{\{1\}}$ for $n \le 8$. Moreover, we know that W acts transitively on these cliques for $n \le 8$, $n \neq 7$ (Proposition 3.2.12). Proposition 3.4.7 shows that there are no cliques of size bigger than eight in $\Gamma_{\{1\}}$, and that there are two orbits of cliques of size seven (which was already known, for example by [Cox30] and [Man86]); it shows that all maximal cliques are of size 7 or 8.

Proposition 3.4.7. *In $\Gamma_{\{1\}}$, the following hold.*

(i) There are only maximal cliques of size 7 and 8.

(ii) There are two orbits of cliques of size 7 in $\Gamma_{\{1\}}$*; one of size 138240, which is given by non-maximal cliques, and one of size 69120, which is given by maximal cliques. A clique of size seven in* $\Gamma_{\{1\}}$ *is maximal if and only if the sum of its vertices is an element in* 2Λ*.*

(iii) There are 17280 cliques of size 8.

Proof. Consider the clique of size six in $\Gamma_{\{1\}}$ given by $\{e_1, \ldots, e_6\}$, where we define

$$\begin{aligned} e_1 &= (1,1,0,0,0,0,0,0)\,, & e_4 &= (1,0,0,0,1,0,0,0) \\ e_2 &= (1,0,1,0,0,0,0,0)\,, & e_5 &= (1,0,0,0,0,1,0,0) \\ e_3 &= (1,0,0,1,0,0,0,0)\,, & e_6 &= (1,0,0,0,0,0,1,0)\,. \end{aligned}$$

Since W acts transitively on the set of cliques of size smaller than 6 in $\Gamma_{\{1\}}$ by Proposition 3.2.12, it follows that every clique of size smaller than 6 in $\Gamma_{\{1\}}$ is contained in a clique of size 6 in $\Gamma_{\{1\}}$. The elements in E that have dot product one with all $e_1, \ldots, e_6$ are given by $c_1 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $c_2 = (1,0,0,0,0,0,0,1)$, and $c_3 = (1,0,0,0,0,0,0,-1)$. Note that $c_1 \cdot c_2 = 1$ and $c_3 \cdot c_1 = c_3 \cdot c_2 = 0$, so $\{e_1, \ldots, e_6, c_1, c_2\}$ is a maximal clique of size 8 in $\Gamma_{\{1\}}$, and $\{e_1, \ldots, e_6, c_3\}$ is a maximal clique of size 7 in $\Gamma_{\{1\}}$. Since W acts transitively on the cliques of size 6 in $\Gamma_{\{1\}}$ by Proposition 3.2.12, all maximal cliques in $\Gamma_{\{1\}}$ are of size 7 or 8. This proves part (i). Moreover, it follows that every non-maximal clique of size 7 is contained in a unique clique of size 8, so there are $\frac{138240}{8} = 17280$ cliques of size 8. This proves part (iii). We will now prove (ii). From part (i) it follows that there exist maximal and non-maximal cliques of size 7 in $\Gamma_{\{1\}}$. It is obvious that they can not be in the same orbit under the action of W. Moreover, there are two orbits of ordered sequences of length 7, hence at most two orbits of cliques of size 7 by Proposition 3.2.12. We conclude that the orbits are given exactly by the maximal cliques and the non-maximal cliques. Since there are 483840 cliques of size 6 (Corollary 3.2.7), from the above it follows that there are $\frac{483840 \cdot 2}{7} = 138240$ non-maximal cliques, and $\frac{483840 \cdot 1}{7} = 69120$ maximal cliques. Now consider the set $\{e_1, \ldots, e_7\}$, where the elements are defined above Lemma 3.3.15. This is a clique of size 7 in $\Gamma_{\{1\}}$, and it is not hard to check that it is maximal. Moreover, we have

$$\sum_{i=1}^{7} e_i = (5,3,3,3,1,1,1,1) \in 2\Lambda.$$

Since W acts transitively on all maximal cliques of size 7 in $\Gamma_{\{1\}}$, for all such cliques the sum of the vertices is an element in 2Λ. On the other hand, consider the set $d = \{d_1, \ldots, d_7\}$ as defined above Lemma 3.3.15. This is a non-maximal clique of size 7 in $\Gamma_{\{1\}}$, since the union of d with the root $\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$ is a clique of size 8 in $\Gamma_{\{1\}}$. Moreover, we have

$$\sum_{i=1}^{7} d_i = (2, 4, 4, 4, 1, -1, -1, -1) \notin 2\Lambda.$$

Since W acts transitively on all non-maximal cliques of size 7 in $\Gamma_{\{1\}}$, for all such cliques the sum of the vertices is not an element in 2Λ. $\square$

Remark 3.4.8. Note that $138240 + 69120 = 207360$, which is the total number of cliques of size 7 by Corollary 3.2.7.

Corollary 3.4.9. *Let K_1 and K_2 be cliques in $\Gamma_{\{1\}}$, and $f\colon K_1 \longrightarrow K_2$ an isomorphism between them. If K_1 and K_2 have size unequal to 7, then f extends to an automorphism of Λ. If K_1 and K_2 have size 7, then f extends if and only if the sum of the vertices of K_1 and the sum of the vertices of K_2 are either both in 2Λ, or both not.*

Proof. Another way of saying that the morphism f extends is that if $\{e_1, \ldots, e_r\}$ is the set of vertices in K_1, then the sequences $(e_1, \ldots, e_r)$ and $(f(e_1), \ldots, f(e_r))$ are conjugate. By Proposition 3.2.12, for $r \leq 8$, $r \neq 7$, there is only one orbit of ordered sequences of length r of roots that have pairwise dot product 1. This implies that f extends to an element in W if K_1, K_2 have size unequal to 7. Furthermore, by the same proposition, there are two orbits of ordered sequences of roots of length 7. By Proposition 3.4.7, there two orbits of cliques of size 7, that are distinguished by whether the sum of the 7 roots is an element in 2Λ or not. We conclude that the two orbits of ordered sequences are distinguished in the same way. This implies that f extends if and only if the sum of the vertices in K_1 and the sum of the vertices in $f(K_1) = K_2$ are both in 2Λ or both not. $\square$

Remark 3.4.10. We know that the cliques of size 7 in $\Gamma_{\{1\}}$ are 6-faces of the $\mathbf{E}_8$ root polytope. We can describe the two orbits of these cliques in this framework as well. A 6-face of the polytope is an intersection of two facets. There are two types of facets of the $\mathbf{E}_8$ root polytope: 7-crosspolytopes and 7-simplices (Proposition 3.2.5). Since the maximal cliques of size 7 in $\Gamma_{\{1\}}$ are not contained in a 7-simplex, these are exactly

the intersections of two 7-crosspolytopes.
Consider the set $c = \{c_1, \ldots, c_7, d_1, \ldots, d_7\}$, defined above Lemma 3.3.15. Note that $d = \{d_1, \ldots, d_7\}$ is a non-maximal clique of size 7 in $\Gamma_{\{1\}}$ that is contained in the 7-crosspolytope with vertices in c, but also in the 7-simplex with vertices $d \cup \left\{\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)\right\}$. It follows that every non-maximal clique of size 7 in $\Gamma_{\{1\}}$ is the intersection of a 7-crosspolytope with a 7-simplex.
From this it also follows that two 7-simplices in the $\mathbf{E}_8$ root polytope never intersect.

Remark – analogy with geometry 3.4.11. Let X be a del Pezzo surface of degree 1 over an algebraically closed field, and I the set of exceptional classes in Pic X. Through the bijection between I and E, cliques in $\Gamma_{\{1\}}$ are related to sets of exceptional classes that are pairwise disjoint. These are called *exceptional sets*, and can be blown down so that we obtain a del Pezzo surface of higher degree (Lemma 1.2.8). Since a del Pezzo surface has degree at most 9 (in which case it is isomorphic to $\mathbb{P}^2$), it is clear that the maximal size of a clique in $\Gamma_{\{1\}}$ is eight. We can also describe the two orbits of size 7 in this setting; cliques that are maximal correspond to exceptional sets that blow down to a del Pezzo surface of degree 8 that is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and cliques that are not maximal correspond to exceptional sets that blow down to a del Pezzo surface of degree 8 that is isomorphic to $\mathbb{P}^1$ blown up in one point [Man86, remark below Corollary 26.8].

3.5 Maximal cliques

In this section we describe all maximal cliques in Γ_c for $c \neq \{-1, 0, 1\}$ (cliques of type IV), and their orbits under the action of W. Note that $\Gamma_{-1,0,1}$ is the graph Γ after removing all edges between roots and their inverses. This means that the maximal cliques in $\Gamma_{-1,0,1}$ are all of size 120: for each root you can either choose the root or its inverse. Therefore there are 2^{120} maximal cliques in $\Gamma_{-1,0,1}$, which gives at least $\left\lceil \frac{2^{120}}{|W|} \right\rceil =$ 1907810427151244719477695595 orbits in the set of maximal cliques under the action of W. Because of the size of these cliques and their orbits, we did not compute the orbits.

In the first two subsections of this section we describe all maximal cliques in $\Gamma_{\{-2\}}, \Gamma_{\{-1\}}, \Gamma_{\{0\}}, \Gamma_{\{1\}}, \Gamma_{\{-2,-1\}}, \Gamma_{\{-2,1\}}, \Gamma_{\{-2,0\}}$, and $\Gamma_{\{-2,-1,0,1\}} = \Gamma$. Cliques in $\Gamma_{\{-2,-1\}}$ and $\Gamma_{\{-2,1\}}$ are monochromatic (Lemma 3.5.5), and maximal cliques in $\Gamma_{\{-2,0\}}$ are in bijection with maximal cliques in $\Gamma_{\{0\}}$ (Lemma 3.5.7). Therefore, everything before Section 3.5.3 follows from results in Section 3.4 and is done without a computer. From Section 3.5.3 onwards, we used `magma` for some computations. The code that we used can be found in [Codb].

Remark 3.5.1. Because of the relation to del Pezzo surfaces, the maximal cliques in $\Gamma_{\{-2,0\}}$ and $\Gamma_{\{-1,0\}}$ are of special interest to us, which is explained in Remark 3.5.4. For these two graphs we have extra results in Sections 3.5.2 and 3.5.3: we compute the structure of the largest cliques in the graphs in Propositions 3.5.9 and 3.5.23, and we show that for these largest cliques, their stabilizer in W acts transitively on the clique itself (Corollaries 3.5.12 and 3.5.25).
Most of the results in Section 3.5.2 were already proven in terms of del Pezzo surfaces by the same author in [Win14]; results 3.5.7 – 3.5.14 correspond to results 4.8, 4.10, 4.11, and 4.16–4.20 in [Win14]. Moreover, Proposition 3.5.24 and Corollary 3.5.25 are the same as Proposition 4.27 and Corollary 4.28 in [Win14]. We decided to repeat these results here for completeness, and to rephrase the results in terms of the roots in $\mathbf{E}_8$. Besides this, the techniques in Sections 3.5.2 and 3.5.3 show how one could prove similar results for graphs with other colors.

The main results of this section are summarized in the tables in Appendix A and Remark 3.6.1.

Notation 3.5.2. To denote cliques of Γ in a compact way, we order the root system E as follows. Roots of the form $\left(\pm\frac{1}{2}, \ldots, \pm\frac{1}{2}\right)$ are ordered lexicographically and denoted by numbers $1-128$; for example, $\left(-\frac{1}{2}, \ldots, -\frac{1}{2}\right)$ is number 1, and $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ number 128. Roots that are permutations of $(\pm1, \pm1, 0, 0, 0, 0, 0, 0)$ are ordered lexicographically and denoted by the numbers $129 - 240$; for example, $(-1, -1, 0, 0, 0, 0, 0, 0)$ is number 129, and $(1, 1, 0, 0, 0, 0, 0, 0)$ is number 240.

The table in Appendix A contains the following information.

Notation 3.5.3.
Graph: a graph Γ_c where c is a set of colors in $\{-2, -1, 0, 1\}$.

K: a clique in Γ_c; we denote vertices by their index as in Notation 3.5.2.

$|K|$: the size of K.

$|W_K|$: the size of the stabilizer of clique K in the group W.

$|\operatorname{Aut}(K)|$: the size of the automorphism group of K as a colored graph.

$\#O$: the number of orbits of the set of all maximal cliques of size $|K|$ in Γ_c under the action of W.

For each graph Γ_c, the list of cliques in Γ_c in the table in Appendix A gives exactly one representative for each orbit of the set of maximal cliques in Γ_c under the action of W. The proofs of these results are in Proposition 3.5.6, Corollary 3.5.16, Proposition 3.5.28, Lemma 3.5.30, Proposition 3.5.33, and Proposition 3.5.35.

The following remark shows the connection between del Pezzo surfaces and cliques in $\Gamma_{\{-2,0\}}$ and $\Gamma_{\{-1,0\}}$.

Remark – analogy with geometry 3.5.4. Let X be a del Pezzo surface of degree 1 over an algebraically closed field, and let I be the set of exceptional classes in Pic X. The question that led us to study the $\mathbf{E}_8$ root system was how many exceptional curves on X go through the same point (see Chapter 4; the following is also stated in Proposition 4.2.4 and Remark 4.2.5). Recall that the linear system $|-2K_X|$ realizes X as a double cover of a cone in $\mathbb{P}^3$, ramified over a sextic curve B that does not contain the vertex of the cone (see Section 1.4.1). There are 120 hyperplanes that are tritangent to B, and such a hyperplane pulls back to the sum of two elements in I that intersect with multiplicity 3. It follows that two elements in I intersecting with multiplicity 3 correspond to curves on X intersecting in 3 points on the ramification curve. Conversely, if an element c in I corresponds to a curve on X that goes through a point P on the ramification curve, then the unique element $c' \in I$ with $c \cdot c' = 3$ corresponds to a curve on X going through P as well.
Through the bijection $I \longrightarrow E,\ c \longmapsto c+K_X$, cliques in Γ that correspond to sets of pairwise intersecting lines on X have edges of colors $-2,-1,0$. Since elements in I with intersection multiplicity 3 correspond to two roots in E with dot product -2, it follows that a set of lines on X that all go through one point on the ramification curve forms a clique in $\Gamma_{\{-2,0\}}$, and a set of lines on X that all go through one point outside the ramification curve forms a clique in $\Gamma_{\{-1,0\}}$. This motivates why we have studied these

two graphs extensively, and especially the biggest cliques in them (with respect to number of vertices).

3.5.1 Maximal cliques in $\Gamma_{\{-2\}}, \Gamma_{\{-1\}}, \Gamma_{\{1\}}, \Gamma_{\{-2,-1\}}, \Gamma_{\{-2,1\}}$, and $\Gamma_{\{-2,-1,0,1\}}$

Lemma 3.5.5. *Cliques in $\Gamma_{\{-2,-1\}}$ and in $\Gamma_{\{-2,1\}}$ are monochromatic.*

Proof. For an element $e \in E$, its inverse $-e$ is the unique element intersecting it with multiplicity -2 (Proposition 3.2.2). Take $e, f \in E$ with $e \cdot f = -1$, then $-e \cdot f = 1$, hence $e, f, -e$ do not form a clique in $\Gamma_{\{-2,-1\}}$. Therefore all cliques in $\Gamma_{\{-2,-1\}}$ are monochromatic. Analogously, the cliques in $\Gamma_{\{-2,1\}}$ are monochromatic. □

Proposition 3.5.6. *For*

$$c \in \{\{-2\}, \{-1\}, \{1\}, \{-2,-1\}, \{-2,1\}, \{-2,-1,0,1\}\},$$

the table in Appendix A gives the complete list of orbits of the maximal cliques in Γ_c, as well as a correct representative for each orbit, the size of its stabilizer in W, and the size of its automorphism group.

Proof. We showed in Section 3.4 that all maximal cliques in $\Gamma_{\{-2\}}$ have size 2, and that they form one orbit of size 120. We also showed that all maximal cliques in $\Gamma_{\{-1\}}$ have size 3, and they form one orbit of size 2240. In Proposition 3.4.7 we showed that there are two orbits of maximal cliques in $\Gamma_{\{1\}}$; one of size 69120, which consists of cliques of size 7, and one of size 17280, which consists of cliques of size 8. For $\Gamma_{\{-2,-1\}}$ and $\Gamma_{\{-2,1\}}$ we proved that all cliques are monochromatic in Lemma 3.5.5, so the maximal cliques and their orbits are found by looking at the monochromatic subgraphs $\Gamma_{\{-2\}}$, $\Gamma_{\{-1\}}$, and $\Gamma_{\{1\}}$.
It is an easy check that for these five graphs, the cliques in the table are correct representatives of the orbits. The sizes of their stabilizers are found by dividing the order of W by the size of their orbit. Since all the cliques in these five graphs are monochromatic, their automorphism group is the permutation group on their vertices.
Finally, note that $\Gamma_{\{-2,-1,0,1\}} = \Gamma$. The only maximal clique in $\Gamma_{\{-2,-1,0,1\}}$ is therefore the whole graph, which forms an orbit of size 1 under the action of W. □

3.5.2 Maximal cliques in $\Gamma_{\{0\}}$ and $\Gamma_{\{-2,0\}}$

The following lemma describes the maximal cliques in $\Gamma_{\{-2,0\}}$.

Lemma 3.5.7. *In $\Gamma_{\{-2,0\}}$, the following hold.*

(i) The maximal size of a clique in $\Gamma_{\{-2,0\}}$ is 16, and there are no maximal cliques of smaller size.

(ii) The set of maximal cliques in $\Gamma_{\{-2,0\}}$ is given by

$$\{\{e_1, \ldots, e_8, -e_1, \ldots, -e_8\} \mid \forall i : e_i \in E;\ \forall i \neq j : e_i \cdot e_j = 0\}.$$

Proof. By Theorem 3.4.6, all maximal cliques in $\Gamma_{\{0\}}$ are of size 8. Let $\{e_1, \ldots, e_8\}$ be a maximal clique in $\Gamma_{\{0\}}$. Then $\{e_1, \ldots, e_8, -e_1, \ldots, -e_8\}$ is a clique in $\Gamma_{\{-2,0\}}$ of size 16. Now assume that $\{c_1, \ldots, c_r\}$ is a clique in $\Gamma_{\{-2,0\}}$ of size bigger than 16. Since edges of color -2 connect a root and its inverse, the clique $\{c_1, \ldots, c_r\}$ contains a subclique of size at least $\lceil \frac{r}{2} \rceil$ with only edges of color 0. But this would give a clique in $\Gamma_{\{0\}}$ of size at least $\left\lceil \frac{17}{2} \right\rceil = 9$, contradicting Theorem 3.4.6. We conclude that the maximal size of a clique in $\Gamma_{\{-2,0\}}$ is 16. Now assume that S is a maximal clique in $\Gamma_{\{-2,0\}}$ of size smaller than 16. Let K be the biggest (with respect to number of vertices) subclique of S with only edges of color 0. Let K' be a maximal clique in $\Gamma_{\{0\}}$ containing K, so K' has size 8. Then the clique consisting of all vertices of K' and their inverses is a clique in $\Gamma_{\{-2,0\}}$ of size 16 that strictly contains S, contradicting the maximality of S. We conclude that there are no maximal cliques of size smaller than 16 in $\Gamma_{\{-2,0\}}$, concluding the proof of (i). Part (ii) is now obvious. $\square$

To show that the group W acts transitively on the maximal cliques in $\Gamma_{\{-2,0\}}$, we use the following lemma, which builds on results in previous sections. Recall the set Y as defined above Lemma 3.3.21.

Lemma 3.5.8. *The following hold.*

(i) For every element $y = (e_1, \ldots, e_4) \in Y$, there is a unique maximal clique in $\Gamma_{\{-2,0\}}$ containing $e_1, \ldots, e_4$.

(ii) Every maximal clique in $\Gamma_{\{-2,0\}}$ contains exactly 896 distinct subsets of four roots $e_1, \ldots, e_4$ such that $(e_1, \ldots, e_4)$ is an element in Y.

Proof.

(i) From Lemma 3.4.2 it follows that an element in Y is contained in a unique clique of size 8 in $\Gamma_{\{0\}}$. But such a clique extends uniquely to a maximal clique in $\Gamma_{\{-2,0\}}$ by adding all inverses of the roots.

(ii) By Lemma 3.5.7, a maximal clique in $\Gamma_{\{-2,0\}}$ consists of eight pairwise orthogonal roots and their inverses. Let K be such a clique. Eight pairwise orthogonal roots in K contain $\binom{8}{4} - 14 = 56$ distinct subsets of four roots that form an element in Y by Corollary 3.4.3. Let $D = \{e_1, e_2, e_3, e_4\}$ be such a subset. If we replace a root in D by its inverse, then the roots in D still form an element in Y. This gives $56 \cdot 2^4 = 896$ distinct subsets of K of that form an element in Y. Since a set of four roots that contains both a root and its inverse never forms an element in Y, these are all of them. □

Let $\mathcal{S}$ be the set of all cliques of size 16 in $\Gamma_{\{-2,0\}}$. By Lemma 3.5.7, this is exactly the set of maximal cliques in $\Gamma_{\{-2,0\}}$. By Lemma 3.5.8 we have a surjective map

$$s\colon Y \longrightarrow \mathcal{S}.$$

Corollary 3.5.9. *The group W acts transitively on $\mathcal{S}$, and we have $|\mathcal{S}| = 2025$.*

Proof. Since the map s is surjective and W acts transitively on Y (Proposition 3.3.29), it follows from Lemma 3.2.14 that W acts transitively on $\mathcal{S}$. From Lemma 3.5.8 it follows that $|\mathcal{S}| = \frac{|Y|}{|896 \cdot 4!|} = 2025$. □

Let K be an element of $\mathcal{S}$, and W_K its stabilizer in W. Now that we fully described all maximal cliques in $\Gamma_{\{-2,0\}}$ and the action of W on the set of these maximal cliques, we finish the study of $\Gamma_{\{-2,0\}}$ by studying the action of W_K on K, and concluding that W acts transitively on cliques of sizes $6, 7, 8$ in $\Gamma_{\{0\}}$ in Proposition 3.5.15. Consider the sets

$$J_1 = \{(e_1, e_2, e_3) \in K^3 \mid e_1 \cdot e_2 = e_1 \cdot e_3 = e_2 \cdot e_3 = 0\},$$

and

$$J_2 = \{(e_1, e_2) \in K^2 \mid e_1 \cdot e_2 = 0\}.$$

Proposition 3.5.10. *The group W_K acts transitively on J_1.*

Proof. From Lemma 3.5.7 we know that K consists of eight pairwise orthogonal roots and their inverses, so we have $|J_1| = 16 \cdot 14 \cdot 12 = 2688$. Fix an element $\iota = (e_1, e_2, e_3)$ in J_1. We want to show that its orbit $W_K\iota$ has size 2688, hence is equal to J_1. Let $W_{K,\iota}$ be the stabilizer in W_K of ι. We have $|W_K\iota| = \frac{|W_K|}{|W_{K,\iota}|}$, and

$$\frac{|W|}{|W_{K,\iota}|} = \frac{|W|}{|W_K|} \cdot \frac{|W_K|}{|W_{K,\iota}|}.$$

By Corollary 3.5.9 we have $\frac{|W|}{|W_K|} = |WK| = 2025$. Moreover, we have

$$\frac{|W|}{|W_{K,\iota}|} = \frac{|W|}{|W_\iota|} \cdot \frac{|W_\iota|}{|W_{\iota,K}|}.$$

By Proposition 3.3.28 we have $\frac{|W|}{|W_\iota|} = |W\iota| = 240 \cdot 126 \cdot 60 = 1814400$. We now compute $\frac{|W_\iota|}{|W_{\iota,K}|} = |W_\iota K|$. From Proposition 3.3.28 we know that there are 24 roots $e \in E$ such that (e_1, e_2, e_3, e) is an element in Y. Since W_ι acts transitively on those 24 roots by Proposition 3.3.29, the orbit $W_\iota K$ contains the cliques $s((e_1, e_2, e_3, e))$ for all 24 roots e. Now fix e and set $y = (e_1, e_2, e_3, e)$, and $L = s(y)$. From Lemma (i) we know that L contains exactly eight roots f such that (e_1, e_2, e_3, f) is an element in Y. Therefore, they determine the same unique clique of size sixteen as e. We conclude that there are $\frac{24}{8} = 3$ different cliques containing ι. So we have $|W_\iota K| \geq 3$, and we find $\frac{|W|}{|W_{K,\iota}|} \geq 1814400 \cdot 3 = 5443200$. It follows that $\frac{|W_K|}{|W_{K,\iota}|} \geq \frac{5443200}{2025} = 2688$. Since on the other hand we have $\frac{|W_K|}{|W_{K,\iota}|} = |W_K\iota| \leq |J_1| = 2688$, we have equality everywhere and we conclude that $W_K\iota = J_1$. This finishes the proof. $\square$

Corollary 3.5.11. *The group W_K acts transitively on J_2.*

Proof. We have a projection map $\lambda\colon J_1 \longrightarrow J_2$ to the first two coordinates. Since K consists of eight pairwise orthogonal roots and their inverses, if we fix two elements e_1, e_2 such that $(e_1, e_2) \in J_2$, there are $16 - 4 = 12$ elements $e \in K$ such that (e_1, e_2, e) is contained in J_1. Therefore, λ is surjective. From Proposition 3.5.10 and Lemma 3.2.14, it follows that W_K acts transitively on J_2. $\square$

Corollary 3.5.12. *The group W_K acts transitively on K.*

Proof. We have a projection map $\lambda\colon J_2 \longrightarrow K$ to the first coordinate. For every element e in K there are 14 elements c such that $(e,c) \in J_2$, so λ is surjective. From Corollary 3.5.11 and Lemma 3.2.14 it follows that W_K acts transitively on K. $\square$

Proposition 3.5.13. *For $n \in \{2,3,5,6,7,8\}$, the group W acts transitively on the set*

$$D_n = \{\{e_1,\ldots,e_n,-e_1,\ldots,-e_n\} \mid \forall i : e_i \in E;\ \forall i \neq j : e_i \cdot e_j = 0\}\,.$$

Proof. For $n = 2,3,5$, this follows from the fact that W acts transitively on the cliques of size n in $\Gamma_{\{0\}}$ (Propositions 3.4.1 and 3.4.6), and the fact that there is a surjective map from the set of cliques in $\Gamma_{\{0\}}$ of size n to D_n. The case $n = 8$ is Corollary 3.5.9. From Proposition 3.5.11, it follows that the stabilizer W_K in W of K acts transitively on the set

$$\{(e_1,e_2,-e_1,-e_2) \in K^4 \mid e_1 \cdot e_2 = 0\}$$

Since K consists of eight pairwise orthogonal roots and their inverses, the cliques of six pairwise orthogonal roots and their inverses in K are the complements of the cliques of two orthogonal roots and their inverses in K, so this implies that W_K acts transitively on the set of cliques of six pairwise orthogonal roots and their inverses in K, too. The statement now follows for $n = 6$ by Corollary 3.5.9. The case $n = 7$ is proved analogously since we showed that W_K acts transitively on K. $\square$

Remark 3.5.14. There are two orbits under the action of W on the set

$$\{\{e_1,\ldots,e_4,-e_1,\ldots,-e_4\} \mid \forall i : e_i \in E;\ \forall i \neq j : e_i \cdot e_j = 0\}\,.$$

Indeed, this follows from Proposition 3.4.1 and the fact that there is a surjective map from the set of cliques of size 4 in $\Gamma_{\{0\}}$ to this set.

As we mentioned before, the fact that W acts transitively on the set of cliques of size r for $1 \leq r \leq 8$ in $\Gamma_{\{0\}}$ is in [DM10]. The following proposition shows how it follows from our results about $\Gamma_{\{-2,0\}}$ as well.

Proposition 3.5.15. *for $n = 6,7,8$, the group W acts transitively on the cliques of size n in $\Gamma_{\{0\}}$.*

Proof. We know that W acts transitively on the set

$$D_n = \{\{e_1,\ldots,e_n,-e_1,\ldots,-e_n\} \mid \forall i : e_i \in E;\ \forall i \neq j : e_i \cdot e_j = 0\}$$

from Proposition 3.5.13. Let F_n be the set of cliques of size n in $\Gamma_{\{0\}}$. We have an obvious map $f\colon F_n \longrightarrow D_n$ which adds adds the inverses to all roots in an element in F_n. Let $D = \{e_1, \dots e_n, -e_1, \dots, -e_n\}$ be an element in D_n and consider its fiber $f^{-1}(D)$ in F_n. This consists of all cliques $\{\pm e_1, \dots, \pm e_n\}$, where for each root either itself or its inverse is chosen. The stabilizer W_D of D acts on $f^{-1}(D)$. Note that for $i \in \{1, \dots, n\}$, the reflection in the hyperplane orthogonal to e_i switches e_i and $-e_i$ and fixes all other roots in D, hence it is an element in W_D. Therefore, W_D acts transitively on $f^{-1}(D)$, and by Lemma 3.2.14, W acts transitively on F_n. □

Corollary 3.5.16. *The table in Appendix A gives the complete list of orbits of the maximal cliques in $\Gamma_{\{0\}}$ and $\Gamma_{\{-2,0\}}$, as well as a correct representative for each orbit, the size of its stabilizer in W, and the size of its automorphism group.*

Proof. Al maximal cliques in $\Gamma_{\{-2,0\}}$ are of size 16 (Lemma 3.5.7) and there is only one orbit of them, of size 2025 (Corollary 3.5.9). It is an easy check that the clique in the table is a representative of this orbit. Its stabilizer size is $\frac{|W|}{|2025|} = 344064$. Its automorphism group is isomorphic to $\mu_2^8 \rtimes S_8$ by Lemma 3.3.30, hence has size $2^8 \cdot 8!$. In Theorem 3.4.6 we showed that all maximal cliques in $\Gamma_{\{0\}}$ have size 8, and that there are 518400 of them. In Proposition 3.5.15 we showed that W acts transitively on the set of these cliques. Therefore the stabilizer of the clique in the table has size $\frac{|W|}{518400} = 1344$. Its automorphism group is the symmetric group on the 8 vertices. □

We finish this subsection by proving Theorem 3.1.4 for maximal cliques in $\Gamma_{\{-2,0\}}$.

Lemma 3.5.17. *Let K_1 and K_2 be two maximal cliques in $\Gamma_{\{-2,0\}}$, and let $f\colon K_1 \longrightarrow K_2$ be an isomorphism between them. Then f extends to an automorphism of Λ if and only if for every subclique S of four pairwise orthogonal roots in K_1, the image $f(S)$ in K_2 is conjugate to S under the action of W.*

Proof. By Corollary 3.5.9, the group W acts transitively on the set of maximal cliques in $\Gamma_{\{-2,0\}}$. Therefore there is an element α in W such that $\alpha(K_1) = K_2$. So $\alpha^{-1} \circ f$ is an element in the automorphism group $\mathrm{Aut}(K_1)$ of K_1. Of course, f extends to an element in W if and only if $\alpha^{-1} \circ f$

does. Moreover, for every set S of four pairwise orthogonal roots, $f(S)$ and $(\alpha^{-1} \circ f)(S)$ are conjugate. We conclude that we can reduce to the case where $K_1 = K_2$, and f is an element in $\mathrm{Aut}(K_1)$. By Lemma 3.5.7, we can choose a subclique $H = \{e_1, \ldots, e_8\}$ of K_1 of eight pairwise orthogonal roots, such that we have $K_1 = \{e_1, \ldots, e_8, -e_1, \ldots, -e_8\}$. Let $\mathrm{Aut}(H)$ be the automorphism group of H as colored graph, and let $(\mathrm{Aut}(K_1))_H$ be the stabilizer of H in $\mathrm{Aut}(K_1)$. Since for every element $e \in K_1$ we have $e \in H$ or $-e \in H$, an element in $\mathrm{Aut}(H)$ determines a unique element in $(\mathrm{Aut}(K_1))_H$, and conversely, every element in $(\mathrm{Aut}(K_1))_H$, when restricted to H, determines a unique element in $\mathrm{Aut}(H)$. So we have an isomorphism $\varphi\colon \mathrm{Aut}(H) \xrightarrow{\sim} (\mathrm{Aut}(K_1))_H$. Let f be an element in $\mathrm{Aut}(K_1)$. Using Lemma 3.3.30, write $f = a \circ r|_{K_1}$, where a is an element in $\varphi(\mathrm{Aut}(H))$, and r is a composition of reflections r_i in the hyperplanes orthogonal to e_i for certain $i \in \{1, \ldots, 8\}$. By definition, $r|_{K_1}$ extends to the element r in W, and $r(S)$ and S are conjugate for all cliques S of four orthogonal roots, so the statement in the lemma is true for f if and only if it is true for a. Of course, if a extends to an automorphism of Λ, then a and $a(S)$ are conjugate for all subcliques S of K_1 of four orthogonal roots. Conversely, assume that $a(S)$ and S are conjugate for all such S. Then in particular, for every subclique S' of size 4 in H, the sets $a|_H(S')$ and S' are conjugate. From Corollary 3.4.5 it follows that $a|_H$ extends to an element in W. Write w for an element in W with $w|_H = a|_H$. Then $w|_{K_1}$ and a are both elements in $(\mathrm{Aut}(K_1))_H$, that are identical on H, hence also on K_1. We conclude that $w|_{K_1}$ and a are the same, so a extends to $w \in W$. This finishes the proof. $\square$

3.5.3 Maximal cliques in $\Gamma_{\{-1,0\}}$

Consider the following twelve elements in E.

$t_1 = (1,1,0,0,0,0,0,0)$; $\quad t_7 = \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$

$t_2 = (0,0,1,1,0,0,0,0)$; $\quad t_8 = \left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$

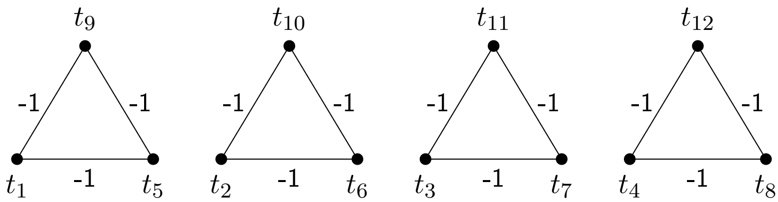
$t_3 = (0,0,0,0,1,1,0,0)$; $\quad t_9 = \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$t_4 = (0,0,0,0,0,0,-1,1)$; $\quad t_{10} = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$

$t_5 = \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$; $\quad t_{11} = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$t_6 = \left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$; $\quad t_{12} = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$

One can easily check that these twelve elements form a clique in $\Gamma_{\{-1,0\}}$, depicted below (where edges of color 0 are not drawn). We call this clique T.



The existence of this clique implies that the maximal size of cliques in $\Gamma_{\{-1,0\}}$ is at least twelve. We will show that this is in fact the maximum. Moreover, we will show that all cliques of size twelve in $\Gamma_{\{-1,0\}}$ are isomorphic, and that W acts transitively on the set of cliques of size twelve (Propositions 3.5.23 and 3.5.24). To describe all maximal cliques of smaller size in $\Gamma_{\{-1,0\}}$ and their orbits under the action of W, we use `magma` for part of the computations.

Lemma 3.5.18. *Take $e_1, e_2, e_3 \in E$ with $e_1 \cdot e_2 = e_2 \cdot e_3 = e_1 \cdot e_3 = -1$. For $e \in E$ with $e \neq e_1, e_2, e_3$, we have $e \cdot e_i \neq 1$ for all $i = 1, 2, 3$ if and only if $e \cdot e_1 = e \cdot e_2 = e \cdot e_3 = 0$.*

Proof. Take $e_1, e_2, e_3 \in E$ with $e_1 \cdot e_2 = e_2 \cdot e_3 = e_1 \cdot e_3 = -1$. Then we have $\|e_1 + e_2 + e_3\| = 0$, so $e_1 + e_2 + e_3 = 0$. For an element $e \in E$ with $e \neq e_1, e_2, e_3$ we have $e \cdot e_i \in \{-2, -1, 0, 1\}$ for $i = 1, 2, 3$, so $e \cdot e_i \neq 1$ for $i = 1, 2, 3$ implies $e \cdot e_i \leq 0$ for $i = 1, 2, 3$. But $e \cdot (e_1 + e_2 + e_3) = e \cdot 0 = 0$, so we have $e \cdot e_i \neq 1$ for $i = 1, 2, 3$ if and only if $e \cdot e_i = 0$ for $i = 1, 2, 3$. $\square$

Lemma 3.5.19. *The maximum size of a clique in $\Gamma_{\{-1,0\}}$ that contains $e_1, e_2, e_3 \in E$ with $e_1 \cdot e_2 = 0$ and $e_1 \cdot e_3 = e_2 \cdot e_3 = -1$, is ten.*

Proof. Define elements $e_1 = (1, 1, 0, 0, 0, 0, 0, 0)$, $e_2 = (0, 0, 1, 1, 0, 0, 0, 0)$, and $e_3 = (-1, 0, -1, 0, 0, 0, 0, 0)$. By Lemma 3.3.33, it is enough to prove that the maximal size of all cliques in $\Gamma_{\{-1,0\}}$ containing e_1, e_2, e_3 is ten. Let A be the set

$$\{e \in E \mid \text{for } i \in \{1, 2, 3\} : e \cdot e_i \in \{-1, 0\}\}.$$

For an element $e = (a_1 \ldots, a_8)$ in A, we have $a_1 + a_2$ in $\{-1, 0\}$, $a_3 + a_4$ in $\{-1, 0\}$, and $-a_1 - a_3$ in $\{-1, 0\}$. This gives the following possibilities for (a_1, a_2, a_3, a_4):

$$\begin{aligned}
(a_1, a_2, a_3, a_4) = &\left(-\tfrac{1}{2}, \pm\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}\right) && \text{(16 roots)}\\
&\left(\tfrac{1}{2}, -\tfrac{1}{2}, \pm\tfrac{1}{2}, -\tfrac{1}{2}\right) && \text{(16 roots)}\\
&\left(\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}\right) && \text{(8 roots)}\\
&(0, -1, 0, -1) && \text{(1 roots)}\\
&(0, 0, 1, -1) && \text{(1 root)}\\
&(1, -1, 0, 0) && \text{(1 root)}\\
&(0, -1, 0, 0) && \text{(8 roots)}\\
&(0, 0, 0, -1) && \text{(8 roots)}\\
&(0, 0, 0, 0) && \text{(24 roots)}
\end{aligned}$$

We conclude that the cardinality of A is 83. As it is too tedious to compute the maximal size of the cliques in $\Gamma_{\{-1,0\}}$ with only vertices in A by hand, we compute this with `magma`. This number is seven, which implies that the maximal size of a clique in $\Gamma_{\{-1,0\}}$ containing e_1, e_2 and e_3 is ten. □

Lemma 3.5.20. *The maximum size of a clique in $\Gamma_{\{-1,0\}}$ that contains a clique of five pairwise orthogonal vertices is ten.*

Proof. Consider the set

$$V_5 = \{\{e_1, \ldots, e_5\} \mid \forall i : e_i \in E;\ \forall i \neq j : e_i \cdot e_j = 0\}.$$

The group W acts transitively on V_5 by Theorem 3.4.6, so it suffices to take

$$\begin{aligned}
e_1 &= (1, 1, 0, 0, 0, 0, 0, 0); & e_4 &= (0, 0, 0, 0, 0, 0, 1, 1);\\
e_2 &= (0, 0, 1, 1, 0, 0, 0, 0); & e_5 &= (0, 0, 0, 0, 0, 0, 1, -1),\\
e_3 &= (0, 0, 0, 0, 1, 1, 0, 0); & &
\end{aligned}$$

and show that a clique in $\Gamma_{\{-1,0\}}$ containing $e_1, \ldots, e_5$ has size at most ten. Let A be the set

$$\{e \in E \mid \text{for } i \in \{1, \ldots, 5\} : e \cdot e_i \in \{-1, 0\}\}.$$

For an element $e = (a_1, \ldots, a_8) \in A$, we have $a_i + a_{i+1} \in \{-1, 0\}$ for $i \in \{1, 3, 5, 7\}$, and $a_7 - a_8 \in \{-1, 0\}$. If e is of the form $\left(\pm\frac{1}{2}, \ldots, \pm\frac{1}{2}\right)$,

then $a_7 + a_8,\ a_7 - a_8 \in \{-1, 0\}$ implies that $a_7 = -\frac{1}{2}$. Moreover, for $i \in \{1, 3, 5\}$, we have either $a_i = a_{i+1} = -\frac{1}{2}$ or $a_i = -a_{i+1}$. This gives three possibilities for each tuple (a_i, a_{i+1}) for $i \in \{1, 3, 5\}$, and a_8 is then determined since an even number of the entries of e should be negative. We find $3^3 = 27$ possibilities.
If e has two non-zero entries that are ± 1, then $a_7 + a_8,\ a_7 - a_8 \in \{-1, 0\}$ implies that either $(a_7, a_8) = (-1, 0)$, or $(a_7, a_8) = (0, 0)$. Moreover, for $i \in \{1, 3, 5\}$ we have $\{a_i, a_{i+1}\} = \{-1, 0\}$ or $\{a_i, a_{i+1}\} = \{-1, 1\}$. It is easy to check that this gives 24 possibilities.
We find that the cardinality of A is 51. As it is too tedious to compute the maximal size of the cliques in $\Gamma_{\{-1,0\}}$ with all vertices in A by hand, we compute this with `magma`. The maximal size of a clique in $\Gamma_{\{-1,0\}}$ with all vertices in A is five, so the maximal size of a clique in $\Gamma_{\{-1,0\}}$ containing $e_1, \dots, e_5$ is ten. □

We recall some known Ramsey numbers.

Theorem 3.5.21. *(Ramsey Numbers). For two integers l, k, let $R(l, k)$ be the least positive integer n such that every undirected graph with n vertices contains either a clique of size four or an independent set of size five. Then we have $R(3, 3) = 6$, $R(3, 4) = 9$, and $R(4, 5) = 25$.*

Proof. See [GRS90, Table 4.1] for $R(3, 3)$ and $R(3, 4)$, and [MR95] for $R(4, 5)$. □

Proposition 3.5.22. *Every clique in $\Gamma_{\{-1,0\}}$ of size bigger than ten contains a subclique of size four with only edges of color 0.*

Proof. Let K be a clique in $\Gamma_{\{-1,0\}}$ of size bigger than ten. Consider the subgraph K' of K whose vertex set consists of all vertices of K, and whose edge set is obtained by taking only the edges in K of color -1. We consider different cases depending on the number of connected components of K'.
If K' has at least four connected components, then we can take four vertices, each from a different connected component, and these vertices form a clique of size four with only edges of color 0 in K.
Now assume that K' has at most three connected components. We first show that every connected component of K' that contains a clique of size three is a clique of size three in itself. To this end, assume that K' contains a clique of size three, given by $\{e_1, e_2, e_3\}$. By Lemma 3.3.9, we have $e_1 + e_2 + e_3 = 0$. If e is another vertex of K', then $e \cdot e_i \in \{-1, 0\}$ for

$i \in \{1,2,3\}$, and $e \cdot (e_1 + e_2 + e_3) = 0$, from which it follows that $e \cdot e_i = 0$ for $i \in \{1,2,3\}$. We conclude that the vertices e_1, e_2, e_3 form a connected component of K'. Since there are at most three connected components by assumption, and K' has more than ten vertices, we conclude that not all components contain a clique of size three. Now remove a vertex from every connected component in K' that is a clique of size three (of which there are at most two), then we are left with a subgraph of K' with at least 9 vertices, and no cliques of size three left. Hence by Theorem 3.5.21, there must be a set of four vertices that are pairwise disjoint in K', meaning that they form a clique with edges of color 0 in K. □

Let V_3, V_4, Z, α, π and Y be as in the diagram above Lemma 3.3.24.

Proposition 3.5.23. *The following hold.*

(i) Let $v = (e_1, e_2, e_3, e_4)$ be an element in V_4. Then e_1, e_2, e_3 and e_4 are contained in a clique of size bigger than ten in $\Gamma_{\{-1,0\}}$ if and only if v is an element of Y.

(ii) Every maximal clique of size at least eleven in $\Gamma_{\{-1,0\}}$ is of the form

$$\left\{ \left\{ \begin{array}{c} e_1, \ldots, e_4, \\ f_1, \ldots, f_4, \\ -e_1 - f_1, \ldots, -e_4 - f_4 \end{array} \right\} \middle| \begin{array}{c} \forall i \neq j : e_i \cdot e_j = f_i \cdot f_j = 0; \\ \forall i : e_i \cdot f_i = -1; \\ \forall i \neq j : e_i \cdot f_j = 0. \end{array} \right\}.$$

(iii) The maximal size of a clique in $\Gamma_{\{-1,0\}}$ is twelve, and there are no maximal cliques of size eleven in $\Gamma_{\{-1,0\}}$.

(iv) For an element $v \in Y$, there are eight cliques of size twelve in $\Gamma_{\{-1,0\}}$ containing the elements of v.

(v) For K a clique of size twelve in $\Gamma_{\{-1,0\}}$, we have

$$|K^4 \cap V_4| = |K^4 \cap Y| = 1944.$$

Proof. Let K be a clique of size bigger than ten in $\Gamma_{\{-1,0\}}$. We know that K contains a subclique of size four with only edges of color 0 from Proposition 3.5.22. Let $\{e_1, e_2, e_3, e_4\}$ be such a subclique in K. Let e be another element in K. By Lemmas 3.5.19 and 3.5.20, there is exactly one i in $\{1,2,3,4\}$ such that $e \cdot e_i = -1$, and $e \cdot e_j = 0$ for $i \neq j \in \{1,2,3,4\}$. It follows that $e \cdot (e_1 + e_2 + e_3 + e_4) = -1$, hence $\sum_{i=1}^{4} e_i \notin 2\Lambda$. It follows from Proposition 3.3.29 that (e_1, e_2, e_3, e_4) is an element in Y. Conversely,

the tuple (t_1, t_2, t_3, t_4) is an element in Y and it is contained in the clique T (page 96), so by Proposition 3.3.29, every element in Y is contained in a clique of size twelve in $\Gamma_{\{-1,0\}}$. This proves (i).
Recall the clique T defined above Lemma 3.5.18. We define the following sets for $i \in \{1, 2, 3, 4\}$.

$$F_i = \left\{ e \in E \;\middle|\; \begin{array}{c} e \cdot t_i = -1, \\ e \cdot t_j = 0 \text{ for } j \in \{1,2,3,4\},\ j \neq i \end{array} \right\}.$$

Let K be a clique in $\Gamma_{\{-1,0\}}$ of size at least eleven. Such a K exists, since the clique T is an example. By Proposition 3.5.22, the clique K contains four vertices that form an element of V_4, and by part (i) this is an element of Y. By Proposition 3.3.29 we can without loss of generality assume that K contains the four vertices t_1, t_2, t_3, t_4. By Lemma 3.5.19 and Lemma 3.5.20, for every element t in $K \setminus \{t_1, t_2, t_3, t_4\}$ there is an $i \in \{1, 2, 3, 4\}$ such that $t \cdot t_i = -1$ and $t \cdot t_j = 0$ for $i \neq j \in \{1, 2, 3, 4\}$. Therefore we have

$$K \setminus \{t_1, t_2, t_3, t_4\} = \bigcup_{i \in \{1,2,3,4\}} K \cap F_i.$$

Fix $i \in \{1, 2, 3, 4\}$. For an element $f \in F_i$ we have $f \cdot t_i = -1$, so by Lemma 3.3.9 there is a unique element $g \in E$ with $f \cdot g = t_i \cdot g = -1$, given by $g = -t_i - f$. Note that this element is also in F_i, since $(-t_i - f) \cdot t_j = 0$ for $j \in \{1, 2, 3, 4\}$ with $j \neq i$. So for $i \in \{1, 2, 3, 4\}$, the set F_i is the union of different sets $\{f, -t_i - f\}$, and we claim that $K \cap F_i$ is contained in one of these sets. To prove this, fix i and $f \in K \cap F_i$. Assume by contradiction that there is an element $h \in (K \cap F_i) \setminus \{f, -t_i - f\}$. Then h is in F_i, so $h \cdot f \neq -1$ by uniqueness of g. But h, f are both elements in K, so this implies $h \cdot f = 0$. But then we have $h \cdot t_i = f \cdot t_i = -1$ and $h \cdot f = 0$, so by Lemma 3.5.19, the clique K has size at most ten, which gives a contradiction. So for $i \in \{1, 2, 3, 4\}$, there are $f_i \in F_i$ such that $K \cap F_i \subseteq \{f_i, -t_i - f_i\}$, and we have

$$K \subseteq \bigcup_{i \in \{1,2,3,4\}} \{t_i, f_i, -t_i - f_i\}.$$

Fix such $f_i \in F_i$ for $i \in \{1, 2, 3, 4\}$. Note that for $i \neq j \in \{1, 2, 3, 4\}$ we have $f_i \cdot f_j = 0$, because if this were not the case then K would contain a triple t_i, f_i, f_j with $t_i \cdot f_i = f_i \cdot f_j = -1$, $f_j \cdot t_i = 0$, which contradicts the fact that K has size bigger than ten by Lemma 3.5.19.

Hence $\bigcup_{i\in\{1,2,3,4\}}\{t_i, f_i, -t_i - f_i\}$ forms a clique in $\Gamma_{\{-1,0\}}$ of the required form, and if K is maximal, it is equal to this clique. This proves part (ii), and part (iii) follows directly.
We proceed by proving (iv). Note that (t_1, t_2, t_3, t_4) is an element in Y. We count the number of cliques of size twelve in $\Gamma_{\{-1,0\}}$ containing $t_1, \ldots, t_4$. By (ii), we know that such a clique is of the form $\bigcup_{i\in\{1,2,3,4\}}\{t_i, f_i, -t_i-f_i\}$, where f_i and $-t_i - f_i$ are elements in F_i for $i \in \{1, 2, 3, 4\}$. By simply considering all elements in E we find

$$F_1 = \left\{ \left(-\tfrac{1}{2}, -\tfrac{1}{2}, a_3, a_4, a_5, a_6, a_7, a_8\right) \;\middle|\; \begin{array}{c} \{a_3, a_4\} = \left\{-\tfrac{1}{2}, \tfrac{1}{2}\right\}, \\ \{a_5, a_6\} = \left\{-\tfrac{1}{2}, \tfrac{1}{2}\right\}, \\ a_7 = a_8 \end{array} \right\}.$$

Since $|F_1| = 8$, there are four choices for the set $\{f_1, -t_1 - f_1\}$. Fix f_1, and write $f_1 = \left(-\frac{1}{2}, -\frac{1}{2}, a_3, \ldots, a_8\right)$. Then f_2, $-t_2 - f_2$ are elements in F_2 that are orthogonal to f_1 by (ii). Again, by considering all elements in E we find

$$F_2 = \left\{ \left(b_1, b_2, -\tfrac{1}{2}, -\tfrac{1}{2}, b_5, b_6, b_7, b_8\right) \;\middle|\; \begin{array}{c} \{b_1, b_2\} = \left\{-\tfrac{1}{2}, \tfrac{1}{2}\right\}, \\ \{b_5, b_6\} = \left\{-\tfrac{1}{2}, \tfrac{1}{2}\right\}, \\ b_7 = b_8 \end{array} \right\}.$$

Let $f = (b_1, \ldots, b_8)$ be an element in F_2. Then f is orthogonal to f_1 if and only if $0 = \sum_{i=5}^{8} a_i b_i = 2(a_5 b_5 + a_7 b_7)$, which holds if and only if $\frac{b_5}{b_7} = -\frac{a_7}{a_5}$. This gives two choices for the tuple (b_5, b_7), and together with the two choices for (b_1, b_2) we find four elements in F_2 that are orthogonal to f_1. This gives two choices for the set $\{f_2, -t_2 - f_2\}$. Fix one. Then $f_3, -t_3 - f_3$, and $f_4, -t_4 - f_4$, are elements in F_3 and F_4 respectively, that are orthogonal to f_1 and f_2. It is an easy check that this determines the sets $\{f_3, -t_3 - f_3\}$ and $\{f_4, -t_4 - f_4\}$ uniquely. So for f_1 we had four choices, for f_2 we had two, and the set $\{f_3, -t_3 - f_3, f_4, -t_4 - f_4\}$ is determined after choosing f_1, f_2. We conclude that there are $4 \cdot 2 = 8$ cliques of size twelve in $\Gamma_{\{-1,0\}}$ containing $t_1, \ldots, t_4$. By Proposition 3.3.29, this holds for every element in Y. This proves (iv).
Let K be a clique of size twelve in $\Gamma_{\{-1,0\}}$. Using the notation in (ii), write

$$K = \{e_1, \ldots, e_4, f_1, \ldots, f_4, -e_1 - f_1, \ldots, -e_4 - f_4\}.$$

It follows from (ii) that the sets of four pairwise orthogonal roots in K are given by

$$\{\{a_1, a_2, a_3, a_4\} \mid a_i \in \{e_i, f_i, -e_i - f_i\} \text{ for } i \in \{1, 2, 3, 4\}\}.$$

This gives $3^4 = 81$ such sets, and these give rise to $81 \cdot 4! = 1944$ elements in $K^4 \cap V_4$. From (i) it follows that $K^4 \cap V_4 = K^4 \cap Y$. This proves (v). $\square$

Proposition 3.5.24. *Let $\mathcal{T}$ be the set of all cliques of size twelve in $\Gamma_{\{-1,0\}}$, and R an element in $\mathcal{T}$. The following hold.*

(i) We have $|\mathcal{T}| = 179200$, and the group W acts transitively on $\mathcal{T}$.

(ii) The stabilizer W_R in W of R acts transitively on $R^4 \cap Y$.

Proof. Let T be the clique $\{t_1, \ldots, t_{12}\}$, as defined above Lemma 3.5.18. Define the set

$$S = \{((e_1, e_2, e_3, e_4), K) \in Y \times \mathcal{T} \mid e_1, \ldots, e_4 \in K\}.$$

We have projections $\lambda\colon S \longrightarrow Y$ and $\mu\colon S \longrightarrow \mathcal{T}$.
From the previous proposition we know that the fibers of λ have cardinality 8, and the fibers of μ have cardinality 1944. Therefore we have $|S| = |Y| \cdot 8 = 348364800$ (Proposition 3.3.29), and $|\mathcal{T}| = \frac{|S|}{1944} = 179200$. We will show that W acts transitively on S, which implies that it acts transitively on $\mathcal{T}$ by the projection μ. Consider the clique $T \in \mathcal{T}$, and set $y = (t_1, t_2, t_3, t_4) \in T^4 \cap Y$. Then (y, T) is in the fiber of λ above y. The stabilizer W_y in W of y acts on this fiber. We show that this action is transitive, that is, that the orbit W_yT is equal to the whole fiber. We have $|W_yT| = \frac{|W_y|}{|W_{y,T}|}$, and $|W_y| = \frac{|W|}{|Wy|} = \frac{|W|}{|Y|} = 16$. Note that t_1, t_2, t_3, t_4 are all orthogonal to the four roots

$$e_1 = (1, -1, 0, 0, 0, 0, 0, 0), \qquad e_2 = (0, 0, 1, -1, 0, 0, 0, 0),$$
$$e_3 = (0, 0, 0, 0, 1, -1, 0, 0), \qquad e_4 = (0, 0, 0, 0, 0, 0, 1, 1).$$

Therefore, for $i \in \{1, 2, 3, 4\}$, the reflection r_i in the hyperplane orthogonal to e_i is contained in the stabilizer W_y. Since the subgroup generated by these four reflections has cardinality 16, we conclude that this is the whole group W_y. We can now compute that for every element r in W_y we have $rT \neq T$, except for the identity and the composition of all four reflections r_1, r_2, r_3, r_4. So $|W_{y,T}| = 2$, and we have $|W_yT| = \frac{|W_y|}{|W_{y,T}|} = \frac{16}{2} = 8$. Since the fiber of λ above y has cardinality 8, we conclude that W_y acts transitively on this fiber. Since W acts transitively on Y, we conclude from Lemma 3.2.14 that W acts transitively on S. Finally, from the surjective projection μ and Lemma 3.2.14, it follows that W acts transitively on $\mathcal{T}$. This proves (i). Since W acts transitively on S, the stabilizer W_R in W

of the clique R acts transitively on the fiber $\mu^{-1}(R)$. Since there is a bijection $\mu^{-1}(R) \longrightarrow R^4 \cap Y$ given by the projection λ, the group W acts transitively on $R^4 \cap Y$ by Lemma 3.2.14. This proves (ii). $\square$

Corollary 3.5.25. *Let R be a clique of size twelve in $\Gamma_{\{-1,0\}}$. Let W_R be its stabilizer in W. Then W_R acts transitively on R.*

Proof. We have a surjective map $R^4 \cap Y \longrightarrow R$ projecting on the first coordinate, so this follows from Lemma 3.2.14 and the previous proposition. $\square$

Now that we described all the largest cliques (with respect to number of vertices) in $\Gamma_{\{-1,0\}}$, we continue to describe all other maximal cliques. Since the size of the stabilizer of a clique is the same for every two cliques that are in the same orbit, we make the following definition.

Definition 3.5.26. The *stabilizer size* of an orbit is the size of the stabilizer of any of the elements in the orbit.

As one can see in the table in Appendix A, for a set c that contains 0 in combination with either -1 or 1, there are many maximal cliques in Γ_c with small stabilizer sizes, which means large orbits. This means that, even though we use `magma` to find all cliques and orbits, computations can become very large and time consuming. Therefore we use the following lemma throughout.

Lemma 3.5.27. *Let H be a finite group acting on a finite set X and consider its induced action on the power set of X. Let A and S be subsets of X and let m denote the number of H-conjugates of A that are contained in S. Then the number of H-conjugates of S that contain A equals*

$$\frac{m \cdot |H_A|}{|H_S|},$$

where H_A and H_S denote the stabilizer subgroups of A and S, respectively.

Proof. Let Z denote the H-subset of the product $HA \times HS$ consisting of all pairs (B, T) with $B \in HA$ and $T \in HS$ satisfying $B \subset T$. The group H acts transitively on the codomains of the projection maps $\pi\colon Z \to HA$ and $\rho\colon Z \to HS$. This implies that all fibers of π have the same size, say r, as the fiber above A, which is the number of H-conjugates of S that contain A, that is, the number that we are looking for. All fibers of ρ have the same size as the fiber above S, which equals m. Hence, we can express the size of Z as both $|HA| \cdot r$ and $|HS| \cdot m$. Since the orbits HA and HS have size $|H|/|H_A|$ and $|H|/|H_S|$, respectively, we find

$$r = \frac{m \cdot |HS|}{|HA|} = \frac{m \cdot |H_A|}{|H_S|}. \qquad \square$$

Note that for $A = \emptyset$, we recover the well-known fact that the length of the orbit of S equals the index $[H : H_S]$.

The following proposition describes all maximal cliques and their orbits in $\Gamma_{\{-1,0\}}$.

Proposition 3.5.28. *For two maximal cliques K_1 and K_2 of the same size in $\Gamma_{\{-1,0\}}$, the following are equivalent.*

(i) K_1 and K_2 are conjugate under the action of W.

(ii) K_1 and K_2 are isomorphic.

(iii) K_1 and K_2 have the same stabilizer size.

(iv) The automorphism groups of K_1 and K_2 have the same cardinality, and, if this cardinality is 16 and K_1 and K_2 have size 9, then K_1 and K_2 both contain a monochromatic clique of size 7 and color 0, or they both do not.

Moreover, the table in Appendix A gives a complete list of representatives of the orbits of the maximal cliques in $\Gamma_{\{-1,0\}}$, as well as for each representative its stabilizer size and the size of its automorphism group.

Proof. The implications (i)⇒(ii), (i)⇒(iii), (i) ⇒(iv), and (ii)⇒(iv) are immediate. We will show (iii)⇒(i) and (iv)⇒(i), which together with the immediate implications prove all equivalences. To this end, we first show that the table is complete and correct as stated. From Propositions 3.5.23 and 3.5.24 we know that the maximal size of all cliques in $\Gamma_{\{-1,0\}}$ is twelve, that there are 179200 cliques of size twelve, and that these cliques form one orbit under the action of W, proving the equivalences for K_1, K_2 of size at least 12. The clique of size 12 in the table is the clique T that is defined above Lemma 3.5.18. The size of its stabilizer in W is $\frac{|W|}{179200} = 3888$. From the description of T we see that its automorphism group is isomorphic to the semidirect product $S_3^4 \rtimes S_4$, where S_4 acts on S_3^4 by permuting the four coordinates. This group has order $6^4 \cdot 24 = 31104$.

To find maximal cliques in $\Gamma_{\{-1,0\}}$ of size smaller than 12, note that there are no maximal cliques in $\Gamma_{\{-1,0\}}$ of size 11 by Proposition 3.5.23, so we only have to look at the cliques of size at most ten. To make computations easier, we first show that every maximal clique in $\Gamma_{\{-1,0\}}$ contains at least one edge of color 0. We know that the only maximal cliques in $\Gamma_{\{-1\}}$ are the cliques of size three. Define the three elements $e_1 = (1,1,0,0,0,0,0,0,0,0)$, $e_2 = (-1,0,1,0,0,0,0,0)$, and $e_3 = (0,-1,-1,0,0,0,0,0)$, then $\{e_1,e_2,e_3\}$ is a maximal clique in $\Gamma_{\{-1\}}$. Note that for $e_4 = (0,0,0,0,0,0,1,1)$, the set $\{e_1,e_2,e_3,e_4\}$ forms a clique in $\Gamma_{\{-1,0\}}$, hence $\{e_1,e_2,e_3\}$ is not a maximal clique in $\Gamma_{\{-1,0\}}$. Since the group W acts transitively on the set of maximal cliques in $\Gamma_{\{-1\}}$ (Corollary 3.3.10), it follows that all maximal cliques in $\Gamma_{\{-1\}}$ are not maximal in $\Gamma_{\{-1,0\}}$. Thus we can assume that the maximal cliques in $\Gamma_{\{-1,0\}}$ contain at least one pair of orthogonal roots. Fix the two roots $c_1 = (1,1,0,0,0,0,0,0)$, $c_2 = (0,0,1,1,0,0,0,0)$. Since W acts transitively on the pairs of orthogonal roots, every maximal clique in $\Gamma_{\{-1,0\}}$ is conjugate to a clique containing c_1, c_2, so by considering only the maximal cliques in $\Gamma_{\{-1,0\}}$ that contain c_1 and c_2, we find representatives for all orbits of the maximal cliques in $\Gamma_{\{-1,0\}}$ under the action of W. This reduces computations, since there are only 136 roots that have dot product -1 or 0 with both c_1 and c_2, which is quickly computed with `magma`, as well as the number of maximal cliques containing c_1, c_2. We find the following.

r	Number of maximal cliques of size r in $\Gamma_{\{-1,0\}}$ containing c_1 and c_2
≤ 7	0
8	261600
9	2779392
10	228408

We now turn to the table in the appendix. One can easily check with `magma` that the sets in the table for $\Gamma_{\{-1,0\}}$ are indeed maximal cliques in $\Gamma_{\{-1,0\}}$; in Remark 3.5.29. For each of these cliques we compute the automorphism groups with `magma`. We see that apart from the cliques

$$L_1 = \{19, 41, 48, 50, 65, 150, 172, 214, 240\}$$

and

$$L_2 = \{41, 48, 50, 55, 65, 78, 178, 214, 240\}$$

of size 9, which both have an automorphism group of size 16, every two cliques of the same size in the table have a different automorphism group. One can check that L_2 contains a subclique with only edges of color zero of size 7, and L_1 does not, so L_1 and L_2 are not isomorphic. This shows that any two cliques of the same size in the table are not isomorphic, and therefore not conjugate.

We claim that every maximal clique in $\Gamma_{\{-1,0\}}$ is conjugate to one of these cliques in the table. To this end, set $A = \{c_1, c_2\}$, and let W_A be the stabilizer of A in W. From Proposition 3.4.1 it follows that $|W_A| = \frac{|W|}{|WA|} = \frac{|W|}{15120} = 46080$. We now show how to proceed for the cliques of size 8, the proof for sizes 9 and 10 goes completely analogously. For each of the five cliques of size 8 in the table we compute the size of its stabilizer (144,128,16,14, and 8) and the number of conjugates of A contained in it (21,20,20,21, and 21, respectively), with `magma`. Lemma 3.5.27 now gives us the number of conjugates of each clique that contain A. This sums up to the number 261600 we find in the table above, proving our claim.

We have showed that the table in the appendix gives exactly one representative for each orbit of the maximal cliques in $\Gamma_{\{-1,0\}}$, so K_1 and K_2 are both conjugate to an element in the table. If either (iii) or (iv) holds, then by looking at the table we see that this implies that K_1 and K_2 are conjugate to the same clique in the table, and in particular, they are conjugate to each other, implying (i). This finishes the proof. $\square$

Remark 3.5.29. In the proof of Proposition 3.5.28 we found 261600 cliques of size 8 in $\Gamma_{\{-1,0\}}$ containig both $c_1 = (1,1,0,0,0,0,0,0)$ and $c_2 = (0,0,1,1,0,0,0,0)$. One can check for any two of them whether they are conjugate with `magma`, but this takes a very long time. To reduce computations, we first sort the cliques by size of their stabilizer. We then go through each set of cliques with the same stabilizer size by taking one clique, and removing all cliques that are conjugate to it from the set.

3.5.4 Maximal cliques of other colors

In this subsection we prove Theorem 3.1.3 and 3.1.4 for all maximal cliques in Γ_c with $c \in \{\{-1,1\},\{-2,-1,1\},\{0,1\},\{-2,-1,0\},\{-2,0,1\}\}$. We make use of `magma` in all cases. The following lemma deals with the cases for which this is straightforward.

Lemma 3.5.30. *For $c \in \{\{-1,1\},\{-2,-1,1\}\}$, and for two maximal* *cliques K_1 and K_2 of the same size in Γ_c, the following are equivalent.*

(i) K_1 and K_2 are conjugate under the action of W.

(ii) K_1 and K_2 are isomorphic.

(iii) K_1 and K_2 have the same stabilizer size.

(iv) The automorphism groups of K_1 and K_2 have the same cardinality.

Moreover, for $c \in \{\{-1,1\},\{-2,-1,1\}\}$, the table in Appendix A gives a complete list of representatives of the orbits of maximal cliques in Γ_c, as well as for each representative its stabilizer size and the size of its automorphism group.

Proof. In these two graphs there are not so many maximal cliques, and we can ask `magma` to compute them, compute the orbits under the action of W, and a representative of each orbit directly. The results are in the table. The size of the stabilizers is found by dividing the order of W by the size of the orbit. The automorphism group of the cliques is also easily found with `magma`. Since cliques of the same size in the table have automorphism groups of different size, they are not isomorphic. The equivalence of the statements (i), (ii), (iii), and (iv) now follows from the table. $\square$

Corollary 3.5.31. *For $c \in \{\{-1,1\},\{-2,-1,1\}\}$, let K_1 and K_2 be two maximal cliques in Γ_c, and $f\colon K_1 \longrightarrow K_2$ an isomorphism between them. Then f extends to an automorphism of Λ.*

Proof. Since K_1 and K_2 are isomorphic, from Lemma 3.5.30 it follows that they are both conjugate to the same clique in the table in de appendix; call this clique H. Then there are elements α, β in W such that we have $\alpha(K_1) = \beta(K_2) = H$. So $\beta \circ f \circ \alpha^{-1}$ is an element in the automorphism group $\mathrm{Aut}(H)$ of H. Of course, f extends to an element in W if and only if $\beta \circ f \circ \alpha^{-1}$ does. We conclude that we can reduce to the case where $K_1 = K_2 = H$, and f is an element in $\mathrm{Aut}(H)$.
For each clique H in the table, we construct the map $W_H \longrightarrow \mathrm{Aut}(H)$ from the stabilizer W_H to the automorphism group $\mathrm{Aut}(H)$ given by restriction in `magma`. For all these cliques, this is a surjective map. It follows that every element in $\mathrm{Aut}(H)$ extends to an element in W. $\square$

The final three cases are much more work, because of the large numbers of maximal cliques and their sizes. The most extreme case is that of maximal cliques of size 29 in $\Gamma_{\{0,1\}}$ and $\Gamma_{\{-2,0,1\}}$; we will treat this separately in Section 3.5.4.

Remark 3.5.32. Recall that the classification of isomorphism classes of maximal cliques in $\Gamma_{\{0,1\}}$ has already been done in [CRS04], where the authors classify all maximal exceptional graphs (Remark 3.1.7). We compare their methods to ours. For maximal cliques in $\Gamma_{\{0,1\}}$ of size unequal to 29, we find the different isomorphism types by showing that each such clique contains a pair of orthogonal roots, fixing a pair (e_1, e_2) of orthogonal roots, and using `magma` to compute the set of all maximal cliques in $\Gamma_{\{0,1\}}$ of size unequal to 29 that contain e_1 and e_2. We cut this set op into smaller sets based on the stabilizer size of the cliques, and in each smaller set we compute with `magma` the different orbits under the action of W. It turns out that each orbit is also a full isomorphism class, and that for each clique K, both the combination of the stabilizer size with the number of pairs or inverse roots contained in K, as well as the combination of the cardinality of the automorphism group with the number of pairs or inverse roots contained in K, are invariants that determine the isomorphism type of K (Proposition 3.5.35). For the maximal cliques of size 29 we do a similar computation: we show that each maximal clique of size 29 contains a monochromatic 5-clique of color 0, or a monochromatic 4-clique of color 1 for which the sum of the corresponding root is a double root in Λ, or a monochromatic 4-clique of color 1 for which this sum is not a double root in Λ. We fix one clique for each of these three types, and use `magma` to compute the set of all maximal cliques in $\Gamma_{\{0,1\}}$ of size 29 that contain at least one of these fixed cliques. We then cut this big set up in smaller sets using for each clique K the stabilizer size and the number of maximal monochromatic subcliques of color 1 of size r, for all $r \in \{1, \dots, 8\}$, that are contained in K. Each smaller set turns out to be an orbit under the action of W, as well as a full isomorphism class (Proposition 3.5.36).
In [CRS04], the authors use a different way to search for all maximal exceptional graphs. They prove that every exceptional graph arises as an extension of an *exceptional star-complement*, and construct a list of 443 graphs that arise as the exceptional star complements for maximal exceptional graphs. In [CRS04, Chapter 6], the authors find all maximal exceptional graphs with a computer search, by extending each of the 443 exceptional star complements. Since an exceptional graph can arise as extensions of different star complements, or as different extensions from the same star complement, being an extension of a certain star complement is not an invariant that differentiates between isomorphism types of graphs.

Therefore the authors of [CRS04] do an isomorphism check in all 443 sets of extensions from the 443 start complements (as an example they state that for one star complement there were 1048580 extensions, giving 457 isomorphism types).
Since we use different methods, it is nice to see that our results coincide, and an alternative approach for finding all orbits of maximal cliques in $\Gamma_{\{0,1\}}$ could be to use the isomorphism types of these graphs that were already known in [CRS04], and compute the orbits per isomorphism type. It is not obvious that this would have been faster, however, since we would still have to check if two cliques are conjugate for *every* two cliques of a certain isomorphism type, which can be many.

Proposition 3.5.33. *For two maximal cliques K_1 and K_2 of the same size in $\Gamma_{\{-2,-1,0\}}$, the following are equivalent.*

(i) K_1 and K_2 are conjugate under the action of W.

(ii) K_1 and K_2 are isomorphic.

(iii) K_1 and K_2 have the same stabilizer size, and, if the stabilizer size is 32 and K_1 and K_2 have size 10, then K_1 and K_2 both contain a pair of inverse roots, or they both do not.

(iv) The automorphism groups of K_1 and K_2 have the same cardinality, and, if this cardinality is 80 and K_1 and K_2 have size 9, or this cardinality is 64 and K_1 and K_2 have size 10, then K_1 and K_2 both contain a pair of inverse roots, or they both do not.

(v) K_1 and K_2 have the same stabilizer size and their automorphism groups have the same cardinality.

Moreover, the table in Appendix A gives a complete list of representatives of the orbits of maximal cliques in $\Gamma_{\{-2,-1,0\}}$, as well as for each representative its stabilizer size and the size of its automorphism group.

Proof. This proof follows the same steps as the proof of Proposition 3.5.28. See also Remark 3.5.29 on how we found the representatives of each orbit that are written in the table.
Cliques in $\Gamma_{\{-2,-1,0\}}$ without an edge of color 0 are monochromatic and not maximal in $\Gamma_{\{-2,-1,0\}}$ (this follows from the results on $\Gamma_{\{-2,-1\}}$, $\Gamma_{\{-2,0\}}$, and $\Gamma_{\{-1,0\}}$). Therefore, to find the maximal cliques in $\Gamma_{\{-2,-1,0\}}$, we only consider cliques that contain two orthogonal roots, and we can choose

these arbitrarily since W acts transitively on the set of pairs of orthogonal roots. Define the roots

$$e_1 = \left(-\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}\right) \text{ and}$$

$$e_2 = \left(-\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right).$$

We find the following.

r	Number of maximal cliques of size r in $\Gamma_{\{-2,-1,0\}}$ containing e_1 and e_2
≤ 7	0
8	192480
9	1961088
10	743536
11	111680
12	8290
13	2100
14–15	0
16	15
≥ 17	0

We turn to the table in the appendix. One can check that all the sets in the table for $\Gamma_{\{-2,-1,0\}}$ are indeed maximal cliques in $\Gamma_{\{-2,-1,0\}}$. For each of these cliques we compute the automorphism group with `magma`. As one can see in the table, except from two cliques

$$L_1 = \{1, 8, 26, 47, 51, 86, 121, 128, 228\},$$

$$L_2 = \{1, 8, 26, 47, 51, 86, 124, 125, 228\}$$

of size 9 that both have an automorphism group of size 80, and two cliques

$$M_1 = \{1, 8, 26, 31, 43, 46, 84, 98, 103, 125\},$$

$$M_2 = \{1, 8, 26, 31, 43, 46, 84, 101, 226, 238\}$$

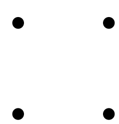
of size 10 that both have an automorphism group of size 64, any two cliques of the same size have different automorphism groups and are therefore not isomorphic. Moreover, L_1 contains the roots 1 and 128, which are each other's inverse, whereas L_2 contains no pairs of inverse roots. And M_1 contains the roots 26 and 103, which are each other's inverse, and M_2

contains no pairs of inverse roots. So also L_1, L_2, M_1 and M_2 are pairwise not isomorphic. We conclude that any two of the cliques in the table are not isomorphic, hence not conjugate.

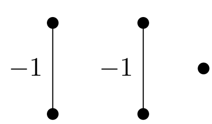
As in the proof of Proposition 3.5.28, we prove for each size r in the table above, using Lemma 3.5.27 and `magma`, the number of maximal cliques of size r containing e_1 and e_2 that are conjugate to one of the cliques in the table in the appendix. This gives exactly the number of maximal cliques of size r containing e_1 and e_2 in the table above. So every maximal clique in $\Gamma_{\{-2,0,1\}}$ containing e_1 and e_2 is conjugate to a clique in the table in the appendix, hence the same holds for every maximal clique in $\Gamma_{\{-2,0,1\}}$. We conclude that the table in the appendix gives a unique representative for each orbit of the set of maximal cliques under the action of W. Finally, for each clique in the table, we compute the size of its stabilizer in W. We see that except for $N_1 = \{1, 8, 26, 31, 43, 86, 106, 115, 224, 234\}$ and $N_2 = \{1, 8, 26, 31, 43, 46, 84, 101, 226, 238\}$, two cliques of the same size in the table have different stabilizer sizes. In N_1, we have roots 43 and 86, and these are each other's inverse; in N_2, there are no two roots that are each other's inverse. Finally, N_1 and N_2 have different automorphism groups.

The equivalence of statements (i) – (v) follows in a similar way as in the proof of Proposition 3.5.28. The implications (i)⇒(ii), (i)⇒(iii), (i)⇒(iv), (i)⇒(v) and (ii)⇒(iv) are immediate. Since both K_1 and K_2 are conjugate to one of the cliques in the table, if any of (iii) – (v) are true, by looking at the table we see that this implies that K_1 and K_2 are conjugate to the same clique in the table, and in particular, they are conjugate to each other, implying (i). This proves that all 5 statements are equivalent. $\square$

We can now prove Theorem 3.1.4 for maximal cliques in $\Gamma_{\{-1,0\}}$ and $\Gamma_{\{-2,-1,0\}}$; the statement is the same for these two graphs. Recall the following graphs that are defined in the introduction, where any two disjoint vertices have an edge of color 0 between them.



A



C_{-1}

Lemma 3.5.34. *Let K_1 and K_2 be two maximal cliques, both in $\Gamma_{\{-1,0\}}$ or both in $\Gamma_{\{-2,-1,0\}}$, and let $f\colon K_1 \longrightarrow K_2$ be an isomorphism between them. The following hold.*

(i) The map f extends to an automorphism of Λ if and only if for every ordered sequence $S = (e_1, \dots, e_r)$ of distinct roots in K_1 such that the colored graph on them induced by Γ is isomorphic to A or C_{-1}, its image $f(S) = (f(e_1), \dots, f(e_r))$ is conjugate to S under the action of W.

(ii) If $S = (e_1, \dots, e_5)$ is a sequence of distinct roots in K_1 such that the colored graph on them induced by Γ is isomorphic to C_{-1} with $e_1 \cdot e_4 = e_2 \cdot e_5 = -1$, then S and $f(S)$ are conjugate under the action of W if and only if both $e = e_1 + e_2 + e_3 - e_4 - e_5$ and $f(e)$ are in the set $\{2f_1 + f_2 \mid f_1, f_2 \in E\}$, or neither are.

Proof. Since K_1 and K_2 are isomorphic, it follows from Propositions 3.5.28 and 3.5.33 that they are both conjugate to the same clique in the table in the appendix; call this clique H. Then there are elements α, β in W such that $\alpha(K_1) = \beta(K_2) = H$, so $\beta \circ f \circ \alpha^{-1}$ is an element in the automorphism group $\mathrm{Aut}(H)$ of H. Of course, f extends to an element in W if and only if $\beta \circ f \circ \alpha^{-1}$ does. Moreover, for every sequence S as in the statement, $f(S)$ and $(\beta \circ f \circ \alpha^{-1})(S)$ are conjugate. We conclude that we can reduce to the case where $K_1 = K_2 = H$, and f is an element in $\mathrm{Aut}(H)$. Let $g\colon W_H \longrightarrow \mathrm{Aut}(H)$ be the map from the stabilizer of H to the automorphism group that restricts elements in W_H to H, and T_H a set of representatives of the classes in the cokernel of g. Since f is a composition of (restrictions of) elements in W_H with an element in T_H, we can reduce further to the case where f is an element in T_H.
For each of the 56 cliques H in the table at $\Gamma_{\{-1,0\}}$ and $\Gamma_{\{-2,-1,0\}}$, we compute the map $g\colon W_H \longrightarrow \mathrm{Aut}(H)$ with `magma`. In all cases, this map is injective. This means that for all cliques with $|W_H| = |\mathrm{Aut}(H)|$, every element in the automorphism group of H extends to a unique automorphism of Λ. We see in the list that this holds for the first five cliques and the 11^{th}, 12^{th}, 15^{th}, and 16^{th} clique in $\Gamma_{\{-1,0\}}$, and the first five cliques and the 8^{th}, 10^{th}, 11^{th}, 13^{th}, 17^{th}, 20^{th}, 23^{rd}, and 24^{th} clique in $\Gamma_{\{-2,-1,0\}}$.
For each clique H of the remaining 34 cliques, we compute the following with the function `CokernelClassesTypeCminus1` [Codb]. First, we create a set T_H of representatives of the classes of the cokernel of the map from W_H to $\mathrm{Aut}(H)$. We then check for each t in T_H, and for all sequences $S = (e_1, e_2, e_3, e_4, e_5)$ of distinct roots in H such that the colored graph on S is isomorphic to C_{-1} with $e_1 \cdot e_4 = e_2 \cdot e_5 = -1$, whether S and $t(S)$ are conjugate. For all t and S for which this is the case, we verify that either $e = e_1 + e_2 + e_3 - e_4 - e_5$ is in the set $F = \{2f_1 + f_2 \mid f_1, f_2 \in E\}$

and $t(e)$ is not, or vice versa. This proves part (ii).
For H equal to the 7^{th}–10^{th}, 13^{th}, 14^{th}, and 18^{th}–23^{rd} clique in $\Gamma_{\{-1,0\}}$ and the 7^{th}, 9^{th}, 12^{th}, 14^{th}, 16^{th}, 18^{th}, 19^{th}, 21^{st}, 22^{nd}, 25^{th}–29^{th}, and 31^{st} clique in $\Gamma_{\{-2,-1,0\}}$, the check we just described gives us for all t in T_H a sequence S with distinct roots in H and graph isomorphic to C_{-1}, such that S and $t(S)$ are not conjugate. For the remaining 7 cliques in the table, we do an almost analogous check with the function `CokernelClassesTypeA` in `magma` [Codb], where S is now a clique whose graph is isomorphic to A. For all 7 cliques H, for all elements in T_H, there exists such an S with S not conjugate to $t(S)$. This finishes the proof of (i). □

Proposition 3.5.35. *For $c \in \{\{0,1\},\{-2,0,1\}\}$, and K_1, K_2 two maximal cliques of the same size $r \neq 29$ in Γ_c, the following are equivalent.*

(i) K_1 and K_2 are conjugate under the action of W.

(ii) K_1 and K_2 are isomorphic.

(iii) K_1 and K_2 have the same stabilizer size, and they contain the same number of pairs of orthogonal roots.

(iv) The automorphism groups of K_1 and K_2 have the same cardinality, and K_1 and K_2 contain the same number of pairs of orthogonal roots.

Moreover, the table in Appendix A gives a complete list of representatives of the orbits of maximal cliques in Γ_c, as well as for each representative its stabilizer size and the size of its automorphism group.

Proof. We show that the table is correct and complete for each c. The steps in the proof are the same as those in the proofs of Propositions 3.5.28 and 3.5.33, and the equivalence of statements (i) – (iv) follows in the same way as in these propositions. See also Remark 3.5.29 on how we found the representatives of each orbit that are written in the table.
• $c = \{0,1\}$
We know that the maximal cliques in $\Gamma_{\{1\}}$ form two orbits; one with cliques of size 7 and one with the cliques of size 8 (Proposition 3.4.7). Note that the clique of size 7 in $\Gamma_{\{1\}}$ in the table is contained in the clique of size 22 in $\Gamma_{\{0,1\}}$, and the clique of size 8 in $\Gamma_{\{1\}}$ is contained in the clique of size 33 in $\Gamma_{\{0,1\}}$. This means that there are no maximal cliques with only edges of color 1 in $\Gamma_{\{0,1\}}$. We fix two orthogonal roots $e_1 = \left(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$, $e_2 = (-1,0,0,0,-1,0,0,0)$. With

`magma` we compute that there are only 136 roots that have dot product 0 or 1 with e_1 and e_2, and we find the following.

r	Number of maximal cliques of size r in $\Gamma_{\{0,1\}}$ containing e_1 and e_2
≤ 21	0
22	3120
23–27	0
28	21120
30	16263276
31	2792800
32	655680
33	105120
34	18800
35	0
36	304
≥ 37	0

For each set K in the table in Appendix A, one can check that it is indeed a maximal clique in $\Gamma_{\{0,1\}}$. We compute the automorphism groups of all cliques. As we see in the table, for all sizes except 30, two cliques of the same size have a different automorphism group, so they are not isomorphic, hence not conjugate. For size 30, all cliques whose automorphism groups have the same cardinality have a different number of pairs of orthogonal roots that they contain; for example, the cliques of size 30 with stabilizer size 48 contain (in order of appearence in the table) 171, 179, 180, 183, 198 subsets of two orthogonal roots. This shows that no two cliques in the table are isomorphic, hence not conjugate. Moreover, using the stabilizer size and the number of subsets of orthogonal roots of each clique K in the list, we can find the number of conjugates of K that contain $\{e_1, e_2\}$ with Lemma 3.5.27. Adding all these numbers up we recover the numbers in the table above, which shows that every maximal clique in $\Gamma_{\{-1,0\}}$ of size unequal to 29 is conjugate to one of the cliques in the list. We conclude that the table in Appendix A is complete. Finally, we see that for each clique in the table, the stabilizer size and the cardinality of the automorphism group is the same. Therefore, by what we showed above, different cliques of the same size and with the same stabilizer size in the table have a different number of subsets of two orthogonal roots.

• $c = \{-2, 0, 1\}$
We start with cliques in $\Gamma_{\{-2,0,1\}}$ containing an edge of color -2. We fix a root e and compute the maximal cliques in $\Gamma_{\{-2,0,1\}}$ containing e and $-e$. We find the following.

r	Number of maximal cliques of size r in $\Gamma_{\{-2,0,1\}}$ containing e and $-e$
≤ 12	0
13	370440
14	250236
15	0
16	77895
17–18	0
19	7019208
20	861840
21	120960
22	44352
23	0
24	4032
25–28	0
≥ 30	0

Since there are no maximal cliques of size bigger than 29 containing an edge of color -2, we conclude that all the maximal cliques in $\Gamma_{\{0,1\}}$ of size at least 29 are also maximal cliques in $\Gamma_{\{-2,0,1\}}$. This leaves us with the maximal cliques in $\Gamma_{\{0,1\}}$ of size 22 and 28. Looking at the table in the appendix, we see that for both sizes there is only one orbit, and it is an easy check that for the listed representatives L_{22} of size 22 and L_{28} of size 28 of both these orbits, there are no roots that can be added to extend the clique in $\Gamma_{\{-2,0,1\}}$. Therefore L_{22} and L_{28} are still maximal in $\Gamma_{\{-2,0,1\}}$. We now turn to the cliques in $\Gamma_{\{-2,0,1\}}$ in the table. First of all, one can check easily with `magma` that these are indeed maximal cliques in $\Gamma_{\{-2,0,1\}}$. For K_1 and K_2 of size 28 or ≥ 30, everything is exactly the same as for $\Gamma_{\{0,1\}}$, and we showed that the proposition holds in these cases. For the other cliques, we see that for all sizes except 13, 19, and 20, two different cliques of the same size have different automorphism groups. For sizes 13, 19, and 20, we compute, completely analogously to what we did for $c = \{0, 1\}$, that the number of subsets of two orthogonal roots in two different cliques whose automorphism groups have the same cardinality is different. For example, the cliques of size 19 whose automorphism group has size 96, contain (in order of appearence in the table) 91, 95, 94, 98, 103 subsets of two orthogonal roots. This proves that all the cliques in the table are pairwise not isomorphic, hence not conjugate. Again using Lemma 3.5.27, we can check that every maximal clique in $\Gamma_{\{-2,0,1\}}$ that is conjugate to one of the cliques in the table, showing that the table is complete. Finally, except for the cliques

$$L_1 = \{1, 8, 12, 14, 15, 20, 22, 23, 36, 38, 39, 128,\\ 136, 137, 138, 139, 149, 160, 169\}$$

and

$$L_2 = \{1, 8, 12, 14, 50, 68, 70, 74, 128, 136, 137, 154,\\ 169, 170, 176, 177, 181, 182, 215\}$$

of size 19, any two different cliques of the same size that have the same stabilizer size have the same cardinality of their automorphism groups as well. We already showed that this means that they contain a different number of pairs of orthogonal roots. We compute that L_1 contains 109 such pairs, and L_2 contains 79. Therefore we can conclude that different cliques of the same size and with the same stabilizer size in the table have a different number of subsets of two orthogonal roots. □

Cliques of size 29 in $\Gamma_{\{0,1\}}$ and $\Gamma_{\{-2,0,1\}}$

Cliques of size 29 in $\Gamma_{\{0,1\}}$

The graph $\Gamma_{0,1}$ contains a surprisingly large number of maximal cliques of size 29, so we will treat this case separately in this section. As before, we say that the stabilizer size of an orbit is the size of the stabilizer of any of the elements in the orbit (Definition 3.5.26).

Proposition 3.5.36. *In the graph $\Gamma_{0,1}$ there are* 62825152320 *maximal cliques of size* 29. *They form* 432 *orbits under the automorphism group W. The multiset of their stabilizer sizes is*

$$\{1^{(8)}, 2^{(81)}, 4^{(107)}, 6^{(5)}, 8^{(50)}, 10, 12^{(41)}, 14^{(2)}, 16^{(28)}, 18^{(2)}, 20^{(5)}, 24^{(28)}, 32^{(4)},\\ 36, 48^{(21)}, 60, 64^{(2)}, 72^{(7)}, 96^{(3)}, 120, 128^{(2)}, 144^{(4)}, 192^{(7)}, 240^{(6)}, 360,\\ 384^{(3)}, 432^{(2)}, 720^{(2)}, 1152^{(2)}, 1440, 1920, 40320, 51840, 103680\},$$

where the superscripts indicate the multiplicity of the elements in the multiset. For two maximal cliques K_1 and K_2 of size 29 in $\Gamma_{\{0,1\}}$, the following are equivalent.

(i) K_1 and K_2 are conjugate under the action of W.

(ii) K_1 and K_2 are isomorphic.

(iii) K_1 and K_2 have the same stabilizer size, and the same number of maximal monochromatic subcliques of color 1 of size r, for all $r \in \{1, \ldots, 8\}$.

(iv) The automorphism groups of K_1 and K_2 have the same cardinality, and K_1 and K_2 have the same number of maximal monochromatic subcliques of color 1 of size r, for all $r \in \{1, \ldots, 8\}$.

Moreover, the table in Appendix B gives a complete list of representatives of the orbits of maximal cliques of size 29 in $\Gamma_{\{0,1\}}$.

The number of cliques mentioned in Proposition 3.5.36 is too large to fit in most computers' memory: even if we were to use only 30 bytes per clique to store the vertices in the clique, then all cliques together would still require close to two terrabytes of storage. Instead of doing this, we will use the fact that each 29-clique contains a monochromatic 5-clique of color 0 or a monochromatic 4-clique of color 1.

Proof. The Ramsey number $R(4,5)$ equals 25 (Theorem 3.5.21). This implies that a 29-clique in $\Gamma_{\{0,1\}}$ contains a 5-clique of edges of color 1 or a 4-clique of edges of color 0. Under the action of the automorphism group W there is only one orbit of 5-cliques with only edges of color 1 (see Proposition 3.2.12); we call these cliques of type $K_5(1)$), and there are two orbits of 4-cliques with pairwise orthogonal roots (see Proposition 3.4.1); we call the 4-cliques of which the sum is a double root of type $K_4^a(0)$ and those of which the sum is not a double of type $K_4^b(0)$). Therefore, if we fix a representative clique for each of these three orbits, then each 29-clique is conjugate to a 29-clique that contains one of our three cliques of size 4 or 5.

We pick the clique $A = \{1, 2, 129, 130, 131\}$ of type $K_5(1)$. There are 109 other vertices that are connected with color 0 or 1 to each of the 5 vertices of A. With `magma`, we count that the graph on these 109 vertices with only edges of color 0 or 1 has exactly $n_1 = 127168449$ maximal cliques of size 24. After adding to each the vertices of A, this yields n_1

maximal 29-cliques that contain A in the graph $\Gamma_{0,1}$. Similarly, for the cliques $B_1 = \{1, 8, 26, 31\}$ and $B_2 = \{1, 8, 26, 43\}$ of type $K_4^a(0)$ and $K_4^b(0)$, respectively, we count with `magma` that there are $n_2 = 16685128$ maximal 29-cliques in $\Gamma_{0,1}$ that contain B_1, and $n_3 = 504$ maximal 29-cliques that contain B_2.

One can easily verify with `magma` that the 432 cliques of size 29 in the table in Appendix B are maximal cliques in $\Gamma_{0,1}$. For each clique K of size 29, for each integer $1 \leq r \leq 8$, we can consider the number χ_r of maximal monochromatic subcliques of K of color 1 of size r. These eight invariants together pin down 430 out of the 432 cliques in the table. Only the sequence $(\chi_1, \chi_2, \ldots, \chi_8) = (0, 0, 0, 0, 0, 4, 138, 17)$ occurs twice: for the 67-th and 299-th cliques in the table. These two cliques have 16 and 18 subcliques of type $K_4^a(0)$, respectively, so they are not isomorphic. We conclude that any two cliques in the table are not isomorphic, hence not conjugate. So there are at least 432 orbits of maximal 29-cliques. We know that there are 483840 cliques of size 5 in $\Gamma_{\{1\}}$ from Corollary 3.2.7, so the stabilizer of A has size $\frac{|W|}{|WA|} = \frac{|W|}{483840} = 1440$. The table also lists for each clique c the number of subcliques of type $K_5(1)$, as well as the stabilizer size, so we can use Lemma 3.5.27 to calculate the number of conjugates of c that contain A. Summing over all these 432 cliques, we obtain exactly the number n_1, so we conclude that all n_1 maximal 29-cliques in $\Gamma_{0,1}$ that contain A are accounted for in these 432 orbits. Similarly, the stabilizers of B_1 and B_2 have sizes 4608 and 384, respectively. The table lists the number of subcliques of type $K_4^a(0)$ and $K_4^b(0)$ for every given clique c, so we can use Lemma 3.5.27 again to calculate the number of conjugates of c that contain B_i for $i = 1, 2$. Summing over all 432 cliques, we find again that all maximal 29-cliques containing B_1 or B_2 are accounted for in these 432 orbits.

We conclude that there are 432 orbits of 29-cliques in $\Gamma_{0,1}$, as claimed, and since no two cliques in the table are isomorphic, this proves (i) $\Leftrightarrow$ (ii). The multiset of stabilizer sizes follows from the table. The length of the orbit of any clique c is $\frac{|W|}{|W_c|}$. Summing over all 432 cliques in the table, we find that the total number of 29-cliques is also as claimed. Finally, as we saw before, the invariant χ_r is different for all cliques except for the 67-th and 299-th cliques in the table. These two cliques have stabilizer size 4 and 8, respectively, so the stabilizer size, together with the χ_r form a set of invariants that uniquely determine each of the 432 orbits of maximal 29-cliques. This proves that (i) is equivalent to (iii). The stabilizer of a clique maps to the automorphism group of this clique as a colored graph.

In all 432 cases, the clique generates a full rank sublattice of our lattice, so this map is injective. It turns out that in all cases, it is in fact a bijection. This proves (iii) $\Leftrightarrow$ (iv). □

Corollary 3.5.37. *Let K_1 and K_2 be two maximal cliques in $\Gamma_{\{0,1\}}$, and $f\colon K_1 \longrightarrow K_2$ an isomorphism between them. Then f extends to a unique automorphism of Λ.*

Proof. Since K_1 and K_2 are isomorphic, it follows from Propositions 3.5.35 and 3.5.36 that they are conjugate to each other; this means that they are both conjugate to the same clique in the tables in de appendix; call this clique H. Then there are elements α, β in W such that we have $\alpha(K_1) = \beta(K_2) = H$, so $\beta \circ f \circ \alpha^{-1}$ is an element in the automorphism group $\mathrm{Aut}(H)$ of H. Of course, f extends to an element in W if and only if $\beta \circ f \circ \alpha^{-1}$ does. We conclude that we can reduce to the case where $K_1 = K_2 = H$, and f is an element in $\mathrm{Aut}(H)$.
In Proposition 3.5.35 we computed the stabilizers and the automorphism groups of all cliques in $\Gamma_{\{0,1\}}$ of size unequal to 29, and we did the same for cliques of size 29 in Proposition 3.5.36. In `magma` we construct for each clique in the table the map between the stabilizer and the automorphism group that is given by restriction. In all cases, this is an isomorphism. We conclude that all automorphisms of the cliques in the table extend to an element in W. □

The table in Appendix B contains the results of the previous proposition, with a representative of each orbit. The notation in the table means the following.

Notation 3.5.38.

K: a clique in $\Gamma_{\{0,1\}}$; we denote vertices by their index as described in Notation 3.5.2.

$|W_K|$: the size of the stabilizer of clique K in the group W.

$\#K_5(1)$: the number of cliques of size 5 with only edges of color 0 in K.

$\#K_4^a(1)$: the number of cliques in K of four roots that sum up to a double root in Λ, with only edges of color 1.

$\#K_4^b(1)$: the number of cliques in K of four roots that do not sum up to a double root in Λ, with only edges of color 1.

Remark 3.5.39. In the proof of Proposition 3.5.36, we found more than 127 million cliques of size 29 that contain $A = \{1, 2, 129, 130, 131\}$. To find that they represent exactly 432 different orbits, one might naively try to just verify for each pair whether they are conjugate. This takes too much time; as described in Remark 3.5.29, we divided the big set into smaller sets according to the stabilizer sizes.

Cliques of size 29 in $\Gamma_{\{-2,0,1\}}$

It is an easy check that all 432 cliques of size 29 in $\Gamma_{\{0,1\}}$ in the table are maximal in $\Gamma_{\{-2,0,1\}}$ as well. We conclude that the orbits of maximal cliques of size 29 in $\Gamma_{\{-2,0,1\}}$ are exactly the 432 that we found in $\Gamma_{\{0,1\}}$, and the orbits of maximal cliques of size 29 that contain an edge of color -2.
As we did in Proposition 3.5.35, we fix a root e and compute all maximal cliques of size 29 in $\Gamma_{\{-2,0,1\}}$ that contain e and $-e$ with `magma`. There are 56 of these, and they form one orbit under the action of the stabilizer W_e of e. Since W acts transitively on pairs of inverse roots, we conclude that all maximal cliques of size 29 in $\Gamma_{\{-2,0,1\}}$ that contain an edge of color -2 are in the same orbit; call this orbit A. One can easily check with `magma` that the clique of size 29 that is written in the table for $\Gamma_{\{-2,0,1\}}$ is maximal, and moreover, it contains the roots 1 and 128, that are each other's inverse. We conclude that it is a representative of A. The stabilizer and automorphism group are computed with `magma`.

We finish with the proof of Theorem 3.1.4 for maximal cliques in $\Gamma_{\{-2,0,1\}}$. This is very similar to the proof of Lemma 3.5.34. Recall the graphs A, C_1, D, and F as defined before Theorem 3.1.4.

Lemma 3.5.40. *Let K_1 and K_2 be two maximal cliques in $\Gamma_{\{-2,0,1\}}$, and $f\colon K_1 \longrightarrow K_2$ an isomorphism between them. The following hold.*

(i) The map f extends to an automorphism of Λ if and only if for every subclique $S = \{e_1, \ldots, e_r\}$ of K_1 that is isomorphic to A, C_1, D, or F, its image $f(S)$ in K_2 is conjugate to S under the action of W.

Let S be a subclique of K_1.

(ii) If S is isomorphic to C_1, then S and $f(S)$ are conjugate if and only if both $\sum_{i=1}^{5} e_i$ and $\sum_{i=1}^{5} f(e_i)$ are in the set $\{2f_1 + f_2 \mid f_1, f_2 \in E\}$, or neither are.

(iii) If S is isomorphic to D, then S and $f(S)$ are conjugate if and only if both $\sum_{i=1}^{5} e_i$ and $\sum_{i=1}^{5} f(e_i)$ are in the set $\{2f_1 + 2f_2 \mid f_1, f_2 \in E\}$, or neither are.

(iv) If S is isomorphic to F, then S and $f(S)$ are conjugate if and only if both $\sum_{i=1}^{5} e_i$ and $\sum_{i=1}^{6} f(e_i)$ are in 2Λ, or neither are.

Proof. This proof is very similar to the proof of Lemma 3.5.34, so we will sketch what we did, and we refer to the other proof for details.
We reduce again to the case $K_1 = K_2 = H$, with H one of the 54 cliques in the list for $\Gamma_{\{-2,0,1\}}$ in the appendix, and f a representative of a class of the cokernel of the map $g\colon W_H \longrightarrow \mathrm{Aut}(H)$, where W_H is the stabilizer of H in W, and $\mathrm{Aut}(H)$ is the automorphism group of H.
For each clique H of those 54 in the table, we check with `magma` that the map $g\colon W_H \longrightarrow \mathrm{Aut}_H$ is injective; for the 13^{th}, 15^{th}, and 17^{th} – 54^{th} cliques it is an isomorphism. It follows that for those cliques, every automorphism extends to an element in W, so we are done. Here we refer to Corollary 3.5.37 for the cliques that are the same as in $\Gamma_{\{0,1\}}$.
For each clique H of the remaining 14 cliques in the list, we do the following in `magma` with the three functions that we name `CokernelClassesTypeF`, `CokernelClassesTypeD`, and `CokernelClassesTypeC1` [Codb]. We construct a set T_H of representatives of the classes of the cokernel of the map from W_H to $\mathrm{Aut}(H)$. We then check for each t in T_H, and for all subcliqes $S = \{e_1, \ldots, e_r\}$ of H that are isomorphic to F (or D, or C_1, respectively), whether S and $t(S)$ are not conjugate. For all t and S for which this is the case, we verify that $\sum_{i=1}^{r} e_i$ is in 2Λ (or in the set $\{2f_1 + 2f_2 \mid f_1, f_2 \in E\}$, or in the set $\{2f_1 + f_2 \mid f_1, f_2 \in E\}$, respectively), and $\sum_{i=1}^{r} t(e_i)$ is not, or vice versa. This proves (ii), (iii), and (iv).
For H equal to the 4^{th}, 8^{th}, 10^{th}, 14^{th}, and 16^{th} clique, for each non-trivial element t in T_H there is a subclique S of H that is isomorphic to F, and such that t and $t(S)$ are not conjugate. Similarly, for each clique H of the remaining 9 cliques in the list, for each non-trivial element t of T_H, there is a subclique S of H that is isomorphic to either C_1, D, or A, and such that S and $t(S)$ are not conjugate. This finishes the proof of (i). □

3.6 Proof of the main theorems

We now put together all the results that form the proofs of Theorem 3.1.3 and Theorem 3.1.4, which are both stated in the Section 3.1.

Proof of Theorem 3.1.3. Part (i) is Proposition 3.4.1 (iii), and part (ii) is Proposition 3.4.7 (ii). We proceed with (iii). Of course, if K_1 and K_2 are conjugate under the action of W, they are isomorphic as colored graphs, since W respects the dot product. Now assume that K_1 and K_2 are isomorphic as colored graphs. We will show that they are conjugate under the action of W. First of all, by Lemma 3.2.13, we can assume that there is a type I, II, III, or IV, that both K_1 and K_2 belong to. Therefore we continue to prove the result per type.
For type I, the results for colors -2 and -1 are at the beginning of Section 3.4; the results for color 0 are in Propositions 3.4.1, 3.4.6 (iii), and 3.5.15, and the results for color 1 are in Proposition 3.2.12.
For type II, from Proposition 3.2.5 we know what the cliques look like, and the results are then in Proposition 3.2.12 and Corollary 3.3.17.
For type III, the results follow from Propositions 3.3.1 and 3.3.2.
Finally, for type IV, the results follow from Propositions 3.5.6, 3.5.28, Lemma 3.5.30, Propositions 3.5.33and 3.5.35, and Section 3.5.4.

Proof of Theorem 3.1.4. By Lemma 3.2.13, we can assume that there is a type I, II, III, or IV, that both K_1 and K_2 belong to. Therefore we continue to prove (i) per type. First of all, if K_1 and K_2 are of type III, then f always extends; this is shown in Corollary 3.3.34.
If K_1 and K_2 are of type I, they are monochromatic. If they have color -2 or -1, then they are of type III (see Section 3.4). For color 0 the proof is in Corollary 3.4.5, and for color 1 in Corollary 3.4.9.
For type II, by Proposition 3.2.5, K_1 and K_2 are either monochromatic of color 0, hence of type I, or they are both sets of the vertices of a 7-crosspolytope, in which case the statement is in Corollary 3.3.32.
If K_1 and K_2 are of type IV, they are maximal cliques in a graph Γ_c, where there are 14 different possibilities for c. For $c \in \{\{-2\}, \{-1\}, \{0\}, \{1\}\}$, the cliques K_1 and K_2 are of type I, which we already covered (note that for K_1 and K_2 *maximal* in $\Gamma_{\{1\}}$, there is always an automorphism extending f!).
For c in $\{\{-2, -1\}, \{-2, 1\}\}$, the cliques K_1 and K_2 are of type I as well (Lemma 3.5.5). For $c = \{-2, 0\}$, the proof is in Lemma 3.5.17. For c in $\{\{-1, 1\}, \{-2, -1, 1\}\}$, an isomorphism of maximal cliques always extends, see Corollary 3.5.31. The same holds for $c = \{0, 1\}$, see Corollary 3.5.37.
For $c \in \{\{-1, 0\}, \{-2, -1, 0\}\}$, the statement is in Lemma 3.5.34.
For $c = \{-2, 0, 1\}$ the statement is Lemma 3.5.40.
Finally, for $c = \{-2, -1, 0, 1\}$ there is one clique, which is Γ itself, and every automorphism of Γ is an element in W. This finishes (i).
Part (ii) follows from Propositions 3.3.29 and 3.4.1 for type A, and it

follows from Propositions 3.2.12 and 3.4.7 for type B. Finally, part (iii) is in Lemma 3.5.34, and part (iv) is in Lemma 3.5.40.

Remark 3.6.1. From Theorem 3.1.4 it follows that for an isomorphism f of two cliques K_1 and K_2 of types I, II, III, or IV, one can determine whether f extends to an automorphism of Λ by checking for all subcliques of K_1 of the form A, B, C_α, D, or F, if f restricted to an associated ordered sequence extends. However, one never has to check all subcliques of those six forms. The following table shows for each type of K_1 and K_2 which subcliques are sufficient to check.

Type	Subtype	All isomorphisms extend	A	B	C_{-1}	C_1	D	F
I	$\Gamma_{\{-2\}}$	x						
I	$\Gamma_{\{-1\}}$	x						
I	$\Gamma_{\{0\}}$		x					
I	$\Gamma_{\{1\}}$			x				
II	k-simplex, $k \leq 7$		x					
II	7-crosspolytope			x				
III	all	x						
IV	$\Gamma_{\{-2\}}$	x						
IV	$\Gamma_{\{-1\}}$	x						
IV	$\Gamma_{\{0\}}$		x					
IV	$\Gamma_{\{1\}}$	x						
IV	$\Gamma_{\{-2,-1\}}$	x						
IV	$\Gamma_{\{-2,0\}}$		x					
IV	$\Gamma_{\{-2,1\}}$	x						
IV	$\Gamma_{\{-1,0\}}$		x		x			
IV	$\Gamma_{\{-1,1\}}$	x						
IV	$\Gamma_{\{0,1\}}$	x						
IV	$\Gamma_{\{-2,-1,0\}}$		x		x			
IV	$\Gamma_{\{-2,-1,1\}}$	x						
IV	$\Gamma_{\{-2,0,1\}}$		x			x	x	x
IV	$\Gamma_{\{-2,-1,0,1\}}$	x						

4

Concurrent exceptional curves on del Pezzo surfaces of degree 1

This chapter is an adaptation of the preprint [vLWb], which is at the moment of this writing submitted for publication. Moreover, part of this chapter is already in the master thesis [Win14] by the same author. We decided to copy those parts here for completion. See Remark 4.1.3 for a comparison with [Win14].

Recall that a del Pezzo surface of degree d over an algebraically closed field contains a fixed number of exceptional curves, depending on d (Table 1.1). The configuration of these curves can play a role in arithmetic questions; we have seen this in Chapter 2. For example, one of the conditions on the point Q that is used to show that the set of rational points on a del Pezzo surface of degree 1 is dense in [SvL14], is for Q not to lie on 6 exceptional curves, if its order is 3 or 5. Another example is found in [STVA14, Corollary 18], where Salgado, Testa and Várilly-Alvarado show that a del Pezzo surface of degree 2 is unirational if and only if it contains a point that is not contained in 4 exceptional curves, and lies outside the ramification curve of the anticanonical map. In this chapter we study the configuration of the exceptional curves on a del Pezzo surface of degree 1, and determine the maximal number of these curves that can go through one point.

4.1 Main results

We call a set of exceptional curves *concurrent* in a point on the surface if that point is contained in all of them. It is well known that on del Pezzo surfaces of degree 3, the number of exceptional curves that are concurrent in a point is at most 3. This can be seen by looking at the graph on the 27 exceptional curves, where two vertices are connected by an edge if the corresponding exceptional curves intersect. For all del Pezzo surfaces of degree 3 this gives the same graph G. A set of concurrent exceptional curves corresponds in this way to a complete subgraph of G, and the maximal size of complete subgraphs in G is 3. On a del Pezzo surface of degree 2, the number of concurrent exceptional curves in a point is at most 4. As in the case for degree 3, this can be derived directly from the intersection graph on the 56 exceptional curves. A geometric argument why 4 is an upper bound is given in [TVAV09], in the proof of Lemma 4.1. An example where this upper bound is reached is given in [STVA14], Example 2.4. For del Pezzo surfaces of degree 1, the situation is more complex. Contrary to the case of del Pezzo surfaces of degree ≥ 2, for char $k \neq 2$, the maximal size of complete subgraphs of the intersection graph on the 240 exceptional curves, which we will show is 16, is not equal to the maximal number of exceptional curves that are concurrent in a point.

Let X be a del Pezzo surface of degree 1 over an algebraically closed field k, and let K_X be the canonical divisor on X. The linear system $|-2K_X|$ gives X the structure of a double cover of a cone Q in $\mathbb{P}^3$, ramified over a sextic curve that is cut out by a cubic surface (Section 1.4.1). Let φ be the morphism associated to this linear system. In this chapter we prove the following two theorems.

Theorem 4.1.1. *Let $P \in X(k)$ be a point on the ramification curve of φ. The number of exceptional curves that go through P is at most ten if char $k \neq 2$, and at most sixteen if char $k = 2$.*

Theorem 4.1.2. *Let $Q \in X(k)$ be a point outside the ramification curve of φ. The number of exceptional curves that go through Q is at most ten if char $k \neq 3$, and at most twelve if char $k = 3$.*

Using the ramification divisor of φ, we obtain with a simple geometrical argument an upper bound of 12 outside characteristic 2 for Theorem 4.1.1, which was pointed out to us by Niels Lubbes. An anonymous referee even suggested that with some more work, this same argument can be improved to give the upper bound of 10 outside characteristic 2. See Remark 4.3.1.

In [SvL14, Example 4.1], for any field of characteristic unequal to 2, 3, or 5, a del Pezzo surface of degree 1 is defined that contains a point outside the ramification curve that is contained in 10 exceptional curves. This shows that the upper bound for char $k \neq 2, 3, 5$ in Theorem 4.1.2 is sharp. In Section 4.5 we show in all characteristics except for characteristic 5 in the case of Theorem 4.1.2, that the upper bounds in Theorems 4.1.1 and 4.1.2 are sharp. Theorems 4.1.1 and 4.1.2 are proved by using results on the automorphism group of the graph on the 240 exceptional curves, and by Propositions 4.3.6 and 4.4.6, which are purely geometrical and show that certain curves in $\mathbb{P}^2$ do not go through the same point.

Remark 4.1.3. Most of the results in Section 4.3 are proved by the same author in the master thesis [Win14]; more specifically, Theorem 4.1.1 and Proposition 4.3.6 are equal to Theorem 1 and Proposition 4.22 in [Win14], and Lemma 4.3.4 is almost the same as Lemma 4.21 in [Win14]. We decided to include these results here for completeness.
In [Win14], Theorem 4.1.2 is stated for char $k = 0$. In this chapter we extend this to a result for all characteristics. Moreover, we added several geometrical arguments (Lemmas 4.4.8 – 4.4.13, Proposition 4.4.15), that heavily reduce the usage of `magma` in the proof of Proposition 4.4.6, which is key to Theorem 4.1.2.
Examples 4.5.1 and 4.5.2 are the same as Exmples 4.24 and 4.23 in [Win14], where it was shown that the upper bounds of Theorem 4.1.1 are sharp in characteristic 0. In Section 4.5 we give extra examples, showing that the upper bounds in Theorem 4.1.1 are sharp in all characteristics, and that the upper bounds in Theorem 4.1.2 are sharp except possibly in characteristic 5.

We use `magma` [BCP97] for our computations, which is the case only in Propositions 4.3.6 and 4.4.6. The proofs of Propositions 4.2.2, 4.4.2, 4.4.3, and 4.4.4 rely on results in Chapter 3 that also make use of `magma`.

We want to thank Niels Lubbes for useful discussions, and Igor Dolgachev for useful comments. We also want to thank an anonymous referee for

giving useful remarks that improved the quality of the paper, and a second anonymous referee for suggesting a shorter proof of the upper bound of 10 outside characteristic 2 on the ramification curve.

4.2 The weighted graph on exceptional classes

We use the same notation as in Definition 1.4.12 and in Chapter 3: we denote the set of exceptional classes in Pic X by I; by G we denote the complete weighted graph whose vertex set is I, and where the weight function is the intersection pairing in Pic X.

When two exceptional curves intersect in a point on X, their corresponding classes in Pic X are connected by an edge of positive weight in G. Therefore, an upper bound on the number of exceptional curves on X that are concurrent in a point is given by the maximal size of cliques in G that have only edges of positive weight. To study these cliques, we use the correspondence between the set I and the root system $\mathbf{E}_8$ as in Remark 1.4.9. In particular, if Γ is the weighted graph where the vertices are the roots in $\mathbf{E}_8$ and the weights are induces by de dot product in $\mathbf{E}_8$, there is an isomorphism of weighted graphs between G and Γ, that sends a vertex c in G to the corresponding vertex $c + K_X$ in Γ, and an edge $d = \{c_1, c_2\}$ in G with weight w to the edge $\delta = \{c_1 + K_X, c_2 + K_X\}$ in Γ with weight $1 - w$ (Remark 1.4.13). The different weights that occur in G are $0, 1, 2,$ and 3, and they correspond to weights $1, 0, -1,$ and -2, respectively, in Γ. From the bijection between Γ and G we immediately obtain the following results.

Lemma 4.2.1. *(i) Let e be an exceptional class. Then there is exactly one exceptional class f with $e \cdot f = 3$, there are 56 exceptional classes f with $e \cdot f = 0$, there are 126 exceptional classes f with $e \cdot f = 1$, and 56 exceptional classes f with $e \cdot f = 2$.*

(ii) For two exceptional classes e_1, e_2 with $e_1 \cdot e_2 = 2$, there is a unique exceptional class f such that $e_1 \cdot f = e_2 \cdot f = 2$.

(iii) For every pair e_1, e_2 of exceptional classes such that $e_1 \cdot e_2 = 1$, there are exactly 60 exceptional classes f with $e_1 \cdot f = e_2 \cdot f = 1$, and 32 exceptional classes f with $e_1 \cdot f = 1$ and $e_2 \cdot f = 0$.

(iv) For e_1, e_2 two exceptional classes with $e_1 \cdot e_2 = 3$, and f a third exceptional class, we have $e_1 \cdot f = 1$ if and only if $e_2 \cdot f = 1$, and $e_1 \cdot f = 0$

if and only if $e_2 \cdot f = 2$.

Proof. Using the fact that two exceptional classes have intersection pairing a if and only if their corresponding roots in E have inner product $1 - a$, we see that (i) is Proposition 4.2.1, (ii) is Lemma 3.3.9, and (iii) is Lemma 3.3.27 and Lemma 3.3.13. Finally, (iv) follows from the fact that two classes e_1, e_2 with $e_1 \cdot e_2 = 3$ correspond to two roots in E with inner product -2, which implies they are each other's inverse as vectors (Proposition 3.2.2). □

We also obtain a first upper bound for the number of exceptional curves that are concurrent in a point on X.

Proposition 4.2.2. *The number of exceptional curves that are concurrent in a point on X is at most 16.*

Proof. Cliques with edges of positive weight in G correspond to cliques with edges of weights $-2, -1, 0$ in Γ. The maximal size of such cliques in Γ is 16 by Proposition 3.5.33 and Appendix A. □

Definition 4.2.3. For an exceptional class e in Pic X, we call the unique exceptional class e' with $e \cdot e' = 3$ its *partner*.

The graph in Figure 4.1 is a translation of Figure 3.1, and summarizes Lemma 4.2.1. Vertices are exceptional classes, and the number in a subset is its cardinality. The number on an edge between two subsets is the intersection pairing of two classes, one from each subset. For $i, j \in \{1, 2, 3\}$, the exceptional class e_i' is the partner of the class e_i, and for $e_i \cdot e_j = 2$, the class $e_{i,j}$ is the unique one that intersects both e_i and e_j with multiplicity 2.
Let φ be the morphism associated to the linear system $|-2K_X|$, which realizes X as a double cover of a cone Q in $\mathbb{P}^3$. We want to distinguish cliques in G corresponding to exceptional curves that intersect in a point on the ramification curve of φ from those intersecting in a point outside the ramification curve of φ. To this end we use Proposition 4.2.4.

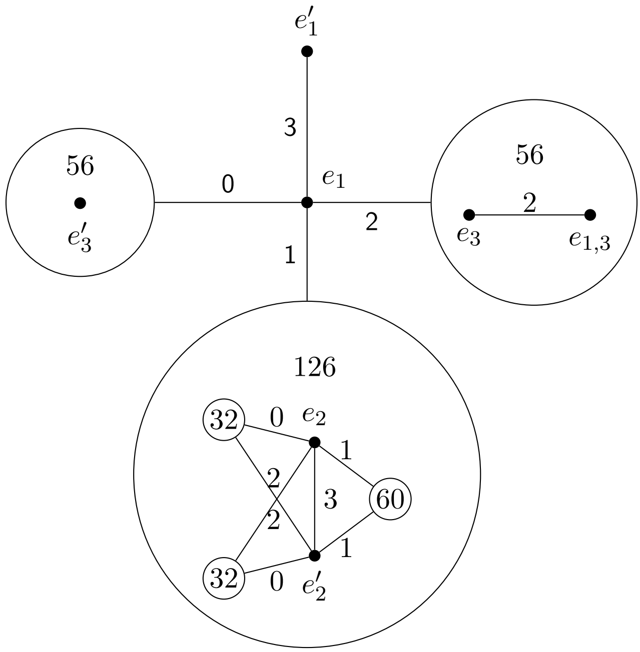


Figure 4.1: Graph G

Proposition 4.2.4.

(i) If e is an exceptional curve on X, then $\varphi(e)$ is a smooth conic, the intersection of Q with a plane in $\mathbb{P}^3$ not containing the vertex of Q. Moreover $\varphi|_e \colon e \longrightarrow \varphi(e)$ is one-to-one.

(ii) If H is a hyperplane section of Q not containing the vertex of Q, then φ^*H has an exceptional curve as component if and only if it has at least three (maybe infinitely near) singular points. If this is the case, then $\varphi^*H = e_1 + e_2$ with e_1, e_2 exceptional curves, and $e_1 \cdot e_2 = 3$. Every exceptional curve arises this way.

Proof. [CO99, Proposition 2.6 and Key-lemma 2.7]. □

Remark 4.2.5. Let e be an exceptional curve on X, and let e' be its partner. Let H be a hyperplane section of Q with $\varphi^*H = e + e'$, which exists by Proposition 4.2.4 (ii). Since $\varphi|_f$ is one-to-one for $f = e, e'$ by part (i) of the same proposition, it follows that $\varphi(e) = \varphi(e') = H$. So every point on H has two preimages under φ, except for the points with a preimage in $e \cap e'$. We conclude that the points where e intersects the ramification curve of φ are exactly the points in $e \cap e'$, hence are also contained in e'. Conversely, if a set of exceptional curves is concurrent in a point P, and this set contains an exceptional curve and its partner, then P lies on the ramification curve of φ.

4.3 Proof of Theorem 4.1.1

In this section we prove Theorem 4.1.1. We first determine which cliques in G may correspond to sets of exceptional curves intersecting on the ramification curve of φ (Remark 4.3.2). We then show that the automorphism group of G acts transitively on certain cliques of that form (Proposition 4.3.3), which allows us to reduce to specific curves on X. In Proposition 4.3.6, which is key to the proof of Theorem 4.1.1, we show that seven curves in $\mathbb{P}^2$ in a specific configuration are not concurrent.

Remark 4.3.1. From Remark 4.2.5 it follows that there is a bijection between planes in $\mathbb{P}^3$ that are tritangent to the branch curve of φ and do not contain the vertex of Q, and pairs of exceptional curves e_1, e_2 with $e_1 \cdot e_2 = 3$. Using this, we can find an upper bound for the number of exceptional curves that are concurrent in a point on the ramification curve. Let P be a point on the branch curve of φ. From Lemma 4.5 in [TVAV09], it follows that over a field of characteristic unequal to 2, there are at most 7 planes that are tangent to the branch curve at P and two other points. Moreover, Niels Lubbes gave us the insight that exactly one of those planes contains the vertex of Q, so we find an upper bound of 6 planes that are tritangent to the branch curve, that contain P, and that do not contain the vertex of Q. This gives an upper bound of 12 exceptional curves that contain the point $\varphi^{-1}(P)$ on the ramification curve of φ, if char $k \neq 2$.
Consider the map $\lambda\colon R \longrightarrow \mathbb{P}^1$, where R is the ramification curve of φ, and $\mathbb{P}^1$ parametrizes the planes through the tangent line to R at $\varphi^{-1}(P)$: λ sends each point x in $R \setminus \varphi^{-1}(P)$ to the unique plane containing x. This map has degree 4, and if char $k \neq 2$, then R is smooth, and λ extends to a morphism. The upper bound of 7 planes that was found in Lemma 4.5 in [TVAV09] comes from the fact that the ramification divisor of λ has degree 14. An anonymous referee gave us the hint that this idea could even be used to give the upper bound of 10 in char $k \neq 2$ directly, by showing that a morphism of degree 4 to $\mathbb{P}^1$ can not have 7 ramification patterns all equal to $(2, 2)$. Therefore there are at most 6 planes that are

tangent to P and two other points on the branch curve of φ. Since one of them is the plane through the vertex of Q, this gives the upper bound of 10 exceptional curves through $\varphi^{-1}(P)$. We are currently working out the details of this argument.

Remark 4.3.2. From Remark 4.2.5 it follows that a maximal set of exceptional curves that are concurrent in a point on the ramification curve consists of exceptional curves and their partners, hence has even size. Moreover, from Lemma 4.2.1 (iv) it follows that such a clique only has edges of weights 1 and 3. We conclude that all cliques in G corresponding to a maximal set of exceptional curves that are concurrent in a point on the ramification curve are of the following form.

$$K_n = \left\{ \{e_1, \ldots, e_n, e_1', \ldots, e_n'\} \;\middle|\; \begin{array}{c} \forall i: \; e_i, e_i' \in I; \; e_i \text{ is the partner of } e_i'; \\ \forall i \neq j : e_i \cdot e_j = e_i \cdot e_j' = e_i' \cdot e_j' = 1 \end{array} \right\}$$

Let W be the group of permutations of I that preserve the intersection pairing, and recall that W is isomorphic to the Weyl group of the $\mathbf{E}_8$ root system (Corollary 1.4.10).

Proposition 4.3.3. *For $n \in \{2, 3, 5, 6, 7, 8\}$, the group W acts transitively on the set K_n.*

Proof. This is Proposition 3.5.13. □

We now set up notation for Lemma 4.3.4; this lemma will be used in Propositions 4.3.6 and 4.4.6. Lemma 4.3.5 is used in Proposition 4.3.6.

Let $\mathbb{P}^2$ be the projective plane over k with coordinates x, y, z, and let $R_1, \ldots, R_9$ be nine points in $\mathbb{P}^2$, with $R_i = (x_i : y_i : z_i)$ for $i \in \{1, \ldots, 9\}$. For $i \in \{1, 2, 3, 4\}$, we define Mon_i to be the decreasing sequence of $r_i = \binom{i+2}{2} = \frac{1}{2}(i+1)(i+2)$ monomials of degree i in x, y, z, ordered lexicographically with $x > y > z$, and for $j \in \{1, \ldots, r_i\}$, let $\mathrm{Mon}_i[j]$ be the j^{th} entry of Mon_i. For $\delta \in \{x, y, z\}$, let Mon_i^δ be the list of derivatives of the entries in Mon_i with respect to δ. We will define matrices M, N, L, H. Note that each row is well defined up to scaling. This means that for all these matrices, the determinant is well defined up to scaling, so asking for the determinant to vanish is well defined.

$$
\begin{aligned}
&M = (a_{i,j})_{i,j\in\{1,2,3\}} && \text{with } a_{i,j} = \mathrm{Mon}_1[j](R_i);\\
&N = (b_{i,j})_{i,j\in\{1,\dots,6\}} && \text{with } b_{i,j} = \mathrm{Mon}_2[j](R_i);\\
&L = (c_{i,j})_{i,j\in\{1,\dots,10\}} && \text{with } c_{i,j} = \begin{cases} \mathrm{Mon}_3[j](R_i) & \text{for } i \leq 8\\ \mathrm{Mon}_3^x[j](R_8) & \text{for } i = 9\\ \mathrm{Mon}_3^z[j](R_8) & \text{for } i = 10 \end{cases}.
\end{aligned}
$$

For $\alpha_7, \alpha_8, \alpha_9 \in \{x, y, z\}$, we define the matrix

$$
H_{\alpha_7,\alpha_8,\alpha_9} = (d_{i,j})_{i,j\in\{1,\dots,15\}},
$$

$$
\text{with } d_{i,j} = \begin{cases} \mathrm{Mon}_4[j](R_i) & \text{for } i \leq 9\\ \mathrm{Mon}_4^{\beta_7}[j](R_7) & \text{for } i = 10\\ \mathrm{Mon}_4^{\gamma_7}[j](R_7) & \text{for } i = 11\\ \mathrm{Mon}_4^{\beta_8}[j](R_8) & \text{for } i = 12\\ \mathrm{Mon}_4^{\gamma_8}[j](R_8) & \text{for } i = 13\\ \mathrm{Mon}_4^{\beta_9}[j](R_9) & \text{for } i = 14\\ \mathrm{Mon}_4^{\gamma_9}[j](R_9) & \text{for } i = 15 \end{cases},
$$

where for $i \in \{7, 8, 9\}$, we have $\{\beta_i, \gamma_i\} = \{x, y, z\} \setminus \{\alpha_i\}$, with $\beta_i > \gamma_i$ with respect to lexicographic ordering.

Lemma 4.3.4. *The following hold.*

(i) The points R_1, R_2, and R_3 are collinear if and only if $\det(M) = 0$.

(ii) The points $R_1, \dots, R_6$ are on a conic if and only if $\det(N) = 0$.

(iii) If the points $R_1, \dots, R_8$ are on a cubic with a singular point at R_8, then $\det(L) = 0$. If $y_8 \neq 0$, then the converse also holds.

(iv) For all $\alpha_7, \alpha_8, \alpha_9$, if the points $R_1, \dots, R_9$ are on a quartic that is singular at R_7, R_8 and R_9, then $\det(H_{\alpha_7,\alpha_8,\alpha_9}) = 0$. If for all i in $\{7, 8, 9\}$, the α_i-coordinate of R_i is non-zero, then the converse also holds.

Proof.

(i) The determinant of M is zero if and only if there is a non-zero element in the nullspace of M, that is, there is a non-zero vector (m_1, m_2, m_3)

such that for all $i \in \{1,2,3\}$, we have $m_1 a_{i,1} + m_2 a_{i,2} + m_3 a_{i,3} = 0$. But this is the case if and only if the line defined by $m_1 x + m_2 y + m_3 z$ contains all three points.

(ii) This proof goes analogously to the proof of (i).

(iii) The determinant of L is zero if and only if there is a non-zero vector $(l_1, \ldots, l_{10})$ in k^{10} such that for all $i \in \{1, \ldots, 10\}$, we have $l_1 c_{i,1} + \cdots + l_{10} c_{i,10} = 0$. This is the case if and only if the cubic C defined by $\lambda = \sum_{i=1}^{10} l_i \mathrm{Mon}_3[i]$ contains all eight points $R_1, \ldots, R_8$, and moreover, the derivatives λ_x, λ_z of λ with respect to x and z vanish in R_8. So if $R_1, \ldots, R_8$ are on a cubic with a singular point at R_8, the determinant of L vanishes. Conversely, if $\det(L) = 0$ and $y_8 \neq 0$, since we have $x\lambda_x + y\lambda_y + z\lambda_z = 3\lambda$, this implies that also the derivative λ_y of λ with respect to y vanishes in R_8, hence C is singular in R_8.

(iv) Take $\alpha_7, \alpha_8, \alpha_9 \in \{x, y, z\}$. The determinant of $H_{\alpha_7,\alpha_8,\alpha_9}$ is zero if and only if there exists a non-zero vector given by $(h_1, \ldots, h_{15})$ such that for all $i \in \{1, \ldots, 15\}$, we have $h_1 d_{i,1} + \cdots + h_{15} d_{i,15} = 0$. This is the case if and only if the quartic K defined by $\lambda = \sum_{i=1}^{15} h_i \mathrm{Mon}_4[i]$ contains $R_1, \ldots, R_9$, and moreover, for $i \in \{7,8,9\}$, the derivatives λ_δ for $\delta \in \{x,y,z\} \setminus \{\alpha_i\}$ vanish in R_i. So if $R_1, \ldots, R_9$ are on a quartic that is singular at R_7, R_8 and R_9, the determinant of $H_{\alpha_7,\alpha_8,\alpha_9}$ vanishes. Conversely, if $\det(H_{\alpha_7,\alpha_8,\alpha_9}) = 0$ and the α_i-coordinate of R_i is non-zero for $i \in \{7,8,9\}$, then, since we have $x\lambda_x + y\lambda_y + z\lambda_z = 4\lambda$, this implies that also λ_{α_i} vanishes in R_i for $i \in \{7,8,9\}$. So K is singular in R_7, R_8, and R_9. $\square$

We recall that k is an algebraically closed field, and $\mathbb{P}^2$ is the projective plane over k.

Lemma 4.3.5. *If $R_1, \ldots, R_7$ are seven distinct points in $\mathbb{P}^2$ such that $R_1, \ldots, R_6$ are in general position, and the line L containing R_1 and R_7 contains none of the other points, then there is a unique cubic containing all seven points that is singular in R_1, which does not contain L.*

Proof. The linear system of cubics containing $R_1, \ldots, R_7$ is at least two-dimensional. Requiring that a cubic in this linear system is singular in R_1 gives two linear conditions, defining a linear subsystem $\mathcal{C}$ of dimension at least 0, so there is at least one cubic containing $R_1, \ldots, R_7$ that is singular at R_1.

Let D be an element of $\mathcal{C}$; we claim that D does not contain the line L that contains R_1 and R_7. Indeed, if D were the union of L and a conic C, then R_1 would be contained in C since it is a singular point of D. Since the points $R_2, \ldots, R_6$ are not on L by assumption, they would also be contained in C, contradicting the fact that $R_1, \ldots, R_6$ are in general position. So D does not contain L. Note that this implies that D is smooth in R_7, since if it were singular, then D would intersect L with multiplicity at least 4, hence D would contain L.
Now assume that there is more than one element in $\mathcal{C}$. Then there are two cubics D_1 and D_2 that contain $R_1, \ldots, R_7$ with a singularity at R_1, and whose defining polynomials are linearly independent. By what we just showed, they are not singular in R_7. For $i = 1, 2$, let l_i be the tangent line to D_i at R_7. If the equations defining l_1 and l_2 are not linearly independent, then there is an element F of $\mathcal{C}$ that is singular in R_7, giving a contradiction. We conclude that the equations defining l_1 and l_2 must be linearly independent. Therefore, there is an element G in $\mathcal{C}$ such that the line L through R_1 and R_7 is the tangent line to G at R_7. But then L intersects G in four points counted with multiplicity, so it is contained in G. This contradicts the fact that G is in $\mathcal{C}$. We conclude that there is a unique cubic through $R_1, \ldots, R_7$ that is singular in R_1, and which does not contain the line through R_1 and R_7. $\square$

Proposition 4.3.6. *Assume that the characteristic of k is not 2. Let $Q_1, \ldots, Q_8$ be eight points in $\mathbb{P}^2$ in general position. For $i \in \{1, 2, 3, 4\}$, let L_i be the line through Q_{2i} and Q_{2i-1}, and for $i, j \in \{1, \ldots, 8\}$, with $i \neq j$, let $C_{i,j}$ be the unique cubic through $Q_1, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_8$ that is singular in Q_j, which exists by Lemma 4.3.5. Assume that the four lines L_1, L_2, L_3 and L_4 are concurrent in a point P. Then the three cubics $C_{7,8}$, $C_{8,7}$, and $C_{6,5}$ do not all contain P.*

Proof. First note that if P were equal to one of the Q_i, then three of the eight Q_i would be on a line, which would contradict the fact that $Q_1, \ldots, Q_8$ are in general position. We conclude that P is not equal to one of the Q_i. Moreover, if P were collinear with any two of the three points Q_1, Q_3, Q_5, say for example with Q_1 and Q_3, then, since P is also contained in L_1 and L_2, it would follow that L_1 and L_2 are equal, giving a contradiction. So Q_1, Q_3, Q_5 and P are in general position.
Let $(x : y : z)$ be the coordinates in $\mathbb{P}^2$. Without loss of generality, after

applying an automorphism of $\mathbb{P}^2$ if necessary, we can define

$$Q_1 = (0:1:1); \qquad Q_3 = (1:0:1)$$
$$Q_5 = (1:1:1); \qquad P = (0:0:1).$$

Then we have the following.

$$L_1 \text{ is the line given by } x = 0;$$
$$L_2 \text{ is the line given by } y = 0;$$
$$L_3 \text{ is the line given by } x = y.$$

Since L_4 contains P, and is unequal to L_1 and L_2, there is an $m \in k^*$ such that L_4 is the line given by $my = x$. Since Q_2, Q_7 and Q_8 are not in L_2, and Q_4 is not in L_1, there are $a, b, c, u, v \in k$ such that

$$Q_2 = (0:1:a); \qquad Q_7 = (m:1:v);$$
$$Q_4 = (1:0:b); \qquad Q_8 = (m:1:c).$$
$$Q_6 = (1:1:u);$$

We define $\mathbb{A}^6$ to be the affine space with coordinate ring T_6 given by $T_6 = k[a, b, c, m, u, v]$. Points in $\mathbb{A}^6$ correspond to configurations of the points $Q_1, \ldots, Q_8$.

Assume by contradiction that $C_{7,8}$, $C_{8,7}$, and $C_{6,5}$ all contain P. This assumption gives polynomial equations in the variables a, b, c, m, u, v, and hence defines an algebraic set A_0 in $\mathbb{A}^6$. We define S_0 to be the algebraic set of all points in $\mathbb{A}^6$ that correspond to the configurations where three of the points $Q_1, \ldots, Q_8$ lie on a line, or six of the points lie on a conic. We want to show that A_0 is contained in S_0, which proves the proposition.

Note that the line containing P and Q_5, which is L_3, does not contain any of the points Q_1, Q_2, Q_3, Q_4, Q_8. From Lemma 4.3.5, after substituting $(R_1, \ldots, R_7) = (Q_5, Q_1, Q_2, Q_3, Q_4, Q_8, P)$, it follows that there is a unique cubic D containing $Q_1, Q_2, Q_3, Q_4, Q_5, Q_8$ and P that is singular in Q_5, and that D does not contain L_3. By uniqueness, D must be equal to $C_{6,5}$, and therefore also contains Q_7. By Lemma 4.3.4, the equation expressing that Q_7 is contained in D (or equivalently, that P is contained in $C_{6,5}$) is given by $\det(L) = 0$, where L is the matrix used in the lemma, associated to the points $(R_1, \ldots, R_8) = (Q_1, Q_2, Q_3, Q_4, Q_7, Q_8, P, Q_5)$. We have

$$\det(L) = -m(m-1)(c-v)(b-1)(a-1)f,$$

where $f = \alpha v + \beta$, with

$$\alpha = a - ac - bc + bm, \quad \beta = b(a-1)m^2 + b(c-2a)m + a(b+c-1).$$

The first five factors of $\det(L)$ define subsets of S_0, and do not correspond to configurations where $Q_1, \ldots, Q_8$ are in general position. Therefore, $C_{6,5}$ contains P if and only if $f = 0$. Define the algebraic set $V = Z(\alpha)$, and let $(a_0, b_0, c_0, m_0, u_0, v_0)$ be an element in $V \cap A_0$. Then we have $\alpha(a_0, b_0, c_0, m_0, u_0, v_0) = f(a_0, b_0, c_0, m_0, u_0, v_0) = 0$, so we find $\beta(a_0, b_0, c_0, m_0, u_0, v_0) = 0$. But α and β do not depent on v, so this implies that we have $f(a_0, b_0, c_0, m_0, u_0, v') = 0$ for every v'. So every element in $V \cap A_0$ corresponds to a configuration of $Q_1, \ldots, Q_8$ such that every point $(m : 1 : v')$ on L_4 is also contained in D. But if this is the case, then D consists of L_4 and a conic, which is singular, since Q_5 is a singular point of D that is not contained in L_4. Since L_4 contains none of the points Q_1, Q_2, Q_3, Q_4, these four points are then on the singular conic, which implies that Q_5 is collinear with at least two other points. We conclude that $V \cap A_0$ is a subset of S_0.
Analogously, the fact that $C_{7,8}$ contains P is expressed by $\det(L') = 0$, where L' is the matrix denoted by L in Lemma 4.3.4 with

$$(R_1, \ldots, R_8) = (Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, P, Q_8).$$

We have

$$\det(L') = -m(u-1)(m-1)(b-1)(a-1)g,$$

where $g = \gamma u + \delta$ with

$$\gamma = bm^3 + (1 - bc - c)m^2 + (c^2 - 2c + 1)m + a(1 - c) + c^2 - c,$$

and

$$\begin{aligned}\delta = -abm^3 &+ (abc + ab + ac - a + b - 2bc)m^2 + \\ &(ab - 2abc + a + 2bc^2 - b - ac^2 + 2c^2 - 2c)m \\ &+ a(bc - b + 2c^2 - 2c) - bc^2 + bc - 2c^3 + 2c^2.\end{aligned}$$

The first five factors of $\det(L')$ correspond to configurations where the eight points are not in general position, so $C_{7,8}$ contains P if and only if $g = 0$. Define $U = Z(\gamma)$. By the same reasoning as for $V \cap A_0$ (now using the fact that D does not contain the line L_3), we have $U \cap A_0 \subseteq S_0$. Set

$$v' = \frac{-\beta}{\alpha} \quad \text{and} \quad u' = \frac{-\delta}{\gamma}.$$

Define $\mathbb{A}^4$ to be the affine space with coordinate ring $T_4 = k[m, a, b, c]$, and let K_4 be its fraction field. Let $Y \subset \mathbb{A}^4$ be the set defined by $\alpha = \gamma = 0$.

Consider the ring homomorphism $\psi\colon T_6 \longrightarrow K_4$ defined by

$$(m, a, b, c, u, v) \longmapsto (m, a, b, c, u', v').$$

This defines a morphism $i\colon \mathbb{A}^4\backslash Y \longrightarrow \mathbb{A}^6\backslash(V \cup U)$, which is a section of the projection $\mathbb{A}^6 \longrightarrow \mathbb{A}^4$ to the first four coordinates. Set $A_0' = A_0 \setminus (V \cup U)$. Then we have $A_0 \subset S_0$ if and only if $A_0' \subseteq S_0$. Moreover, A_0' is contained in $Z(f, g)$, and since f and g are linear in v and u respectively, we have $i^{-1}(A_0') \cong A_0'$. Set $A_1 = i^{-1}(A_0')$ and $S_1 = i^{-1}(S_0)$, then $A_0' \subseteq S_0$ is equivalent to $A_1 \subseteq S_1$.
Let L'' be the matrix denoted by L in Lemma 4.3.4 with

$$(R_1, \ldots, R_8) = (Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, P, Q_7).$$

Similarly to $C_{7,8}$, the fact that $C_{8,7}$ contains P is expressed by the vanishing of the determinant of L''. We compute this determinant and write it in terms of the coordinates of $\mathbb{A}^4$ using ψ. We find the expression

$$-2abm(m-1)^3(b-1)(a-1)(a+b-1)f_1f_2f_3, \tag{4.1}$$

with

$$f_1 = ac - a + bcm - bm^2 - c^2 + cm + c - m,$$

$$f_2 = abm^2 - 2abm + ab - ac^2 + 2ac - a - bc^2 + 2bcm - bm^2,$$

and

$$\begin{aligned} f_3 = {} & abcm^2 - 2abcm + abc - abm^3 + abm^2 + abm - ab - ac^2m + 2ac^2 \\ & + acm^2 - 3ac - am^2 + am + a + 2bc^2m - bc^2 - 3bcm^2 + bc + bm^3 \\ & + bm^2 - bm - 2c^3 + 3c^2m + 3c^2 - cm^2 - 4cm - c + m^2 + m. \end{aligned}$$

Expression (4.1) defines the set A_1 in $\mathbb{A}^4$. Since char $k \neq 2$, we have $(4.1) = 0$ if and only if at least one of the non-constant factors of (4.1) equals zero. We show that all non-constant factors of expression (4.1) define components of S_1. If $a = 0$, then Q_2, Q_3 and Q_5 are contained in the line given by $x - z = 0$. Similarly, $b = 0$ implies that Q_1, Q_4 and Q_5 are on the line given by $y - z = 0$, and $a + b - 1 = 0$ implies that Q_2, Q_4, and Q_5 are on the line given by $bx + ay - z = 0$. If $m = 0$ then $L_4 = L_2$, and $m = 1$ implies $L_4 = L_3$, so in both cases there are four points on a line. If $a = 1$ or $b = 1$, then two of the eight points would be the same. Set

$(R_1, \ldots, R_6) = (Q_3, \ldots, Q_8)$, and let N be the corresponding matrix from Lemma 4.3.4. We compute the determinant of N and find that $f_1 f_2 f_3$ divides $\det(N)$. This means that f_1, f_2, as well as f_3 define components of S_1, more specifically, they define configurations where $Q_3, \ldots, Q_8$ are on a conic. We conclude that all irreducible components of A_1 are contained in S_1, which finishes the proof. □

Remark 4.3.7. Note that, theoretically, we could have proved Proposition 4.3.6 with a computer, by checking that A_0 is contained in S_0 using Groebner bases. However, in practice, this turned out to be too big for `magma` to do.

We can now prove Theorem 4.1.1. We use the following notation.

Notation 4.3.8. Let $P_1, \ldots, P_8$ be eight points in general position in $\mathbb{P}^2$ such that X is isomorphic to $\mathbb{P}^2$ blown up these points. For $i \in \{1, \ldots, 8\}$, let E_i be the class in Pic X corresponding to the exceptional curve above P_i, and let L be the class in Pic X corresponding to the pullback of a line in $\mathbb{P}^2$ that does not contain any of the points $P_1, \ldots, P_8$.

Recall that a maximal set of exceptional curves that are concurrent in a point on the ramification curve consists of curves and their partners (Remark 4.3.2).

Proof of Theorem 4.1.1. First note that by Proposition 4.2.2, the number of exceptional curves through any point in X is at most sixteen in all characteristics; this proves the case char $k = 2$.
Now assume char $k \neq 2$. Consider the clique $K = \{e_1, \ldots, e_6, e_1', \ldots, e_6'\}$ in G, where

$$\begin{aligned}
e_1 &= L - E_1 - E_2;\\
e_2 &= L - E_3 - E_4;\\
e_3 &= L - E_5 - E_6;\\
e_4 &= L - E_7 - E_8;\\
e_5 &= 3L - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - 2E_8;\\
e_6 &= 3L - E_1 - E_2 - E_3 - E_4 - 2E_5 - E_7 - E_8,
\end{aligned}$$

and e_i' is the partner of e_i, for all $i \in \{1, \ldots, 6\}$. By Remark 1.2.7, the classes $e_1, \ldots, e_4$ correspond to the strict transforms of the four lines through P_i and P_{i+1} for $i \in \{1, 3, 5, 7\}$, and e_5, e_6, e_5' correspond to the

strict transforms of the unique cubics through the points $P_1, \ldots, P_6, P_8$, and the points $P_1, \ldots, P_5, P_7, P_8$, and the points $P_1, \ldots, P_6, P_7$, respectively, that are singular in P_8, and P_5, and P_7, respectively.
Now let K' be a clique in G with only edges of weights 1 and 3, consisting of at least six sets of an exceptional class with its partner. Let $\{\{f_1, f_1'\}, \ldots, \{f_6, f_6'\}\}$ be a set of six such sets in K'. Since W acts transitively on the set of cliques of six exceptional classes and their partners by Proposition 4.3.3, after changing the indices and interchanging f_i's with their partner if necessary, there is an element $w \in W$ such that $f_i = w(e_i)$ and $f_i' = w(e_i')$ for $i \in \{1, \ldots, 6\}$. For $i \in \{1, \ldots, 8\}$, set $E_i' = w(E_i)$. Since the E_i' are pairwise disjoint, by Lemma 1.2.8 we can blow down $E_1', \ldots, E_8'$ to points $Q_1, \ldots, Q_8 \in \mathbb{P}^2$ that are in general position, such that X is isomorphic to the blow-up of $\mathbb{P}^2$ at $Q_1, \ldots, Q_8$, and E_i' is the class in Pic X corresponding to the exceptional curve above Q_i for all i. By Remark 1.2.9, the sequence $(E_1', \ldots, E_8')$ induces a bijection between the exceptional curves on X and the 240 vectors in Proposition 1.2.6, such that the element f_i corresponds to the class of the strict transform of the line through Q_{2i-1} and Q_i for $i \in \{1, \ldots, 4\}$, the elements f_5 and f_6 correspond to the classes of the strict transforms of the unique cubics through the points $Q_1, \ldots, Q_6, Q_8$ and $Q_1, \ldots, Q_5, Q_7, Q_8$, respectively, that are singular in Q_8 and Q_5 respectively, and f_i' is the unique class in I intersecting f_i with multiplicity three for all i. From Proposition 4.3.6 it follows that the curves on X corresponding to $f_1, \ldots, f_6, f_5'$ and f_6' are not concurrent.
We conclude that a set of at least six exceptional curves and their partners is never concurrent. Since any maximal set of exceptional curves going through the same point on the ramification curve forms a clique consisting of curves and their partners, hence of even size, we conclude that this maximum is at most ten. $\square$

4.4 Proof of Theorem 4.1.2

In this section we prove Theorem 4.1.2. The structure of the proof is similar to that of Theorem 4.1.1; we first determine the cliques in G that possibly come from a set of exceptional curves that are concurrent outside the ramification curve of φ (Remark 4.4.1), and show that their maximal size is 12 (Proposition 4.4.2). Then we show that the group W acts transitively on these cliques of size 12 (Proposition 4.4.3) and 11 (Proposition 4.4.4), and finally we show that ten curves in $\mathbb{P}^2$ in a specific configuration are not concurrent in Proposition 4.4.6. This final proposition is again key to the proof of Theorem 4.1.2.

Remark 4.4.1. From Remark 4.2.5 we know that cliques in G corresponding to exceptional curves that intersect each other in a point outside the ramification curve have no edges of weight 3. We conclude that these cliques contain only edges of weights 1 and 2.

Proposition 4.4.2. *The maximal size of cliques in G with only edges of weights 1 and 2 is 12, and there are no maximal cliques with only edges of weights 1 and 2 of size 11.*

Proof. We use the correspondence with the graph Γ in Chapter 3, where the corresponding cliques have only edges of colors -1 and 0; the statement is Proposition 3.5.23. □

Proposition 4.4.3. *The group W acts transitively on the set of cliques of size 12 in G with only edges of weights 1 and 2.*

Proof. This is Proposition 3.5.24. □

Proposition 4.4.4. *The group W acts transitively on the set of cliques of size 11 in G with only edges of weights 1 and 2.*

Proof. By Proposition 4.4.2, any clique of size 11 with only edges of weights 1 and 2 is contained in a clique of size 12 with only edges of weights 1 and 2. By Corollary 3.5.25, for such a clique K of size 12, the stabilizer W_K acts transitively on K, which implies that W_K also acts transitively on the set of cliques of size 11 within K. Since W acts transitively on the set of all cliques of size 12 with only edges of weights 1 and 2 by Proposition 4.4.3, the statement follows. □

Now that we know which cliques in G to look at and what their maximal size is, we show that ten curves in $\mathbb{P}^2$ in a specific configuration are not concurrent in Proposition 4.4.6.

Remark 4.4.5. It is well known that two distinct points in $\mathbb{P}^2$ define a unique line, and five points in $\mathbb{P}^2$ in general position define a unique conic. Now let $R_1, \ldots, R_8$ be eight distinct points in $\mathbb{P}^2$ in general position. The linear system $\mathcal{Q}$ of quartics in $\mathbb{P}^2$ has dimension 14. For three distinct points $R_i, R_j, R_l \in \{R_1, \ldots, R_8\}$, requiring a quartic to contain $R_1, \ldots, R_8$ and be singular in in R_i, R_j, R_l gives $8+3\cdot 2 = 14$ linear relations. Since the eight points are in general position, the 14 linear conditions are linearly independent, so this gives a zero-dimensional linear subsystem of $\mathcal{Q}$. Hence there is a unique quartic containing all eight points that is singular in R_i, R_j, R_l.

Let $R_1, \ldots, R_8$ be eight points in $\mathbb{P}^2$ in general position. Remark 4.4.5 allows us to define the following curves.

L_1 is the line through R_1 and R_2;

L_2 is the line through R_3 and R_4;

C_1 is the conic through $R_1,\ R_3,\ R_5,\ R_6$ and R_7;

C_2 is the conic through $R_1,\ R_4,\ R_5,\ R_6$ and R_8;

C_3 is the conic through $R_2,\ R_3,\ R_5,\ R_7$ and R_8;

C_4 is the conic through $R_2,\ R_4,\ R_6,\ R_7$ and R_8;

D_1 is the quartic through all eight points, singular in $R_1,\ R_7$ and R_8;

D_2 is the quartic through all eight points, singular in $R_2,\ R_5$ and R_6;

D_3 is the quartic through all eight points, singular in $R_3,\ R_6$ and R_8;

D_4 is the quartic through all eight points, singular in $R_4,\ R_5$ and R_7.

Proposition 4.4.6. *Assume that the characteristic of k is not 3. Then the ten curves $L_1,\ L_2,\ C_1, \ldots C_4,\ D_1, \ldots, D_4$ are not concurrent.*

Remark 4.4.7. As in the case of Proposition 4.3.6, in theory we could prove Proposition 4.4.6 with a computer by using Groebner bases, but in practice, this is undoable since the computations become too big (see also Remark 4.3.7). In the case of Proposition 4.4.6 the computations become even bigger, since we now have 10 curves to check, four of which are of degree 4, in contrast to the 7 curves of degrees at most 3 in Proposition 4.3.6.

Before we write down the proof of Proposition 4.4.6, we make some reductions. In $\mathbb{P}^2$, we can choose four points in general position. Fix these and call them Q_1, Q_5, Q_6, and R. We are interested in those configurations of five points Q_2, Q_3, Q_4, Q_7 and Q_8 in $\mathbb{P}^2$ such that the following 11 conditions hold.

0) The points $Q_1, \ldots, Q_8$ are in general position.
1) There is a line through R, Q_1, Q_2.
2) There is a line through R, Q_3, Q_4.
3) There is a conic through $R, Q_1, Q_3, Q_5, Q_6, Q_7$.
4) There is a conic through $R, Q_1, Q_4, Q_5, Q_6, Q_8$.
5) There is a conic through $R, Q_2, Q_3, Q_5, Q_7, Q_8$.
6) There is a conic through $R, Q_2, Q_4, Q_6, Q_7, Q_8$.
7) There is a quartic through all nine points, singular in Q_1, Q_7, Q_8.
8) There is a quartic through all nine points, singular in Q_2, Q_5, Q_6.
9) There is a quartic through all nine points, singular in Q_3, Q_6, Q_8.
10) There is a quartic through all nine points, singular in Q_4, Q_5, Q_7.

We will prove Proposition 4.4.6 by showing that there are no such configurations: all of the configurations satisfying 1–10 violate condition 0.

We consider the space $(\mathbb{P}^2)^5$. Within this space, we define the following two sets.

$$Y = \left\{(Q_2, Q_3, Q_4, Q_7, Q_8) \in (\mathbb{P}^2)^5 \ \mid \text{conditions 1–5 are satisfied}\right\}.$$

$$S = \left\{(Q_2, Q_3, Q_4, Q_7, Q_8) \in (\mathbb{P}^2)^5 \ \mid \text{three of } Q_1, \ldots, Q_8 \text{ are collinear}\right\}.$$

Note that for an element $(Q_2, Q_3, Q_4, Q_7, Q_8)$ in S, condition 0 is violated. Let F_1 be the linear system of conics through R, Q_1, Q_5, Q_6. Note that this is a one-dimensional linear system that is isomorphic to $\mathbb{P}^1$. Let F_2 be the linear system of lines through R, which is also isomorphic to $\mathbb{P}^1$. We will show that there is a bijection between $Y \setminus S$ and a subset of $F_1^2 \times F_2^3$ in Proposition 4.4.15. We start with two lemmas.

Lemma 4.4.8. *If $(Q_2, Q_3, Q_4, Q_7, Q_8)$ is a point in $Y \setminus S$, then we have $Q_i \neq R$ for $i = 2, 3, 4, 7, 8$.*

Proof. Take a point $Q = (Q_2, Q_3, Q_4, Q_7, Q_8)$ in $Y \setminus S$. Since Q is an element of Y, by condition 1 the points R, Q_1, Q_2 are on a line. That means that if $R = Q_i$ for $i = 3, 4, 7, 8$, the points Q_i, Q_1, Q_2 would be on a line, contradicting the fact that Q is not in S. Moreover, by condition 2, the points R, Q_3, Q_4 are on a line, so if $R = Q_2$ then Q_2, Q_3, Q_4 are on a line, again contradicting the fact that Q is not in S. □

The following result is well known, but we include a proof, as we could not find a reference for this exact statement.

Lemma 4.4.9. *If $S_1, \ldots, S_5$ are five distinct points in $\mathbb{P}^2$, such that the four points $S_1, \ldots, S_4$ are in general position, then there is a unique conic containing $S_1, \ldots, S_5$, which is irreducible if all five points are in general position.*

Proof. The linear system of conics containing $S_1, \ldots, S_4$ is one-dimensional and has only these four points as base points. Requiring for a conic in this linear system to contain the point S_5 gives a linear condition, and since S_5 is different from $S_1, \ldots, S_4$, this condition defines a linear subspace of dimension at least zero. If there were two distinct conics in this subspace, they would intersect in 5 distinct points, so they would have a common component, which is a line. Since no 4 of the points $S_1, \ldots, S_5$ are collinear, there are at most 3 of the 5 points on this line. But then the other two points uniquely determine the second component of both conics, contradicting that they are distinct. We conclude that there is a unique conic containing $S_1, \ldots, S_5$. If, moreover, S_5 is such that all five points are in general position, then no three of them are collinear by definition, so the unique conic containing them cannot contain a line, hence it is irreducible. □

Notation 4.4.10. Let $(Q_2, Q_3, Q_4, Q_7, Q_8)$ be a point in $Y \setminus S$. Note that by condition 3, there is a conic through the points R, Q_1, Q_3, Q_5, Q_6, and Q_7, and by Lemma 4.4.9 it is unique, since R, Q_1, Q_5, Q_6 are in general position. We call this conic A_1. By the same reasoning and condition 4, there is a unique conic containing the points $R, Q_1, Q_4, Q_5, Q_6, Q_8$. We call this conic A_2. By Lemma 4.4.8, the points Q_3, Q_7, Q_8 are all different from R, so we can define the line M_1 through X and Q_3, the line M_2 through R and Q_7, and the line M_3 through R and Q_8.

Recall that F_1 is the linear system of conics through R, Q_1, Q_5, Q_6, and F_2 the linear system of lines through R. We define a map

$$\varphi\colon Y \setminus S \longrightarrow F_1^2 \times F_2^3,$$
$$(Q_2, Q_3, Q_4, Q_7, Q_8) \longmapsto (A_1, A_2, M_1, M_2, M_3).$$

Note that φ is well defined by the definitions of A_1, A_2, M_1, M_2, M_3 in Notation 4.4.10. We want to describe its image. To this end, define the set

$$U = \left\{ (B_1, B_2, N_1, N_2, N_3) \in F_1^2 \times F_2^3 \;\middle|\; \begin{array}{c} B_1, B_2 \text{ irreducible} \\ B_1 \neq B_2 \\ N_1, N_2 \text{ not tangent to } B_1 \\ N_1, N_3 \text{ not tangent to } B_2 \\ N_1 \neq N_2, N_3 \\ Q_1, Q_5, Q_6 \notin N_1, N_2, N_3 \end{array} \right\}.$$

Lemma 4.4.11. *The image of φ is contained in U.*

Proof. Take a point $Q = (Q_2, Q_3, Q_4, Q_7, Q_8) \in Y \setminus S$ and consider its image under φ given by $\varphi(Q) = (A_1, A_2, M_1, M_2, M_3)$. Since Q is not in S, by Lemma 4.4.9, the conics A_1 and A_2 are unique and irreducible. Moreover, if they were equal to each other, then they would both contain the points R, Q_3, Q_4, which are collinear by condition 2, contradicting the fact that they are irreducible.
The line M_1 is tangent to A_1 only if R is equal to Q_3, the line M_2 is tangent to A_1 only if R is equal to Q_7, and the line M_3 is tangent to A_2 only if R is equal to Q_8, all of which are impossible by Lemma 4.4.8. Note that by condition 2, the line M_1 contains Q_4, so M_1 is tangent to A_2 only if $R = Q_4$, which is again impossible by Lemma 4.4.8. If M_2 or M_3 were equal to M_1, then either Q_7 or Q_8 is contained in M_1, which also contains the points R, Q_3, Q_4. But this can not be true since Q is not in S. If M_1 or M_2 contained any of the points Q_1, Q_5, Q_6, then this line would have three points in common with A_1, which implies that A_1 contains a line, contradicting the fact that A_1 is irreducible. Similarly, if M_3 contained Q_1, Q_5, or Q_6, then A_2 would contain M_3, contradicting the irreducibility of A_2. $\square$

We want to define an inverse to φ. We set up the following notation for a point in U.

Notation 4.4.12. Let $u = (B_1, B_2, N_1, N_2, N_3)$ be a point in U. Since the conics B_1 and B_2 are irreducible, they do not contain any of the lines N_1, N_2, N_3, and moreover, since N_1, N_2 are not tangent to B_1, and N_1, N_3

are not tangent to B_2, we can define the following five points in $\mathbb{P}^2$.

$$S_3 = \text{the point of intersection of } B_1 \text{ with } N_1 \text{ that is not } X.$$
$$S_4 = \text{the point of intersection of } B_2 \text{ with } N_1 \text{ that is not } X.$$
$$S_7 = \text{the point of intersection of } B_1 \text{ with } N_2 \text{ that is not } X.$$
$$S_8 = \text{the point of intersection of } B_2 \text{ with } N_3 \text{ that is not } X.$$

Lemma 4.4.13. *Let $u = (B_1, B_2, N_1, N_2, N_3)$ be a point in U. Define the points S_3, S_4, S_7, S_8 as in Notation 4.4.12. There is a unique conic through R, S_3, Q_5, S_7, and S_8, which does not contain the line through R and Q_1.*

Proof. Note that S_3 and S_7 are different from R by definition, and they are different from Q_1, Q_5, Q_6 since Q_1, Q_5, Q_6 are not contained in N_1, nor in N_2, by definition of U. If S_3 were equal to S_7, then N_1 and N_2 would both contain R and S_3, hence they would be equal, contradicting the fact that u is an element of U. So R, S_3, Q_5, S_7 are all distinct, and since they are all contained in B_1, they are in general position because B_1 is irreducible. We will show that S_8 is different from any of these four points. By definition, S_8 is different from R. If S_8 were equal to S_3, then B_1 and B_2 would both contain R, Q_1, Q_5, Q_6 and S_3. But since S_3 is different from R, Q_1, Q_5, Q_6, there is a unique conic through these five points by Lemma 4.4.9. So this would imply $B_1 = B_2$, contradicting the fact that u is in U. Hence S_8 is different from S_3, and similarly, S_8 is different from S_7. Finally, S_8 is different from Q_5, since the line N_3 does not contain Q_5. We conclude that by Lemma 4.4.9, there is a unique conic C through the points R, S_3, Q_5, S_7, and S_8. Note that R, S_3, Q_5, S_7 are all distinct from Q_1. If C contained the line L through R and Q_1, then C would be the union of two lines (one of them being L). This means that either L would contain one of the points S_3, Q_5, S_7, or the points S_3, Q_5, S_7 are all on the second line. But since R, Q_1, S_3, Q_5, S_7 are all in B_1, which is irreducible, both of these cases would be a contradiction. We conclude that C does not contain L. □

Notation 4.4.14. Let $u = (B_1, B_2, N_1, N_2, N_3)$ be a point in U, and let S_3, S_4, S_7, S_8 be the corresponding points as in Notation 4.4.12. We define a fifth point S_2 to be the point of intersection of the conic through R, S_3, Q_5, S_7, S_8 with the line through R and Q_1, that is not R. Note that S_2 is well defined by Lemma 4.4.13.

Using Notations 4.4.12 and 4.4.14, for any point u in U we have now defined an element $(S_2, S_3, S_4, S_7, S_8)$ of $(\mathbb{P}^2)^5$, and it is easy to see that for such a point conditions 1–5 are satisfied, hence it is an element of Y. This leads us to define the following map.

$$\psi\colon U \longrightarrow Y,$$
$$(B_1, B_2, N_1, N_2, N_3) \longmapsto (S_2, S_3, S_4, S_7, S_8).$$

Let T be the set $\psi^{-1}(S)$.

Proposition 4.4.15. *The map $\psi|_{U\setminus T}\colon U \setminus T \longrightarrow Y \setminus S$ is a bijection, with inverse given by φ.*

Proof. Let $u = (B_1, B_2, N_1, N_2, N_3)$ be an element in $U \setminus T$. Write $\psi(u) = (S_2, S_3, S_4, S_7, S_8)$ and $\varphi(\psi(u)) = (B_1', B_2', N_1', N_2', N_3')$. Since $\psi(u)$ is not in S by definition of T, no three of the points $Q_1, Q_5, Q_6, S_2, S_3, S_4, S_7, S_8$ are collinear. Therefore, B_1' and B_2' are the unique and irreducible conics through Q_1, S_3, Q_5, Q_6, S_7 and through Q_1, S_4, Q_5, Q_6, S_8, respectively, by Lemma 4.4.9. Since B_1 and B_2 both contain Q_1, Q_5, Q_6, and B_1 contains S_3 and S_7 and B_2 contains S_4 and S_8 by definition of $\psi(u)$, we conclude that $B_1' = B_1$ and $B_2' = B_2$. The line N_1' is defined as the line containing R and S_3, which are both contained in N_1 as well by definition. We conclude that $N_1' = N_1$, and similarly $N_2' = N_2$, and $N_3' = N_3$. We conclude that $\varphi(\psi(u)) = u$. This proves injectivity of $\psi|_{U\setminus T}$. We now prove surjectivity. Take $Q = (Q_2, Q_3, Q_4, Q_7, Q_8) \in Y\setminus S$; write $\varphi(Q) = (A_1, A_2, M_1, M_2, M_3)$ and $\psi(A_1, A_2, M_1, M_2, M_3) = (Q_2', Q_3', Q_4', Q_7', Q_8')$. The point Q_3' is defined by taking the second point of intersection of A_1 with the line M_1 through R and Q_3. Since A_1 is irreducible ($\varphi(Q)$ is in U by Lemma 4.4.11), it does not contain M_1, so $Q_3' = Q_3$. Similarly, we have $Q_7' = Q_7$, $Q_4' = Q_4$, and $Q_8' = Q_8$. Therefore there is a unique conic C containing the points R, Q_3, Q_5, Q_7, Q_8 by Lemma 4.4.13. Since there is a conic through R, Q_3, Q_5, Q_7, Q_8 and Q_2 by condition 5, we conclude that C contains Q_2 by uniqueness. Since the line L through R and Q_1 is not contained in C by Lemma 4.4.13, and since L contains Q_2 by condition 1, it follows that Q_2 is the second point of intersection of L and C. Hence $Q_2' = Q_2$. We conclude that $\psi(\varphi(Q)) = Q$, and hence $\varphi(Q)$ is contained in $U \setminus T$, and $\psi|_{U\setminus T}$ is surjective.

Since $\psi_{U\setminus T}\colon U \setminus T \longrightarrow Y \setminus S$ is a bijection and we showed that for all elements $u \in U \setminus T$ we have $\varphi(\psi(u)) = u$, we conclude that φ is the inverse function. □

We now prove Proposition 4.4.6. The computations are verified in `magma`; see [Codc] for the code. Recall that we fixed eight points $R_1, \dots, R_8$ in general position $\mathbb{P}^2$ and ten curves $L_1, L_2, C_1, \dots, C_4, D_1, \dots, D_4$, above Proposition 4.4.6.

Proof of Proposition 4.4.6. We assume that these ten curves contain a common point P, and will show that this contradicts the fact that $R_1, \dots, R_8$ are in general position. First note that if P were equal to one of the eight points $R_1, \dots, R_8$, then one of the conics would contain six of the eight points, which would contradict the fact that $R_1, \dots, R_8$ are in general position. Moreover, if P and any two of the three points R_1, R_5, R_6 lie on a line L, then the conic C_1 would intersect L in P and the two points. But this implies that C_1 is not irreducible, and since C_1 contains five of the points $R_1, \dots, R_8$, this implies that at least three of them are collinear, contradicting the fact that $R_1, \dots, R_8$ are in general position. We conclude that R_1, R_5, R_6 and P are in general position.
Let $(x : y : z)$ be the coordinates in $\mathbb{P}^2$. Without loss of generality, after applying an automorphism of $\mathbb{P}^2$ if necessary, we can choose R_1, R_5, R_6, and P to be any four points in general position in $\mathbb{P}^2$. We now distinguish between char $k \neq 2$ and char $k = 2$.

Assume char $k \neq 2$. Set

$$R_1 = (1 : 0 : 1); \qquad R_6 = (0 : -1 : 1);$$
$$R_5 = (0 : 1 : 1); \qquad P = (-1 : 0 : 1).$$

It follows that the line L_1, which contains R_1 and P, is given by $y = 0$. The linear system of quadrics through R_1, R_5, R_6 and P is generated by two linearly independent quadrics, and we take these to be $x^2 + y^2 - z^2$ and xy. Let $l, m \in k$ be such that

$$C_1 \text{ is given by } x^2 + y^2 - z^2 = 2lxy;$$
$$C_2 \text{ is given by } x^2 + y^2 - z^2 = 2mxy.$$

Since R_3, R_4, R_7, and R_8 are not contained in L_1, there are $s, t, u \in k$ such that

the line L_2 is given by $sy = x + z$;
the line L_3 through P and R_7 is given by $ty = x + z$;
the line L_4 through P and R_8 is given by $uy = x + z$.

We want to show that all possible configurations of the five points R_2, R_3, R_4, R_7, and R_8 in $\mathbb{P}^2$ such that all ten curves contain P, are such that

$R_1, \ldots, R_8$ are not in general position. By Proposition 4.4.15, all configurations of R_2, R_3, R_4, R_7, R_8 such that L_1, L_2, C_1, C_2, C_3 contain the point P and no three of the points $R_1, \ldots, R_8$ are collinear are given in terms of the conics C_1 and C_2 and the lines L_2, L_3, L_4. By computing the appropriate intersections we find

$$\begin{aligned} R_3 &= \left(-s^2+1 : 2l-2s : 2ls-s^2-1\right); \\ R_4 &= \left(-s^2+1 : 2m-2s : 2ms-s^2-1\right); \\ R_7 &= \left(-t^2+1 : 2l-2t : 2lt-t^2-1\right); \\ R_8 &= \left(-u^2+1 : 2m-2u : 2mu-u^2-1\right). \end{aligned}$$

By Lemma 4.4.13, there is a unique conic containing R_3, R_5, R_7, R_8, and P, and we compute a defining polynomial and find

$$\begin{gathered} \Big(2l^2u+2l^2-2lmu-2lm-lsu-ls-ltu-lt+lu^2+2lu+l+mst \\ +ms+mt-2mu-m+st-su-tu+u^2\Big)\,x^2+\Big(2l^2u^2+2l^2u \\ +2lmst-2lmsu-2lmtu-2lmu-lstu+lst-lsu+ls-ltu+lt \\ +2lu^2+lu+l+mstu+mst-msu-ms-mtu-mt-mu-m\Big)\,xy \\ +2(u+1)(l+1)(l-m)xz+\Big(lstu+lst+lu^2+lu-mstu-msu-mtu \\ -mu+st-su-tu+u^2\Big)\,y^2+(u+1)(t+1)(s+1)(l-m)yz+\Big(lsu \\ +ls+ltu+lt-lu^2+l-mst-ms-mt-m-st+su+tu-u^2\Big)\,z^2. \end{gathered}$$

Intersecting this conic with the line L_1 gives besides P the point R_2, and we find

$$\begin{aligned} R_2 = (&-(lsu+ls+ltu+lt-lu^2+l-mst-ms-mt-m \\ &-st+su+tu-u^2) : 0 : (2l^2-2lm-ls-lt)(u+1)+lu^2 \\ &+2lu+l+mst+ms+mt-2mu-m+st-su-tu+u^2). \end{aligned}$$

We define $\mathbb{A}^5$ to be the affine space with coordinate ring $T_5 = k[l, m, s, t, u]$. Following all the above, points in $\mathbb{A}^5$ correspond to configurations of the points $R_1, \ldots, R_8$. The fact that the ten curves contain P gives polynomial equations in these five variables, and hence defines an algebraic set A_0 in $\mathbb{A}^5$. We define S_0 to be the algebraic set of all points in $\mathbb{A}^5$ that correspond to the configurations where the points $R_1, \ldots, R_8$ are not in

general position. We want to show that A_0 is contained in S_0, which would prove the proposition. In what follows we will show that indeed every component of A_0 is contained in S_0.
Note that by construction of $R_1, \dots, R_8$, the curves L_1, L_2, C_1, C_2, C_3 contain P. We will add conditions for $C_4, D_1, \dots, D_4$ to contain P, too. We start with C_4. The equation expressing that P is contained in C_4, is given by $\det(N) = 0$, where N is the matrix in Lemma 4.3.4 corresponding to $(R_2, R_4, R_6, R_7, R_8, P)$. This determinant is given by

$$\det(N) = 16(u+1)(t+1)(s+1)(s-u)(m-u)(m-s)(l-t)(l-m)f_1 f_2,$$

where

$$\begin{aligned} f_1 = l^2u + l^2 - lmu - lm - lsu - ls - ltu - lt + lu^2 + lu + mst + ms \\ + mt - mu + st - su - tu + u^2, \end{aligned}$$

and

$$f_2 = at^2 + btu + cu^2 + dt + eu + f,$$

with

$$\begin{aligned} a &= (s+1)(m-1)(m+1), & b = d = -e &= 2s(m-1)(l+1), \\ c &= (s-1)(l-1)(l+1), & f &= (l-m)(ls - l - ms - m + 2s). \end{aligned}$$

Let $F_2 \subset \mathbb{A}^5$ be the affine variety given by $f_2 = 0$. Every component of A_0 is contained in one of the components of the algebraic set given by $\det(N) = 0$. With `magma` it is an easy check that apart from f_2, all non-constant factors of $\det(N)$ define configurations of $R_1, \dots, R_8$ where three of the points are collinear (see [Codc]; $f_1 = 0$ corresponds to R_2, R_3, R_4 being collinear), and hence they define components of S_0. Therefore, it suffices to prove that $A_0 \cap F_2$ is contained in S_0.
Since f_2 is quadratic in t and u, the projection π from F_2 to the affine space $\mathbb{A}^3$ with coordinates l, m, s has fibers that are (possibly non-integral) affine conics. Let Δ be the discriminant of the quadratic form that is the homogenisation of f_2 with respect to t and u, which is given by

$$\Delta = 4acf - ae^2 - b^2 f + bde - cd^2;$$

the singular fibers of π lie exactly above the points $(l, m, s) \in \mathbb{A}^3$ for which $\Delta = 0$. We compute the factorization of Δ in $\mathbb{Z}[l, m, s]$, and find

$$\Delta = 4(s-1)(s+1)(m-1)(m+1)(l-1)(l+1)(l-m)g,$$

with $g = ls - l - ms - m + 2s$. All non-constant factors of Δ except for g, when viewed as elements of T_5, define components of S_0 in $\mathbb{A}^5$. Therefore, the fibers under π above the zero sets of these factors in $\mathbb{A}^3$ are contained in S_0. We will show that the same holds for the inverse image under π of the zero set $Z(g) \subset \mathbb{A}^3$ of g, which is given by the zero set $Z(f_2, g)$ in $\mathbb{A}^5$. Note that we can write

$$f_2 = (s-1)(l+1)(u-t)a_1 + (t-1)ga_2,$$

with $a_1 = (l-1)(u+1)-(m+1)(t-1)$ and $a_2 = (l+1)(u+1)-(m+1)(t+1)$. Therefore, the set $Z(f_2, g)$ is given by $g = (s-1)(l+1)(u-t)a_1 = 0$, so $Z(f_2, g)$ is the union of four algebraic sets:

$$Z(f_2, g) = Z(g, s-1) \cup Z(g, l+1) \cup Z(g, u-t) \cup Z(g, a_1) \subset \mathbb{A}^5.$$

Note that $s-1, l+1$, and $u-t$ define components of S_0, so the first three terms in this union are contained in S_0. With `magma`, we check that the irreducible polynomial $\gamma = (m-u)(l-1)g + (l-s)(m-1)a_1$ corresponds to a configuration where the six points $R_3, \ldots, R_8$ are contained in a conic, and hence it defines a component of S_0. Since γ is contained in the ideal in $\mathbb{Z}[l, m, s, t, u]$ generated by g and a_1, it follows that $Z(g, a_1)$ is also contained in S_0. We conclude that all the singular fibers of π lie in S_0.

The generic fiber $F_{2,\eta}$ of π is a conic in the affine plane $\mathbb{A}^2$ with coordinates t and u over the function field $k(l, m, s)$, where l, m, s are transcendentals. This fiber contains the point $(t, u) = (l, m)$. We can parametrize $F_{2,\eta}$ with a parameter v by intersecting it with the line M given by $v(t-l) = (u-m)$, which intersects $F_{2,\eta}$ in the point (l, m) and a second intersection point that we associate to v. Consider the open subset $F_2' \subset F_2$ given by the complement in F_2 of the singular fibers of π and the hyperplane section defined by $t - l = 0$, so $F_2 \setminus F_2' \subset S_0$. In what follows, we use the idea of this parametrization to construct an isomorphism between F_2' and an open subset of the affine space $\mathbb{A}^4$ with coordinates l, m, s, v.

Consider the ring $T_5^v = k[l, m, s, t, v]$, and let φ be the map $\varphi\colon T_5 \longrightarrow T_5^v$ that sends u to $v(t-l) + m$ and l, m, s, t to themselves. Then we have $\varphi(f_2) = (t-l)(\alpha t + \beta)$, where

$$\alpha = l^2sv^2 - l^2v^2 - 2lmsv + 2lsv + m^2s + m^2 - 2msv - sv^2 + 2sv - s + v^2 - 1,$$

and

$$\begin{aligned}\beta = l^3sv^2 - l^3v^2 - 2l^2msv + 2l^2mv + lm^2s - lm^2 - 2lmsv - lsv^2 \\ + 2lsv - ls + lv^2 + l + 2m^2s - 2mv + 2sv - 2s.\end{aligned}$$

The map φ induces a birational morphism $\psi\colon \mathbb{A}^5_v \longrightarrow \mathbb{A}^5$, where $\mathbb{A}^5_v$ is the affine space with coordinate ring T_5^v. Moreover, ψ is an isomorphism on the complements of the zero sets of $t-l$ in its domain and codomain. Set

$$G = Z(\alpha t + \beta) \setminus Z(t-l) \subset \mathbb{A}^5_v,$$

then ψ induces an isomorphism $G \cong F_2 \setminus Z(t-l)$. In particular, ψ induces an isomorphism from $G \setminus Z(\Delta)$ to F_2'. We want to show that $G \setminus Z(\alpha\Delta)$ equals $G \setminus Z(\Delta)$; to do this it suffices to show that $\psi(G \cap Z(\alpha))$ is contained in a union of singular fibers of π. Note that we have $G \cap Z(\alpha) = G \cap Z(\alpha, \beta)$. Let $(l_0, m_0, s_0, t_0, v_0)$ be a point in $G \cap Z(\alpha, \beta)$, then, since α and β do not depend on t, the point (l_0, m_0, s_0, t, v_0) is contained in $Z(\alpha t + \beta)$ for all t. It follows that the fiber on F_2 in $\mathbb{A}^2(t, u)$ under π above the point $(l_0, m_0, s_0) \in \mathbb{A}^3$ contains the line $u = v_0(t - l_0) + m_0$, hence is singular. Moreover, this fiber contains the point $\psi((l_0, m_0, s_0, t_0, v_0))$. We conclude that $\psi(G \cap Z(\alpha))$ is contained in a union of singular fibers of F_2. It follows that

$$\psi(G \setminus Z(\alpha\Delta)) = \psi(G \setminus Z(\Delta)) = F_2'.$$

Consider the ring $T_4 = k[l, m, s, v]$, and let K_4 be its field of fractions. Consider the ring homomorphism $\rho\colon T_5^v \longrightarrow K_4$ that sends t to $\frac{-\beta}{\alpha}$, and l, m, s, v to themselves. This induces a birational map

$$i\colon \mathbb{A}^4 \longrightarrow Z(\alpha t + \beta) \subset \mathbb{A}^5_v,$$

where $\mathbb{A}^4$ is the affine space with coordinate ring T_4. The map i induces an isomorphism from $\mathbb{A}^4 \setminus Z(\alpha)$ to $Z(\alpha t + \beta) \setminus Z(\alpha)$; this isomorphism sends the zero set of Δ in $\mathbb{A}^4 \setminus Z(\alpha)$ to the zero set of Δ in $Z(\alpha t + \beta) \setminus Z(\alpha)$, and the zero set of $t - l$ in $Z(\alpha t + \beta) \setminus Z(\alpha)$ corresponds to the zero set of $\alpha l + \beta$ in $\mathbb{A}^4 \setminus Z(\alpha)$. Hence, we have an isomorphism

$$\mathbb{A}^4 \setminus Z(\alpha\Delta(\alpha l + \beta)) \cong G \setminus Z(\alpha\Delta).$$

We conclude that we have an isomorphism

$$\psi \circ i\colon \mathbb{A}^4 \setminus Z(\alpha\Delta(\alpha l + \beta)) \longrightarrow F_2'.$$

Recall that our aim is to show that $A_0 \cap F_2$ is contained in S_0. Since we showed that all components of $F_2 \setminus F_2'$ are contained in S_0, we have $A_0 \cap F_2 \subset S_0$ if and only if $A_0 \cap F_2' \subset S_0$. Moreover, after setting

$$A_1 = i^{-1}(\psi^{-1}(A_0 \cap F_2')) \text{ and } S_1 = i^{-1}(\psi^{-1}(S_0 \cap F_2')),$$

showing $A_0 \subseteq S_0$ is equivalent to showing $A_1 \subseteq S_1$.
For i in $\{1,2,3,4\}$, the expression stating that P is contained in D_i is given by $\det(H_i) = 0$, where H_i is the matrix denoted by $H_{\alpha_7,\alpha_8,\alpha_9}$ in Lemma 4.3.4 associated to

$$\begin{aligned}
&(R_2, R_3, R_4, R_5, R_6, R_1, R_7, R_8) \text{ for } i = 1;\\
&(R_1, R_3, R_4, R_7, R_8, R_2, R_5, R_6) \text{ for } i = 2;\\
&(R_1, R_2, R_4, R_5, R_7, R_3, R_6, R_8) \text{ for } i = 3;\\
&(R_1, R_2, R_3, R_6, R_8, R_4, R_5, R_7) \text{ for } i = 4,
\end{aligned}$$

where we set $\alpha_7 = x$, $\alpha_8 = \alpha_9 = y$ for $i \in \{1,2\}$, and $\alpha_7 = \alpha_8 = \alpha_9 = y$ for $i \in \{3,4\}$. For $i \in \{1,2,3,4\}$, let $B_i \subset F_2 \subset \mathbb{A}^5$ be the locus of points corresponding to configurations of $R_1, \ldots, R_8$ such that D_i contains P. Then we have $A_0 \cap F_2 = \bigcap_{i=1}^4 B_i$, so $A_0 \cap F_2' = \bigcap_{i=1}^4 (B_i \cap F_2')$, and hence $A_1 = \bigcap_{i=1}^4 i^{-1}(\psi^{-1}(B_i \cap F_2'))$. Note that B_i is defined by $f_2 = \det(H_i) = 0$. For $i \in \{1,2,3,4\}$, we compute the determinant of H_i and its factorization in $\mathbb{Z}[l,m,s,t,u]$ in `magma`. For all i, this factorization has a constant factor that is a power of 2, and there is exactly one irreducible factor h_i that does not define a component of S_0; it follows that $Z(f_2, h_i) \setminus S_0 = B_i \setminus S_0$. Note that for $i \in \{1,2,3,4\}$, the set $i^{-1}(\psi^{-1}(Z(f_2,h_i) \setminus Z(\alpha\Delta(t-l)))$ is defined in $\mathbb{A}^4 \setminus Z(\alpha\Delta(\alpha l + \beta))$ by the numerator of $\rho(\varphi(h_i))$; we compute the factorization of this numerator in $\mathbb{Z}[l,m,s,v]$. Again, for all i, this factorization has as constant factor a power of 2, and contains exactly one irreducible factor that does not define a component of S_1; we call this factor g_i. It follows that for $i \in \{1,2,3,4\}$, the set $i^{-1}(\psi^{-1}(B_i \setminus S_0))$ is contained in $Z(g_i)$, so $A_1 \setminus S_1$ is contained in $Z(g_1, g_2, g_3, g_4)$. Computing g_1, g_2, g_3, g_4 takes `magma` over an hour, and these polynomials are too big to write down here; you can find them in [Codd]. Set

$$\begin{aligned}
\delta = (ls - l - ms - m + 2s)^2(l-m)(l-s)(l+1)(m-1)(s+1)\cdot \\
(l-1)(m+1)(s-1)v^2.
\end{aligned}$$

We check that all factors of $\delta \in \mathbb{Z}[l,m,s,v]$ define components of S_1 (the first factor corresponds to both R_2, R_3, R_5 and R_2, R_4, R_6 being collinear). We will show that δ is contained in the ideal $\mathcal{I}$ of T_4 generated by g_1, g_2, g_3, and g_4. We use a Gröbner basis for $\mathcal{I}$ to check this. In `magma`, we define the ideal $\mathcal{I}$ in the ring T_4 with $k = \mathbb{Q}$ with the ordering $s > v > m > l$. With the function `G,b:=GroebnerBasis(I:ReturnDenominators)` we compute the reduced Gröbner basis G for $\mathcal{I}$; after using this function, `magma` uses G as a generator set for $\mathcal{I}$. We then use G to check that δ is contained in

$\mathcal{I}$, again over $\mathbb{Q}$. This finishes the proof for char $k = 0$; We continue the proof for char $k = p > 0$ with $p \neq 2, 3$.
The element δ can be written as a linear combination of the elements in G with coefficients in T_4. Let C be the set of these coefficients (obtained by the function `Coordinates(I,f)`). In the proces of computing G, `magma` makes divisions by integers, which are stored in the set b. Let $\mathcal{P}$ be the set containing the prime divisors of all elements in b, and all prime divisors of the denominators of the coefficients of the elements in G, and all prime divisors of the denominators of the coefficients of the elements in C. Then for a prime $p \notin \mathcal{P}$, the reductions modulo p of the elements in G are well defined. Moreover, since $\mathcal{P}$ contains all prime divisors of the elements in b, the reductions modulo p of the elements in G still form a Gröbner basis for the ideal $\mathcal{J}$ generated by the reductions modulo p of g_1, g_2, g_3, g_4. Finally, the reduction modulo p of δ is contained in $\mathcal{J}$, since the prime divisors of the denominators of the coefficients of the elements in C are in $\mathcal{P}$. This finishes the proof for char $k = p > 0$ with $p \neq 2, 3$, $p \notin \mathcal{P}$.
For all finitely many $p \in \mathcal{P} \setminus \{2, 3\}$, let $\overline{T_4}$ be the ring $\mathbb{F}_p[l, m, s, v]$, let $\overline{\delta}$ be the reduction of δ modulo p, and for $i \in \{1, 2, 3, 4\}$, let $\overline{g_i}$ be the reduction of g_i modulo p; then it is a quick check in `magma` that $\overline{\delta}$ is contained in the ideal $(\overline{g_1}, \overline{g_2}, \overline{g_3}, \overline{g_4})$ of $\overline{T_4}$. We conclude that for char $k \neq 2, 3$, the set $A_1 \setminus S_1$ is contained in the union of the varieties defined by the factors of δ, so $A_1 \setminus S_1$ is a subset of S_1. We conclude that A_1 is contained in S_1. This finishes the proof for char $k \neq 2$.

<u>Assume char $k = 2$.</u>
Since the points R_1, R_5, R_6, P as defined in the previous case are not in general position over a field of characteristic 2, we redefine these points here. The proof then goes completely analogous to the previous case; see [Codc] for the code in `magma` where we verify everything over the field $k = \mathbb{F}_2$ of two elements. Set

$$R_1 = (1:0:1); \qquad R_6 = (0:1:1);$$
$$R_5 = (0:1:0); \qquad P = (1:0:0).$$

These four points are in general position in $\mathbb{P}^2$. We take $z^2 + xz + yz$ and xy for the two generators of the linear system of quadrics through R_1, R_5, R_6 and P.
We now do all the steps as in the previous case, and everything works analogously. In fact, checking that all singular fibers of the analog of π from the previous case are contained in the analog of S_0 can be done even more directly in `magma` than as described in the previous case. We

obtain again an algebraic set $A_1 \subset \mathbb{A}^4$, where $\mathbb{A}^4$ is the affine space over $\mathbb{F}_2$ with coordinates l, m, s, v, and A_1 is the algebraic set corresponding to the configurations where the ten curves $L_1, L_2, C_1, \ldots, C_4, D_1, \ldots, D_4$ all contain the point P. Again, we want to show that A_1 is contained in S_1, where $S_1 \subset \mathbb{A}^4$ is the algebraic set defined by the polynomials that correspond to the eight points $R_1, \ldots, R_8$ not being in general position. Completely analogously to the case char $k \neq 2$, from the conditions that P is contained in D_1, D_2, D_3, D_4, we now obtain four polynomials g_1, g_2, g_3, g_4 in $\mathbb{F}_2[l, m, s, v]$ (see [Codd]). Again, we have $A_1 \setminus S_1 \subset Z(g_1, g_2, g_3, g_4)$. Set

$$\delta = (ls+ms+m+s)(lv+m+1)(l+m)(l+s)(m+s)(l+1)(m+1)m^3(s+1)lvs.$$

It is a quick check with `magma` that δ is contained in $\mathcal{I}$. Moreover, it is again a quick check that all factors of δ correspond to three points being collinear, and hence define a component of S_1. We conclude again that A_1 is contained in S_1. $\square$

We can now prove Theorem 4.1.2. Recall Notation 4.3.8.

Proof of Theorem 4.1.2. Recall that every set of exceptional curves without partners corresponds to a clique in G with only edges of weights 1 and 2, so by Lemma 4.4.2, the number of exceptional curves that are concurrent in a point outside the ramification curve of φ is at most twelve. This proves the case char $k = 3$.
Now assume that char $k \neq 3$. Consider the eleven classes in C given by

$$\begin{aligned}
e_1 &= L - E_1 - E_2;\\
e_2 &= L - E_3 - E_4;\\
e_3 &= 2L - E_1 - E_3 - E_5 - E_6 - E_7;\\
e_4 &= 2L - E_1 - E_4 - E_5 - E_6 - E_8;\\
e_5 &= 2L - E_2 - E_3 - E_5 - E_7 - E_8;\\
e_6 &= 2L - E_2 - E_4 - E_6 - E_7 - E_8;\\
e_7 &= 4L - 2E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - 2E_7 - 2E_8;\\
e_8 &= 4L - E_1 - 2E_2 - E_3 - E_4 - 2E_5 - 2E_6 - E_7 - E_8;\\
e_9 &= 4L - E_1 - E_2 - 2E_3 - E_4 - E_5 - 2E_6 - E_7 - 2E_8;\\
e_{10} &= 4L - E_1 - E_2 - E_3 - 2E_4 - 2E_5 - E_6 - 2E_7 - E_8;\\
e_{11} &= 5L - 2E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - E_6 - E_7 - 2E_8;
\end{aligned}$$

It is straightforward to check that they form a clique with only edges of weights 1 and 2 in G. By Remark 1.2.7, we know that $e_1, \ldots, e_{10}$ correspond to the classes in Pic X of the strict transforms of the curves $L_1, L_2, C_1, \ldots, C_4, D_1, \ldots, D_4$, defined as above Proposition 4.4.6 with respect to P_i instead of R_i for $i \in \{1, \ldots, 8\}$.
Let $K = \{c_1, \ldots, c_{11}\}$ be a clique of size eleven in G with only edges of weights 1 and 2. By Proposition 4.4.4, after changing the indices if necessary, there is an element $w \in W$ such that $c_i = w(e_i)$ for i in $\{1, \ldots, 11\}$. Set $E_i' = w(E_i)$. Then, since the E_i' are pairwise disjoint, by Lemma 1.2.8 we can blow down $E_1', \ldots, E_8'$ to points $Q_1, \ldots, Q_8$ in $\mathbb{P}^2$ that are in general position, such that X is isomorphic to the blow-up of $\mathbb{P}^2$ at $Q_1, \ldots, Q_8$, and E_i' is the class in Pic X that corresponds to the exceptional curve above Q_i for all i. By the bijection in Remark 1.2.7, the elements $c_1, \ldots, c_{10}$ are the classes that correspond to the strict transforms of $L_1, L_2, C_1, \ldots, C_4, D_1, \ldots, D_4$ defined as above Proposition 4.4.6 with respect to Q_i instead of R_i for $i \in \{1, \ldots, 8\}$. Since char $k \neq 3$, it follows from Proposition 4.4.6 that the curves corresponding to $c_1, \ldots, c_{10}$ are not concurrent. We conclude that the number of concurrent exceptional curves in a point outside the ramification curve of φ is less than eleven. □

4.5 Examples

4.5.1 On the ramification curve

This section contains examples that show that the upper bounds in Theorem 4.1.1 are sharp. Example 4.5.1 is a del Pezzo surface over a field of characteristic 2 with 16 concurrent exceptional curves, Example 4.5.2 is a del Pezzo surface over any field of characteristic unequal to $2, 3, 5, 7, 11, 13, 17, 19$ with 10 concurrent exceptional curves, and Example 4.5.3 contains examples of ten concurrent exceptional curves on del Pezzo surfaces in the remaining 7 characteristics.

Example 4.5.1. Set $f = x^5 + x^2 + 1 \in \mathbb{F}_2[x]$, and let $F \cong \mathbb{F}_2[x]/(f)$ be the finite field of 32 elements defined by adjoining a root α of f to $\mathbb{F}_2$.

Define the following eight points in $\mathbb{P}^2_F$.

$$\begin{aligned} Q_1 &= (0:1:1); & Q_5 &= (1:1:1);\\ Q_2 &= (0:1:\alpha^{19}); & Q_6 &= (\alpha^{20}:\alpha^{20}:\alpha^{16});\\ Q_3 &= (1:0:1); & Q_7 &= (\alpha^{24}:\alpha^{25}:1);\\ Q_4 &= (1:0:\alpha^5); & Q_8 &= (\alpha^{30}:1:\alpha^5). \end{aligned}$$

With `magma` we check that the determinants of the appropriate matrices in Lemma 4.3.4 are all non-zero, so these eight points are in general position. Therefore, the blow-up of $\mathbb{P}^2$ in $\{Q_1, \ldots, Q_8\}$ is a del Pezzo surface S. We have the following four lines in $\mathbb{P}^2$.

$$\begin{aligned} &\text{The line } L_1 \text{ through } Q_1 \text{ and } Q_2, \text{ which is given by } x = 0;\\ &\text{the line } L_2 \text{ through } Q_3 \text{ and } Q_4, \text{ which is given by } y = 0;\\ &\text{the line } L_3 \text{ through } Q_5 \text{ and } Q_6, \text{ which is given by } x = y;\\ &\text{the line } L_4 \text{ through } Q_7 \text{ and } Q_8, \text{ which is given by } y = \alpha x. \end{aligned}$$

Let $C_{i,j}$ be the unique cubic through $Q_1, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_8$ that is singular in Q_j. Set $(R_1, \ldots, R_8) = (Q_1, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8, Q_2)$, and let L be the corresponding matrix from Lemma 4.3.4. Then the equation defining $C_{1,2}$ is the determinant of L', where L' is equal to L after replacing the first row by Mon_3. Similarly, we compute the defining equations of $C_{3,4}$, $C_{5,6}$, $C_{7,8}$ and $C_{8,7}$, and find the following.

$$C_{1,2}\colon x^3+\alpha^{24}x^2y+\alpha^{28}x^2z+\alpha^{30}xy^2+\alpha^9xyz+\alpha^{26}xz^2+\alpha^{13}y^3+\alpha^6yz^2 = 0$$

$$C_{3,4}\colon x^3+\alpha^{12}x^2y+\alpha^4xy^2+\alpha^{11}xyz+\alpha^{21}xz^2+y^3+\alpha^{23}y^2z+\alpha^{12}yz^2 = 0$$

$$\begin{aligned} C_{5,6}\colon x^3 + \alpha^4x^2y + \alpha^{28}x^2z + \alpha^{25}xy^2 + \alpha^{20}xyz + \alpha^{26}xz^2 + \alpha^{17}y^3&\\ + \alpha^9y^2z + \alpha^{29}yz^2 &= 0 \end{aligned}$$

$$\begin{aligned} C_{7,8}\colon x^3 + \alpha x^2y + \alpha^{28}x^2z + \alpha^{17}xy^2 + \alpha^{10}xyz + \alpha^{26}xz^2 + \alpha^{16}y^3&\\ + \alpha^8y^2z + \alpha^{28}yz^2 &= 0 \end{aligned}$$

$$\begin{aligned} C_{8,7}\colon x^3 + \alpha^{26}x^2y + \alpha^{28}x^2z + \alpha^{19}xy^2 + \alpha^{10}xyz + \alpha^{26}xz^2 + \alpha^{16}y^3&\\ + \alpha^8y^2z + \alpha^{28}yz^2 &= 0 \end{aligned}$$

Let $e_1, \ldots, e_8$ be the strict transforms of the eight curves

$$L_1, \ldots, L_4, C_{1,2}, C_{3,4}, C_{5,6}, C_{7,8},$$

and let c_8 be the strict transform of $C_{8,7}$. Since these nine curves all contain the point $(0:0:1)$, the exceptional curves $e_1, \ldots, e_8, c_8$ are concurrent in a point P on S. Let ψ be the morphism associated to the linear system $|-2K_S|$. Since $e_8 \cdot c_8 = 3$, the point P lies on the ramification curve of ψ by Remark 4.2.5. Therefore, by the same remark, for $i \in \{1, \ldots, 7\}$, the partners of $e_1, \ldots, e_7$ contain P, too. We conclude that there are sixteen exceptional curves on S that are concurrent in P.

Example 4.5.2. Let k be a field of characteristic unequal to $2, 3, 5, 7, 11, 13, 17, 19$. Define the following eight points in $\mathbb{P}^2_k$.

$$\begin{aligned}
Q_1 &= (0:1:1); & Q_5 &= (1:1:1);\\
Q_2 &= (0:5:3); & Q_6 &= (4:4:5);\\
Q_3 &= (1:0:1); & Q_7 &= (-2:2:1);\\
Q_4 &= (-1:0:1); & Q_8 &= (2:-2:1).
\end{aligned}$$

With `magma` we compute the determinants of the matrices in Lemma 4.3.4 that determine whether three of the points are on a line, or six of the points are on a conic, or seven of them are on a cubic that is singular at one of them. These determinants are non-zero for char $k \neq 2, 3, 5, 7, 11, 13, 17, 19$, so the points are in general position. Therefore, the blow-up of $\mathbb{P}^2_k$ in $\{Q_1, \ldots, Q_8\}$ is a del Pezzo surface S. We define the lines L_1, L_2, L_3 as in Example 4.5.1. We define L_4 to be the line containing Q_7 and Q_8, which is given by $x = -y$.
Let $C_{7,8}$ be the unique cubic through $Q_1, \ldots, Q_6, Q_8$ that is singular in Q_8, and $C_{8,7}$ the unique cubic through $Q_1, \ldots, Q_7$ that is singular in Q_7. As in Example 4.5.1 we compute the defining equations for $C_{7,8}$ and $C_{8,7}$, and we find

$$\begin{aligned}
C_{7,8}\colon x^3 - \tfrac{3}{4}x^2y - \tfrac{31}{12}xy^2 + \tfrac{10}{3}xyz - xz^2 - y^3 + \tfrac{8}{3}y^2z - \tfrac{5}{3}yz^2 &= 0,\\
C_{8,7}\colon x^3 + \tfrac{13}{4}x^2y + \tfrac{43}{4}xy^2 - 14xyz - xz^2 + 15y^3 - 40y^2z + 25yz^2 &= 0.
\end{aligned}$$

On S, we define the four exceptional curves $e_1, \ldots, e_4$ to be the strict transforms of $L_1, \ldots, L_4$, and e_5, e_5' the strict transforms of $C_{7,8}$ and $C_{8,7}$, respectively. Since $L_1, \ldots, L_4, C_{7,8}, C_{8,7}$ all contain the point $(0:0:1)$, the six exceptional curves $e_1, \ldots, e_5, e_5'$ are concurrent in a point P in S.

Let ψ be the morphism associated to the linear system $|-2K_S|$. By Remark 4.2.5, since $e_5 \cdot e_5' = 3$, the point P lies on the ramification curve of ψ, and for $i \in \{1, \dots, 4\}$, the partners of $e_1, \dots, e_4$ contain P, too. We conclude that there are ten exceptional curves on S that are concurrent in P.

Example 4.5.3. For $p \in \{3, 5, 7, 11, 13, 17, 19\}$, we construct a del Pezzo surface over a field of characteristic p with ten exceptional curves that are concurrent in a completely analogous way to the one in Example 4.5.2.
Let p be a prime, and $\mathbb{F}_p$ be the finite field of p elements. Let $f_p \in \mathbb{F}_p[x]$ be an irreducible polynomial. Let α be a root of f_p, and $\mathbb{F} \cong \mathbb{F}_p[x]/(f_p)$ the field extension of $\mathbb{F}_p$ obtained by adjoining α to $\mathbb{F}_p$. For $a, b, c, m, u, v \in \mathbb{F}$, define the following eight points in $\mathbb{P}^2_{\mathbb{F}}$.

$$\begin{aligned} Q_1 &= (0:1:1); & Q_5 &= (1:1:1); \\ Q_2 &= (0:1:a); & Q_6 &= (1:1:c); \\ Q_3 &= (1:0:1); & Q_7 &= (m:1:u); \\ Q_4 &= (1:0:b); & Q_8 &= (m:1:v). \end{aligned}$$

Let x, y, z be the coordinates of $\mathbb{P}^2_{\mathbb{F}}$. We define again the lines L_1, L_2, L_3 as in Example 4.5.1, and the line L_4 by $x = my$. Note that $L_1, \dots, L_4$ all contain the point $(0:0:1)$. Let $C_{7,8}$ be the unique cubic through $Q_1, \dots, Q_6, Q_8$ that is singular in Q_8, and $C_{8,7}$ the unique cubic through $Q_1, \dots, Q_7$ that is singular in Q_7. For all fixed $(p, f_p, a, b, c, m, u, v)$ that we describe below, we check as we did in Example 4.5.2 that the eight points are in general position, and compute the defining equations for $C_{7,8}$ and $C_{8,7}$. In all cases, the point $(0:0:1)$ is also contained in $C_{7,8}$ and $C_{8,7}$, and as in Example 4.5.2 this implies that there are 10 exceptional curves on the del Pezzo surface obtained by blowing up $\mathbb{P}^2_{\mathbb{F}}$ in $Q_1, \dots, Q_8$, that are concurrent in a point on the ramification curve.

- For $p = 3$ we take

$$f_p = x^3 + 2x + 1, \quad (a, b, c, m, u, v) = (\alpha, \alpha^{20}, \alpha^{15}, \alpha^8, \alpha^2, \alpha^{12}).$$

- For $p = 5$ we take

$$f_p = x^2 + 4x + 2, \quad (a, b, c, m, u, v) = (\alpha^{19}, \alpha^{11}, \alpha^{10}, \alpha^{21}, \alpha^3, \alpha^{14}).$$

- For $p = 7$ we take

$$f_p = x^2 + 6x + 3, \quad (a, b, c, m, u, v) = (3, \alpha^{45}, \alpha^{35}, \alpha^4, \alpha^{46}, \alpha^9).$$

- For $p = 11$ we take
$$f_p = x^2 + 7x + 2, \quad (a, b, c, m, u, v) = (\alpha^{106}, \alpha^{94}, 4, \alpha^{62}, \alpha^{111}, \alpha^{6}).$$
- For $p = 13$ we take
$$f_p = x^2 + 12x + 2, \quad (a, b, c, m, u, v) = (\alpha^{161}, \alpha^{156}, \alpha^{83}, \alpha^{94}, \alpha^{132}, \alpha^{146}).$$
- For $p = 17$ we take
$$f_p = x^2 + 16x + 3, \quad (a, b, c, m, u, v) = (\alpha^{74}, \alpha^{166}, \alpha^{64}, \alpha^{24}, \alpha^{178}, \alpha^{250}).$$
- For $p = 19$, we take $\mathbb{F} = \mathbb{F}_{19}$, and $(a, b, c, m, u, v) = (2, 2, 14, 8, 7, 12)$.

All these examples are generated in `magma` by generating random values for the elements a, b, c, m, u, v in each case, until the points defined by the values are in general position.

4.5.2 Outside the ramification curve

In this section we give examples that show that the upper bound in Theorem 4.1.2 is sharp. Example 4.5.4 gives a del Pezzo surface of degree one over a field of characteristic 3 with twelve exceptional curves that are concurrent in a point outside the ramification curve. In Example 4.5.5 we give a del Pezzo surface over a field of characteristic unequal to 5 that contains ten exceptional curves that are concurrent in a point outside the ramification curve. This surface is isomorphic to the one in Example 4.1 in [SvL14] if the characteristic of k is unequal to 2 and 3. We do not give an example in characteristic 5, since we have not found one; it might very well be that the maximum in this case is less than ten.

Example 4.5.4. Let $f = x^3 + 2x + 1$ be a polynomial in $\mathbb{F}_3[x]$. Let α be a root of f, and let $\mathbb{F} \cong \mathbb{F}_3[x]/f$ be the field of 27 elements obtained by adjoining α to $\mathbb{F}_3$. Let $\mathbb{P}^2_{\mathbb{F}}$ be the projective plane over $\mathbb{F}$, and define the following eight points in this plane.

$$\begin{aligned} Q_1 &= (1 : 0 : 1); & Q_5 &= (0 : 1 : 1);\\ Q_2 &= (\alpha^{20} : 0 : \alpha^{18}); & Q_6 &= (0 : 2 : 1);\\ Q_3 &= (\alpha^{6} : \alpha^{23} : \alpha^{2}); & Q_7 &= (\alpha^{9} : \alpha^{23} : 2);\\ Q_4 &= (\alpha^{15} : \alpha^{19} : \alpha^{18}); & Q_8 &= (\alpha^{24} : \alpha^{7} : \alpha^{5}). \end{aligned}$$

With `magma` we check that no three of these points are on a line, no six of them are on a conic, and no seven of them are on a cubic that is singular

at one of them, by checking that the appropriate determinants of the matrices in Lemma 4.3.4 are non-zero. Therefore, the blow-up of $\mathbb{P}^2_{\mathbb{F}}$ in these eight points is a del Pezzo surface S of degree one.
Let L_1 be the line containing Q_1 and Q_2, which is given by $y = 0$. Let L_2 be the line containing Q_3 and Q_4, which is given by $\alpha^{23}y = x + z$. For five points $Q_{i_1}, \ldots, Q_{i_5}$ we find the equation of the conic containing these points by computing the determinant of the matrix N in Lemma 4.3.4, with $(R_2, \ldots, R_6) = (Q_{i_1}, \ldots, Q_{i_5})$, and where the first row is replaced by the list Mon_2. We obtain the following conics in $\mathbb{P}^2_{\mathbb{F}}$.

$C_1\colon x^2 + \alpha^7 xy + y^2 + 2z^2 = 0$, containing Q_1, Q_3, Q_5, Q_6, Q_7.
$C_2\colon x^2 + \alpha^{16} xy + y^2 + 2z^2 = 0$, containing Q_1, Q_4, Q_5, Q_6, Q_8.
$C_3\colon x^2+\alpha^{25}xz+\alpha^{16}y^2+\alpha^{11}yz+\alpha^{15}z^2 = 0$, containing Q_2, Q_3, Q_5, Q_7, Q_8.
$C_4\colon x^2+\alpha^9xy+\alpha^{25}xz+\alpha^{20}y^2+\alpha^6yz+\alpha^{15}z^2 = 0$, cont. Q_2, Q_4, Q_6, Q_7, Q_8.

Similarly, we compute defining equations for the quartics D_1, D_2, D_3, D_4 containing all the eight points with singularities in Q_1, Q_7, Q_8, and Q_2, Q_5, Q_6, and Q_3, Q_6, Q_8, and Q_4, Q_5, Q_7, respectively. We find

$$D_1\colon \alpha^4x^4 + \alpha^{11}x^3y + \alpha^{12}x^3z + \alpha^{24}x^2y^2 + \alpha^{10}x^2yz + \alpha^{16}x^2z^2 + \alpha^{16}xy^3 + \alpha^{21}xy^2z + \alpha^{17}xyz^2 + \alpha^{25}xz^3 + \alpha^6y^4 + \alpha^{12}y^3z + \alpha^{25}yz^3 + \alpha^{19}z^4 = 0,$$

$$D_2\colon \alpha^{14}x^4 + x^3y + \alpha^{16}x^3z + \alpha^4x^2y^2 + \alpha^4x^2yz + \alpha^{21}x^2z^2 + \alpha^{25}xy^3 + \alpha^{16}xy^2z + \alpha^{12}xyz^2 + \alpha^3xz^3 + \alpha^5y^4 + \alpha^5y^2z^2 + \alpha^5z^4 = 0,$$

$$D_3\colon \alpha^{21}x^4 + \alpha^4x^3y + \alpha^{20}x^3z + \alpha^9x^2y^2 + \alpha^{19}x^2yz + \alpha^3x^2z^2 + \alpha^{21}xy^3 + \alpha^{11}xy^2z + \alpha^2xyz^2 + \alpha^7xz^3 + \alpha^2y^4 + \alpha^{17}y^3z + \alpha y^2z^2 + \alpha^4yz^3 + \alpha^{23}z^4 = 0,$$

$$D_4\colon \alpha^{19}x^4 + \alpha^{22}x^3y + \alpha^{18}x^3z + \alpha^{20}x^2y^2 + \alpha^{21}x^2yz + \alpha x^2z^2 + \alpha^2xy^3 + \alpha^{20}xy^2z + \alpha^{10}xyz^2 + \alpha^5xz^3 + \alpha^{23}y^4 + \alpha^{20}y^3z + \alpha^3y^2z^2 + \alpha^7yz^3 + \alpha^{21}z^4 = 0.$$

Finally, in a similar way we compute the defining equations of the quintics G_1 and G_2, which contain all eight points and are singular in $Q_1, Q_2, Q_3, Q_4, Q_5, Q_8$, and $Q_1, Q_2, Q_3, Q_4, Q_6, Q_7$, respectively. We obtain

$$\begin{aligned} G_1\colon \alpha x^5 + \alpha^8x^4y + 2x^4z + \alpha^{21}x^3y^2 + \alpha^{20}x^3yz + \alpha^{23}x^3z^2 + \alpha^5x^2y^3 \\ + \alpha^{25}x^2y^2z + \alpha^{22}x^2yz^2 + \alpha^7x^2z^3 + \alpha^{25}xy^4 + \alpha^{12}xy^3z + 2xy^2z^2 \\ + \alpha^{25}xyz^3 + \alpha^2xz^4 + \alpha^{21}y^5 + \alpha^6y^4z + \alpha^8y^3z^2 + \alpha y^2z^3 + \alpha^5z^5 = 0, \end{aligned}$$

$$\begin{aligned} G_2\colon & \alpha^4x^5+\alpha^{11}x^4y+\alpha^{16}x^4z+\alpha^7x^3y^2+\alpha^{16}x^3yz+x^3z^2+\alpha x^2y^3 \\ & +\alpha^{25}x^2y^2z+\alpha^2x^2yz^2+\alpha^{10}x^2z^3+\alpha^{17}xy^3z+\alpha^{15}xy^2z^2+\alpha^8xyz^3 \\ & +\alpha^5xz^4+\alpha^{14}y^5+\alpha^{16}y^4z+\alpha^{11}y^3z^2+\alpha^{10}y^2z^3+\alpha^{25}yz^4+\alpha^8z^5=0. \end{aligned}$$

Now consider the point $P=(2:0:1)$ in $\mathbb{P}^2_{\mathbb{F}}$. It is an easy check that P is contained in all twelve curves $L_1,L_2,C_1,\ldots,C_4,D_1,\ldots,D_4,G_1,G_2$. Therefore, the twelve exceptional curves on S that are the strict transforms of these twelve curves in $\mathbb{P}^2_{\mathbb{F}}$ are concurrent in a point Q on S. Let ψ be the morphism associated to the linear system $|-2K_S|$. Since none of the twelve exceptional curves intersect each other with multiplicity 3, the point Q is outside the ramification curve of ψ.

Example 4.5.5. Let k be a field of characteristic unequal to 5. For β an element in k^*, let S be the del Pezzo surface of degree one in $\mathbb{P}(2,3,1,1)$ with coordinates x,y,z,w over k given by

$$y^2+(\beta+1)xyw+\beta yw^3=x^3+\beta x^2w^2-z^5w.$$

For char $k\neq 2,3$, this surface is isomorphic to the surface in [SvL14, Example 4.1]. The blow-up of S in the point $(1:1:0:0)$ has the structure of an elliptic surface over $\mathbb{P}^1$ with coordinates z,w. The fiber above $z=0$ contains a point of order 5, which is given by $Q=(0:0:0:1)$; in fact, the cubic curve $E:y^2+(\beta+1)xy+\beta y=x^3+\beta x^2$ is the universal elliptic curve over the modular curve $Y_1(5)=\operatorname{Spec}(k[\beta,1/\Delta(E)])$ with $\Delta(E)=-\beta^5(\beta^2+11\beta-1)$ that parametrizes elliptic curves over extensions of k with a point of order 5 [CE11, Proposition 8.2.8].
Choose β such that S is smooth in all characteristics; for example, we can set $\beta=2$ in characteristic 11, and $\beta=1$ in all other characteristics. Let ρ,σ be elements of a field extension of k such that $\rho^2=\rho+1$, and $(\beta+\rho^5)\sigma^5=1$. Consider the curve $C_{\rho,\sigma}$ in $\mathbb{P}(2,3,1,1)$ defined by

$$\begin{aligned} x&=\sigma^2z^2w^4+\rho\sigma zw^5,\\ y&=-\sigma^3z^3w^3+(\rho+1)\sigma^2z^2w^4. \end{aligned}$$

Then $C_{\rho,\sigma}$ is an exceptional curve in S, defined over $k(\rho,\sigma)$. It is easy to see that Q is contained in $C_{\rho,\sigma}$. There are ten pairs (ρ,σ), so we conclude that there are ten exceptional curves through Q over a field extension of k. Finally, let φ be the morphism associated to $|-2K_S|$. Since the points on the ramification curve of φ are exactly the points on S that are 2-torsion on their fiber, we conclude that Q is outside the ramification curve.

5

Exceptional curves and torsion points

The del Pezzo surface of degree 1 in Example 4.5.5 contains a point that is contained in the intersection of 10 exceptional curves, and whose corresponding point on the elliptic surface associated to the del Pezzo surface (obtained by blowing up the base point of the anticanonical linear system, see Section 1.4.3) is torsion on its fiber. This example comes from [SvL14, Section 4], where we find several examples of a point on a del Pezzo surface of degree 1 that is contained in the intersection of at least 6 exceptional curves, and, in all cases, corresponds to a point that is torsion on its fiber. Moreover, we do not know any example of a point that is contained in more than 6 exceptional curves and that corresponds to a point that is *not* torsion on its fiber. A natural question is therefore whether a point on a del Pezzo surface of degree 1 that is contained in 'many' exceptional curves always corresponds to a point that is torsion on its fiber (where 'many' of course needs to be specified). In this final and short chapter we give a positive answer to this question where we take 'many' to be 9 (Theorem 5.1.1), using results from Chapter 3. We also show that if we take 'many' to be 6, the answer to this question is negative in most characteristics, by providing a counterexample that comes from Chapter 4 (Example 5.1.5). Computations were done in `magma` [BCP97], and the code that we used can be found in [Code].

5.1 Main results

Let S be a del Pezzo surface of degree 1 with canonical divisor K_S, and let $\mathcal{E}$ be the associated elliptic surface obtained by blowing up the basepoint $\mathcal{O}$ of the linear system $|-K_S|$. For a point P in $S \setminus \{\mathcal{O}\}$, we denote by $P_{\mathcal{E}}$ the corresponding point on $\mathcal{E}$, and by the fiber of $P_{\mathcal{E}}$ we mean the fiber of the elliptic fibration $\mathcal{E} \longrightarrow \mathbb{P}^1$ that contains $P_{\mathcal{E}}$. The main result of this chapter is the following.

Theorem 5.1.1. *If at least 9 exceptional curves on S are concurrent in a point P, then $P_{\mathcal{E}}$ is torsion on its fiber.*

Remark 5.1.2. For del Pezzo surfaces of degree 2, the situation is simpler, and a result similar to our theorem is known [Kuw05, Proposition 7.1]. A del Pezzo surface of degree 2 is a double cover of $\mathbb{P}^2$ ramified along a smooth quartic curve. On such a surface, a point is contained in at most 4 exceptional curves, and this happens exactly when its projection to $\mathbb{P}^2$ is in the intersection of 4 bitangents of the quartic curve. In [Kuw05], Kuwata gives a construction for an elliptic surface by blowing up twice on the del Pezzo surface, and he shows that for a point contained in 4 exceptional curves, the corresponding point on the elliptic surface is torsion on its fiber. The situation in Theorem 5.1.1 is more complex, since there are a priori many different sets of 9 or more exceptional curves on a del Pezzo surface of degree 1 that can be concurrent in a point.

Remark 5.1.3. Theorem 5.1.1 seems intuitively true by the following argument, which was pointed out to us by several people. Let P be a point on S that is contained in at least 9 exceptional curves, say $L_1, \ldots, L_n$. These curves correspond to sections $\tilde{L}_1, \ldots, \tilde{L}_n$ of the elliptic surface $\mathcal{E}$ associated to S (Remark 1.4.20), which in turn correspond to elements in the Mordell–Weil group of $\mathcal{E}$ (i.e., the Mordell–Weil group of the generic fiber, which is an elliptic curve over the function field $k(t)$ of $\mathbb{P}^1$). This Mordell–Weil group has rank at most 8 over k (Remark 1.4.17), so in this group there must be a relation $a_1\tilde{L}_1 + \cdots + a_n\tilde{L}_n = 0$, where $a_1, \ldots, a_n \in \mathbb{Z}$ are not all zero. Since all n exceptional curves contain the point P, this specializes to $(a_1 + \cdots + a_n)P_{\mathcal{E}} = 0$ on the fiber of P on $\mathcal{E}$. If one reasons too quickly, it seems that this proves that $P_{\mathcal{E}}$ is torsion of order dividing $a_1 + \cdots + a_n$ on its fiber. However, it might be the case that $a_1 + \cdots + a_n = 0$, so this does not prove Theorem 5.1.1. The key part in our proof is therefore that we show, using results from Chapter 3, that there is always a relation between $\tilde{L}_1, \ldots, \tilde{L}_n$ in the Mordell–Weil group of $\mathcal{E}$ that specializes to a *non-trivial* relation on the fiber of $P_{\mathcal{E}}$; see Lemma 5.2.2.

Remark 5.1.4. Recall that S can be embedded in the weighted projective space $\mathbb{P}(2,3,1,1)$ as the set of solutions to the equation

$$y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6 = 0, \tag{5.1}$$

where $a_i \in k[z,w]$ is homogeneous of degree i for each i in $\{1,\ldots,6\}$. The linear system $|-2K_S|$ of the bi-anticanonical divisor of S induces a morphism φ, which is the composition of the projection to $\mathbb{P}(2,1,1)$ and the 2-uple embedding in $\mathbb{P}^3$; this morphism realizes S as a double cover of a cone in $\mathbb{P}^3$ ramified over a sextic curve (see also Section 1.4.1). It follows that points on S that are on the ramification curve of φ correspond to points on $\mathcal{E}$ that are 2-torsion on their fiber.

The following example shows that if S is defined over a field of characteristic 0, for a point P on S that is contained in 6 exceptional curves, the point $P_{\mathcal{E}}$ is not guaranteed to be torsion on its fiber.

Example 5.1.5. Let k be a field of characteristic 0, and consider the eight points in $\mathbb{P}^2_k$ given by

$$\begin{aligned}
&P_1 = (1:0:1); && P_2 = (889:0:823);\\
&P_3 = (2600:101:2551); && P_4 = (325:12:287);\\
&P_5 = (0:1:1); && P_6 := (0:-1:1);\\
&P_7 = (4005:2464:3499); && P_8 = (195:22:-113).
\end{aligned}$$

We check that these points are in general position, by verifying that the determinants of the matrices in Lemma 3.3.12 that determine whether three of the points are on a line, or six of the points are on a conic, or seven of them are on a cubic that is singular at one of them, are non-zero. Let X be the blow-up of $\mathbb{P}^2$ in these points, which is a del Pezzo surface of degree 1. Let L_1 be the line through P_1 and P_2, which is given by $y=0$, and let L_2 be the line through P_3 and P_4, which is given by $51y = x+z$. Finally, let C_1 be the conic through P_1,P_3,P_5,P_6,P_7, let C_2 be the conic through P_1,P_4,P_5,P_6,P_8, let C_3 be the conic through P_2,P_3,P_5,P_7,P_8, and C_4 the conic through P_2,P_4,P_6,P_7, and P_8. Note that $L_1,L_2,C_1,\ldots,C_4$ are 6 of the 10 curves in Proposition 4.4.6; using the proof of this proposition, we chose $P_1,\ldots,P_8$ such that these 6 curves are concurrent in a point. The conics $C_1, \ldots, C_4$ are defined by the following equations.

$$\begin{aligned}
&C_1 \colon x^2 + y^2 - z^2 = xy;\\
&C_2 \colon x^2 + y^2 - z^2 = 6xy;\\
&C_3 \colon 823x^2 - 1884xy - 66xz - 3739y^2 + 4628yz - 889z^2;\\
&C_4 \colon 823x^2 - 4038xy - 66xz + 3139y^2 + 2250yz - 889z^2.
\end{aligned}$$

Indeed, the curves $L_1, L_2, C_1, \ldots, C_4$ all contain the point $(-1 : 0 : 1)$, so the strict transforms of these six curves, which are exceptional curves on X, are concurrent in a point P on X. Let $\mathcal{C}$ be the pencil of cubics through $P_1, \ldots, P_8$. This has a unique base point, which is

$$B = (3453493845425 : -16508630016087 : 20919196389638).$$

The fiber of $P_{\mathcal{E}}$ on the elliptic surface $\mathcal{E}$ is given by the element of $\mathcal{C}$ that contains P, and it is an elliptic curve with base point B. With `magma` it is quick to check that the point $P_{\mathcal{E}}$ is non-torsion on its fiber; see [Code] for the code that we used.

Remark 5.1.6. The previous example also holds if the characteristic of k is p for all but a finite number of primes p. In fact, the only characteristics for which this does not hold are the ones for which $P_1, \ldots, P_8$ are not in general position, which form a set of 42 primes. Using the proof of Proposition 4.4.6, it is not hard to generate similar examples that hold in some of those 42 characteristics; for example, the eight points in $\mathbb{P}^2$ given by

$$\begin{aligned}
&Q_1 = (1 : 0 : 1); && Q_5 = (0 : 1 : 1);\\
&Q_2 = (-236857 : 0 : 402962); && Q_6 = (0 : -1 : 1);\\
&Q_3 = (666 : 5 : -301); && Q_7 = (-2337353334 : 1829935 : 2432407789);\\
&Q_4 = (222 : 5 : 143); && Q_8 = (-101872359 : 3659870 : 141722269);
\end{aligned}$$

are in general position in all but 55 characteristics, and this gives, together with Example 5.1.5, an example of six exceptional curves that are concurrent in a point P such that $P_{\mathcal{E}}$ is not torsion on its fiber for each characteristic except for 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 71, 101, and 113.

From Theorem 5.1.1 and Example 5.1.5 it is clear that there are still open questions: if a point P on S is contained in 7 exceptional curves, is the point $P_{\mathcal{E}}$ then torsion on its fiber? And what about points contained in 8 exceptional curves? We have not yet found a proof nor a counterexample to these questions.

5.2 Proof of the main theorem

In this section we prove Theorem 5.1.1. We first describe a pairing on the Mordell–Weil group of $\mathcal{E}$, and use this pairing to state and prove two lemmas.

Let $L_1, \ldots, L_n$ be at least 9 exceptional curves on S that are concurrent in a point P that lies outside the ramification curve of φ. Let $\tilde{L}_1, \ldots, \tilde{L}_n$ be the corresponding sections on $\mathcal{E}$. Let $\langle \cdot, \cdot \rangle_h$ be the symmetric and bilinear pairing on the Mordell–Weil group of $\mathcal{E}$ as defined in [Shi90, Theorem 8.4]; that is, for C_1, C_2 in $E(k(t))$, we have $\langle C_1, C_2 \rangle_h = -(\varphi_h(C_1) \cdot \varphi_h(C_2))$, where $\varphi_h \colon E(k(t)) \longrightarrow \operatorname{Pic} \mathcal{E}$ is the map given in [Shi90, Lemmas 8.1 and 8.2], and $\cdot$ is the intersection pairing in the Picard group of $\mathcal{E}$. We call $\langle \cdot, \cdot \rangle_h$ the *height pairing* on $E(k(t))$.

Lemma 5.2.1. *For two exceptional curves in Pic S, the height pairing of the corresponding sections in the Mordell–Weil group of $\mathcal{E}$ is the same as the dot product of the roots in the root system $\boldsymbol{E}_8$ associated to these exceptional curves under the bijection in Remark 1.4.9.*

Proof. Let C_1, C_2 be two sections of $\mathcal{E}$ that are strict transforms of exceptional curves c_1, c_2 in S. Since $\mathcal{E}$ has no reducible fibers, by [Shi90, Lemma 8.1] we have

$$\varphi_h(C_1) \cdot \varphi_h(C_2) = ([C_1] - [\tilde{\mathcal{O}}] - F) \cdot ([C_2] - [\tilde{\mathcal{O}}] - F),$$

where $[C_1], [C_2], [\tilde{\mathcal{O}}]$ are the classes of C_1, C_2, and the zero section, respectively, and F is the class of a fiber. This gives

$$\varphi_h(C_1) \cdot \varphi_h(C_2) = [C_1] \cdot [C_2] - 1,$$

where we use that the zero section is an exceptional curve, and it is disjoint from C_1 and C_2. We conclude that we have $\langle C_1, C_2 \rangle_h = 1 - [C_1] \cdot [C_2]$. Since C_1, C_2 are disjoint from $\tilde{\mathcal{O}}$, the intersection pairing of C_1 and C_2 in Pic $\mathcal{E}$ is the same as the intersection pairing of c_1 and c_2 in Pic S. The statement now follows from the bijection in Remark 1.4.9. □

Let M be the *height pairing matrix* of $L_1, \dots, L_n$, that is, M is the $n \times n$ matrix with $M_{ij} = \langle \tilde{L}_i, \tilde{L}_j \rangle_h$ for $i, j \in \{1, \dots, n\}$.

Lemma 5.2.2. *The kernel of the matrix M contains a vector $(a_1, \dots, a_n)$ in $\mathbb{Z}^n$ with $a_1 + \cdots + a_n \neq 0$.*

Proof. Recall the complete weighted graphs G and Γ as defined in Definition 1.4.12. Since P lies outside the ramification curve of φ, the exceptional curves $L_1, \dots, L_n$ correspond to a clique of size n in G that is contained in a maximal clique in G with only edges of weights 1 and 2 (Remark 4.2.5), which corresponds to a maximal clique C in $\Gamma_{\{-1,0\}}$ by the bijection given in Remark 1.4.13. Since $n \geq 9$, the clique C has size at least 9. The table in Appendix A contains all isomorphism types of maximal cliques in $\Gamma_{\{-1,0\}}$ of size at least 9 (Proposition 3.5.28); there are 11 maximal cliques of size 9, which we call $\alpha_1, \dots, \alpha_{11}$ in the order that they appear in the table, there are 6 maximal cliques of size 10, which we call $\beta_1, \dots, \beta_6$ in the order that they appear in the table, and there is 1 maximal clique of size 12, which we call γ. For each of these 18 cliques, whose elements correspond to roots in $\mathbf{E}_8$, we compute its Gram matrix, which is the matrix where the entry (i, j) is the dot product of the roots corresponding to the i-th and j-th vertex in the clique after choosing an ordering on the vertices. With `magma` we find the generators for the kernels of these matrices (see [Code]). The results are in Table 5.1. Let r be the number of vertices of C, and let N be the Gram matrix of C; then the kernel of N is equal to one of the 18 kernels in the table, after rearranging the order of the vertices in C if necessary. Since $n \geq 9$, we see from Table 5.1 that for any subset of n vertices in C, there is a vector $(a_1, \dots, a_r)$ in the kernel of N which is 0 outside the entries corresponding to the n vertices, and such that $a_1 + \cdots + a_r \neq 0$. By Lemma 5.2.1, this gives a vector in the kernel of M as claimed. $\square$

Proof of Theorem 5.1.1. Let P be a point on S. If P is contained in the ramification curve of the morphism induced by the linear system of the bi-anticanonical divisor, then $P_{\mathcal{E}}$ is torsion (Remark 5.1.4), and we are done. Now assume that P is not contained in this ramification curve, and that there is a set of at least 9 exceptional curves that are concurrent in P. Let $K_1, \dots, K_n$ be the corresponding sections of $\mathcal{E}$, and let N be the height pairing matrix of these sections. Let $(a_1, \dots, a_n) \in \mathbb{Z}^n$ be a vector in the kernel of N such that $a_1 + \cdots + a_n \neq 0$, which exists by

Clique	Basis for the kernel
α_1	$\{(1,1,0,0,0,0,1,0,1),(0,0,1,1,1,1,0,2,0)\}$
α_2	$\{(1,0,1,0,0,1,0,0,1),(0,0,0,1,1,0,1,0,0)\}$
α_3	$\{(1,1,1,0,0,1,0,0,1),(0,0,0,1,1,0,1,1,0)\}$
α_4	$\{(1,1,0,1,0,0,1,0,1),(0,0,1,0,0,1,0,1,0)\}$
α_5	$\{(2,1,1,0,2,0,0,1,1),(0,0,0,1,0,1,1,0,0)\}$
α_6	$\{(1,1,1,1,1,1,1,1,1)\}$
α_7	$\{(1,1,1,0,1,1,1,1,1)\}$
α_8	$\{(0,1,1,2,2,2,1,1,0)\}$
α_9	$\{(2,1,1,1,1,2,2,2,2))\}$
α_{10}	$\{(2,2,0,3,1,4,2,3,1)\}$
α_{11}	$\{(6,3,1,4,4,2,2,5,3)\}$
β_1	$\{(1,0,1,0,0,2,1,0,0,1),(0,1,0,1,2,0,0,1,1,0)\}$
β_2	$\{(1,1,0,0,0,0,0,0,1,1),(0,0,0,0,1,1,1,1,0,0)\}$
β_3	$\{(1,1,0,1,0,0,0,1,0,1),(0,0,1,0,1,1,1,0,1,0)\}$
β_4	$\{(1,1,0,1,0,1,0,0,1,1),(0,0,0,0,1,0,1,1,0,0)\}$
β_5	$\{(1,1,0,0,0,0,0,0,1,1),(0,0,1,1,1,1,2,2,0,0)\}$
β_6	$\{(2,1,3,0,2,0,2,0,1,1),(0,0,0,1,0,1,0,1,0,0)\}$
γ	$\{(1,1,0,0,0,0,0,0,0,0,0,1),(0,0,1,0,0,1,0,0,0,0,1,0),$ $(0,0,0,1,0,0,0,1,1,0,0,0),(0,0,0,0,1,0,1,0,0,1,0,0)\}$

Table 5.1: Bases

Lemma 5.2.2. Then we have for all $i \in \{1,\dots,n\}$ we have that

$$a_1\langle K_i, K_1\rangle_h + \cdots + a_n\langle K_i, K_n\rangle_h = 0,$$

and since the height pairing is bilinear this implies

$$\langle K_i, a_1K_1 + a_2K_2 + \cdots + a_nK_n\rangle_h = 0 \text{ for all } i \in \{1,\dots,n\}, \tag{5.2}$$

which implies

$$\langle a_1K_1 + a_2K_2 + \cdots + a_nK_n, a_1K_1 + a_2K_2 + \cdots + a_nK_n\rangle_h = 0.$$

From the latter we conclude that $a_1K_1 + a_2K_2 + \cdots + a_nK_n$ is torsion in the Mordell–Weil group of $\mathcal{E}$ [Shi90, Theorem 8.4], and since the torsion subgroup is trivial [Shi90, Theorem 10.4], we conclude that

$$a_1K_1 + a_2K_2 + \cdots + a_nK_n = 0.$$

Since for all i in $\{1, \ldots, n\}$, the section K_i contains the point $P_{\mathcal{E}}$, we have, on the fiber of $P_{\mathcal{E}}$, the equality $(a_1 + \cdots + a_n)P_{\mathcal{E}} = 0$. Since $a_1 + \cdots + a_n \neq 0$, this implies that $P_{\mathcal{E}}$ is torsion on its fiber. $\square$

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Appendices

Appendix A

Orbits of maximal cliques

The following pages contain a table summarizing part of the results of Section 3.5. We recall the Notation 3.5.3.

Notation.

Graph: a graph Γ_c *where* c *is a set of colors in* $\{-2,-1,0,1\}$.

K*: a clique in* Γ_c*; we denote vertices by their index as written below.*

$|K|$*: the size of* K*.*

$|W_K|$*: the size of the stabilizer of clique* K *in the group* W*.*

$|\operatorname{Aut}(K)|$*: the size of the automorphism group of* K *as a colored graph.*

#O*: the number of orbits of the set of all maximal cliques of size* $|K|$ *in* Γ_c *under the action of* W*.*

Roots of the form $\left(\pm\frac{1}{2},\ldots,\pm\frac{1}{2}\right)$ are ordered lexicographically and denoted by numbers $1-128$; for example, $\left(-\frac{1}{2},\ldots,-\frac{1}{2}\right)$ is number 1, and $\left(\frac{1}{2},\ldots,\frac{1}{2}\right)$ number 128. Permutations of $(\pm1,\pm1,0,0,0,0,0,0)$ are ordered lexicographically and denoted by the numbers $129-240$; for example, $(-1,-1,0,0,0,0,0,0)$ is number 129, and $(1,1,0,0,0,0,0,0)$ is number 240.

Graph	$\|K\|$	$\#O$	$\|W_K\|$	$\|\mathrm{Aut}(K)\|$	K
$\Gamma_{\{-2\}}$	2	1	5806080	2	$\{1, 128\}$
$\Gamma_{\{-1\}}$	3	1	311040	6	$\{1, 32, 240\}$
$\Gamma_{\{0\}}$	8	1	1344	40320	$\{1, 8, 26, 31, 43, 46, 52, 53\}$
$\Gamma_{\{1\}}$	7	1	10080	5040	$\{1, 2, 3, 5, 129, 130, 131\}$
	8	1	40320	40320	$\{1, 2, 3, 4, 129, 130, 131, 132\}$
$\Gamma_{\{-2,-1\}}$	2	1	5806080	2	$\{1, 128\}$
	3	1	311040	6	$\{1, 32, 240\}$
$\Gamma_{\{-2,0\}}$	16	1	344064	10321920	$\{25, 32, 51, 54, 75, 78, 97, 104, 130, 144, 177, 181, 188, 192, 225, 239\}$
$\Gamma_{\{-2,1\}}$	2	1	5806080	2	$\{1, 128\}$
	7	1	10080	5040	$\{1, 2, 3, 5, 129, 130, 131\}$
	8	1	40320	40320	$\{1, 2, 3, 4, 129, 130, 131, 132\}$
$\Gamma_{\{-1,0\}}$	8	5	144	144	$\{41, 48, 50, 78, 144, 187, 214, 240\}$
			128	128	$\{12, 17, 41, 71, 170, 193, 214, 240\}$
			16	16	$\{12, 17, 40, 41, 71, 86, 214, 240\}$
			14	14	$\{12, 23, 41, 50, 70, 168, 214, 240\}$
			8	8	$\{7, 41, 48, 50, 75, 86, 214, 240\}$
	9	11	64	192	$\{3, 6, 41, 48, 50, 55, 214, 227, 240\}$
			48	96	$\{19, 41, 48, 50, 75, 146, 193, 214, 240\}$
			40	80	$\{12, 23, 41, 50, 67, 86, 163, 214, 240\}$
			30	60	$\{12, 23, 40, 41, 50, 65, 86, 214, 240\}$
			24	48	$\{19, 41, 48, 50, 70, 75, 193, 214, 240\}$
			18	18	$\{12, 23, 41, 50, 163, 168, 214, 227, 240\}$
			16	16	$\{19, 41, 48, 50, 65, 150, 172, 214, 240\}$
			8	16	$\{41, 48, 50, 55, 65, 78, 178, 214, 240\}$
			4	8	$\{3, 41, 48, 50, 55, 66, 152, 214, 240\}$
			2	2	$\{3, 41, 48, 50, 55, 72, 77, 214, 240\}$
			1	1	$\{7, 41, 48, 50, 68, 78, 85, 214, 240\}$
	10	6	192	1152	$\{41, 48, 50, 55, 66, 152, 178, 184, 214, 240\}$
			128	256	$\{3, 6, 41, 48, 50, 55, 76, 77, 214, 240\}$
			100	200	$\{12, 23, 40, 41, 50, 67, 77, 86, 214, 240\}$
			36	72	$\{6, 19, 41, 48, 50, 65, 76, 192, 214, 240\}$
			32	64	$\{3, 6, 41, 48, 50, 55, 76, 85, 214, 240\}$
			18	36	$\{19, 41, 48, 50, 65, 76, 86, 192, 214, 240\}$
	12	1	3888	31104	$\{11, 22, 36, 46, 49, 69, 74, 84, 184, 196, 214, 240\}$
$\Gamma_{\{-1,1\}}$	3	2	311040	6	$\{55, 80, 173\}$
			103680	2	$\{84, 88, 194\}$
	7	4	10080	5040	$\{118, 126, 191, 195, 213, 224, 237\}$
			1440	720	$\{8, 24, 32, 113, 129, 138, 151\}$

Graph	$\lvert K\rvert$	$\#O$	$\lvert W_K\rvert$	$\lvert\mathrm{Aut}(K)\rvert$	K
			480	240	{42, 72, 103, 120, 136, 193, 237}
			288	144	{37, 39, 53, 74, 167, 235, 238}
	8	5	40320	40320	{33, 41, 49, 57, 132, 133, 134, 142}
			5040	5040	{6, 7, 21, 24, 135, 148, 193, 201}
			1440	1440	{24, 32, 99, 107, 139, 152, 195, 213}
			1152	1152	{34, 63, 111, 114, 180, 182, 196, 203}
			720	720	{33, 91, 98, 101, 148, 151, 153, 154}
$\Gamma_{\{0,1\}}$	22	1	1344	1344	{1, 2, 3, 5, 9, 12, 17, 29, 33, 38, 51, 129, 130, 131, 132, 133, 134, 135, 136, 144, 158, 173}
	28	1	336	336	{1, 2, 3, 5, 7, 9, 13, 14, 15, 17, 25, 29, 33, 43, 45, 53, 129, 130, 131, 132, 133, 134, 136, 137, 139, 140, 149, 157}
	29	432			separate section
	30	25	3840	3840	{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 17, 18, 20, 22, 26, 33, 129, 130, 131, 132, 133, 134, 136, 137, 143, 144, 145, 146, 149}
			1152	1152	{1, 2, 3, 4, 5, 6, 8, 9, 11, 13, 15, 17, 19, 21, 23, 26, 33, 129, 130, 131, 132, 133, 134, 136, 137, 140, 141, 145, 146, 149}
			720	720	{1, 2, 3, 4, 5, 6, 8, 9, 11, 13, 15, 17, 19, 21, 23, 26, 33, 129, 130, 131, 132, 133, 134, 135, 136, 138, 139, 143, 144, 147}
			192	192	{1, 2, 3, 5, 7, 8, 9, 13, 14, 17, 22, 29, 33, 37, 39, 53, 65, 129, 130, 131, 132, 133, 134, 136, 144, 147, 157, 160, 166, 184}
			72	72	{1, 2, 3, 5, 6, 8, 9, 13, 14, 15, 17, 22, 29, 33, 37, 39, 53, 65, 129, 130, 131, 132, 133, 134, 136, 144, 147, 157, 160, 181}
			64	64	{1, 2, 3, 5, 6, 7, 8, 9, 14, 17, 22, 29, 33, 129, 130, 131, 132, 133, 134, 136, 139, 143, 144, 146, 147, 149, 157, 160, 166, 169}
			48	48	{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 13, 17, 18, 19, 21, 26, 33, 129, 130, 131, 132, 133, 134, 135, 136, 143, 144, 145, 146, 147}
			48	48	{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 17, 18, 19, 20, 26, 33, 129, 130, 131, 132, 133, 134, 135, 136, 138, 139, 143, 144, 147}
			48	48	{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 13, 15, 17, 19, 21, 26, 33, 129, 130, 131, 132, 133, 134, 135, 136, 138, 139, 143, 144, 147}
			48	48	{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 14, 15, 17, 33, 36, 37, 42, 129, 130, 131, 132, 133, 134, 135, 136, 138, 139, 140, 143, 158}
			48	48	{1, 2, 3, 4, 5, 6, 8, 9, 11, 13, 17, 18, 19, 21, 23, 26, 33, 129, 130, 131, 132, 133, 134, 136, 138, 139, 140, 143, 147, 149}
			32	32	{1, 2, 3, 4, 5, 6, 8, 9, 10, 17, 18, 26, 33, 129, 130, 131, 132, 133, 134, 135, 136, 143, 144, 145, 146, 147, 156, 157, 165, 166}
			24	24	{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 13, 17, 18, 19, 21, 26, 33, 129, 130, 131, 132, 133, 134, 136, 137, 138, 139, 143, 144, 149}
			16	16	{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 15, 17, 19, 20, 21, 26, 33, 129, 130, 131, 132, 133, 134, 135, 136, 138, 139, 143, 144, 147}
			16	16	{1, 2, 3, 4, 5, 6, 8, 9, 10, 15, 17, 33, 35, 36, 37, 38, 42, 129, 130, 131, 132, 133, 134, 135, 136, 138, 139, 140, 143, 158}
			12	12	{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 17, 18, 20, 22, 26, 33, 129, 130, 131, 132, 133, 134, 135, 136, 138, 143, 144, 145, 147}
			12	12	{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 17, 18, 20, 22, 26, 33, 129, 130, 131, 132, 133, 134, 135, 136, 138, 139, 143, 144, 147}
			12	12	{1, 2, 3, 4, 5, 6, 8, 9, 10, 15, 17, 33, 35, 36, 37, 38, 42, 129, 130, 131, 132, 133, 134, 135, 136, 138, 140, 143, 156, 158}
			10	10	{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 17, 18, 22, 26, 33, 36, 38, 65, 129, 130, 131, 132, 133, 134, 136, 143, 144, 149, 157, 165}
			8	8	{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 13, 17, 18, 19, 22, 26, 33, 129, 130, 131, 132, 133, 134, 135, 136, 138, 143, 144, 145, 147}
			8	8	{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 15, 17, 33, 35, 36, 37, 42, 129, 130, 131, 132, 133, 134, 135, 136, 139, 140, 143, 157, 158}
			4	4	{1, 2, 3, 5, 7, 8, 9, 10, 11, 13, 17, 18, 19, 21, 27, 29, 33, 129, 130, 131, 132, 133, 134, 135, 136, 139, 140, 143, 146, 147}
			4	4	{1, 2, 3, 5, 7, 9, 10, 11, 13, 17, 18, 19, 21, 33, 37, 41, 65, 129, 130, 131, 132, 133, 134, 136, 143, 146, 147, 155, 174, 179}
			4	4	{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 17, 18, 19, 21, 27, 33, 129, 130, 131, 132, 133, 134, 135, 136, 139, 140, 143, 146, 147}
			4	4	{1, 2, 3, 5, 7, 9, 10, 11, 13, 17, 18, 19, 33, 37, 41, 65, 129, 130, 131, 132, 133, 134, 136, 143, 146, 147, 155, 156, 174, 179}
	31	7	480	480	{1, 3, 5, 9, 11, 17, 19, 25, 27, 33, 65, 67, 73, 75, 81, 83, 89, 129, 132, 134, 145, 147, 158, 167, 174, 179, 183, 184, 187, 199, 208}
			120	120	{1, 3, 5, 9, 11, 15, 17, 19, 25, 27, 65, 67, 73, 75, 83, 89, 129, 132, 134, 143, 144, 145, 147, 148, 153, 156, 158, 174, 179, 183, 187}
			72	72	{1, 3, 5, 9, 11, 17, 18, 19, 21, 23, 25, 27, 33, 49, 51, 81, 89, 129, 132, 133, 134, 141, 144, 145, 147, 165, 167, 174, 179, 183, 208}
			48	48	{1, 3, 5, 9, 11, 17, 19, 21, 23, 25, 27, 49, 65, 81, 89, 129, 132, 133, 134, 144, 145, 146, 147, 148, 158, 165, 167, 174, 179, 183, 208}

Graph	$\lvert K\rvert$	$\#O$	$\lvert W_K\rvert$	$\lvert\mathrm{Aut}(K)\rvert$	K
			16	16	$\{1,3,5,9,11,13,15,17,19,21,25,27,31,41,67,73,89,129,132,134,145,147,158,167,174,179,183,184,187,199,208\}$
			12	12	$\{1,3,5,9,11,13,15,17,19,21,25,27,31,41,67,73,89,129,132,134,145,146,147,148,158,167,174,179,183,199,208\}$
			8	8	$\{1,3,5,9,11,13,15,17,19,21,25,27,31,41,67,73,89,129,132,134,145,147,148,158,167,174,179,183,187,199,208\}$
	32	3	144	144	$\{2,3,5,9,12,17,20,26,27,29,65,74,75,82,83,89,129,132,136,143,145,146,147,149,153,160,173,174,176,181,184,201\}$
			48	48	$\{1,2,3,4,5,7,9,10,11,12,13,15,16,35,41,43,75,129,130,132,134,136,143,145,147,155,156,157,158,160,162,164\}$
			24	24	$\{1,3,5,9,11,17,18,19,20,21,23,25,26,27,29,33,49,57,129,131,132,133,134,135,136,140,141,145,146,147,174,208\}$
	33	1	96	96	$\{1,2,3,4,5,6,7,8,9,10,11,12,14,17,18,20,33,34,129,130,131,132,133,134,135,136,137,143,144,145,146,155,156\}$
	34	2	2880	2880	$\{1,2,5,6,8,14,17,18,22,33,34,36,37,38,42,50,53,54,129,130,131,132,133,136,137,139,142,155,157,166,169,170,181,182\}$
			720	720	$\{1,2,3,4,5,6,8,9,12,14,15,17,20,22,23,26,33,36,38,129,130,131,132,133,134,136,137,138,139,143,144,149,160,169\}$
	36	1	40320	40320	$\{1,2,3,4,5,6,7,8,9,10,11,12,17,18,19,20,33,34,35,36,129,130,131,132,133,134,135,136,137,138,143,144,145,155,156,165\}$
$\Gamma_{\{-2,-1,0\}}$	8	4	144	144	$\{1,8,26,31,43,54,227,240\}$
			128	128	$\{1,8,26,47,83,102,226,238\}$
			16	16	$\{1,8,26,47,83,110,226,233\}$
			8	8	$\{1,8,26,31,43,54,228,239\}$
	9	9	80	80	$\{1,8,26,47,51,86,121,128,228\}$
			64	192	$\{1,8,26,31,43,46,84,85,240\}$
			40	80	$\{1,8,26,47,51,86,124,125,228\}$
			28	28	$\{1,8,26,47,51,86,110,121,236\}$
			24	48	$\{1,8,26,31,43,54,100,125,227\}$
			18	18	$\{1,8,26,47,51,86,110,124,232\}$
			12	12	$\{1,8,26,31,43,86,106,115,125\}$
			4	8	$\{1,8,26,31,43,46,84,113,237\}$
			1	1	$\{1,8,26,31,43,54,100,113,238\}$
	10	10	288	576	$\{1,8,26,47,51,77,121,128,185,229\}$
			192	1152	$\{1,8,26,31,43,46,52,53,227,240\}$
			100	200	$\{1,8,26,47,51,86,91,125,222,228\}$
			64	64	$\{1,8,26,31,43,46,84,98,103,125\}$
			60	120	$\{1,8,26,47,51,86,91,128,218,228\}$
			48	96	$\{1,8,26,31,43,86,101,106,115,128\}$
			32	32	$\{1,8,26,31,43,86,106,115,224,234\}$
			32	64	$\{1,8,26,31,43,46,84,101,226,238\}$
			18	36	$\{1,8,26,31,43,54,100,109,119,227\}$
			4	4	$\{1,8,26,31,43,46,84,98,117,238\}$
	11	5	2304	2304	$\{1,8,26,31,43,54,98,103,121,128,227\}$
			648	2592	$\{1,8,26,47,51,77,108,121,185,213,229\}$
			192	384	$\{1,8,26,31,43,54,100,101,121,128,227\}$
			32	64	$\{1,8,26,31,43,46,52,85,98,103,238\}$
			72	144	$\{1,8,26,31,43,54,100,109,113,128,227\}$
	12	3	3888	31104	$\{1,8,26,47,51,77,91,108,185,213,218,229\}$

Graph	$\lvert K\rvert$	$\#O$	$\lvert W_K\rvert$	$\lvert\mathrm{Aut}(K)\rvert$	K
			1024	3072	$\{1, 8, 26, 31, 43, 46, 84, 85, 98, 103, 121, 128\}$
			512	1024	$\{1, 8, 26, 31, 43, 46, 84, 85, 98, 103, 226, 238\}$
	13	1	1536	9216	$\{1, 8, 26, 31, 43, 46, 52, 53, 76, 77, 83, 86, 240\}$
	16	1	344064	10321920	$\{1, 8, 26, 31, 43, 46, 52, 53, 76, 77, 83, 86, 98, 103, 121, 128\}$
$\Gamma_{\{-2,-1,1\}}$	6	1	622080	12	$\{24, 33, 96, 105, 131, 238\}$
	14	1	20160	10080	$\{11, 41, 88, 118, 135, 159, 169, 175, 183, 186, 194, 200, 210, 234\}$
	16	1	80640	80640	$\{11, 22, 43, 54, 75, 86, 107, 118, 156, 172, 174, 178, 191, 195, 197, 213\}$
$\Gamma_{\{-2,0,1\}}$	13	7	1536	9216	$\{1, 8, 31, 40, 71, 98, 128, 136, 150, 164, 178, 191, 205\}$
			768	4608	$\{1, 8, 31, 40, 98, 128, 136, 137, 150, 164, 178, 191, 205\}$
			512	1024	$\{1, 8, 31, 98, 128, 136, 137, 149, 150, 162, 163, 171, 172\}$
			384	768	$\{1, 8, 12, 14, 15, 128, 136, 137, 138, 139, 154, 169, 215\}$
			384	768	$\{1, 8, 12, 38, 47, 82, 128, 136, 137, 152, 160, 161, 171\}$
			192	384	$\{1, 8, 31, 39, 45, 51, 98, 128, 136, 137, 149, 162, 172\}$
			64	128	$\{1, 8, 12, 23, 38, 47, 82, 128, 136, 137, 152, 160, 171\}$
	14	4	15360	30720	$\{1, 27, 43, 51, 57, 59, 128, 136, 138, 140, 141, 142, 177, 192\}$
			3072	18432	$\{1, 12, 23, 30, 45, 47, 77, 79, 99, 106, 117, 128, 163, 199\}$
			192	384	$\{1, 29, 30, 47, 54, 78, 82, 93, 117, 128, 152, 191, 198, 209\}$
			64	128	$\{1, 46, 70, 72, 79, 83, 100, 101, 103, 128, 160, 172, 228, 231\}$
	16	3	344064	10321920	$\{1, 22, 46, 57, 72, 83, 107, 128, 138, 153, 160, 177, 192, 209, 216, 231\}$
			336	336	$\{1, 39, 40, 43, 51, 53, 55, 115, 128, 141, 142, 169, 192, 216, 218, 219\}$
			192	384	$\{1, 16, 38, 42, 68, 70, 74, 77, 102, 106, 113, 128, 160, 182, 215, 228\}$
	19	29	2880	2880	$\{1, 8, 12, 14, 15, 20, 22, 23, 36, 38, 39, 128, 136, 137, 138, 139, 149, 160, 169\}$
			2880	5760	$\{1, 8, 12, 14, 50, 68, 70, 74, 128, 136, 137, 154, 169, 170, 176, 177, 181, 182, 215\}$
			2304	2304	$\{1, 8, 12, 14, 23, 24, 39, 40, 68, 70, 128, 136, 137, 151, 152, 162, 163, 169, 170\}$
			1440	1440	$\{1, 8, 12, 15, 16, 20, 23, 24, 36, 39, 40, 70, 128, 136, 137, 139, 151, 162, 171\}$
			384	384	$\{1, 8, 12, 15, 23, 24, 38, 40, 68, 70, 128, 136, 137, 150, 152, 162, 163, 169, 171\}$
			144	144	$\{1, 8, 12, 14, 23, 24, 39, 40, 68, 128, 136, 137, 138, 151, 152, 162, 163, 169, 170\}$
			120	120	$\{1, 8, 12, 14, 20, 24, 36, 39, 128, 136, 137, 138, 139, 149, 151, 161, 162, 169, 170\}$
			96	96	$\{1, 8, 12, 14, 15, 22, 26, 38, 45, 128, 136, 137, 138, 139, 149, 152, 160, 161, 169\}$
			96	96	$\{1, 8, 12, 14, 15, 24, 40, 128, 136, 137, 138, 139, 150, 151, 152, 161, 162, 163, 169\}$
			96	96	$\{1, 8, 12, 14, 20, 23, 38, 39, 128, 136, 137, 138, 139, 149, 152, 160, 162, 169, 170\}$
			96	96	$\{1, 8, 12, 14, 39, 40, 128, 136, 137, 138, 139, 151, 152, 160, 161, 162, 163, 169, 170\}$
			96	96	$\{1, 8, 12, 15, 16, 22, 23, 24, 36, 39, 40, 68, 70, 128, 136, 137, 151, 163, 171\}$
			72	72	$\{1, 8, 12, 14, 15, 22, 24, 38, 40, 128, 136, 137, 138, 139, 150, 152, 161, 163, 169\}$
			48	48	$\{1, 8, 12, 14, 20, 22, 39, 40, 128, 136, 137, 138, 139, 151, 152, 160, 161, 169, 170\}$
			32	32	$\{1, 8, 12, 14, 15, 20, 22, 23, 36, 38, 40, 128, 136, 137, 138, 139, 150, 160, 169\}$
			32	32	$\{1, 8, 12, 14, 23, 39, 128, 136, 137, 138, 139, 149, 151, 152, 160, 162, 163, 169, 170\}$
			32	32	$\{1, 8, 12, 14, 23, 24, 39, 68, 128, 136, 137, 138, 149, 151, 152, 162, 163, 169, 170\}$
			24	24	$\{1, 8, 12, 14, 15, 23, 36, 38, 40, 128, 136, 137, 138, 139, 150, 160, 162, 163, 169\}$

Graph	$\lvert K\rvert$	$\#O$	$\lvert W_K\rvert$	$\lvert\mathrm{Aut}(K)\rvert$	K
			24	24	$\{1,8,12,14,15,24,38,40,128,136,137,138,139,150,152,161,162,163,169\}$
			24	24	$\{1,8,12,14,23,24,38,39,68,128,136,137,138,149,152,162,163,169,170\}$
			20	20	$\{1,8,12,14,20,24,39,128,136,137,138,139,149,151,152,161,162,169,170\}$
			16	16	$\{1,8,12,14,15,20,23,38,40,128,136,137,138,139,150,152,160,162,169\}$
			16	16	$\{1,8,12,14,23,40,128,136,137,138,139,150,151,152,160,162,163,169,170\}$
			12	12	$\{1,8,12,14,15,22,23,38,40,128,136,137,138,139,150,152,160,163,169\}$
			8	8	$\{1,8,12,14,15,20,23,36,38,40,128,136,137,138,139,150,160,162,169\}$
			8	8	$\{1,8,12,14,15,23,38,40,128,136,137,138,139,150,152,160,162,163,169\}$
			8	8	$\{1,8,12,14,15,23,24,38,40,128,136,137,138,139,150,152,162,163,169\}$
			8	8	$\{1,8,12,14,20,24,38,39,128,136,137,138,139,149,152,161,162,169,170\}$
			8	8	$\{1,8,12,14,20,39,40,128,136,137,138,139,151,152,160,161,162,169,170\}$
	20	5	192	192	$\{1,8,12,14,15,16,20,22,23,36,38,39,128,136,137,138,139,140,149,160\}$
			128	128	$\{1,8,12,14,15,16,20,23,38,40,128,136,137,138,139,140,150,152,160,162\}$
			96	96	$\{1,8,12,14,15,16,20,38,39,40,128,136,137,138,139,140,152,160,161,162\}$
			16	16	$\{1,8,12,14,15,16,20,22,38,39,40,128,136,137,138,139,140,152,160,161\}$
			16	16	$\{1,8,12,14,15,16,20,22,38,39,128,136,137,138,139,140,149,152,160,161\}$
	21	1	48	48	$\{1,8,12,14,15,16,22,38,39,45,128,136,137,138,139,140,149,152,160,161,163\}$
	22	3	1440	1440	$\{1,26,30,42,46,58,74,78,90,106,114,121,128,177,182,185,197,198,200,207,218,231\}$
			1344	1344	$\{1,2,3,5,9,12,17,29,33,38,51,129,130,131,132,133,134,135,136,144,158,173\}$
			144	144	$\{1,12,26,28,36,42,43,44,50,52,57,58,59,106,128,176,177,178,197,206,217,230\}$
	24	1	1440	1440	$\{1,8,12,14,15,16,38,39,40,45,46,47,128,136,137,138,139,140,152,160,161,162,163,164\}$
	28			**as in 0,1**	
	29	433	103680	103680	$\{1,8,12,14,15,16,36,38,39,40,42,43,44,45,46,47,128,136,137,138,139,140,142,160,161,162,163,164,215\}$
				rest	**as in 0,1**
	30			**as in 0,1**	
	31			**as in 0,1**	
	32			**as in 0,1**	
	33			**as in 0,1**	
	34			**as in 0,1**	
	36			**as in 0,1**	
Γ	240	1	$\lvert W\rvert$	$\lvert W\rvert$	Γ

Appendix B

Maximal cliques of size 29 in $\Gamma_{\{0,1\}}$

The following pages contain a table summarizing the results in Proposition 3.5.36. We recall Notation 3.5.38.

Notation.

K: a clique in $\Gamma_{\{0,1\}}$; we denote vertices by their index as written below.

$|W_K|$: the size of the stabilizer of clique K in the group W.

$\#K_5(1)$: the number of cliques of size 5 with only edges of color 0 in K.

$\#K_4^a(1)$: the number of cliques in K of four roots that sum up to a double root in Λ, with only edges of color 1.

$\#K_4^b(1)$: the number of cliques in K of four roots that do not sum up to a double root in Λ, with only edges of color 1.

Roots of the form $\left(\pm\frac{1}{2},\dots,\pm\frac{1}{2}\right)$ are ordered lexicographically and denoted by numbers $1-128$; for example, $\left(-\frac{1}{2},\dots,-\frac{1}{2}\right)$ is number 1, and $\left(\frac{1}{2},\dots,\frac{1}{2}\right)$ number 128. Permutations of $(\pm1,\pm1,0,0,0,0,0,0)$ are ordered lexicographically and denoted by the numbers $129-240$; for example, $(-1,-1,0,0,0,0,0,0)$ is number 129, and $(1,1,0,0,0,0,0,0)$ is number 240.

Nr.	$\lvert W_K \rvert$	$\#K_5(1)$	$\#K_4^a(0)$	$\#K_4^b(0)$	K
1	12	1176	36	0	{1,2,3,4,5,9,17,19,34,37,41,49,65,66,129,130,131,132,133, 146,148,155,156,158,165,166,167,173,174}
2	48	2352	8	0	{1,2,3,4,5,6,8,9,10,12,14,17,18,20,22,26,129,130,131,132, 133,134,137,138,143,144,145,148,149}
3	12	2008	12	0	{1,2,3,4,5,6,8,9,10,17,34,42,66,129,130,131,132,133,137,143, 145,146,150,155,156,157,160,165,166}
4	8	1256	32	0	{1,2,3,4,5,6,9,12,14,17,34,42,50,65,66,129,130,131,132,133, 145,146,150,155,156,157,160,169,173}
5	96	1288	36	0	{1,2,3,4,5,6,9,10,17,19,21,25,35,37,41,65,129,130,131,132, 133,134,143,147,148,158,165,166,173}
6	16	1256	28	0	{1,2,3,4,5,6,9,17,19,25,37,41,129,130,131,132,133,134,135, 143,146,147,148,156,158,165,166,167,173}
7	1	800	45	0	{1,2,3,4,5,7,9,10,13,14,16,34,37,38,41,43,129,130,131,132, 133,138,139,142,155,158,159,160,161}
8	4	816	49	0	{1,2,3,4,5,7,9,10,12,13,15,16,35,38,41,43,129,130,131,132, 133,139,142,155,157,158,159,160,161}
9	1	800	45	0	{1,2,3,4,5,7,9,11,12,13,14,16,34,39,41,43,129,130,131,132, 133,139,142,155,157,158,159,160,161}
10	4	800	45	0	{1,2,3,4,5,7,9,13,14,16,34,37,38,39,41,43,129,130,131,132, 133,138,142,155,156,158,159,160,161}
11	2	792	43	0	{1,2,3,4,5,7,9,12,14,16,34,36,38,39,41,43,129,130,131,132, 133,139,142,155,157,158,159,160,161}
12	2	808	47	0	{1,2,3,4,5,7,9,12,13,14,15,16,38,39,41,43,129,130,131,132, 133,139,142,155,157,158,159,160,161}
13	12	832	53	0	{1,2,3,5,7,9,11,12,13,14,16,34,39,41,43,45,129,130,131,132, 133,135,137,142,155,156,157,159,161}
14	4	808	47	0	{1,2,3,5,7,9,11,12,14,16,34,36,39,41,43,45,129,130,131,132, 133,135,137,138,142,155,156,159,161}
15	4	1440	25	0	{1,2,3,4,5,6,8,9,17,20,34,38,42,65,66,129,130,131,132,133, 143,146,150,155,156,160,165,166,169}
16	4	1400	28	0	{1,2,3,4,5,6,9,17,19,25,34,35,37,65,129,130,131,132,133,134, 143,144,148,156,158,165,166,167,173}
17	2	1392	26	0	{1,2,3,4,5,6,9,17,19,35,37,41,129,130,131,132,133,134,135, 143,147,148,155,156,158,165,166,167,173}
18	16	976	42	0	{1,2,3,5,9,13,17,19,34,35,37,41,49,66,129,130,131,132,133, 135,148,155,156,158,166,167,173,174,179}
19	16	1240	32	0	{1,2,3,4,5,6,7,9,17,19,21,41,49,65,66,129,130,131,132,133, 145,146,147,148,155,158,165,166,167}
20	2	632	52	0	{1,2,3,5,7,9,10,19,21,34,41,49,65,66,129,130,131,132,133, 145,146,148,155,158,159,165,166,174,179}
21	2	624	50	0	{1,2,3,5,7,9,10,13,19,21,34,35,41,49,65,66,129,130,131,132, 133,145,148,155,158,159,166,174,179}
22	4	624	50	0	{1,2,3,5,7,9,11,13,18,19,21,34,41,49,65,66,129,130,131,132, 133,145,146,148,155,159,167,174,179}
23	4	640	54	0	{1,2,3,5,6,7,9,10,19,25,35,37,41,49,65,66,129,130,131,132, 133,147,148,156,158,159,165,166,179}
24	16	616	48	0	{1,2,3,5,7,11,18,19,25,34,37,49,65,66,129,130,131,132,133, 144,146,148,156,159,165,167,168,174,179}
25	4	632	52	0	{1,2,3,5,7,10,18,19,25,35,37,49,65,66,129,130,131,132,133, 144,147,148,156,159,165,166,168,174,179}
26	32	648	56	0	{1,2,3,5,6,11,18,19,25,35,37,49,65,66,129,130,131,132,133, 144,147,148,156,159,165,167,168,173,179}
27	48	680	64	0	{1,2,3,4,5,13,21,25,37,49,65,66,129,130,131,132,133,144,146, 147,148,157,158,159,166,167,168,173,174}
28	2	656	50	0	{1,2,3,5,7,9,12,14,16,34,36,38,39,41,43,45,129,130,131,132, 133,135,138,142,155,156,158,159,161}
29	2	648	48	0	{1,2,3,5,7,9,11,14,15,16,36,37,39,41,43,45,129,130,131,132, 133,135,137,139,142,155,157,159,161}
30	1	664	52	0	{1,2,3,5,7,9,10,12,15,16,35,36,38,41,43,45,129,130,131,132, 133,135,138,139,142,155,158,159,161}
31	4	664	52	0	{1,2,3,5,7,9,10,12,15,16,35,36,38,41,43,45,129,130,131,132, 133,137,138,139,142,155,159,160,161}
32	4	648	48	0	{1,2,3,5,7,9,10,14,15,16,36,37,38,41,43,45,129,130,131,132, 133,135,138,140,142,156,158,159,161}
33	8	680	56	0	{1,2,3,5,7,9,10,15,16,35,36,37,38,41,43,45,129,130,131,132, 133,138,139,140,143,158,159,160,161}
34	4	656	50	0	{1,2,3,5,7,9,10,12,15,16,35,36,38,41,43,45,129,130,131,132, 133,137,139,140,142,157,159,160,161}

Nr.	$\lvert W_K \rvert$	$\#K_5(1)$	$\#K_4^a(0)$	$\#K_4^b(0)$	K
35	16	664	52	0	{1,2,3,5,7,18,20,22,23,24,36,38,40,49,51,53,129,130,131,132, 133,135,141,144,165,166,167,168,170}
36	2	656	50	0	{1,2,3,5,7,18,19,20,24,34,35,36,40,49,51,53,129,130,131,132, 133,138,139,141,144,167,168,169,170}
37	12	688	58	0	{1,2,3,4,5,21,22,23,24,37,38,39,40,49,50,51,129,130,131,132, 133,135,139,141,144,166,167,168,170}
38	48	1416	34	0	{1,2,3,4,5,9,13,17,34,35,37,41,49,65,129,130,131,132,133, 134,148,155,156,157,158,166,167,173,174}
39	48	1184	36	0	{1,2,3,4,5,6,9,17,25,35,37,41,49,65,129,130,131,132,133,134, 147,148,156,157,158,165,166,167,173}
40	2	952	38	0	{1,2,3,4,5,6,9,12,17,18,22,36,38,42,50,65,66,129,130,131, 132,133,149,150,155,157,165,169,173}
41	2	968	42	0	{1,2,3,4,5,6,9,12,14,17,18,36,38,42,50,65,66,129,130,131, 132,133,149,150,155,156,157,169,173}
42	2	960	40	0	{1,2,3,4,5,6,9,12,14,17,20,34,38,42,50,65,66,129,130,131, 132,133,146,150,155,156,160,169,173}
43	1	952	38	0	{1,2,3,4,5,6,9,14,17,20,36,42,50,65,66,129,130,131,132,133, 145,149,150,155,156,160,166,169,173}
44	2	960	40	0	{1,2,3,4,5,6,9,14,17,22,36,42,50,65,66,129,130,131,132,133, 145,149,150,155,157,160,166,169,173}
45	4	640	56	0	{1,2,3,5,7,9,10,19,21,35,41,49,65,66,129,130,131,132,133, 145,147,148,155,158,159,165,166,174,179}
46	2	616	50	0	{1,2,3,5,7,9,11,19,21,34,41,49,65,66,129,130,131,132,133, 145,146,148,155,158,159,165,167,174,179}
47	8	624	52	0	{1,2,3,5,7,9,19,21,34,41,49,66,129,130,131,132,133,135,145, 146,148,155,158,159,165,166,167,174,179}
48	2	624	52	0	{1,2,3,5,7,9,10,19,21,25,34,35,41,49,65,66,129,130,131,132, 133,145,148,158,159,165,166,174,179}
49	24	656	60	0	{1,2,3,5,6,9,10,19,21,25,35,37,41,49,65,66,129,130,131,132, 133,147,148,158,159,165,166,173,179}
50	2	624	52	0	{1,2,3,5,7,11,13,18,25,35,49,66,129,130,131,132,133,135,144, 145,147,148,156,157,159,167,168,174,179}
51	2	616	50	0	{1,2,3,5,7,10,11,18,19,25,37,49,65,66,129,130,131,132,133, 144,146,147,148,156,159,165,168,174,179}
52	4	624	52	0	{1,2,3,5,7,11,18,25,35,49,65,66,129,130,131,132,133,144,145, 147,148,156,157,159,165,167,168,174,179}
53	4	640	56	0	{1,2,3,4,5,10,11,13,21,25,37,49,65,66,129,130,131,132,133, 144,146,147,148,157,158,159,168,173,174}
54	4	480	56	0	{1,2,3,5,10,13,21,25,35,49,65,66,129,130,131,132,133,144,145, 147,148,157,158,159,166,168,173,174,179}
55	4	488	58	0	{1,2,3,5,11,13,19,25,37,49,65,66,129,130,131,132,133,144,146, 147,148,156,158,159,167,168,173,174,179}
56	48	504	62	0	{1,2,3,5,11,13,25,35,37,49,65,66,129,130,131,132,133,144,147, 148,156,157,158,159,167,168,173,174,179}
57	1152	1032	44	0	{1,2,3,4,9,10,11,17,18,21,25,33,34,35,41,49,67,69,129,130, 131,134,135,145,157,165,173,174,175}
58	4	376	62	0	{1,2,3,5,11,18,21,25,37,41,66,67,129,130,131,132,135,143,146, 147,148,157,159,165,167,168,173,174,179}
59	36	384	64	0	{1,2,3,5,11,18,21,25,34,37,41,65,66,67,129,130,131,132,143, 146,148,157,159,165,167,168,173,174,179}
60	16	392	66	0	{1,2,3,5,19,21,25,34,37,41,66,67,129,130,131,132,135,143,146, 148,158,159,165,166,167,168,173,174,179}
61	20	368	60	0	{1,2,3,5,10,19,25,37,41,49,66,67,129,130,131,132,135,146,147, 148,156,158,159,165,166,168,173,174,179}
62	96	424	74	0	{1,2,3,4,5,10,13,19,21,25,37,41,49,65,66,67,129,130,131,132, 146,147,148,158,159,166,168,173,174}
63	1440	488	90	0	{1,2,3,4,5,13,21,25,37,41,49,65,66,67,129,130,131,132,146, 147,148,157,158,159,166,167,168,173,174}
64	24	360	58	0	{1,2,3,9,11,13,18,19,37,49,66,69,129,130,131,133,135,144,146, 147,148,155,156,159,167,173,174,175,179}
65	240	1888	20	0	{1,2,3,4,5,6,9,10,17,34,35,37,41,65,129,130,131,132,133,134, 143,148,155,156,157,158,165,166,173}
66	4	1744	18	0	{1,2,3,4,5,6,8,9,11,12,15,17,19,20,23,26,129,130,131,132, 133,134,137,138,139,143,144,148,149}
67	4	1776	16	0	{1,2,3,4,5,6,8,9,12,13,17,18,19,22,23,26,129,130,131,132, 133,134,135,138,143,144,145,147,148}
68	4	1616	20	0	{1,2,3,4,5,6,8,9,14,17,18,20,21,22,23,26,129,130,131,132, 133,134,139,140,143,146,147,148,149}

Nr.	$\lvert W_K\rvert$	$\#K_5(1)$	$\#K_4^a(0)$	$\#K_4^b(0)$	K
69	24	1632	24	0	{1,2,3,4,5,6,8,9,12,17,20,33,34,38,42,50,66,129,130,131, 132,133,137,146,155,156,160,165,169}
70	192	2264	7	84	{1,2,3,4,5,6,8,9,10,11,17,18,21,26,129,130,131,132,133,134, 135,143,144,145,146,147,148,157,165}
71	8	872	42	0	{1,2,3,4,5,9,11,13,17,19,21,34,35,37,41,49,65,66,129,130, 131,132,133,148,155,158,167,173,174}
72	4	880	44	0	{1,2,3,4,5,9,11,13,17,21,35,37,41,49,65,66,129,130,131,132, 133,147,148,155,157,158,167,173,174}
73	12	1344	33	0	{1,2,3,4,5,9,13,17,34,35,41,49,65,129,130,131,132,133,134, 145,148,155,156,157,158,166,167,173,174}
74	8	736	48	0	{1,2,3,5,7,9,18,19,21,35,41,49,65,66,129,130,131,132,133, 145,147,148,155,159,165,166,167,174,179}
75	16	1952	12	0	{1,2,3,4,5,6,8,9,11,12,14,17,18,20,23,26,129,130,131,132, 133,134,135,137,138,143,144,145,148}
76	4	1584	20	0	{1,2,3,4,5,6,9,10,11,17,18,21,25,35,37,65,129,130,131,132, 133,134,143,144,147,148,157,165,173}
77	8	1592	22	0	{1,2,3,4,5,6,9,10,11,17,19,21,25,35,37,65,129,130,131,132, 133,134,143,144,147,148,158,165,173}
78	240	1208	40	0	{1,2,3,5,7,8,9,11,12,15,33,35,36,39,43,45,129,130,131,132, 133,138,139,140,142,158,159,160,161}
79	6	1232	34	0	{1,2,3,4,5,9,13,17,18,34,35,41,49,65,66,129,130,131,132,133, 145,148,155,156,157,166,167,173,174}
80	8	520	53	0	{1,2,3,5,7,18,22,23,24,36,37,38,40,49,51,53,129,130,131,132, 133,135,138,141,144,165,167,168,170}
81	2	520	53	0	{1,2,3,5,7,18,19,22,24,34,36,37,40,49,51,53,129,130,131,132, 133,138,139,141,144,167,168,169,170}
82	4	536	57	0	{1,2,3,5,7,18,20,22,24,34,36,38,40,49,51,53,129,130,131,132, 133,138,139,141,144,167,168,169,170}
83	4	528	55	0	{1,2,3,5,7,18,24,34,35,36,37,38,40,49,51,53,129,130,131,132, 133,138,139,141,144,167,168,169,170}
84	2	528	55	0	{1,2,3,5,7,18,23,24,35,36,37,38,40,49,51,53,129,130,131,132, 133,138,141,144,165,167,168,169,170}
85	4	528	55	0	{1,2,3,5,7,20,22,24,34,36,38,39,40,49,51,53,129,130,131,132, 133,138,141,144,165,167,168,169,170}
86	16	544	59	0	{1,2,3,5,6,18,19,23,24,35,36,37,40,49,50,53,129,130,131,132, 133,138,139,141,144,167,168,169,170}
87	2	536	57	0	{1,2,3,4,5,21,22,23,24,37,38,39,40,49,50,51,129,130,131,132, 133,135,138,141,144,165,167,168,170}
88	20	528	55	0	{1,2,3,4,5,19,22,23,24,36,37,39,40,49,50,51,129,130,131,132, 133,135,139,141,144,166,167,168,170}
89	24	552	61	0	{1,2,3,4,5,23,24,35,36,37,38,39,40,49,50,51,129,130,131,132, 133,138,139,141,144,167,168,169,170}
90	8	560	63	0	{1,2,3,4,5,21,22,23,24,37,38,39,40,49,50,51,129,130,131,132, 133,138,139,141,144,167,168,169,170}
91	240	608	75	0	{1,2,3,5,9,19,20,23,27,35,36,39,43,67,68,129,130,131,132, 138,143,151,162,165,167,169,174,176,184}
92	12	784	49	0	{1,2,3,5,7,19,20,21,22,24,34,39,40,49,51,53,129,130,131,132, 133,135,137,144,155,165,166,168,170}
93	8	600	52	0	{1,2,3,5,11,18,19,25,37,49,65,66,129,130,131,132,133,144,146, 147,148,156,159,165,167,168,173,174,179}
94	4	608	54	0	{1,2,3,5,10,11,13,25,35,49,65,66,129,130,131,132,133,144,145, 147,148,156,157,158,159,168,173,174,179}
95	240	1568	35	0	{1,2,3,5,8,9,12,14,15,17,20,22,23,26,27,29,129,130,131,132, 133,134,137,139,140,141,146,148,149}
96	12	1344	32	0	{1,2,3,4,5,6,9,17,25,34,35,37,41,65,129,130,131,132,133,134, 143,148,156,157,158,165,166,167,173}
97	8	1520	26	0	{1,2,3,4,5,6,9,17,19,34,35,37,41,65,129,130,131,132,133,134, 143,148,155,156,158,165,166,167,173}
98	12	992	40	0	{1,2,3,4,5,9,13,17,19,34,35,37,41,49,65,66,129,130,131,132, 133,148,155,156,158,166,167,173,174}
99	2	1632	20	0	{1,2,3,4,5,6,9,11,17,25,34,35,65,129,130,131,132,133,134, 143,144,145,148,156,157,158,165,167,173}
100	2	1632	20	0	{1,2,3,4,5,6,9,10,17,19,25,34,35,41,65,129,130,131,132,133, 134,143,145,148,156,158,165,166,173}
101	4	1808	15	0	{1,2,3,4,5,6,8,9,13,14,17,18,21,22,23,26,129,130,131,132, 133,134,137,138,139,143,144,148,149}
102	8	720	48	0	{1,2,3,5,7,9,13,18,19,21,25,35,41,49,65,66,129,130,131,132, 133,145,147,148,159,166,167,174,179}

Nr.	$\vert W_K\vert$	$\#K_5(1)$	$\#K_4^a(0)$	$\#K_4^b(0)$	K
103	2	1072	34	0	{1,2,3,4,5,6,9,11,13,17,18,19,21,35,37,41,65,66,129,130, 131,132,133,143,147,148,155,167,173}
104	1	1080	36	0	{1,2,3,4,5,6,9,10,11,17,19,37,41,49,65,66,129,130,131,132, 133,146,147,148,155,156,158,165,173}
105	2	1072	34	0	{1,2,3,4,5,6,9,10,11,13,17,19,35,37,41,49,65,66,129,130, 131,132,133,147,148,155,156,158,173}
106	4	1088	38	0	{1,2,3,4,5,6,7,9,10,17,19,21,35,37,41,49,65,66,129,130,131, 132,133,147,148,155,158,165,166}
107	2	496	56	0	{1,2,3,5,7,11,13,18,25,34,35,49,65,66,129,130,131,132,133, 144,145,148,156,157,159,167,168,174,179}
108	4	488	54	0	{1,2,3,5,7,13,18,25,35,49,65,66,129,130,131,132,133,144,145, 147,148,156,157,159,166,167,168,174,179}
109	8	504	58	0	{1,2,3,5,7,11,13,18,25,35,37,49,65,66,129,130,131,132,133, 144,147,148,156,157,159,167,168,174,179}
110	4	512	60	0	{1,2,3,4,5,10,13,21,25,35,37,49,65,66,129,130,131,132,133, 144,147,148,157,158,159,166,168,173,174}
111	4	1376	28	0	{1,2,3,4,5,6,9,17,18,21,35,41,65,66,129,130,131,132,133,143, 145,147,148,155,157,165,166,167,173}
112	12	1376	28	0	{1,2,3,4,5,6,7,9,10,11,13,17,35,37,41,49,65,66,129,130,131, 132,133,147,148,155,156,157,158}
113	4	1168	36	0	{1,2,3,4,5,6,9,17,35,37,41,49,65,66,129,130,131,132,133,147, 148,155,156,157,158,165,166,167,173}
114	4	736	46	0	{1,2,3,5,7,9,11,21,34,35,41,49,65,66,129,130,131,132,133, 145,148,155,157,158,159,165,167,174,179}
115	2	744	48	0	{1,2,3,5,7,9,21,34,35,41,49,66,129,130,131,132,133,135,145, 148,155,157,158,159,165,166,167,174,179}
116	8	760	52	0	{1,2,3,5,7,9,10,19,21,25,35,37,41,66,129,130,131,132,133, 135,143,147,148,158,159,165,166,174,179}
117	2	744	48	0	{1,2,3,5,7,9,10,19,21,25,35,41,49,66,129,130,131,132,133, 135,145,147,148,158,159,165,166,174,179}
118	8	1568	22	0	{1,2,3,4,5,6,8,9,11,14,15,17,20,21,23,26,129,130,131,132, 133,134,137,138,139,143,144,148,149}
119	4	1992	12	0	{1,2,3,4,5,6,8,9,12,14,17,18,20,22,23,26,129,130,131,132, 133,134,135,137,139,143,144,146,148}
120	2	784	45	0	{1,2,3,4,5,7,9,12,14,15,16,36,38,39,41,43,129,130,131,132, 133,137,139,142,155,157,159,160,161}
121	2	784	45	0	{1,2,3,5,7,9,13,14,16,34,37,38,39,41,43,45,129,130,131,132, 133,135,138,142,155,156,158,159,161}
122	16	792	47	0	{1,2,3,5,7,9,12,16,34,35,36,38,39,41,43,45,129,130,131,132, 133,135,138,139,142,155,158,159,161}
123	4	792	47	0	{1,2,3,5,7,9,10,15,16,35,36,37,38,41,43,45,129,130,131,132, 133,135,139,142,155,157,158,159,161}
124	4	792	47	0	{1,2,3,5,7,9,10,15,16,35,36,37,38,41,43,45,129,130,131,132, 133,135,138,140,143,156,158,159,161}
125	24	416	64	0	{1,2,3,5,11,13,18,25,35,37,49,65,66,129,130,131,132,133,144, 147,148,156,157,159,167,168,173,174,179}
126	4	400	60	0	{1,2,3,5,10,13,19,25,35,37,49,65,66,129,130,131,132,133,144, 147,148,156,158,159,166,168,173,174,179}
127	16	432	68	0	{1,2,3,5,6,11,19,25,37,41,49,65,66,129,130,131,132,133,146, 147,148,156,158,159,165,167,168,173,179}
128	4	392	58	0	{1,2,3,5,11,18,25,37,41,66,67,129,130,131,132,135,143,146, 147,148,156,157,159,165,167,168,173,174,179}
129	4	408	62	0	{1,2,3,5,11,18,21,25,34,37,41,66,67,129,130,131,132,135,143, 146,148,157,159,165,167,168,173,174,179}
130	4	408	62	0	{1,2,3,5,11,18,21,25,35,37,41,66,67,129,130,131,132,135,143, 147,148,157,159,165,167,168,173,174,179}
131	16	416	64	0	{1,2,3,5,21,25,34,37,41,66,67,129,130,131,132,135,143,146, 148,157,158,159,165,166,167,168,173,174,179}
132	192	480	80	0	{1,2,3,4,5,13,19,21,25,37,41,49,65,66,67,129,130,131,132, 146,147,148,158,159,166,167,168,173,174}
133	128	416	64	0	{1,2,3,4,9,10,21,25,37,41,49,67,69,129,130,131,134,135,146, 147,148,157,158,159,165,166,173,174,175}
134	64	384	56	0	{1,2,3,9,11,13,18,19,35,37,49,66,69,129,130,131,133,135,144, 147,148,155,156,159,167,173,174,175,179}
135	720	728	60	0	{1,2,3,5,9,19,21,25,35,37,41,49,65,66,129,130,131,132,133, 147,148,158,159,165,166,167,173,174,179}
136	12	624	60	0	{1,2,3,5,7,9,10,19,21,25,35,37,41,49,65,66,129,130,131,132, 133,147,148,158,159,165,166,174,179}

Nr.	$\lvert W_K\rvert$	$\#K_5(1)$	$\#K_4^a(0)$	$\#K_4^b(0)$	K
137	4	592	52	0	{1,2,3,5,7,9,18,21,25,35,41,49,65,66,129,130,131,132,133, 145,147,148,157,159,165,166,167,174,179}
138	16	608	56	0	{1,2,3,5,11,13,25,49,65,66,129,130,131,132,133,144,145,146, 147,148,156,157,158,159,167,168,173,174,179}
139	24	592	52	0	{1,2,3,5,11,13,18,25,34,41,66,67,129,130,131,132,135,143,145, 146,148,156,157,159,167,168,173,174,179}
140	192	1344	33	0	{1,2,3,5,9,10,17,34,35,37,49,65,66,129,130,131,132,133,144, 148,155,156,157,158,165,166,173,174,179}
141	20	1808	15	0	{1,2,3,4,5,6,9,10,17,20,33,42,66,129,130,131,132,133,137, 143,145,146,149,155,156,160,165,166,173}
142	8	1960	12	0	{1,2,3,4,5,6,8,9,11,12,13,17,18,19,23,26,129,130,131,132, 133,134,137,143,144,145,146,148,149}
143	16	664	52	0	{1,2,3,5,7,11,13,18,25,49,66,129,130,131,132,133,135,144,145, 146,147,148,156,157,159,167,168,174,179}
144	18	648	48	0	{1,2,3,5,7,10,11,13,18,19,25,37,49,65,66,129,130,131,132, 133,144,146,147,148,156,159,168,174,179}
145	24	664	52	0	{1,2,3,4,5,10,11,13,19,21,25,37,49,65,66,129,130,131,132, 133,144,146,147,148,158,159,168,173,174}
146	432	696	60	0	{1,2,3,4,5,6,11,13,25,35,37,41,49,65,66,129,130,131,132,133, 147,148,156,157,158,159,167,168,173}
147	4	680	52	0	{1,2,3,4,5,7,9,10,14,15,16,36,37,38,41,43,129,130,131,132, 133,138,139,142,155,158,159,160,161}
148	2	664	48	0	{1,2,3,4,5,7,9,12,13,14,16,34,38,39,41,43,129,130,131,132, 133,138,139,142,155,158,159,160,161}
149	6	672	50	0	{1,2,3,5,7,9,11,12,14,16,34,36,39,41,43,45,129,130,131,132, 133,135,137,139,142,155,157,159,161}
150	4	672	50	0	{1,2,3,5,7,9,10,12,16,34,35,36,38,41,43,45,129,130,131,132, 133,138,139,140,143,158,159,160,161}
151	72	712	60	0	{1,2,3,4,5,6,9,10,15,16,35,36,37,38,41,42,129,130,131,132, 133,138,139,140,142,158,159,160,161}
152	8	760	48	0	{1,2,3,5,7,9,11,13,19,21,34,41,49,66,129,130,131,132,133, 135,145,146,148,155,158,159,167,174,179}
153	48	1704	25	0	{1,2,3,4,5,6,8,9,12,14,17,18,20,22,23,26,129,130,131,132, 133,134,137,138,139,140,141,148,149}
154	4	1640	18	0	{1,2,3,4,5,6,8,9,10,17,36,42,66,129,130,131,132,133,137,143, 145,149,150,155,156,157,160,165,166}
155	8	1648	20	0	{1,2,3,4,5,6,9,10,14,17,20,33,36,42,66,129,130,131,132,133, 137,143,145,149,155,156,160,166,173}
156	4	1408	24	0	{1,2,3,4,5,6,9,11,17,21,25,35,37,41,129,130,131,132,133,134, 135,143,147,148,157,158,165,167,173}
157	12	1200	34	0	{1,2,3,4,5,6,9,11,13,17,19,21,35,37,41,66,129,130,131,132, 133,135,143,147,148,155,158,167,173}
158	24	1032	41	0	{1,2,3,5,7,8,9,11,15,33,35,36,37,39,43,45,129,130,131,132, 133,137,138,139,140,142,159,160,161}
159	12	1592	22	0	{1,2,3,4,5,6,9,13,17,18,34,35,41,65,66,129,130,131,132,133, 143,145,148,155,156,157,166,167,173}
160	24	1600	24	0	{1,2,3,4,5,6,9,10,17,22,34,36,42,65,66,129,130,131,132,133, 143,145,150,155,157,160,165,166,173}
161	2	1080	37	0	{1,2,3,4,5,9,14,17,34,36,42,50,66,129,130,131,132,133,137, 145,150,155,156,157,160,166,169,173,176}
162	4	1080	37	0	{1,2,3,4,5,8,9,14,17,20,22,38,42,50,66,129,130,131,132,133, 137,146,149,150,155,160,166,169,176}
163	8	1088	39	0	{1,2,3,5,9,10,11,13,17,34,35,37,49,65,66,129,130,131,132, 133,144,148,155,156,157,158,173,174,179}
164	12	1072	35	0	{1,2,3,5,9,12,14,17,18,20,22,36,38,50,66,129,130,131,132, 133,137,144,149,150,155,169,173,176,181}
165	72	1088	39	0	{1,2,3,5,7,8,9,11,12,16,34,35,36,39,43,45,129,130,131,132, 133,135,137,138,142,155,156,159,161}
166	4	768	44	0	{1,2,3,4,5,7,9,11,18,21,37,41,49,66,129,130,131,132,133,135, 146,147,148,155,157,159,165,167,174}
167	2	776	46	0	{1,2,3,4,5,7,9,13,18,19,37,41,49,66,129,130,131,132,133,135, 146,147,148,155,156,159,166,167,174}
168	8	800	52	0	{1,2,3,4,5,7,9,10,13,18,21,35,41,49,65,66,129,130,131,132, 133,145,147,148,155,157,159,166,174}
169	2	768	44	0	{1,2,3,4,5,7,9,10,13,18,19,37,41,49,65,66,129,130,131,132, 133,146,147,148,155,156,159,166,174}
170	12	1168	38	0	{1,2,3,5,8,9,12,14,17,18,20,22,36,38,42,50,66,129,130,131, 132,133,137,149,150,155,169,176,181}

Nr.	$\lvert W_K\rvert$	$\#K_5(1)$	$\#K_4^a(0)$	$\#K_4^b(0)$	K
171	12	1776	19	0	{1,2,3,4,5,6,8,9,11,12,14,15,17,20,23,26,129,130,131,132, 133,134,137,138,139,143,144,148,149}
172	14	1288	28	0	{1,2,3,4,5,6,9,11,17,18,19,35,37,41,66,129,130,131,132,133, 135,143,147,148,155,156,165,167,173}
173	48	1304	32	0	{1,2,3,4,5,6,7,9,10,11,17,18,19,35,37,41,49,65,66,129,130, 131,132,133,147,148,155,156,165}
174	240	1568	30	0	{1,2,3,4,5,6,7,9,17,25,34,35,37,65,129,130,131,132,133,134, 143,144,148,156,157,158,165,166,167}
175	12	1936	14	0	{1,2,3,4,5,6,8,9,11,12,13,14,17,18,23,26,129,130,131,132, 133,134,135,137,143,144,145,146,148}
176	48	896	50	0	{1,2,3,5,7,9,10,17,19,21,35,37,41,49,65,66,129,130,131,132, 133,147,148,155,158,165,166,174,179}
177	4	1216	30	0	{1,2,3,4,5,6,9,13,17,18,19,21,35,41,65,66,129,130,131,132, 133,143,145,147,148,155,166,167,173}
178	2	1224	32	0	{1,2,3,4,5,6,9,17,18,19,21,35,37,41,65,66,129,130,131,132, 133,143,147,148,155,165,166,167,173}
179	16	1760	19	0	{1,2,3,4,5,6,9,17,18,34,35,41,65,129,130,131,132,133,134, 143,145,148,155,156,157,165,166,167,173}
180	2	1240	30	0	{1,2,3,4,5,6,9,13,17,18,21,35,41,66,129,130,131,132,133,135, 143,145,147,148,155,157,166,167,173}
181	2	1240	30	0	{1,2,3,4,5,6,9,17,18,19,35,37,41,66,129,130,131,132,133,135, 143,147,148,155,156,165,166,167,173}
182	24	1264	36	0	{1,2,3,4,5,6,7,9,10,13,17,18,21,35,41,49,65,66,129,130,131, 132,133,145,147,148,155,157,166}
183	4	1248	32	0	{1,2,3,4,5,6,7,9,10,17,18,19,35,37,41,49,65,66,129,130,131, 132,133,147,148,155,156,165,166}
184	16	1440	25	0	{1,2,3,4,5,6,9,17,20,36,42,65,66,129,130,131,132,133,143, 145,149,150,155,156,160,165,166,169,173}
185	24	1440	25	0	{1,2,3,4,5,6,9,10,12,14,17,20,22,36,38,42,66,129,130,131, 132,133,137,143,149,150,155,160,173}
186	32	1512	26	0	{1,2,3,4,5,6,9,17,34,41,49,65,129,130,131,132,133,134,145, 146,148,155,156,157,158,165,166,167,173}
187	12	1480	27	0	{1,2,3,4,5,6,9,10,17,20,22,33,36,38,42,65,66,129,130,131, 132,133,143,149,155,160,165,166,173}
188	2	1464	23	0	{1,2,3,4,5,6,9,17,20,33,38,42,66,129,130,131,132,133,137, 143,146,149,155,156,160,165,166,169,173}
189	192	808	46	0	{1,2,3,4,9,10,13,17,18,21,25,33,34,35,41,49,67,69,129,130, 131,134,135,145,157,166,173,174,175}
190	12	920	42	0	{1,2,3,5,7,9,10,14,16,34,36,37,38,41,43,45,129,130,131,132, 133,134,138,142,155,156,158,159,160}
191	8	928	44	0	{1,2,3,5,7,9,11,14,16,34,36,37,39,41,43,45,129,130,131,132, 133,134,137,142,155,156,157,159,160}
192	4	920	42	0	{1,2,3,5,7,9,12,16,34,35,36,38,39,41,43,45,129,130,131,132, 133,135,142,155,156,157,158,159,161}
193	8	928	44	0	{1,2,3,5,7,9,10,11,15,16,35,36,37,41,43,45,129,130,131,132, 133,135,138,139,142,155,158,159,161}
194	144	1216	36	0	{1,2,3,4,5,7,9,11,13,34,37,41,49,66,129,130,131,132,133,135, 146,148,155,156,157,158,159,167,174}
195	72	1200	32	0	{1,2,3,4,5,7,9,10,11,13,21,34,35,37,41,49,65,66,129,130, 131,132,133,148,155,157,158,159,174}
196	8	1408	28	0	{1,2,3,4,5,6,9,10,17,19,25,34,35,37,41,65,129,130,131,132, 133,134,143,148,156,158,165,166,173}
197	2	1400	26	0	{1,2,3,4,5,6,9,17,19,25,34,35,37,41,129,130,131,132,133,134, 135,143,148,156,158,165,166,167,173}
198	24	1392	28	0	{1,2,3,4,5,6,9,17,19,21,35,37,41,65,129,130,131,132,133,134, 143,147,148,155,158,165,166,167,173}
199	4	1032	38	0	{1,2,3,4,5,7,9,11,13,17,21,34,35,37,41,49,65,66,129,130, 131,132,133,148,155,157,158,167,174}
200	24	1048	42	0	{1,2,3,4,5,7,9,13,17,21,35,37,41,49,65,66,129,130,131,132, 133,147,148,155,157,158,166,167,174}
201	24	1152	40	0	{1,2,3,5,7,9,17,19,34,35,37,41,49,66,129,130,131,132,133, 135,148,155,156,158,165,166,167,174,179}
202	4	1424	24	0	{1,2,3,4,5,6,9,10,11,17,21,25,34,35,37,65,129,130,131,132, 133,134,143,144,148,157,158,165,173}
203	48	1920	16	0	{1,2,3,4,5,6,9,10,11,13,17,18,19,21,25,35,37,65,129,130, 131,132,133,134,143,144,147,148,173}
204	2	1424	26	0	{1,2,3,4,5,6,9,10,17,19,21,25,34,35,37,65,129,130,131,132, 133,134,143,144,148,158,165,166,173}

Nr.	$\|W_K\|$	$\#K_5(1)$	$\#K_4^a(0)$	$\#K_4^b(0)$	K
205	2	1416	24	0	{1,2,3,4,5,6,9,11,17,19,21,25,34,35,37,65,129,130,131,132, 133,134,143,144,148,158,165,167,173}
206	48	1232	40	0	{1,2,3,5,9,17,34,35,37,41,49,66,129,130,131,132,133,135,148, 155,156,157,158,165,166,167,173,174,179}
207	2	920	40	0	{1,2,3,4,5,6,9,13,17,19,21,35,41,49,65,66,129,130,131,132, 133,145,147,148,155,158,166,167,173}
208	12	944	46	0	{1,2,3,4,5,7,9,13,17,18,21,35,41,49,65,66,129,130,131,132, 133,145,147,148,155,157,166,167,174}
209	16	928	42	0	{1,2,3,4,5,7,9,11,13,17,18,37,41,49,65,66,129,130,131,132, 133,146,147,148,155,156,157,167,174}
210	12	1696	22	0	{1,2,3,4,5,6,9,17,34,35,37,41,129,130,131,132,133,134,135, 143,148,155,156,157,158,165,166,167,173}
211	24	520	61	0	{1,2,3,5,11,21,25,41,49,65,66,129,130,131,132,133,145,146, 147,148,157,158,159,165,167,168,173,174,179}
212	16	496	55	0	{1,2,3,4,9,10,18,21,25,33,37,41,49,67,69,129,130,131,134, 135,146,147,157,159,165,166,173,174,175}
213	8	1072	38	0	{1,2,3,4,5,6,9,13,17,21,34,35,41,49,65,66,129,130,131,132, 133,145,148,155,157,158,166,167,173}
214	48	1000	45	0	{1,2,3,5,7,9,11,15,16,35,36,37,39,41,43,45,129,130,131,132, 133,135,138,139,140,142,158,159,161}
215	24	1416	31	0	{1,2,3,4,5,6,9,12,14,15,17,20,22,23,25,26,129,130,131,132, 133,134,137,138,139,143,144,148,149}
216	48	1368	32	0	{1,2,3,4,5,6,9,10,17,18,35,37,41,49,65,66,129,130,131,132, 133,147,148,155,156,157,165,166,173}
217	4	1072	36	0	{1,2,3,4,5,6,9,11,17,19,21,35,37,41,65,66,129,130,131,132, 133,143,147,148,155,158,165,167,173}
218	4	1064	34	0	{1,2,3,4,5,6,9,17,19,21,37,41,49,66,129,130,131,132,133,135, 146,147,148,155,158,165,166,167,173}
219	8	1080	38	0	{1,2,3,4,5,6,7,9,13,17,21,35,41,49,65,66,129,130,131,132, 133,145,147,148,155,157,158,166,167}
220	20	1568	20	0	{1,2,3,4,5,6,9,17,19,34,35,41,129,130,131,132,133,134,135, 143,145,148,155,156,158,165,166,167,173}
221	384	1376	40	0	{1,2,3,5,8,9,12,14,17,20,22,26,36,38,42,50,66,129,130,131, 132,133,137,149,150,160,169,176,181}
222	1	1112	33	0	{1,2,3,4,5,6,9,17,20,36,38,42,66,129,130,131,132,133,137, 143,149,150,155,156,160,165,166,169,173}
223	2	1120	35	0	{1,2,3,4,5,6,9,12,17,18,20,36,38,42,50,65,66,129,130,131, 132,133,149,150,155,156,165,169,173}
224	4	1128	37	0	{1,2,3,4,5,6,9,14,17,22,34,36,42,50,66,129,130,131,132,133, 137,145,150,155,157,160,166,169,173}
225	4	1120	35	0	{1,2,3,4,5,6,9,12,14,17,20,22,34,36,42,50,66,129,130,131, 132,133,137,145,150,155,160,169,173}
226	24	2112	12	0	{1,2,3,4,5,6,8,9,11,13,17,18,19,21,23,26,129,130,131,132, 133,134,135,138,143,144,145,147,148}
227	12	1136	35	0	{1,2,3,4,5,6,7,9,12,14,15,16,36,38,39,41,129,130,131,132, 133,134,135,138,142,155,156,158,159}
228	2	1128	33	0	{1,2,3,4,5,6,7,9,11,14,15,16,36,37,39,41,129,130,131,132, 133,134,137,138,142,155,156,159,160}
229	12	1120	31	0	{1,2,3,4,5,6,7,9,11,12,15,16,35,36,39,41,129,130,131,132, 133,134,137,139,142,155,157,159,160}
230	8	1128	33	0	{1,2,3,4,5,6,7,9,12,14,16,34,36,38,39,41,129,130,131,132, 133,135,138,142,155,156,158,159,161}
231	4	1592	22	0	{1,2,3,4,5,6,9,17,34,35,41,129,130,131,132,133,134,135,143, 145,148,155,156,157,158,165,166,167,173}
232	24	1600	24	0	{1,2,3,4,5,6,8,9,10,12,15,17,19,20,22,26,129,130,131,132, 133,134,137,138,139,140,143,148,149}
233	2	1080	34	0	{1,2,3,4,5,6,9,13,17,18,19,35,37,41,65,66,129,130,131,132, 133,143,147,148,155,156,166,167,173}
234	12	1104	40	0	{1,2,3,4,5,6,9,10,17,19,21,35,37,41,65,66,129,130,131,132, 133,143,147,148,155,158,165,166,173}
235	16	1080	34	0	{1,2,3,4,5,6,9,10,13,17,18,19,35,37,41,49,65,66,129,130, 131,132,133,147,148,155,156,166,173}
236	2	1416	27	0	{1,2,3,4,5,6,9,17,25,34,35,41,65,129,130,131,132,133,134, 143,145,148,156,157,158,165,166,167,173}
237	4	1416	27	0	{1,2,3,4,5,6,8,9,12,15,17,19,20,22,23,26,129,130,131,132, 133,134,138,139,140,143,147,148,149}
238	2	808	43	0	{1,2,3,4,5,7,9,10,11,14,16,34,36,37,41,43,129,130,131,132, 133,138,139,142,155,158,159,160,161}

Nr.	$\lvert W_K \rvert$	$\#K_5(1)$	$\#K_4^a(0)$	$\#K_4^b(0)$	K
239	4	808	43	0	{1,2,3,4,5,7,9,10,12,13,16,34,35,38,41,43,129,130,131,132, 133,138,139,142,155,158,159,160,161}
240	4	824	47	0	{1,2,3,4,5,7,9,10,12,15,16,35,36,38,41,43,129,130,131,132, 133,138,139,142,155,158,159,160,161}
241	8	808	43	0	{1,2,3,4,5,7,9,11,13,14,16,34,37,39,41,43,129,130,131,132, 133,137,138,142,155,156,159,160,161}
242	48	528	62	0	{1,2,3,5,6,11,13,18,25,35,37,49,65,66,129,130,131,132,133, 144,147,148,156,157,159,167,168,173,179}
243	14	504	56	0	{1,2,3,4,5,10,11,21,25,35,37,49,65,66,129,130,131,132,133, 144,147,148,157,158,159,165,168,173,174}
244	2	1272	28	0	{1,2,3,4,5,6,9,13,17,19,34,35,41,65,66,129,130,131,132,133, 143,145,148,155,156,158,166,167,173}
245	2	1280	30	0	{1,2,3,4,5,6,9,17,22,34,36,42,66,129,130,131,132,133,137, 143,145,150,155,157,160,165,166,169,173}
246	4	1288	32	0	{1,2,3,4,5,6,9,10,17,20,36,38,42,65,66,129,130,131,132,133, 143,149,150,155,156,160,165,166,173}
247	24	1288	32	0	{1,2,3,4,5,6,8,9,17,34,42,50,66,129,130,131,132,133,137,145, 146,150,155,156,157,160,165,166,169}
248	48	760	55	0	{1,2,3,5,19,20,21,22,23,33,39,40,49,50,51,53,129,130,131, 132,133,138,139,141,144,167,168,169,170}
249	8	888	42	0	{1,2,3,5,7,9,10,11,13,19,21,34,35,37,41,49,65,66,129,130, 131,132,133,148,155,158,159,174,179}
250	12	896	44	0	{1,2,3,5,7,9,11,13,21,34,35,41,49,66,129,130,131,132,133, 135,145,148,155,157,158,159,167,174,179}
251	2	1288	28	0	{1,2,3,4,5,6,9,12,14,17,18,20,33,36,38,42,50,65,66,129,130, 131,132,133,149,155,156,169,173}
252	12	1304	32	0	{1,2,3,4,5,6,9,12,14,17,20,22,33,34,36,42,50,65,66,129,130, 131,132,133,145,155,160,169,173}
253	8	1288	28	0	{1,2,3,4,5,6,9,14,17,22,33,36,42,50,66,129,130,131,132,133, 137,145,149,155,157,160,166,169,173}
254	4	1256	32	0	{1,2,3,4,5,6,9,12,15,17,19,20,22,23,25,26,129,130,131,132, 133,134,137,138,139,140,143,148,149}
255	10	1248	30	0	{1,2,3,4,5,6,9,12,13,14,15,17,22,23,25,26,129,130,131,132, 133,134,137,138,140,141,145,148,149}
256	4	896	42	0	{1,2,3,4,5,6,9,11,13,17,19,21,35,37,41,49,65,66,129,130, 131,132,133,147,148,155,158,167,173}
257	8	896	42	0	{1,2,3,5,7,9,11,19,21,34,35,41,49,66,129,130,131,132,133, 135,145,148,155,158,159,165,167,174,179}
258	12	2128	10	0	{1,2,3,4,5,6,8,9,11,13,17,18,19,21,23,26,129,130,131,132, 133,134,135,137,143,144,145,146,148}
259	12	744	47	0	{1,2,3,5,7,9,12,14,15,16,36,38,39,41,43,45,129,130,131,132, 133,137,138,139,142,155,159,160,161}
260	4	752	49	0	{1,2,3,5,7,9,16,34,35,36,37,38,39,41,43,45,129,130,131,132, 133,138,140,142,156,158,159,160,161}
261	4	1344	30	0	{1,2,3,4,5,6,9,17,25,34,35,41,49,65,129,130,131,132,133,134, 145,148,156,157,158,165,166,167,173}
262	24	1200	36	0	{1,2,3,4,5,7,9,13,17,21,34,35,37,41,49,66,129,130,131,132, 133,135,148,155,157,158,166,167,174}
263	8	1584	22	0	{1,2,3,4,5,6,9,10,17,18,25,35,37,65,129,130,131,132,133,134, 143,144,147,148,156,157,165,166,173}
264	12	1584	22	0	{1,2,3,4,5,6,9,17,19,21,25,34,35,37,129,130,131,132,133,134, 135,143,144,148,158,165,166,167,173}
265	4	760	47	0	{1,2,3,5,7,9,11,14,15,16,36,37,39,41,43,45,129,130,131,132, 133,135,138,139,142,155,158,159,161}
266	8	776	51	0	{1,2,3,5,7,9,10,15,16,35,36,37,38,41,43,45,129,130,131,132, 133,135,138,139,140,142,158,159,161}
267	240	1208	40	0	{1,2,3,5,7,9,11,17,19,34,35,37,41,49,66,129,130,131,132,133, 135,148,155,156,158,165,167,174,179}
268	1920	1608	40	0	{1,2,3,5,9,17,34,35,37,41,49,65,129,130,131,132,133,134,148, 155,156,157,158,165,166,167,173,174,179}
269	2	896	40	0	{1,2,3,4,5,6,9,13,17,19,35,37,41,49,65,66,129,130,131,132, 133,147,148,155,156,158,166,167,173}
270	2	912	44	0	{1,2,3,4,5,7,9,13,17,18,35,37,41,49,65,66,129,130,131,132, 133,147,148,155,156,157,166,167,174}
271	2	904	42	0	{1,2,3,4,5,7,9,13,17,19,34,37,41,49,65,66,129,130,131,132, 133,146,148,155,156,158,166,167,174}
272	2	904	42	0	{1,2,3,4,5,7,9,13,18,21,37,41,49,66,129,130,131,132,133,135, 146,147,148,155,157,159,166,167,174}

Nr.	$\|W_K\|$	$\#K_5(1)$	$\#K_4^a(0)$	$\#K_4^b(0)$	K
273	12	920	46	0	{1,2,3,5,7,9,10,19,34,35,37,49,65,66,129,130,131,132,133, 144,148,155,156,158,159,165,166,174,179}
274	4	968	38	0	{1,2,3,4,5,6,7,9,12,13,16,34,35,38,39,41,129,130,131,132, 133,138,139,142,155,158,159,160,161}
275	16	960	36	0	{1,2,3,4,5,6,7,9,11,15,16,35,36,37,39,41,129,130,131,132, 133,137,139,142,155,157,159,160,161}
276	32	992	44	0	{1,2,3,4,5,6,7,9,10,15,16,35,36,37,38,41,129,130,131,132, 133,138,139,142,155,158,159,160,161}
277	48	1008	48	0	{1,2,3,4,5,7,8,9,10,15,16,35,36,37,38,43,129,130,131,132, 133,138,139,142,155,158,159,160,161}
278	4	1240	32	0	{1,2,3,4,5,6,9,13,17,34,35,41,49,65,66,129,130,131,132,133, 145,148,155,156,157,158,166,167,173}
279	12	784	46	0	{1,2,3,5,6,7,9,10,21,25,34,35,41,49,65,66,129,130,131,132, 133,145,148,157,158,159,165,166,179}
280	4	912	42	0	{1,2,3,4,5,7,9,11,13,18,37,41,49,66,129,130,131,132,133,135, 146,147,148,155,156,157,159,167,174}
281	2	904	40	0	{1,2,3,4,5,7,9,18,19,21,37,41,49,66,129,130,131,132,133,135, 146,147,148,155,159,165,166,167,174}
282	4	912	42	0	{1,2,3,4,5,7,9,10,13,18,19,21,41,49,65,66,129,130,131,132, 133,145,146,147,148,155,159,166,174}
283	4	912	42	0	{1,2,3,4,5,7,9,10,13,18,35,37,41,49,65,66,129,130,131,132, 133,147,148,155,156,157,159,166,174}
284	8	912	42	0	{1,2,3,4,5,7,9,11,13,21,34,37,41,49,65,66,129,130,131,132, 133,146,148,155,157,158,159,167,174}
285	48	928	46	0	{1,2,3,4,5,7,9,10,13,21,35,37,41,49,65,66,129,130,131,132, 133,147,148,155,157,158,159,166,174}
286	16	928	46	0	{1,2,3,4,5,7,9,13,21,37,41,49,65,66,129,130,131,132,133,146, 147,148,155,157,158,159,166,167,174}
287	4	1800	17	0	{1,2,3,4,5,6,9,10,17,25,34,35,65,129,130,131,132,133,134, 143,144,145,148,156,157,158,165,166,173}
288	48	1896	16	0	{1,2,3,4,5,6,9,11,13,17,34,35,37,41,129,130,131,132,133,134, 135,143,148,155,156,157,158,167,173}
289	51840	1728	45	0	{1,2,3,4,10,11,12,18,19,20,28,34,35,36,44,52,72,129,130,131, 135,137,138,145,156,165,175,177,178}
290	16	384	66	0	{1,2,3,5,10,19,21,25,37,41,49,65,66,67,129,130,131,132,146, 147,148,158,159,165,166,168,173,174,179}
291	432	384	66	0	{1,2,3,5,10,19,21,25,35,37,41,49,65,66,67,69,129,130,131, 147,148,158,159,165,166,168,173,174,179}
292	24	368	62	0	{1,2,3,4,9,10,18,21,25,35,37,41,49,65,67,69,129,130,131,134, 147,148,157,159,165,166,173,174,175}
293	18	360	60	0	{1,2,3,9,11,18,19,37,41,49,66,69,129,130,131,133,135,146,147, 148,155,156,159,165,167,173,174,175,179}
294	103680	3528	0	0	{1,2,3,4,5,6,7,9,10,11,13,17,18,19,21,25,129,130,131,132, 133,134,135,143,144,145,146,147,148}
295	2	1064	38	0	{1,2,3,4,5,6,9,10,17,19,35,37,41,49,65,66,129,130,131,132, 133,147,148,155,156,158,165,166,173}
296	2	1056	36	0	{1,2,3,4,5,6,9,11,13,17,19,21,37,41,49,66,129,130,131,132, 133,135,146,147,148,155,158,167,173}
297	2	1056	36	0	{1,2,3,4,5,6,9,17,21,35,41,49,65,66,129,130,131,132,133,145, 147,148,155,157,158,165,166,167,173}
298	2	1768	16	0	{1,2,3,4,5,6,8,9,11,12,13,14,17,18,23,26,129,130,131,132, 133,134,135,138,143,144,145,147,148}
299	8	1776	18	0	{1,2,3,4,5,6,8,9,11,14,17,18,20,21,23,26,129,130,131,132, 133,134,135,138,143,144,145,147,148}
300	2	928	40	0	{1,2,3,4,5,6,9,10,11,17,19,21,35,37,41,49,65,66,129,130, 131,132,133,147,148,155,158,165,173}
301	4	928	40	0	{1,2,3,4,5,6,9,11,17,19,37,41,49,66,129,130,131,132,133,135, 146,147,148,155,156,158,165,167,173}
302	4	920	38	0	{1,2,3,4,5,6,9,13,17,19,37,41,49,66,129,130,131,132,133,135, 146,147,148,155,156,158,166,167,173}
303	8	936	42	0	{1,2,3,4,5,6,9,11,13,17,19,21,34,35,41,49,65,66,129,130, 131,132,133,145,148,155,158,167,173}
304	1152	2112	13	0	{1,2,3,4,5,6,9,17,18,34,41,65,66,129,130,131,132,133,143, 145,146,148,155,156,157,165,166,167,173}
305	4	1088	35	0	{1,2,3,4,5,6,9,12,14,17,36,38,42,50,66,129,130,131,132,133, 137,149,150,155,156,157,160,169,173}
306	12	1104	39	0	{1,2,3,4,5,8,9,12,14,17,34,38,42,50,66,129,130,131,132,133, 137,146,150,155,156,157,160,169,176}

Nr.	$\|W_K\|$	$\#K_5(1)$	$\#K_4^a(0)$	$\#K_4^b(0)$	K
307	24	1096	37	0	{1,2,3,4,5,7,9,10,13,16,34,35,37,38,41,43,129,130,131,132, 133,139,142,155,157,158,159,160,161}
308	16	1096	37	0	{1,2,3,4,5,7,9,11,13,14,16,34,37,39,41,43,129,130,131,132, 133,142,155,156,157,158,159,160,161}
309	4	1104	35	0	{1,2,3,4,5,6,9,12,17,20,22,36,38,42,66,129,130,131,132,133, 137,143,149,150,155,160,165,169,173}
310	12	1112	37	0	{1,2,3,4,5,6,9,17,20,22,36,38,42,65,66,129,130,131,132,133, 143,149,150,155,160,165,166,169,173}
311	2	1104	35	0	{1,2,3,4,5,6,9,12,14,17,36,42,50,65,66,129,130,131,132,133, 145,149,150,155,156,157,160,169,173}
312	8	1096	33	0	{1,2,3,4,5,6,8,9,12,17,22,36,38,42,50,65,66,129,130,131, 132,133,149,150,155,157,160,165,169}
313	4	512	55	0	{1,2,3,5,18,19,23,24,35,36,37,40,49,50,51,53,129,130,131, 132,133,139,141,144,166,167,168,169,170}
314	2	512	55	0	{1,2,3,5,19,22,23,24,36,37,39,40,49,50,51,53,129,130,131, 132,133,135,141,144,165,166,167,168,170}
315	4	528	59	0	{1,2,3,5,20,22,23,24,36,38,39,40,49,50,51,53,129,130,131, 132,133,135,141,144,165,166,167,168,170}
316	8	520	57	0	{1,2,3,5,20,22,23,24,36,38,39,40,49,50,51,53,129,130,131, 132,133,135,138,139,141,144,167,168,170}
317	16	552	65	0	{1,2,3,5,6,19,20,21,23,24,35,39,40,49,50,53,129,130,131,132, 133,141,142,165,166,167,168,169,170}
318	4	520	57	0	{1,2,3,5,11,15,20,27,35,36,39,67,68,129,130,131,132,138,143, 144,151,156,160,162,167,171,174,176,184}
319	48	520	57	0	{1,2,3,4,7,12,16,18,28,34,44,52,66,68,72,129,130,131,137, 145,148,150,156,159,161,170,171,175,178}
320	8	944	40	0	{1,2,3,4,5,6,9,12,14,17,20,22,34,36,38,42,50,65,66,129,130, 131,132,133,150,155,160,169,173}
321	1	944	40	0	{1,2,3,4,5,6,9,12,17,20,22,36,38,42,50,65,66,129,130,131, 132,133,149,150,155,160,165,169,173}
322	4	936	38	0	{1,2,3,4,5,6,9,12,17,22,36,38,42,50,66,129,130,131,132,133, 137,149,150,155,157,160,165,169,173}
323	8	960	44	0	{1,2,3,4,5,8,9,10,17,20,22,36,38,42,50,65,66,129,130,131, 132,133,149,150,155,160,165,166,176}
324	8	960	44	0	{1,2,3,4,5,8,9,10,14,17,22,36,38,42,50,65,66,129,130,131, 132,133,149,150,155,157,160,166,176}
325	4	952	42	0	{1,2,3,4,5,9,12,14,17,20,34,38,42,50,66,129,130,131,132,133, 137,146,150,155,156,160,169,173,176}
326	4	944	40	0	{1,2,3,4,5,9,12,14,17,22,34,38,42,50,66,129,130,131,132,133, 137,146,150,155,157,160,169,173,176}
327	4	944	40	0	{1,2,3,4,5,7,9,13,16,34,35,37,38,39,41,43,129,130,131,132, 133,137,139,142,155,157,159,160,161}
328	4	944	40	0	{1,2,3,4,5,7,9,14,16,34,36,37,38,39,41,43,129,130,131,132, 133,135,139,142,155,157,158,159,161}
329	4	952	42	0	{1,2,3,4,5,7,9,13,14,16,34,37,38,39,41,43,129,130,131,132, 133,139,142,155,157,158,159,160,161}
330	6	768	46	0	{1,2,3,4,5,7,9,13,18,41,49,66,129,130,131,132,133,135,145, 146,147,148,155,156,157,159,166,167,174}
331	2	760	44	0	{1,2,3,4,5,7,9,18,21,41,49,66,129,130,131,132,133,135,145, 146,147,148,155,157,159,165,166,167,174}
332	4	768	46	0	{1,2,3,4,5,7,9,13,18,19,21,35,41,49,65,66,129,130,131,132, 133,145,147,148,155,159,166,167,174}
333	1	768	46	0	{1,2,3,4,5,7,9,13,18,37,41,49,65,66,129,130,131,132,133,146, 147,148,155,156,157,159,166,167,174}
334	2	776	48	0	{1,2,3,4,5,7,9,10,18,21,35,37,41,49,65,66,129,130,131,132, 133,147,148,155,157,159,165,166,174}
335	4	776	48	0	{1,2,3,4,5,7,9,13,18,21,35,37,41,49,65,66,129,130,131,132, 133,147,148,155,157,159,166,167,174}
336	4	784	50	0	{1,2,3,5,7,9,10,19,21,34,35,49,65,66,129,130,131,132,133, 144,145,148,155,158,159,165,166,174,179}
337	12	760	44	0	{1,2,3,5,7,9,10,19,25,37,41,66,129,130,131,132,133,135,143, 146,147,148,156,158,159,165,166,174,179}
338	24	1960	16	0	{1,2,3,4,5,6,8,9,12,14,17,18,20,22,23,26,129,130,131,132, 133,134,137,138,139,143,144,148,149}
339	48	976	43	0	{1,2,3,5,9,12,14,17,20,22,36,38,42,50,66,129,130,131,132, 133,137,149,150,155,160,169,173,176,181}
340	144	632	60	0	{1,2,3,5,11,13,25,41,49,65,66,129,130,131,132,133,145,146, 147,148,156,157,158,159,167,168,173,174,179}

Nr.	$\lvert W_K \rvert$	$\#K_5(1)$	$\#K_4^a(0)$	$\#K_4^b(0)$	K
341	72	600	52	0	{1,2,3,4,9,10,18,21,25,33,35,41,49,67,69,129,130,131,134, 135,145,147,157,159,165,166,173,174,175}
342	360	288	65	0	{1,2,3,5,10,11,18,21,25,35,37,41,49,65,66,67,69,129,130,131, 147,148,157,159,165,168,173,174,179}
343	40320	448	105	0	{1,2,3,4,13,21,25,37,41,49,65,66,67,129,130,131,132,146,147, 148,157,158,159,166,167,168,173,174,175}
344	384	320	73	0	{1,2,3,4,9,13,18,21,25,35,37,41,49,65,66,67,69,129,130,131, 147,148,157,159,166,167,173,174,175}
345	96	288	65	0	{1,2,3,11,13,18,21,35,41,49,65,66,69,129,130,131,133,145,147, 148,155,157,159,167,168,173,174,175,179}
346	64	864	44	0	{1,2,3,5,7,9,11,14,15,33,36,37,39,41,43,45,129,130,131,132, 133,137,138,139,140,142,159,160,161}
347	72	1792	18	0	{1,2,3,4,5,6,7,9,10,17,18,25,35,37,129,130,131,132,133,134, 135,143,144,147,148,156,157,165,166}
348	8	1256	32	0	{1,2,3,4,5,6,9,11,17,18,21,35,41,66,129,130,131,132,133,135, 143,145,147,148,155,157,165,167,173}
349	4	1240	28	0	{1,2,3,4,5,6,9,13,17,18,19,35,41,66,129,130,131,132,133,135, 143,145,147,148,155,156,166,167,173}
350	4	1248	30	0	{1,2,3,4,5,6,9,10,11,17,18,19,21,35,37,41,65,66,129,130, 131,132,133,143,147,148,155,165,173}
351	72	1600	30	0	{1,2,3,4,5,6,9,17,34,35,37,41,49,65,129,130,131,132,133,134, 148,155,156,157,158,165,166,167,173}
352	12	728	52	0	{1,2,3,5,7,9,18,19,21,25,35,37,41,49,65,66,129,130,131,132, 133,147,148,159,165,166,167,174,179}
353	2	640	50	0	{1,2,3,5,7,9,12,14,15,16,36,38,39,41,43,45,129,130,131,132, 133,135,138,139,142,155,158,159,161}
354	4	640	50	0	{1,2,3,5,7,9,12,15,16,35,36,38,39,41,43,45,129,130,131,132, 133,135,139,142,155,157,158,159,161}
355	24	680	60	0	{1,2,3,5,7,9,10,15,16,35,36,37,38,41,43,45,129,130,131,132, 133,138,139,142,155,158,159,160,161}
356	2	656	54	0	{1,2,3,5,7,9,10,15,16,35,36,37,38,41,43,45,129,130,131,132, 133,137,138,139,140,142,159,160,161}
357	8	648	52	0	{1,2,3,5,7,18,20,23,24,35,36,38,40,49,51,53,129,130,131,132, 133,135,138,141,144,165,167,168,170}
358	4	640	50	0	{1,2,3,5,7,20,24,34,35,36,38,39,40,49,51,53,129,130,131,132, 133,141,144,165,166,167,168,169,170}
359	720	848	60	0	{1,2,3,4,5,6,9,10,19,21,25,35,37,41,65,66,129,130,131,132, 133,143,147,148,158,159,165,166,173}
360	2	1048	36	0	{1,2,3,4,5,6,9,17,19,35,37,41,49,66,129,130,131,132,133,135, 147,148,155,156,158,165,166,167,173}
361	2	1056	38	0	{1,2,3,4,5,6,9,17,19,21,35,37,41,49,65,66,129,130,131,132, 133,147,148,155,158,165,166,167,173}
362	12	1072	42	0	{1,2,3,4,5,7,9,17,19,34,37,41,49,66,129,130,131,132,133,135, 146,148,155,156,158,165,166,167,174}
363	2	1056	38	0	{1,2,3,4,5,7,9,13,17,34,37,41,49,66,129,130,131,132,133,135, 146,148,155,156,157,158,166,167,174}
364	8	1056	38	0	{1,2,3,4,5,7,9,13,34,37,41,49,65,66,129,130,131,132,133,146, 148,155,156,157,158,159,166,167,174}
365	4	1232	30	0	{1,2,3,4,5,6,9,17,25,35,37,41,129,130,131,132,133,134,135, 143,147,148,156,157,158,165,166,167,173}
366	8	1240	32	0	{1,2,3,4,5,6,9,17,19,21,25,34,35,37,41,65,129,130,131,132, 133,134,143,148,158,165,166,167,173}
367	4	480	54	0	{1,2,3,5,7,10,13,25,35,49,65,66,129,130,131,132,133,144,145, 147,148,156,157,158,159,166,168,174,179}
368	8	496	58	0	{1,2,3,5,7,10,25,35,37,49,65,66,129,130,131,132,133,144,147, 148,156,157,158,159,165,166,168,174,179}
369	4	496	58	0	{1,2,3,5,11,13,18,19,25,35,37,49,65,66,129,130,131,132,133, 144,147,148,156,159,167,168,173,174,179}
370	2	488	56	0	{1,2,3,5,10,11,21,25,35,49,65,66,129,130,131,132,133,144,145, 147,148,157,158,159,165,168,173,174,179}
371	2	496	58	0	{1,2,3,5,10,19,25,35,37,49,65,66,129,130,131,132,133,144,147, 148,156,158,159,165,166,168,173,174,179}
372	4	512	62	0	{1,2,3,4,5,13,25,35,37,49,65,66,129,130,131,132,133,144,147, 148,156,157,158,159,166,167,168,173,174}
373	24	544	70	0	{1,2,3,5,6,11,19,25,35,37,41,49,65,66,129,130,131,132,133, 147,148,156,158,159,165,167,168,173,179}
374	48	496	62	0	{1,2,3,5,11,21,25,35,41,49,65,66,129,130,131,132,133,145,147, 148,157,158,159,165,167,168,173,174,179}

Nr.	$\|W_K\|$	$\#K_5(1)$	$\#K_4^a(0)$	$\#K_4^b(0)$	K
375	4	1008	42	0	{1,2,3,4,5,9,13,17,21,34,35,37,41,49,65,66,129,130,131,132, 133,148,155,157,158,166,167,173,174}
376	2	1592	20	0	{1,2,3,4,5,6,8,9,11,12,13,17,18,19,23,26,129,130,131,132, 133,134,137,138,139,143,144,148,149}
377	12	1600	22	0	{1,2,3,4,5,6,8,9,11,12,13,14,17,18,23,26,129,130,131,132, 133,134,138,139,143,144,147,148,149}
378	16	1600	22	0	{1,2,3,4,5,6,7,9,10,17,19,25,34,35,37,65,129,130,131,132, 133,134,143,144,148,156,158,165,166}
379	4	1104	35	0	{1,2,3,4,5,6,9,12,17,18,38,42,50,65,66,129,130,131,132,133, 146,149,150,155,156,157,165,169,173}
380	8	1104	35	0	{1,2,3,4,5,6,9,14,17,20,34,42,50,65,66,129,130,131,132,133, 145,146,150,155,156,160,166,169,173}
381	8	1288	28	0	{1,2,3,4,5,6,7,9,11,12,14,16,34,36,39,41,129,130,131,132, 133,134,135,139,142,155,157,158,159}
382	192	1568	24	0	{1,2,3,4,5,6,9,17,34,41,49,65,66,129,130,131,132,133,145, 146,148,155,156,157,158,165,166,167,173}
383	4	1064	37	0	{1,2,3,4,5,9,12,14,17,20,34,36,38,42,50,65,66,129,130,131, 132,133,150,155,156,160,169,173,176}
384	4	1072	39	0	{1,2,3,4,5,9,14,17,22,34,36,38,42,50,66,129,130,131,132,133, 137,150,155,157,160,166,169,173,176}
385	24	1160	40	0	{1,2,3,5,8,9,12,17,20,22,36,38,42,50,66,129,130,131,132,133, 137,149,150,155,160,165,169,176,181}
386	12	1784	17	0	{1,2,3,4,5,6,9,10,11,17,34,49,65,129,130,131,132,133,134, 144,145,146,148,155,156,157,158,165,173}
387	4	768	48	0	{1,2,3,4,5,7,9,13,18,21,41,49,65,66,129,130,131,132,133,145, 146,147,148,155,157,159,166,167,174}
388	2	760	46	0	{1,2,3,4,5,7,9,18,21,37,41,49,65,66,129,130,131,132,133,146, 147,148,155,157,159,165,166,167,174}
389	144	816	60	0	{1,2,3,5,7,9,10,19,21,35,37,49,65,66,129,130,131,132,133, 144,147,148,155,158,159,165,166,174,179}
390	12	752	44	0	{1,2,3,5,7,9,11,18,19,21,34,35,41,49,65,66,129,130,131,132, 133,145,148,155,159,165,167,174,179}
391	60	768	50	0	{1,2,3,4,5,10,13,21,25,49,65,66,129,130,131,132,133,144,145, 146,147,148,157,158,159,166,168,173,174}
392	4	1824	15	0	{1,2,3,4,5,6,8,9,10,17,20,34,36,42,66,129,130,131,132,133, 137,143,145,150,155,156,160,165,166}
393	20	1008	40	0	{1,2,3,5,7,9,34,35,41,49,65,66,129,130,131,132,133,145,148, 155,156,157,158,159,165,166,167,174,179}
394	6	1568	24	0	{1,2,3,4,5,6,9,10,17,25,34,35,37,65,129,130,131,132,133,134, 143,144,148,156,157,158,165,166,173}
395	2	1560	22	0	{1,2,3,4,5,6,9,11,17,21,34,35,37,41,129,130,131,132,133,134, 135,143,148,155,157,158,165,167,173}
396	144	1128	45	0	{1,2,3,4,5,8,9,12,17,20,34,38,42,50,66,129,130,131,132,133, 137,146,150,155,156,160,165,169,176}
397	16	1096	37	0	{1,2,3,4,5,8,9,14,17,22,34,38,42,50,66,129,130,131,132,133, 137,146,150,155,157,160,166,169,176}
398	8	1448	23	0	{1,2,3,4,5,6,8,9,14,17,20,34,36,38,42,65,66,129,130,131, 132,133,143,150,155,156,160,166,169}
399	2	1456	25	0	{1,2,3,4,5,6,9,17,20,22,33,36,38,42,66,129,130,131,132,133, 137,143,149,155,160,165,166,169,173}
400	2	1456	25	0	{1,2,3,4,5,6,9,12,17,22,33,36,38,42,66,129,130,131,132,133, 137,143,149,155,157,160,165,169,173}
401	2	1448	23	0	{1,2,3,4,5,6,9,17,20,34,36,42,66,129,130,131,132,133,137, 143,145,150,155,156,160,165,166,169,173}
402	128	2136	8	112	{1,2,3,4,5,6,8,9,10,12,17,18,22,26,129,130,131,132,133,134, 135,143,144,145,146,147,148,157,165}
403	192	2344	6	0	{1,2,3,4,5,6,8,9,10,12,14,17,18,20,22,26,129,130,131,132, 133,134,135,143,144,145,146,147,148}
404	48	1800	17	0	{1,2,3,4,5,6,7,9,10,11,17,34,35,49,65,129,130,131,132,133, 134,144,145,148,155,156,157,158,165}
405	12	832	46	0	{1,2,3,5,7,9,18,19,21,25,41,49,65,66,129,130,131,132,133, 145,146,147,148,159,165,166,167,174,179}
406	24	848	50	0	{1,2,3,5,7,9,11,19,21,25,35,37,41,49,65,66,129,130,131,132, 133,147,148,158,159,165,167,174,179}
407	384	2696	4	0	{1,2,3,4,5,6,8,9,10,11,13,17,18,19,21,26,129,130,131,132, 133,134,135,143,144,145,146,147,148}
408	2	752	48	0	{1,2,3,5,7,9,10,21,34,35,41,49,65,66,129,130,131,132,133, 145,148,155,157,158,159,165,166,174,179}

Nr.	$\lvert W_K \rvert$	$\#K_5(1)$	$\#K_4^a(0)$	$\#K_4^b(0)$	K
409	4	768	52	0	{1,2,3,5,7,9,10,19,21,34,35,37,41,49,65,66,129,130,131,132, 133,148,155,158,159,165,166,174,179}
410	24	752	48	0	{1,2,3,5,7,9,10,18,19,21,41,49,65,66,129,130,131,132,133, 145,146,147,148,155,159,165,166,174,179}
411	2	744	46	0	{1,2,3,5,7,9,11,13,19,21,34,35,41,49,65,66,129,130,131,132, 133,145,148,155,158,159,167,174,179}
412	2	1264	30	0	{1,2,3,4,5,6,9,12,17,22,34,36,42,65,66,129,130,131,132,133, 143,145,150,155,157,160,165,169,173}
413	2	1264	30	0	{1,2,3,4,5,6,9,17,20,22,34,36,42,65,66,129,130,131,132,133, 143,145,150,155,160,165,166,169,173}
414	4	1264	30	0	{1,2,3,4,5,6,8,9,12,17,18,20,22,36,38,42,50,65,66,129,130, 131,132,133,149,150,155,165,169}
415	8	1272	32	0	{1,2,3,4,5,6,9,10,17,18,20,36,38,42,50,65,66,129,130,131, 132,133,149,150,155,156,165,166,173}
416	4	1752	18	0	{1,2,3,4,5,6,8,9,11,12,14,17,18,20,23,26,129,130,131,132, 133,134,137,138,139,143,144,148,149}
417	8	800	45	0	{1,2,3,4,5,9,10,14,17,20,22,36,38,42,50,65,66,129,130,131, 132,133,149,150,155,160,166,173,176}
418	16	800	45	0	{1,2,3,4,5,9,12,14,17,20,22,34,38,42,50,65,66,129,130,131, 132,133,146,150,155,160,169,173,176}
419	32	800	45	0	{1,2,3,5,7,18,19,20,22,24,34,36,40,49,51,53,129,130,131,132, 133,138,141,144,165,167,168,169,170}
420	192	832	53	0	{1,2,3,5,6,18,19,20,23,24,35,36,40,49,50,53,129,130,131,132, 133,138,141,144,165,167,168,169,170}
421	12	632	52	0	{1,2,3,5,19,20,21,23,33,35,39,40,49,50,51,53,129,130,131, 132,133,137,138,139,141,144,168,169,170}
422	72	664	60	0	{1,2,3,4,5,19,20,21,22,23,33,39,40,49,50,51,129,130,131,132, 133,138,139,141,142,167,168,169,170}
423	4	1392	26	0	{1,2,3,4,5,6,9,11,13,17,18,19,21,35,41,66,129,130,131,132, 133,135,143,145,147,148,155,167,173}
424	8	1400	28	0	{1,2,3,4,5,6,9,10,17,18,19,35,37,41,65,66,129,130,131,132, 133,143,147,148,155,156,165,166,173}
425	2	1216	32	0	{1,2,3,4,5,6,9,10,11,17,35,37,41,49,65,66,129,130,131,132, 133,147,148,155,156,157,158,165,173}
426	16	1216	32	0	{1,2,3,4,5,6,7,9,10,17,19,21,41,49,65,66,129,130,131,132, 133,145,146,147,148,155,158,165,166}
427	4	1216	32	0	{1,2,3,4,5,6,7,9,17,19,21,35,41,49,65,66,129,130,131,132, 133,145,147,148,155,158,165,166,167}
428	12	1760	16	0	{1,2,3,4,5,6,8,9,11,12,13,14,17,18,23,26,129,130,131,132, 133,134,135,137,138,139,143,144,148}
429	16	1520	28	0	{1,2,3,4,5,6,9,17,34,35,41,49,65,129,130,131,132,133,134, 145,148,155,156,157,158,165,166,167,173}
430	2	904	42	0	{1,2,3,4,5,9,12,14,17,22,34,36,38,42,50,65,66,129,130,131, 132,133,150,155,157,160,169,173,176}
431	6	920	46	0	{1,2,3,4,5,9,14,17,22,36,38,42,50,65,66,129,130,131,132,133, 149,150,155,157,160,166,169,173,176}
432	120	1888	20	0	{1,2,3,4,5,6,8,9,11,13,15,17,19,21,23,26,129,130,131,132, 133,134,135,138,139,143,144,147,148}

Summary

This thesis contains results on the arithmetic and geometry of del Pezzo surfaces of degree 1. These are exactly the smooth surfaces in the weighted projective space $\mathbb{P}(2,3,1,1)$ with coordinates x, y, z, w given by an equation of the form

$$y^2 + a_1(z,w)xy + a_3(z,w)y = x^3 + a_2(z,w)x^2 + a_4(z,w)x + a_6(z,w),$$

where $a_i \in k[z,w]$ is homogeneous of degree i. Such a surface contains 240 curves with negative self-intersection, called exceptional curves.

In Chapter 1 we give the necessary background, assuming the reader is familiar with algebraic geometry. Two main points that we cover are the elliptic surface that is constructed from a del Pezzo surface of degree 1 by blowing up the base point of the anticanonical linear system, and the connection between the exceptional curves on a del Pezzo surface of degree 1 and the $\mathbf{E}_8$ root system.

In Chapter 2, which is joint work with Julie Desjardins, we prove that for a del Pezzo surface S over a number field k, of the form

$$y^2 = x^3 + Az^6 + Bw^6$$

with $A, B \in k$ non-zero, the set $S(k)$ of k-rational points on S is dense with respect to the Zariski topology if and only if S contains a point with non-zero z, w coordinates such that the corresponding point on the elliptic surface constructed from S lies on a smooth fiber and is non-torsion on that fiber. We do this by constructing an infinite family of multisections, and showing that at least one of them has infinitely many k-rational points. This is the first result that gives necessary and sufficient conditions for the

set of k-rational points of this family to be Zariski-dense, where k is any number field.

In Chapter 3, which is an adaptation of the preprint [vLWa], we study the action of the Weyl group W_8 on the $\mathbf{E}_8$ root system. The 240 roots in $\mathbf{E}_8$ are in one-to-one correspondence with the 240 exceptional curves on a del Pezzo surface of degree 1, and we use results from this chapter in Chapters 4 and 5. However, this chapter is also interesting for the reader that wants to know about the $\mathbf{E}_8$ root system without any interest in del Pezzo surfaces of degree 1. We define the complete weighted graph Γ where each vertex represents a root, and two vertices are connected by an edge of weight w if the corresponding roots have dot product w. The group of symmetries of Γ is the Weyl group W_8. We prove that for a large class of subgraphs of Γ, any two subgraphs from this class are isomorphic if and only if there is a symmetry of Γ that maps one to the other. We also give invariants that determine the isomorphism type of a subgraph. Moreover, we show that for two isomorphic subgraphs G_1, G_2 from this class that do not contain one of 7 specific subgraphs, any isomorphism between G_1 and G_2 extends to a symmetry of the whole graph Γ. These results reduce computations on the graph Γ significantly.

In Chapter 4, which is an adaptation of the preprint [vLWb], we study the configurations of the 240 exceptional curves on a del Pezzo surface of degree 1, using results from Chapter 3. We prove that a point on a del Pezzo surface of degree 1 is contained in at most 16 exceptional curves in characteristic 2, at most 12 exceptional curves in characteristic 3, and at most 10 exceptional curves in all other characteristics. We give examples that show that the upper bounds are sharp in all characteristics, except possibly in characteristic 5.

Finally, in Chapter 5 we show that if at least 9 exceptional curves intersect in a point on a del Pezzo surface S of degree 1, the corresponding point on the elliptic surface constructed from S is torsion on its fiber. This is less trivial than some experts thought. We use a list of all possible configurations of at least 9 pairwise intersecting exceptional curves computed in Chapter 3, and with an example from Chapter 4 we show that the analogue statement is false for 6 or fewer exceptional curves.

Samenvatting

Dit proefschrift bevat resultaten over de meetkunde en arithmetiek van del Pezzo oppervlakken van graad 1. Dit zijn de gladde oppervlakken in de gewogen projectieve ruimte $\mathbb{P}(2,3,1,1)$ met coördinaten x, y, z, w gegeven door een vergelijking van de vorm

$$y^2 + a_1(z,w)xy + a_3(z,w)y = x^3 + a_2(z,w)x^2 + a_4(z,w)x + a_6(z,w),$$

waar $a_i \in k[z,w]$ homogeen van graad i is. Zo'n oppervlak bevat 240 krommen met negatieve zelfdoorsnijding, zogenaamde exceptionele krommen.

In Hoofdstuk 1 staat de voorkennis die nodig is voor de rest van het proefschrift. We nemen aan dat de lezer bekend is met algebraïsche meetkunde. Twee belangrijke onderwerpen die we hier behandelen zijn het elliptisch oppervlak dat ontstaat door het basispunt van het antikanonieke lineaire systeem op een del Pezzo oppervlak van graad 1 op te blazen, en de relatie tussen the exceptionele krommen op een del Pezzo oppervlak van graad 1 en het wortelsysteem $\mathbf{E}_8$.

Hoofdstuk 2 komt voort uit een samenwerking met Julie Desjardins. We bewijzen in dit hoofdstuk dat voor een del Pezzo oppervlak S van graad 1 over een getallenlichaam k, van de vorm

$$y^2 = x^3 + Az^6 + Bw^6$$

met $A, B \in k$ ongelijk aan nul, de verzameling $S(k)$ van k-rationale punten op S dicht ligt in de Zariksi topologie dan en slechts dan als S een punt met z, w coördinaten ongelijk aan nul bevat, zodanig dat het corresponderende punt op het elliptisch oppervlak dat uit S wordt geconstrueerd op een gladde vezel ligt en daar niet torsie is. We doen dit door een oneindige familie van multisecties te construeren, en te laten zien dat ten minste één van deze multisecties oneindig veel k-rationale punten bevat. Dit is het eerste resultaat dat zowel noodzakelijke als voldoende voorwaarden geeft voor het dicht liggen van de verzameling k-rationale punten op deze familie oppervlakken, waarbij k een willekeurig getallenlichaam is.

Hoofdstuk 3 is een aangepaste versie van het artikel [vLWa]. In dit hoofdstuk bestuderen we de werking van de Weyl groep W_8 op het wortelsysteem $\mathbf{E}_8$. De 240 wortels in $\mathbf{E}_8$ hebben een één-op-één relatie met de 240 exceptionele krommen op een del Pezzo oppervlak van graad 1, en we gebruiken resultaten uit dit hoofdstuk in Hoofdstukken 4 en 5. Toch is dit hoofdstuk ook apart te lezen, en interessant voor de lezer zonder interesse in del Pezzo oppervlakken maar met een interesse in $\mathbf{E}_8$. We definiëren de gewogen graaf Γ, waarin elk knooppunt een wortel vertegenwoordigt, en twee knooppunten verbonden zijn door een tak van gewicht w dan en slechts dan als de twee bijbehorende wortels inproduct w hebben. De groep van symmetrieën van Γ is de Weyl groep W_8. We bewijzen dat voor een grote klasse van deelgrafen van Γ, twee deelgrafen isomorf zijn dan en slechts dan als er een symmetrie van Γ is die de ene deelgraaf op de andere afbeeldt. We geven ook invarianten die het isomorfismetype van een deelgraaf vastleggen. Daarnaast laten we zien dat voor twee isomorfe deelgrafen G_1 en G_2 uit deze klasse die niet een van 7 specifieke deelgrafen bevatten, elk isomorfisme tussen G_1 en G_2 uitbreidt tot een symmetrie van Γ. Deze resultaten kunnen gebruikt worden om berekeningen in de graaf Γ drastisch te reduceren.

Hoofdstuk 4 is een aangepaste versie van het artikel [vLWb]. In dit hoofdstuk bestuderen we de configuraties van de 240 exceptionele krommen op een del Pezzo oppervlak van graad 1, waarbij we resultaten uit Hoofdstuk 3 gebruiken. We bewijzen dat een punt op een del Pezzo oppervlak van graad 1 bevat is in ten hoogste 16 exceptionele krommen in karakteristiek 2, in ten hoogste 12 exceptionele krommen in karakteristiek 3, en in ten hoogste 10 exceptionele krommen in alle andere karakteristieken. We geven bovendien voorbeelden waarmee we laten zien dat deze bovengrenzen worden behaald in alle karakteristieken behalve misschien in karakteristiek 5.

Tot slot bewijzen we in Hoofdstuk 5 dat als ten minste 9 exceptionele krommen allemaal in een punt op een del Pezzo oppervlak S van graad 1 snijden, dat het corresponderende punt op het elliptisch oppervlak geconstrueerd uit S torsie op zijn vezel is. Dit blijkt niet zo triviaal te zijn als sommige experts dachten. We gebruiken een lijst die in Hoofdstuk 3 is gemaakt van alle mogelijke configuraties van ten minste 9 exceptionele krommen die elkaar paarsgewijs snijden, en met een voorbeeld uit Hoofdstuk 4 laten we zien dat de equivalente stelling niet waar is voor 6 of minder exceptionele krommen.

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Curriculum Vitae

Rosa Winter was born in Amsterdam on 24th October 1988. After completing her high school education and spending eight months doing volunteer work in Zambia, she started studying mathematics at Universiteit Leiden in 2008.

She received her bachelor degree in 2011, and started the ALGANT master program in pure mathematics, spending the first year at Università degli Studi di Padova, and the second year at Universiteit Leiden. She wrote her thesis under the supervision of Ronald van Luijk and graduated cum laude in 2014.

After receiving her master degree, Rosa started the traineeship Eerst de Klas. During the next two years she combined teaching mathematics at the Calandlyceum in Amsterdam with professional development and an internship at the Dutch company Attero. In 2015 she obtained her teaching degree in mathematics (eerstegraads onderwijsbevoegdheid) cum laude from the Universiteit van Amsterdam.

In 2016, she started her PhD at Universiteit Leiden under the supervision of Ronald van Luijk and Martin Bright. This thesis summarizes the studies that were done in the four years after.

Since September 2020, Rosa has been working as a postdoc at the Max Planck Institut für Mathematik in den Naturwissenschaften in Leipzig, Germany.