

PAPER

# Distribution of the Wigner–Smith time-delay matrix for chaotic cavities with absorption and coupled Coulomb gases

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# Distribution of the Wigner–Smith time-delay matrix for chaotic cavities with absorption and coupled Coulomb gases

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## Abstract

Within the random matrix theory approach to quantum scattering, we derive the distribution of the Wigner–Smith time delay matrix  $Q$  for a chaotic cavity with uniform absorption, coupled via  $N$  perfect channels. In the unitary class  $\beta = 2$  we obtain a compact expression for the distribution of the full matrix in terms of a matrix integral. In the other symmetry classes we derive the joint distribution of the eigenvalues. We show how the large  $N$  properties of this distribution can be analysed in terms of two interacting Coulomb gases living on two different supports. As an application of our results, we study the statistical properties of the Wigner time delay  $\tau_W = \text{tr}[Q]/N$  in the presence of absorption.

Keywords: random matrix theory, quantum scattering, Coulomb gas

(Some figures may appear in colour only in the online journal)

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## 1. Introduction

The scattering of waves (quantum or classical) in complex systems has been a very active field of research, both from the theoretical and experimental sides. This interest is motivated by applications in diverse fields, such as nuclear physics [1, 2], coherent quantum transport [3], chaotic billiards [4] and propagation of electromagnetic waves in random media [5]. In this context, the central object is the scattering matrix  $\mathcal{S}(\varepsilon)$  which relates the amplitudes of incoming and outgoing waves in the different scattering channels at a given energy  $\varepsilon$ . The number  $N$  of open channels is fixed by the energy  $\varepsilon$  (for example, it is given by transverse quantisation in a wave guide connected to a cavity). In an ideal system without losses or gains, the conservation of the particle number imposes that the scattering matrix is unitary. This matrix can also satisfy other constraints, depending on the symmetries of the system. The different symmetry classes are labelled by the Dyson index  $\beta$  [6, 7] (see also the review [3]). In the absence of time-reversal symmetry ( $\beta = 2$ ), the only constraint is the unitarity. If time-reversal symmetry is preserved ( $\beta = 1$ ),  $\mathcal{S}$  must additionally be symmetric. The last index  $\beta = 4$  corresponds to the breaking of spin-rotation symmetry (in the presence of strong spin-orbit coupling). In this case,  $\mathcal{S}$  can be represented by a quaternionic self-dual unitary matrix.

Another important matrix, which has attracted a lot of attention, is the Wigner–Smith time delay matrix  $\mathcal{Q} = -i\hbar\mathcal{S}^\dagger\partial_\varepsilon\mathcal{S}$  [8, 9] (in the following we set  $\hbar = 1$ ). This Hermitian matrix contains information about the temporal aspect of the scattering process. The diagonal elements  $Q_{ii}$  are called *injectances* and correspond to the contribution of the  $i$ th scattering mode to the density of states [10]. The eigenvalues of  $\mathcal{Q}$ , which we denote  $\{\tau_1, \dots, \tau_N\}$ , are called proper time delays. Finally, the Wigner time delay, defined as the trace of the Wigner–Smith matrix,

$$\tau_W = \frac{1}{N}\text{tr}\mathcal{Q} = \frac{1}{N}\sum_{i=1}^N Q_{ii} = \frac{1}{N}\sum_{i=1}^N \tau_i, \quad (1)$$

plays an important role in many applications, as it is related to the density of states of the open system (see the review [10]).

For complex systems which exhibit chaotic dynamics, random matrix theory (RMT) provides a powerful framework to characterise the statistical properties of the matrices aforementioned [3, 11, 12]. The distribution of the scattering matrix  $\mathcal{S}(\varepsilon)$  at a given energy  $\varepsilon$  has been obtained in the three symmetry classes  $\beta = 1, 2$  and  $4$  using two different methods: either from a maximal entropy principle (this is called the stochastic approach [13, 14]), or by assuming that the Hamiltonian of the closed system can be described by a random matrix (Hamiltonian approach [15, 16]). In the universal regime where RMT is expected to apply, the two approaches are equivalent [17]. The resulting distribution, known as the Poisson kernel, is a cornerstone of the application of RMT to quantum transport (see the review [3] and references therein).

The Wigner–Smith matrix  $\mathcal{Q}$  is obtained from the energy derivative of  $\mathcal{S}$ . Therefore it is not sufficient to know the distribution of  $\mathcal{S}$  at a given energy: one should also get information about the energy dependence. Different methods have been introduced to tackle this more complex question [18–23]. The joint distribution of the proper time delays  $\{\tau_n\}$ , for perfectly coupled chaotic cavities, has been shown to be related to the Laguerre ensemble of RMT [22, 23],

$$\mathcal{P}(\{\gamma_i = \tau_H/\tau_i\}) \propto \prod_{i < j} |\gamma_i - \gamma_j|^\beta \prod_{n=1}^N \gamma_n^{\frac{\beta N}{2}} e^{-\frac{\beta}{2}\gamma_n}, \quad (2)$$

where  $\tau_H = 2\pi/\Delta$  is the Heisenberg time, and  $\Delta$  the mean level spacing of the closed system. This joint distribution has been used as a starting point to study many quantities involving the proper time delays, such as the Wigner time delay  $\tau_W$  [24–26].

However in real experiments, absorption is always present to some level. This leads to losses, which are one source of decoherence in quantum systems. In particular, the absorption needs to be accounted for to properly describe the results of some experiments [27]. The strength of the absorption is characterised by the absorption time  $\tau_a$ , which measures the mean time a wave can spend in the system before being absorbed. It is convenient to introduce the dimensionless absorption rate<sup>1</sup>  $\gamma = \tau_d/\tau_a$ , where  $\tau_d = \tau_H/N$  is the dwell time inside the system. In the following, all the times will be expressed in units of the Heisenberg time  $\tau_H$  (i.e. we set  $\tau_H = 1$ ).

In the presence of absorption, the scattering matrix becomes sub-unitary. It is thus often referred to as a reflection matrix, since it encodes the reflection of the fraction of the wave that is not absorbed by the system. In the following, we will denote this matrix  $r_\gamma$ . The Wigner–Smith matrix  $\mathcal{Q}$  in the presence of absorption measures the deficit of unitarity of the reflection matrix:  $r_\gamma^\dagger r_\gamma = \mathbb{1}_N - \gamma N \mathcal{Q}$  [29]. These two matrices are thus related, and one can study either one or the other.

Many results have been obtained on the matrices  $r_\gamma$  and  $\mathcal{Q}$  in the presence of absorption. The joint distribution of the eigenvalues of  $r_\gamma^\dagger r_\gamma$  has been found for  $N = 1$  [30] or  $N = 2$  channels [28]. For higher number of channels, this distribution is known only in the limits of strong [32] and weak absorption [30]. Exact expressions for the mean density of eigenvalues of  $r_\gamma^\dagger r_\gamma$  have been derived for any number of channels [29], and reduce to simpler expressions in the large  $N$  limit [31]. Concerning the matrix  $r_\gamma$  itself, its distribution has been obtained for  $N = 1$  in the presence of tunnel coupling [33] or direct processes [34]. We can also mention that another important matrix, the Wigner reaction matrix  $K = i(r_\gamma - \mathbb{1})/(r_\gamma + \mathbb{1})$  has been extensively studied<sup>2</sup>. The distribution of its diagonal entries [37–39], and recently the one of its off-diagonal elements [40], has been found. For reviews of the different results and their applications, see for instance [41, 42].

Despite all these efforts, the distribution of the matrix  $\mathcal{Q}$  (or the joint distribution of its eigenvalues) for any absorption rate  $\gamma$  is still unknown. The aim of this paper is to provide this distribution.

### 1.1. Summary of the main results

Our main results are about the distribution of the Wigner–Smith matrix  $\mathcal{Q}$  in a chaotic absorbing cavity (with absorption rate  $\gamma$ ), perfectly coupled to  $N$  channels. The distribution is more

<sup>1</sup>In the literature, the dimensionless absorption rate is either defined as  $\tau_H/\tau_a$  [28, 29] or  $\tau_d/\tau_a$  [30, 31]. Here we prefer the latter as it will be more convenient to study the limit  $N \rightarrow \infty$ .

<sup>2</sup>In the context of electromagnetic cavities,  $K$  is related to the impedance matrix of the system [35, 36].

conveniently expressed in terms of the inverse matrix  $\Gamma = (N\mathcal{Q})^{-1}$  (we rescale by a factor  $N$  as the eigenvalues of  $\mathcal{Q}$  behave as  $\mathcal{O}(N^{-1})$  for large  $N$ ).

If time-reversal symmetry is broken (unitary class  $\beta = 2$ ), we show that the distribution of the matrix  $\Gamma$  has the compact form

$$P(\Gamma) \propto e^{-N\text{tr}\Gamma} \int_0^{\gamma \mathbb{1}_N} dT \det(\mathbb{1}_N \otimes \Gamma - T \otimes \mathbb{1}_N) e^{-N\text{tr}T}, \quad \Gamma > \gamma \mathbb{1}_N, \quad (3)$$

where the notation indicates that the integral runs over complex Hermitian matrices  $T$  with eigenvalues in  $[0, \gamma]$ , and  $\otimes$  denotes the Kroenecker product of two matrices. The eigenvalues of  $\Gamma$  are constrained to be larger than the absorption rate  $\gamma$ . This restriction indicates that the presence of absorption forbids the appearance of large time delays (small eigenvalues of  $\Gamma$ ) since waves that remain in the system for too long will be absorbed. In a different context, the distribution of  $\mathcal{Q}$  for arbitrary tunnel coupling (but no absorption), was also expressed in terms of an integral over a  $N \times N$  Hermitian matrix [43].

The distribution (3) will be derived in section 2.1, by first obtaining the distribution of the reflection matrix  $r_\gamma$ . The method used to obtain this distribution is difficult to extend to the other symmetry classes due to the additional constraints satisfied by the matrix  $r_\gamma$  when  $\beta = 1$  or 4. We will thus present in section 2.2 a different derivation, which focuses on the eigenvalues of  $\mathcal{Q}$  and is valid in the three symmetry classes ( $\beta = 1, 2$  or 4). We obtain the joint distribution of eigenvalues  $\{\Gamma_n\}$  of  $\Gamma = (N\mathcal{Q})^{-1}$  as

$$\begin{aligned} \mathcal{P}(\{\Gamma_n\}) &\propto \prod_{i<j} |\Gamma_i - \Gamma_j|^\beta \prod_{n=1}^N e^{-\frac{\beta N}{2} \Gamma_n} \\ &\times \int_0^\gamma dt_1 \cdots dt_{N_t} \prod_{i<j} |t_i - t_j|^{\frac{4}{\beta}} \prod_{n=1}^{N_t} \left( [t_n(\gamma - t_n)]^{\frac{2}{\beta}-1} e^{-N t_n} \prod_{m=1}^N (\Gamma_m - t_n) \right), \end{aligned} \quad (4)$$

where  $N_t = \beta N/2$  and  $\Gamma_n > \gamma$ . These two results can be shown to be equivalent for  $\beta = 2$  by diagonalising the matrices  $\Gamma$  and  $T$  in equation (3). Nevertheless, we still give the distribution in terms of the full matrix  $\Gamma$  for  $\beta = 2$  as the expression is more compact.

We further show how our results (3) and (4) can be used to study the distribution of the Wigner time delay (1). We develop a modified Coulomb gas technique to compute the cumulants of  $\tau_W$  in the limit of large number  $N$  of channels. In the two regimes of weak and strong absorption, we obtain respectively

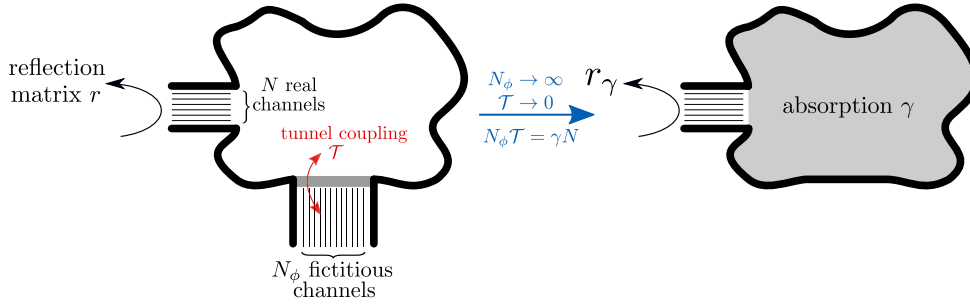
$$\langle \tau_W \rangle \simeq \frac{1}{N}(1 - \gamma), \quad \text{Var}(\tau_W) \simeq \frac{4}{\beta N^4}(1 - 6\gamma), \quad \text{for } \gamma \ll 1, \quad (5)$$

$$\langle \tau_W \rangle \simeq \frac{1}{\gamma N} \left( 1 - \frac{1}{\gamma} \right), \quad \text{Var}(\tau_W) \simeq \frac{2}{\beta(N\gamma)^4}, \quad \text{for } \gamma \gg 1. \quad (6)$$

The expansions of the first cumulant in these two limits is consistent with the known expression  $\langle \tau_W \rangle = 1/(N(\gamma + 1))$  valid for any absorption and large number of channels [31]. Furthermore, we show that in the regime of weak absorption, the higher order cumulants can be obtained from the cumulants at  $\gamma = 0$  (see equation (85)).

## 1.2. Outline of the paper

The paper is organised as follows. Section 2 is mainly devoted to the derivation of the distribution of the Wigner–Smith matrix  $\mathcal{Q}$ . We first show how to obtain the distribution of the



**Figure 1.** The model for chaotic cavities with absorption. The cavity is connected to  $N$  real channels via a perfect contact, and to  $N_\phi$  fictitious channels with tunnel probability  $\mathcal{T}$ . In the limit  $N_\phi \rightarrow \infty$  and  $\mathcal{T} \rightarrow 0$  with  $N_\phi \mathcal{T} = \gamma N$ , this model describes a cavity with uniform absorption, with rate  $\gamma$ .

full matrix in the unitary case ( $\beta = 2$ ), equation (3), in section 2.1. In section 2.2 we obtain the joint distribution of eigenvalues (4), valid in the three symmetry classes, starting from the results of [44, 45]. We show in section 2.3 how the Coulomb gas method can be adapted to handle the distribution (4) in the limit of many open channels. These results on the distribution of  $\mathcal{Q}$  are used in section 3 to study the cumulants of the Wigner time delay (1) in the presence of absorption.

## 2. Distribution of the Wigner–Smith matrix

Let us consider a chaotic cavity perfectly coupled to  $N$  scattering channels. In the presence of absorption, with rate  $\gamma$ , the  $N \times N$  Wigner–Smith matrix  $\mathcal{Q}$  is related to the reflection matrix  $r_\gamma$  as [29]

$$r_\gamma^\dagger r_\gamma = \mathbb{1}_N - \gamma N \mathcal{Q}. \quad (7)$$

This relation shows that  $\mathcal{Q}$  measures the deficit of unitarity of the reflection matrix. In particular, when there is no absorption ( $\gamma = 0$ ) the reflection matrix becomes unitary. In the limit of weak absorption, relation (7) has been used to obtain the distribution of the reflection eigenvalues from the distribution of  $\mathcal{Q}$  (without absorption) [30]. In this paper, we will follow the opposite route: we will first obtain the distribution of the reflection matrix for chaotic absorbing cavities, and then deduce the distribution of  $\mathcal{Q}$  from (7).

The absorption is modelled by introducing  $N_\phi$  fictitious channels, coupled with tunnel probability  $\mathcal{T}$  [46, 47]. In the limit of many fictitious channels  $N_\phi \rightarrow \infty$  and weak coupling  $\mathcal{T} \rightarrow 0$  with fixed product

$$N_\phi \mathcal{T} = \gamma N, \quad (8)$$

this model describes a cavity with uniform absorption rate  $\gamma$ <sup>3</sup> [28]. This is illustrated in figure 1. In practice, we will take this double scaling limit by defining  $\mathcal{T}$  in terms of  $N_\phi$  using (8) for  $N_\phi \rightarrow \infty$  (ensuring that  $\mathcal{T} < 1$ ). The full system (real and fictitious channels) can be described by a  $(N + N_\phi) \times (N + N_\phi)$  unitary scattering matrix  $\mathcal{S}$ .

<sup>3</sup> Alternatively one could shift the energy in a model *without* absorption along the imaginary axis  $E_\gamma = E + i\gamma N/2$  to introduce absorption. The two procedures are equivalent [28].

Assuming that the dynamics inside the cavity is chaotic, we can follow the approach of random matrix theory [3, 12]. The scattering matrix  $\mathcal{S}$  is thus taken as random, with distribution known as the Poisson kernel [13, 14, 17],

$$P(\mathcal{S}) \propto |\det(\mathbb{1} - \bar{\mathcal{S}}^* \mathcal{S})|^{-2-\beta(N+N_\phi-1)}, \quad (9)$$

where  $\bar{\mathcal{S}}$  is the mean scattering matrix. If we label the first  $N$  lines and columns of  $\mathcal{S}$  to correspond to the real channels, and the remaining ones to the  $N_\phi$  fictitious channels,  $\bar{\mathcal{S}}$  takes the form

$$\bar{\mathcal{S}} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{1-\mathcal{T}} \mathbb{1}_{N_\phi} \end{pmatrix}, \quad (10)$$

where  $\mathcal{T}$  is the tunnel coupling to the fictitious channels. The zero in the top-left block is a  $N \times N$  matrix, which corresponds to the fact that the real channels are perfectly coupled to the cavity. We can also decompose the scattering matrix into reflection and transmission blocks:

$$\mathcal{S} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}. \quad (11)$$

The  $N \times N$  top-left block is the reflection matrix  $r$  from the real channels. In the limit of infinite number of (weakly coupled) fictitious channels  $N_\phi \rightarrow \infty$ , this block becomes the reflection matrix of the absorbing cavity:

$$r \xrightarrow[N_\phi \rightarrow \infty]{} r_\gamma. \quad (12)$$

It can then be related to the Wigner–Smith matrix by equation (7). Our aim is to obtain the distribution of this matrix.

In the situation studied in this paper, the  $N$  real channels are equivalent. This means that there is no preferred basis: the matrix  $\mathcal{Q}$  is invariant under unitary transformations  $\mathcal{Q} \rightarrow U^\dagger \mathcal{Q} U$  (this is clear on the distribution (3), derived below in section 2.1). The consequence is that the eigenvalues and eigenvectors of  $\mathcal{Q}$  are statistically uncorrelated, and we can focus on the eigenvalues  $\{\tau_n\}$  of  $\mathcal{Q}$  only. Equivalently, thanks to the relation (7), we can consider the eigenvalues  $\{R_n\}$  of  $r_\gamma^\dagger r_\gamma$ . The joint distribution of the eigenvalues of  $r^\dagger r$  for finite  $N$ ,  $N_\phi$  and any tunnel coupling  $\mathcal{T}$  is known [44, 45]. We can thus use this result to obtain the joint distribution of the reflection eigenvalues  $\{R_n\}$  in the absorbing situation. This will be done in section 2.2. However, we will first present in section 2.1 a derivation of the distribution of the full reflection matrix  $r$  in the unitary case  $\beta = 2$ . Besides providing an alternative derivation to the one given in [44, 45], the procedure described in section 2.1 has the advantage to consider the full reflection matrix  $r$  (eigenvalues and eigenvectors), and could in principle be extended to a situation where the  $N$  real channels are not equivalent.

### 2.1. Unitary class

Let us first consider the case of broken time-reversal symmetry, which corresponds to the Dyson index  $\beta = 2$ . We start from the distribution of the unitary matrix  $\mathcal{S}$  (9). In terms of the block decomposition (11), it becomes only a function of the  $N_\phi \times N_\phi$  bottom-right block  $r'$ ,

$$P(\mathcal{S}) \propto \left| \det(\mathbb{1}_{N_\phi} - \sqrt{1-\mathcal{T}} r') \right|^{-2(N+N_\phi)} \stackrel{\text{def}}{=} P_0(r'). \quad (13)$$

Our goal is to obtain the distribution of the  $N \times N$  reflection block  $r$ . It can formally be written as

$$P(r) \propto \int dr' dt dt' P_0(r') \delta(r^\dagger r + t^\dagger t - \mathbb{1}_N) \delta(r'^\dagger r + t'^\dagger t - \mathbb{1}_{N_\phi}) \delta(r^\dagger t' + t^\dagger r'), \quad (14)$$

where the  $\delta$ -functions impose the unitarity of  $\mathcal{S}$ , and the integration measures  $dr'$ ,  $dt$  and  $dt'$  are the Lebesgue measures over the spaces of  $N_\phi \times N_\phi$ ,  $N_\phi \times N$  and  $N \times N_\phi$  complex matrices respectively. For example

$$dr' = \prod_{i=1}^{N_\phi} \prod_{j=1}^{N_\phi} d\text{Re}(r'_{ij}) d\text{Im}(r'_{ij}). \quad (15)$$

The idea is to perform all the integrals in (14) in order to obtain a form which is convenient to take the limit  $N_\phi \rightarrow \infty$ .

#### Getting rid of the Dirac delta-functions

The first step to evaluate the integrals in equation (14) is to perform the integral over the  $N \times N_\phi$  matrix  $t'$ . Let us make the change of variables

$$t' = -(r^\dagger)^{-1} X, \quad (16)$$

where  $X$  is the new matrix variable of size  $N \times N_\phi$ . The Jacobian of this transformation is [48]

$$dt' = \det(r^\dagger r)^{-N_\phi} dX. \quad (17)$$

The integral (14) thus becomes

$$P(r) \propto (\det r^\dagger r)^{-N_\phi} \int dr' dt dX P_0(r') \delta(r^\dagger r + t^\dagger t - \mathbb{1}_N) \times \delta(r'^\dagger r' + X^\dagger (r^\dagger r)^{-1} X - \mathbb{1}_{N_\phi}) \delta(t^\dagger r' - X). \quad (18)$$

The last  $\delta$ -function straightforwardly cancels the integral over  $X$ , so we obtain

$$P(r) \propto (\det r^\dagger r)^{-N_\phi} \int dr' dt P_0(r') \delta(r^\dagger r + t^\dagger t - \mathbb{1}_N) \delta(r'^\dagger (\mathbb{1}_{N_\phi} + t(r^\dagger r)^{-1} t^\dagger) r' - \mathbb{1}_{N_\phi}). \quad (19)$$

In the second  $\delta$ -function appears the matrix  $\mathbb{1}_{N_\phi} + t(r^\dagger r)^{-1} t^\dagger$ , which is Hermitian and positive (all its eigenvalues are positive). Therefore, we can make the change of variables

$$r' = (\mathbb{1}_{N_\phi} + t(r^\dagger r)^{-1} t^\dagger)^{-1/2} Y. \quad (20)$$

The corresponding Jacobian is [48]

$$dr' = \det(\mathbb{1}_{N_\phi} + t(r^\dagger r)^{-1} t^\dagger)^{-N_\phi} dY = \det(\mathbb{1}_N + (r^\dagger r)^{-1} t^\dagger t)^{-N_\phi} dY, \quad (21)$$

where we have used Sylvester's identity. Using also that

$$(\mathbb{1}_{N_\phi} + t(r^\dagger r)^{-1} t^\dagger)^{-1} = \mathbb{1}_{N_\phi} - t(r^\dagger r + t^\dagger t)^{-1} t^\dagger, \quad (22)$$

we deduce

$$P(r) \propto \int dY dt \det(r^\dagger r + t^\dagger t)^{-N_\phi} P_0((\mathbb{1}_{N_\phi} - t(r^\dagger r + t^\dagger t)^{-1} t^\dagger)^{1/2} Y) \times \delta(r^\dagger r + t^\dagger t - \mathbb{1}_N) \delta(Y^\dagger Y - \mathbb{1}_{N_\phi}). \quad (23)$$



The combination  $r^\dagger r + t^\dagger t$  which appear both in the determinant and in the argument of  $P_0$  can be replaced by the identity thanks to the first  $\delta$ -function. The second  $\delta$ -function imposes that  $Y$  is unitary, therefore

$$P(r) \propto \int_{\text{U}(N_\phi)} d\mu(Y) \int dt P_0((\mathbb{1}_{N_\phi} - tt^\dagger)^{1/2} Y) \delta(r^\dagger r + t^\dagger t - \mathbb{1}_N), \quad (24)$$

where  $d\mu(Y)$  denotes the Haar measure on the unitary group. This expression involves both the combination  $t^\dagger t$  and  $tt^\dagger$ . The first matrix is of size  $N \times N$ , while the second has dimension  $N_\phi \times N_\phi$ . Since we want eventually to take the limit  $N_\phi \rightarrow \infty$ , we can assume that  $N_\phi > N$ . The matrix  $tt^\dagger$  thus has the same eigenvalues as  $t^\dagger t$ , plus a series of  $N_\phi - N$  eigenvalues equal to zero. Therefore, there exists a unitary matrix  $U$  such that

$$tt^\dagger = U \begin{pmatrix} t^\dagger t & 0 \\ 0 & 0 \end{pmatrix} U^\dagger. \quad (25)$$

Since  $d\mu(U^\dagger Y U) = d\mu(Y)$ , the integrals (24) become

$$P(r) \propto \int_{\text{U}(N_\phi)} d\mu(Y) \int dt P_0 \left( \begin{pmatrix} (\mathbb{1}_N - t^\dagger t)^{1/2} & 0 \\ 0 & \mathbb{1}_{N_\phi - N} \end{pmatrix} Y \right) \delta(r^\dagger r + t^\dagger t - \mathbb{1}_N). \quad (26)$$

We can now make the last change of variables

$$T = t^\dagger t. \quad (27)$$

This change of variables is not one-to-one, as  $T$  defines  $t$  up to a unitary matrix. Nevertheless, the Lebesgue measure  $dt$  can be expressed in terms of  $T$  and a unitary matrix  $V$ , uniformly distributed over  $\text{U}(N_\phi)$  [48]:

$$dt = 2^{-N} \det(T)^{N_\phi - N} dT d\mu(V). \quad (28)$$

Integration over  $V$  yields a constant (the volume of the unitary group), and we can straightforwardly integrate over the Hermitian matrix  $T$  to obtain

$$P(r) \propto \det(\mathbb{1}_N - r^\dagger r)^{N_\phi - N} \int_{\text{U}(N_\phi)} d\mu(Y) P_0 \left( \begin{pmatrix} (r^\dagger r)^{1/2} & 0 \\ 0 & \mathbb{1}_{N_\phi - N} \end{pmatrix} Y \right). \quad (29)$$

Replacing  $P_0$  by its expression (13) gives

$$P(r) \propto \det(\mathbb{1}_N - r^\dagger r)^{N_\phi - N} \int_{\text{U}(N_\phi)} \frac{d\mu(Y)}{|\det(\mathbb{1}_{N_\phi} - A(r)Y)|^{2(N+N_\phi)}}, \quad (30)$$

where we have introduced the Hermitian matrix

$$A(r) = \sqrt{1 - \mathcal{T}} \begin{pmatrix} (r^\dagger r)^{1/2} & 0 \\ 0 & \mathbb{1}_{N_\phi - N} \end{pmatrix}. \quad (31)$$

We have reduced the original integral over the  $(N + N_\phi) \times (N + N_\phi)$  unitary matrix  $\mathcal{S}$  to an integral over the  $N_\phi \times N_\phi$  unitary matrix  $Y$ . However, this expression (30) is not convenient to take the limit  $N_\phi \rightarrow \infty$ , as the integration domain depends explicitly on  $N_\phi$ . We will now evaluate this last integral.

*Evaluation of the integral over the unitary group*

Integrals of the type

$$\int_{U(N_\phi)} \frac{d\mu(Y)}{|\det(\mathbb{1}_{N_\phi} - AY)|^{2n}} \quad (32)$$

have been studied in [49], using the theory of Schur functions [50]. However, they have been computed for  $0 \leq n \leq N_\phi$ , while in equation (30) we have  $n = N_\phi + N > N_\phi$ . The idea to evaluate this integral in this domain is given in the appendix of [28]: we make the change of variables<sup>4</sup>

$$Y = A - \sqrt{\mathbb{1} - A^2} U (1 - AU)^{-1} \sqrt{\mathbb{1} - A^2}, \quad (33)$$

where  $A$  is Hermitian and  $U$  is unitary. The Jacobian of the change of variables (33) is [28, 48]

$$d\mu(Y) \propto \det(\mathbb{1}_{N_\phi} - A^2)^{-N_\phi} |\det(\mathbb{1}_{N_\phi} - AY)|^{2N_\phi} d\mu(U), \quad (34)$$

which cancels out the power  $2N_\phi$  in the denominator of (30). Furthermore, since

$$\det(\mathbb{1}_{N_\phi} - AY) = \det(\mathbb{1}_{N_\phi} - A^2) \det(\mathbb{1}_{N_\phi} - AU)^{-1}, \quad (35)$$

the remaining power of the determinant changes sign. Therefore, the integral in (30) can be expressed as

$$\begin{aligned} \int_{U(N_\phi)} \frac{d\mu(Y)}{|\det(\mathbb{1}_{N_\phi} - AY)|^{2(N+N_\phi)}} &\propto \det(\mathbb{1}_{N_\phi} - A^2)^{-N_\phi - 2N} \\ &\times \int_{U(N_\phi)} d\mu(U) |\det(\mathbb{1}_{N_\phi} - AU)|^{2N}. \end{aligned} \quad (36)$$

The integral on the r.h.s has been computed in [49], with no restriction on the values of  $N$  and  $N_\phi$ :

$$\int_{U(N_\phi)} d\mu(U) |\det(\mathbb{1}_{N_\phi} - AU)|^{2N} \propto \int dZ \det(\mathbb{1}_N + Z^\dagger Z)^{-N_\phi - 2N} \det(\mathbb{1} + Z^\dagger Z \otimes A^2), \quad (37)$$

where the integral runs over the  $N \times N$  matrix  $Z$  with  $N^2$  independent complex entries, and  $\otimes$  denotes the Kronecker product of two matrices. In order to simplify this expression, we first introduce the matrix  $X = Z^\dagger Z$ . We can perform this change of variables in the integral (37) similarly as we did with the matrix  $t^5$  (27) and (28). We obtain

$$\int_{U(N_\phi)} d\mu(U) |\det(\mathbb{1}_{N_\phi} - AU)|^{2N} \propto \int dX \det(\mathbb{1}_N + X)^{-N_\phi - 2N} \det(\mathbb{1} + X \otimes A^2), \quad (38)$$

where the integral now runs over the Hermitian and positive matrix  $X$ . Combining equations (36) and (38), we can express the distribution of the reflection matrix (30) as

$$\begin{aligned} P(r) &\propto \det(\mathbb{1}_N - r^\dagger r)^{N_\phi - N} \det(\mathbb{1}_{N_\phi} - A(r)^2)^{-N_\phi - 2N} \\ &\times \int dX \det(\mathbb{1}_N + X)^{-N_\phi - 2N} \det(\mathbb{1} + X \otimes A(r)^2). \end{aligned} \quad (39)$$

<sup>4</sup>This is a well known change of variables in the context of quantum scattering, as it is the one that relates the scattering matrix of a cavity with perfect couplings (which would be here  $U$ ) to the one with arbitrary coupling (here  $Y$ ). The couplings are described by the matrix  $A$  [17].

<sup>5</sup>In (28),  $t$  was a  $N_\phi \times N$  matrix, while  $Z$  is now  $N \times N$ . Therefore we must set  $N_\phi = N$  in the Jacobian (28).

Substituting the expression of the matrix  $A(r)$  (31), this becomes

$$P(r) \propto \det(\mathbb{1}_N - r^\dagger r)^{-3N} \det\left(\mathbb{1}_N + \mathcal{T} \frac{r^\dagger r}{\mathbb{1}_N - r^\dagger r}\right)^{-N_\phi - 2N} \\ \times \int \frac{dX}{\det(\mathbb{1}_N + X)^{3N}} \det(\mathbb{1} + (1 - \mathcal{T})X \otimes r^\dagger r) \det\left(\mathbb{1} - \mathcal{T} \frac{X}{\mathbb{1}_N + X}\right)^{N_\phi - N}. \quad (40)$$

From this distribution of the reflection matrix  $r$ , one can deduce the joint distribution of the eigenvalues of  $r^\dagger r$ . By also diagonalising  $X = X^\dagger > 0$ , one can recover from the distribution (40) the joint distribution of reflection eigenvalues derived in [45]. The main difference with our derivation is that we are dealing with the full reflection matrix, and not only the eigenvalues of  $r^\dagger r$ . Besides providing a more compact expression for the distribution, our approach is also more natural to analyse the situation where the  $N$  channels are not equivalent (and thus the eigenvalues and eigenvectors no longer decouple).

The distribution (40) is well suited to derive the distribution of the reflection matrix  $r_\gamma$  in the presence of absorption. Indeed, the dimension of the integration domain depends only on the number  $N$  of real channels, and not on the number  $N_\phi$  of fictitious channels which we introduced to model the absorption. The parameter  $N_\phi$  only appears in the power of some determinants, and in the tunnel coupling  $\mathcal{T}$  (8). Therefore, we can now take the limit  $N_\phi \rightarrow \infty$ . Using the famous identity  $\log \det = \text{tr} \log$ , we deduce

$$\det\left(\mathbb{1}_N + \frac{\gamma N}{N_\phi} \frac{r^\dagger r}{\mathbb{1}_N - r^\dagger r}\right) \xrightarrow{N_\phi \rightarrow \infty} e^{-\gamma N \text{tr}\left[\frac{r^\dagger r \gamma}{\mathbb{1}_N - r^\dagger r \gamma}\right]} = e^{\gamma N^2} e^{-\gamma N \text{tr}[(\mathbb{1}_N - r_\gamma^\dagger r_\gamma)^{-1}]}, \quad (41)$$

and similarly for the determinant involving  $X$ . Finally, we obtain the distribution of the reflection matrix  $r_\gamma$  (i.e. the scattering matrix of the absorbing cavity):

$$P(r) \propto \det(\mathbb{1}_N - r_\gamma^\dagger r_\gamma)^{-3N} e^{-\gamma N \text{tr}[(\mathbb{1}_N - r_\gamma^\dagger r_\gamma)^{-1}]} \int dX \frac{\det(\mathbb{1} + X \otimes r_\gamma^\dagger r_\gamma)}{\det(\mathbb{1}_N + X)^{3N}} e^{-\gamma N \text{tr}\left[\frac{X}{\mathbb{1}_N + X}\right]}. \quad (42)$$

We can further simplify this expression by introducing the variable

$$T = \frac{X}{\mathbb{1}_N + X}, \quad (43)$$

which is associated to the following Jacobian [48]:

$$dX = \frac{dT}{\det(\mathbb{1}_N - T)^{2N}}. \quad (44)$$

This transformation yields our final result for the reflection matrix:

$$P(r) \propto \det(\mathbb{1}_N - r_\gamma^\dagger r_\gamma)^{-3N} e^{-\gamma N \text{tr}[(\mathbb{1}_N - r_\gamma^\dagger r_\gamma)^{-1}]} \\ \times \int_0^{\mathbb{1}_N} dT \det(\mathbb{1} - T \otimes (\mathbb{1}_N - r_\gamma^\dagger r_\gamma)) e^{-\gamma N \text{tr} T}, \quad (45)$$

where the notation indicates that the integration is performed over Hermitian matrices  $T$  with eigenvalues in  $[0, 1]$ . This distribution is the extension of the uniform distribution of the scattering matrix for  $\gamma = 0$  to the absorbing situation  $\gamma > 0$ .

Before using our result (45) to derive the distribution of the Wigner–Smith matrix  $\mathcal{Q}$ , let us check that this distribution properly reproduces the different limits which are known. In the limit of strong absorption  $\gamma \rightarrow +\infty$ , the exponentials in (45) strongly suppress the distribution for  $r_\gamma^\dagger r_\gamma$  away from zero. Therefore, we can drop the contribution of the integral, and expand

$(\mathbb{1}_N - r_\gamma^\dagger r_\gamma)^{-1} \simeq \mathbb{1}_N + r_\gamma^\dagger r_\gamma$ . The distribution of  $r_\gamma$  thus reduces to  $P(r_\gamma) \propto e^{-\gamma N \text{tr}(r_\gamma^\dagger r_\gamma)}$ , which coincides with the result of [32]. In the converse limit of weak absorption  $\gamma \rightarrow 0$ , the matrix  $r_\gamma$  is weakly sub-unitary, therefore  $r_\gamma^\dagger r_\gamma$  is close to the identity. We can thus drop the integral in the distribution (45), and we recover the result of [30]. Finally, for  $N = 1$  and any absorption rate  $\gamma$ , our result reduces to the distribution given in [30].

*Distribution of the Wigner–Smith matrix*

The Wigner–Smith matrix  $\mathcal{Q}$  is directly related to the reflection matrix  $r_\gamma$  via (7). As in the well studied case without absorption ( $\gamma = 0$ ), the distribution is more conveniently expressed in terms of the inverse matrix<sup>6</sup>

$$\Gamma = (N\mathcal{Q})^{-1} = \gamma(\mathbb{1}_N - r_\gamma^\dagger r_\gamma)^{-1}. \tag{46}$$

Since  $r_\gamma$  is sub-unitary (the absorption causes losses) the eigenvalues of  $r_\gamma^\dagger r_\gamma$  are in  $[0, 1]$ , therefore those of  $\Gamma$  are larger than the absorption rate  $\gamma$ . This implies that the eigenvalues of  $\mathcal{Q}$ , the proper time delays  $\{\tau_n\}$ , are smaller than  $1/(N\gamma)$ . The effect of the absorption is thus to introduce an upper cutoff, which forbids the existence of arbitrarily large time delays.

From the distribution of the reflection matrix  $r_\gamma$  (45), we can deduce<sup>7</sup>

$$P(\Gamma) \propto e^{-N\text{tr}\Gamma} \int_0^{\gamma\mathbb{1}_N} dT \det(\mathbb{1}_N \otimes \Gamma - T \otimes \mathbb{1}_N) e^{-N\text{tr}T}, \quad \Gamma > \gamma\mathbb{1}_N, \tag{47}$$

where we have rescaled the matrix  $T$  in the integral by a factor  $\gamma$ , in order to obtain a more symmetric expression. This distribution is the first central result of this paper.

First, let us notice that in the limit of no absorption  $\gamma \rightarrow 0$ , the distribution becomes

$$P(\Gamma) \propto \det(\Gamma \otimes \mathbb{1}_N) e^{-N\text{tr}\Gamma} = (\det \Gamma)^N e^{-N\text{tr}\Gamma}, \tag{48}$$

which is the celebrated Wishart–Laguerre distribution of the inverse Wigner–Smith matrix at zero absorption [22, 23].

The distribution (47) is invariant under unitary transformations  $\Gamma \rightarrow U\Gamma U^\dagger$ , with  $U \in \text{U}(N)$ . This is expected since the  $N$  real channels are equivalent, therefore there is no preferred basis. The consequence is that the eigenvalues and eigenvectors of  $\Gamma$  become statistically independent, and the eigenvectors are uniformly distributed. We can therefore integrate over the eigenvectors, and deduce the joint distribution of eigenvalues  $\{\Gamma_n\}$ <sup>8</sup>,

$$\mathcal{P}(\{\Gamma_n\}) \propto \prod_{i < j} (\Gamma_i - \Gamma_j)^2 \prod_{i=1}^N e^{-N\Gamma_i} \int_0^{\gamma\mathbb{1}_N} dT \prod_{i=1}^N \det(\Gamma_i \mathbb{1}_N - T) e^{-N\text{tr}T}. \tag{49}$$

The integral over the matrix  $T$  is also invariant under unitary transformations  $T \rightarrow UTU^\dagger$ , therefore we can also reduce it to an integral over the eigenvalues  $\{t_n\}$  only:

<sup>6</sup> We also rescale by a factor  $N$  for convenience, since the eigenvalues of  $\mathcal{Q}$  scale with  $N$  as  $\mathcal{O}(N^{-1})$ .

<sup>7</sup> For this, we need the Jacobian of (46). First, we introduce  $R = r_\gamma^\dagger r_\gamma$ , which is associated to the Jacobian  $dr_\gamma = 2^{-N} dR d\mu(U)$ , where  $U$  is a Haar distributed unitary matrix of size  $N$  which can be integrated over (see the discussion between equations (27) and (28)). Then we have  $\Gamma = \gamma(1 - R)^{-1}$ , thus  $dR \propto (\det \Gamma)^{-2N} d\Gamma$  [48].

<sup>8</sup> The Jacobian of the eigendecomposition involves the Vandermonde determinant  $\prod_{i < j} |\Gamma_i - \Gamma_j|^\beta$  [48], which is well known in RMT [51, 52].

$$\mathcal{P}(\{\Gamma_n\}) \propto \prod_{i<j} (\Gamma_i - \Gamma_j)^2 \prod_{i=1}^N e^{-N\Gamma_i} \int_0^\gamma dt_1 \cdots dt_N \prod_{i<j} ((t_i - t_j)^2 (\Gamma_i - t_j)) \prod_{i=1}^N e^{-Nt_i T}. \quad (50)$$

In this form, the integral over the eigenvalues  $\{t_n\}$  can be performed using Andréief’s identity [53, 54], which gives

$$\mathcal{P}(\{\Gamma_n\}) \propto \prod_{i<j} (\Gamma_i - \Gamma_j)^2 \prod_{i=1}^N e^{-N\Gamma_i} \det \left[ \int_0^\gamma dt t^{m+m-2} e^{-Nt} \prod_{i=1}^N (\Gamma_i - t) \right]_{1 \leq n, m \leq N}. \quad (51)$$

This expression is useful to obtain exact expressions for the joint distribution of eigenvalues  $\{\Gamma_n\}$  for small number  $N$  of channels. For example, for  $N = 1$ , this yields

$$\mathcal{P}(\Gamma) \propto e^{-\Gamma} \int_0^\gamma dt (\Gamma - t) e^{-t} = e^{-\Gamma} (\Gamma(1 - e^{-\gamma}) + e^{-\gamma}(\gamma + 1) - 1), \quad (52)$$

which coincides with the known result [29]. However, equation (51) is not well suited to analyse the limit of large number  $N$  of channels. We will present in section 2.3 a method, based on the Coulomb gas technique, which is more convenient in this case. But before that, we now extend the result (50) to the other symmetry classes  $\beta = 1$  or 4.

### 2.2. General case

In section 2.1 we have obtained the distribution of the Wigner–Smith matrix  $\mathcal{Q}$  in the unitary case. The derivation that we have presented is difficult to extend to the other symmetry classes due to the presence of additional constraints on the scattering matrix (11). For instance, if  $\beta = 1$  the scattering matrix is symmetric:  $\mathcal{S}^T = \mathcal{S}$ . This constraint needs to be taken into account and leads to additional complications. Instead, we will follow a different approach, by focusing on the joint distribution of the eigenvalues of  $\mathcal{Q}$ . This alternative approach is valid in all three symmetry classes. As we have seen in the case  $\beta = 2$  discussed above, since the  $N$  channels are equivalent, the eigenvectors of  $\mathcal{Q}$  are statistically independent from the eigenvalues and are uniformly distributed. Therefore, determining the joint distribution of the eigenvalues is sufficient to fully characterise the matrix  $\mathcal{Q}$ .

Our starting point is the joint distribution of the reflection eigenvalues  $\{R_n\}_{n=1, \dots, N}$  for a chaotic cavity perfectly coupled to  $N$  channels and  $N_\phi > N$  other channels via a tunnel barrier  $\mathcal{T}$  [45]:

$$\begin{aligned} \mathcal{P}(\{R_n\}) \propto & \prod_{i<j} |R_i - R_j|^\beta \prod_{n=1}^N (1 - R_n)^{\frac{\beta}{2} - 1 + \frac{\beta}{2}(N_\phi - N)} (1 - (1 - \mathcal{T})R_n)^{-1 - \frac{\beta}{2}(2N + N_\phi - 1)} \\ & \times \int_0^1 dt_1 \cdots dt_{N_t} \prod_{i<j} |t_i - t_j|^{\frac{4}{\beta}} \prod_{n=1}^{N_t} \left( [t_n(1 - t_n)]^{\frac{2}{\beta} - 1} \prod_{m=1}^{N_\phi} (1 - t_n(1 - (1 - \mathcal{T})R_m)) \right), \end{aligned} \quad (53)$$

where  $N_t = \beta N/2$  and  $R_n = 1$  for  $n > N$ . There is a duality between the cases  $\beta = 1$  and  $\beta = 4$ : the distribution of the reflection eigenvalues for  $\beta = 1$  is given in terms of an integral for  $\beta = 4$ , and vice versa. This type of duality has also been found for integrals over the Ginibre ensembles [55]. Additionally, for  $\beta = 1$  the dimension  $N_t = N/2$  of the integral in (53) restricts the number of channels to even numbers. The problem of finding a similar representation valid for odd number of channels is still open.

We can obtain the distribution of the reflection eigenvalues in the presence of absorption from (53) as we did in the previous section: we set the tunnel coupling  $\mathcal{T} = \gamma N/N_\phi$ , equation (8), and let  $N_\phi \rightarrow \infty$ . This gives

$$\begin{aligned} \mathcal{P}(\{R_n\}) &\propto \prod_{i < j} |R_i - R_j|^\beta \prod_{n=1}^N (1 - R_n)^{\beta - 2 - 3\beta N/2} e^{-\frac{\beta N}{2} \gamma \frac{R_n}{1 - R_n}} \\ &\times \int_0^1 dt_1 \cdots dt_{N_i} \prod_{i < j} |t_i - t_j|^{\frac{4}{\beta}} \prod_{n=1}^{N_i} \left( [t_n(1 - t_n)]^{\frac{2}{\beta} - 1} e^{-\gamma N t_n} \prod_{m=1}^N (1 - t_n(1 - R_m)) \right). \end{aligned} \quad (54)$$

This is the analogous of equation (45), valid in the three symmetry classes, but this time expressed in terms of the eigenvalues of  $r_\gamma^\dagger r_\gamma$ .

From the joint distribution of the reflection eigenvalues in the presence of absorption (54), we now deduce the distribution of the eigenvalues of  $\mathcal{Q}$ , the proper time delays  $\{\tau_n\}$ . Similarly to the case  $\beta = 2$  discussed in section 2.1, it is more convenient to work with the rescaled inverse time delays

$$\Gamma_n = \frac{1}{N\tau_n} = \frac{\gamma}{1 - R_n}. \quad (55)$$

This relation is the analogous of (46), but expressed in terms of the eigenvalues. Performing this change of variables in the distribution (54), and rescaling the integration variables by a factor  $\gamma$ , we obtain

$$\begin{aligned} \mathcal{P}(\{\Gamma_n\}) &\propto \prod_{i < j} |\Gamma_i - \Gamma_j|^\beta \prod_{n=1}^N e^{-\frac{\beta N}{2} \Gamma_n} \\ &\times \int_0^\gamma dt_1 \cdots dt_{N_i} \prod_{i < j} |t_i - t_j|^{\frac{4}{\beta}} \prod_{n=1}^{N_i} \left( [t_n(\gamma - t_n)]^{\frac{2}{\beta} - 1} e^{-N t_n} \prod_{m=1}^N (\Gamma_m - t_n) \right). \end{aligned} \quad (56)$$

This joint distribution is the second central result of this paper. It is the extension to  $\beta = 1$  and  $\beta = 4$  of the distribution (50) which we derived above for  $\beta = 2$ . In the limit of weak absorption  $\gamma \rightarrow 0$ , equation (56) reduces to the well-known Wishart–Laguerre distribution of the inverse proper time delays [22, 23], equation (2), as it should.

### 2.3. Coulomb gas description

The representation (56) can be used to obtain exact expressions for the distribution of the time delays in the case of a few open channels  $N = 1, 2, \dots$ . In the converse situation of many channels  $N \rightarrow \infty$ , the Coulomb gas method has proved to be a powerful tool to study different quantities involving the eigenvalues of random matrices, such as the mean density or linear statistics (quantities of the form  $\sum_n f(\lambda_n)$ , where the  $\lambda_n$ 's are the eigenvalues of a matrix and  $f$  is any given function, not necessarily linear<sup>9</sup>). We first recall the main ideas of this technique, and show how to adapt it in the case where the joint distribution of eigenvalues involves an integral over a domain whose dimension scale as  $\mathcal{O}(N)$ , as in (56). This formalism will be useful to study the statistical properties of the Wigner time-delay (1) (which is a linear statistics) in section 3.

The Coulomb gas method has been developed for invariant ensembles of random matrices, with a joint distribution of eigenvalues  $\{\lambda_n\}$  of the form

<sup>9</sup>The name linear comes from the fact that there are no products of different eigenvalues.

$$\mathcal{P}(\lambda_1, \dots, \lambda_N) \propto \prod_{i < j} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^N e^{-NV(\lambda_i)}, \quad (57)$$

where  $V$  is a function, called the potential, that diverges sufficiently fast at infinity to ensure that the distribution can be normalised (the factor  $N$  in the exponent ensures that the eigenvalues are of order 1 when  $N \rightarrow \infty$ ). This function depends on the ensemble of random matrices under consideration (for example  $V(x) = x^2$  for the Gaussian ensembles). The idea of the Coulomb gas technique is to write the distribution (57) as a Gibbs weight [56]

$$\mathcal{P}(\lambda_1, \dots, \lambda_N) \propto e^{-\frac{\beta N^2}{2} E_0(\{\lambda_n\})}, \quad (58)$$

where we introduced

$$E_0(\{\lambda_n\}) = -\frac{1}{N^2} \sum_{i \neq j} \ln |\lambda_i - \lambda_j| + \frac{1}{N} \sum_{i=1}^N V(\lambda_i). \quad (59)$$

This function can be interpreted as the energy of a gas of particles located at positions  $\lambda_i$ , placed in a potential  $V(\lambda)$ , with logarithmic repulsion (hence the name *Coulomb gas*). For large  $N$ , the distribution (58) becomes peaked near the minimum of the energy (59). This observation leads to important simplifications which allow to solve many problems analytically in this limit. For a recent overview, we refer to the introduction of [57] and the PhD theses [58–60].

In our situation, we cannot apply directly the Coulomb gas method as described above, since the distribution of eigenvalues  $\Gamma_n = 1/(N\tau_n)$  (56) involves a multiple integral over of domain of dimension  $\mathcal{O}(N)$ . Therefore, we need to adapt this method by writing the joint distribution (56) in the form

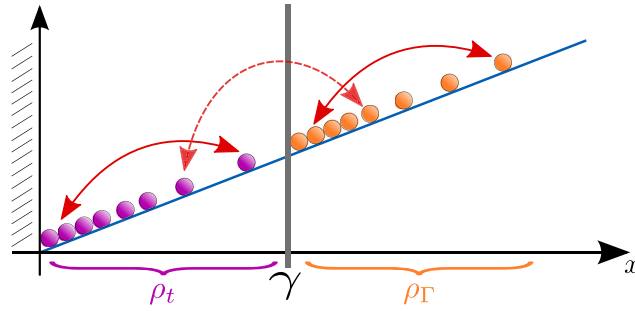
$$\mathcal{P}(\{\Gamma_n\}) \propto \int_0^\gamma dt_1 \cdots dt_{\frac{\beta N}{2}} e^{-\frac{\beta N^2}{2} E(\{\Gamma_n\}, \{t_n\})}, \quad (60)$$

where we introduced the energy

$$\begin{aligned} E(\{\Gamma_n\}, \{t_n\}) &= \frac{1}{N} \sum_{i=1}^N \Gamma_i - \frac{1}{N^2} \sum_{i \neq j} \ln |\Gamma_i - \Gamma_j| \\ &+ \frac{2}{\beta N} \sum_{n=1}^{\beta N/2} t_n - \frac{4}{\beta^2 N^2} \sum_{n \neq m} \ln |t_i - t_j| - \frac{2}{\beta N^2} \left( \frac{2}{\beta} - 1 \right) \sum_{n=1}^{\beta N/2} \ln [t_n(\gamma - t_n)] \\ &- \frac{2}{\beta N^2} \sum_{i=1}^N \sum_{n=1}^{\beta N/2} \ln(\Gamma_i - t_n). \end{aligned} \quad (61)$$

This expression can be interpreted as the energy of two Coulomb gases:

- A first gas of  $N$  particles, located at positions  $\Gamma_i > \gamma$ . These are the particles of interest, as they are related to the eigenvalues of the Wigner–Smith matrix  $\mathcal{Q}$ . This gas is placed in a linear potential  $V(x) = x$  and exhibits logarithmic repulsion between the particles (first line of equation (61));
- A second gas composed of  $\beta N/2$  particles, at positions  $t_i \in [0, \gamma]$ . The particles in this gas also repel logarithmically, and are placed in the potential  $\tilde{V}(t) = t - \frac{1}{N} \left( \frac{2}{\beta} - 1 \right) \ln [t(\gamma - t)]$ , which becomes linear for  $N \rightarrow \infty$  (second line of equation (61)).



**Figure 2.** The Coulomb gases associated to the energy (61), or equivalently to the continuous version (64). Two log-gases are placed in a linear confining potential. The first gas is confined on  $[0, \gamma]$ , while the other one is restricted to  $[\gamma, +\infty)$ . The particles of different gases repel logarithmically, with an interaction weaker (dashed arrow) by a factor 2 compared to the repulsion within each gas (solid arrows). For large  $N$ , the two gases can be described by the continuous densities  $\rho_t$  and  $\rho_\Gamma$ .

These two gases interact with each other logarithmically, but the strength of this repulsion is half of the one within each gas (see the discussion in section 3). This interpretation is illustrated in figure 2.

**2.3.1. Continuous formulation.** Instead of working with the sets of eigenvalues, it is more convenient to introduce the two empirical densities

$$\rho_\Gamma(x) = \frac{1}{N} \sum_{n=1}^N \delta(x - \Gamma_n) \quad \text{and} \quad \rho_t(x) = \frac{2}{\beta N} \sum_{n=1}^{\beta N/2} \delta(x - t_n), \quad (62)$$

both normalised to unity. In the limit  $N \rightarrow \infty$ , these densities can be replaced by continuous ones. Since  $t_n \in [0, \gamma]$ , the support of  $\rho_t$  is contained in  $[0, \gamma]$ . Similarly, the support of  $\rho_\Gamma$  is contained in  $[\gamma, +\infty)$ . For  $N \gg 1$ , the distribution of eigenvalues (60) can be replaced by a weight over the set densities:

$$\mathcal{P}(\{\Gamma_n\}) d\Gamma_1 \cdots d\Gamma_N \longrightarrow \mathcal{D}\rho_\Gamma \delta\left(\int \rho_\Gamma - 1\right) \int \mathcal{D}\rho_t e^{-\frac{\beta N^2}{2} \mathcal{E}[\rho_\Gamma, \rho_t]} \delta\left(\int \rho_t - 1\right), \quad (63)$$

where the  $\delta$ -functions ensure that both densities are normalised, and the energy functional  $\mathcal{E}$  is the continuous version of (61):

$$\begin{aligned} \mathcal{E}[\rho_\Gamma, \rho_t] = & \int dx \rho_\Gamma(x) x - \int dx dx' \rho_\Gamma(x) \rho_\Gamma(x') \ln|x - x'| \\ & + \int dt \rho_t(t) t - \int dt dt' \rho_t(t) \rho_t(t') \ln|t - t'| \\ & - \int dx dt \rho_\Gamma(x) \rho_t(t) \ln(x - t). \end{aligned} \quad (64)$$

We have neglected the subleading  $\frac{1}{N}$  corrections, and in particular the entropy which arises when replacing the discrete sets of eigenvalues by the continuous densities [56, 61, 62]. We will now use this formulation to study the Wigner time delay.



### 3. Moments of the Wigner time delay

As an application of our results (47) and (56) for the distribution of the Wigner–Smith matrix  $\mathcal{Q}$ , we study the statistical properties of the Wigner time delay

$$\tau_W = \frac{1}{N} \text{tr} \mathcal{Q} = \frac{1}{N} \sum_{n=1}^N \tau_n = \frac{1}{N^2} \sum_{n=1}^N \frac{1}{\Gamma_n} \quad (65)$$

in the presence of absorption. (We have used (55) to express  $\tau_W$  in terms of the eigenvalues  $\{\Gamma_n\}$ .) Since the eigenvalues  $\Gamma_n$  are of order 1, the Wigner time delay scales as  $N^{-1}$  for large  $N$ . In order to work with quantities which do not scale with  $N$ , we introduce the rescaled variable

$$s = N\tau_W = \frac{1}{N} \sum_{n=1}^N \frac{1}{\Gamma_n}. \quad (66)$$

We follow an approach similar to [26], where the distribution of  $\tau_W$  was derived for  $\gamma = 0$  in the large- $N$  limit. However, instead of the distribution we focus on the moment generating function of the random variable (66), at fixed absorption rate  $\gamma$ ,

$$G_\gamma(\mu) = \left\langle e^{-\frac{\beta N^2}{2} \mu s} \right\rangle = \int d\Gamma_1 \cdots d\Gamma_N \mathcal{P}(\{\Gamma_n\}) e^{-\frac{\beta N^2}{2} \frac{\mu}{N} \sum_n 1/\Gamma_n}, \quad (67)$$

where  $\langle \cdots \rangle$  denotes the average with respect to the joint distribution (56) and we multiplied the argument  $\mu$  by a factor  $\beta N^2/2$  to coincide with the scaling of the energy. We can replace the integration over the eigenvalues  $\{\Gamma_n\}$  by a functional integral over the density  $\rho_\Gamma$ , as prescribed by equation (63):

$$G_\gamma(\mu) = \frac{\int \mathcal{D}\rho_\Gamma \delta\left(\int \rho_\Gamma - 1\right) \int \mathcal{D}\rho_t e^{-\frac{\beta N^2}{2} (\mathcal{E}[\rho_\Gamma, \rho_t] + \mu \int dx \rho_\Gamma(x)/x)} \delta\left(\int \rho_t - 1\right)}{\int \mathcal{D}\rho_\Gamma \delta\left(\int \rho_\Gamma - 1\right) \int \mathcal{D}\rho_t e^{-\frac{\beta N^2}{2} \mathcal{E}[\rho_\Gamma, \rho_t]} \delta\left(\int \rho_t - 1\right)}, \quad (68)$$

where the denominator ensures that  $G_\gamma(0) = 1$ , which follows from the normalisation of the distribution. For  $N \gg 1$ , we can estimate these integrals by a saddle point method. They are dominated by the densities  $\rho_\Gamma, \rho_t$  which minimise the energy  $\mathcal{E}$  under the constraints imposed by the  $\delta$ -functions. We can find this minimum by introducing Lagrange multipliers  $\mu_0^{(\Gamma)}$  and  $\mu_0^{(t)}$ . For the numerator, we thus need to find the minimum of

$$\mathcal{F}[\rho_\Gamma, \rho_t; \mu] \stackrel{\text{def}}{=} \mathcal{E}[\rho_\Gamma, \rho_t] + \mu_0^{(t)} \left( \int \rho_t(t) dt - 1 \right) + \mu_0^{(\Gamma)} \left( \int \rho_\Gamma(x) dx - 1 \right) + \mu \int \frac{\rho_\Gamma(x)}{x} dx. \quad (69)$$

This can be done by taking the functional derivatives of this expression with respect to  $\rho_\Gamma(x)$  and  $\rho_t(t)$ . This gives two coupled equations for these densities. As usual in random matrix theory, it is more convenient to work with the derivatives of these equations (with respect to  $x$  for the equation  $\delta\mathcal{F}/\delta\rho_\Gamma(x) = 0$  and to  $t$  for  $\delta\mathcal{F}/\delta\rho_t(t) = 0$ ). This gives the set of two coupled equations:

$$\int dx' \frac{\rho_\Gamma(x')}{x-x'} + \frac{1}{2} \int dt \frac{\rho_t(t)}{x-t} = \frac{1}{2} - \frac{\mu}{2x^2} \quad \text{for } x \in \text{Supp}(\rho_\Gamma), \quad (70)$$

$$\int dt' \frac{\rho_t(t')}{t-t'} + \frac{1}{2} \int dx \frac{\rho_\Gamma(x)}{t-x} = \frac{1}{2} \quad \text{for } t \in \text{Supp}(\rho_t), \quad (71)$$

where  $\int$  denotes a principal value integral. These two equations can be interpreted as the force balance for the two Coulomb gases. Let us look for example at equation (70). On the right-hand-side, the first term is the force coming from the linear confining potential (this is why it is also present in the second equation (71)). The second term, proportional to the argument  $\mu$  of the generating function, acts as an additional force. Since we probe the statistics of a quantity (66) that only involves the eigenvalues  $\Gamma_n$ , this term is not present in (71). On the left-hand-side of (70), the first term is the force felt by the particle at point  $x$  from the repulsion of all the other particles in the *same* gas. The second term is the force felt by the same particle from the repulsion of the *other* gas. The factor  $\frac{1}{2}$  shows explicitly that the inter-gas interaction is weaker than the intra-gas one. This makes a crucial difference with the situation previously studied in the literature where the interaction between the gases is the same as within each gas, see for instance [57, 63–65] (in these papers the interaction is the same since the two gases come from one global Coulomb gas cut in two parts).

To illustrate the impact of this factor  $\frac{1}{2}$ , let us look at the situation  $\mu = 0$ . The solution  $\rho_\Gamma$  of (70) and (71) is the typical density of eigenvalues  $\{\Gamma_n\}$  (which is also the density that dominates the denominator in (68)). This density was derived in [31]<sup>10</sup>, and is expressed in terms of a cubic root. On the other hand, equations similar to (70) and (71), but with the same factor in front of the two integrals were studied in [65] (there the number of eigenvalues of Wishart matrices larger than  $\gamma$  is studied). In this case, the density is expressed in terms of a square root, as it is often the case in random matrix theory [51, 52].

Let us denote  $\rho_\Gamma^*(x; \mu)$  and  $\rho_t^*(x; \mu)$  the solutions of the saddle point equations (70) and (71). We can then estimate the generating function (68) as<sup>11</sup>

$$G_\gamma(\mu) \underset{N \rightarrow \infty}{\sim} e^{-\frac{\beta N^2}{2} \Phi_\gamma(\mu)}, \tag{72}$$

where

$$\Phi_\gamma(\mu) = \mathcal{E}[\rho_\Gamma^*(x; \mu), \rho_t^*(x; \mu)] + \mu \int \frac{dx}{x} \rho_\Gamma^*(x; \mu) - \mathcal{E}[\rho_\Gamma^*(x; 0), \rho_t^*(x; 0)]. \tag{73}$$

In this expression, the last term comes from the denominator of (68). Equation (72) shows that the cumulants generating function  $G_\gamma(\mu)$  takes a large deviations form, with a large deviations function  $\Phi_\gamma(\mu)$ . In order to compute this function, one should in principle compute the double integrals in the energy functional (64). However, it is simpler to use the thermodynamic identity [60, 66]

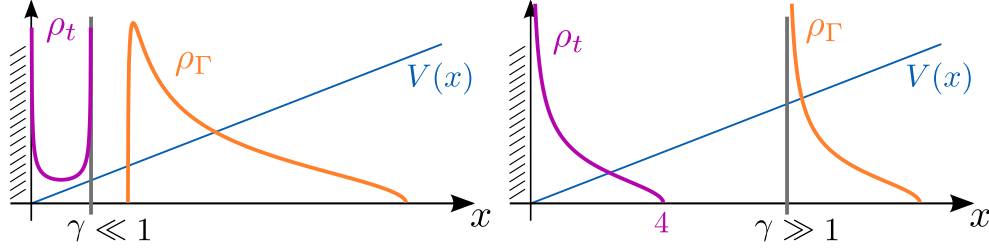
$$\frac{d\Phi_\gamma}{d\mu} = \int \frac{dx}{x} \rho_\Gamma^*(x; \mu), \tag{74}$$

which is the analogous for the generating function of another thermodynamic identity introduced in the computation of the distribution of linear statistics [67, 68]. This identity allows us to easily study the cumulants of  $s$ . Indeed, the cumulant generating function is

$$\ln G_\gamma(\mu) = \sum_{k=1}^{\infty} \frac{1}{k!} \left( -\frac{\beta N^2 \mu}{2} \right)^k \langle s^k \rangle_c \simeq -\frac{\beta N^2}{2} \Phi_\gamma(\mu), \tag{75}$$

<sup>10</sup> The result for any finite number of channels  $N$  is given in [29].

<sup>11</sup> The notation  $X \underset{N \rightarrow \infty}{\sim} e^{-\frac{\beta N^2}{2} Y}$  means that  $\lim_{N \rightarrow \infty} \frac{-2}{\beta N^2} \ln X = Y$ .



**Figure 3.** Sketch of the densities of the two Coulomb gases, solutions of (70) and (71) in the two limiting cases. Left: regime of weak absorption ( $\gamma \ll 1$ ). Right: regime of strong absorption ( $\gamma \gg 1$ ).

where we have denoted  $\langle s^k \rangle_c$  the  $k$ th cumulant of  $s$ . The cumulants can thus be obtained by differentiating  $\Phi_\gamma$  with respect to  $\mu$ . For example, the first two cumulants are

$$\langle s \rangle_c = \langle s \rangle \simeq \left. \frac{d\Phi_\gamma}{d\mu} \right|_{\mu=0}, \quad (76)$$

$$\langle s^2 \rangle_c = \text{Var}(s) \simeq -\frac{2}{\beta N^2} \left. \frac{d^2\Phi_\gamma}{d\mu^2} \right|_{\mu=0}. \quad (77)$$

These derivatives can be conveniently computed from the thermodynamic identity (74).

We have reduced the problem of finding the cumulants of the Wigner time delay to finding the solutions of the saddle point equations (70) and (71). This is however a complex problem, in particular due to the difference of prefactors in front of the integrals (as discussed above). Henceforth, we will study the two limiting cases of weak and strong absorption, in which the problem simplifies.

### 3.1. Weak absorption

Let us first focus on the regime of weak absorption  $\gamma \ll 1$ . In this limit the left gas, described by the density  $\rho_t$ , is confined in a small interval  $[0, \gamma]$ , as illustrated in figure 3 (left). Therefore, the force balance for this gas (71) is dominated by the repulsion within the density  $\rho_t$ . Both the effect of the other gas and of the confining potential can thus be neglected. This will allow us to find the solution  $\rho_t^*$  of (71), which can then be used to solve (70).

Let us formalise this discussion by expanding the force balance equations (70) and (71) in powers of  $\gamma$ . First, we introduce the rescaled density (also normalised to unity)

$$\tilde{\rho}_t(u) = \gamma \rho_t(\gamma u), \quad u \in [0, 1], \quad (78)$$

in terms of which equation (71) becomes

$$\frac{1}{\gamma} \int_0^1 du' \frac{\tilde{\rho}_t(u')}{u-u'} = \frac{1}{2} + \frac{1}{2} \int dx \frac{\rho_\Gamma(x)}{x-\gamma u} = \frac{\gamma}{2} + \frac{\gamma}{2} \sum_{n=0}^{\infty} (\gamma u)^n \int dx \frac{\rho_\Gamma(x)}{x^{n+1}}. \quad (79)$$

We can solve this equation in terms of the constants  $\int dx \rho_\Gamma(x)/x^{n+1}$ , which should be determined self-consistently later (however as we shall see, these terms will not contribute at the leading order in  $\gamma$ ). The first of these terms is  $\int dx \rho_\Gamma(x)/x$ , which is the rescaled Wigner time delay  $s$ . At leading order in  $\gamma$ , equation (79) becomes

$$\int_0^1 du' \frac{\tilde{\rho}_t(u')}{u-u'} = \frac{\gamma}{2}(1+s) + \mathcal{O}(\gamma^2), \quad u \in [0,1]. \quad (80)$$

This integral equation can be solved explicitly by using an inversion formula due to Tricomi [69] (see appendix). This procedure yields

$$\tilde{\rho}_t(u) = \frac{1}{\pi\sqrt{u(1-u)}} \left[ 1 + \frac{\gamma}{2}(1+s) \left( \frac{1}{2} - u \right) + \mathcal{O}(\gamma^2) \right]. \quad (81)$$

We can now use this result in the equation for the density  $\rho_\Gamma$  (70), which becomes

$$\begin{aligned} \int dx' \frac{\rho_\Gamma(x')}{x-x'} &= \frac{1}{2} \left( 1 - \frac{\mu}{x^2} - \int_0^1 du \frac{\tilde{\rho}_t(u)}{x-\gamma u} \right) \\ &= \frac{1}{2} \left[ 1 - \frac{1}{x} - \frac{\mu + \gamma/2}{x^2} + \mathcal{O}(\gamma^2) \right]. \end{aligned} \quad (82)$$

Remarkably, the unknown parameter  $s = \int dx \rho_\Gamma(x)/x$  cancels at first order in  $\gamma$ . As we could expect, this last integral equation is similar to the one studied in the case without absorption ( $\gamma = 0$ ) [26]. At leading order, the effect of the absorption is simply to shift the argument  $\mu$  of the generating function by  $\gamma/2$ . Therefore, we can easily relate the cumulant generating function for  $\gamma > 0$  to the one at  $\gamma = 0$ ,

$$\Phi_\gamma(\mu) = \Phi_0(\mu + \gamma/2) + \mathcal{O}(\gamma^2), \quad (83)$$

which we can equivalently express as

$$\Phi_\gamma(\mu) = \Phi_0(\mu) + \frac{\gamma}{2} \frac{d\Phi_0}{d\mu}(\mu) + \mathcal{O}(\gamma^2). \quad (84)$$

From this identity between the two generating functions, we can straightforwardly deduce a relation between the cumulants of the Wigner time delay with and without absorption:

$$\boxed{\langle s^n \rangle_c = \langle s^n \rangle_c^{(\gamma=0)} - \frac{\gamma\beta N^2}{4} \langle s^{n+1} \rangle_c^{(\gamma=0)} + \mathcal{O}(\gamma^2).} \quad (85)$$

We recall that this relation holds for large  $N$ . Equation (85) expresses the  $n$ th order cumulant of the rescaled Wigner time delay at weak absorption  $\gamma > 0$  in terms of the  $n$ th and  $(n+1)$ th order cumulants at zero absorption. The cumulants  $\langle s^n \rangle_c^{(\gamma=0)}$  at zero absorption being known [25], we can straightforwardly apply the relation (85) to deduce the ones in the presence of weak absorption. For the first two cumulants, this yields the expressions (5) given in the introduction.

### 3.2. Strong absorption

We now analyse the regime of strong absorption  $\gamma \gg 1$ . In this case, the right gas is pushed towards the right by the wall at  $x = \gamma$ , as shown in figure 3 (right). The interaction between the two gases thus becomes weak compared to the one within each gas.

To formalise this, let us shift the density  $\rho_\Gamma$ :

$$\tilde{\rho}_\Gamma(x) = \rho_\Gamma(x + \gamma). \quad (86)$$

Making the substitution  $x = \gamma + y$  into the force balance equation (70), we obtain

$$\int dy' \frac{\tilde{\rho}_\Gamma(y')}{y - y'} = \frac{1}{2} - \frac{\mu}{2(y + \gamma)^2} - \frac{1}{2} \int dt \frac{\rho_t(t)}{y + \gamma - t}. \quad (87)$$

We see on this expression that if we expand in powers of  $1/\gamma$ , the term proportional to  $\mu$  will be of order  $\mathcal{O}(\gamma^{-2})$ , subleading compared to the last term. In order to compensate this fact, we rescale the parameter  $\mu$  by a factor  $\gamma$ :

$$\mu = \gamma \tilde{\mu}. \quad (88)$$

In terms of this new parameter, equation (87) becomes

$$\int dy' \frac{\tilde{\rho}_\Gamma(y')}{y - y'} = \frac{1}{2} - \frac{\tilde{\mu}}{2\gamma} - \frac{1}{2\gamma} + \mathcal{O}(\gamma^{-2}), \quad (89)$$

where we have used that  $\rho_t$  is normalised to unity. We can use Tricomi's theorem (see appendix) to solve this equation. Let us denote  $[a, b]$  the support of  $\tilde{\rho}_\Gamma$ , we obtain

$$\tilde{\rho}_\Gamma(y) = \frac{1}{\pi \sqrt{(y-a)(b-y)}} \left[ 1 + \frac{1}{2} \left( 1 - \frac{\tilde{\mu} + 1}{\gamma} \right) \left( \frac{a+b}{2} - y \right) + \mathcal{O}(\gamma^{-2}) \right]. \quad (90)$$

Since there is no repulsion from  $y = 0$ , we have  $a = 0$ . The value of  $b$  can be determined by imposing that the density vanishes at  $y = b$ . This means that the bracket in (90) is zero for  $y = b$ . Hence

$$b = 4 + \frac{4(1 + \tilde{\mu})}{\gamma} + \mathcal{O}(\gamma^{-2}), \quad (91)$$

and the density (90) takes the form

$$\tilde{\rho}_\Gamma(y) = \frac{1}{2\pi} \sqrt{\frac{b-y}{y}} \left( 1 - \frac{\tilde{\mu} + 1}{\gamma} + \mathcal{O}(\gamma^{-2}) \right). \quad (92)$$

From the thermodynamic identity (74), we deduce the expression of the cumulant generating function in terms of  $\tilde{\rho}_\Gamma$ :

$$\frac{d\Phi_\gamma}{d\mu} = \frac{1}{\gamma} \frac{d\Phi_\gamma}{d\tilde{\mu}} = \int_0^b \frac{dy}{y + \gamma} \tilde{\rho}_\Gamma(y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\gamma^{n+1}} \int dy \tilde{\rho}_\Gamma(y) y^n. \quad (93)$$

Since  $\tilde{\rho}_\Gamma$  is normalised to unity, the first term of this series is  $1/\gamma$ . We thus have

$$\frac{1}{\gamma} \frac{d\Phi_\gamma}{d\tilde{\mu}} = \frac{1}{\gamma} - \frac{1}{\gamma^2} \int dy \tilde{\rho}_\Gamma(y) y + \frac{1}{\gamma^3} \int dy \tilde{\rho}_\Gamma(y) y^2 + \mathcal{O}(\gamma^{-4}). \quad (94)$$

Using the expression of the density  $\tilde{\rho}_\Gamma$  (90), we obtain

$$\frac{1}{\gamma} \frac{d\Phi_\gamma}{d\tilde{\mu}} = \frac{1}{\gamma} - \frac{1}{\gamma^2} + \frac{1 - \tilde{\mu}}{\gamma^3} + \mathcal{O}(\gamma^{-4}). \quad (95)$$

From this expansion, we deduce the first two cumulants by taking derivatives of the generating function,

$$\langle s \rangle = \left. \frac{d\Phi_\gamma}{d\mu} \right|_{\mu=0} = \left. \frac{1}{\gamma} \frac{d\Phi_\gamma}{d\tilde{\mu}} \right|_{\tilde{\mu}=0} = \frac{1}{\gamma} - \frac{1}{\gamma^2} + \frac{1}{\gamma^3} + \mathcal{O}(\gamma^{-4}), \quad (96)$$

$$\text{Var}(s) = -\frac{2}{\beta N^2} \left. \frac{d^2 \Phi_\gamma}{d\mu^2} \right|_{\mu=0} = -\frac{2}{\beta N^2 \gamma^2} \left. \frac{d^2 \Phi_\gamma}{d\tilde{\mu}^2} \right|_{\tilde{\mu}=0} = \frac{2}{\beta N^2 \gamma^4} + \mathcal{O}(\gamma^{-5}), \quad (97)$$

which correspond to the expression (6) given in the introduction.

#### 4. Conclusion

We have considered the scattering of waves by a chaotic absorbing cavity, perfectly coupled to  $N$  channels. Within the random matrix theory framework, we have derived the distribution of the Wigner–Smith time delay matrix  $\mathcal{Q}$  for any absorption rate  $\gamma$ . This result thus extends the one of Brouwer *et al* [22, 23] obtained for zero absorption  $\gamma = 0$ , to any absorption  $\gamma > 0$ . Our distribution is expressed either in terms of an integral over a  $N \times N$  positive Hermitian matrix, or over its eigenvalues. Although providing the distribution of  $\mathcal{Q}$  in the presence of absorption in most situations, our derivation should still be extended to yield the distribution for odd number of channels in the orthogonal class  $\beta = 1$ .

We have shown how our distribution can be interpreted, in the limit of many channels  $N \rightarrow \infty$ , in terms of two interacting Coulomb gases. We have applied this formalism to analyse the cumulants of the Wigner time delay  $\tau_W = \text{tr} \mathcal{Q}/N$ . In particular, we have obtained the first cumulants in the two limits of weak ( $\gamma \rightarrow 0$ ) and strong ( $\gamma \rightarrow \infty$ ) absorption. Furthermore, we have established a relation between the cumulants of  $\tau_W$  at weak absorption and the ones at zero absorption.

It would be interesting to see if one could derive the expression of the cumulants of  $\tau_W$  for any number  $N$  of channels from our new distribution, thus extending the results known at  $\gamma = 0$  [25]. The double Coulomb gas method that we have introduced in this paper could probably be extended to find the full distribution of the Wigner time delay in the presence of absorption (for  $N \gg 1$ ), as it was done for the case without absorption [26]. The situation is however more complex here as it requires a more detailed analysis of the saddle point equations (70) and (71). Our technique could also be used to study other linear statistics involving the Wigner–Smith matrix, such as trace of higher powers  $\text{tr} \mathcal{Q}^k$  and their correlators, which have been computed for  $\gamma = 0$  [70].

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#### Appendix. Tricomi’s theorem

Tricomi’s theorem gives an explicit form for the solution of integral equations of the type

$$\oint dx' \frac{f(x')}{x - x'} = g(x), \quad x \in \text{Supp}(f), \quad (\text{A.1})$$

where  $f$  denotes a principal value integral. If we assume that the support of the solution  $f$  has a compact support  $[a, b]$ , it can be expressed as [69]

$$f(x) = \frac{1}{\pi\sqrt{(x-a)(b-x)}} \left[ A + \int \frac{dt}{\pi} \frac{\sqrt{(t-a)(b-t)}}{t-x} g(t) \right], \quad (\text{A.2})$$

where  $A = \int_a^b f(x) dx$  is a constant. In the situation considered in this paper, the function  $f$  is a density of eigenvalues, normalised to unity, thus  $A = 1$ .

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