Bayesian inference for Gaussian models: Inverse problems and evolution equations

Yan, D.

Author: Yan, D.
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Appendix

Appendix A

Mathematical Tools

In this appendix we collect the mathematical elements, mainly from operator theory, that serve as the underlying language and building blocks for this thesis. They are from well established fields and can be found in textbooks and monographs. Hence, results will be present, and proofs are referred to the literature.

Operators are ubiquitous in this thesis, as one main component of the Gaussian linear model, the transform $\mathcal{A}$, is an operator. In particular, compact operators is of great importance, which is demonstrated by the following examples. First, the ill-posedness in a large class of linear inverse problems is characterised as the compactness of transform operators. Second, a Gaussian measure is a proper probability measure (instead of a generalised stochastic process) only when its covariance operator is of trace class, which is necessarily compact. Third, an element in a compact space can be well approximated by a finite-dimensional subspace, and the error estimate is closely related to the compactness. Besides the aforementioned cases, there are other places where the compactness is leveraged.

In this section, we collect the necessary information on operator theory, with special attention to compact operators. All the materials are standard and can be found in many textbooks, e.g. [102].

First let us summarize the common notations for operators. Let X, Y be normed spaces over the field $\mathbb{R}$. A linear operator $\mathcal{T}$ from X to Y is a linear mapping from the *domain* of $\mathcal{T}$, i.e. a subspace of X denoted by $\operatorname{Dom}\mathcal{T}$, into Y. The image of $\mathcal{T}$ is called *range*, i.e. $\operatorname{Ran}\mathcal{T} = \mathcal{T}(\operatorname{Dom}\mathcal{T}) = \{\mathcal{T}f : f \in \operatorname{Dom}\mathcal{T}\}$. A linear operator from X to $\mathbb{R}$ is a linear *functional*. The notation $\mathcal{T} : X \to Y$ is understood as $\operatorname{Dom}\mathcal{T} = X$ and $\operatorname{Ran}\mathcal{T} \subseteq Y$, unless the domain is given explicitly.

An operator is *injective* precisely when $\mathcal{T}f = 0$ implies $f = 0$. For an injective operator, the *inverse* $\mathcal{T}^{-1}$ of $\mathcal{T}$ is given by

$$\operatorname{Dom}\mathcal{T}^{-1} = \operatorname{Ran}\mathcal{T}, \quad \mathcal{T}^{-1}g = f, \quad \text{for } g = \mathcal{T}f \in \operatorname{Ran}\mathcal{T}.$$

The space of bounded linear operators form X to Y is denoted as $B(X, Y)$, i.e.

$$B(X,Y) := \left\{ \mathcal{T} : X \to Y \mid \text{linear and } \|\mathcal{T}\|_{X\to Y} := \sup_{h\in X:\|h\|\le 1} \|\mathcal{T}h\|_Y < \infty \right\},$$

where $\|\cdot\|_{X\to Y}$ is the operator norm and $\|\cdot\|$ may be used if no danger. If $X = Y$, we write $L(X)$.

Definition A.1 (Adjoints)**.** Let X, Y be Banach spaces. The *adjoint* $\mathcal{T}^*$ of a densely defined (not necessarily bounded) linear operator $\mathcal{T} : X \to Y$ is the operator uniquely determined by

$$\begin{aligned} T^* y^* &= x^*, \\ \langle y^*, Tx \rangle &= \langle x^*, x \rangle, \quad \forall x \in \operatorname{Dom} \mathcal{T}. \end{aligned}$$

An densely defined operator $\mathcal{S} : X \to X$ is *self-adjoint* if $\operatorname{Dom}\mathcal{S} = \operatorname{Dom}\mathcal{S}^*$ and $\langle \mathcal{S}h, g \rangle = \langle h, \mathcal{S}^* g \rangle$, for all $h, g \in \operatorname{Dom}\mathcal{S}$.

Remark A.2. If X and Y are Hilbert spaces, the dual spaces X^* and Y^* can be identified with the original space by Riesz representation theorem. If the operator $\mathcal{T} : X \to Y$ is bounded, then the definition above is equivalent to the standard definition of adjoints on Hilbert spaces, that there exists a unique operator $\mathcal{T}^* : Y \to X$ such that

$$\langle \mathcal{T}x, y \rangle_Y = \langle x, \mathcal{T}^* y \rangle_X,$$

for all $x \in X$ and $y \in Y$.

Definition A.3 (Positivity)**.** An operator $\mathcal{T}$ on a Hilbert space is called *positive*, denoted by $\mathcal{T} \geq 0$, if $\langle \mathcal{T}h, h \rangle \geq 0$, for all $h \in \operatorname{Dom}\mathcal{T}$. For two positive operators $\mathcal{S}, \mathcal{T}$, we write $\mathcal{S} \geq \mathcal{T}$ if $\operatorname{Dom}\mathcal{S} \subset \operatorname{Dom}\mathcal{T}$ and $\mathcal{S} - \mathcal{T} \geq 0$ on $\operatorname{Dom}\mathcal{S}$. We also write $\mathcal{S} = \mathcal{T}$ if $\mathcal{S} \geq \mathcal{T}$ and $\mathcal{T} \geq \mathcal{S}$.

If the above properties hold up to independent constants, then we use the notations $\mathcal{S} \lesssim \mathcal{T}$ and $\mathcal{S} \simeq \mathcal{T}$.

A.1 Miscellaneous Lemmas

In this section, we collect a few useful lemmas.

The following lemma is known the bounded linear transform (BLT) theorem,(see Theorem I.7, [81]).

Lemma A.4 (BLT theorem)**.** *Let $\mathcal{T}$ be a bounded linear operator from $(X, \|\cdot\|_X)$ to a complete normed space Y. Then there exists a unique bounded extension $\widetilde{\mathcal{T}}$ of $\mathcal{T}$ from the completion of X under $\|\cdot\|_X$ to Y.*

The following lemma is a direct consequence of Hahn-Banach theorem.

Lemma A.5. *Given a normed space $(E, \|\cdot\|)$ with its topological dual E^*, the following holds*

$$\|x\| = \sup_{f \in U(E^*)} |\langle f, x \rangle|.$$

Using positivity, we have another characterisation of operator norms on Hilbert spaces.

Lemma A.6. *Let $\mathcal{T}$ be an positive element in $B(H)$. Then,*

$$\|\mathcal{T}\|= \sup_{\|x\|\leq 1} \langle \mathcal{T}x, x\rangle.$$

The following result provides the soundness to Gelfand triples.

Lemma A.7. *Let G and H be two Banach spaces such that G is a dense subset of H, and the embedding $\iota : G \to H, g \mapsto g$ is continuous. Then, the following hold.*

(i) The inclusion mapping $\widetilde{\iota} : H^* \to G^*,\ \ell \mapsto \ell|_G$, where $\ell|_G$ is the restriction of ℓ to set G, is continuous. In particular,

$$\langle \ell, g\rangle_{H^*\times H} = \langle \ell|_G, g\rangle_{G^*\times G}, \quad \forall \ell \in H^*,\ \forall g \in G. \tag{A.1}$$

(ii) H^* is dense in G^*, if G is reflexive.

In particular, if H is a Hilbert space and G is reflexive, we have

$$G \subset H = H^* \subset G^*.$$

Proof. First we show the continuity of $\widetilde{\iota}$. Notice that for all $g \in G$, $\|g\|_H \lesssim \|g\|_G$, because of the continuity of ι. For any $\ell \in H^*$, we have

$$|\ell(g)| \lesssim \|h\|_H \|g\|_G.$$

Let $\widetilde{\ell}$ be the restriction of ℓ to the subset $G \subset H$. Then, $\widetilde{\ell} \in G^*$ such that

$$\widetilde{\ell}(g) = \ell(g), \quad \forall g \in G, \tag{A.2}$$

and

$$\|\widetilde{\ell}\|_{G^*} \leq \|\ell\|_{X^*}, \quad \forall \ell \in H^*.$$

In addition, $\widetilde{\ell} = 0$ implies $\ell = 0$. This is because of (A.2) and the density of G in H. Hence the inclusion mapping $\widetilde{\iota} : \ell \to \widetilde{\ell}$ is injective and continuous, and (A.1) holds.

Now we are going to show that H^* is dense in G^* by contradiction. If the statement is not true, then the closure of H^* in G^* is a proper closed subspace of G^*. By Hahn-Banach theorem, there exists a non-zero functional $\varphi_g \in (G^*)^*$ such that $\varphi_g(\widetilde{\ell}) = 0$ for all $\widetilde{\ell} \in \widetilde{\iota}(H^*) \subset G^*$. Because of reflexivity, the functional can be identified with an element $g \in G$, such that $\varphi_g(\widetilde{\ell}) = \widetilde{\ell}(g) = 0$, for all $\widetilde{\ell} \in \widetilde{\iota}(H^*) \subset G^*$. Due to (A.1), $\ell(g) = 0$, for all $\ell \in H^*$. Since $g \in G \subset H$, it implies that $g = 0$, which contradicts to $\varphi_g \neq 0$.

□

An embedding of Hilbert spaces naturally gives rise to an isometric isomorphism, which is useful in several occasions in this thesis. Meanwhile, it also shares some similar flavour of Lemma A.7.

Lemma A.8. *Assume that Hilbert space H is a dense subspace of Hilbert space X such that $\|h\|_H \geq \|h\|_X$, for all $h \in H$, and let the canonical embedding be*

$$\iota : H \to X, \ h \mapsto h.$$

Then,

$$\mathcal{U} = (\iota\iota^*)^{-1/2} : \mathrm{Dom}\,\mathcal{U} \subset X \to X,$$

where $\mathrm{Dom}\,\mathcal{U} = H$, *is an isometric isomorphism, i.e.* $\|\mathcal{U}h\|_X = \|h\|_H$.

Proof. This proof is adopted from Theorem IV.1.12, [63].

Since ι is compact, so is $\mathcal{S} = \iota\iota^*$. Furthermore, $\mathcal{S} : X \to X$ is self-adjoint and positive, and $\mathrm{Ran}\,\mathcal{S} \subset H$. We can define a self-adjoint operator $\mathcal{T} = \mathcal{S}^{-1}$ on domain $\mathrm{Dom}\,\mathcal{T} = \mathrm{Ran}\,\mathcal{S} \subset H$, such that

$$\langle h, g \rangle_H = \langle \mathcal{T}h, g \rangle_X, \tag{A.3}$$

for all $h \in \mathrm{Dom}\,\mathcal{T}$ and $g \in H$.

Using spectral theorem, define an operator $\mathcal{U} = (\mathcal{T})^{1/2}$, whose domain $\mathrm{Dom}\,\mathcal{U}$ is the closure of $\mathrm{Dom}\,\mathcal{T}$ with respect to the norm

$$\|\mathcal{U}h\|_X = \sqrt{\langle \mathcal{T}h, h \rangle_X} = \|h\|_H.$$

The domain $\mathrm{Dom}\,\mathcal{U}$ a closed set in H. We are going to show in fact $\mathrm{Dom}\,\mathcal{U} = H$ by contradiction. Assume $\mathrm{Dom}\,\mathcal{U} \subsetneq H$. Then by Hahn-Banach theorem, there exists an element $h_0 \in H$ such that $\langle g, h_0 \rangle_H = 0$ for all $g \in \mathrm{Dom}\,\mathcal{U}$, and in particular, all $g \in \mathrm{Dom}\,\mathcal{T}$. Due to (A.3), we have $\langle \mathcal{U}g, h_0 \rangle_X = 0$, for all $g \in \mathrm{Dom}\,\mathcal{U}$. Since $\mathrm{Ran}\,\mathcal{U} = X$, we conclude that $h_0 = 0$, which leads to a contradiction. □

A.2 Pseudo-Inverse

Let H be a Hilbert space and G be a normed space.

Definition A.9. Let $\mathcal{T} \in B(H, G)$, and $\mathrm{Ker}\,\mathcal{T} = \{h \in H \mid \mathcal{T}h = 0\}$. The *pseudo-inverse* is defined as:

$$\mathcal{T}^{-1} := \left(\mathcal{T}|_{(\mathrm{Ker}\,\mathcal{T})^\perp}\right)^{-1} : \mathcal{T}\big((\mathrm{Ker}\,\mathcal{T})^\perp\big) = \mathcal{T}(H) \to (\mathrm{Ker}\,\mathcal{T})^\perp,$$

which is bijective by construction.

Remark A.10. For $g \in \mathcal{T}(H)$, one can let $\mathcal{T}^{-1}g \in H$ be the solution of operator equation $\mathcal{T}h = g$ with the minimal norm. This gives an equivalent definition of pseudo-inverse.

When $\mathcal{T}$ has a genuine inverse, it induces a inner product on its image, as shown in the following lemma.

Lemma A.11. *Suppose that $\mathcal{T} : H \to G$ is an injective linear operator. Then, the range $\mathcal{T}(H) : \mathrm{Ran}\,\mathcal{T} \subset G$ of $\mathcal{T}$ is a Hilbert space equipped with the inner product*

$$\langle x, y \rangle_{\mathcal{T}(H)} := \left\langle \mathcal{T}^{-1}x, \mathcal{T}^{-1}y \right\rangle_H, \quad x, y \in \mathcal{T}(H). \tag{A.4}$$

In addition, $\mathcal{T} : H \to T$ is bounded and its adjoint is $\mathcal{T}^* = \mathcal{T}^{-1}$.

Proof. Since $\mathcal{T}$ is injective, the inner product given above is well-defined. Let $\{x_n\}$ be a Cauchy sequence in T. Then $x_n = \mathcal{T}h_n$ with $h_n \in H$ and $\{h_n\}$ is Cauchy as well because $\|h_m - h_n\|_H = \|x_m - x_n\|_T$. Since H is Hilbert, there exists a h such that $h_n \to h$. Therefore, there exists a vector $x = \mathcal{T}h \in T$ and

$$\|h_n - h\|_H = \|x_n - x\|_T \to 0, \quad \text{as } n \to \infty.$$

The boundedness and the adjoint follow directly from the construction. □

The previous lemma has the following implications on the pseudo-inverse.

Corollary A.12. *Let $\mathcal{T} \in B(H)$ and $\mathcal{T}^{-1}$ be the pseudo-inverse of $\mathcal{T}$. Then, the following statements hold.*

(i) $\mathcal{T}(H)$ is an Hilbert space equipped with the inner product induced by $\mathcal{T}$ given in (A.4).

(ii) If $\{e_k\}_k$ is an orthonormal basis of $(\operatorname{Ker}\mathcal{T})^\perp$, then $\{\mathcal{T}e_k\}_k$ is an orthonormal basis for the Hilbert space $\mathcal{T}(H)$ in (i).

The proof of the following lemma is given in Proposition C.0.5, [68].

Lemma A.13. *Let H_1, H_2 and G be Hilbert spaces, and let $\mathcal{T}_1 \in B(H_1, G)$ and $\mathcal{T}_2 \in B(H_2, G)$. If $\|\mathcal{T}_1^* g\|_1 = \|\mathcal{T}_2^* g\|_2$ for all $g \in G$, then for all*

$$g \in \operatorname{Ran}\mathcal{T}_1 = \operatorname{Ran}\mathcal{T}_2,$$

we have $\|\mathcal{T}_1^{-1} g\|_1 = \|\mathcal{T}_2^{-1} g\|_2$.

Proposition A.14. *Let $\mathcal{T} \in B(H, G)$ and $\mathcal{Q} = \mathcal{T}\mathcal{T}^* \in B(G)$. Then, for all*

$$g \in \operatorname{Ran}\mathcal{Q}^{1/2} = \operatorname{Ran}\mathcal{T},$$

we have $\|\mathcal{Q}^{-1/2} g\| = \|\mathcal{T}^{-1} g\|_H$.

Proof. Since $\mathcal{Q}$ is self-adjoint, the square root is defined via the spectral theorem. Furthermore, for all $g \in G$, we have

$$\left\|(\mathcal{Q}^{1/2})^* g\right\|^2 = \left\|\mathcal{Q}^{1/2} g\right\|^2 = \langle g, \mathcal{Q}g\rangle = \|\mathcal{T}^* g\|_H.$$

The rest follows from the previous lemma. □

A.3 Compact Operators

Let H, G be two Hilbert spaces. We denote the inner product of H by $\langle\cdot,\cdot\rangle_H$, or simply $\langle\cdot,\cdot\rangle$ when there is no confusion.

Definition A.15 (Compact operators)**.** An operator $\mathcal{T} : H \to G$ is compact if for any bounded sequence $\{h_n\}$ in H, the sequence $\{\mathcal{T}h_n\}$ in G contains a convergent subsequence. The space of compact operators in $L(H, G)$, equipped with the operator norm, is denoted by $S_\infty(H, G)$.

If $\mathcal{T} : H \to G$ is an compact operator, then $\mathcal{T}^*\mathcal{T}$ is compact, self-adjoint, and *non-negative*, i.e. $\langle \mathcal{T}^*\mathcal{T}h, h\rangle \geq 0$ for all $h \in H$. The *absolute value* of $\mathcal{T}$ is defined with the equality $|\mathcal{T}|= (\mathcal{T}^*\mathcal{T})^{1/2}$, where $(\mathcal{T}^*\mathcal{T})^{1/2}$ is the unique non-negative square root of $\mathcal{T}^*\mathcal{T}$ (see Theorem 7.4 in [102]). The positive eigenvalues of $|\mathcal{T}|$ are called the *singular values* of $\mathcal{T}$. In fact, singular values $\{s_j(\mathcal{T})\}$ encode great information about the operator $\mathcal{T}$.

Theorem A.16 (Singular Value Decomposition (Theorem 7.6, [102]))**.** *Let $\mathcal{T} : H \to G$ be a compact operator and $\{s_j\}$ denote the (possibly finite) non-decreasing sequence of the singular values of $\mathcal{T}$. There exists orthonormal sequence (h_j) from H and (g_j) from G such that for all $h \in H$ and $g \in G$,*

$$\mathcal{T}h = \sum_j s_j \langle h, h_j\rangle g_j, \qquad \mathcal{T}^*g = \sum_j s_j \langle g, g_j\rangle h_j,$$
$$|\mathcal{T}|h = \sum_j s_j \langle h, h_j\rangle h_j, \quad |\mathcal{T}^*|g = \sum_j s_j \langle g, g_j\rangle g_j.$$

The (h_j) and (g_j) are the eigenvectors of $|\mathcal{T}|$ and $|\mathcal{T}^*|$, respectively. In particular, $\mathcal{T}, |\mathcal{T}|, \mathcal{T}^*$ and $|\mathcal{T}^*|$ have the same singular values.

With the help of singular values, we can define the following spaces of operators.

Definition A.17 (Schatten Class)**.** For a compact operator $\mathcal{T} : H \to G$, the *p-Schatten norm*, $p \in [1, \infty)$, is defined with its singular values $\{s_j(\mathcal{T})\}$,

$$\|\mathcal{T}\|_p := \left(\sum_j |s_j(\mathcal{T})|^p \right)^{1/p}.$$

We denote by $S_p(H, G)$ the set of compact operators with finite p-Schatten norm.

Remark A.18. It is not difficult to show that $\|\mathcal{T}\|_\infty = s_1(\mathcal{T})$ from the definition.

The p-Schatten norms are authentic norms satisfying the triangle inequality. S_p spaces are similar to L^p spaces. For example, the spaces S_p are Banach spaces and in particular, S_2 is a Hilbert space. A version of the Hölder inequality also holds in S_p spaces. These properties are summarised in the following propositions.

Proposition A.19 (Lemma 10 and 14, XI.9, [25])**.** *Let $1 \leq p \leq p' \leq \infty$ and let $1/p + 1/q = 1$.*

(i) For $\mathcal{T}, \mathcal{U} \in S_p$, we have $\|\mathcal{T} + \mathcal{U}\|_p \leq \|\mathcal{T}\|_p + \|\mathcal{U}\|_p$.

(ii) S_p is complete under the norm $\|\cdot\|_p$. S_2 is an inner product space.

(iii) $S_p \subset S_{p'}$ and $\|h\|_p \geq \|h\|_{p'}$, if $h \in S_p$.

(iv) For $\mathcal{T} \in S_p$ and $\mathcal{U} \in S_q$, then $\|\mathcal{T}\mathcal{U}\|_1 \leq \|\mathcal{T}\|_p \|\mathcal{U}\|_q$.

Proposition A.20 (Theorem 7.8, [102])**.**

(i) If $\mathcal{T} \in S_p$ and $\mathcal{U} \in S_q$ $(p, q \in [1, \infty))$ and $1/r = 1/p + 1/q$, then

$$\|\mathcal{T}\mathcal{U}\|_r \leq 2^{1/r} \|\mathcal{T}\|_p \|\mathcal{U}\|_q.$$

(ii) If $\mathcal{T} \in L(H_1, H_2)$ and $\mathcal{U} \in S_p(H_0, H_1)$, then

$$\|\mathcal{T}\mathcal{U}\|_p \leq \|\mathcal{T}\| \|\mathcal{U}\|_p.$$

For $\mathcal{T} \in S_p(H_1, H_2)$ and $\mathcal{U} \in L(H_0, H_1)$, the corresponding assertion also holds.

Schatten class contains two important sets of compact operators.

- $S_2(H, G)$ is identical to the set of *Hilbert-Schmidt* operators. Namely, for any Hilbert-Schmidt operator $\mathcal{T} : H \to G$,

$$\|\mathcal{T}\|_2 = \|\mathcal{T}\|_{HS} := \sum_{i \in \mathcal{I}} \|\mathcal{T}\varphi_i\|_G^2 < \infty,$$

  where $\{\varphi_i\}_{i \in \mathcal{I}}$ is an orthonormal basis in H.

- $S_1(H, G)$ is identical to the set of *trace class* operators. Namely, for any operator $\mathcal{T} : H \to G$ of trace class, we have

$$\|\mathcal{T}\|_1 = \operatorname{Trace} \mathcal{T} := \sum_{i \in \mathcal{I}} \langle \sqrt{\mathcal{T}^*\mathcal{T}} \varphi_i, \varphi_i \rangle_H < \infty,$$

  where $\{\varphi_i\}_{i \in \mathcal{I}}$ is an orthonormal basis in H.

The two equalities above are obvious. Since the Hilbert-Schmidt norm and trace are both independent of the basis $\{\varphi_i\}_{i \in \mathcal{I}}$, the equalities are obtained by taking the basis to be the eigenbasis of $|\mathcal{T}|$. Furthermore, the following can be derived directly from (iii) in the previous proposition,

$$\|\mathcal{T}\| \leq \|\mathcal{T}\|_{HS} = \|\mathcal{T}^*\|_{HS} \leq \operatorname{Trace} \mathcal{T},$$

where the equality of Hilbert-Schmidt norms can be found in Theorem 6.9 in [102].

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