Bayesian inference for Gaussian models: Inverse problems and evolution equations

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Chapter 9

Linear Evolution Equations

Stochastic partial differential equations (SPDEs) provide a powerful toolkit to study dynamical systems with stochastic nature. In the last few decades, it has gained much interest due to the wide range of applications in physics and finance. One way to study SPDEs is using the theory of linear evolution equations in infinite-dimensional spaces. This analytical approach treats dynamical systems as vector-valued ordinary differential functions, whose state spaces are function spaces. In this chapter we collect the basic results from this approach. For a detailed treatment on the subject, we refer to the standard reference [23] and the more recent monographs [68, 85].

First we recall some standard definitions from stochastic processes. Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and a positive number $T < \infty$.

Definition 9.1. With an index set $\mathfrak{I} \subset \mathbb{R}_0^+$, a *filtration* $\{\mathcal{F}_t\}_{\mathfrak{I}}$ is a family of sub σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_s \subset \mathcal{F}_t$ if $s < t$. The following conditions are known as the *usual conditions*.

(i) Completeness: $A \in \mathcal{F}_0$, for all $A \in \mathcal{F}$ such that $\mathbb{P}(A) = 0$.

(ii) Right continuity: $\mathcal{F}_t = \mathcal{F}_{t+} := \cap_{s\in\mathfrak{I}:s>t}\mathcal{F}_s$, for all $t \in \mathfrak{I}$.

A filtration that satisfies the usual conditions is also called *normal*.

Definition 9.2. Let $(E, \|\cdot\|)$ be a separable Banach space. An E-valued process $\{M(t) : t \geq 0\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$ is a $\mathcal{F}_t$-*martingale* if

(i) $\mathbb{E}\|M(t)\| < \infty$, for all $t \geq 0$,

(ii) $M(t)$ is $\mathcal{F}_t$-measurable for all $t \geq 0$,

(iii) and $\mathbb{E}(M(t)|\mathcal{F}_s) = M(s)$ almost surely in $\mathbb{P}$, for all $0 \leq s \leq t < \infty$.

In addition, denote the space of all E-valued continuous square integrable martingales by $\mathcal{M}_T^2(E)$, which is a Banach space equipped with the norm

$$\|M\|_{\mathcal{M}_T^2} := \sup_{t\in[0,T]} (\mathbb{E}\|M(t)\|^2)^{1/2} = (\mathbb{E}\|M(T)\|^2)^{1/2}.$$

For the proof, see, e.g. Proposition 3.10, [23].

9.1 $\mathcal{Q}$-Wiener Processes

The characteristic differing stochastic evolution systems from the deterministic ones is the appearance of stochastic noise. As the state of the system is infinite dimensional, the noise is also expected to be infinite dimensional. $\mathcal{Q}$-Wiener processes, a generalization of classical Wiener processes to vector-valued processes, will be used to model the infinite dimensional noise.

Let U be a *separable* infinite-dimensional Hilbert space, and $\mathcal{Q} \in L(U)$ be a bounded linear operator on U satisfying, for any $u, v \in U$,

(i) nonnegative: $\langle \mathcal{Q}u, u\rangle \geq 0$,

(ii) symmetric: $\langle \mathcal{Q}u, v\rangle = \langle u, \mathcal{Q}v\rangle$,

(iii) nuclear: $\operatorname{Trace} \mathcal{Q} < \infty$.

Definition 9.3. On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a (standard) $\mathcal{Q}$-Wiener process is a U-valued process $\{W(t) : \ t \in [0, T]\}$ satisfying

(i) $W(0) = 0$,

(ii) W is continuous almost surely in $\mathbb{P}$,

(iii) W has independent increments, i.e.

$$W(t_1), W(t_2) - W(t_1), \cdots, W(t_n) - W(t_{n-1}),$$

are independent, for all $0 \leq t_1 < \cdots t_n \leq T$,

(iv) for all $0 \leq s \leq t \leq T$, $W(t) - W(s)$ is distributed as a centred Gaussian on U with covariance operator $(t-s)\mathcal{Q}$.

$\mathcal{Q}$ is called the covariance operator of $\mathcal{Q}$-Wiener process.

Remark 9.4. For a $\mathcal{Q}$-Wiener process $\{W(t) : t \in [0, T]\}$, there always exists a normal filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$ such that:

1. $W(t)$ is adapted to $\{\mathcal{F}_t\}_{0\leq t\leq T}$, i.e. $W(t)$ is $\mathcal{F}_t$-measurable for all $0 \leq t \leq T$,
2. and $W(t) - W(s)$ is independent of $\mathcal{F}_s$ for all $0 \leq s \leq t \leq T$.

See Proposition 2.1.13 in [68].

Similar to the Karhunen-Loève expansion of Gaussian elements, the $\mathcal{Q}$-Wiener process has a concrete representation.

Proposition 9.5 (Presentation of $\mathcal{Q}$-Wiener Process). *Let $\{e_i\}_{i\in\mathbb{N}}$ be the eigenfunctions of $\mathcal{Q}$ with the corresponding eigenvalues $\{q_i\}_{i\in\mathbb{N}}$. A U-valued process $W(t)$ is $\mathcal{Q}$-Wiener if and only if there exists a sequence of independent ordinary Wiener processes $\{W_i : i \in \mathfrak{I}\}$ with $\mathfrak{I} = \{i : q_i > 0\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that*

$$W(t) = \sum_{i\in\mathfrak{I}} \sqrt{q_i} e_i W_i(t), \quad t \in [0, T], \tag{9.1}$$

which converges in $L^2(\mathbb{P})$, and always has a $\mathbb{P}$-a.s. continuous version, i.e.

$$\mathbb{P}[W(t) \in C([0,T], U)] = 1.$$

Proof. See Proposition 2.1.10 in [68]. □

9.2 Stochastic Integrals in Hilbert Spaces

A *H-valued* stochastic integral with respect to $W^Q(t)$ can be developed in a similar manner as for the scalar valued ordinary stochastic integral. We outline the construction of the integral and summarize a few useful results. A detailed treatment can be found in Chapter 2 in [68] and Section 4.2 & 4.3 in [23].

Similar to ordinary stochastic integrals, the construction relies on a space of martingales and a class of elementary processes. Recall that the space $\mathcal{M}_T^2(H)$ of all H-valued continuous square integrable martingales $\{M(t) : t \in [0,T]\}$ is a Banach space equipped with the norm

$$\|M\|_{\mathcal{M}_T^2} := \sup_{t \in [0,T]} (\mathbb{E}\|M(t)\|^2)^{1/2} = (\mathbb{E}\|M(T)\|^2)^{1/2}.$$

The standard machinery to construct stochastic integrals can be summarised as follows.

(I) For a class $\mathcal{E}$ of elementary processes, define a linear mapping

$$\mathrm{Int} : \mathcal{E} \ni \Phi \to \int_0^t \Phi(s)\, dW(s) =: \Phi \cdot W(t),$$

which is an element in $\mathcal{M}_T^2(H)$.

(II) Find a norm on $\mathcal{E}$ such that $\mathrm{Int} : \mathcal{E} \to \mathcal{M}_T^2(H)$ is an isometry. Since $\mathcal{M}_T^2(H)$ is complete, Int is extended to the abstract completion $\overline{\mathcal{E}}$ of $\mathcal{E}$. Furthermore, an explicit representation is found for $\mathcal{E}$.

(III) Further extend the integral to local martingale by localization.

For completeness of the exposition, we also discuss some detail on the procedure. The readers familiar with ordinary stochastic integrals will immediately notice the similarity. Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a normal filtration, and denote $dt \otimes \mathbb{P}$ by $\mathbb{P}_T$, where dt is the Lebesgue measure. Let W^Q be the Q-Wiener process introduced in Section 9.1. In this section, $L(U,H)$ denotes the space of the bounded operators from U to H.

(I) An $L(U,H)$-valued process $\Phi(t), t \in [0,T]$ with normal filtration is *elementary* if

$$\Phi(t) = \sum_{m=0}^{k-1} \Phi_m \, \mathbb{1}_{(t_m, t_{m+1}]}(t), \quad t \in [0,T],$$

for some $0 < t_0 < \cdots < t_k = T$, $\Phi_m : \Omega \to L(U,H)$ is $\mathcal{F}_{t_m}$-measurable with respect to the strong Borel σ-algebra on $L(U,H)$, $0 \le m \le k-1$, and each

Φ_m takes only a finite number of values in $L(U, H)$. The stochastic integral of elementary processes is defined by

$$\mathrm{Int}(\Phi)(t) := \int_0^t \Phi(s)\, dW(s) := \sum_{m=0}^{k-1} \Phi_m(W(t_{m+1} \wedge t) - W(t_m \wedge t)), \quad t \in [0, T],$$

and it belongs to $\mathcal{M}_T^2(H)$.

(II) Recall that for $\Phi \in L(U, H)$, $\Phi \circ \mathcal{Q}^{1/2}$ is a Hilbert-Schmidt operator in $S_2(U, H)$ (see Section A.3). The previous stochastic integral satisfies the following *Itô isometry*,

$$\left\| \int_0^\cdot \Phi(s)\, dW(s) \right\|_{\mathcal{M}_T^2}^2 = \mathbb{E}\left(\int_0^T \|\Phi(s) \circ \mathcal{Q}^{1/2}\|_{HS}^2\, ds \right) =: \|\Phi\|_T^2. \tag{9.2}$$

Notice that $\|\cdot\|_T$ is only a semi-norm on $\mathcal{E}$, and two elementary processes belonging to one equivalent class are not necessarily $\mathbb{P}_T$-a.e. equal, because the equivalence only occurs on $\mathcal{Q}^{1/2}(U)$ $\mathbb{P}_T$-a.e.

Because of the Itô isometry,

$$\mathrm{Int} : (\mathcal{E}, \|\cdot\|_T) \to (\mathcal{M}_T^2, \|\cdot\|_{\mathcal{M}_T^2})$$

is an isomorphism, and consequently there uniquely exists an isometric extension of mapping Int to the abstract completion $\bar{\mathcal{E}}$ of $\mathcal{E}$ with respect to $\|\cdot\|_T$.

Now we prepare for the explicit presentation of $\bar{\mathcal{E}}$. Let $U_0 := \mathcal{Q}^{1/2}(U)$, which is a Hilbert space with the induced inner product $\langle \cdot, \cdot \rangle_0 := \langle \mathcal{Q}^{-1/2} \cdot, \mathcal{Q}^{-1/2} \cdot \rangle$ (see Lemma A.11). We denote the space of all Hilbert-Schmidt operators [1] from U_0 to H by L_2^0, which is a separable Hilbert space equipped with the norm

$$\|\Phi\|_{L_2^0}^2 = \|\Phi \mathcal{Q}^{1/2}\|_{HS}^2 = \mathrm{Trace}[(\Phi \mathcal{Q}^{1/2})(\Phi \mathcal{Q}^{1/2})^*]. \tag{9.3}$$

In particular, the space $L(U, H)$ can be embedded into L_2^0 by restricting the domain of operators to U_0. Let $L(U, H)_0 = \{T|_{U_0} : T \in L(U, H)\}$ denote the space of the restricted operators. Then, $L(U, H)_0 \subset L_2^0$ and

$$\|\Phi\|_T = \left[\mathbb{E} \int_0^T \|\Phi(s)\|_{L_2^0}^2\, ds \right]^{1/2},$$

for $\Phi \in \mathcal{E}$.

The final claim is that $\bar{\mathcal{E}}$ is given by

$$\begin{aligned} \mathcal{N}_W^2(0, T; H) :=& \{\Phi : [0, T] \times \Omega \to L_2^0 \quad | \quad \Phi \text{ is predictable and } \|\Phi\|_T < \infty\} \\ =& L^2([0, T] \times \Omega, dt \otimes \mathbb{P}; L_2^0), \end{aligned}$$

which we also write $\mathcal{N}_W^2(H)$ and $\mathcal{N}_W^2$ for brevity. It is not difficult to see $\mathcal{E} \subset \mathcal{N}_W^2$. In addition, because $\mathcal{N}_W^2$ is in fact a Bochner space (see Section 1.2.b in [49]), it is complete. The preceding claim is proved by showing that $\mathcal{E}$ is a dense subset of $\mathcal{N}_W^2$ (Proposition 2.3.8 in [68]).

[1] The consistent notation is $S_2^{\mathbb{U}}$ with $\mathbb{U} = \mathcal{Q}^{1/2}(U)$. Here we adopt the conventional notations from the SPDE literature.

(III) The integrals can be further extended to local martingales with the following class of integrands,

$$\mathcal{N}_W(0,T;H) := \left\{ \Phi : [0,T] \times \Omega \to L_2^0 \mid \Phi \text{ is predictable and} \right.$$
$$\left. \mathbb{P}\left(\int_0^T \|\Phi(s)\|_{L_2^0}^2 \, ds < \infty \right) = 1 \right\}.$$

The detail is omitted here, since we will only consider the martingale case. We only remark that the obvious relation $\mathcal{N}_W^2(0,T;H) \subset \mathcal{N}_W(0,T;H)$, and some results below will be stated with the more general space.

Following the procedure described above, the isometric extension of Int to $\mathcal{N}_W^2$,

$$\mathrm{Int} : \Phi \in \mathcal{N}_W^2 \mapsto \int_0^t \Phi(s) \, dW^Q(s), \quad t \in [0,T],$$

defines the stochastic integral.

Remark 9.6. Notice that the stochastic integral is defined with a class of predictable processes. However, in this thesis we will only consider the case of deterministic integrands.

Not surprisingly, the H-valued stochastic integral shares many similar properties as the ordinary stochastic integral. For example, the integration is interchangeable with other linear operations, as stated in Lemma 9.7 below.

Lemma 9.7 (Lemma 2.4.1 in [68]). *Let $\Phi \in \mathcal{N}_W(H)$ and $\mathcal{T} \in L(H, \widetilde{H})$, where $\widetilde{H}$ is another separable Hilbert space. Then the process $\mathcal{T}\Phi(t), t \in [0,T]$, is an element of $\mathcal{N}_W(\widetilde{H})$, and*

$$\mathcal{T}\left(\int_0^T \Phi(s) \, dW(s) \right) = \int_0^T \mathcal{T}(\Phi(s)) \, dW(s)$$

$\mathbb{P}$-almost surely.

In addition, the stochastic integral also admits a series representation, as in the following lemma.

Lemma 9.8 (Proposition 2.4.5 in [68]). *If $\Phi \in \mathcal{N}_W^2(H)$, then*

$$\int_0^t \Phi(s) \, dW(s) = \sum_{i \in \mathbb{N}} \sqrt{q_i} \int_0^t \Phi(s)(e_i) \, dW_i(s), \quad t \in [0,T],$$

$\mathbb{P}$-almost surely, where q_i, e_i, W_i are as in Proposition 9.5 and the sum on the right-hand side converges in $L^2(\mathbb{P})$ and realises in $C([0,T], U)$ almost surely.

For the detailed properties of stochastic integrals, we refer to Section 2.4 in [68] and Section 4.3 in [23].

9.3 Extension of Stochastic Integrals

In the previous section, we have developed the integration theory under the condition that the covariance $\mathcal{Q}$ is a trace class operator. In this section we are going to relax the condition, and consequently, the class of integrators of the stochastic integral is extended.

9.3.1 Cylindrical Wiener Process

We first extend the H-valued Wiener process.

Recall that the Wiener process in Section 9.1 admits the representation

$$W(t) = \sum_{k\in\mathbb{N}} \sqrt{q_k} W_k(t) e_k, \quad t \in [0,T],$$

where $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis for $U_0 = \mathcal{Q}^{1/2}(U)$ and $W_k(t), k \in \mathbb{N}$, are independent real-valued Wiener processes. The convergence of the series in $L^2(\mathbb{P})$ is due to the fact that the inclusion $U_0 \subset U$ is Hilbert-Schmidt (see Proposition 2.1.10, [68]).

Now let $\mathcal{Q}$ be an operator satisfying the properties in Section 9.1 but not necessarily nuclear, that is, $\mathrm{Trace}\,\mathcal{Q} = \sum_k q_k < \infty$ is not required. Let $U_0 = \mathcal{Q}^{1/2}(U)$ with the induced inner product and $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis for U_0. Let U_1 be another Hilbert space such that there exists a Hilbert-Schmidt embedding $\mathcal{J} : U_0 \to U_1$.

Remark 9.9. The space U_1 always exists. For example, let $U_1 = U$ and define

$$\mathcal{J} : U_0 \to U, \quad f \mapsto \sum_{k\in\mathbb{N}} \rho_k \langle f, e_k\rangle_U e_k,$$

with a sequence $\{\rho_k\}_{k\in\mathbb{N}}$ such that $\sum_k \rho_k^2 < \infty$. Then, $\mathcal{J}$ is injective and Hilbert-Schmidt, c.f. Section 3.3.

Then, there exists a Wiener process on the larger Hilbert space U_1, associated with the previously introduced $\mathcal{Q}$.

Proposition 9.10. *Under the previously introduced condition, define $\mathcal{Q}_1 := \mathcal{J}\mathcal{J}^*$. Then, $\mathcal{Q}_1$ is nonnegative, symmetric and nuclear. The series*

$$W(t) = \sum_{k\in\mathbb{N}} W_k(t) \mathcal{J} e_k, \quad t \in [0,T],$$

converges in $\mathcal{M}_T^2(U_1)$ and defines $\mathcal{Q}_1$-Wiener process on U_1. Furthermore,

$$\mathcal{Q}_1^{1/2}(U_1) = \mathcal{J}(U_0),$$

and for all $u_0 \in U_0$,

$$\|u_0\|_0 = \|\mathcal{Q}_1^{-1/2}\mathcal{J}u_0\|_1 =: \|\mathcal{J}u_0\|_{\mathcal{Q}^{1/2}(U_1)},$$

i.e. $\mathcal{J} : U_0 \to \mathcal{Q}^{1/2}(U_1)$ is an isometry.

The process $\{W(t) : t \in [0,T]\}$ introduced in Proposition 9.10 is called a *cylindrical Wiener process* with *covariance* $\mathcal{Q}$ in U (while it does not realise in U).

9.3.2 Stochastic Integral with Cylindrical Wiener Process

Let $W(t)$ be a cylindrical Wiener process with covariance $\mathcal{Q}$ as previously introduced. Known from Section 9.2, a process $\Phi(t)$ is integrable with respect to $W(t)$ if it is predictable, $\Phi(t) \in L_2(\mathcal{Q}_1^{1/2}(U_1), H)$ for all $t \in [0,T]$, and

$$\mathbb{P}\left(\int_0^T \|\Phi(s)\|^2_{L_2(\mathcal{Q}_1^{1/2}(U_1),H)}\, ds < \infty\right) = 1.$$

We adopt all the notations from Proposition 9.10 and recall $\mathcal{Q}_1^{1/2}(U_1) = \mathcal{J}(U_0)$. Then by the polarisation identity, for all $u, v \in U_0$,

$$\langle \mathcal{J}u, \mathcal{J}v\rangle_{\mathcal{Q}_1^{1/2}(U)} = \langle u, v\rangle_0,$$

which implies that $\mathcal{J}e_k$ is an orthonormal basis for $\mathcal{Q}_1^{1/2}(U_1)$. Consequently, we have

$$\Phi \in L_2^0 = L_2(\mathcal{Q}^{1/2}(U), H) \Longleftrightarrow \Phi \circ \mathcal{J}^{-1} \in L_2(\mathcal{Q}_1^{1/2}(U_1), H),$$

because

$$\|\Phi\|^2_{L_2^0} = \sum_k \langle \Phi e_k, \Phi e_k\rangle = \sum_k \langle \Phi \circ \mathcal{J}^{-1}\mathcal{J}e_k, \Phi \circ \mathcal{J}^{-1}\mathcal{J}e_k\rangle = \|\Phi \circ \mathcal{J}^{-1}\|^2_{L_2(\mathcal{Q}_1(U_0),H)}.$$

Then, for the cylindrical process $W(t)$, the integral can be defined as

$$\int_0^t \Phi(s)\, dW(s) := \int_0^t \Phi(s) \circ \mathcal{J}^{-1}\, dW(s), \quad t \in [0,T].$$

Apparently, the previously defined stochastic integral holds with the same class of integrands from Section 9.2. Therefore, the stochastic integral has been extended to cylindrical Wiener processes.

We end this section with several remarks. First, the integral is actually independent from the space U_1. This is because the Itô isometry (9.2) together with (9.3) is independent from the bigger space U_1. Second, when $\operatorname{Trace} \mathcal{Q} < \infty$, we can simply take $\mathcal{J}$ to be the identity map $\mathrm{id} : U_0 \to U$, and then the definition coincides with the integral developed Section 9.2. In both statements above, the important fact is that the stochastic integral is uniquely defined with the space $\mathcal{Q}^{1/2}(U)(= \mathcal{Q}_1^{1/2}(U_1))$, which is the reproducing kernel Hilbert space of the Gaussian measure $\mathcal{N}_U(0, \mathcal{Q})$ (the radonified Gaussian measure $\mathcal{N}_{U_1}(0, \mathcal{Q}_1)$, see Section 3.3).

9.4 Deterministic Evolution Equations

We start with introducing the dynamical systems without noise. Let H be a separable Hilbert space. An evolution equation in H is given by

$$\begin{cases} u'(t) + \mathcal{L}u(t) = f(t), \\ u(0) = u_0 \in H, \end{cases} \tag{9.4}$$

where $u : [0,T] \to H$ is a H-valued function, $\mathcal{L} : D(\mathcal{L}) \subset H \to H$ is a linear operator, unbounded in general, with a dense domain $D(\mathcal{L})$ in H, and $u'(t)$ is the strong derivative of $u(t)$, i.e.

$$u'(t) = \lim_{h\to 0} \frac{u(t+h) - u(t)}{h},$$

where the limit is taken in the topology of H.

The existence and uniqueness of (9.4) with the initial condition U_0 is known as the *deterministic* abstract (nonhomogeneous) Cauchy problem. It can be answered in the language of C_0-semigroup theory. When $\mathcal{L}$ is a infinitesimal generator of a C_0-semigroup $S(\cdot)$ in H and $U_0 \in L^p([0,T];H), p \in [1,\infty]$, the *strict* solution u of problem (9.4), i.e. a function u that belongs to $W^{1,p}([0,T];H) \cap L^p([0,T];D(\mathcal{L}))$ and satisfies (9.4), is given by the following *variation of constant formula*,

$$u(t) = S(t)U_0 + \int_0^t S(t-s)f(s)\,ds. \tag{9.5}$$

See Chapter 4 in [75] for the details.

9.5 Solutions of SPDEs

We now formally define the model (III.1). Assume that all Hilbert spaces in this section are separable. Recall the model

$$dX(t) + \mathcal{L}X(t)\,dt = f(t)\,dt + \frac{1}{\sqrt{n}} B\,dW^{\mathcal{Q}}(t), \tag{9.6}$$

with the initial condition $X(0) = u \in H$.

The items in (9.6) are defined as follows. $f : [0,T] \to H$ is a H-valued measurable function, $\mathcal{L} : D(\mathcal{L}) \subset H \to H$ is a densely defined unbounded linear operator, $B : U \to H$ is a linear operator, $W^{\mathcal{Q}}$ is a $\mathcal{Q}$-Wiener process, and u is a deterministic vector in H.

One key component in the development of SPDE is the stochastic integral with respect to $\mathcal{Q}$-Wiener process in Hilbert spaces, covered in Section 9.2. In addition, some language of functional analysis is also standard in this field. To ease the reading effort, we collect the minimal material on the syntax of functional analysis in Appendix A.

Concrete examples covered by the above model are the parabolic evolution equations in a bounded domain $\mathcal{D} \subset \mathbb{R}^d$ with a smooth boundary $\partial\mathcal{D}$, with time interval $[0,T]$. Time-space domain and boundary is denoted by $\mathcal{D}_T := [0,T] \times \mathcal{D}$ and $\Gamma_T := [0,T] \times \partial\mathcal{D}$ respectively. Let $\mathcal{L}$ be a strongly elliptic differential operator of order $2m$. That is, given

$$\mathcal{L}(x,D) = \sum_{|k|_1 \le 2m} a_k(x) D^k,$$

where $k = (k_1, \cdots, k_d)$ are multi-indices, and $D^k := D_{x_1}^{k_1} \cdots D_{x_d}^{k_d}$, its principle part $\mathcal{L}(x,D)' = \sum_{|k|_1 = 2m} a_k(x) D^k$ satisfies

$$(-1)^m \mathcal{L}(x,z)' \ge c|z|^{2m}.$$

We are going to introduce the assumptions which guarantee the unique existence of a solution for the model (9.6). We start with the items characterizing the deterministic dynamics.

Suppose $f \in L^2([0,T];H)$, where H is a Hilbert space of functions defined on a compact domain $\mathcal{D} \subset \mathbb{R}^d$. Assume that there exists a densely defined positive self-adjoint operator, $\Lambda : D(\Lambda) \subset H \to H$ that has an eigensystem, i.e. eigenfunctions $\{\varphi_k\}_{k\in\mathbb{N}^d}$ with corresponding eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}^d}$.

Remark 9.11 (Index of eigensystem)**.** By Proposition 5.12 in [84], the existence of eigensystem is equivalent to the existence of purely discrete spectrum. Hence the index set can be chosen $\mathbb{N}$. Instead, we use $\mathbb{N}^d$ to make the formula more naturally fit later use, and occasionally we switch back to $\mathbb{N}$, e.g. when using spectral integral representation as in the next paragraph.

If $\mathcal{L}$ is a positive function $g(\Lambda)$ of the operator Λ, $\mathcal{L}$ admits a spectral integral representation in terms of the spectral measure E_λ of Λ, i.e. for any $h \in D(\mathcal{L})$,

$$\mathcal{L}h = g(\Lambda)h = \int f(\lambda)\, dE_\lambda h = \int_0^\infty f(\lambda)\, dE_\lambda h,$$

where the last equality is due to the positivity of $\mathcal{L}$. Because of Remark 9.11, upon reindexing, the spectral integral can be written into the series form,

$$\int_0^\infty g(\lambda)\, dE_\lambda h = \sum_{k\in\mathbb{N}^d} g(\lambda_k)\langle\varphi_k, h\rangle\varphi_k.$$

We will mainly use the spectral integral form in the latter text as it slightly shortens the notation. By Theorem 6.14 in [84], $-\mathcal{L}$ generates a strongly continuous contraction semigroup $S(t)$, $0 \le t < \infty$, which also admits a spectral integral representation, i.e.

$$e^{-t\mathcal{L}} := S(t) = \int_0^\infty e^{-tg(\lambda)}\, E_\lambda = \sum_{k\in\mathbb{N}^d} e^{-tg(\lambda_k)}\langle\varphi_k, \cdot\rangle\varphi_k. \tag{9.7}$$

The discussion above can be summarized as follows.

Assumption 9.12. We impose the following conditions on the model (9.6). The function f is in $L^2([0,T];H)$, where H is a Hilbert space of functions defined on a compact domain $\mathcal{D} \subset \mathbb{R}^d$. There exists a densely defined positive self-adjoint operator, $\Lambda : D(\Lambda) \subset H \to H$ that has an eigensystem, i.e. eigenfunctions $\{\varphi_k\}_{k\in\mathbb{N}^d}$ with corresponding eigenvalues $\{\lambda_k = |k|_1\}_{k\in\mathbb{N}^d}$. Furthermore, the operator $\mathcal{L} = \Lambda^{(\nu)}$ is a function of Λ defined via the eigensystem, i.e. for all φ_k,

$$\mathcal{L}\varphi_k = \left(\sum_{i=1}^d k_i^\nu\right)\varphi_k =: \ell_k\varphi_k,$$

where the constant $\nu \in \mathbb{N}$.

Remark 9.13. Recall $|\cdot|_p$ is the p-norm on $\mathbb{R}^d$. From Lemma 2.19, for any $q \in [0, \infty]$,

$$\ell_k = |\ell^{\boldsymbol{\nu}}|_1 \simeq_d |k^{\boldsymbol{\nu}}|_2 \simeq_d |k|_q^{\nu},$$

where the constant is universally between 1 and d. We will frequently use this equivalence in the subsequent sections.

Now we move to the stochastic part. To facilitate the statistical investigation, we impose the following relationship between the operator $\mathcal{L}$ characterizing the deterministic dynamics and the operators B and $\mathcal{Q}$ expressing the stochastic propagation.

Assumption 9.14. Let $\{\lambda_k, \varphi_k\}_{k \in \mathbb{N}^d}$ and $\{q_k, e_k\}_{k \in \mathbb{N}^d}$ be the eigensystems of Λ and $\mathcal{Q}$, respectively. In addition to Assumption 9.12, we assume that the operator B in (9.6) is linear[2] from U to H and is diagonalized by $\{e_k; \varphi_k\}$, i.e. for all $k \in \mathbb{N}^d$, $Be_k = b_k \varphi_k$. Furthermore, we assume that for the eigenvalues b_k, q_k and λ_k, the following conditions hold:

(i) Initial condition. With a positive constant $d/2 < \mu < \infty$,

$$\frac{b_k^2 q_k}{\ell_k} \simeq_d |k|^{-2\mu}. \tag{9.8}$$

(ii) Drift. The operator B is bounded and with $p = \mu - \nu/2 \geq 0$, where μ is given in (i),

$$b_k^2 q_k \simeq_d |k|^{-2p}, \tag{9.9}$$

where the involved constant is independent of k.

Remark 9.15. The case (ii) in Assumption 9.14 is a consequence of (i) and Assumption 9.12. Situation (i) can be relaxed to $b_k^2 q_k / \ell_k$ being bounded from above and below by a polynomial decay of $|k|$. This will not affect the results for the recovery of initial condition, because the severe ill-posedness induces exponential decay, see Section 10.3.1. However, for the recovery of the drift, the rate will then depend on both of the upper and lower bounds. To simplify the exposition, we confine ourselves to the special case (9.9).

Under Assumption 9.14, with $\|T\|_{L_2^0}^2 = \mathrm{Trace}[T\mathcal{Q}^{1/2}(T\mathcal{Q}^{1/2})^*]$, we have

$$\begin{aligned}
&\int_0^T \|S(t)B\|_{L_2^0}^2 \, dt \\
&= \int_0^T \mathrm{Trace}[S(t)B\mathcal{Q}B^*S^*(t)] \, dr = \sum_{i \in \mathbb{N}^d} \frac{b_i^2 q_i}{2\ell_i}(1 - e^{-2\ell_i T}) \\
&\leq \sum_{i \in \mathbb{N}^d} \frac{b_i^2 q_i}{\ell_i} \simeq_d \sum_{k \in \mathbb{N}^d} |k|^{-2\mu} \simeq \sum_{j \in \mathbb{N}} \sum_{|k|_1 = j} j^{-2\mu} \\
&\simeq \sum_{j \in \mathbb{N}} \left(\binom{j+d}{d} - \binom{j+d-1}{d} \right) j^{-2\mu} \simeq \sum_{j \in \mathbb{N}} j^{d-1-2\mu} < \infty.
\end{aligned} \tag{9.10}$$

[2] Not necessarily bounded.

We now introduce the concept of solutions to SPDEs. According to Theorem 5.4 in [23], under Assumption 9.12 and the assumption that the semigroup $S(t)$ satisfies (9.10), the unique analytical[3] *weak* solution of (9.6) is given by the *stochastic* variation of constant formula, for $t \in [0, T]$,

$$X(t) = S(t)u + \int_0^t S(t-s)f(s)\,ds + \frac{1}{\sqrt{n}}\int_0^t S(t-s)B\,dW^{\mathcal{Q}}(s). \tag{9.11}$$

Remark 9.16. (9.11) is defined as the *mild solution* of (9.6), see page 161 in [23]. However, under Assumption 9.12 and (9.10), the weak solution and mild solution coincide (Theorem 6.7, [23]). For SPDEs, there are several concepts of solutions, which are not always equivalent. For details, we refer to the relevant sections in [23] and Appendix G in [68].

Using the covariance formula given in Theorem 5.2 in [23], the covariance of the stochastic integral is given by

$$\begin{aligned}&\operatorname{Trace}\mathbb{C}\mathrm{ov}\left(\int_0^t S(t-s)B\,dW^{\mathcal{Q}}(s)\right) = \operatorname{Trace}\left(\int_0^t S(t-s)B\mathcal{Q}B^*S^*(t-s)ds\right)\\ &= \sum_{k\in\mathbb{N}^d}\frac{b_k^2 q_k}{2\ell_k}\frac{1-e^{-2\ell_k T}}{e^{2\ell_k T}} \le \sum_{k\in\mathbb{N}^d}\frac{b_k^2 q_k}{\ell_k} < \infty,\end{aligned}$$

under Assumption 9.14. As a consequence, the stochastic component in the weak solution is a proper Gaussian process in H. This is in contrast to the well-known white noise model, which is almost surely not in H.

Now we discuss the series representation of (9.11). Let $\{\varphi_i\}_{i\in\mathbb{N}^d}$ be the orthonormal basis of H from assumption 9.12. Under Assumptions 9.12 and 9.14, the three items in (9.11) can be represented as follows.

Since $u \in H$, we have

$$u = \sum_{k\in\mathbb{N}^d} u_k\varphi_k, \quad \text{and} \quad S(t)u = \sum_{k\in\mathbb{N}^d} e^{-t\ell_k}u_k\varphi_k.$$

Since $L^2([0,T];H) \cong H \otimes L^2([0,T];\mathbb{R})$ (see Section 2.3.1), the drift term can be written in the form of $f(t) = \sum_{k\in\mathbb{N}^d}\varphi_k f_i(t)$. Consequently,

$$\int_0^t S(t-s)f(s)\,ds = \sum_{k\in\mathbb{N}^d}\varphi_k\int_0^t e^{-\ell_k(t-s)}f_k(s)\,ds.$$

Regarding the stochastic integral in (9.11), by Lemma 9.8, it admits the following representation

$$\sum_{k\in\mathbb{N}^d}\frac{b_k\sqrt{q_k}}{\sqrt{n}}\varphi_k\int_0^t e^{-\ell_k(t-s)}\,dW_k(s), \tag{9.12}$$

where $W_k, k \in \mathbb{N}^d$, are independent standard real-valued Wiener processes.

[3]It means that the notion 'weak' is in the PDE sense.

In summary, (9.11) admits the series representation

$$X^{(n)}(t) = \sum_{k \in \mathbb{N}^d} X_k^{(n)}(t) \varphi_k \tag{9.13}$$

with

$$X_k^{(n)}(t) = e^{-t\ell_k} u_k + \int_0^t e^{-(t-s)\ell_k} f_k(s)\, ds + \frac{b_k \sqrt{q_k}}{\sqrt{n}} \int_0^t e^{-(t-s)\ell_k}\, dW_k(s),$$

where $u_k \in \mathbb{R}$ and $f_k \in L^2[0,T]$ are from

$$u = \sum_{k \in \mathbb{N}^d} u_k \varphi_k \quad \text{and} \quad f(t) = \sum_{k \in \mathbb{N}^d} \varphi_k f_k(t)$$

respectively, and W_k are independent Wiener process on $[0,T]$.

9.6 Notes

Assumption 9.14 imposes constraints on the structure of the stochastic integral, which is necessary to obtain solutions as H-valued processes. Consider the following example. Recall that the (negative) Laplacian $-\Delta$ in $\mathbb{R}^d$ has eigenvalues λ_k of the order $|k|^2$. As a consequence, a stochastic heat equation with space-time white noise, i.e. $U = H$ and $B = \mathcal{Q} = I$, does not have a H-valued weak solution when the underlying domain is not one-dimensional. In general, in order to obtain a regular (H-valued) solution, the regularity property of the composition of $S(t)B\mathcal{Q}^{1/2}$ needs to compensate the deterioration with growth of dimensions.

On the other hand, the requirement on B and $\mathcal{Q}$ to obtain well defined statistics is not as restrictive as to obtain unique solutions. While the solution to (9.6) only exists in a larger space than H, meaningful statistics for the terms $S(t)u$ and $\int_0^t S(t-s)f(s)\, ds$ of interest still can be obtained, see [51] for the detail.

In the end, the $\mathcal{Q}$-Wiener process on U might be considered superfluous, as the stochastic integral can always be treated as $\int_0^t S(t-s)\, d\widetilde{W}(s)$, where $\widetilde{W}$ is a $\widetilde{\mathcal{Q}} = B\mathcal{Q}B^*$-Wiener process on H.