Bayesian inference for Gaussian models: Inverse problems and evolution equations

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Chapter 6

Inverse Problems with Continuous Observations in Smoothness Scales

6.1 Introduction

In this chapter, we study the following linear inverse problem in white noise,

$$Y^{(n)} = \mathcal{A}f + \frac{1}{\sqrt{n}}\xi, \tag{6.1}$$

where $\mathcal{A}$ is characterised in Section 5.2, and ξ is an isonormal Gaussian process (see Definition 3.28 and Section 4.2.1).

This chapter is organised as follows. We present a general posterior contraction theorem in Section 6.2, which is based on a testing approach using the estimator from Section 5.3. The result does not depend on any conjugacy of priors. Then, the general theorem is applied to two examples of priors: series priors in Section 6.3 and Gaussian priors in Section 6.4. Since the simple Gaussian prior is not fully adaptive, we introduce Gaussian mixture priors to obtain adaptation in Section 6.5. It is noteworthy that the priors are defined in terms of the scale, rather than the operator. In other words, the operator and the prior are assumed related, but only indirectly, through the scale. In this arrangement, priors can be chosen directly related to common bases (e.g., splines or wavelets bases) and function spaces, rather than to the operator through its singular value decomposition. Section 6.6 contains the proofs. we conclude this chapter with the discussion of several extensions of the present work in Section 6.7.

6.2 General Result

In this section we present a general theorem on posterior contraction. We form the posterior distribution $\Pi_n(\cdot \mid Y^{(n)})$ as in (4.1), given a prior Π on the space $H = H_0$ and an observation $Y^{(n)}$, whose conditional distribution given f is determined by the model (6.1). We study this random distribution under the assumption that $Y^{(n)}$ follows the model (6.1) for a given 'true' function $f = f_0$, which we assume to be an element of H_β in a given smoothness scale $(H_s)_{s\in\mathbb{R}}$, as in Definition 2.1.

The result is based on an extension of the testing approach of [35] to the inverse problem (6.1). The inverse problem is handled with the help of the Galerkin method, which is a well known strategy in numerical analysis to solve the operator equation $y = \mathcal{A}f$ for f, in particular for differential and integral operators. The Galerkin method has several variants, which are useful depending on the properties of the operator involved. Here we use the least squares method, which is of general application; for other variants and background, see e.g., [57]. In Section 5.3 we have given a self-contained derivation of the necessary inequalities, exactly in our framework. We note that the Galerkin method only appears as a tool to state and derive a posterior contraction rate. In our context it does not enter into the solution of the inverse problem, which is achieved through the Bayesian method.

Let $W_j = \mathcal{A}V_j \subset G$ be the image under $\mathcal{A}$ of a finite-dimensional approximation space V_j linked to the smoothness scale $(H_s)_{s\in\mathbb{R}}$ as in Assumption 2.3, and let $Q_j : G \to W_j$ be the orthogonal projection onto W_j. If $\mathcal{A} : H \to G$ is injective, then $\mathcal{A}$ is a bijection between the finite-dimensional vector spaces V_j and W_j, and hence for every $f \in H$ there exists $f^{(j)} \in V_j$ such that $\mathcal{A}f^{(j)} = Q_j\mathcal{A}f$. The element $f^{(j)}$ is called the *Galerkin solution* to Af in V_j. By the projection theorem in Hilbert spaces it is characterized by the property that $f^{(j)} \in V_j$ together with the orthogonality relations

$$\langle \mathcal{A}f^{(j)}, w\rangle_0 = \langle \mathcal{A}f, w\rangle_0, \qquad w \in W_j. \tag{6.2}$$

The idea of the Galerkin inversion is to project the (complex) object $\mathcal{A}f$ onto the finite-dimensional space W_j, and next find the inverse image $f^{(j)}$ of the projection, in the finite-dimensional space V_j, as in the diagram:

$$\begin{tikzcd}
H_0 \ni f \arrow[r, "\mathcal{A}"] & \mathcal{A}f \in G \arrow[d, "Q_j"'] \\
V_j \ni f^{(j)} & Q_j\mathcal{A}f \in W_j \arrow[l, "\mathcal{A}^{-1}"']
\end{tikzcd}$$

Clearly the Galerkin solution to an element $f \in V_j$ is f itself, but in general $f^{(j)}$ is an approximation to f, which will be better for increasing j, but increasingly complex. The following theorem uses a dimension $j = j_n$ that balances approximation to complexity, where the complexity is implicitly determined by a testing criterion.

Theorem 6.1. *For smoothness classes $(H_s)_{s\in\mathbb{R}}$ as in Definition 2.1, assume that $\|\mathcal{A}f\|_0 \simeq \|f\|_{-\gamma}$ for some $\gamma > 0$, and let $f^{(j)}$ denote the Galerkin solution to $\mathcal{A}f$ relative to linear subspaces V_j associated to $(H_s)_{s\in\mathbb{R}}$ as in Assumption 2.3. Let $f_0 \in H_\beta$ for some $\beta \in (0, S)$, and for $\eta_n \geq \varepsilon_n \downarrow 0$ such that $n\varepsilon_n^2 \to \infty$, and $j_n \in \mathbb{N}$*

such that $j_n \to \infty$, and some $c > 0$, assume

$$j_n \leq cn\varepsilon_n^2, \tag{6.3}$$

$$\eta_n \geq \frac{\varepsilon_n}{\delta(j_n, \gamma)}, \tag{6.4}$$

$$\eta_n \geq \delta(j_n, \beta). \tag{6.5}$$

Consider prior probability distributions Π on H_0 satisfying

$$\Pi(f : \|\mathcal{A}f - \mathcal{A}f_0\| < \varepsilon_n) \geq e^{-n\varepsilon_n^2}, \tag{6.6}$$

$$\Pi(f : \|f^{(j_n)} - f\|_0 > \eta_n) \leq e^{-4n\varepsilon_n^2}. \tag{6.7}$$

Then the posterior distribution in the model (6.1) contracts at the rate η_n at f_0, i.e. for a sufficiently large constant M we have $\Pi_n(f : \|f - f_0\|_0 > M\eta_n \mid Y^{(n)}) \to 0$, in probability under the law of $Y^{(n)}$ given by (6.1) with $f = f_0$.

Proof. The Kullback-Leibler divergence and variation between the distributions of $Y^{(n)}$ under two functions f and f_0 are given by $n\|\mathcal{A}f - \mathcal{A}f_0\|^2/2$ and twice this quantity, respectively. (E.g., Lemma 8.30 in [35].) Therefore the neighbourhoods $B_{n,2}(f_0, \varepsilon)$ in (8.19) of [35] contain the ball $\{f \in H_0 : \|\mathcal{A}f - \mathcal{A}f_0\| \leq \varepsilon\}$. By assumption (6.6) this has prior mass at least $e^{-n\varepsilon_n^2}$.

Because the quotient of the left sides of (6.6) and (6.7) is $o(e^{-2n\varepsilon_n^2})$, the posterior probability of the set $\{f : \|f^{(j_n)} - f\|_0 > \eta_n\}$ tends to zero, by Theorem 8.20 in [35].

By a variation of Theorem 8.22 in [35] it is now sufficient to show the existence of tests τ_n such that, for some $M > 0$,

$$P_{f_0}^{(n)}\tau_n \to 0, \qquad \sup_{\substack{f:\|f-f_0\|_0 > M\eta_n, \\ \|f^{(j_n)}-f\|_0 \leq \eta_n}} P_f^{(n)}(1 - \tau_n) \leq e^{-4n\varepsilon_n^2}.$$

Indeed, in the case that the prior mass condition (8.20) in Theorem 8.22 of [35] can be strengthened to (8.22), as is the case in our setup in view of (6.6), it suffices to verify (8.24) only for a single value of j. Furthermore, we can apply Theorem 8.22 with the metrics $d_n(f, g) = \|f - g\|_0\, \varepsilon_n/\eta_n$ in order to reduce the restriction $d_n(\theta, \theta_{n,0}) > M\varepsilon_n$ to $\|f - f_0\|_0 > M\eta_n$.

Fix any orthonormal basis $(\bar{\psi}_i)_{i<j}$ of $W_j = \mathcal{A}V_j$ and define

$$\begin{aligned}\bar{Y}_j = \sum_{i<j} Y_{\bar{\psi}_i}^{(n)} \bar{\psi}_i &= \sum_{i<j} \langle \mathcal{A}f, \bar{\psi}_i\rangle \bar{\psi}_i + \frac{1}{\sqrt{n}} \sum_{i<j} \xi_{\bar{\psi}_i} \bar{\psi}_i \\ &= Q_j \mathcal{A}f + \frac{1}{\sqrt{n}} \bar{\xi}_j,\end{aligned}$$

where $\bar{\xi}_j := \sum_{i<j} \xi_{\bar{\psi}_i} \bar{\psi}_i$. The latter is a "standard normal vector in the finite-dimensional space W_j": because $(\xi_{\bar{\psi}_i})_{i<j}$ are i.i.d. standard normal variables, the variable $\langle \bar{\xi}_j, w\rangle = \sum_{i<j} \xi_{\bar{\psi}_i} \langle \bar{\psi}_i, w\rangle$ is $N(0, \|Q_j w\|^2)$-distributed, for every $w \in G$.

Let the operator $R_j : G \mapsto V_j$ be defined as $R_j = \mathcal{A}^{-1} Q_j$, where $\mathcal{A}^{-1}$ is the inverse of $\mathcal{A}$, which is well defined on the range $W_j = \mathcal{A}V_j$ of Q_j. Then by definition $R_j\mathcal{A}f$ is equal to the Galerkin solution $f^{(j)}$ to $\mathcal{A}f$. By the preceding display $R_j\bar{Y}_j$ is a well-defined Gaussian random element in V_j, satisfying

$$R_j\bar{Y}_j = f^{(j)} + \frac{1}{\sqrt{n}}R_j\bar{\xi}_j. \tag{6.8}$$

The variable $R_j\bar{\xi}_j$ is a Gaussian random element in V_j with strong and weak second moments

$$\mathbb{E}\|R_j\bar{\xi}_j\|_0^2 \leq \|R_j\|^2\mathbb{E}\|\bar{\xi}_j\|^2 = \|R_j\|^2\mathbb{E}\sum_{i<j}\xi_{\bar{\psi}_i}^2 = \|R_j\|^2(j-1) \lesssim \frac{j}{\delta(j,\gamma)^2},$$

$$\sup_{\|f\|_0\leq 1}\mathbb{E}\langle R_j\bar{\xi}_j, f\rangle_0^2 = \sup_{\|f\|_0\leq 1}\mathbb{E}\langle\bar{\xi}_j, R_j^*f\rangle^2 = \sup_{\|f\|_0\leq 1}\|Q_jR_j^*f\|^2 \leq \|R_j^*\|^2 \lesssim \frac{1}{\delta(j,\gamma)^2}.$$

In both cases the inequality on $\|R_j\| = \|R_j^*\|$ at the far right side follows from (5.7).

The first inequality shows that the first moment $\mathbb{E}\|R_j\bar{\xi}_j\|_0$ of the variable $\|R_j\bar{\xi}_j\|_0$ is bounded above by $\sqrt{j}/\delta(j,\gamma)$. By Borell's inequality (e.g. Lemma 3.1 in [67] and subsequent discussion), applied to the Gaussian random variable $R_j\bar{\xi}_j$ in H_0, we see that there exist positive constants a and b such that, for every $t>0$,

$$\Pr\Big(\|R_j\bar{\xi}_j\|_0 > t + a\frac{\sqrt{j}}{\delta(j,\gamma)}\Big) \leq e^{-bt^2\delta(j,\gamma)^2}.$$

For $t = 2\sqrt{n}\eta_n/\sqrt{b}$ and η_n, ε_n and j_n satisfying (6.3), (6.4) and (6.5) this yields, for some $a_1 > 0$,

$$\Pr\Big(\|R_{j_n}\bar{\xi}_{j_n}\|_0 > a_1\sqrt{n}\eta_n\Big) \leq e^{-4n\varepsilon_n^2}. \tag{6.9}$$

We apply this to bound the error probabilities of the tests

$$\tau_n = 1\{\|R_{j_n}\bar{Y}_{j_n} - f_0\|_0 \geq M_0\eta_n\}, \tag{6.10}$$

where M_0 is a given constant, to be determined.

Under f_0, the decomposition (6.8) is valid with $f = f_0$, and hence $R_j\bar{Y}_j - f_0 = n^{-1/2}R_j\bar{\xi}_j + f_0^{(j)} - f_0$. By the triangle inequality it follows that $\tau_n = 1$ implies that $n^{-1/2}\|R_{j_n}\bar{\xi}_{j_n}\|_0 \geq M_0\eta_n - \|f_0^{(j)} - f_0\|_0$. By (5.9) the assumption that $f_0 \in H_\beta$ implies that $\|f_0^{(j)} - f_0\|_0 \leq M_1\delta(j,\beta)$, for some M_1, which at $j = j_n$ is further bounded by $M_1\eta_n$, by assumption (6.5). Hence the probability of an error of the first kind satisfies

$$P_{f_0}^{(n)}\tau_n \leq \Pr\Big(\frac{1}{\sqrt{n}}\|R_{j_n}\bar{\xi}_{j_n}\|_0 \geq (M_0 - M_1)\eta_n\Big).$$

For $M_0 - M_1 > a_1$, the right side is bounded by $e^{-4n\varepsilon_n^2}$, by (6.9).

Under f the decomposition (6.8) gives that $R_j\bar{Y}_j - f_0 = n^{-1/2}R_j\bar{\xi}_j + f^{(j)} - f_0$. By the triangle inequality $\tau_n = 0$ implies that $n^{-1/2}\|R_{j_n}\bar{\xi}_{j_n}\|_0 \geq \|f^{(j_n)} - f_0\|_0 - M_0\eta_n$. For f such that $\|f - f_0\|_0 > M\eta_n$ and $\|f - f^{(j_n)}\|_0 \leq \eta_n$, we have $\|f^{(j_n)} - f_0\| \geq (M-1)\eta_n$. Hence the probability of an error of the second kind satisfies

$$P_f^{(n)}(1-\tau_n) \leq \Pr\Big(\frac{1}{\sqrt{n}}\|R_{j_n}\bar{\xi}_{j_n}\|_0 \geq (M - 1 - M_0)\eta_n\Big),$$

For $M - 1 - M_0 > a_1$, this is bounded by $e^{-4n\varepsilon_n^2}$, by (6.9).

We can first choose M_0 large enough so that $M_0 - M_1 > a_1$, and next M large enough so that $M - 1 - M_0 > a_1$, to finish the proof. $\square$

Inequality (6.6) is the usual *prior mass condition* for the 'direct problem' of estimating $\mathcal{A}f$ (see [33]). It determines the rate of contraction ε_n of the posterior distribution of $\mathcal{A}f$ to $\mathcal{A}f_0$. The rate of contraction η_n of the posterior distribution of f is slower due to the necessity of (implicitly) inverting the operator $\mathcal{A}$. The theorem shows that the rate η_n depends on the combination of the prior, through (6.7), and the inverse problem, through the various approximation rates.

Remark 6.2. It would be possible to obtain the theorem as a corollary of Theorem 2.1 in [58]. We would take the sets $\mathcal{S}_n$ in the latter high-level result equal to the sets $\{f : \|f^{(j_n)} - f\|_0 > \eta_n\}$ appearing in (6.7). To verify the conditions of [58] for this choice, most of the preceding proof would be needed. Since the next theorem appears not to be a consequence of this approach, and its proof uses the preceding proof, we have given a direct proof instead.

The theorem applies to a true function f_0 that is 'smooth' of order β (i.e., $f_0 \in H_\beta$). For a prior that is constructed to give an optimal contraction rate for multiple values of β simultaneously, the theorem may not give the best result. The following theorem refines Theorem 6.1 by considering a mixture prior of the form

$$\Pi = \int \Pi_\tau \, dQ(\tau), \tag{6.11}$$

where Π_τ is a prior on H, for every given 'hyperparameter' τ running through some measurable space, and Q is a prior on this hyperparameter. The idea is to *adapt* the prior to multiple smoothness levels through the hyperparameter τ.

Theorem 6.3. *Consider the setup and assumptions of Theorem 6.1 with a prior of the form (6.11). Assume that (6.3), (6.4), (6.5) and (6.6) hold, but replace (6.7) by the pair of conditions, for numbers $\eta_{n,\tau}$ and $C > 0$ and every τ,*

$$\Pi_\tau(f : \|f - f_0\|_0 < 2\eta_{n,\tau}) \leq e^{-4n\varepsilon_n^2}, \quad \forall \tau \text{ with } \eta_{n,\tau} \geq C\eta_n, \tag{6.12}$$

$$\Pi_\tau(f : \|f^{(j_n)} - f\|_0 > \eta_{n,\tau}) \leq e^{-4n\varepsilon_n^2}. \tag{6.13}$$

Then the posterior distribution in the model (6.1) contracts at the rate η_n at f_0, i.e. for a sufficiently large constant M we have $\Pi_n(f : \|f - f_0\|_0 > M\eta_n \mid Y^{(n)}) \to 0$, in probability under the law of $Y^{(n)}$ given by (6.1) with $f = f_0$.

Proof. We take the parameter of the model as the pair (f, τ), which receives the joint prior given by $f \mid \tau \sim \Pi_\tau$ and $\tau \sim Q$. With abuse of notation, we denote this prior also by Π. The likelihood still depends on f only, but the joint prior gives rise to a posterior distribution on the pair (f, τ), which we also denote by $\Pi_n(\cdot \mid Y^{(n)})$, by a similar abuse of notation.

By (6.11) and eqs. (6.12) and (6.13),

$$\Pi((f,\tau) : \eta_{n,\tau} \geq C\eta_n, \|f - f_0\|_0 < 2\eta_{n,\tau}) \leq e^{-4n\varepsilon_n^2},$$
$$\Pi((f,\tau) : \|f^{(j_n)} - f\|_0 > \eta_{n,\tau}) \leq e^{-4n\varepsilon_n^2}.$$

In view of (6.6) and Theorem 8.20 in [35], the posterior probabilities of the two sets in the left sides tend to zero. As in the proof of Theorem 6.1, we can apply a variation of Theorem 8.22 in [35] to see that it is now sufficient to show the existence of tests τ_n such that, for some $M \geq 2C$,

$$P_{f_0}^{(n)}\tau_n \to 0, \qquad \sup_{\substack{(f,\tau):\|f-f_0\|_0>M\eta_n\vee 2\eta_{n,\tau},\\ \|f^{(j_n)}-f\|_0\leq\eta_{n,\tau}}} P_f^{(n)}(1-\tau_n) \leq e^{-4n\varepsilon_n^2}.$$

(Note that $M\eta_n \vee 2\eta_{n,\tau} = M\eta_n$ if $\eta_{n,\tau} < C\eta_n$ and $M \geq 2C$.) We use the tests defined in (6.10), as in the proof of Theorem 6.1. The latter proof shows that the tests are consistent. We adapt the bound on the power, as follows.

By the triangle inequality $\tau_n = 0$ implies that, for (f,τ) with $\|f - f_0\|_0 > M\eta_n \vee 2\eta_{n,\tau}$ and $\|f^{(j_n)} - f\|_0 \leq \eta_{n,\tau}$,

$$\begin{aligned} n^{-1/2}\|R_{j_n}\bar{\xi}_{j_n}\|_0 &\geq \|f^{(j_n)} - f_0\|_0 - M_0\eta_n \geq \|f - f_0\|_0 - \|f^{(j_n)} - f\|_0 - M_0\eta_n \\ &\geq M\eta_n \vee 2\eta_{n,\tau} - \eta_{n,\tau} - M_0\eta_n \geq (M/2 - M_0)\eta_n. \end{aligned}$$

Hence by (6.9) the probability of an error of the second kind is bounded by $e^{-4n\varepsilon_n^2}$, for M sufficiently large that $M/2 - M_0 > a_1$. ☐

In a typical application of the preceding theorem the priors Π_τ for τ such that $\eta_{n,\tau} \geq C\eta_n$ will be the priors on 'rough' functions, with 'intrinsic' contraction rate $\eta_{n,\tau}$ slower than η_n. These 'bad' priors do not destroy the overall contraction rate, because they put little mass near the true function f_0, by condition (6.12). It is necessary to address these priors explicitly in the conditions, because they will typically fail the approximation condition (6.7), which must be relaxed to (6.13). A further generalization might be to allow the truncation levels j_n to depend on τ, but this will not be needed for our examples.

Inspection of the proof shows that the posterior probability of the sets $\{\tau : \eta_{n,\tau} \gtrsim C\eta_n\}$ tends to zero. This means that the posterior correctly disposes of the models that are 'too rough', for the given true function f_0. In general there is no similar protection against models that are too smooth, but this does not affect the contraction rate.

6.3 Random Series Priors

Suppose that $\{\phi_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of $H = H_0$ that gives optimal approximation relative to the scale of smoothness classes $(H_s)_{s\in\mathbb{R}}$ in the sense that the linear spaces $V_j = \mathsf{Span}\{\phi_i\}_{i<j}$ satisfy Assumption 2.3. Consider a prior defined as the law of the random series

$$f = \sum_{i=1}^{M} f_i\phi_i, \tag{6.14}$$

where M is a random variable in $\mathbb{N}$ independent from the independent random variables $f_1, f_2, \ldots$ in $\mathbb{R}$.

Condition 6.4 (Random series prior). (i) The probability density function p_M of M satisfies, for some positive constants b_1, b_2,

$$e^{-b_1 k} \lesssim p_M(k) \lesssim e^{-b_2 k}, \qquad \forall k \in \mathbb{N}.$$

(ii) The variable f_i has density $p(\cdot/\kappa_i)/\kappa_i$, for a given probability density p on $\mathbb{R}$ and a constant $\kappa_i > 0$ such that, for some $C > 0$ and $w > 0$, $\alpha, \beta_0 > 0$,

$$p(x) \gtrsim e^{-C|x|^w}, \tag{6.15}$$

$$i^{-\beta_0/d}(\log i)^{-1/w} \lesssim \kappa_i \lesssim i^{\alpha}. \tag{6.16}$$

Priors of this type were studied in [4, 80], and applied to inverse problems in the SVD framework in [80] (see Section 3.1 of the latter paper for discussion). For Gaussian variables f_j and degenerate M the series (6.14) is a Gaussian process, and has been more widely studied, but we focus here on the non-Gaussian case. Since the basis $(\phi_i)_{i\in\mathbb{N}}$ used in the prior is linked to the smoothness class $(H_s)_{s\in\mathbb{R}}$, rather than to the operator $\mathcal{A}$, the prior is not restricted to the SVD framework. Of course, in the theorem below we do require the operator to be smoothing in the same smoothness scale, thus maintaining a link between prior and operator.

The assumption on the density p_M is mild and is satisfied, for instance, by the Poisson distribution. The assumption on the density p is mild as well, and is satisfied by many distributions with full support in $\mathbb{R}$, including the Gaussian and Laplace distributions. The parameter β_0 in (6.16) must be a lower bound on the smoothness of the true parameter f_0. Apart from this, condition (6.16) is also very mild, and allows the scale parameters κ_i to tend both to zero or to infinity.

The preceding random series prior is not conjugate to the inverse problem (6.1). In general the resulting posterior distribution will not have a closed form expression, but must be computed using simulation, such as Markov chain Monte Carlo, or approximated using an optimisation method, such as variational approximation. However, the contraction rate of the posterior distribution can be established without the help of an explicit expression for the posterior distribution, as shown in the following theorem.

Theorem 6.5 (Random Series Prior). *Let $(\phi_i)_{i\in\mathbb{N}}$ be an orthonormal basis of H_0 such that the spaces $V_j = \mathsf{Span}\{\phi_i\}_{i<j}$ satisfy Assumption 2.3 with $\delta(j,s) = j^{-s/d}$ relative to smoothness classes $(H_s)_{s\in\mathbb{R}}$ as in Definition 2.1. Assume that $\|\mathcal{A}f\| \simeq \|f\|_{-\gamma}$ for some $\gamma > 0$, and let $f_0 \in H_\beta$ for some $\beta \in (0, S)$. Then, for the random series prior defined in (6.14) and satisfying Condition 6.4 with $\beta_0 \leq \beta$, and sufficiently large $\underline{M} > 0$, for $\tau = (\beta+\gamma)(1+2\gamma/d)/(2\beta+2\gamma+d)$,*

$$\Pi_n\Big(f : \|f - f_0\|_0 > \underline{M} n^{-\beta/(2\beta+2\gamma+d)}(\log n)^\tau \mid Y^{(n)}\Big) \overset{P_{f_0}^{(n)}}{\to} 0.$$

The rate $n^{-\beta/(2\beta+2\gamma+d)}$ is known to be the minimax rate of estimation of a β-regular function on a d-dimensional domain, in an inverse problem with inverse parameter γ (see, e.g., [19]). The assumption that $\delta(j,s) = j^{-s/d}$ places the setup of the theorem in this setting, and hence the rate of contraction obtained in the preceding theorem is the minimax rate up to a logarithmic factor. The rate is adaptive to the regularity of β of the true parameter, which is not used in the construction of the prior, apart from the assumption that $\beta \geq \beta_0$. (See [34] and Chapter 10 in [35] for general discussion of adaptation in the Bayesian sense.)

The proof of the theorem is deferred to Section 6.6; it will be based on Theorem 6.1.

Example 6.6 (Wavelet basis). Let p be a standard normal density, p_M a standard Poisson probability mass function, and set the scaling parameters κ_i equal to 1 (no scaling).

Consider an S-regular orthonormal wavelet basis $\{\phi_{j,k}\}$ for the space of square-integrable functions on the d-dimensional torus $(0, 2\pi]^d$. We can renumber the index (j, k) into $\mathbb{N}$ by ordering the basis functions by their multiresolution levels, $2^{jd} + k$, and next construct the random series prior (6.14).

An S-regular orthonormal wavelet basis is known to correspond to the scale of Sobolev spaces up to smoothness level S. Therefore, by Theorem 6.5, the contraction rate of the posterior distribution is $n^{-\beta/(2\beta+2\gamma+d)}$ times a logarithmic factor whenever the operator is smoothing relative to the Sobolev scale and the true function f_0 belongs to the Sobolev space of order β, for $\beta_0 \leq \beta < S$. Thus the posterior distributions are adaptive up to a logarithmic factor to the scale of Sobolev spaces of orders between β_0 and S.

For increasing $\beta \geq S$ the rate given by the theorem still improves. However, the 'regularity' β defined by the scale $(H_s)_{s\in\mathbb{R}}$ may then not coincide with the Sobolev scale.

6.4 Gaussian Priors

If the function f in (6.1) is equipped with a Gaussian prior, then the corresponding posterior distribution will be Gaussian as well. Furthermore, the posterior mean will then be equal to the solution found by the method of Tikhonov-type regularization (see e.g. [30, 59, 88]). Although this allows to study the posterior mean and the full posterior distribution by direct methods, in this section we derive the rate of posterior contraction from the general result Theorem 6.1. An advantage of this approach is that the proof can be extended to mixtures of Gaussian priors. Taking mixtures is important to obtain optimal recovery rates for true functions of different smoothness levels. See Section 6.5.

A Gaussian prior on the Hilbert space $H = H_0$ is determined by a mean, which we shall take equal to zero, and a covariance operator. To connect the prior to a smoothness scale $(H_s)_{s\in\mathbb{R}}$ as defined in Definition 2.1, it is natural to assume that the latter forms a *Hilbert scale*, which may be viewed a smoothness scale with additional structure. For reference we include a short summary on Hilbert scales. Extended discussions of Hilbert scales in the context of regularization theory can be found e.g. in Chapter 8 of [29], and a general treatment of the subject in [62].

Centred Gaussian distributions on a separable Hilbert space correspond bijectively to covariance operators. By definition a random variable F with values in H_0 is Gaussian if $\langle F, g\rangle_0$ is normally distributed, for every $g \in H_0$, and it has zero mean if these variables have zero means. The variances of these variables can then be written as

$$\mathbb{E}\langle F, g\rangle_0^2 = \langle Cg, g\rangle_0,$$

for a linear operator $C : H_0 \to H_0$, called the *covariance operator*. A covariance operator C is necessarily self-adjoint, nonnegative, and of *trace class*, i.e., $\sum_{i\in\mathbb{N}}\langle C\phi_i, \phi_i\rangle < \infty$, for some (and then every) orthonormal basis $(\phi_i)_{i\in\mathbb{N}}$ of H_0; and every operator with these properties generates a Gaussian distribution.

In the setting of a Hilbert scale $(H_s)_{s\in\mathbb{R}}$ generated by the operator L it is natural to choose a Gaussian prior with covariance operator of the form $L^{-2\alpha}$, for some $\alpha > 0$. If L^{-1} has eigenvalues λ_j, then this operator is of trace class if $\sum_{j\in\mathbb{N}}\lambda_j^{-2\alpha} < \infty$. Thus α must be chosen big enough for the Gaussian prior to exist as a 'proper' prior on H_0. For instance, if $\lambda_j \simeq j^{-1/d}$, then every choice $\alpha > d/2$ yields a proper prior.

This leads to the following theorem on posterior contraction rates for Gaussian priors, the proof of which is given in Section 6.6.

Theorem 6.7 (Gaussian Prior). *Consider a Hilbert scale $(H_s)_{s\in\mathbb{R}}$ generated by an operator L as in the preceding such that $L^{-1} : H_0 \to H_0$ is compact with eigenvalues λ_j satisfying $\lambda_j \simeq j^{-1/d}$. Suppose the operator $\mathcal{A} : H_0 \to G$ satisfies $\|\mathcal{A}f\| \simeq \|f\|_{-\gamma}$, assume that $f_0 \in H_\beta$, for some $\beta > 0$, and let the prior be zero-mean Gaussian with covariance operator $L^{-2\alpha}$, for some $\alpha > d/2$. Then the posterior distribution satisfies, for sufficiently large $M > 0$,*

$$\Pi_n\Big(f : \|f - f_0\|_0 > Mn^{-((\alpha-d/2)\wedge\beta)/(2\alpha+2\gamma)} \mid Y^{(n)}\Big) \overset{P_{f_0}^{(n)}}{\to} 0.$$

If F is distributed according to the prior in the preceding theorem, then L^sF is also zero-mean Gaussian distributed, with covariance operator $L^{2s-2\alpha}$, which has eigenvalues $j^{-(2\alpha-2s)/d}$. For $s < \alpha - d/2$, this operator is of trace class and hence L^sF is a proper random variable in H_0. In other words, the distribution of F gives probability 1 to $L^{-s}H_0 = H_s$, for every $s < \alpha - d/2$. The prior in the preceding theorem can therefore be interpreted as being 'almost' of regularity $\alpha - d/2$. The rate $n^{-((\alpha-d/2)\wedge\beta)/(2\alpha+2\gamma)}$ is therefore comparable to the rate obtained in Theorem 3.5 in [80] and Theorem 4.1 in [59] (without scaling parameter), except that the parameter α in the latter references is denoted presently by $\alpha - d/2$.

An improvement of the present theorem is that the covariance operator of the Gaussian prior is not directly linked to the operator $\mathcal{A}$, but only weakly so by (5.3). For example, we may construct a prior by a random series (see Theorem I.23 in Appendix I.6, [35]), in any basis corresponding to the smoothness scale. We illustrate this below by using the wavelet basis for an inverse problem given by a differential operator, after first noting that the singular value setup is covered as well.

Example 6.8 (SVD). The scale of smoothness classes constructed in Example 2.6 and Example 5.2 is the Hilbert scale attached to the operator L given by $Lf = \sum_{i\in\mathbb{N}} b_i f_i \phi_i$ defined on the domain of functions $f = \sum_{i\in\mathbb{N}} f_i\phi_i$, with $\sum_{i\in\mathbb{N}} b_i^2 f_i^2 < \infty$. Under assumption (5.4) this operator can also be expressed as $L = (\mathcal{A}^*\mathcal{A})^{-1/(2\gamma)}$, and depends on the operator $\mathcal{A}$ through its eigenfunctions. A Gaussian prior with covariance operator $L^{-2\alpha}$ corresponds to modelling the coefficients f_i relative to the basis ϕ_i as independent zero-mean normal variables F_i with variances $b_i^{-2\alpha}$. This follows, because in that case $\mathbb{E}\langle F, g\rangle_0^2 = \sum_{i\in\mathbb{N}} b_i^{-2\alpha} g_i^2 = \langle L^{-2\alpha} g, g\rangle_0^2$, for every $g \in H_0$.

Thus in this case the prior coincides with the ones in the literature studied under the SVD framework, e.g. [59, 60]. In the present more general setting L need not be directly linked to $\mathcal{A}$, except that the operator must possess the smoothing property Assumption 5.1.

Example 6.9 (Sobolev scales, wavelet prior)**.** Let $\{\phi_{j,k}\}_{(j,k)\in\Lambda}$, be an S-regular orthonormal wavelet basis in $L^2(\mathbb{T})$, on $\mathbb{T} := (0, 2\pi]$. Let $f_{j,k} = \int_{\mathbb{T}} f(x)\phi_{j,k}(x)\,dx$ be the wavelet coefficients of a function f. By Parseval's identity, the map $U : f \mapsto \{f_{j,k}\}$ is a unitary operator $U : L^2(\mathbb{T}) \to \ell^2(\Lambda)$. The multiplication operator $m : \{f_{j,k}\} \mapsto \{2^j f_{j,k}\}$ on $\ell^2(\Lambda)$ has s-th power given by $m^s : \{f_{j,k}\} \mapsto \{2^{js} f_{j,k}\}$. Then $L := U^* m U$ has s-th power $L^s := U^* m^s U$ and generates a Hilbert scale $(H_s)_{s\in\mathbb{R}}$. For $f \in H_s$, we have

$$\|f\|^2_{H_s(\mathbb{T})} = \sum_{j=0}^{\infty} 2^{2js} \sum_{k=0}^{2^j-1} f_{j,k}^2.$$

This norm can be shown to be equivalent to the standard Sobolev norm, for $0 \le s < S$.

The Gaussian prior with covariance operator $L^{-2\alpha}$ can be represented by a random series of the form

$$F = \sum_{(j,k)\in\Lambda} F_{j,k}\phi_{j,k},$$

where $F_{j,k} \sim \mathcal{N}(0, 2^{-2j\alpha})$ are independent random variables. This prior corresponds to the Hilbert scale, but does not refer to an operator $\mathcal{A}$. For instance, the eigenbasis of the operator in Example 5.4 is the Fourier basis (see [57]), and not the wavelet basis. Thus we have constructed a Gaussian prior that is not related to the eigenbasis, but attains the same contraction rate.

It may be noted that the scale $(H_s)_{s\in\mathbb{R}}$ is well defined for every $s \in \mathbb{R}$, and with the preceding prior Theorem 6.7 is applicable to the full scale, and gives a contraction rate relative to the scale, which is optimal when $\beta = \alpha - d/2$. However, the scale agrees with the Sobolev scale only for $\beta < S$, and hence the optimality is in the Sobolev sense only if $\beta < S$. This restriction is typical when working with an approximation scheme such as wavelets or splines. One can of course choose a suitably large value of S, or may mix over multiple wavelet bases, as in the next section.

As mentioned in Section 6.1, there are many works on Bayesian inverse problems with Gaussian priors. The setup of the preceding theorem is similar to [1, 30], arguably closer to [1]. While we mainly treat the white noise case, our results can be extended to cover the noise structure in [1], and hence also cover the model in [30]. On the other hand, we differ from [1] in the following sense. First, unlike Assumption 3.1 in [1], our characterization of the smoothing property of the operator $\mathcal{A}$, i.e. Assumption 5.1, is simple, and in principle, our setup can also be extended to severely ill-posed problems, see Section 6.7. Second, our proof strategy is different, as we do not use Gaussian conjugacy, which is the main tool in [1]. This also allows us to obtain posterior contraction rates for non-conjugate priors in Section 6.3, and for Gaussian mixtures in Section 6.5.

6.5 Gaussian Mixtures

The posterior contraction rate resulting from a zero-mean Gaussian prior with covariance operator $L^{-2\alpha}$, as considered in Section 6.4, is equal to the minimax rate $n^{-\beta/(2\beta+2\gamma+d)}$ (see [19]) only when $\alpha - d/2 = \beta$, i.e., when the prior smoothness $\alpha - d/2$ matches the true smoothness β. By mixing over Gaussian priors of varying smoothness the minimax rate can often be obtained simultaneously for a range of values β (cf. [61], [98], [89]). In this section we consider mixtures of the mean-zero Gaussian priors with covariance operators $\tau^2 L^{-2\alpha}$ over the 'hyperparameter' τ. Thus the prior Π is the distribution of τF, where F is a zero-mean Gaussian variable in H_0 with covariance operator $L^{-2\alpha}$, as in Section 6.4, and τ is an independent scale parameter. The variable $1/\tau^a$ may be taken to possess a Gamma distribution for some given $0 < a \leq 2$, or, more generally, should satisfy the following mild condition.

Condition 6.10. The distribution Q of τ has support $[0, \infty)$ and satisfies

$$\begin{cases} -\log Q((t, 2t)) \lesssim t^{-2}, & \text{as } t \downarrow 0, \\ -\log Q((t, 2t)) \lesssim t^{d/(\alpha - d/2)}, & \text{as } t \to \infty. \end{cases}$$

Theorem 6.11 (Gaussian mixture prior)**.** *Consider a Hilbert scale $(H_s)_{s\in\mathbb{R}}$ generated by an operator L as in the preceding such that $L^{-1} : H_0 \to H_0$ is compact with eigenvalues λ_j satisfying $\lambda_j \simeq j^{-1/d}$. Suppose the operator $\mathcal{A} : H_0 \to G$ satisfies $\|\mathcal{A}f\| \simeq \|f\|_{-\gamma}$, assume that $f_0 \in H_\beta$, for some $\beta \in (0, \alpha]$, and let the prior be a mixture of the zero-mean Gaussian distributions with covariance operators $\tau^2 L^{-2\alpha}$ over the parameters τ equipped with a prior satisfying Condition 6.10, for some $\alpha > d/2$. Then the posterior distribution satisfies, for sufficiently large $M > 0$,*

$$\Pi_n\Big(f : \|f - f_0\|_0 > M n^{-\beta/(2\beta+2\gamma+d)} \mid Y^{(n)}\Big) \overset{P_{f_0}^{(n)}}{\to} 0.$$

The proof is given in Section 6.6.

6.6 Proofs

6.6.1 Proof of Theorem 6.5

The theorem is a corollary to Theorem 6.1 and uses arguments as in the proof of Proposition 3.2 in [80].

First we determine ε_n to satisfy the prior mass condition (6.6) of the direct problem. Let P_j be the projection onto the linear span of the first $j - 1$ basis elements ϕ_i. By the assumption on $\mathcal{A}$ and the triangle inequality, for any $i_n \in \mathbb{N}$,

$$\begin{aligned}\|\mathcal{A}f - \mathcal{A}f_0\| &\lesssim \|f - f_0\|_{-\gamma} \lesssim \|f - P_{i_n} f_0\|_{-\gamma} + \|P_{i_n} f_0 - f_0\|_{-\gamma} \\ &\lesssim \|f - P_{i_n} f_0\|_{-\gamma} + \delta(i_n, \gamma)\delta(i_n, \beta)\|f_0\|_\beta, \end{aligned} \qquad (6.17)$$

by (2.4), if $0 \leq \beta, \gamma < S$. Here $\delta(i_n, \gamma)\delta(i_n, \beta) = i_n^{-(\gamma+\beta)/d} \simeq \varepsilon_n$ if $i_n \simeq \varepsilon_n^{-d/(\gamma+\beta)}$.

By the orthogonality of the basis (ϕ_i), the function ϕ_j is orthogonal to the space V_j spanned by $(\phi_i)_{i<j}$. Hence $P_j \phi_j = 0$, so that $\|\phi_j\|_{-\gamma} \leq \delta(j, \gamma)\|\phi_j\|_0 \lesssim j^{-\gamma/d}$, for every j, by (2.4). Consequently, for $f = \sum_{i=1}^{i_n-1} f_i \phi_i \in V_{i_n}$ and $f_0 = \sum_i f_{0,i}\phi_i$, by the triangle inequality,

$$\|f - P_{i_n} f_0\|_{-\gamma} \lesssim \sum_{i=1}^{i_n-1} |f_i - f_{0,i}| i^{-\gamma/d}.$$

It follows that there exists a constant $a > 0$ such that

$$\begin{aligned}\Pi(f : \|f - P_{i_n} f_0\|_{-\gamma} < a\varepsilon) &\geq \Pi\Big(((f_i), M) : \sum_{i=1}^{i_n-1} |f_i - f_{0,i}| i^{-\gamma/d} < \varepsilon, M = i_n - 1\Big) \\ &\geq \prod_{i=1}^{i_n} \Pi\left(f_i : |f_i - f_{0,i}| < \frac{\varepsilon\, i^{\gamma/d}}{i_n}\right) \Pi(M = i_n - 1) \\ &\geq \prod_{i=1}^{i_n} \int_0^{\varepsilon\, i^{\gamma/d}/(\kappa_i i_n)} p\Big(x + \frac{f_{0,i}}{\kappa_i}\Big)\, dx\; e^{-b_1 i_n},\end{aligned}$$

in view of Condition 6.4. By (6.15) of the latter assumption, the integral $\int_0^r p(x + \mu)\, dx$ is bounded below by a constant times $re^{-C(r+|\mu|)^w}$. It follows that for ε such that $\varepsilon\, i^{\gamma/d}/(\kappa_i i_n) \leq 1$, for $i \leq i_n$, the preceding display is lower bounded by a multiple of

$$\varepsilon^{i_n} \Big[\prod_{i=1}^{i_n} \frac{i^{\gamma/d}}{\kappa_i i_n}\Big] \exp\Big[-C \sum_{i=1}^{i_n} \Big(1 + \frac{|f_{0,i}|}{\kappa_i}\Big)^w\Big]\, e^{-b_1 i_n}.$$

By (6.16), we have $i^{\gamma/d}/\kappa_i \gtrsim (1/i)^{\gamma/d-\alpha}$, which is bounded below by 1 if $\gamma/d - \alpha \geq 0$ and by $(1/i_n)^{\alpha-\gamma/d}$ otherwise, and hence always by $(1/i_n)^\alpha$. This shows that the first term in square brackets is bounded below by $(a_2/i_n^{\alpha+1})^{i_n}$, for some $a_2 > 0$. Since $f_0 \in H_\beta$, by assumption, the norm duality (2.1) gives that $|f_{0,i}| = |\langle f_0, \phi_i\rangle_0| \leq \|f_0\|_\beta \|\phi_i\|_{-\beta} \lesssim i^{-\beta/d}$. Together with (6.16) this gives that $|f_{0,i}|/\kappa_i \lesssim i^{(\beta_0-\beta)/d}(\log i)^{1/w} \leq (\log i)^{1/w}$, whence minus the exponent in the second term in square brackets is bounded by a multiple of $i_n(1 + (\log i_n)^{1/w})^w$. We conclude that there exists a constant $a_3 > 0$ such that

$$\Pi(f : \|f - P_{i_n} f_0\|_{-\gamma} < a\varepsilon) \geq \varepsilon^{i_n} e^{-a_3 i_n \log i_n} e^{-b_1 i_n},$$

for every $\varepsilon > 0$ such that $\varepsilon\, i^{\gamma/d}/(\kappa_i i_n) \leq 1$, for every $i \leq i_n$. Since $i^{\gamma/d}/\kappa_i \lesssim i^{(\gamma+\beta_0)/d}(\log i)^{1/w}$, again by (6.16), a sufficient condition for the latter is that $\varepsilon\, i_n^{(\gamma+\beta_0)/d}(\log i_n)/i_n \leq 1$.

Combining this with (6.17), we see that (6.6) is satisfied for ε_n such that there exists i_n with

$$i_n^{-(\gamma+\beta)/d} \lesssim \varepsilon_n, \qquad i_n \log i_n \lesssim n\varepsilon_n^2, \qquad \varepsilon_n i_n^{(\gamma+\beta_0)/d}(\log i_n) \leq i_n.$$

This leads to the rates

$$\varepsilon_n \simeq (\log n/n)^{(\beta+\gamma)/(2\beta+2\gamma+d)}, \qquad i_n \simeq (n/\log n)^{d/(2\beta+2\gamma+d)}.$$

(The third requirement is easily satisfied and remains inactive.) We can choose a sufficiently large proportionality constant in $\simeq$ when defining ε_n, so that (6.6) is satisfied for ε_n, since the left and right sides of (6.6) are increasing and decreasing in ε_n, respectively.

Since the Galerkin projection $f^{(j)}$ is equal to f itself if $f \in V_j$, we have that $\|f^{(j_n)} - f\|_0 = 0$ for the random series $f = \sum_{i=1}^M f_i\phi_i$ if $M < j_n$. By (ii) of Condition 6.4 it follows that, for some $b_2' > 0$ and every $\eta_n > 0$,

$$\Pi(f : \|f^{(j_n)} - f\|_0 > \eta_n) \leq \Pi(M \geq j_n) \leq e^{-b_2' j_n}.$$

Hence (6.7) is satisfied for $j_n = n\varepsilon_n^2/(4b_2')$. Thus we choose

$$j_n \simeq n^{d/(2\beta+2\gamma+d)}(\log n)^{(2\beta+2\gamma)/(2\beta+2\gamma+d)},$$

with a sufficiently large constant in $\simeq$. Then (6.3) is satisfied and it remains to solve η_n from (6.4) and (6.5). This leads to the inequalities

$$\eta_n \geq \varepsilon_n j_n^{\gamma/d} \simeq n^{-\beta/(2\beta+2\gamma+d)}(\log n)^{(1+2\gamma/d)(\beta+\gamma)/(2\beta+2\gamma+d)},$$
$$\eta_n \geq j_n^{-\beta/d} \simeq n^{-\beta/(2\beta+2\gamma+d)}(\log n)^{-\beta(2\beta+2\gamma)/((2\beta+2\gamma+d)d)}.$$

The rate is the maximum of the rates at the right hand sides, which coincides with the first rate. This concludes the proof.

6.6.2 Proof of Theorem 6.7

The theorem is a corollary to Theorem 6.1. The main tasks are to determine ε_n satisfying the prior mass condition (6.6) of the direct problem, and next to identify η_n from the prior mass condition (6.7) and the other conditions.

The first task is achieved in the following lemma.

Lemma 6.12. *Under the assumptions of Theorem 6.7, for $f_0 \in H_\beta$, as $\varepsilon \downarrow 0$,*

$$-\log\Pi(f : \|\mathcal{A}f - \mathcal{A}f_0\| < \varepsilon) \lesssim \begin{cases} \varepsilon^{-d/(\alpha+\gamma-d/2)}, & \text{if } d/2 < \alpha \leq \beta + d/2, \\ \varepsilon^{-(2\alpha-2\beta)/(\beta+\gamma)}, & \text{if } \alpha > \beta + d/2. \end{cases} \tag{6.18}$$

Proof. Since by assumption $\|\mathcal{A}f - \mathcal{A}f_0\| \simeq \|f - f_0\|_{-\gamma}$, the probability in the left side is the decentered small ball probability $\Pi(f : \|f - f_0\|_{-\gamma} < a\varepsilon)$ of the Gaussian random variable F distributed according to the prior and viewed as map into $H_{-\gamma} \supset H_0$, for some $a > 0$. Because F has covariance operator $L^{-2\alpha}$ as a map in H_0, its reproducing kernel Hilbert space (or Cameron-Martin space) $\mathbb{H}$ (which does not depend on its range space) is equal to the range of $L^{-\alpha}$ under the norm $\|L^{-\alpha}h\|_{\mathbb{H}}= \|h\|_0$ (see e.g., Example I.14 of [35]). Since $L^{-\alpha}: H_0 \to H_\alpha$ is a norm isometry, by (iii) of Proposition 2.8, this is the Hilbert space H_α with its natural norm $\|\cdot\|_\alpha$. The left side of (6.18) is therefore up to constants equivalent to

$$\inf_{h\in H_\alpha:\|h-f_0\|_{-\gamma}<\varepsilon}\|h\|_\alpha^2 - \log \Pi(\|f\|_{-\gamma}< \varepsilon). \tag{6.19}$$

See [64, 65, 99], or Section 11.2, in particular, Proposition 11.19 in [35].

By (2.4) $\|P_jf_0 - f_0\|_{-\gamma}\lesssim \delta(j,\gamma)\delta(j,\beta)\|f_0\|_\beta$, which is bounded above by ε for $j \simeq \varepsilon^{-d/(\beta+\gamma)}$. Thus for this value of j the first term in (6.19) is bounded above by

$$\|P_jf_0\|_\alpha \lesssim \begin{cases} \|P_jf_0\|_\beta, & \text{if } \alpha \le \beta,\\ 1/\delta(j,\alpha-\beta)\|P_jf_0\|_\beta, & \text{if } \alpha > \beta \end{cases}$$

by (2.11). Here $\|P_jf_0\|_\beta \le \|P_jf_0 - f_0\|_\beta + \|f_0\|_\beta \le (\delta(j,0)+1)\|f_0\|_\beta$, by (2.10). It follows that the contribution of the decentering in (6.19) is of order 1 if $\alpha \le \beta$ and is bounded above by a term of order $\varepsilon^{-2(\alpha-\beta)/(\beta+\gamma)}$ if $\alpha > \beta$.

By Lemma 2.22, the metric entropy $\log N(\varepsilon, \{f \in H_\alpha : \|f\|_\alpha \le 1\}, \|\cdot\|_{-\gamma})$ is of the order $\varepsilon^{-d/(\alpha+\gamma)}$. Hence, by [64] (see Lemma 6.2 in [100]),

$$-\log \Pi(\|f\|_{-\gamma}< \varepsilon) \simeq \varepsilon^{-d/(\alpha+\gamma-d/2)}.$$

Finally, the assertion of the lemma follows from discussion by cases. ☐

It follows that (6.6) is satisfied for

$$\varepsilon_n \ge n^{-(\beta\wedge(\alpha-d/2)+\gamma)/(2\alpha+2\gamma)}. \tag{6.20}$$

The next step of the proof is to bound the prior probability in (6.7).

Lemma 6.13. *Under the assumptions of Theorem 6.7, there exist $a, b > 0$, such that for every $j \in \mathbb{N}$ and $t > 0$,*

$$\Pi(f : \|f^{(j)} - f\|_0 > t + aj^{1/2-\alpha/d}) \le e^{-bt^2j^{2\alpha/d}}.$$

Proof. We have $f^{(j)} - f = (R_j\mathcal{A} - I)f$, for $R_j = \mathcal{A}^{-1}Q_j$. Therefore, the probability on the left concerns the random variable $(R_j\mathcal{A} - I)F$, if F is a variable distributed according to the prior Π. Since F is zero-mean normal with covariance operator $L^{-2\alpha}$, this variable is zero-mean Gaussian with covariance operator $(R_j\mathcal{A} - I)L^{-2\alpha}(R_j\mathcal{A} - I)^*$. We shall compute the weak and strong second moments of the variable $(R_j\mathcal{A} - I)F$, and next apply Borell's inequality for the norm of a Gaussian variable to obtain the exponential bound.

Because $\langle (R_j\mathcal{A} - I)F, g\rangle_0 = \langle F, (R_j\mathcal{A} - I)^*g\rangle_0$ is zero-mean Gaussian with variance $\|L^{-\alpha}(R_j\mathcal{A} - I)^*g\|_0^2 = \|(R_j\mathcal{A} - I)^*g\|_{-\alpha}^2$, the weak second moment of $(R_j\mathcal{A} - I)F$ is given by

$$\sup_{\|g\|_0\le 1} \mathbb{E}\langle (R_j\mathcal{A} - I)F, g\rangle_0^2 = \sup_{\|g\|_0\le 1} \|(R_j\mathcal{A} - I)^*g\|_{-\alpha}^2.$$

By the norm duality (2.1), the right side is equal to

$$\sup_{\|g\|_0\le 1}\sup_{\|f\|_\alpha\le 1}\langle f,(R_j\mathcal{A}-I)^*g\rangle_0^2\le\sup_{\|f\|_\alpha\le 1}\|(R_j\mathcal{A}-I)f\|_0^2\lesssim\delta(j,\alpha)^2.$$

in view of (5.9).

The strong second moment of the Gaussian variable $(R_j\mathcal{A}-I)F$ is equal to the trace of its covariance operator. As $\mathrm{Trace}(S^*S)=\sum_i\|S\phi_i\|^2=\sum_i\sum_j\langle S\phi_i,\phi_j\rangle^2=\sum_i\|S^*\phi_i\|^2$, for any orthonormal basis (ϕ_i) and operator S, we have

$$\mathbb{E}\|(R_j\mathcal{A}-I)F\|_0^2=\sum_{i\in\mathbb{N}}\|(R_j\mathcal{A}-I)L^{-\alpha}\phi_i\|_0^2.$$

For the orthonormal basis of eigenfunctions of L^{-1} and V_j the span of the first $j-1$ of these eigenfunctions, as in Proposition 2.9, $L^{-\alpha}V_j\subset V_j$, and hence $(R_j\mathcal{A}-I)L^{-\alpha}\phi_i$ vanishes for $i<j$. For $i\ge j$ the latter element is the difference $g^{(j)}-g$ of the Galerkin solution $g^{(j)}$ to $g=L^{-\alpha}\phi_i$. Therefore, by (5.9) the preceding display is bounded above by a multiple of

$$\sum_{i\ge j}\delta(i,\alpha)^2\|L^{-\alpha}\phi_i\|_\alpha^2=\sum_{i\ge j}\delta(i,\alpha)^2\|\phi_i\|_0^2\lesssim j^{1-2\alpha/d},$$

where we used the estimate $\sum_{i>j}i^{-b}\le j^{1-b}/(b-1)$, for $b>1$.

Since the first moment of $\|(R_j\mathcal{A}-I)F\|_0$ is bounded by the root of its second moment, the lemma follows by Borell's inequality (see e.g. Lemma 3.1 and subsequent discussion in [67]). □

For $t^2=4n\varepsilon_n^2/(bj_n^{2\alpha/d})$ and $j=j_n$ the bound in the preceding lemma becomes $e^{-4n\varepsilon_n^2}$. Hence (6.7) is satisfied for

$$\eta_n\gtrsim\sqrt{n}\varepsilon_n j_n^{-\alpha/d}+j_n^{1/2-\alpha/d}.$$

Here we choose ε_n the minimal solution that satisfies the direct prior mass condition (6.6), given in (6.20). Next we solve for η_n under the constraints (6.4) and (6.5). The first of these constraints, $j_n\le n\varepsilon_n^2$, shows that the first term on the right side of the preceding display always dominates the second term. Therefore, we obtain the requirements $j_n\le n\varepsilon_n^2$ and

$$\begin{aligned}
&\eta_n\ge\sqrt{n}\,n^{-(\beta\wedge(\alpha-d/2)+\gamma)/(2\alpha+2\gamma)}j_n^{-\alpha/d},\\
&\eta_n\ge n^{-(\beta\wedge(\alpha-d/2)+\gamma)/(2\alpha+2\gamma)}j_n^{\gamma/d},\\
&\eta_n\ge j_n^{-\beta/d}.
\end{aligned}$$

Depending on the relation between α and $\beta+d/2$, two situations need to be discussed separately.

(i) $\alpha\le\beta+d/2$. We choose $j_n\simeq n^{d/(2\alpha+2\gamma)}=n\varepsilon_n^2$ and then see that the first two requirements in the preceding display both reduce to $\eta_n\ge n^{-(\alpha-d/2)/(2\alpha+2\gamma)}$, while the third becomes $\eta_n\ge n^{-\beta/(2\alpha+2\gamma)}$ and becomes inactive.

(ii) $\alpha>\beta+d/2$. We choose $j_n\simeq n^{d/(2\alpha+2\gamma)}\le n\varepsilon_n^2$, and then see that all three requirements reduce to $\eta_n\ge n^{-\beta/(2\alpha+2\gamma)}$.

Finally, we apply Theorem 6.1 to complete the proof.

6.6.3 Proof of Theorem 6.11

Let Π_τ denote the zero-mean Gaussian distribution on H with covariance operator $\tau^2 L^{-2\alpha}$ (where $\alpha > d/2$).

Lemma 6.14. *Under the assumptions of Theorem 6.11, for $f_0 \in H_\beta$ and $\beta \leq \alpha$, as $\varepsilon \downarrow 0$,*

$$-\log \Pi_\tau(f : \|\mathcal{A}f - \mathcal{A}f_0\| < \varepsilon) \lesssim \frac{1}{\tau^2}\left(\frac{1}{\varepsilon}\right)^{(2\alpha-2\beta)/(\beta+\gamma)} + \left(\frac{\tau}{\varepsilon}\right)^{d/(\alpha+\gamma-d/2)}.$$

Lemma 6.15. *Under the assumptions of Theorem 6.11, for $f_0 \in H_\beta$ and $\beta \leq \alpha$, as $\varepsilon \downarrow 0$,*

$$-\log \Pi_\tau(f : \|f\|_0 < \varepsilon) \gtrsim \left(\frac{\tau}{\varepsilon}\right)^{d/(\alpha-d/2)}.$$

Lemma 6.16. *Under the assumptions of Theorem 6.11, there exist $a, b > 0$ such that, for every $j \in \mathbb{N}$ and $x, \tau > 0$,*

$$\Pi_\tau(f : \|f^{(j)} - f\|_0 > \tau x + \tau a j^{1/2-\alpha/d}) \leq e^{-bx^2 j^{2\alpha/d}}$$

Proofs. The proof of the first lemma follows the same lines as the proof of Lemma 6.12, except that now the Cameron-Martin space of the measure Π_τ on $H_{-\gamma}$ is H_α equipped with the norm $\|\cdot\|_{\mathbb{H}} = \frac{1}{\tau}\|\cdot\|_\alpha$ rather than its natural norm. The second lemma follows similarly, but considers the centered probability only. The third lemma is immediate from Lemma 6.13 as Π_τ is the law of τF, for F the Gaussian variable with the law Π as in the latter lemma, and the map $f \mapsto f^{(j)} - f$ is linear. □

As preparation for the proof of Theorem 6.11, we first show that the minimax rate can be obtained by a Gaussian prior with the deterministic scaling, dependent on β, given by

$$\tau_n = n^{(\alpha-d/2-\beta)/(2\beta+2\gamma+d)}. \tag{6.21}$$

Theorem 6.17. *Assume the conditions on the Hilbert scale, the forward operator A and the true parameter f_0 in Theorem 6.7 hold. Suppose that the priors Π are zero-mean Gaussian with covariance operators $\tau_n^2 L^{-2\alpha}$ with τ_n as given in (6.21) and $\alpha > d/2$. Then for $\beta \leq \alpha$, the posterior distribution satisfies, for sufficiently large $M > 0$,*

$$\Pi_n\Big(f : \|f - f_0\|_0 > M n^{-\beta/(2\beta+2\gamma+d)} \mid Y^{(n)}\Big) \overset{P_{f_0}^{(n)}}{\to} 0.$$

Proof. The theorem is a corollary to Theorem 6.1. The proof follows the same lines as the proof of Theorem 6.7. By Lemma 6.14, inequality (6.6) is satisfied for

$$\varepsilon_n \gtrsim n^{-(\beta+\gamma)/(2\beta+2\gamma+d)}.$$

By Lemma 6.16, inequality (6.7) is satisfied for

$$\eta_n \gtrsim \tau_n(\sqrt{n}\varepsilon_n j_n^{-\alpha/d} + j_n^{1/2-\alpha/d}).$$

We choose $j_n \simeq n\varepsilon_n^2$, and the minimal solution $\varepsilon_n = n^{-(\beta+\gamma)/(2\beta+2\gamma+d)}$ to the second last display. It is then straightforward to verify that (6.4), (6.5) and (6.7) are satisfied for $\eta_n \simeq n^{-\beta/(2\beta+2\gamma+d)}$. ☐

Theorem 6.11 is a corollary of Theorem 6.3, with the choices

$$\begin{aligned}\eta_n &\simeq n^{-\beta/(2\beta+2\gamma+d)}, & \varepsilon_n &\simeq n^{-(\beta+\gamma)/(2\beta+2\gamma+d)},\\ j_n &\simeq n\varepsilon_n^2 = n^{d/(2\beta+2\gamma+d)}.\end{aligned}$$

Conditions (6.3), (6.4), and (6.5) are satisfied for these choices. It remains to verify (6.6), and eqs. (6.12) and (6.13).

For ease of notation, for the moment, define η_n and ε_n as in the preceding display, with exact equality (i.e., with the constant set equal 1). Let τ_n be the 'optimal' scaling rate defined in (6.21).

Verification of (6.6). For $\tau \simeq \tau_n$ and $\varepsilon \simeq \varepsilon_n$ as given and $\beta \leq \alpha$, both terms in the right side of Lemma 6.14 are of the order $n\varepsilon_n^2$. The lemma yields, for $\tau_n \leq \tau \leq 2\tau_n$ and some constant $a_1 > 0$,

$$-\log \Pi_\tau(f : \|Af - Af_0\| < \varepsilon_n) \leq a_1 n\varepsilon_n^2.$$

This shows that

$$\begin{aligned}\Pi(f : \|Af - Af_0\| < \varepsilon_n) &= \int_0^\infty \Pi_\tau(f : \|Af - Af_0\| < \varepsilon_n)\, dQ(\tau)\\ &\geq e^{-a_1 n\varepsilon_n^2} Q(\tau_n, 2\tau_n).\end{aligned}$$

If $\alpha - d/2 < \beta$, then $\tau_n \to 0$, and Condition 6.10 on Q gives that

$$-\log Q(\tau_n, 2\tau_n) \lesssim \tau_n^{-2} = n^{(2\beta-2\alpha+d)/(2\beta+2\gamma+d)} \leq n^{d/(2\beta+2\gamma+d)} = n\varepsilon_n^2,$$

if $\beta \leq \alpha$. If $0 < \beta < \alpha - d/2$, then $\tau_n \to \infty$, and Condition 6.10 on Q gives that

$$\begin{aligned}-\log Q(\tau_n, 2\tau_n) \lesssim \tau_n^{d/(\alpha-d/2)} &= n^{(d(\alpha-d/2-\beta)/(\alpha-d/2)(2\beta+2\gamma+d))}\\ &\leq n^{d/(2\beta+2\gamma+d)} = n\varepsilon_n^2.\end{aligned}$$

Finally if $\alpha - d/2 = \beta$, then $\tau_n = 1$ and $Q(\tau_n, 2\tau_n) \gtrsim 1$. Thus in all three cases $Q(\tau_n, 2\tau_n)$ is bounded below by a power of $e^{-n\varepsilon_n^2}$. Combining this with the preceding, we see that $\Pi(f : \|Af - Af_0\| \leq \varepsilon_n) \geq e^{-a_2 n\varepsilon_n^2}$, for some positive constant a_2, which we can take bigger than 1. Then (6.6) is satisfied for ε_n equal to $\sqrt{a_2}$ times the current ε_n.

Verification of (6.12). Lemma 6.15 gives that

$$\Pi_\tau(f : \|f - f_0\|_0 < 2\eta_{n,\tau}) \leq \Pi_\tau(f : \|f\|_0 < 2\eta_{n,\tau}) \leq e^{-a_3(\tau/\eta_{n,\tau})^{d/(\alpha-d/2)}},$$

for some constant a_3. This is bounded above by $e^{-4a_2 n\varepsilon_n^2}$ if

$$\eta_{n,\tau} = 2a_4\tau\, n^{(d/2-\alpha)/(2\beta+2\gamma+d)} = 2a_4\tau\, \eta_n/\tau_n,$$

for a sufficiently small constant $a_4 > 0$.

Verification of (6.13). Choosing $x = a_4\eta_n/\tau_n = \eta_{n,\tau}/(2\tau)$ in Lemma 6.16, we see that the left side of (6.13) is bounded above by $e^{-4a_2n\varepsilon_n^2}$ if j_n satisfies

$$a j_n^{1/2-\alpha/d} \leq a_4\eta_n/\tau_n, \qquad \text{and} \qquad b a_4^2(\eta_n/\tau_n)^2 j_n^{2\alpha/d} \geq 4a_2 n\varepsilon_n^2.$$

Both inequalities become equalities for j_n of the order $j_n \simeq n^{d/(2\beta+2\gamma+d)}$, as indicated at the beginning of the proof. Since $1/2-\alpha/d < 0$ and $2\alpha/d > 0$, the left side of the first inequality is decreasing in j_n and the left side of second inequality is increasing. Thus both inequalities are satisfied for $j_n = a_5 n^{d/(2\beta+2\gamma+d)}$ and a sufficiency large constant a_5.

Finally we choose ε_n and j_n in Theorem 6.3 equal to $\sqrt{a_2}$ and a_5 times the orders indicated at the beginning of the proof. Then (6.3) is satisfied, and (6.4) and (6.5) are satisfied if η_n is chosen of the indicated order times a sufficiently large constant.

6.7 Discussion and Comments

In this section we comment on the present setup and discuss directions in which the results in this chapter can be extended.

Coloured Noise

We have examined the case that the noise ξ in model (6.1) is white noise. Statistical estimation in the case that the noise is a proper centred Gaussian random element in G, as studied in [30], is easier in terms of minimax rates (if in both cases the noise is scaled to the same unit), as this would imply that the noise is less variable. By inspection of our proofs one sees that the concentration inequalities that drive the testing criterion remain valid if the covariance operator of the noise is bounded above by the identity, as is assumed in [1, 7]. As a consequence, the proof of Theorem 6.1 goes through and the theorem remains valid, as do the corollaries in the later sections. However, for truly coloured noise the result may be suboptimal, as one may expect a faster posterior contraction rate, which will incorporate the decrease of the noise variance in certain directions. The methods of the present chapter can be adapted to this case as long as the covariance operator fits the scale of smoothness classes, as in [30]. A sharp result in full generality may be difficult to attain, as it will be the outcome of the interaction of the directions of decrease in the noise, the true parameter and the prior.

Approximation Numbers of Embeddings

In the corollaries to the main result we have assumed that the approximation numbers $\delta(j,s)$ of the canonical embedding $\iota : H_s \to H_0$ are of polynomial order $j^{-s/d}$. This order matches the approximation numbers of Sobolev spaces on d-dimensional, bounded domains, and seems common. Other decay rates do arise, e.g., an exponential rate in severely ill-posed problems (as in the heat equation considered in [60]), or a logarithmic rate (as in [14]). The general Theorem 6.1 remains valid, but its corollaries must be adapted. For Gaussian priors in logarithmic or exponential scales, this is relatively straightforward using the general theory of approximation numbers, which relates these to singular values and metric entropy. Some results can be found in Section 10.4.